



Regularity of the Free Boundary in $\operatorname{div}(a(x)\nabla u(x, y)) = -(h(x)\gamma(u))_x$ with $h'(x) < 0$

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Abstract: A free boundary problem of type $\operatorname{div}(a(x)\nabla u) = -(h(x)\gamma(u))_x$ with $h_x < 0$ is considered. A regularity of the free boundary as a curve $y = \Phi(x)$ is established using a local monotony $bu_x - u_y < 0$ close to free boundary points.

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1. Introduction

Our interest in this work came from the lubrication model studied by M. Chambat and G. Bayada in [2] where the pressure p is a periodic function solution of

$$\operatorname{div}(h^3(x)\nabla p) = (h(x)\gamma)_x; \quad h(x) = 1 + \alpha \cos x; \quad \alpha \in (0, 1)$$

and $0 \leq \gamma(p) \leq 1$ with $\gamma(p) = 1$ on $[p > 0]$. The authors established the existence of a solution and the uniqueness under regularity assumptions on the free boundary.

Recent works [4–6], showed a continuity of the free boundary in

$$\operatorname{div}(a(x, y)\nabla u) = -(h(x, y)\gamma(u))_x \quad \text{when } h_x(x, y) \geq 0.$$

This monotony on h led to a monotony of γ which allowed the characterization of the free boundary as a function $x = \phi(y)$. The continuity of ϕ is established under assumptions relating $a(x, y)$ with $h(x, y)$ in [4], and under $C_{loc}^{0,\alpha}$ regularity on a in [5]. These work brought answers to the Lubrication free boundary problem in half of the domain since

$$h'(x) = -\alpha \sin(x) \text{ is negative on } (0, \pi) \text{ and positive on } (\pi, 2\pi).$$

In an attempt to explore the situation where $h_x < 0$, we assume, in this paper, that a and h are independent of y . We look for a monotone solution u in the y -direction. The free boundary is then defined as a function $y = \Phi(x)$. We establish its

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continuity by using techniques developed in [1, 9] for obstacle problems where a solution is more regular at the free boundary points than it is in our case. The idea is to construct a cone with a vertex at a free boundary point while controlling a part of $[u > 0]$ in that cone.

For simplicity, we set the domain

$$\Omega = (0, 1) \times (0, 1), \quad \text{denote: } \Gamma_0 = (0, 1) \times \{0\}, \quad \Gamma_1 = (0, 1) \times \{1\},$$

and formulate the problem as:

$$(P) \left\{ \begin{array}{l} \text{Find } (u, \gamma) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ (i) \quad u \geq 0, \quad 0 \leq \gamma \leq 1, \quad u(\gamma - 1) = 0 \quad \text{a.e. in } \Omega \\ (ii) \quad u = \varphi \quad \text{on } \partial\Omega \\ (iii) \quad \int_{\Omega} (a(x)\nabla u + \gamma h(x)e_x) \cdot \nabla \xi \, dx dy = 0 \quad \forall \xi \in H_0^1(\Omega) \end{array} \right.$$

where $e_x = (1, 0)$, $\varphi \in C^{0,1}(\overline{\Omega})$ with

$$\varphi(x, y) = \left\{ \begin{array}{ll} 0 & \text{on } \Gamma_0, \\ u_a & \text{on } \Gamma_1, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \theta_0(y) & \text{on } \{0\} \times [0, 1] \\ \theta_1(y) & \text{on } \{1\} \times [0, 1] \end{array} \right.$$

with θ_i being regular and nondecreasing functions satisfying $0 \leq \theta_i(y) \leq u_a$, $i = 1, 2$, and u_a is a positive constant.

The function h is $C^2([0, 1])$ and satisfies for some positive constants $\bar{h}$ and λ :

$$|h(x)| \leq \bar{h}, \quad -\bar{h} \leq h'(x) < -\lambda < 0, \quad |h''(x)| \leq \bar{h} \quad \text{for } x \in [0, 1]. \quad (1)$$

The matrix a depends only on the x -variable and satisfies:

$$a \in W^{2,\infty}(0, 1) \cap C^{1,1}[0, 1] \quad (2)$$

$$m|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq M|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad m > 0, \quad M > 0. \quad (3)$$

The existence of a solution to (P) follows the proof in [3].

