A Nonlinear Two Phase Fluid Flow Through a Porous Medium in Presence of a Well

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Abstract. We study a flow of fresh and salt water in a two dimensional axially symmetric coastal aquifer with a well on the central axis. The flow is governed by a nonlinear Darcy's law. We also show the behaviour of the solution when the out flow of salt water at well goes to 0.

Introduction

When fresh water is stored in aquifers in coastal areas, as a result of sea water intrusion, one may encounter situations where wells pump back the fresh water from regions above a fresh-salt interface.
As a result of the pumping and the pressure distribution established in the fluids, the interface below a well will rise, at first slowly and later faster, towards that well. The interface will eventually touch the well in a cusp-like form (see [18]). This situation will be investigated in this paper.
H.W. Alt and J.V. Duijn studied in [5] the solutions for flows in a bounded axial symmetric reservoir of constant thickness, with only one well on the central axis. The model which they considered was described by a linear Darcy's law.
In this paper, we are interested with the same model in the two dimensional case and where the flow is governed by a nonlinear Darcy's law (see [12], [14], [15]). For a weak power-law exponent (i.e. $0 < m < 1$), we look for our solution in a classical Sobolev space. When the power-law exponent is greater than 1 (i.e. $m > 1$), the problem is still open.
In section 1, we state the problem and present the weak formulation of the model. We prove the existence of monotone solutions and establish some properties. In section 2, we show that the interface is a continuous curve. On the axis it ends up in the well and on the lateral boundary in a well-defined point. The last section is devoted to study the limit problem obtained when the outflow of salt water at

well goes to 0. We prove that the limit problem has a unique monotone solution with a decreasing continuous free boundary.
For more information about this subject, we refer the reader to [3], [4], [5], [6], [18], [12], [13], [16].

1 Statement of the problem

1.1 Strong formulation

Let $\tilde{\Omega} = (-a, a) \times (0, 1)$ $(a > 0)$ be a porous medium where a well W is located at the position $(0, h)$, $(0 < h < 1)$. The porous medium is saturated by fresh and salt water which are being extracted through W with respective discharge Q_f and Q_s. The flow is governed by the following nonlinear Darcy law :

$$\frac{\mu}{k}|v|^{m-1}v + \nabla p + \gamma e_z = 0 \qquad \text{in } \tilde{\Omega} \quad (m > 0) \tag{1.1}$$

where v denotes the fluid velocity, p its pressure, e_z the unit vertical vector of $I\!R^2$ i.e. $e_z = (0, 1)$, μ is the dynamic viscosity of the two fluids, $k > 0$ is the constant permeability of the medium and γ is the constant specific weight ($\gamma = \gamma_f$ for fresh water and $\gamma = \gamma_s$ for salt water with $0 < \gamma_f < \gamma_s$).
Moreover, we have the fluid balance equation (see [11]) :

$$div(v) = -2Q\delta_W \qquad \text{in } \tilde{\Omega} \tag{1.2}$$

where $Q = Q_f + Q_s$ is the total production rate of the fluid and δ_W is the Dirac distribution at W.
We assume the upper and lower boundary of $\tilde{\Omega}$ to be impervious i.e.

$$v.e_z = 0 \qquad \text{on } [z = 0] \cup [z = 1]. \tag{1.3}$$

At the lateral sides of $\tilde{\Omega}$, the flow is assumed to be horizontal only i.e.

$$v.e_z = 0 \qquad \text{on } [x = a] \cup [x = -a]. \tag{1.4}$$

Now, due to the symmetry of $\tilde{\Omega}$ with respect to the axis $[x = 0]$, we expect a symmetry of the unknown (p, γ). So we can restrict our study to the domain $\Omega = (0, a) \times (0, 1)$. Furthermore, we deduce from (1.2) the existence of a stream function $\psi : \Omega \to I\!R$ such that :

$$v = Rot\psi = (-\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x}) \qquad \text{in } \Omega. \tag{1.5}$$

Let us make the following normalization :

$$\begin{array}{rcl} \psi : & = & \psi.\big(\mu/(k(\gamma_s - \gamma_f))\big)^{1/m}, \qquad Q, Q_s \text{ and } Q_f \text{ similar} \\ \gamma : & = & (\gamma - \gamma_f)/(\gamma_s - \gamma_f). \end{array}$$

Then from (1.1), (1.2) and (1.5), we deduce

$$div(|\nabla\psi|^{q-2}\nabla\psi+\gamma e_x)=0 \qquad \text{in } \Omega \qquad \text{with } q=m+1. \tag{1.6}$$

Using (1.3), we deduce that ψ is constant on $[z=0]\cup[z=1]$. Set

$$\psi(x,0)=c_1 \quad \text{and} \quad \psi(x,1)=c_2 \qquad \text{for } x\in(0,a).$$

Since $p(x,z)=p(-x,z)$ in $\tilde{\Omega}$, we get if everything is smooth, $\dfrac{\partial p}{\partial x}(0,z)=0$ for all $z\neq h$. This leads by (1.1) and (1.5) to

$$\frac{\partial\psi}{\partial z}(0,z)=0 \qquad \text{for } z\neq h. \tag{1.7}$$

Thus if we assume that ψ is continuous at $(0,0)$ and $(0,1)$, we get

$$\psi(0,z)=c_1 \quad \text{for } 0\leq z<h, \qquad \psi(0,z)=c_2 \quad \text{for } h<z\leq 1.$$

Let us take $c_1=0$ and determine c_2. For this purpose, we consider the rectangle $C_\varepsilon^\delta=(-\varepsilon,\varepsilon)\times(h-\delta,h+\delta)$ for δ and ε positive and small enough. Let $\zeta\in\mathcal{D}(\tilde{\Omega})$ be an even function with respect to x such that : $\zeta=1$ in $\overline{C}_\varepsilon^\delta$. Using (1.2) and the symmetry of (p,γ), we get

$$-2Q=-2\int_\Omega v.\nabla\zeta=-2\int_{\partial(\Omega\setminus C_\varepsilon^\delta)} v.\nu\zeta$$

where ν is the outward unit vector normal to $\partial(\Omega\setminus C_\varepsilon^\delta)$.
By (1.5) and (1.7), we then get :

$$-2Q=-2\Big([\psi(0,h+\delta)-\psi(\varepsilon,h+\delta)]+[\psi(\varepsilon,h+\delta)-\psi(\varepsilon,h-\delta)]+[\psi(\varepsilon,h-\delta)-\psi(0,h-\delta)]\Big).$$

Letting ε go to 0 with δ fixed, we obtain

$$\psi(0,h+\delta)-\psi(0,h-\delta)=Q.$$

So $c_2=Q$ and we have

$$\begin{cases}\psi(x,0)=0, \qquad \psi(x,1)=Q \qquad \text{for } x\in[0,a]\\ \psi(0,z)=0 \quad \text{for } 0\leq z<h, \qquad \psi(0,z)=Q \quad \text{for } h<z\leq 1.\end{cases} \tag{1.8}$$

From (1.4), we have :

$$\frac{\partial\psi}{\partial x}(a,z)=0 \qquad \text{for } z\in(0,1). \tag{1.9}$$

Since we assume that the two fluids are unmixed, we have : $v.\nu=0$ on the interface Γ of fresh and salt water (ν is the unit normal to Γ). So ψ is constant along Γ.

To insure that the prescribed salt water discharge Q_s is being extracted from the reservoir $\tilde{\Omega}$, we take

$$\psi = Q_s \qquad \text{on } \Gamma. \tag{1.10}$$

By (1.6), (1.8), (1.9), the weak and the strong maximum principles, we get :

$$0 < \psi < Q_s \quad \text{in } \Omega_s \qquad \text{and} \qquad Q_s < \psi < Q \quad \text{in } \Omega_f$$

where Ω_s (resp. Ω_f) is the subset of Ω containing salt (resp. fresh) water. Therefore we are led to consider the following problem :

$$(SF) \qquad \begin{cases} \text{Find } (\psi, \gamma) \text{ such that } : & \\ div(|\nabla\psi|^{q-2}\nabla\psi + \gamma e_x) = 0 & \text{in } \Omega \\ \qquad \psi = 0 & \text{on AOW} \\ \qquad \psi = Q & \text{on WCB} \\ \qquad \frac{\partial \psi}{\partial x} = 0 & \text{on AB} \\ \qquad \gamma \in 1 - H(\psi - Q_s). & \end{cases}$$

where H is the Heaviside graph. For the location of the points A, O, B and C, we refer to Figure 1.

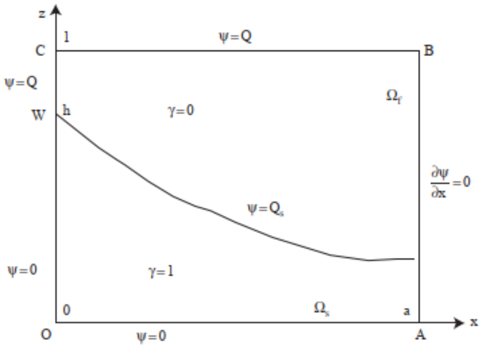

Figure 1

1.2 Weak formulation

The natural space in which we have to look for ψ is $W^{1,q}(\Omega)$. However since ψ is not continuous at the point W and due to the Sobolev imbedding $W^{1,q}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $q > 2$, we have to restrict ourselves to the case $1 < q < 2$. The case $q = 2$ is considered by H.W. Alt and V. Duijn in [6].

So throughout this paper, we assume that q lies in $(1, 2)$. First, we prove the following result :

Lemma 1.1. *There exists a function $\psi_0 \in W^{1,q}(\Omega)$ such that :*

$$\begin{cases} \psi_0 = 0 & \text{on } AOW \\ \psi_0 = Q & \text{on } WCB \\ \frac{\partial \psi_0}{\partial x} = 0 & \text{on } AB. \end{cases}$$

Proof. Let $s \in (\frac{3}{2}, 2]$. We consider the following problem :

$$(P_0) \qquad \begin{cases} \text{Find } p_0 \text{ such that } : \\ div(|\nabla p_0|^{s-2}\nabla p_0) = 2Q\delta_W & \text{in } \mathcal{D}'(\tilde{\Omega}) \\ \qquad p_0(a,z) = p_0(-a,z) = 0 & \text{for } z \in (0,1) \\ \qquad |\nabla p_0|^{s-2}\nabla p_0.e_z = 0 & \text{for } z \in \{0,1\}. \end{cases}$$

To solve (P_0), we consider the following approximated problem :

$$(P_n) \qquad \begin{cases} \text{Find } p_n \in W^{1,s}(\tilde{\Omega}) \text{ satisfying } : \\ i) \quad p_n = 0 \qquad \text{for } x = \pm a \\ ii) \quad \int_{\tilde{\Omega}} |\nabla p_n|^{s-2}\nabla p_n.\nabla\zeta = \int_{\tilde{\Omega}} f_n\zeta \\ \qquad\qquad \forall \zeta \in W^{1,s}(\tilde{\Omega}) \text{ such that } \zeta = 0 \text{ for } x = \pm a \end{cases}$$

where $f_n = \dfrac{2Qn^2}{\pi}\chi\big(B(W, \frac{1}{n})\big)$, $n \geq 1$, $B(W, \frac{1}{n})$ is the open ball with center W and radius $1/n$ and $\chi(E)$ denotes the characteristic function of the set E. It is well known (see [28]) that (P_n) has a unique solution $p_n \in W^{1,s}(\tilde{\Omega})$. Moreover p_n is symmetric with respect to x.
Arguing as in [10], where a similar problem with Dirichlet Boundary conditions is considered, one can prove that (p_n) converges to an element p_0 in any $W^{1,t}(\tilde{\Omega})$ with $t \in [1, 2(s-1))$ and p_0 is a solution of problem (P_0). Note that we have :

$$\begin{cases} i) \quad p_0 \text{ is a non constant s-harmonic function in } \tilde{\Omega} \setminus W \\ ii) \quad p_0 \in C^1(\tilde{\Omega} \setminus W) \quad \text{(see [25], [17], [21], [33])} \\ iii) \quad p_0 \in C^1(\overline{\tilde{\Omega}} \setminus \{W, A, B, A', B'\}) \\ \qquad \text{with } A' = (-a, 0), \quad B' = (-a, 1) \quad \text{(see [27]).} \end{cases}$$

From $i)$ and since Ω is a simply connected domain, we deduce (see [7]) that there exists an s′-harmonic function $\psi_0 : \Omega \to I\!R$ with $\dfrac{1}{s'} + \dfrac{1}{s} = 1$ such that

$$Rot\psi_0 = |\nabla p_0|^{s-2}\nabla p_0 \qquad \text{in } \Omega. \tag{1.11}$$

So we have $|\nabla\psi_0| = |\nabla p_0|^{s-1} \in L^{t/(s-1)}(\Omega)$. Then if we choose $t \in \big(\max(1, q(s-1)), 2(s-1)\big)$, we obtain

$$\nabla\psi_0 \in L^q(\Omega). \tag{1.12}$$

Now for the boundary conditions satisfied by ψ_0, we first have : $\dfrac{\partial p_0}{\partial z}(a,z) = 0$ since $p_0(a,z) = 0$ then by (1.11), we get

$$\frac{\partial\psi_0}{\partial x} = 0 \qquad \text{on } AB. \tag{1.13}$$

We also have by (1.11) and the Neuman condition satisfied by p_0 : $\dfrac{\partial\psi_0}{\partial x} = 0$ on $OA \cup CB$. Hence ψ_0 is constant on OA and CB. Using the symmetry of p_0, we obtain $\dfrac{\partial p_0}{\partial x}(0,z) = 0$ for all $z \neq h$ and then by (1.11), we get $\dfrac{\partial\psi_0}{\partial z}(0,z) = 0$ for all $z \in (0,h) \cup (h,1)$ which means that ψ_0 is constant on OW and CW.
Now using the continuity of ψ_0 at O and C, we can choose

$$\psi_0 = 0 \qquad \text{on } AOW \tag{1.14}$$

and we deduce as in section 1.1 that

$$\psi_0 = Q \qquad \text{on } WCB. \tag{1.15}$$

Combining (1.12), (1.14) and the Poincaré-inequality, we get $\psi_0 \in W^{1,q}(\Omega)$. □

Using the strong formulation (SF) and Lemma 1.1, we are led to the following weak formulation of our problem :

$$(P) \quad \begin{cases} \text{Find } (\psi,\gamma,\gamma_N) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \times L^\infty(\Gamma_N) \text{ satisfying :} \\ i) \quad \int_\Omega (|\nabla\psi|^{q-2}\nabla\psi + \gamma e_x).\nabla\zeta = \int_{\Gamma_N} \gamma_N \zeta \\ \qquad\qquad \forall\zeta \in W^{1,q}(\Omega) \text{ such that } \zeta = 0 \text{ on } \Gamma_D \\ ii) \quad \gamma \in 1 - H(\psi - Q_s) \text{ a.e. in } \Omega, \quad \gamma_N \in 1 - H(\psi - Q_s) \text{ a.e. on } \Gamma_N \\ iii) \quad \psi = \psi_0 \qquad \text{on } \Gamma_D \end{cases}$$

with $\Gamma_N = AB$, $\Gamma_D = AOCB$. Then we have

Theorem 1.2. *There exists a solution (ψ,γ,γ_N) of (P) satisfying :*

$$\partial_z\psi \geq 0, \quad \partial_z\gamma \leq 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{and} \quad \partial_z\gamma_N \leq 0 \quad \text{in } \mathcal{D}'(\Gamma_N). \tag{1.16}$$

Proof. For $\varepsilon > 0$, we introduce the following approximated problem :

$$(P_\varepsilon) \quad \begin{cases} \text{Find } \psi_\varepsilon \in W^{1,q}(\Omega) \text{ such that :} \\ i) \quad \int_\Omega (|\nabla\psi_\varepsilon|^{q-2}\nabla\psi_\varepsilon + (1 - H_\varepsilon(\psi_\varepsilon - Q_s))e_x).\nabla\zeta \\ \qquad = \int_{\Gamma_N} (1 - H_\varepsilon(\psi_\varepsilon - Q_s))\zeta \\ \qquad\qquad \forall\zeta \in W^{1,q}(\Omega) \text{ such that } \zeta = 0 \text{ on } \Gamma_D \\ ii) \quad \psi_\varepsilon = \psi_0 \qquad \text{on } \Gamma_D \end{cases}$$

where $H_\varepsilon(s) = \min(1, \dfrac{s^+}{\varepsilon})$.
Arguing as in [14], [12], we can prove that there exists a unique solution ψ_ε of (P_ε) satisfying

$$\partial_z\psi_\varepsilon \geq 0 \qquad \text{in } \mathcal{D}'(\Omega)$$

and up to a subsequence, we have

$$\psi_\varepsilon \longrightarrow \psi \qquad \text{in } W^{1,q}(\Omega)$$

$$\big(1-H_\varepsilon(\psi_\varepsilon-Q_s),1-H_\varepsilon(\psi_\varepsilon-Q_s)\big)\rightharpoonup(\gamma,\gamma_N)\qquad\text{in } L^{q'}(\Omega)\times L^{q'}(\Gamma_N).$$

Then we verify that (ψ,γ,γ_N) is a solution of (P). By the monotonicity of ψ_ε and H_ε, we get (1.16). □

Proposition 1.3. *Let (ψ,γ,γ_N) be a solution of (P). We have :*

$$\Delta_q\psi+\partial_x\gamma=div(|\nabla\psi|^{q-2}\nabla\psi+\gamma e_x)=0\qquad in\ \mathcal{D}'(\Omega)\tag{1.17}$$

$$0\le\psi\le Q\qquad in\ \Omega\tag{1.18}$$

$$\psi\in C^{0,\alpha}_{loc}(\overline{\Omega}\setminus\{W\})\qquad for\ 0<\alpha<1.\tag{1.19}$$

Proof. i) (1.17) is a consequence of $(P)i)$ by taking $\zeta\in\mathcal{D}(\Omega)$.
ii) (1.18) is obtained by taking ψ^- and $(\psi-Q)^+$ as test functions in $(P)i)$.
iii) First, we deduce (see [20], [23], [32]) that

$$\psi\in C^{0,\alpha}_{loc}(\Omega\cup\Gamma_D\setminus\{O,W,C\})\qquad\text{for } 0<\alpha<1.$$

It remains to prove the $C^{0,\alpha}$ continuity of ψ on $\overline{\Gamma}_N\cup\{O,C\}=[AB]\cup\{O,C\}$.

