Computing Derivatives: First Tools

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

e.g. $\frac{d}{dx}(x^3) = 3x^{(3-1)} = 3x^2$

Please check carefully this matches variable differentiating wrt

Product Rule: $\frac{d}{dz}[fg] = f\frac{dg}{dt} + g\frac{df}{dt}$

e.g. $\frac{d}{dt}\left[t^2\cos(t)\right] = t^2\frac{d}{dt}[\cos(t)] + \cos(t)\underline{\underline{\frac{d}{dt}[t^2]}}$

$= t^2[-\sin(t)] + \cos(t)\underline{\underline{(2t)}}$

$= \boxed{-t^2\sin(t) + 2t\cos(t)}$

$$\boxed{\frac{d}{dt}[t^2] = 2t^{2-1} = 2t^1 = 2t}$$

Quotient Rule:

$$\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g\frac{d}{dz}[f] - f\frac{d}{dz}[g]}{g^2}$$

$$\frac{d}{dx}\left[\frac{x^2}{\cos(x)}\right] = \frac{\cos(x)\frac{d}{dx}[x^2] - x^2\frac{d}{dx}[\cos(x)]}{(\cos(x))^2}$$

$$= \frac{(\cos(x))(2x) - x^2(-\sin(x))}{(\cos(x))^2}$$

$$= \boxed{\frac{2x\cos(x) + x^2\sin(x)}{(\cos(x))^2}}$$

Computing Derivatives: Polynomials

Key Tools:

① Power Rule: $\frac{d}{dx}(x^n) = nx^{(n-1)}$

New!

② "Linearity" of Differentiation:

$$\frac{d}{dx}[cf] = c\frac{d}{dx}[f]$$

has to be a constant or must use product rule

AND

$$\frac{d}{dx}[f+g] = \frac{d}{dx}[f] + \frac{d}{dx}[g]$$

must be addition or again must use product rule

Can combine to differentiate polynomials "term by term"

Example:

The constant e

$$\frac{d}{dx}[5x^3 + \pi x^2 - e]$$

$$= \frac{d}{dx}[5x^3] + \frac{d}{dx}[\pi x^2] - \underbrace{\frac{d}{dx}[e]}_{=0}$$

=0 because the derivative of every constant is zero

$$= 5\frac{d}{dx}[x^3] + \pi\frac{d}{dx}[x^2] - 0$$

$$= 5[3x^{\overset{2}{3-1}}] + \pi[2x^{\overset{1}{2-1}}]$$

$$= \boxed{15x^2 + 2\pi x}$$

Differentiation with "Funny Powers" Example

$$\frac{d}{dz}\left[\sqrt{z} + \frac{16}{\sqrt[3]{z}} + \sqrt{2}\, z^{\pi}\right]$$

Rewrite as powers

$$= \frac{d}{dz}\left[z^{1/2} + 16 z^{-1/3} + \sqrt{2}\, z^{\pi}\right]$$

Use "linearity" to "break apart"
Pull out constants

$$= \frac{d}{dz}\left[z^{1/2}\right] + 16\frac{d}{dz}\left[z^{-1/3}\right] + \sqrt{2}\,\frac{d}{dz}\left[z^{\pi}\right]$$

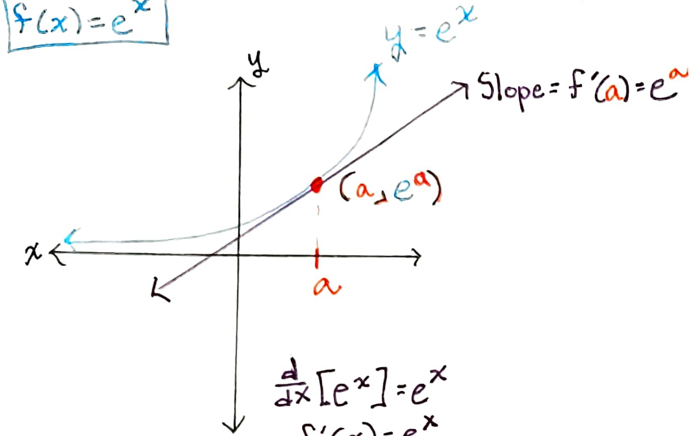
Careful use of parantheses &
1 step at a time

$$= \left[\frac{1}{2} z^{\overbrace{(\frac{1}{2}-1)}^{\frac{1}{2}-\frac{2}{2}=\frac{-1}{2}}}\right] + 16\left[\left(-\frac{1}{3}\right) z^{\overbrace{(-1/3-1)}^{\frac{-1}{3}-\frac{3}{3}=\frac{-4}{3}}}\right] + \sqrt{2}\left[\pi z^{(\pi-1)}\right]$$

$$= \boxed{\frac{1}{2} z^{-1/2} - \frac{16}{3} z^{-4/3} + \sqrt{2}\,\pi z^{(\pi-1)}}$$