We introduce, for $\epsilon \in (0, \min(1, u_a))$, the penalization problem:

$$(P_\epsilon) \left\{ \begin{array}{l} \text{Find } u_\epsilon^\eta \in H^1(\Omega) \text{ such that :} \\ (i) \quad u_\epsilon = \varphi \quad \text{on } \partial\Omega \\ (ii) \quad \int_{\Omega} (a(x)\nabla u_\epsilon + h(x)H_\epsilon(u_\epsilon)e_x) \cdot \nabla \xi \, dx dy = 0 \quad \forall \xi \in H_0^1(\Omega) \end{array} \right.$$

with

$$H_\epsilon(t) = \left\{ \begin{array}{ll} 0 & \text{if } t < 0 \\ t/\epsilon & \text{if } 0 \leq t \leq \epsilon \\ 1 & \text{if } t > \epsilon \end{array} \right.$$

We show, as in [3], that there exists a unique solution for (P_ϵ) satisfying:

$$u_\epsilon \rightharpoonup u \quad \text{in } H^1(\Omega), \quad H_\epsilon(u_\epsilon) \rightharpoonup \gamma \quad \text{in } L^2(\Omega)$$

and that (u, γ) is a solution of (P) .

Taking u_ϵ^- (resp. $(u_\epsilon - u_a)^+$) as a test functions in (P_ϵ) , shows that $u_\epsilon \geq 0$ (resp. $u_\epsilon \leq u_a$). Then, comparing $u_\epsilon^\eta = u_\epsilon(x, y + \eta)$ with u_ϵ as in [7], we obtain $(u_\epsilon)_y \geq 0$ and finally get

$$0 \leq u \leq u_a, \quad \frac{\partial u}{\partial y} \geq 0 \quad \text{a.e in } \Omega. \quad (4)$$

In all what follows, we consider only monotone solutions of (P) . As a consequence, we deduce that:

- $\forall (x_0, y_0) \in [u > 0] = [u(x, y) > 0] \cap \Omega, \exists \delta > 0$ such that $u(x, y) > 0$ for $(x, y) \in B_\delta(x_0, y_0) \cup (x_0 - \delta, x_0 + \delta) \times [y_0, 1]$
- $\Phi : (0, 1) \rightarrow [0, 1]$ is well defined by $\Phi(x) = \inf\{y \in (0, 1) / u(x, y) > 0\}$ and is upper semi-continuous (u.s.c) on $(0, 1)$.
- $[u > 0] = [y > \Phi(x)]$.

Now, we list some properties of the solutions of (P) . We have

- $\text{div}(a(x)\nabla u) = -(h\gamma)_x$ in $\mathcal{D}'(\Omega)$.
- $u \in C_{loc}^{0,\alpha}(\Omega \cup \Gamma_0 \cup \Gamma_1)$ ([8, Theorem 8.24, p 202]).
- $[u > 0]$ is an open set.
- If $a \in C^{1,1}[0, 1]$ and $h \in C^2(0, 1)$, then $u \in C_{loc}^2([u > 0])$ ([8, Theorem 8.10, p 186]).
- $\text{div}(a(x)\nabla u) \geq -(h)_x \chi([u > 0])$ in $\mathcal{D}'(\Omega)$.
- $\text{div}(a(x)\nabla u) \geq 0$ and $(h\chi)_x \leq 0$ in $\mathcal{D}'(\Omega)$.

Remark 1.1. *The above last inequalities are obtained by taking $\pm(H_\epsilon(u)\xi)$, $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$ as a test function in (P) . The $C_{loc}^{0,\alpha}$ regularity holds because $h\gamma \in L^q(\Omega)$ for $q > 2$.*

The main result of this paper is the following:

Theorem 1.2. *Assume the interior of the set $(0, 1) \cap [\Phi(x) > 0]$ non empty. Then Φ is continuous at each interior point of $(0, 1) \cap [\Phi(x) > 0]$.*

To prove the theorem, we work close to a free boundary point P_0 . We construct a half cone with vertex at P_0 . This is possible by establishing a local monotony $b \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \leq 0$.

2. Local Monotony

For the purpose of clarity, we establish the monotonicity result through the following steps.

Step 1: First, we have

Lemma 2.1.

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } [u > 0]. \quad (5)$$

Proof. We have

$$\text{div}(a(x)\nabla \left(\frac{\partial u}{\partial y}\right)) = \frac{\partial}{\partial y} \left(\text{div}(a(x)\nabla u) \right) = \frac{\partial}{\partial y} (h'(x)) = 0 \quad \text{in } [u > 0] \quad \text{and} \quad \frac{\partial u}{\partial y} \geq 0.$$

By the strong maximum principle ([8, Theorem 9.6 p.225]), one has

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } [u > 0] \quad \text{or} \quad \frac{\partial u}{\partial y} = 0 \quad \text{in } [u > 0].$$

But if $\frac{\partial u}{\partial y} = 0$ in $[u > 0]$, then $u = u(x) = u(x, 1) = u_a$ since $u \in C^0(\Omega \cup \Gamma_1)$. This leads to $0 = \operatorname{div}(a(x)\nabla u) = -h'(x) > 0$ in $[u > 0]$ which is not possible. $\square$