Case of the point O : Let $\tilde\psi$ and $\tilde\gamma$ defined by :

$$\tilde\psi(x,z)=\begin{cases}\psi(x,z) & \text{in } \Omega\\ -\psi(-x,z) & \text{in } \tilde\Omega\setminus\Omega\end{cases}\qquad\text{and}\qquad\tilde\gamma(x,z)=\begin{cases}\gamma(x,z) & \text{a.e. in } \Omega\\ -\gamma(-x,z) & \text{a.e. in } \tilde\Omega\setminus\Omega\end{cases}$$

It is clear that $(\tilde\psi,\tilde\gamma)\in W^{1,q}(\tilde\Omega)\times L^\infty(\tilde\Omega)$. Furthermore we deduce from i) that :

$$\Delta_q\tilde\psi+\tilde\gamma_x=0\qquad\text{in } \mathcal{D}'(\tilde\Omega).\tag{1.20}$$

Now, since we have $\tilde\psi(x,0)=0$ for $x\in(-a,a)$, we deduce from (1.20) (see [20]) that $\tilde\psi$ is $C^{0,\alpha}$ near $[z=0]$. Thus ψ is $C^{0,\alpha}$ near O.

Case of the point C : Let $\Omega_h=(-a,a)\times(h,1)$. Then we have $(\widetilde{\psi-Q})\in W^{1,q}(\Omega_h)$ and

$$\Delta_q(\widetilde{\psi-Q})+\tilde\gamma_x=0\qquad\text{in } \mathcal{D}'(\Omega_h).\tag{1.21}$$

Since $(\widetilde{\psi-Q})(x,1)=0$ for $x\in(-a,a)$, we deduce from (1.21) (see [20]) that $(\widetilde{\psi-Q})$ is $C^{0,\alpha}$ near $[z=1]$. Thus ψ is $C^{0,\alpha}$ near C.

Case of $\overline{\Gamma}_N=[AB]$: Set $\Omega^*=(0,2a)\times(0,1)$ and define ψ^* and γ^* by :

$$\psi^*(x,z)=\begin{cases}\psi(x,z) & \text{in } \Omega\\ \psi(2a-x,z) & \text{in } \Omega^*\setminus\Omega\end{cases}$$

$$\text{and}\qquad\gamma^*(x,z)=\begin{cases}\gamma(x,z) & \text{a.e. in } \Omega\\ -\gamma(2a-x,z) & \text{a.e. in } \Omega^*\setminus\Omega\end{cases}$$

then $(\psi^*, \gamma^*) \in W^{1,q}(\Omega^*) \times L^\infty(\Omega^*)$ and satisfy :

$$\Delta_q \psi^* + (\gamma^* - 2\gamma_N \chi(\Omega))_x = 0 \qquad \text{in } \mathcal{D}'(\Omega^*). \tag{1.22}$$

Since we have $\psi^*(x, 0) = 0$ and $\psi^*(x, 1) = 0$ for $x \in (0, 2a)$, we deduce from (1.22) (see [32], [20]) that $\psi^* \in C^{0,\alpha}_{loc}(\Omega^* \cup [z = 0] \cup [z = 1])$. Thus ψ is $C^{0,\alpha}_{loc}(\Omega \cup \overline{\Gamma}_N)$. □

Corollary 1.4. *Let (ψ, γ, γ_N) be a solution of (P) that satisfies (1.16). We have :*

i) $[\psi > Q_s]$ *and* $[\psi < Q_s]$ *are open connected sets where ψ is not constant.*

ii) $\psi \in C^{1,1}_{loc}([\psi > Q_s])$ *and* $\psi \in C^{1,1}_{loc}([\psi < Q_s])$.

iii) ψ *is analytic in* $[\psi > Q_s] \setminus S$ *(resp.* $[\psi < Q_s] \setminus S$*) where* $S = \{\, (x, z) \in \Omega \setminus [\psi = Q_s] \,/\, \nabla\psi(x, z) = 0 \,\}$.

iv) S *is a discrete set.*

Proof. i) $[\psi > Q_s]$ is open by continuity of ψ. Let (x_1, z_1), $(x_2, z_2) \in [\psi > Q_s]$. By monotonicity of ψ we have

$$\psi(x_1, z) > Q_s \quad \text{for } z_1 \le z \le 1 \qquad \text{and} \qquad \psi(x_2, z) > Q_s \quad \text{for } z_2 \le z \le 1.$$

Since we have $\psi(x, 1) = Q > Q_s$, we deduce by continuity of ψ that there exists a small $\varepsilon > 0$ such that

$$\psi > Q_s \qquad \text{in } [x_1 - \varepsilon, x_2 + \varepsilon] \times [1 - \varepsilon, 1].$$

Then the arc $\{x_1\} \times [z_1, 1 - \varepsilon] \cup [x_1, x_2] \times \{1 - \varepsilon\} \cup \{x_2\} \times [1 - \varepsilon, z_2] \subset [\psi > Q_s]$. Now assume that ψ is constant in $[\psi > Q_s]$. Then $\psi = Q$ in $[\psi > Q_s]$. If $\partial[\psi > Q_s] \cap \Omega = \emptyset$ then $[\psi > Q_s] = \Omega$ which contradicts the fact that $\psi = 0$ on OA. So $\partial[\psi > Q_s] \cap \Omega \neq \emptyset$ and $\psi = Q_s$ on this set. Contradiction.
In the same way, we prove that $[\psi < Q_s]$ is also a domain on which ψ is not constant.

ii) In $[\psi > Q_s]$ (resp. $[\psi < Q_s]$), we have $\gamma = 0$ (resp. $\gamma = 1$) then by (1.17) we get $\Delta_q \psi = 0$ in $\mathcal{D}'([\psi > Q_s])$ (resp. $\mathcal{D}'([\psi < Q_s])$) from which we deduce ii) (see [2]).

iii) (see [26]).

iv) (see [2], [30]). □

2 Study of the free boundary

In this section, we shall be concerned only with monotone solutions of (P) in the sense of (1.16). For these kind of solutions, we will study the corresponding free boundary $\Gamma = [\psi = Q_s]$.

Using the continuity of ψ on OA and CB, one can define g_1, $g_2 : (0,a) \to I\!R$ by :

$$g_1(x) = \sup\{\, z/\, \psi(x,z) < Q_s \,\} \qquad \text{and} \qquad g_2(x) = \inf\{\, z/\, \psi(x,z) > Q_s \,\}.$$

Then we deduce from the continuity and the monotonicity of ψ :

Proposition 2.1.

i) g_1 *(resp.* g_2*) is lower (resp. upper) semi-continuous in* $(0,a)$.

ii) $[\psi(x,z) < Q_s] = [z < g_1(x)], \qquad [\psi(x,z) > Q_s] = [z > g_1(x)].$

iii) $\Gamma = [\psi = Q_s] = [g_1(x) \le z \le g_2(x)].$

The main result of this section is the following :

Theorem 2.2. *We have :*

i) $g_1 = g_2 = g, \qquad \Gamma = [z = g(x)]$ *and* $g : (0,a) \to I\!R$ *is continuous*

ii) $\displaystyle\lim_{x \to 0^+} g(x) = h$

iii) $\displaystyle\lim_{x \to a_-} g(x) = u_s \in (0,1).$

The proof of this theorem will be established through some lemmas. The first one (Lemma 2.3) is a non-oscillation lemma. Lemma 2.4 and Lemma 2.5 are used to eliminate possible vertical segments of the free boundary.
Note that similar lemmas are proved by H.W. Alt and V. Duijn in [6] in the case where $q = 2$. Our proof follows their ideas, however, we have to overcome some difficulties due to the nonlinearity of the q-Laplacian operator and through lack of the regularity of ψ near points of the set S.

Lemma 2.3. *Let* $0 < z_1 < z_2 < 1$, $0 < x_1 < x_2 < x_3 < x_4 < a$. *Set*

$$l_i = \{x_i\} \times [z_1, z_2] \qquad i = 1,2,3,4.$$

Suppose that $\psi - Q_s < 0$ *on* l_1, $\psi - Q_s > 0$ *on* l_2, $\psi - Q_s < 0$ *on* l_3 *and* $\psi - Q_s > 0$ *on* l_4. *Then*

$$z_2 - z_1 = O(x_4 - x_1)$$

i.e. $z_2 - z_1 \to 0$ *when* $x_4 - x_1 \to 0$.

Proof.

Step 1. By continuity of ψ, there exists a small $0 < \delta < (z_2 - z_1)/4$ such that $\psi > Q_s + \alpha$ in $U_\delta = [x_2, x_2 + \delta] \times [z_1, z_1 + \delta]$ for some $\alpha > 0$ small enough. We claim that there exists $(\xi_1, z_1') \in U_\delta$ such that

$$\partial_z \psi(\xi_1, z_1') > 0.$$

Indeed, otherwise, we have $\psi = \psi(x)$ in U_δ and since $\Delta_q\psi = 0$ in U_δ, we obtain $\psi = \alpha x + \beta$ in U_δ. By analytic continuation (see Corollary 1.4), we get $\psi = \alpha x + \beta$ in $[\psi > Q_s] \setminus S$ which leads to $\psi = \alpha x + \beta$ in $[\psi > Q_s]$. Since $\psi(x, 1) = Q$, we get $\psi = Q$ in $[\psi > Q_s]$ which is not possible by Corollary 1.4 i).

Step 2. Consider the following ordinary differential equation :

$$\begin{cases} X'(t) = \frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}(X(t)) = f(X(t)) \\ X(0) = (\xi_1, z_1') = X_0. \end{cases}$$

The function $Y \longmapsto \dfrac{Y}{\sqrt{1+|Y|^2}}$ is Lipschitz continuous from $I\!R^2$ to $I\!R^2$ and $\nabla\psi \in C^{0,1}_{loc}([\psi > Q_s])$, then there exists a unique maximal solution $X : I_1 = (a_1, b_1) \longrightarrow [\psi > Q_s]$, $t \longmapsto X(t) = (X_1(t), X_2(t))$ (see [31]). We denote by $\gamma^+(X_0) = \{\, X(t) \,/\, t \geq 0 \,\}$ the positive semi-orbite of X. We distinguish two cases:

1^{st} *case* : $b_1 < \infty$

Since f is bounded, $\lim\limits_{t \to b_-} X(t) = X(b_-)$ exists and then $X(b_-) \in \partial[\psi > Q_s]$. So $X(b_-) \in \partial\Omega$ or $\psi(X(b_-)) = Q_s$. But we cannot have the second situation since ψ is nondecreasing along the trajectory of X. Indeed we have :

$$\frac{d}{dt}\big(\psi(X(t))\big) = \frac{|\nabla\psi|^2}{\sqrt{1+|\nabla\psi|^2}}(X(t)) \geq 0 \qquad \text{and} \qquad \psi(X_0) > Q_s + \alpha.$$

Thus $X(b_-) \in \partial\Omega$ and $\gamma^+(X_0)$ leaves the rectangle

$$R_1 = (x_1, x_3) \times (z_1, z_2)$$

from the top since we have $\psi(x_1, z) < Q_s$, $\psi(x_3, z) < Q_s$ $\forall z \in [z_1, z_2]$ and $X_2(t) > z_1$ $\forall t > 0$.

2^{nd} *case* : $b_1 = +\infty$

- If $\overline{\gamma^+(X_0)} \not\subset [\psi > Q_s]$ then there exists $X_* \in \overline{\gamma^+(X_0)} \cap (\partial[\psi > Q_s])$. If $X_* \in (\partial[\psi > Q_s]) \cap \Omega$ then $\psi(X_*) = Q_s$ which is impossible since $\psi(X_*) = \lim\limits_{n \to +\infty} \psi(X(t_n)) \geq Q_s + \alpha > Q_s$. So $X_* \in (\partial[\psi > Q_s]) \cap \partial\Omega$ and $\gamma^+(X_0)$ leaves R_1 from the top.

- If $\overline{\gamma^+(X_0)} \subset [\psi > Q_s]$ then $\gamma^+(X_0)$ is relatively compact in $[\psi > Q_s]$. Then it follows that the positive limit set of X_0 defined by

$$\omega(X_0) = \{\, Y \in [\psi > Q_s] \,/\, \exists t_k \to +\infty \quad \text{such that} \quad X(t_k) \to Y \,\}$$

is nonempty, compact, connected and (see [1])

$$X(t) \longrightarrow \omega(X_0) \qquad \text{when } t \to +\infty.$$

Note that $\overline{\gamma^+(X_0)} = \gamma^+(X_0) \cup \omega(X_0)$. Moreover, we have

$$\omega(X_0) \subset \{\, Y \in \overline{\gamma^+(X_0)} \,/\, \nabla\psi(Y) = 0 \,\} \subset S.$$

By Corollary 1.4 iv), we deduce that $\omega(X_0)$ is reduced to a unique point X_* and $\lim_{t\to+\infty} X(t) = X_*$.

Let us denote $\gamma_1^+ = \gamma^+(X_0), \quad X^1(t) = X(t, X_0) = X(t), \quad X_*^1 = X_*, \quad X_0^1 = X_0$. We have

$$\psi(X_*^1) \geq \psi(X_0^1) > Q_s + \alpha, \qquad \nabla\psi(X_*^1) = 0.$$

Let $U(X_*^1)$ be a neighbourhood of X_*^1 on which $\psi > Q_s+\alpha$. Let J be the connected component of the set $E = \{x \in [X_{*1}^1, x_3] \,/\, \psi(x, X_{*2}^1) > Q_s + \alpha\,\}$ containing X_{*1}^1. We have $J = [X_{*1}^1, \overline{x}[$ with $\psi(\overline{x}, X_{*2}^1) = Q_s + \alpha$.