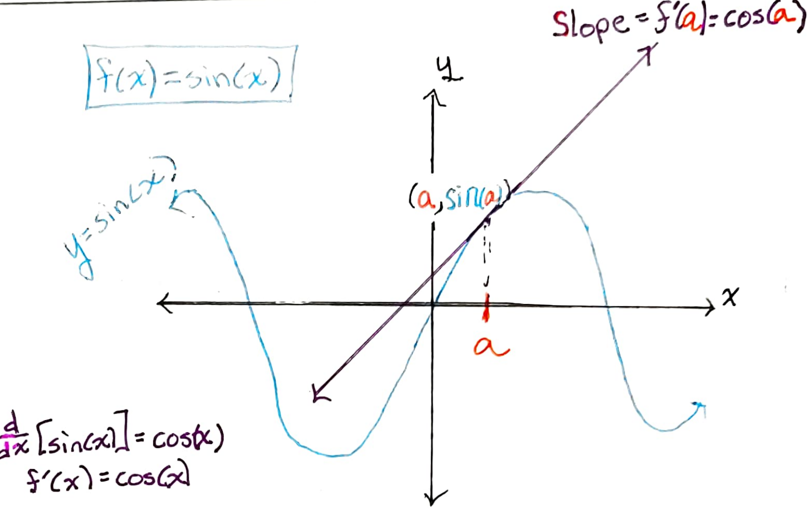
A Few (More) Good Derivatives to Know

$f(x)=e^x$



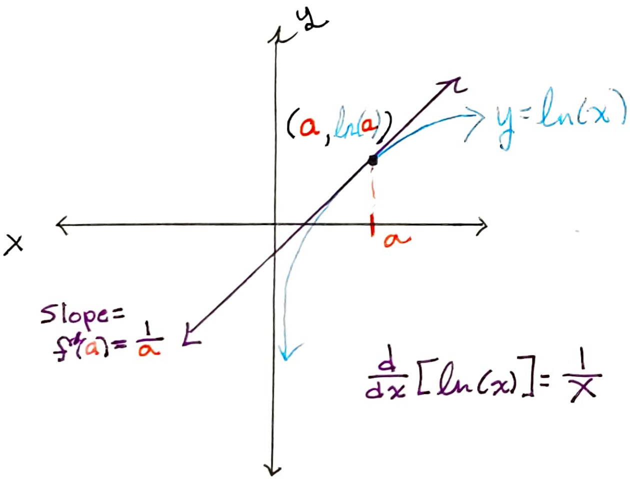
$$\frac{d}{dx}[e^x]=e^x$$
$$f'(x)=e^x$$

$f(x)=\sin(x)$



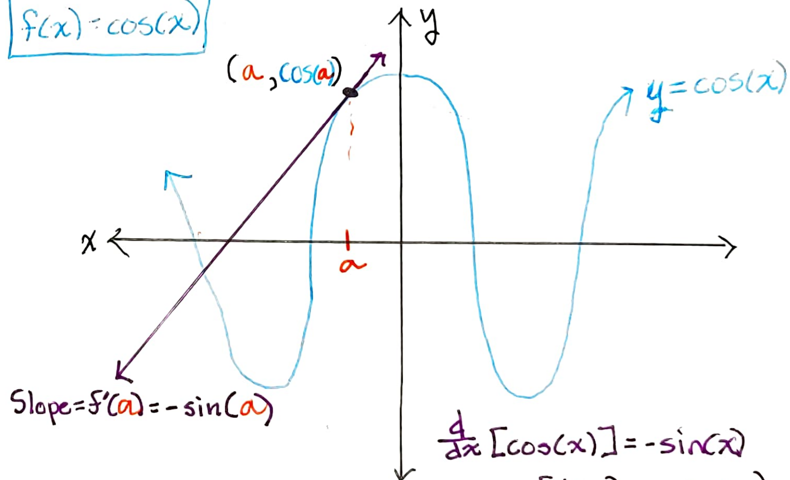
$$\frac{d}{dx}[\sin(x)]=\cos(x)$$
$$f'(x)=\cos(x)$$

$f(x)=\ln(x)$



$$\frac{d}{dx}[\ln(x)]=\frac{1}{x}$$

$f(x)=\cos(x)$



$$\frac{d}{dx}[\cos(x)]=-\sin(x)$$
$$f'(x)=-\sin(x)$$

Chain Rule by Example

Chain Rule: $\frac{d}{dx}\big[\underbrace{f\circ g}_{f(u(x))}\big] = \frac{d}{du}[f(u)]\cdot\frac{d}{dx}[u(x)]$

$$\left\{\begin{array}{l} \overbrace{y = f(u)}^{\text{outer}} \\ \underbrace{u = g(x)}_{\text{inner}} \end{array}\right\}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Ex: $\frac{d}{dx}\big[\overbrace{\cos}^{\text{Outside}}(\overbrace{3x^7+1}^{\text{Inside}})\big]$

$$= \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= [-\sin(u)]\cdot[21x^6]$$