Step 2: Next, let $x_0 \in (0, 1)$ with $y_0 = \Phi(x_0) > 0$. Set $\epsilon_0 = (1 - y_0)/6$, $\delta_0 = \min(x_0, 1 - x_0)/6$. Since Φ is u.s.c, then for $\epsilon \in (0, \epsilon_0)$, $\exists \delta \in (0, \delta_0)$ such that $\Phi(x) < \Phi(x_0) + \epsilon$ for any $x \in (x_0 - \delta, x_0 + \delta)$. Using the continuity of u up to the boundary $y = 1$, we can find $0 < \rho < 1 - (y_0 + 3\epsilon_0)$ such that $u > 0$ on $[x_0 - \delta, x_0 + \delta] \times [1 - \rho, 1]$. Set

$$F = [x_0 - \delta/2, x_0 + \delta/2] \times [y_0 + 2\epsilon, 1 - \rho]$$

$$G = (x_0 - \delta, x_0 + \delta) \times \left(y_0 + \epsilon, 1 - \frac{\rho}{2}\right)$$

Note that: $G \subset [u > 0] = [y > \Phi(x)]$.

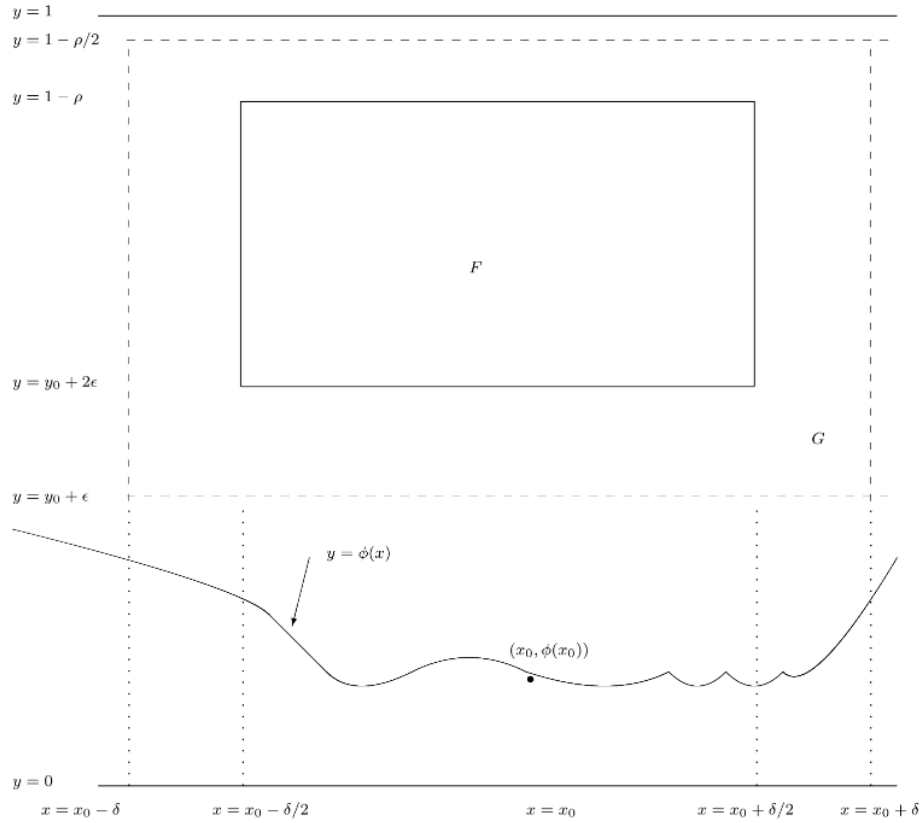


Figure 1: Set F and set G

Step 3: Since $u \in C^2([u > 0])$, then we have in G :

$$\begin{aligned} \operatorname{div}(a(x)\nabla u) &= -h'(x) \\ \operatorname{div}\left(a(x)\nabla\left(\frac{\partial u}{\partial y}\right)\right) &= 0 \end{aligned}$$

$$\operatorname{div} \left(a(x) \nabla \left(\frac{\partial u}{\partial x} \right) \right) = \frac{\partial}{\partial x} \left(\operatorname{div}(a(x) \nabla u) \right) - \left(\frac{\partial a_{11}}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial a_{22}}{\partial x} \frac{\partial^2 u}{\partial y^2} + \frac{\partial(a_{12} + a_{21})}{\partial x} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 a_{11}}{\partial x^2} \frac{\partial u}{\partial x} + \frac{\partial^2 a_{12}}{\partial x^2} \frac{\partial u}{\partial y} \right).$$