Now assume that $\partial_x\psi(x, X_{*2}^1) \geq 0 \ \forall x \in (X_{*1}^1, \overline{x})$. Then $Q_s + \alpha = \psi(\overline{x}, X_{*2}^1) \geq \psi(X_{*1}^1, X_{*2}^1) = \psi(X_*^1) > Q_s + \alpha$ and we get a contradiction. So there exists $\xi_2' \in (X_{*1}^1, \overline{x})$ such that $\partial_x\psi(\xi_2', X_{*2}^1) < 0$. Moreover we have $\psi(\xi_2', X_{*2}^1) > Q_s + \alpha$ since $\xi_2' \in J$. By continuity, there exists a neighbourhood V of (ξ_2', X_{*2}^1) such that $\psi > Q_s + \alpha$ and $\partial_x\psi < 0$ in V. As in step 1, there exists $(\overline{\xi}_2, \overline{z}_2) \in V \cap [z > X_{*2}^1]$ such that $\partial_z\psi(\overline{\xi}_2, \overline{z}_2) > 0$. Hence we have found a point $(\overline{\xi}_2, \overline{z}_2)$ such that

$$\begin{cases} \psi > Q_s + \alpha \quad \text{on } [X_{*1}^1, \overline{\xi}_2] \times \{X_{*2}^1\} \cup \{\overline{\xi}_2\} \times [X_{*2}^1, \overline{z}_2] = H_1 \cup V_1 \\ \partial_x\psi < 0 \quad \text{on } \{\overline{\xi}_2\} \times [X_{*2}^1, \overline{z}_2] = V_1 \\ \partial_z\psi(\overline{\xi}_2, \overline{z}_2) > 0. \end{cases}$$

Let us consider now the differential equation :

$$\begin{cases} X'(t) = f(X(t)) \\ X(0) = (\overline{\xi}_2, \overline{z}_2) = X_0^2. \end{cases}$$

There exists a unique maximal solution $X(., X_0^2)$ defined on $I_2 = (a_2, b_2)$. We distinguish two cases :

- $b_2 < +\infty$: then the trajectory γ_2^+ leaves R_1 from the top.

- $b_2 = +\infty$:

  . if $\overline{\gamma_2^+} \not\subset [\psi > Q_s]$ then γ_2^+ leaves R_1 from the top.

  . if $\overline{\gamma_2^+} \subset [\psi > Q_s]$ then $\omega(X_0^2) = \{X_*^2\}$ with $\psi(X_*^2) > Q_s+\alpha$ and $\nabla\psi(X_*^2) = 0$.

Assume that we can construct a sequence $(X_*^n)_{n\in\mathbb{N}^*} \subset R_1$ of such points. Note that the sequence (X_*^n) is not stationary since (X_{*2}^n) is an increasing sequence. Since $\overline{R}_1$ is compact, there exists a subsequence $(X_*^{n_k})$ which converges to a point $X^* \in \overline{R}_1$ satisfying

$$\psi(X^*) \geq Q_s + \alpha > Q_s \quad \text{and} \quad \nabla\psi(X^*) = 0 \qquad \text{i.e. } X^* \in S.$$

This contradicts the fact that S is a discrete set. So the above process is necessarily finite and there exists $n_0 \geq 1$ such that $\gamma_{n_0}^+$ leaves R_1 from the top. We denote by $\{(\tilde{\xi}_1, z_2)\} = \gamma_{n_0}^+ \cap \overline{R}_1$.

Step 3. Set $R_2 = [x_2, x_4] \times [z_1, z_2]$.

Arguing as in step 1, we can find (ξ_2, z_2') in $U_{\delta'} = [x_3 - \delta', x_3] \times [z_2 - \delta', z_2]$ a small neighbourhood of (x_3, z_2) $(0 < \delta' < (z_2 - z_4)/4)$ satisfying for some $\alpha' > 0$

$$\psi(\xi_2, z_2') < Q_s - \alpha' \qquad \text{and} \qquad \partial_z \psi(\xi_2, z_2') > 0.$$

We then consider the following differential equation :

$$\begin{cases} Y'(t) = -f(Y(t)) \\ Y(0) = (\xi_2, z_2') = Y_0^1. \end{cases}$$

There exists a unique maximal solution $Y(., Y_0) = Y^1(.) : (c_1, d_1) \longrightarrow [\psi < Q_s]$, $t \longmapsto Y^1(t) = (Y_1^1(t), Y_2^1(t))$. Then if

- $d_1 < +\infty$: as in the 1^{st} case of step 2, the positive trajectory $\gamma_1^{+'}$ of Y^1 leaves R_2 from the bottom.

- $d_1 = +\infty$:

. if $\overline{\gamma_1^{+'}} \not\subset [\psi < Q_s]$ then $\gamma_1^{+'}$ leaves R_2 from the bottom.

. if $\overline{\gamma_1^{+'}} \subset [\psi < Q_s]$ then $\lim\limits_{t \to +\infty} Y^1(t) = Y_*^1$ with $\psi(Y_*^1) < Q_s - \alpha'$ and $\nabla \psi(Y_*^1) = 0$.

Arguing as in the 2^{nd} case of step 2, one can find a point $(\underline{\xi}_2, \underline{z}_2)$ such that :

$$\begin{cases} \underline{z}_2 < Y_{*2}^1 \\ \psi < Q_s - \alpha' \quad \text{on } [Y_{*1}^1, \underline{\xi}_2] \times \{Y_{*2}^1\} \cup \{\underline{\xi}_2\} \times [Y_{*2}^1, \underline{z}_2] = H_1' \cup V_1' \\ \partial_x \psi > 0 \quad \text{on } \{\underline{\xi}_2\} \times [Y_{*2}^1, \underline{z}_2] = V_1' \\ \partial_z \psi(\underline{\xi}_2, \underline{z}_2) > 0. \end{cases}$$

Then we consider the maximal solution $Y(., Y_0^2) = Y^2(.)$ with $Y_0^2 = (\underline{\xi}_2, \underline{z}_2)$. If $\gamma_2^{+'}$ does not leave $\overline{R}_2$, we can repeat the above process. But as in step 2, one can prove that there exists $n_1 \geq 1$ such that $\gamma_{n_1}^{+\,\prime}$ leaves $\overline{R}_2$ from the bottom. We denote by $\{(\tilde{\xi}_2, z_1)\} = \gamma_{n_1}^{+\prime} \cap \overline{R}_1$.

Step 4. We denote by R (see Figure 2) the region delimited in the bottom by $z = z_1'$, in the top by $z = z_2'$ and laterally by Σ and Σ' given by :

$$\begin{aligned} \Sigma &= \overline{\gamma}_1^+ \cup (H_1 \cup V_1) \cup ... \cup (\gamma_{n_0}^+ \cap [z \leq z_2']) \\ \Sigma' &= \overline{\gamma}_1^{+\prime} \cup (H_1' \cup V_1') \cup ... \cup (\gamma_{n_1}^{+\prime} \cap [z \geq z_1']). \end{aligned}$$

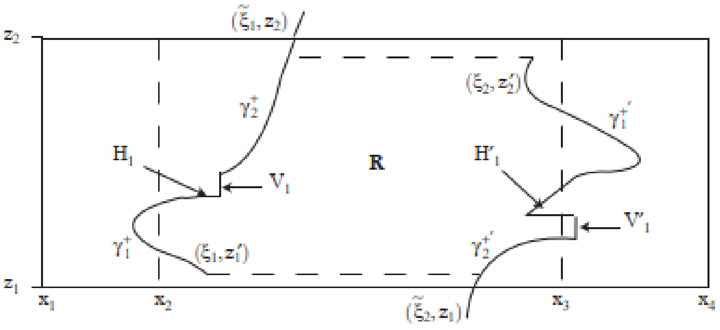

Figure 2

Clearly ∂R is locally Lipschitz.
Let $\eta \in \mathcal{D}(z_1', z_2')$ such that $supp\eta \subset [z_1'+\varepsilon, z_2'-\varepsilon]$ ($\varepsilon > 0$ small enough), $0 \leq \eta \leq 1$ and for $0 < \rho << \varepsilon$ small, we consider

$$d_\rho(x,z) = \min\Big(1, \frac{d((x,z), \partial R)}{\rho}\Big).$$

Then $\eta d_\rho \chi(R)$ is a suitable test function for (P) and we have :

$$\int_R |\nabla\psi|^{q-2}\nabla\psi.\nabla(\eta d_\rho) = -\int_R \eta\gamma\partial_x d_\rho.$$

Note that for small ρ, we have by continuity of ψ and since $\Sigma \subset [\psi > Q_s]$ and $\Sigma' \subset [\psi < Q_s]$:

$$\begin{array}{lll} \psi(x,z) & > & Q_s \quad \text{for } (x,z) \in R \text{ such that } d((x,z),\Sigma) \leq \rho \text{ and then } \gamma = 0 \text{ a.e.} \\ \psi(x,z) & < & Q_s \quad \text{for } (x,z) \in R \text{ such that } d((x,z),\Sigma') \leq \rho \text{ and then } \gamma = 1 \text{ a.e.} \end{array}$$

So we have

$$-\int_R \eta\gamma\partial_x d_\rho = \int_{[d(.,\Sigma')\leq\rho]} \partial_x\big(\eta(z)(1-d_\rho)\big) = \int_{\Sigma'} \eta(z)\nu_x$$

and then

$$\int_{\Sigma'} \eta(z)\nu_x = \int_R d_\rho|\nabla\psi|^{q-2}\partial_z\psi.\partial_z\eta + \int_R \eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho. \tag{2.1}$$

The second term of the right-hand side of (2.1) can be written as :

$$\int_R \eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho = \int_{[d(.,\Sigma)\leq\rho]} \eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \tag{2.2}$$

$$+\int_{[d(.,\Sigma')\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho = I^1_\rho + I^2_\rho.$$

We have for I^1_ρ :

$$\begin{aligned} I^1_\rho &= \sum_{i=1}^{n_0-1}\Big(\int_{[d(.,\gamma_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \\ &+ \int_{[d(.,\gamma_i)>\rho]\cap[d(.,H_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \\ &+ \int_{[d(.,\gamma_{i+1})>\rho]\cap[d(.,V_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho\Big) \\ &+ \int_{[d(.,\gamma_{n_0})\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho. \end{aligned} \tag{2.3}$$

Since $\partial_x\psi < 0$ on V_i and $\psi\in C^1([\psi > Q_s])$, it follows that $\partial_x\psi < 0$ in $[d(.,V_i)\le\rho]$ for ρ small enough and then :

$$\int_{[d(.,\gamma_{i+1})>\rho]\cap[d(.,V_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho = \frac{1}{\rho}\int_{[d(.,\gamma_{i+1})>\rho]\cap[d(.,V_i)\le\rho]}\eta|\nabla\psi|^{q-2}\partial_x\psi \le 0. \tag{2.4}$$

Moreover $\partial_z\psi\ge 0$, then we have

$$\int_{[d(.,\gamma_i)>\rho]\cap[d(.,H_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho = \frac{1}{\rho}\int_{[d(.,\gamma_i)>\rho]\cap[d(.,H_i)\le\rho]}\eta|\nabla\psi|^{q-2}(-\partial_z\psi) \le 0. \tag{2.5}$$

Now, we have

$$\begin{aligned} \int_{[d(.,\gamma_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho &= \int_{[d(.,\gamma_i)\le\rho]\cap[d(.,\{X^i_0,X^i_*\})>\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \\ &+ \int_{[d(.,\gamma_i)\le\rho]\cap[d(.,X^i_0)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \\ &+ \int_{[d(.,\gamma_i)\le\rho]\cap[d(.,X^i_*)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \end{aligned} \tag{2.6}$$

and

$$\Big|\int_{[d(.,\gamma_i)\le\rho]\cap[d(.,X^i_0)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho\Big| \le \frac{c}{\rho}|B(X^i_0,\rho)| = \pi c\rho$$

where $c = \max\{|\nabla\psi|^{q-1}(x,z)\,/\,d((x,z),\Sigma)\le\rho_0\}$ for a fixed small ρ_0. So

$$\lim_{\rho\to 0}\int_{[d(.,\gamma_i)\le\rho]\cap[d(.,X^i_0)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho = 0. \tag{2.7}$$

In the same way, we get

$$\lim_{\rho\to 0}\int_{[d(.,\gamma_i)\le\rho]\cap[d(.,X_*^i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho=0. \tag{2.8}$$

For the first term of the right-hand side of (2.6), we have :

$$\begin{aligned}\int_{[d(.,\gamma_i)\le\rho]\cap[d(.,\{X_0^i,X_*^i\})>\rho]}&\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho\\ =-\frac{1}{\rho}\int_{[d(.,\gamma_i)\le\rho]\cap[d(.,\{X_0^i,X_*^i\})>\rho]}&\eta|\nabla\psi|^{q-2}\nabla\psi.\nu(y(x,z))\quad=J_\rho^i\end{aligned}$$

where $\nu(y(x,z))$ is the outward unit normal to γ_i on the point $y(x,z)$ defined by : $d((x,z),y(x,z))=d((x,z),\gamma_i)$. Indeed, since $\nabla\psi$ does not vanish on $\overline{\gamma}_i\setminus\{X_*^i\}$, ψ is analytic in a neighbourhood of $\overline{\gamma}_i\setminus\{X_*^i\}$. We then deduce that X^i is analytic on $[0,+\infty)$ (see [24]) and $d(.,\gamma_i)$ is C^∞ in a neighbourhood of the set $[d(.,\gamma_i)\le\rho]\cap[d(.,\{X_0^i,X_*^i\})>\rho]$ where $\nabla d((x,z),\gamma_i)=-\nu(y(x,z))$ (see [22]).
Moreover γ_i is a curve of the form $\{(f_i(z),z),\ z\in X_2^i([0,+\infty))\}$ with $f_i(z)=(X_1^i o(X_2^i)^{-1})(z)$. In fact $(X_2^i)'(t)\ge 0\ \forall t\ge 0$ since $\partial_z\psi\ge 0$ and vanishes only on isolated points since X_2^i is analytic. So X_2^i is increasing. Thus $(X_2^i)^{-1}$ is well defined, continuous on $X_2^i([0,+\infty))$ and C^∞ on $X_2^i([0,+\infty))\setminus T$ with $T=\{t\in[0,+\infty),\ (X_2^i)'(t)=0\}$ which is a discrete set. It follows that

$$\nu(y(x,z))=\nu(f_i(z),z)=\frac{1}{\sqrt{1+f_i'^2(z)}}(-e_x+f_i'(z)e_z)\qquad\text{for } z\in X_2^i([0,+\infty))\setminus T$$

and

$$|J_\rho^i|\le\frac{1}{\rho}\int_{X_{02}^i}^{X_{*2}^i}\int_{f_i(z)}^{f_i(z)+\rho/\sqrt{1+f_i'^2(z)}}|\nabla\psi|^{q-2}|\nabla\psi(x,z).\nu(f_i(z),z)|.$$

By Lebesgue's theorem, we get

$$\begin{aligned}\lim_{\rho\to 0}\int_{X_{02}^i}^{X_{*2}^i}\ \tfrac{1}{\rho}\ \int_{f_i(z)}^{f_i(z)+\rho/\sqrt{1+f_i'^2(z)}}&|\nabla\psi|^{q-2}|\nabla\psi(x,z).\nu(f_i(z),z)|\\ =\int_{X_{02}^i}^{X_{*2}^i}&|\nabla\psi|^{q-2}|\nabla\psi(f_i(z),z).\nu(f_i(z),z)|=0\end{aligned}$$

since by construction, we have $\nabla\psi(f_i(z),z).\nu(f_i(z),z)=0$ a.e. Then $\lim_{\rho\to 0}J_\rho^i=0$ and by (2.6)-(2.8), we deduce that

$$\lim_{\rho\to 0}\int_{[d(.,\gamma_i)\le\rho]}\eta|\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho=0. \tag{2.9}$$

Now, using (2.3)-(2.5) and (2.9), we get $\limsup_{\rho\to 0} I_\rho^1 = 0$. In the same way, we prove that $\limsup_{\rho\to 0} I_\rho^2 = 0$. This leads by (2.2) to

$$\limsup_{\rho\to 0} \int_R \eta |\nabla\psi|^{q-2}\nabla\psi.\nabla d_\rho \leq 0.$$

Letting $\rho \to 0$ in (2.1), we obtain

$$\int_{\Sigma'} \eta(z)\nu_x \leq \int_R |\nabla\psi|^{q-2}\partial_z\psi.\partial_z\eta.$$