Not done yet, not until u disappears & everything's in terms of x

$$= [-\sin(3x^7+1)]\cdot[21x^6]$$

$$= \boxed{-21x^6\sin(3x^7+1)}$$

Outer: $\overbrace{f(u)=\cos(u)}^{y}$

$$\frac{dy}{du} = \frac{d}{du}[\cos(u)]$$

$$\frac{dy}{du} = -\sin(u)$$

Inner: $u(x) = 3(x)^7+1$

$$\frac{du}{dx} = \frac{d}{dx}[3(x)^7+1] = 21x^6$$

Practice Identifying "Inside" and "Outside" In Chain Rule

① $h(t) = (\overbrace{\cos(t)}^{\text{Inside}})^2$

Inside $= \cos(t)$

Outside $= (u)^2 = u^2$

② $p(y) = e^{\overbrace{\sin(y)}^{\text{Inside}}}$

Inside $= \sin(y)$

Outside $= -e^{(u)} = e^u$

③ $g(x) = \dfrac{1}{x^2 - e^x}$ } Inside

Inside $= x^2 - e^x$

Outside $= \dfrac{1}{(u)} = \dfrac{1}{u}$

④ $s(t) = e^{\overbrace{2t}^{\text{Inside}}}$

Inside $= 2x$

Outside $= e^{(u)} = e^u$

Alternatively:

$s(t) = e^{2t} = (\overbrace{e^t}^{\text{Inside}})^2$

Inside $= e^t$

Outside $= (u)^2 = u^2$

Combining Product Rule and Chain Rule Example 1

Example 1: $\frac{d}{dx}\left[\sin(xe^x)\right]$, $f(u)$, $u = xe^x$

$$= \underbrace{\frac{d}{du}\left[\sin(u)\right]}\cdot\underbrace{\frac{d}{dx}\left[xe^x\right]}_{\text{Product Rule}}$$

$$= \left[\cos(u)\right]\cdot\left[x\cdot\underbrace{\frac{d}{dx}(e^x)}_{e^x} + e^x\cdot\underbrace{\frac{d}{dx}(x)}_{1}\right]$$

$$= \left[\cos(xe^x)\right]\cdot\left[xe^x + e^x\right]$$

$$= \underbrace{(xe^x + e^x)}_{\text{Derivative of inside sits here on outside}}\underbrace{\cos}_{\text{Took derivative of outside}}\underbrace{(xe^x)}_{\text{Left inside unchanged}}$$

Take It Slow!

Combining Product Rule and Chain Rule Example 2

Plan of Attack:

1. Apply product rule
2. In taking derivatives, carefully use chain rule (on side)

Example 2: $\frac{d}{dx}\left[\sqrt{x^2+1}\cdot e^{\sin(x)}\right]$

$$\sqrt{x^2+1}\,\frac{d}{dx}\left[e^{\sin(x)}\right]+e^{\sin(x)}\frac{d}{dx}\Big[\overbrace{\sqrt{x^2+1}}^{(x^2+1)^{1/2}}\Big]$$

$$=\sqrt{x^2+1}\left[e^{\sin(x)}\cdot\cos(x)\right]+e^{\sin(x)}\left[x(x^2+1)^{-1/2}\right]$$

$$\frac{d}{dx}\left[e^{\sin(x)}\right]$$

Outside: e^u
Inside: $\sin(x)$

$$\frac{d}{du}\left[e^u\right]\cdot\frac{d}{dx}\left[\sin(x)\right]$$

$$e^u\cdot\cos(x)=e^{\sin(x)}\cdot\cos(x)$$

$$\frac{d}{dx}\left[(x^2+1)^{1/2}\right]$$

Outside: $u^{1/2}$
Inside: x^2+1

$$=\frac{d}{du}\left[u^{1/2}\right]\cdot\frac{d}{dx}\left[x^2+1\right]$$

$$=\frac{1}{2}u^{-1/2}\cdot 2x$$

$$=\frac{1}{2}\left[x^2+1\right]^{-1/2}\cdot 2x=x\left[x^2+1\right]^{-1/2}$$

Higher Derivatives Introduced

2^{nd} Derivative of f:

$$f''(x) = \underbrace{\frac{d}{dx}}_{\text{derivative of}}\left(\underbrace{\frac{d}{dx}(f)}_{1^{st}\text{ derivative of } f}\right)$$

SO taking derivative (wrt x) twice!

n^{th} Derivative of f:

$$\underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\frac{d}{dx}\right.\right.}_{\text{Taking derivative (wrt } x\text{) } n \text{ times!}}\left.\left.(f)\right)\right)$$

Taking derivative (wrt x) n times!

More Notation:

SUPPOSE $y = f(x)$

n^{th} derivative written

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Example: SUPPOSE $y = f(x) = 2x^3$

- $f'(x) = \frac{d}{dx}(f) = \frac{d}{dx}(2x^3) = 6x^2$
- $f''(x) = \frac{d}{dx}(f') = \frac{d}{dx}(6x^2) = 12x$
- $f^{(3)}(x) = \frac{d}{dx}(f'') = \frac{d}{dx}(12x) = 12$

Each $f^{(n)}(x)$ is a function until some x value is subbed in to get out a number