Using the assumptions on a and h , we deduce that the function

$$w = u + \tau \frac{\partial u}{\partial x} - E \frac{\partial u}{\partial y} \quad (6)$$

satisfies

$$\begin{aligned} \operatorname{div}(a(x) \nabla w) &= \operatorname{div}(a(x) \nabla u) + \tau \operatorname{div} \left(a(x) \nabla \left(\frac{\partial u}{\partial x} \right) \right) - E \operatorname{div} \left(a(x) \nabla \left(\frac{\partial u}{\partial y} \right) \right) \\ &= -h'(x) + \tau \operatorname{div} \left(a(x) \nabla \left(\frac{\partial u}{\partial x} \right) \right) - 0 \geq \lambda - |\tau| C_1 \quad \text{in } G \end{aligned}$$

where $C_1 = C(\bar{h}, |a|_{1,1}, |u|_{C^2(\bar{G})}, |a|_{W^{2,\infty}})$ is a constant depending on ϵ . Thus, for $|\tau| < \frac{\lambda}{2C_1}$, we have

$$\operatorname{div}(a(x) \nabla w) > \frac{\lambda}{2} \quad \text{in } G. \quad (7)$$

Step 4: Now, let $\zeta \in C^\infty(\mathbb{R}^2)$ satisfying $\zeta = 0$ on F , $0 \leq \zeta \leq 1$ and $\zeta \geq 1$ on ∂G . We have

$$\operatorname{div}(a(x) \nabla \zeta) = a_{11} \frac{\partial^2 \zeta}{\partial x^2} + a_{22} \frac{\partial^2 \zeta}{\partial y^2} + (a_{12} + a_{21}) \frac{\partial^2 \zeta}{\partial x \partial y} + \frac{\partial a_{11}}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial a_{12}}{\partial x} \frac{\partial \zeta}{\partial y}$$

Then,

$$|\operatorname{div}(a(x) \nabla \zeta)| \leq C_2 = \max_{\bar{G}} |\operatorname{div}(a(x) \nabla \zeta)|$$

We choose $\mu \in \left(0, \frac{\lambda}{2C_2}\right)$ so that

$$\mu \operatorname{div}(a(x) \nabla \zeta) \leq \frac{\lambda}{2} \quad \text{in } G$$

From (7), we deduce that

$$\operatorname{div}(a(x) \nabla (w - \mu \zeta)) \geq 0 \quad \text{in } G. \quad (8)$$

Step 5: Set

$$\begin{aligned} k &= \max_{\bar{G}} |u| + \max_{\bar{G}} |\nabla u|, & \tau_0 &= \frac{\mu}{1+k}, \\ E_0 &= \frac{k}{\beta(\tau_0)}, & \beta(\tau_0) &= \min_{[u \geq \tau_0] \cap \bar{G}} \left(\frac{\partial u}{\partial y} \right) \end{aligned}$$

By the extreme value theorem, the minimum value $\beta(\tau_0)$ is attained if the closed bounded set $[u \geq \tau_0] \cap \bar{G} \neq \emptyset$ and is strictly positive by Lemma 2.1; that is

$$\beta(\tau_0) = \frac{\partial u}{\partial y}(m_0) > 0; \quad m_0 \in [u \geq \tau_0] \cap \bar{G}.$$

To show that $w - \mu \zeta \leq 0$ on ∂G , we discuss the following:

- If $\partial G \cap [u \geq \tau_0] = \emptyset$, then $u < \tau_0$ on ∂G . As a consequence, we have

$$\begin{aligned} w &= u + \tau \frac{\partial u}{\partial x} - E \frac{\partial u}{\partial y} \leq \tau_0 + |\tau| k \quad \text{since } \frac{\partial u}{\partial y} \geq 0 \\ &\leq \tau_0 + \tau_0 k = \mu \quad \text{on } \partial G \quad \text{if } |\tau| < \tau_0. \end{aligned}$$

- If $\partial G \cap [u \geq \tau_0] \neq \emptyset$, then

- on $\partial G \cap [u < \tau_0]$, we have, as in the previous case $w \leq \mu$ on $\partial G \cap [u < \tau_0]$ if $|\tau| < \tau_0$.
- on $\partial G \cap [u \geq \tau_0]$,

$$\begin{aligned} w &= u + \tau \frac{\partial u}{\partial x} - E \frac{\partial u}{\partial y} \leq k + |\tau|k - E\beta(\tau_0) \quad \text{since } \frac{\partial u}{\partial y} \geq \beta(\tau_0) > 0 \\ &\leq k + \tau_0 k - E_0\beta(\tau_0) \quad \text{for } E \geq E_0 \\ &= \tau_0 k \leq \tau_0(1+k) = \mu \quad \text{on } \partial G \cap [u \geq \tau_0] \quad \text{if } |\tau| < \tau_0 \quad \text{and } E \geq E_0. \end{aligned}$$

Step 6: Now, since $\operatorname{div}(a(x)\nabla(w - \mu\zeta)) \geq 0$ in G and $w - \mu\zeta \leq 0$ on ∂G , we deduce, by the maximum principle, that $w - \mu\zeta \leq 0$ in G . In particular, we have $w \leq 0$ in F . Hence

$$\tau \frac{\partial u}{\partial x} - E \frac{\partial u}{\partial y} \leq -u < 0 \quad \text{in } F \quad \text{if } |\tau| < \tau_0 \quad \text{and } E \geq E_0.$$