Note that we have $\int_{\Sigma'} \eta(z)\nu_x = \int_{z_1'}^{z_2'} \eta(z)dz$ and if we choose $\eta = 1$ in $(z_1' + 2\varepsilon, z_2' - 2\varepsilon)$ with $\varepsilon < \frac{z_2' - z_1'}{8}$, we get

$$\frac{z_2' - z_1'}{2} \leq \int_R |\nabla\psi|^{q-2}\partial_z\psi.\partial_z\eta.$$

Finally by the choice of z_1', z_2', we have $z_2' - z_1' \geq \frac{z_2 - z_1}{2}$ and then

$$z_2 - z_1 \leq 4\int_R |\nabla\psi|^{q-2}\partial_z\psi.\partial_z\eta.$$

□

Lemma 2.4. *Let $m_0 = (x_0, z_0) \in \Omega$, $\rho > 0$ such that $B(m_0, \rho) \subset \Omega$. Set $S = \{x_0\} \times (z_0 - \rho, z_0 + \rho)$. Then we cannot have the following situation :*

$$\psi = Q_s \quad \textit{on } S \quad \textit{and} \quad \psi \neq Q_s \quad \textit{in } B(m_0,\rho) \setminus S.$$

Proof.

i) Assume that we have $\psi > Q_s$ in $B(m_0,\rho) \setminus S$. Then $\gamma = 0$ a.e. in $B(m_0,\rho)$ and $\Delta_q \psi = 0$ in $B(m_0,\rho)$. We deduce that $\Delta_q(\psi - Q_s) = 0$ in $B(m_0,\rho)$. But since we have $\psi - Q_s \geq 0$ in $B(m_0,\rho)$ and $\psi - Q_s = 0$ in S, we get by the strong maximum principle : $\psi - Q_s = 0$ in $B(m_0,\rho)$ which contradicts the fact that $\psi > Q_s$ in $B(m_0,\rho) \setminus S$.

ii) Assume that we have $\psi < Q_s$ in $B(m_0,\rho) \setminus S$. Then as in i), $\gamma = 1$ a.e. in $B(m_0,\rho)$ and $\Delta_q\psi = -\partial_x\gamma = 0$ in $B(m_0,\rho)$. So we have

$$\begin{cases} \Delta_q(Q_s - \psi) = 0 & \text{in } B(m_0,\rho) \\ Q_s - \psi \geq 0 & \text{in } B(m_0,\rho) \\ Q_s - \psi = 0 & \text{on } S \end{cases}$$

which leads by the strong maximum principle to $Q_s - \psi = 0$ in $B(m_0,\rho)$. We get a contradiction with $\psi < Q_s$ in $B(m_0,\rho)\setminus S$.

iii) Assume that we have $\psi < Q_s$ in $B^-(m_0,\rho) = B(m_0,\rho)\cap[x<x_0]$ and $\psi > Q_s$ in $B^+(m_0,\rho) = B(m_0,\rho)\cap[x>x_0]$. Let $0<\delta<\dfrac{\rho}{2}$ and set $\psi_\delta(x,z) = \psi(x,z-\delta)$. We have

$$\begin{aligned}\int_{B(m_0,\rho/2)} |\nabla\psi_\delta|^{q-2}\nabla\psi_\delta.\nabla\zeta \quad &= -\int_{B(m_0,\rho/2)}\gamma(x,z-\delta)\partial_x\zeta(x,z)\\ = -\int_{B^-(m_0,\rho/2)}\partial_x\zeta(x,z) \quad &= -\int_{B(m_0,\rho/2)}\gamma(x,z)\partial_x\zeta(x,z)\end{aligned}$$

since, we have $\gamma = 1$ in $B^-(m_0,\rho)$ and $\gamma = 0$ in $B^+(m_0,\rho)$ a.e. Then we obtain

$$\int_{B(m_0,\rho/2)} |\nabla\psi_\delta|^{q-2}\nabla\psi_\delta.\nabla\zeta = \int_{B(m_0,\rho/2)} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta. \tag{2.10}$$

Now, set $\varphi = \psi - \psi_\delta$, $\varphi_t = t\psi + (1-t)\psi_\delta$ for $t\in[0,1]$ and note that ψ and $\psi_\delta \in C^1(\overline{B^\pm(m_0,\rho/2)})$. We have by (2.10)

$$\begin{aligned} 0 &= \int_{B(m_0,\rho/2)}(|\nabla\psi|^{q-2}\nabla\psi - |\nabla\psi_\delta|^{q-2}\nabla\psi_\delta).\nabla\zeta\\ &= \int_{B^+(m_0,\rho/2)}\Big(\int_0^1 \frac{d}{dt}\big(|\nabla\varphi_t|^{q-2}\nabla\varphi_t\big)dt\Big).\nabla\zeta\\ &+ \int_{B^-(m_0,\rho/2)}\Big(\int_0^1 \frac{d}{dt}\big(|\nabla\varphi_t|^{q-2}\nabla\varphi_t\big)dt\Big).\nabla\zeta\end{aligned}$$

which can be written

$$\int_{B(m_0,\rho/2)} a(x,z)\nabla\varphi.\nabla\zeta = 0 \tag{2.11}$$

with

$$a(x,z) = (a_{ij}(x,z))_{1\le i,j\le 2}, \qquad a_{ij}(x,z) = \int_0^1 \frac{\partial A^i}{\partial h_j}(\nabla\varphi_t)dt$$

where $A(h) = |h|^{q-2}h$, $h\in I\!R^2$ and $(x,z)\in B^+(m_0,\rho/2)\cup B^-(m_0,\rho/2)$.

Now arguing as in [15], we verify that we have :

$$(q-1)\lambda(x,z)|y|^2 \le a(x,z)y.y \le \lambda(x,z)|y|^2 \qquad \text{a.e. } (x,z)\in B(m_0,\rho/2), \quad \forall y\in I\!R^2 \tag{2.12}$$

with

$$\lambda(x,z) = \int_0^1 |\nabla\varphi_t|^{q-2}(x,z)dt. \tag{2.13}$$

First, since ψ and $\psi_\delta\in C^1(\overline{B^+(m_0,\rho/2)})$, there exists $c_0>0$ such that $|\nabla\psi|$, $|\nabla\psi_\delta| \le c_0/2$ in $\overline{B^+(m_0,\rho/2)}$. So $|\nabla\varphi_t| \le |\nabla\psi| + |\nabla\psi_\delta| \le c_0$ and $\lambda(x,z)\ge c_0^{q-2}$ in $B^+(m_0,\rho/2)$ since $q<2$.

In the same way, we have $\lambda(x,z)\geq c_1^{q-2}$ in $B^-(m_0,\rho/2)$ for some $c_1>0$.
Next, we have $\Delta_q(Q_s-\psi)=0$, $Q_s-\psi>0$ in $B^-(m_0,\rho/2)$ and $Q_s-\psi=0$ on $\partial B^-(m_0,\rho/2)\cap S$. By the Hopf maximum principle (see [33]), we deduce that $\nabla(Q_s-\psi)\neq 0$ or $\nabla\psi\neq 0$ on $\partial B^-(m_0,\rho/2)\cap S$. In particular we have $\nabla\psi(m_0)\neq 0$.
We distinguish two cases :
- Assume that $\nabla(\psi-\psi_\delta)(m_0)=0$ then $\nabla\varphi_t(m_0)=\nabla\psi(m_0)\neq 0$ $\forall t\in[0,1]$. By continuity of $\nabla\psi$ there exists $\rho_1\in(0,\rho/2)$ such that :

$$|\nabla\varphi_t(x,z)|\geq c_0' \qquad \forall (x,z)\in B^-(m_0,\rho_1) \qquad \forall t\in[0,1]$$

where c_0' is a positive constant. We then deduce that $\lambda(x,z)\leq (c_0')^{q-2}$ in $B^-(m_0,\rho_1)$.
- Assume that $\nabla(\psi-\psi_\delta)(m_0)\neq 0$ then by continuity of $\nabla\psi$ there exists $\rho_1\in(0,\rho/2)$, $c_0'>0$ such that :

$$|\nabla(\psi-\psi_\delta)(x,z)|\geq c_0' \qquad \forall (x,z)\in B^-(m_0,\rho_1).$$

We have

$$\begin{aligned}|\nabla\varphi_t| &= |\nabla\psi_\delta+t\nabla\psi|\\ &= |\nabla\varphi|\Big(\big(t+\frac{\nabla\psi_\delta.\nabla\varphi}{|\nabla\varphi|^2}\big)^2+\frac{|\nabla\psi_\delta|^2.|\nabla\varphi|^2-(\nabla\psi_\delta.\nabla\varphi)^2}{|\nabla\varphi|^4}\Big)^{1/2}\\ &\geq |\nabla\varphi|.\Big|t+\frac{\nabla\psi_\delta.\nabla\varphi}{|\nabla\varphi|^2}\Big|\end{aligned}$$

thus

$$|\nabla\varphi_t|^{q-2}\leq|\nabla\varphi|^{q-2}\Big|t+\frac{\nabla\psi_\delta.\nabla\varphi}{|\nabla\varphi|^2}\Big|^{q-2}\leq (c_0')^{q-2}|t-k(x,z)|^{q-2}$$

with $k(x,z)=-\dfrac{\nabla\psi_\delta.\nabla\varphi}{|\nabla\varphi|^2}$. Then

$$\lambda(x,z)\leq (c_0')^{q-2}\int_0^1|t-k(x,z)|^{q-2}dt \qquad \forall (x,z)\in B^-(m_0,\rho_1).$$

To prove that the above integral is bounded, we distinguish 3 cases :
1^{st} *case* : If $k(x,z)\leq 0$, then we have for $(x,z)\in B^-(m_0,\rho_1)$

$$\begin{aligned}\int_0^1|t-k(x,z)|^{q-2}dt &= \frac{1}{q-1}\Big((1-k(x,z))^{q-1}-(-k(x,z))^{q-1}\Big)\\ &\leq \frac{1}{q-1}(1-k(x,z))^{q-1}\leq\frac{1}{q-1}\Big(1+\frac{|\nabla\psi_\delta|}{|\nabla\varphi|}\Big)^{q-1}\\ &\leq \frac{1}{q-1}\Big(1+\frac{c_1}{2c_0'}\Big)^{q-1}.\end{aligned}$$

2^{nd} *case* : If $k(x,z)\in(0,1)$, then one have for $(x,z)\in B^-(m_0,\rho_1)$

$$\int_0^1|t-k(x,z)|^{q-2}dt = \int_0^{k(x,z)}(k(x,z)-t)^{q-2}dt+\int_{k(x,z)}^1(t-k(x,z))^{q-2}dt$$

$$\begin{aligned} &\leq \frac{1}{q-1}\big((k(x,z))^{q-2}+(1-k(x,z))^{q-1}\big)\\ &\leq \frac{2}{q-1}. \end{aligned}$$

3^{rd} *case* : If $k(x,z)\geq 1$ then we have for $(x,z)\in B^-(m_0,\rho_1)$

$$\begin{aligned} \int_0^1 |t-k(x,z)|^{q-2}dt &= \frac{1}{q-1}\big((k(x,z))^{q-1}-(k(x,z)-1)^{q-1}\big)\\ &\leq \frac{1}{q-1}(k(x,z))^{q-1}\leq \frac{1}{q-1}\Big(\frac{c_1}{2c_0'}\Big)^{q-1}. \end{aligned}$$

So there exists a positive constant c_3 such that

$$\lambda(x,z)\leq c_3 \qquad \forall (x,z)\in B^-(m_0,\rho_1).$$

In the same way, since we have

$$\begin{cases} \Delta_q(\psi-Q_s)=0 & \text{in } B^+(m_0,\rho/2)\\ \psi-Q_s>0 & \text{in } B^+(m_0,\rho/2)\\ \psi-Q_s=0 & \text{on } \partial B^+(m_0,\rho/2)\cap S, \end{cases}$$

then by the Hopf maximum principle, we deduce that $\nabla(\psi-Q_s)\neq 0$ or $\nabla\psi\neq 0$ on $\partial B^+(m_0,\rho/2)\cap S$. In particular we have $\nabla\psi(m_0)\neq 0$. Proceeding as above, we prove the existence of $\rho_2\in(0,\rho/2)$ such that

$$\lambda(x,z)\leq c_3' \qquad \forall (x,z)\in B^+(m_0,\rho_2).$$

Hence we get

$$\lambda_1=\min(c_0^{q-2},c_1^{q-2})\leq \lambda(x,z)\leq \max(c_3,c_3')=\lambda_2 \qquad \text{in } B(m_0,\min(\rho_1,\rho_2))$$

which proves using (2.12) the coerciveness of the matrix $a(x,z)$ in $B(m_0,\rho_3)$ with $\rho_3=\min(\rho_1,\rho_2)$. So, we have proved that φ satisfies :

$$\begin{cases} div(a(x,z)\nabla\varphi)=0 & \text{in } \mathcal{D}'(B(m_0,\rho_3))\\ \varphi\geq 0 & \text{in } B(m_0,\rho_3)\\ \varphi=0 & \text{on } B(m_0,\rho_3)\cap S\\ \varphi\in W^{1,\infty}(B(m_0,\rho_3)). \end{cases}$$

Hence by the strong maximum principale for linear elliptic equations (see [22]), we obtain $\varphi=0$ in $B(m_0,\rho_3)$ and consequently

$$\psi=\psi_\delta \quad \text{in} \quad B(m_0,\rho_3).$$

Letting δ go to 0, we get $\partial_z\psi=0$ and $\psi(x,z)=\theta(x)$ in $B(m_0,\rho_3)$. Since $\Delta_q\psi=0$ in $B^+(m_0,\rho_3)$, we get $\theta(x)=\alpha x+\beta$ in $B^+(m_0,\rho_3)$. Moreover by the monotonicity of ψ, we have $\psi>Q_s$ in $D=B^+(m_0,\rho_3)\cup(x_0,x_0+\rho_3)\times(z_0,1)$. Since ψ is analytic in $D \setminus S$, we obtain by unique continuation $\psi(x,z) = \alpha x + \beta$ in D. Using the boundary data of ψ at the segment CB, we get $\alpha = 0$, $\beta = Q$ and $\psi(x,z) = Q$ in D. This contradicts the fact that $\psi(m_0) = Q_s < Q$. □

Lemma 2.5. *Assume that there exists* $x_0 \in (0,a)$, $z_1, z_2 \in (0,1)$ *with* $z_1 < z_2$ *such that :*

$$\begin{cases} \psi(x_0,z) > Q_s & \forall z \in (z_2,1) \\ \psi(x_0,z) = Q_s & \forall z \in [z_1,z_2] \\ \psi(x_0,z) < Q_s & \forall z \in (0,z_1). \end{cases}$$

Then $\psi = Q_s$ *in* $(0,x_0) \times [z_1,z_2]$.

Proof. Let $z_{T'}, z_T \in [z_1,z_2]$ such that $z_{T'} < z_T$. Set $T = (x_0, z_T)$ and $T' = (x_0, z_{T'})$. Then one of the following situations hold :

$$\begin{array}{lllll} i) & \exists \varepsilon_1 > 0 & \text{such that} & \psi \geq Q_s & \text{in} \quad \overline{B^-(T,\varepsilon_1)}, \\ ii) & \exists \varepsilon_2 > 0 & \text{such that} & \psi \leq Q_s & \text{in} \quad \overline{B^-(T',\varepsilon_2)}. \end{array}$$

Indeed, if not, we will have

$$\begin{array}{lll} \forall \varepsilon > 0, & \exists m_\varepsilon & \in \overline{B^-(T,\varepsilon)} \text{ such that } \psi(m_\varepsilon) < Q_s \\ & \exists m'_\varepsilon & \in \overline{B^-(T',\varepsilon)} \text{ such that } \psi(m'_\varepsilon) > Q_s. \end{array}$$

Note that $m_\varepsilon \notin \overline{B^-(T,\varepsilon)} \cap [x = x_0]$ and $m'_\varepsilon \notin \overline{B^-(T',\varepsilon)} \cap [x = x_0]$.
Hence we obtain a sequence $m_n = (x_n, z_n) \to T$ such that $x_n < x_0$ and $\psi(m_n) < Q_s$ and a sequence $m'_n = (x'_n, z'_n) \to T'$ such that $x'_n < x_0$ and $\psi(m'_n) > Q_s$. But by Lemma 2.3, these sequences cannot exist simultaneously.

Let us assume that i) holds. Then we have

$$\psi = Q_s \qquad \text{in } B^-(T,\varepsilon_1) \cap [z \leq z_T] = V. \tag{2.14}$$

Indeed, if not, there exists $m = (x_m, z_m) \in V$ such that $\psi(m) > Q_s$. By continuity of ψ, there exists $\varepsilon > 0$ such that $\psi(x,z) > Q_s$ $\forall (x,z) \in B(m,\varepsilon) \subset V$. This leads by $(P)ii)$ to $\gamma = 0$ in $B(m,\varepsilon)$.
Set $\mathcal{C} = B(m,\varepsilon) \cup (x_m, x_0) \times (z_m - \varepsilon, z_m + \varepsilon)$. Let $\zeta \in \mathcal{D}(\mathcal{C})$, $\zeta \geq 0$ and $\eta > 0$. Then $\min(\zeta, \dfrac{\psi - Q_s}{\eta})$ is a test function for (P) and we have

$$\int_{\mathcal{C}\cap[\eta\zeta \leq \psi - Q_s]} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta + \frac{1}{\eta}\int_{\mathcal{C}\cap[\eta\zeta > \psi - Q_s]} |\nabla\psi|^q = -\int_{\mathcal{C}} \gamma\partial_x \min\Big(\zeta, \frac{\psi - Q_s}{\eta}\Big) = 0$$

then

$$\int_{\mathcal{C}\cap[\eta\zeta \leq \psi - Q_s]} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta \leq 0.$$

Letting $\eta \to 0$, we get $\Delta_q \psi \geq 0$ in $\mathcal{D}'(\mathcal{C})$. By (1.17) we deduce that $\partial_x \gamma \leq 0$ in $\mathcal{D}'(\mathcal{C})$. Thus $\gamma = 0$ in $\mathcal{C}$ since we have $\gamma = 0$ in $B(m,\varepsilon)$ and $\gamma \geq 0$ in Ω. Using

(1.17), we get $\Delta_q\psi = 0$ in $\mathcal{C}$ and by the strong maximum principle, $\psi(x,z) > Q_s$ in $\mathcal{C}$.

Consider two points $P \neq P'$ of $\overline{\mathcal{C}} \cap [x = x_0]$ with $z_P > z_{P'}$. Arguing as above for T and T', one of the following situations holds :

$$\begin{array}{lllll} a) & \exists\varepsilon_1' > 0 & \text{such that} & \psi \geq Q_s & \text{in} \quad \overline{B^+(P,\varepsilon_1')}, \\ b) & \exists\varepsilon_2' > 0 & \text{such that} & \psi \leq Q_s & \text{in} \quad \overline{B^+(P',\varepsilon_2')}. \end{array}$$

- Suppose that a) holds. We have $\psi \geq Q_s$ in $B(P,\varepsilon_1')$ then $\Delta_q\psi \geq 0$ and $\partial_x\gamma \leq 0$ in $\mathcal{D}'(B(P,\varepsilon_1'))$. But since $\gamma = 0$ in $B^-(P,\varepsilon_1')$, we have $\gamma = 0$ in $B(P,\varepsilon_1')$. We deduce that ψ is q-harmonic in $B(P,\varepsilon_1')$ and by the strong maximum principle, $\psi > Q_s$ in $B(P,\varepsilon_1')$. This contradicts the fact that $\psi = Q_s$ in $B(P,\varepsilon_1') \cap [x = x_0]$.

- Suppose that b) holds. We have $\psi \leq Q_s$ in $\overline{B^+(P,\varepsilon_2')}$. Assume that $\psi \not\equiv Q_s$ in $\overline{B^+(P,\varepsilon_2')}$. Let $m' = (x_{m'}, z_{m'}) \in B^+(P,\varepsilon_2')$ such that $\psi(m') < Q_s$. By continuity of ψ, we have $\psi < Q_s$ in $B(m',\varepsilon_3)$. Denote by $\mathcal{C}'$ the set $\big(B(m',\varepsilon_3) \cup [x \geq x_{m'}]\big) \cap B^+(P,\varepsilon_2')$ and let $\zeta \in \mathcal{D}(\mathcal{C}')$, $\zeta \geq 0$ and $\eta > 0$. Then $\min(\zeta, \dfrac{Q_s - \psi}{\eta})$ is a test function for (P). We proceed as above and we get $\Delta_q\psi \leq 0$ in $\mathcal{D}'(\mathcal{C}')$. By (1.17) we deduce that $\partial_x\gamma \geq 0$ in $\mathcal{D}'(\mathcal{C}')$. Thus $\gamma = 1$ in $\mathcal{C}'$ since we have $\gamma = 1$ in $B(m',\varepsilon_3)$ and $\gamma \leq 1$ in Ω. Using (1.17), we get $\Delta_q\psi = 0$ in $\mathcal{C}'$ and by the strong maximum principle, $\psi(x,z) < Q_s$ in $\mathcal{C}'$.

Set $\{m''\} = \big(\partial B^+(P',\varepsilon_2') \setminus [x = x_0]\big) \cap [z = z_{m'}]$. Using the monotonicity of ψ and γ in the z-direction, we obtain

$$\psi < Q_s \qquad \text{and} \qquad \gamma = 1 \qquad \text{in } (x_{m'}, x_{m''}) \times (0, z_{m'}).$$

We distinguish again two cases :

<u>1^{st} case</u> : $\exists B(P',\varepsilon_4)$ such that $\psi = Q_s$ in $B^+(P',\varepsilon_4)$. Then $\psi \geq Q_s$ in $B(P',\varepsilon_4)$ and we obtain as in a), $\psi > Q_s$ in $B(P',\varepsilon_4)$ which is a contradiction.

<u>2^{nd} case</u> : $\forall\delta > 0, \exists m_\delta \in B^+(P',\delta)$ such that $\psi(m_\delta) < Q_s$.
Let $M \in B^+(P',\varepsilon_2') \cap [z < z_{P'}]$ and set $\delta_0 = \min\Big(\dfrac{x_M - x_0}{2}, \dfrac{z_{P'} - z_M}{2}\Big)$. By assumption, there exists $m_{\delta_0} \in B^+(P',\delta_0)$ such that $\psi(m_{\delta_0}) < Q_s$. Then as previously, we get $\psi < Q_s$ in $(x_{m_{\delta_0}}, x_{m'_{\delta_0}}) \times (0, z_{m_{\delta_0}})$ where $\{m'_{\delta_0}\} = \big(\partial B^+(P',\varepsilon_2') \setminus [x = x_0]\big) \cap [z = z_{\delta_0}]$. It is easy to verify that $M \in (x_{m_{\delta_0}}, x_{m'_{\delta_0}}) \times (0, z_{m_{\delta_0}})$ and then $\psi(M) < Q_s$. So we have proved that

$$\psi < Q_s \qquad \text{in } B^+(P',\varepsilon_2') \cap [z < z_{P'}]$$

which leads to a contradiction with Lemma 2.4. The claim (2.14) is proved.

Now, by monotonicity of ψ, we have $\psi \leq Q_s$ in $(x_0 - \varepsilon_1, x_0) \times (0, z_T)$. So $\psi \leq Q_s$ in a left neighbourhood of T'. Arguing as we do when i) holds, one can prove that

$\psi = Q_s$ in a small left neighbourhood of T'. Then $\psi = Q_s$ in $B^-(T', \varepsilon_1) \cap [z \geq z_{T'}]$. We deduce by monotonicity of ψ that $\psi = Q_s$ in $(x_0 - \varepsilon_1, x_0) \times (z_{T'}, z_T)$. Since T and T' are arbitrary chosen, we get

$$\psi = Q_s \qquad \text{in } (x_0 - \varepsilon_1, x_0) \times [z_1, z_2]. \tag{2.15}$$

In the same way, we prove (2.15) when ii) holds.

It remains to prove that $\psi = Q_s$ in $(0, x_0) \times [z_1, z_2]$.
Set $I = \{\varepsilon \in (0, x_0)$ such that $\psi = Q_s$ in $(x_0 - \varepsilon, x_0) \times [z_1, z_2]\}$. I is a bounded nonempty set. Let $\rho = \sup I$. We have $0 < \rho \leq x_0$.
First we verify that $\psi = Q_s$ in $(x_0 - \rho, x_0) \times [z_1, z_2]$.
Suppose that there exists $(x, z) \in (x_0 - \rho, x_0) \times [z_1, z_2]$ such that $\psi(x, z) \neq Q_s$. By continuity there exists $0 < \delta < \min(x_0 - x, x - x_0 + \rho)$ such that $\psi(x, z) \neq Q_s$ in $B((x, z), \delta)$. Let $\varepsilon \in I$. We necessarily have $x_0 - \varepsilon > x - \delta$ i.e. $\varepsilon < x_0 - x + \delta = \rho_\delta$. Since $\delta < x - x_0 + \rho$, we deduce that $\varepsilon < \rho_\delta < \rho$ which contradicts $\rho = \sup I$.
Next, if $\rho < x_0$, set $z_1' = g_1(x_0 - \rho)$ and $z_2' = g_2(x_0 - \rho)$. Then we have

$$\begin{cases} \psi > Q_s & \text{on } \{x_0 - \rho\} \times (z_2', 1] \\ \psi = Q_s & \text{on } \{x_0 - \rho\} \times [z_1', z_2'] \\ \psi < Q_s & \text{on } \{x_0 - \rho\} \times [0, z_1'). \end{cases}$$

Arguing as above, we deduce that $\psi = Q_s$ on $(x_0 - \rho - \eta, x_0) \times [z_1', z_2']$ for some $\eta > 0$. Then $\psi = Q_s$ on $(x_0 - \rho - \eta, x_0) \times [z_1, z_2]$ which contradicts $\rho = \sup I$. □

Proof of Theorem 2.2.

i) Assume that there exists $x_0 \in (0, a)$ such that $g_1(x_0) < g_2(x_0)$. Then we have :

$$\begin{cases} \psi(x_0, z) > Q_s & \text{for } z > g_2(x_0) \\ \psi(x_0, z) = Q_s & \text{for } g_1(x_0) \leq z \leq g_2(x_0) \\ \psi(x_0, z) < Q_s & \text{for } z < g_1(x_0). \end{cases}$$

From Lemma 2.5, we deduce that $\psi = Q_s$ in $(0, x_0) \times [g_1(x_0), g_2(x_0)]$. But this contradicts the continuity of ψ in $\overline{\Omega} \setminus \{W\}$ (see (1.19)) and the boundary conditions on the segment OC. Hence $g_1 = g_2 = g$ and g is continuous since it is lower and upper semi-continuous on (0,a) by Proposition 2.1i).

ii) Set $l_1 = \liminf\limits_{x \to 0+} g(x)$ and $l_2 = \limsup\limits_{x \to 0+} g(x)$. We have $l_1 \leq l_2$ and there exists in $(0, a)$ two sequences (x_n^1) and (x_n^2) satisfying : $\lim\limits_{n \to +\infty} x_n^i = 0$ and $\lim\limits_{n \to +\infty} g(x_n^i) = l_i$ $(i = 1, 2)$. Assume that $l_i \neq h$. Since ψ is continuous at the point $(0, l_i)$ (see (1.19)), we have $\psi(0, l_i) = \lim\limits_{n \to +\infty} \psi(x_n^i, g(x_n^i)) = Q_s$ since we have for all n, $\psi(x_n^i, g(x_n^i)) = Q_s$. But this contradicts the boundary condition of ψ on OC. So $l_1 = l_2 = h$.

iii) Set $l_1 = \liminf\limits_{x \to a^-} g(x)$ and $l_2 = \limsup\limits_{x \to a-} g(x)$. We have $l_1 \leq l_2$ and there exists in $(0, a)$ subsequences (x_n^1) and (x_n^2) satisfying : $\lim\limits_{n \to +\infty} x_n^i = a$ and $\lim\limits_{n \to +\infty} g(x_n^i) = l_i$

$(i=1,2)$. If $l_i \in \{0,1\}$ then $\psi(a,l_i) \in \{0,Q\}$. But by (1.19), we have $\psi(a,l_i) = \lim\limits_{n\to+\infty} \psi(x_n^i, g(x_n^i)) = Q_s$ since we have for all n, $\psi(x_n^i, g(x_n^i)) = Q_s$ and we get a contradiction. So $l_i \in (0,1)$.
Assume now that $l_1 < l_2$. Let $0 < \varepsilon < (l_2 - l_1)/2$, there exists $n_0 \geq 1$ such that $\forall n \geq n_0\ g(x_n^1) < l_1 + \varepsilon$ and $g(x_n^2) > l_2 - \varepsilon$. We get

$$\begin{array}{ll} \text{for} & z > l_1 + \varepsilon > g(x_{n_0}^1), \quad \psi(x_{n_0}^1, z) > Q_s \\ \text{and} & \quad \text{for } z < l_2 - \varepsilon < g(x_{n_1}^2), \quad \psi(x_{n_1}^2, z) < Q_s, \end{array}$$

where $n_1 > n_0$ and $x_{n_1}^2 > x_{n_0}^1$. Let $n_3 > n_2 > n_1$ such that $x_{n_3}^2 > x_{n_2}^1 > x_{n_1}^2 > x_{n_0}^1$. Then we have

$$\begin{array}{ll} \text{for} & z > l_1 + \varepsilon > g(x_{n_2}^1), \quad \psi(x_{n_2}^1, z) > Q_s \\ \text{and} & \quad \text{for } z < l_2 - \varepsilon < g(x_{n_3}^2), \quad \psi(x_{n_3}^2, z) < Q_s \end{array}$$

and we get a contradiction with Lemma 2.3. Hence $l_1 = l_2 = u_s \in (0,1)$. □

Corollary 2.6. *We have*

i) $\gamma = \chi([\psi < Q_s])$ *a.e. in* Ω

ii) $\gamma_N = \chi([z < u_s])$ *a.e. in* $(0,1)$.

Proof. i) We have $\gamma \in 1 - H(\psi - Q_s)$, then $\gamma = \chi([\psi < Q_s])$ a.e. in $\Omega \setminus [\psi = Q_s]$. But since the set $[\psi = Q_s] = [x = g(z)]$ is of Lebesgue measure zero, i) holds.

ii) We have $\gamma_N \in 1 - H(\psi - Q_s)$ a.e. in AB, then $\gamma_N = \chi([\psi < Q_s])$ a.e. in $(0,1) \setminus ([\psi = Q_s] \cap AB) = (0,1) \setminus \{u_s\}$ which we can write $\gamma_N = \chi([z < u_s])$ a.e. in $(0,1)$. □

3 Asymptotic behaviour when $Q_s \to 0$

As in section 2, we consider only monotone solutions of (P) in the sense of (1.16). In order to show the dependence on the quantity Q_s, we will denote in all this section, a solution (ψ, γ, γ_N) of (P) by $(\psi_s, \gamma_s, \gamma_{Ns})$ and (P) by (P_s). Then we have :

Theorem 3.1. *Let* $(\psi_s, \gamma_s, \gamma_{Ns})$ *be a solution of* (P_s). *Then we have :*

$$\psi_s \longrightarrow \psi \qquad \text{in } W^{1,q}(\Omega) \tag{3.1}$$

$$\gamma_s \rightharpoonup \gamma \qquad \text{in } L^{q'}(\Omega) \tag{3.2}$$

$$\gamma_{Ns} \rightharpoonup \gamma_N \qquad \text{in } L^{q'}(\Gamma_N) \tag{3.3}$$

where (ψ, γ, γ_N) *is a solution of*

$$(P)\qquad \begin{cases} \textit{Find } (\psi,\gamma,\gamma_N) \in W^{1,q}(\Omega)\times L^\infty(\Omega)\times L^\infty(\Gamma_N) \textit{ satisfying :}\\ i)\quad \int_\Omega (|\nabla\psi|^{q-2}\nabla\psi + \gamma e_x).\nabla\zeta = \int_{\Gamma_N}\gamma_N\zeta\\ \qquad\qquad \forall\zeta\in W^{1,q}(\Omega) \textit{ such that } \zeta = 0 \textit{ on } \Gamma_D\\ ii)\quad \gamma\in 1-H(\psi) \textit{ a.e. in } \Omega, \qquad \gamma_N \in 1-H(\psi) \textit{ a.e. on } \Gamma_N\\ iii)\quad \psi=\psi_0 \qquad \textit{on } \Gamma_D. \end{cases}$$

Moreover, we have

$$0 \le \psi \le Q_f. \tag{3.4}$$

Proof. Take $\psi_s - \psi_0$ as a test function in (P_s). Since by $(P_s)ii)$, γ_s and γ_{Ns} are bounded and ψ_0 does not depend on Q_s, we see that $|\nabla\psi_s|_{L^q(\Omega)}$ is bounded and by (1.18), we get

$$|\psi_s|_{1,q} \le c. \tag{3.5}$$

Up to a subsequence, we have (3.2), (3.3) and the following convergences :

$$\psi_s \rightharpoonup \psi \qquad \text{in } W^{1,q}(\Omega) \tag{3.6}$$

$$\psi_s \longrightarrow \psi \qquad \text{in } L^q(\Omega) \text{ and a.e. in } \Omega \tag{3.7}$$

$$\psi_s \longrightarrow \psi \qquad \text{in } L^q(\partial\Omega) \text{ and a.e. in } \partial\Omega \tag{3.8}$$

First, using (1.18) and (3.7), we get (3.4). Next, by $(P_s)ii)$, $1-\gamma_s \in H(\psi - Q_s)$ and by (3.2), $1-\gamma_s \rightharpoonup 1-\gamma$ in $L^{q'}(\Omega)$. Using (3.7), we get $1-\gamma\in H(\psi)$ since H is a maximal monotone graph (see [8]). In the same way, we have $1-\gamma_N \in H(\psi)$ a.e. in Γ_N. Hence $(P)ii)$ holds.