Setting $b = \frac{\tau}{E}$ and $b_0 = \frac{\tau_0}{E_0}$, we finally established, through the 6 steps, the following result:

Theorem 2.2. *There exists a constant $b_0 > 0$ depending on x_0, y_0, ϵ such that*

$$b \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} < 0 \quad \text{in } F \quad \text{for } |b| < b_0.$$

3. Set of Directions

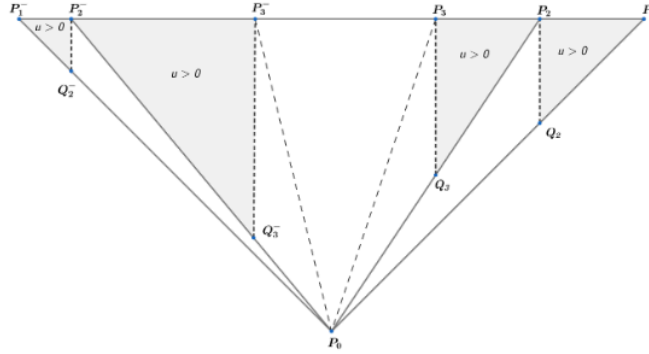


Figure 2: Lines P_0P_n and $P_n^-P_0$

In this section, we describe how we form particular lines passing through the vertex $P_0 = (x_0, y_0)$ and crossing the horizontal line $y = y_0 + 2\epsilon = y_c$.

Line segment P_0P_1 : It is possible to find $x_1 \in (x_0, x_0 + \delta/4)$ such that

$$\frac{x_1 - x_0}{y_c - y_0} = \frac{x_1 - x_0}{2\epsilon} = \kappa b_0 < b_0$$

It suffices, to choose

$$\kappa = \frac{x_1 - x_0}{2\epsilon} \cdot \frac{1}{b_0} \quad \text{such that} \quad |x_1 - x_0| = 2\epsilon b_0 \kappa < \frac{\delta}{4}; \quad \kappa \in \left(0, \min\left(\frac{1}{2}, \frac{\delta}{8\epsilon b_0}\right)\right).$$

Set $P_1 = (x_1, y_\epsilon)$ and $b_1 = \kappa b_0$. The line joining P_0 and P_1 has the slope

$$\frac{y_\epsilon - y_0}{x_1 - x_0} = b_1^{-1}$$

and can be described by the following two parameterizations:

$$P_0P_1 : \begin{cases} x = x_1 - tb_1 \\ y = y_\epsilon - t \end{cases} \quad t \in [0, 2\epsilon] \quad \text{or} \quad \begin{cases} x = x_0 + sb_1 \\ y = y_0 + s \end{cases} \quad s \in [0, 2\epsilon]$$

Define the function

$$f_{b_1}(t) = u(x_1 - tb_1, y_\epsilon - t).$$

We have

$$f_{b_1}(0) = u(x_1, y_\epsilon) > 0, \quad f'_{b_1}(0) = \left(-b_1 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)(x_1, y_\epsilon) < 0$$

By continuity, we have $f'_{b_1}(t) < 0$ for t small. Let

$$t_2 = \sup U; \quad U = \{t \in [0, 2\epsilon] \text{ such that } f_{b_1}(s) > 0 \ \forall \ s \in [0, t)\}.$$

Since $U \neq \emptyset$ and bounded, then t_2 is well defined and by continuity of f_{b_1} , we have $f_{b_1}(t_2) = 0$. Note that $t_2 > \epsilon$ since $u > 0$ in $(x_0 - \delta, x_0 + \delta) \times [y_0 + \epsilon, y_\epsilon]$. If $t_2 = 2\epsilon$, then $u > 0$ along the line segment $P_0P_1 \setminus \{P_0\}$. If $t_2 \neq 2\epsilon$, set

$$x_2 = x_1 - t_2 b_1; \quad y_2 = y_\epsilon - t_2; \quad Q_2 = (x_2, y_2),$$

then form the second line segment.