We also deduce $(P)iii)$ from (3.8) and $(P_s)iii)$. To verify $(P)i)$, it suffices to establish the strong convergence (3.1). So take $\psi_s - \psi$ as a test function in $(P_s)i)$, we obtain :

$$\int_\Omega |\nabla\psi_s|^q = \int_\Omega |\nabla\psi_s|^{q-2}\nabla\psi_s.\nabla\psi - \int_\Omega \gamma_s(\psi_s-\psi)_x + \int_{\Gamma_N}\gamma_{Ns}(\psi_s-\psi). \tag{3.9}$$

By (3.3) and (3.8), we have

$$\lim_{Q_s\to 0}\int_{\Gamma_N}\gamma_{Ns}(\psi_s-\psi) = 0. \tag{3.10}$$

We also have

$$\lim_{Q_s\to 0}\int_\Omega \gamma_s(\psi_s-\psi)_x = 0. \tag{3.11}$$

Indeed, we have

$$\int_\Omega \gamma_s(\psi_s-\psi)_x = \int_\Omega \gamma_s(\psi_s-Q_s)_x + \int_\Omega \gamma_s(Q_s-\psi)_x$$

$$\begin{aligned}
&= -\int_\Omega \gamma_s(\psi_s - Q_s)_x^- - \int_\Omega \gamma_s\psi_x \quad \text{since } \gamma_s(\psi_s-Q_s)_x^+ = 0 \text{ a.e. in } \Omega \\
&= -\int_{\partial\Omega} (\psi_s - Q_s)^-\nu_x - \int_\Omega \gamma_s\psi_x \\
&\to -\int_{\partial\Omega} \psi^-\nu_x - \int_\Omega \gamma\psi_x = 0
\end{aligned}$$

since by $(P)ii)$ and (3.4), we have $\gamma\psi_x = 0$ and $\psi^- = 0$ a.e. in Ω. Now by (3.10), (3.11), we deduce from (3.9) by applying the Hölder inequality

$$\limsup_{Q_s\to 0} |\nabla\psi_s|_{L^q(\Omega)} \leq |\nabla\psi_s|_{L^q(\Omega)}.$$

Then (see [9]) $\nabla\psi_s \longrightarrow \nabla\psi$ in $L^q(\Omega)$. □

As a consequence of (3.1)-(3.3) and (1.16), we have

Corollary 3.2. *Let (ψ,γ,γ_N) be a limit of $(\psi_s,\gamma_s,\gamma_{Ns})$, then we have*

$$\partial_z\psi \geq 0, \quad \partial_z\gamma \leq 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{and} \quad \partial_z\gamma_N \leq 0 \quad \text{in } \mathcal{D}'(\Gamma_N). \tag{3.12}$$

Moreover, arguing as in Proposition 3.1, we get

Proposition 3.3. *Let (ψ,γ,γ_N) be a solution of (P). We have*

$$\Delta_q\psi + \partial_x\gamma = div(|\nabla\psi|^{q-2}\nabla\psi + \gamma e_x) = 0 \qquad \text{in } \mathcal{D}'(\Omega) \tag{3.13}$$

$$0 \leq \psi \leq Q_f \qquad \text{in } \Omega \tag{3.14}$$

$$\psi \in C^{0,\alpha}_{loc}(\overline{\Omega}\setminus\{W\}) \qquad \text{for } 0<\alpha<1. \tag{3.15}$$

Now we have

Proposition 3.4. *Let (ψ,γ,γ_N) be a solution of (P). We have*

$$\Delta_q\psi \geq 0, \qquad \partial_x\gamma \leq 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.16}$$

Proof. Let $\zeta \in \mathcal{D}(\Omega)$, $\zeta \geq 0$ and $\eta > 0$. Then $\min(\zeta, \dfrac{\psi}{\eta})$ is a test function for (P) and we argue as in the proof of Lemma 2.5. □

We also have similar properties of Corollary 1.4 :

Corollary 3.5.

i) $[\psi > 0]$ *is an open connected set where* ψ *is not constant.*

ii) $\psi \in C^{1,1}_{loc}([\psi > 0])$.

iii) ψ *is analytic in* $[\psi > 0] \setminus S$ *where* $S = \{\, (x,z) \in \Omega \setminus [\psi = 0] \,/\, \nabla\psi(x,z) = 0 \,\}$.

iv) S *is a discrete set.*

In all what follows, we only consider solutions of (P) obtained as a limit of solutions of (P_s).

3.1 Study of the free boundary

We define the free boundary by $\Gamma = \partial([\psi > 0]) \cap \Omega$. As in section 2, one can define $g : (0,a) \to I\!R$ by : $g(x) = \inf\{\, z / \,\psi(x,z) > 0 \,\}$ and verify that :

$$\begin{array}{ll} i) & g \text{ is upper semi-continuous on } (0,a). \\ ii) & [\psi(x,z) > 0] = [z > g(x)] \text{ and } [\psi = 0] = [z \le g(x)]. \end{array}$$

We have

Theorem 3.6. *g is a decreasing function on $(0,a)$.*

Proof. 1^{st} *step. Let* $(x_0,z_0) \in \Omega$ *such that* $\psi(x_0,z_0) > 0$. *Then* $\psi > 0$ *in* $[x_0,a) \times [z_0,1)$.
Indeed, since ψ is continuous, there exists $\varepsilon > 0$ such that $\psi > 0$ in $B((x_0,z_0),\varepsilon)$. So $\gamma = 0$ in $B((x_0,z_0),\varepsilon)$. But since $0 \le \gamma \le 1$ and $\partial_x\gamma \le 0$, we get : $\gamma = 0$ in $B((x_0,z_0),\varepsilon) \cup (x_0,a) \times (z_0-\varepsilon, z_0+\varepsilon) = \mathcal{C}$ and then $\Delta_q\psi = 0$ in $\mathcal{C}$. Moreover $\psi \ge 0$ in $\mathcal{C}$. By the strong maximum principle, we deduce that $\psi > 0$ in $\mathcal{C}$ and using the monotonicity of ψ, we get $\psi > 0$ in $[x_0,a) \times [z_0,1)$.

2^{nd} *step. g is a non-increasing function.*
Indeed, let $x_0 \in (0,a)$ and $\delta > 0$. We have for $\varepsilon > 0$ small enough, $\psi(x_0, g(x_0)+\varepsilon) > 0$ and by the 1^{st} step, $\psi(x_0+\delta, g(x_0)+\varepsilon) > 0$. Then $g(x_0)+\varepsilon > g(x_0+\delta)$ $\forall\varepsilon > 0$ small. Letting $\varepsilon \to 0$, we obtain $g(x_0) \ge g(x_0+\delta)$.

3^{rd} *step. g is a decreasing function.*
Assume that there exists $0 < x_0 < x_1 < a$ such that $g(x_0) = g(x_1)$. Since g is non-increasing, then $g(x) = g(x_0)$ $\forall x \in [x_0,x_1]$. We then have

$$\begin{cases} \psi(x,z) = 0 & \text{for } (x,z) \in [x_0,x_1] \times [0, g(x_0)] \\ \psi(x,z) > 0 & \text{for } (x,z) \in [x_0,x_1] \times (g(x_0), 1]. \end{cases}$$

Set $D = (x_0,x_1) \times (0,1)$, $D^0 = (x_0,x_1) \times (0, g(x_0))$ and $D^+ = (x_0,x_1) \times (g(x_0),1)$. In D^0, we have $\gamma_x = -\Delta_q\psi = 0$ since $\psi = 0$, then $\gamma = \gamma(z)$. Let $\zeta \in \mathcal{D}(D)$. We have

$$\int_D |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta = -\int_D \gamma\zeta_x = -\int_{D^0} \gamma(z)\zeta_x = -\int_{\partial D^0} \gamma(z)\zeta\nu_x = 0.$$

So, ψ is a q-harmonic function in D satisfying $\psi \geq 0$ in D, $\psi > 0$ in D^+ and $\psi = 0$ in D^0. This leads to a contradiction with the strong maximum principle. $\square$

Theorem 3.7.

g is a continuous function on $(0, a)$,

$$\lim_{x\to 0^+} g(x) = g(0^+) \leq h \textit{ and } \lim_{x\to a^-} g(x) = g(a^-) < 1 \textit{ exist.}$$

To prove this theorem, we need two lemmas.

Lemma 3.8. *We have $\gamma = 1$ a.e. in $[\psi = 0]$.*

Proof. Let $Q_s^1 \leq Q_s^2$ and $\psi_{s\varepsilon}^1$ (resp. $\psi_{s\varepsilon}^2$) be the solution of $(P_\varepsilon(Q_s^1))$ (resp. $(P_\varepsilon(Q_s^2))$). We note that $\psi_{s\varepsilon}^1 - Q_s^1$ and $\psi_{s\varepsilon}^2 - Q_s^2$ satisfy the same equation with $\psi_{s\varepsilon}^1 - Q_s^1 \geq \psi_{s\varepsilon}^2 - Q_s^2$ on Γ_D. Arguing as in [12], we can prove that $\psi_{s\varepsilon}^1 - Q_s^1 \geq \psi_{s\varepsilon}^2 - Q_s^2$ in Ω. Letting ε go to 0, we obtain $\psi_s^1 - Q_s^1 \geq \psi_s^2 - Q_s^2$ in Ω.
Letting Q_s^1 go to 0, we get $\psi \geq \psi_s^2 - Q_s^2$ in Ω. Hence, we have $\psi_s^2 - Q_s^2 \leq 0$ a.e. in $[\psi = 0]$ and then $\gamma_s^2 = 1$ a.e. in $[\psi = 0]$. Letting $Q_s^2 \to 0$, we obtain $\gamma = 1$ a.e. in $[\psi = 0]$. $\square$

Lemma 3.9. *Let $m_0 = (x_0, z_0)$ and $\rho > 0$ such that $B(m_0, \rho) \subset \Omega$. Set $S_{eg} = \{x_0\} \times (z_0 - \rho, z_0 + \rho)$. Then we cannot have the following situation :*

$$\psi = 0 \qquad \textit{in } B^- \cup S_{eg} \qquad \textit{and} \qquad \psi > 0 \qquad \textit{in } B^+$$

where $B^- = B(m_0, \rho) \cap [x < x_0]$ and $B^+ = B(m_0, \rho) \cap [x > x_0]$.

Proof. Let $0 < \delta < \rho/2$ and set $\psi_\delta(x, z) = \psi(x, z - \delta)$. Using Lemma 3.8 and arguing as in the proof of (2.10), we obtain

$$\int_{B(m_0,\rho/2)} |\nabla\psi_\delta|^{q-2}\nabla\psi_\delta.\nabla\zeta = \int_{B(m_0,\rho/2)} |\nabla\psi|^{q-2}\nabla\psi.\nabla\zeta$$

from which we deduce by using the same notation as in the proof of Lemma 2.4,

$$\int_{B^+(m_0,\rho/2)} a(x, z)\nabla\varphi.\nabla\zeta = 0.$$

Since $\psi = \psi_\delta = 0$ on $B^-(m_0, \rho/2)$, we deduce by setting $a(x, z) = I_2$ in $B^-(m_0, \rho/2)$, the following equality

$$\int_{B(m_0,\rho/2)} a(x, z)\nabla\varphi.\nabla\zeta = 0.$$

Now arguing as in the proof of Lemma 2.4, we verify that the matrix $a(x, z)$ is strictly elliptic in $B(m_0, \rho/2)$. Then by applying the strong maximum principle for linear P.D.E, we get $\varphi = 0$ in $B(m_0, \rho/2)$ since we have $\varphi \geq 0$ in $B(m_0, \rho/2)$ and $\varphi = 0$ in $B^-(m_0, \rho/2)$. So $\psi = \psi_\delta$ in $B(m_0, \rho/2)$ and we conclude as in the proof of Lemma 2.4. $\square$

Proof of Theorem 3.7. By monotonicity of g, we deduce that $g(0^+)$, $g(a^-)$ exist and for all $x_0 \in (0, a)$, $l^- = \lim\limits_{x \to x_0^-} g(x)$, $l^+ = \lim\limits_{x \to x_0^+} g(x)$ also exist and satisfy $l^- \geq l^+$.
Assume that $l^- > l^+$, then we have by monotonicity of g, $g(x) \geq l^-$ $\forall x < x_0$ and $g(x) \leq l^+$ $\forall x > x_0$. We deduce :

$$\begin{array}{rcll} \psi(x,z) & = & 0 & \forall (x,z) \in (0, x_0) \times (0, l^-) \qquad \text{and} \\ \psi(x,z) & > & 0 & \forall (x,z) \in (x_0, a) \times (l^+, 1). \end{array}$$

We have by continuity of ψ, $\psi = 0$ on $S_{eg} = \{x_0\} \times (l^+, l^-)$. This contradicts Lemma 3.9. Hence $l^- = l^+$ and g is continuous at the point x_0.
Assume that $g(0^+) > h$. Let $(x_n)_n$ be a sequence such that $\lim\limits_{n \to +\infty} x_n = 0$ and $\lim\limits_{n \to +\infty} g(x_n) = g(0^+)$. We have $\psi(x_n, g(x_n)) = 0$. So $0 = \lim\limits_{n \to +\infty} \psi(x_n, g(x_n)) = \psi(0, g(0^+)) = Q_f$ which is impossible.
Now we have $\psi(a, 1) = Q > 0$, then $\psi(x,z) > 0$ $\forall (x,z) \in (a - \varepsilon, a) \times (1 - \varepsilon, 1)$ for some $\varepsilon > 0$, then $g(x) < 1 - \varepsilon$ $\forall x \in (a - \varepsilon, a)$. Thus $g(a^-) \leq 1 - \varepsilon < 1$. $\square$

Corollary 3.10. *We have*

i) $\Gamma = [z = g(x)]$.

ii) $\gamma = \chi([\psi = 0]) = \chi([z \leq g(x)])$ *a.e. in* Ω.

iii) $\gamma_N = \chi([\psi = 0]) = \chi([z < g(a^-)])$ *a.e. in* $(0,1)$.

Proof. i) is a consequence of the definition of Γ and the continuity of g.

ii) Note that we have $\Omega = [\psi > 0] \cup Int([\psi = 0]) \cup \Gamma$ where $Int(E)$ denotes the interior of the set E. Since Γ is of Lebesgue's measure zero, ii) holds as a consequence of Lemma 3.8.

iii) Set $\alpha = g(a^-)$. Let $z_0 \in (\alpha, 1)$ and $\varepsilon > 0$ such that $I_\varepsilon = (z_0 - \varepsilon, z_0 + \varepsilon) \subset (\alpha, 1)$. Let $\eta > 0$ such that $\alpha + \eta < z_0 - \varepsilon$. Then there exists $\delta > 0$ such that $\forall x \in (a - \delta, a)$, we have $\alpha \leq g(x) < \alpha + \eta < z_0 - \varepsilon$. So we have $\forall (x,z) \in (a - \delta, a) \times I_\varepsilon = \Delta$, $\psi(x,z) > 0$ and $\gamma = 0$ a.e. in Δ. Let $\zeta \in \mathcal{D}((a - \delta, a + \delta) \times I_\varepsilon)$, we have

$$\int_\Delta |\nabla\psi|^{q-2} \nabla\psi . \nabla\zeta = \int_{\Gamma_N \cap \overline{\Delta}} \gamma_N \zeta$$

from which we deduce

$$\gamma_N = |\nabla\psi|^{q-2} \partial_x \psi \qquad \text{a.e. on } \Gamma_N \cap \overline{\Delta}.$$

We distinguish two cases :
. If $\psi(a,z)>0$ on I_ε, then $\gamma_N=0$ a.e. on I_ε.
. If $\exists z_1\in I_\varepsilon$ such that $\psi(a,z_1)=0$ then by monotonicity of ψ, we have $\psi(a,z)=0$ $\forall z\in(z_0-\varepsilon,z_1]$. Since ψ is a non-negative q-harmonic function in Δ, we deduce by the Hopf's maximum principle (see [33]) that $\nabla\psi\neq 0$ on $\{a\}\times(z_0-\varepsilon,z_1)$. But since $\psi(a,z)=0$ $\forall z\in(z_0-\varepsilon,z_1]$, we have $\partial_z\psi=0$ on $\{a\}\times(z_0-\varepsilon,z_1)$. Then $\partial_x\psi\neq 0$. More precisely, we have $\partial_x\psi<0$ on $\{a\}\times(z_0-\varepsilon,z_1)$ since ψ reaches its minimal value on this set. We get a contradiction with $\gamma_N\geq 0$. So we have $\psi(a,z)>0$ in I_ε and $\gamma_N=0$ a.e. in I_ε.
Thus $\gamma_N=0$ a.e. in $(\alpha,1)$.

Now if $\alpha=0$ then iii) holds. Assume that $\alpha>0$. Take $z_0\in(0,\alpha)$ and $\varepsilon>0$ such that $I_\varepsilon\subset(0,\alpha)$. Since g is a decreasing function, we have $g(x)\geq\alpha$ $\forall x\in(0,a)$. We deduce that : $\forall(x,z)\in(0,a)\times I_\varepsilon=\Delta$, $g(x)\geq\alpha>z_0+\varepsilon>z$. Then $\psi=0$ and $\gamma=1$ a.e. in Δ. Moreover, taking $\zeta\in\mathcal{D}((0,a+1)\times I_\varepsilon)$ as a test function in (P), we obtain

$$\int_\Delta \zeta_x=\int_{\Gamma_N\cap\overline{\Delta}}\gamma_N\zeta.$$

Hence $\gamma_N=1$ a.e. in I_ε. So we have proved that $\gamma_N=1$ a.e. on $(0,\alpha)$ and iii) holds. □

3.2 Comparison principle

Theorem 3.11. *Let $(\psi_1,\gamma_1,\gamma_N^1)$ and $(\psi_2,\gamma_2,\gamma_N^2)$ be two solutions of (P). Then we have for $i=1$, 2*

$$T_i(\zeta)=\int_\Omega\big((|\nabla\psi_i|^{q-2}\nabla\psi_i-|\nabla\psi_m|^{q-2}\nabla\psi_m)+(\gamma_i-\gamma_M)e_x\big).\nabla\zeta=0\qquad\forall\zeta\in\mathcal{D}(I\!R^2),$$

where $\psi_m=\min(\psi_1,\psi_2)$ and $\gamma_M=\max(\gamma_1,\gamma_2)$.

To prove the theorem, we will need several lemmas. First, we introduce the inverted function of g_i denoted by $g_i^{-1}:(a_i,b_i)\longrightarrow(0,a)$ where $a_i=g_i(a^-)$, $b_i=g_i(0^+)$ and $f_i:[0,1]\longrightarrow(0,a)$ defined by :

$$f_i(z)=\begin{cases}a & \text{for } z\in[0,a_i]\\ g_i^{-1}(z) & \text{for } z\in(a_i,b_i)\\ 0 & \text{for } z\in[b_i,1].\end{cases}$$

Then f_i is a continuous and nonincreasing function on $[0,1]$. We also have $[\psi_i=0]=[x\leq f_i(z)]$ and $[\psi_i>0]=[x>f_i(z)]$. We define f_M by :

$$f_M(z)=\max(f_i,f_j)(z)=\begin{cases}a & \text{if } z\in[0,a_M]\\ \max(g_i^{-1}(z),g_j^{-1}(z)) & \text{if } z\in(a_M,\min(b_i,b_j))\\ g_i^{-1}(z) & \text{if } z\in(b_i,b_M)\text{ and } b_i\leq b_j\\ g_j^{-1}(z) & \text{if } z\in(b_j,b_M)\text{ and } b_j\leq b_i\\ 0 & \text{if } z\in(b_M,1)\end{cases}$$

with $a_M = \max(a_i, a_j)$ and $b_M = \max(b_i, b_j)$. Next, we have

Lemma 3.12. *Under the assumptions of Theorem 3.11, we have :*

$$T_i(\zeta) \leq \int_{(a_i,b_i)\cap I} \zeta(f_i(z), z)dz \qquad \forall \zeta \in \mathcal{D}(I\!R^2), \quad \zeta \geq 0$$

where $I = \{z \in (0,1) \,/\, f_i(z) < f_M(z)\}$.

Proof. Let $\zeta \in \mathcal{D}(I\!R^2)$, $\zeta \geq 0$ and $\varepsilon > 0$, then if we take $\xi = \min\Big(\zeta, \dfrac{\psi_i - \psi_m}{\varepsilon}\Big)$ as a test function in (P), we obtain :

$$\int_\Omega \big((|\nabla\psi_i|^{q-2}\nabla\psi_i - |\nabla\psi_j|^{q-2}\nabla\psi_j) + (\gamma_i - \gamma_j)e_x\big).\nabla\xi = \int_{\Gamma_N} (\gamma_i^N - \gamma_j^N)\xi.$$

Since $\gamma_i^N \in 1 - H(\psi_i)$ and H is a maximal monotone graph, we have $(\gamma_i^N - \gamma_j^N).(\psi_i - \psi_j) \leq 0$ a.e. in Γ_N. Then

$$\int_{\Gamma_N} (\gamma_i^N - \gamma_j^N)\xi = \int_{\Gamma_N\cap[\psi_i>\psi_m]} (\gamma_i^N - \gamma_j^N)\xi = \int_{\Gamma_N\cap[\psi_i>\psi_j]} (\gamma_i^N - \gamma_j^N)\xi \leq 0.$$

We deduce that

$$\int_\Omega \big((|\nabla\psi_i|^{q-2}\nabla\psi_i - |\nabla\psi_m|^{q-2}\nabla\psi_m) + (\gamma_i - \gamma_M)e_x\big).\nabla\xi \leq 0$$

which we can write as :

$$\begin{aligned}\int_{\Omega\cap[\psi_i-\psi_m\geq\varepsilon\zeta]} &\big(|\nabla\psi_i|^{q-2}\nabla\psi_i - |\nabla\psi_m|^{q-2}\nabla\psi_m\big).\nabla\zeta + \int_\Omega (\gamma_i - \gamma_M)e_x.\nabla\zeta \\ &\leq \int_\Omega (\gamma_i - \gamma_M)\Big(\zeta - \tfrac{\psi_i - \psi_m}{\varepsilon}\Big)^+_x .\end{aligned}$$

Remark that on $[\psi_m > 0]$, we have $\psi_i > 0$ and $\psi_j > 0$ and then $\gamma_i = \gamma_j = \gamma_M = 0$. We also have on $[\psi_i = 0]$, $\psi_m = 0$ and then $\gamma_i = \gamma_M = 1$. So the right-hand side of the above inequality can be written

$$\begin{aligned}\int_\Omega (\gamma_i - \gamma_M)\Big(\zeta - \frac{\psi_i - \psi_m}{\varepsilon}\Big)^+_x &= \int_{\Omega\cap[\psi_i>0]\cap[\psi_m=0]} -\Big(\zeta - \frac{\psi_i}{\varepsilon}\Big)^+_x \\ &= -\int_I \Big(\int_{f_i(z)}^{f_M(z)} \Big(\zeta - \frac{\psi_i}{\varepsilon}\Big)^+_x dx\Big)dz \\ &\leq \int_{I\cap(a_i,b_i)} \zeta(f_i(z), z) + \int_{I\cap(b_i,1)} \Big(\zeta(0,z) - \frac{Q_f}{\varepsilon}\Big)^+.\end{aligned}$$

Combining the last two inequalities and letting $\varepsilon \to 0$, the lemma follows. □

Lemma 3.13. *Under the assumptions of Theorem 3.11, we have :*

$$T_i(\zeta) \leq \int_{b_i}^{b_M} \zeta(0,z)dz \qquad \forall \zeta \in \mathcal{D}(I\!R^2), \quad \zeta \geq 0.$$

Proof. Let $\delta > 0$ and $\alpha_\delta(x,z) = \Big(1 - \dfrac{d((x,z),A_m)}{\delta}\Big)^+$ with $A_m = [\psi_m > 0]$. We have $1 - \alpha_\delta = 0$ on $\overline{A}_m$.
Let $\zeta \in \mathcal{D}(I\!R^2)$, $\zeta \geq 0$ and write $T_i(\zeta) = T_i(\alpha_\delta \zeta) + T_i((1-\alpha_\delta)\zeta)$. By the previous lemma, which is still true for $\zeta \in W^{1,q}(\Omega) \cap C^0(\overline{\Omega})$ $\zeta \geq 0$, we have

$$T_i(\alpha_\delta \zeta) \leq \int_{I \cap (a_i,b_i)} (\alpha_\delta \zeta)(f_i(z),z).$$

Note that for $z \in I \cap (a_i,b_i)$, $(f_i(z),z) \notin \overline{A}_m$, then by Lebesgue's theorem, we have

$$\lim_{\delta \to 0} \int_{I \cap (a_i,b_i)} (\alpha_\delta \zeta)(f_i(z),z) = 0. \tag{3.17}$$

Moreover, we have

$$\begin{aligned} T_i((1-\alpha_\delta)\zeta) &= \int_\Omega \big(|\nabla \psi_i|^{q-2} \nabla \psi_i + \gamma_i e_x\big).\nabla((1-\alpha_\delta)\zeta) \\ &- \int_\Omega \big(|\nabla \psi_m|^{q-2} \nabla \psi_m + \gamma_M e_x\big).\nabla((1-\alpha_\delta)\zeta) \\ &= I_1^\delta - I_2^\delta. \end{aligned}$$

First, we have

$$I_2^\delta = \int_{\Omega \setminus \overline{A}_m} \big(|\nabla \psi_m|^{q-2} \nabla \psi_m + \gamma_M e_x\big).\nabla((1-\alpha_\delta)\zeta) = \int_{[\psi_m = 0]} ((1-\alpha_\delta)\zeta)_x.$$

Since we have $[\psi_m > 0] = [x > f_M(z)]$ and $[\psi_m = 0] = [x \leq f_M(z)]$, one can write

$$\begin{aligned} I_2^\delta &= \int_0^{b_M} \int_0^{f_M(z)} ((1-\alpha_\delta)\zeta)_x \\ &= \int_0^{a_M} ((1-\alpha_\delta)\zeta)(a,z) - \int_0^{b_M} ((1-\alpha_\delta)\zeta)(0,z). \end{aligned} \tag{3.18}$$

Next, let $\eta > 0$ and set $d_\eta(x,z) = \min\Big(1, \dfrac{d((x,z),B_M OA)}{\eta}\Big)$ where $B_M = (0,b_M)$. Then $d_\eta(1-\alpha_\delta)\zeta$ is a test function for (P). So we have

$$\int_\Omega \big(|\nabla \psi_i|^{q-2} \nabla \psi_i + \gamma_i e_x\big).\nabla(d_\eta(1-\alpha_\delta)\zeta) = \int_{\Gamma_N} \gamma_N^i d_\eta (1-\alpha_\delta)\zeta. \tag{3.19}$$

Note that $\min\Big(\dfrac{\psi_i}{\varepsilon},(1-d_\eta)(1-\alpha_\delta)\zeta\Big)$ is also a test function ($\varepsilon>0$) for (P). We obtain

$$\int_{\Omega\cap[\psi_i>\varepsilon(1-d_\eta)(1-\alpha_\delta)\zeta]} |\nabla\psi_i|^{q-2}\nabla\psi_i.\nabla\big((1-d_\eta)(1-\alpha_\delta)\zeta\big) + \frac{1}{\varepsilon}\int_{\Omega\cap[\psi_i\le\varepsilon(1-d_\eta)(1-\alpha_\delta)\zeta]} |\nabla\psi_i|^q = 0$$

since we have by Corollary 3.10 ii) and iii),

$$\int_\Omega \gamma_i e_x.\nabla\min\Big(\frac{\psi_i}{\varepsilon},(1-d_\eta)(1-\alpha_\delta)\zeta\Big)=0$$

$$\text{and}\qquad \int_{\Gamma_N}\gamma_N^i\min\Big(\frac{\psi_i}{\varepsilon},(1-d_\eta)(1-\alpha_\delta)\zeta\Big)=0.$$

Then

$$\int_{\Omega\cap[\psi_i>\varepsilon(1-d_\eta)(1-\alpha_\delta)\zeta]} |\nabla\psi_i|^{q-2}\nabla\psi_i.\nabla\big((1-d_\eta)(1-\alpha_\delta)\zeta\big)\le 0$$

and by letting $\varepsilon\to0$, we get

$$\int_\Omega |\nabla\psi_i|^{q-2}\nabla\psi_i.\nabla\big((1-d_\eta)(1-\alpha_\delta)\zeta\big)\le 0. \tag{3.20}$$

Adding (3.19) and (3.20), we deduce

$$\int_\Omega|\nabla\psi_i|^{q-2}\nabla\psi_i.\nabla((1-\alpha_\delta)\zeta)+\int_\Omega\gamma_i e_x.\nabla(d_\eta(1-\alpha_\delta)\zeta)\le\int_{\Gamma_N}\gamma_N^i d_\eta(1-\alpha_\delta)\zeta. \tag{3.21}$$

Now, since we have $d_\eta(x,z)=\min\Big(1,\dfrac{\inf(x,z)}{\eta}\Big)\ \forall(x,z)\in[x\le f_M(z)]$, the second term of (3.21) can be written as

$$\int_\Omega\gamma_i e_x.\nabla(d_\eta(1-\alpha_\delta)\zeta)=\int_\Omega\gamma_i d_\eta((1-\alpha_\delta)\zeta)_x+\frac{1}{\eta}\int_0^{b_i}\Big(\int_0^{\min(\eta,z,f_i(z))}(1-\alpha_\delta)\zeta dx\Big)dz.$$

By letting $\eta\to0$ in the above equality, we get, since $d_\eta\to1$ in $\overline{\Omega}\setminus B_MOA$,

$$\lim_{\eta\to0}\int_\Omega\gamma_i e_x.\nabla(d_\eta(1-\alpha_\delta)\zeta)=\int_\Omega\gamma_i e_x.\nabla((1-\alpha_\delta)\zeta)+\int_0^{b_i}(1-\alpha_\delta)\zeta(0,z)dz. \tag{3.22}$$

We also have

$$\lim_{\eta\to0}\int_{\Gamma_N}\gamma_N^i d_\eta(1-\alpha_\delta)\zeta=\int_{\Gamma_N}\gamma_N^i(1-\alpha_\delta)\zeta=\int_0^{a_i}((1-\alpha_\delta)\zeta)(a,z)dz. \tag{3.23}$$

By (3.21)-(3.23), we get

$$I_1^\delta \leq \int_0^{a_i} ((1-\alpha_\delta)\zeta)(a,z)dz - \int_0^{b_i} (1-\alpha_\delta)\zeta(0,z)dz. \tag{3.24}$$

Hence

$$T_i(\zeta) \leq \int_{I\cap(a_i,b_i)} (\alpha_\delta\zeta)(f_i(z),z) - \int_{a_i}^{a_M} ((1-\alpha_\delta)\zeta)(a,z)dz + \int_{b_i}^{b_M} ((1-\alpha_\delta)\zeta)(0,z)dz.$$

Letting $\delta \to 0$ and using (3.17), we obtain

$$T_i(\zeta) \leq \int_{b_i}^{b_M} \zeta(0,z)dz - \int_{a_i}^{a_M} \zeta(a,z)dz \leq \int_{b_i}^{b_M} \zeta(0,z)dz.$$

□

Lemma 3.14. *Under the assumptions of Theorem 3.11, we have :*

$$T_i(\zeta) \leq 0 \qquad \forall \zeta \in \mathcal{D}(I\!R^2 \setminus \{B_i\}), \quad \zeta \geq 0$$

with $B_i = (0, b_i)$.