Line segment P_0P_2 : Set $P_2 = (x_2, y_\epsilon)$. Then, we have

$$\begin{aligned} \frac{x_2 - x_0}{y_\epsilon - y_0} &= \frac{1}{2\epsilon}((x_1 - t_2 b_1) - x_0) \\ &= \frac{1}{2\epsilon}((x_0 + 2\epsilon b_1 - t_2 b_1) - x_0) \text{ since } x_1 = x_0 + 2\epsilon b_1 \\ &= \left(\frac{2\epsilon - t_2}{2\epsilon} \right) b_1 = b_2 < b_1 < b_0. \end{aligned}$$

The line joining P_0 and P_2 has the slope

$$\frac{y_\epsilon - y_0}{x_2 - x_0} = b_2^{-1}$$

and can be described by the following two parameterizations:

$$P_0P_2 : \begin{cases} x = x_2 - tb_2 \\ y = y_\epsilon - t \end{cases} \quad t \in [0, 2\epsilon] \quad \text{or} \quad \begin{cases} x = x_0 + sb_2 \\ y = y_0 + s \end{cases} \quad s \in [0, 2\epsilon]$$

Define the function

$$f_{b_2}(t) = u(x_2 - tb_2, y_\epsilon - t).$$

We have

$$f_{b_2}(0) = u(x_2, y_\epsilon) > 0, \quad f'_{b_2}(0) = \left(-b_2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)(x_2, y_\epsilon) < 0$$

By continuity $f'_{b_2}(t) < 0$ for small t . Let $t_3 \in (0, 2\epsilon]$ such that

$$f_{b_2}(t_3) = 0 \quad \text{and} \quad f_{b_2}(t) > 0 \quad \text{for } t \in [0, t_3).$$

Note that $t_3 > \epsilon$ since $u > 0$ in $(x_0 - \delta, x_0 + \delta) \times [y_0 + \epsilon, y_\epsilon]$. If $t_3 = 2\epsilon$, then $u > 0$ along the line segment $P_0P_2 \setminus \{P_0\}$. If $t_3 \neq 2\epsilon$, set

$$x_3 = x_2 - t_3 b_2; \quad y_3 = y_\epsilon - t_3; \quad Q_3 = (x_3, y_3),$$

then we form the line segment P_0P_3 .

Line segment P_0P_n : By repeating the previous process, we will obtain a sequence of points $P_n = (x_n, y_\epsilon)$ and $Q_{n+1} = (x_{n+1}, y_\epsilon - t_{n+1})$ satisfying:

$$\begin{array}{ll} x_1 - x_0 = 2\epsilon b_1 = 2\epsilon b_0 \kappa & x_1 - x_2 = t_2 b_1 \geq \epsilon b_1 \\ x_2 - x_0 = 2\epsilon b_2 & x_2 - x_3 = t_3 b_2 \geq \epsilon b_2 \\ \vdots & \vdots \\ x_n - x_0 = 2\epsilon b_n & x_n - x_{n+1} = t_{n+1} b_n \geq \epsilon b_n \end{array}$$

The sequence (x_n) is convergent since it is decreasing and bounded. Indeed, we have

$$0 \leq 2\epsilon b_{n+1} \leq 2\epsilon b_n \leq 2\epsilon b_0 \kappa \leq \epsilon b_0 \quad \text{and} \quad t_{n+1} \geq \epsilon.$$

We also have

$$\begin{aligned} 2\epsilon b_0 \kappa &= x_1 - x_0 \geq x_1 - x_{n+1} \\ &= (x_1 - x_2) + (x_2 - x_3) + \dots + (x_n - x_{n+1}) \\ &\geq \epsilon b_1 + \epsilon b_2 + \dots + \epsilon b_n = s_n \geq 0 \end{aligned}$$

(s_n) is a bounded increasing sequence. Thus convergent. As a consequence

$$\epsilon b_n = s_n - s_{n-1} \longrightarrow 0 \quad \text{and} \quad x_n = x_0 + 2\epsilon b_n \longrightarrow x_0 \quad \text{as } n \longrightarrow +\infty.$$

Similarly, we obtain a sequence of points $P_1^-, P_2^-, \dots, P_n^-, \dots$ to the left of P_0 .

Line segment $P_1^-P_0$: With $x_1^- - x_0 = -2\epsilon \kappa b_0$, we have

$$\frac{x_1^- - x_0}{y_\epsilon - y_0} = \frac{x_1^- - x_0}{2\epsilon} = -\kappa b_0 > -b_0, \quad x_1^- \in (x_0 - \delta/4, x_0)$$

Since

$$\kappa = \frac{x_1^- - x_0}{2\epsilon} \cdot \frac{1}{-b_0} \quad \text{is such that} \quad |x_1^- - x_0| = 2\epsilon b_0 \kappa < \frac{\delta}{4} \quad \text{for } \kappa \in \left(0, \min\left(\frac{1}{2}, \frac{\delta}{8\epsilon b_0}\right)\right).$$

Set $P_1^- = (x_1^-, y_\epsilon)$ and $b_1 = \kappa b_0$. The line joining P_1^- and P_0 has the slope

$$\frac{y_\epsilon - y_0}{x_1^- - x_0} = -b_1^{-1}$$

and can be described by the following two parameterizations:

$$P_1^- P_0 : \begin{cases} x = x_1^- + tb_1 \\ y = y_c - t \end{cases} \quad t \in [0, 2\epsilon] \quad \text{or} \quad \begin{cases} x = x_0 - sb_1 \\ y = y_0 + s \end{cases} \quad s \in [0, 2\epsilon]$$

Define the function

$$g_{b_1}(t) = u(x_1^- + tb_1, y_c - t).$$

We have

$$g_{b_1}(0) = u(x_1^-, y_c) > 0, \quad g'_{b_1}(0) = \left(b_1 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right)(x_1^-, y_c) < 0$$

By continuity $g'_{b_1}(t) < 0$ for small t . Let $t_2^- \in (0, 2\epsilon]$ such that $g_{b_1}(t_2^-) = 0$ and $g_{b_1}(t) > 0$ for $t \in [0, t_2^-)$. Note that $t_2^- > \epsilon$ since $u > 0$ in $(x_0 - \delta, x_0 + \delta) \times [y_0 + \epsilon, y_c]$. If $t_2^- = 2\epsilon$, then $u > 0$ along the line segment $P_1^- P_0 \setminus \{P_0\}$. If $t_2^- \neq 2\epsilon$, set $x_2^- = x_1^- + t_2^- b_1$; $y_2^- = y_c - t_2^-$; $Q_2^- = (x_2^-, y_2^-)$, then we form the following line segment $P_2^- P_0$ with $P_2^- = (x_2^-, y_c)$.

Line segment $P_n^- P_0$: We obtain a sequence of points $P_n^- = (x_n^-, y_c)$ and $Q_{n+1}^- = (x_{n+1}^-, y_c - t_{n+1}^-)$ satisfying:

$$0 \leq x_0 - x_n^- = 2\epsilon b_n \leq 2\epsilon b_1 = 2\epsilon b_0 \kappa \leq \epsilon b_0; \quad x_{n+1}^- - x_n^- = t_{n+1}^- b_n \geq \epsilon b_n$$

and $x_n^- = x_0 - 2\epsilon b_n \rightarrow x_0$ as $n \rightarrow +\infty$.

4. Proof of Continuity

Proof of Theorem 1.2. Let x_0 be in the interior of $(0, 1) \cap [\Phi(x) > 0]$. Define $\epsilon_0, \delta_0, \rho$, and the sets F and G as in Step 2 of section 2. Then, construct points $P_1^-, P_2^-, \dots, P_n^-, \dots$ to the left of P_0 and points $P_1, P_2, \dots, P_n, \dots$ to the right of P_0 (as in section 3) such that $u > 0$ above the line segments $[P_i^- Q_{i+1}^-)$ and $[P_i Q_{i+1})$; $i = 1, 2, \dots, n, \dots$

For $x \in (x_0 - \frac{\delta}{8}, x_0 + \frac{\delta}{8}) \setminus \{x_0\}$, there exists $n \geq 1$ such that $x_{n+1} \leq x < x_n$ or $x_n^- < x \leq x_{n+1}^-$ since $x_n = x_0 + 2\epsilon b_n$ and $x_n^- = x_0 - 2\epsilon b_n$. Now, because $u(x, \Phi(x)) = 0$, the point $(x, \Phi(x))$ will be under the line segment

$$[P_n Q_{n+1}) : \quad x - x_0 = b_n(y - y_0) \quad \text{or} \quad [P_n^- Q_{n+1}^-) : \quad x - x_0 = -b_n(y - y_0).$$

We discuss two situations:

i). $\Phi(x) \geq y_0 = \Phi(x_0)$. We have

$$\Phi(x) - \Phi(x_0) < b_n^{-1}(x - x_0) \quad \text{if} \quad x_{n+1} \leq x < x_n$$

$$\Phi(x) - \Phi(x_0) < b_n^{-1}(x_0 - x) \quad \text{if} \quad x_n^- < x \leq x_{n+1}^-$$

* For $x_{n+1} \leq x < x_n$, we have

$$\begin{aligned} 0 < x - x_0 &= (x - x_n) + (x_n - x_0) \\ &\leq (x_n - x_{n+1}) + (x_n - x_0) \\ &= t_n b_n + 2\epsilon b_n \\ b_n^{-1}(x - x_0) &\leq t_n + 2\epsilon \leq 2\epsilon + 2\epsilon = 4\epsilon \end{aligned}$$

* For $x_n^- < x \leq x_{n+1}^-$, we have

$$\begin{aligned} 0 < x_0 - x &= (x_0 - x_{n+1}^-) + (x_{n+1}^- - x) \\ &\leq (x_0 - x_{n+1}^-) + (x_{n+1}^- - x_n^-) \\ &= 2\epsilon b_{n+1} + t_{n+1}^- b_n \\ b_n^{-1}(x_0 - x) &\leq 2\epsilon b_n^{-1} \cdot b_{n+1} + t_{n+1}^- \leq 2\epsilon + 2\epsilon = 4\epsilon \text{ since } b_{n+1} \leq b_n. \end{aligned}$$

Thus $\Phi(x) - \Phi(x_0) \leq 4\epsilon$.

ii). $\Phi(x) < y_0 = \Phi(x_0)$.