Proof. If $B_i = B_M$, the lemma follows from Lemma 3.13. Assume then $B_i \neq B_M$ i.e. $b_i < b_M$.
Let $\zeta \in \mathcal{D}(I\!R^2 \setminus \{B_i\})$, $\zeta \geq 0$. Then there exist $\rho_0 > 0$, $\rho_0^2 < a^2 + (b_i - a_i)^2$ such that $\zeta = 0$ on $B(B_i, \rho_0)$ and a unique point $P_i = (x_i, g(x_i)) \in \partial B(B_i, \rho_0) \cap [z = g_i(x)]$ (see Figure 3).

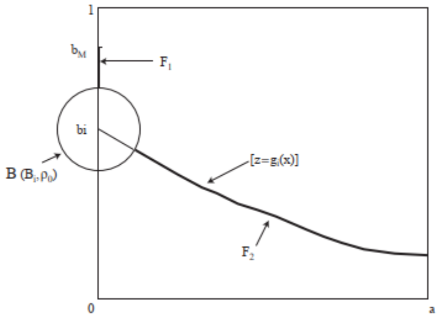

Figure 3

Set $F_1 = \{0\} \times [b_i + \rho_0, b_M]$ and $F_2 = \{(x, g_i(x)),\ x \in [x_i, a]\}$. For $\rho > 0$, consider for $i = 1, 2$ $d_\rho^i(x,z) = \min\left(1, \dfrac{d((x,z),F_i)}{\rho}\right)$. Since F_1, F_2 are closed sets and $F_1 \cap F_2 = \emptyset$ then $d_\rho^1 + d_\rho^2 \neq 0$.

We have $\zeta = \dfrac{d_\rho^1}{d_\rho^1 + d_\rho^2}\zeta + \dfrac{d_\rho^2}{d_\rho^1 + d_\rho^2}\zeta = \zeta_1 + \zeta_2$ where $\zeta_1, \zeta_2 \in W^{1,q}(\Omega) \cap C^0(\overline{\Omega})$, $\zeta_1, \zeta_2 \geq 0$, $\zeta_1 = 0$ on $\{0\} \times [b_i, b_M]$ and $\zeta_2 = 0$ on $[z = g_i(x)]$. Using lemmas 3.12, 3.13, which are still true for $\zeta \in W^{1,q}(\Omega) \cap C^0(\overline{\Omega})$ $\zeta \geq 0$, we get $T_i(\zeta) = T_i(\zeta_1) + T_i(\zeta_2) \leq 0$. □

Lemma 3.15. *Under the assumptions of Theorem 3.11, we have :*

$$T_i(\zeta) \leq 0 \qquad \forall \zeta \in \mathcal{D}(\mathbb{R}^2), \quad \zeta \geq 0.$$

Proof. Recall that we have $W_0^{1,q}(\mathbb{R}^2 \setminus \{B_i\}) = \overline{\mathcal{D}(\mathbb{R}^2 \setminus \{B_i\})}$. Moreover $\{B_i\}$ is $(1,2)$-polar (see [19]). Since $q' \geq 2$ then $\{B_i\}$ is $(1,q')$-polar and we have (see [29]) : $W_0^{1,q}(\mathbb{R}^2 \setminus \{B_i\}) = W^{1,q}(\mathbb{R}^2)$.

Now, let $\zeta \in W^{1,q}(\mathbb{R}^2)$, $\zeta \geq 0$. There exist $\zeta_n \in \mathcal{D}(\mathbb{R}^2 \setminus \{B_i\})$, $\zeta_n \geq 0$ such that $\lim\limits_{n\to\infty} \zeta_n = \zeta$ in $W_0^{1,q}(\mathbb{R}^2 \setminus \{B_i\})$. By Lemma 3.14, we have $T_i(\zeta_n) \leq 0$ $\forall n$. By letting $n \to \infty$, the lemma follows. □

Proof of Theorem 3.11. Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geq 0$. Let $M = \sup\limits_{\mathbb{R}^2} \zeta$ and $\xi \in \mathcal{D}(\mathbb{R}^2)$ such that $\xi = 1$ on $\overline{\Omega}$. Since $\xi.(M - \zeta) \in \mathcal{D}(\mathbb{R}^2)$ and $\xi.(M - \zeta) \geq 0$, we obtain $T_i(\xi.(M - \zeta)) \leq 0$ which can be written $T_i(\zeta) \geq 0$. So we have $T_i(\zeta) = 0$ $\forall \zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geq 0$ and by density the result holds for $\zeta \geq 0$ in $W^{1,q}(\mathbb{R}^2)$. If $\zeta \in W^{1,q}(\Omega)$, we write $\zeta = \zeta^+ - \zeta^-$. Then $T_i(\zeta) = T_i(\zeta^+) - T_i(\zeta^-) = 0$. In particular, we have $T_i(\zeta) = 0$ $\forall \zeta \in \mathcal{D}(\mathbb{R}^2)$. □

Corollary 3.16. *Under the assumptions of Theorem 3.11,* $(\psi_m, \gamma_M, \gamma_M^N)$ *is a solution of* (P) *with* $\gamma_M^N = \max(\gamma_i^N, \gamma_j^N) = \gamma_i^N = \gamma_j^N$.

Proof. We deduce from Theorem 3.11 that

$$\int_\Omega \left(|\nabla \psi_m|^{q-2} \nabla \psi_m + \gamma_M e_x\right).\nabla \zeta = \int_{\Gamma_N} \gamma_i^N \zeta \qquad \forall \zeta \in W^{1,q}(\Omega), \quad \zeta = 0 \text{ on } \Gamma_D.$$

In particular for $\zeta = 0$ on $\Omega \setminus (0,a) \times (0,a_M)$, we have

$$\int_{(0,a)\times(0,a_M)} \zeta_x = \int_0^{a_i} \zeta(a,z)$$

since $(\psi_m, \gamma_M) = (0,1)$ a.e. in $(0,a) \times (0,a_M)$. Then

$$\int_0^{a_M} \zeta(a,z) = \int_0^{a_i} \zeta(a,z)$$
$$\forall \zeta \in W^{1,q}(\Omega), \qquad \zeta = 0 \text{ on } \Gamma_D, \quad \zeta = 0 \text{ on } \Omega \setminus (0,a) \times (0,a_M).$$

We deduce that $a_M = a_i$. Hence $\gamma_1^N = \gamma_2^N = \gamma_M^N$ and $(P)i)$ is satisfied by $(\psi_m, \gamma_M, \gamma_M^N)$. □

3.3 Uniqueness of the solution

Theorem 3.17. *There exists a unique solution of problem* (P).

Proof. Let $(\psi_1, \gamma_1, \gamma_N^1)$, $(\psi_2, \gamma_2, \gamma_N^2)$ be two solutions of (P). Let $x_0 \in (0,a)$ and $\varepsilon_0 \in (0, \min(x_0, a-x_0, 1-h))$. Since $g_1(x), g_2(x) \le h \ \forall x \in (0,a)$, we have $\psi_1(x,z), \psi_2(x,z) > 0 \ \forall (x,z) \in B((x_0,1),\varepsilon_0) \cap \Omega$. Then $\Delta_q \psi_1 = \Delta_q \psi_2 = \Delta_q \psi_m = 0$ in $B((x_0,1),\varepsilon_0) \cap \Omega$. Moreover $\psi_1 = \psi_2 = \psi_m = Q_f$ on $B((x_0,1),\varepsilon_0) \cap \partial\Omega$. Then $\psi_1, \psi_2, \psi_m \in C^1(B((x_0,1),\varepsilon_0) \cap \overline{\Omega})$ (see [27], [33]). We deduce that $\partial_x \psi_1(x,1) = \partial_x \psi_m(x,1) = 0 \ \forall x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$. From Theorem 3.11, we deduce by taking $\zeta = 0$ on $\Omega \setminus B$:

$$|\nabla \psi_1|^{q-2} \nabla \psi_1 . \nu = |\nabla \psi_m|^{q-2} \nabla \psi_m . \nu \qquad \text{on } B((x_0,1),\varepsilon_0) \cap \partial\Omega.$$

Then $|\partial_z \psi_1|^{q-2} \partial_z \psi_1 = |\partial_z \psi_m|^{q-2} \partial_z \psi_m$ on $B((x_0,1),\varepsilon_0) \cap \partial\Omega$ and we deduce that : $\partial_z \psi_1(x,1) = \partial_z \psi_m(x,1) \ \forall x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$.
Since we have $\Delta_q(Q_f - \psi_1) = 0$, $Q_f - \psi_1 \ge 0$ in $B((x_0,1),\varepsilon_0) \cap \Omega$ and $Q_f - \psi_1 = 0$ on $B((x_0,1),\varepsilon_0) \cap \partial\Omega$, it follows by the strong maximum principle (see [33]) that $\nabla \psi_1 \ne 0$ on $B((x_0,1),\varepsilon_0) \cap \partial\Omega$ i.e. $\partial_z \psi_1(x,1) \ne 0 \ \forall x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$.
Now set $\psi_t = t\psi_1 + (1-t)\psi_m \ \forall t \in [0,1]$. We have :

$$\nabla \psi_t(x,1) = \nabla \psi_1(x,1) = \nabla \psi_m(x,1) \ne 0 \qquad \forall x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0), \quad \forall t \in [0,1]$$

and by continuity of $\nabla \psi_1$ and $\nabla \psi_m$, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that :

$$\nabla \psi_t(x,z) \ne 0 \qquad \forall (x,z) \in \overline{B((x_0,1),\varepsilon_1) \cap \Omega}, \quad \forall t \in [0,1].$$

Let $\zeta \in \mathcal{D}(B((x_0,1),\varepsilon_1))$, then we have :

$$\int_{B((x_0,1),\varepsilon_1)\cap\Omega} (|\nabla \psi_1|^{q-2} \nabla \psi_1 - |\nabla \psi_m|^{q-2} \nabla \psi_m) . \nabla \zeta = 0$$

which can be written

$$\int_{B((x_0,1),\varepsilon_1)} a(x,z) \nabla \varphi . \nabla \zeta = 0 \qquad \forall \zeta \in \mathcal{D}(B((x_0,1),\varepsilon_1))$$

with $\varphi = \psi_1 - \psi_m$ in $B((x_0,1),\varepsilon_1) \cap \Omega$ and $\varphi = 0$ in $B((x_0,1),\varepsilon_1) \setminus \Omega$,

$$a(x,z) = \begin{cases} \Big(\displaystyle\int_0^1 \frac{\partial A^k}{\partial h_l} (\nabla \psi_t(x,z)) dt \Big)_{1 \le k,l \le 2} & \text{in } B((x_0,1),\varepsilon_1) \cap \Omega \\ I_2 & \text{in } B((x_0,1),\varepsilon_1) \setminus \Omega \end{cases}$$

where A is defined in the proof of Lemma 2.4. Now, as in [15], we verify that we have :

$$(q-1)\lambda(x,z)|y|^2 \le a(x,z)y.y \le \lambda(x,z)|y|^2 \qquad \text{a.e. in } B((x_0,1),\varepsilon_1) \cap \Omega, \quad \forall y \in I\!R^2$$

with $\lambda(x,z) = \displaystyle\int_0^1 |\nabla\varphi_t|^{q-2}(x,z)dt$. Set

$$\begin{array}{rcl}\lambda_0 & = & \min\big\{\, |\nabla\varphi_t|^{q-2}(x,z), \quad (x,z) \in \overline{B((x_0,1),\varepsilon_1)\cap\Omega},\ t\in[0,1]\,\big\},\\ \lambda_1 & = & \max\big\{\, |\nabla\varphi_t|^{q-2}(x,z), \quad (x,z) \in \overline{B((x_0,1),\varepsilon_1)\cap\Omega},\ t\in[0,1]\,\big\}.\end{array}$$

Then $0 < \lambda_0 \le \lambda_1 < +\infty$ and

$$\begin{array}{ll}(q-1)\min(1,\lambda_0)|y|^2 & \le a(x,z)y.y \le \max(1,\lambda_1)|y|^2\\ \forall (x,z)\in B((x_0,1),\varepsilon_1), & \quad \forall y \in I\!R^2.\end{array}$$

So φ satisfies :

$$\left\{\begin{array}{ll}\varphi \in W^{1,\infty}(B((x_0,1),\varepsilon_1)) & \\ \varphi \ge 0 \quad \text{in } B((x_0,1),\varepsilon_1) & \text{and} \quad \varphi = 0 \quad \text{in } B((x_0,1),\varepsilon_1)\setminus\Omega\\ div(a(x,z)\nabla\varphi) = 0 & \text{in } \mathcal{D}'(B((x_0,1),\varepsilon_1)).\end{array}\right.$$

By the strong maximum principle, we deduce that $\varphi = 0$ in $B((x_0,1),\varepsilon_1)$. Thus

$$\psi_1 = \psi_m \qquad \text{in } B((x_0,1),\varepsilon_1)\cap\Omega.$$

Now, on $[\psi_m > 0] = [\psi_1 > 0]\cap[\psi_m > 0]$, ψ_1 and ψ_m are q-harmonic, so they are analytic on $[\psi_m > 0]\setminus Z$ where $Z = \{(x,z)\in[\psi_m>0] \,/\, \nabla\psi_1 = \nabla\psi_m = 0\}$. Since ψ_1 and ψ_m are not constant on $[\psi_m > 0]$, the set Z is discrete. Now since we have $\psi_1 = \psi_m$ in $B((x_0,1),\varepsilon_1)\cap\Omega \subset [\psi_m > 0]$, we deduce by analytic continuation that $\psi_1 = \psi_m$ in $[\psi_m > 0]$.

Let us show that $[\psi_1 > 0] = [\psi_m > 0]$. Indeed, since $[\psi_m > 0]$ is a nonempty open subset of the connected set $[\psi_1 > 0]$, it suffices to verify that $[\psi_m > 0]$ is a closed subset of $[\psi_1 > 0]$. So let $(p_n)_n$ be a sequence of elements of $[\psi_m > 0]$ converging to p in $[\psi_1 > 0]$. We have $\psi_1(p_n) = \psi_m(p_n)\ \forall n$. Letting $n\to\infty$, we get $\psi_1(p) = \psi_m(p)$ then $\psi_m(p) > 0$ and $p\in[\psi_m > 0]$.

As a consequence, we have $[\psi_m = 0] = \Omega\setminus[\psi_m > 0] = \Omega\setminus[\psi_1 > 0] = [\psi_1 = 0]$ and $\psi_1 = \psi_m = 0$ on $[\psi_1 = 0] = [\psi_m = 0]$. So $\psi_1 = \psi_m$ in Ω and by Corollary 3.10, we get $\gamma_1 = \gamma_M$, $\gamma_1^N = \gamma_M^N$.

Arguing as above, we prove the same result for ψ_2. Hence $(\psi_1,\gamma_1,\gamma_1^N) = (\psi_2,\gamma_2,\gamma_2^N)$.
□

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