Set $x'_0 = x$ and $y'_0 = \Phi(x) = y$. The point $P'_0 = (x'_0, y'_0)$ will play the role of (x_0, y_0) in section 3 and we will form with the same process the sequence of points $P_1^-, P_2^-, \dots, P_n^-, \dots$ to the left of P'_0 and points $P'_1, P'_2, \dots, P'_n, \dots$ to the right of P'_0 . It is sufficient that we start with P'_1 (resp. P_1^-) such that the corresponding b'_1 (resp. b_1^-) is less than b_0 .

Line segments $P'_0 P'_1$ and $P_1^- P'_0$: We choose $P'_1 = (x'_1, y_c) = P_1 = (x_1, y_c)$ and $P_1^- = (x_1^-, y_c) = P_1^- = (x_1^-, y_c)$. This ensures that we remain working on the interval $[x_1^-, x_1]$ and x_0 is in this interval. We have

$$\begin{aligned} |x_0 - x'_0| &\leq 2\epsilon b_1 \text{ since } |x_0 - x_1| = |x_1^- - x_0| = 2\epsilon b_1 \\ |x'_1 - x'_0| &= |x_1 - x_0 + x_0 - x'_0| \leq |x_1 - x_0| + |x_0 - x'_0| \leq 4\epsilon b_1 \text{ since } x_1 = x_0 + 2\epsilon b_1 \\ y_c - y'_0 &\geq y_c - y_0 = 2\epsilon \text{ since } y'_0 < y_0 \\ b'_1 &= \frac{x'_1 - x'_0}{y_c - y'_0}. \end{aligned}$$

So

$$|b'_1| = \frac{|x'_1 - x'_0|}{y_c - y'_0} \leq \frac{4\epsilon b_1}{2\epsilon} = 2b_1 = 2\kappa b_0 < b_0 \iff \kappa < \frac{1}{2}$$

which is satisfied by the first choice of κ . Similarly, we have

$$-b_1^{-'} = \frac{x_1^- - x'_0}{y_c - y'_0}; \quad |b_1^{-'}| = \frac{|x_1^- - x'_0|}{y_c - y'_0} = \frac{|x_0 - 2\epsilon b_1 - x'_0|}{y_c - y'_0} \leq \frac{4\epsilon b_1}{2\epsilon} = 2b_1.$$

Arguing as in i), we obtain

$$\Phi(x_0) - \Phi(x) \leq \max\left(\frac{1}{b_q^{-'}}, \frac{1}{b'_m}\right) |x - x_0| \leq 4\epsilon,$$

depending if x_0 is to the right of x ; $x'_{m+1} \leq x_0 < x'_m$, or x_0 is to the left of x ; $x_q^{-'} < x_0 \leq x'_{q+1}$. Finally, we have

$$|\Phi(x) - \Phi(x_0)| \leq 4\epsilon \text{ for } |x - x_0| < \frac{\delta}{8}.$$

This completes the proof of the continuity of Φ .

References

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- [1] H. W. Alt, *The fluid flow Through Porous Media. Regularity of the free Surface*, Manuscripta Math., 21(1977), 255-272.
 - [2] Guy Bayada and Michèle Chambat, *Nonlinear variational formulation for a cavitation problem in lubrication*, Journal of Mathematical Analysis and Applications, 90(2)(1982), 286-298.
 - [3] M. Chipot, *Variational Inequalities and Flow in Porous Media*, Springer-Verlag New York Inc, (1984).

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- [4] M. Chipot, *On the Continuity of the Free Boundary in some Class of Dimensional Problems*, Interfaces Free Bound., 3(1)(2001), 81-99.
 - [5] S. Challal and A. Lyaghfour, *Continuity of the Free Boundary in Problems of type $\operatorname{div}(a(x)\nabla u) = -(\chi(u)h(x))_{x_1}$* , Nonlinear Analysis: Theory, Methods & Applications, 62(2)(2005), 283-300.
 - [6] S. Challal and A. Lyaghfour, *On the Continuity of the Free Boundary in Problems of type $\operatorname{div}(a(X)\nabla u) = -\operatorname{div}(\chi(u)H(X))$* , Differential and Integral Equations, 19(5)(2006), 481-516.
 - [7] S. Challal and A. Lyaghfour, *A stationary flow of fresh and salt groundwater in a heterogeneous coastal aquifer*, Bollettino della Unione Matematica Italiana, 8(2)(2000), 505-533.
 - [8] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, (1983).
 - [9] J. F. Rodrigues, *Obstacle problems in Mathematical physics*, North Holland, (1987).