Investigating the Plethysm Coefficients For Schur Functions

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Abstract

Plethysm is an operation on symmetric functions, which holds an important position in representation theory and algebraic combinatorics . However, efficient methods for finding the coefficients that emerge when plethysms of Schur functions are expanded using the Schur basis remains a challenge. The main goal of this thesis is to explore and suggest solutions, for the problem of determining plethysm coefficients.

This study aims to introduce methods for computing these plethysm coefficients offering formulas where feasible and investigating their combinatorial and algebraic characteristics. This research seeks to employ theoretical and computational tools analysis in order to enhance our comprehension of plethysm and contribute to the wider domain of symmetric function theory.

Key words: Symmetric function, Schur function, Plethysm, Young tableau.

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Notations

The following are some useful symbols and their meanings

Symbols	**Meaning**
$\mathbb{N}$	set of non-negative integers.
$\mathbb{Q}$	set of rational numbers.
$\mathrm{SSYT}_{(\underbrace{k,k,\ldots,k}_{n})}$	semi-standard Young tableaux with content k 1's, k 2's, k 3's up to k n's

Chapter ONE

Introduction

1.1 Background

Schur functions constitutes one of the bases of the vector space of symmetric functions [22, 25]. These family of symmetric functions were first studied by Jacobi, Trudie, Naegelsbach and Kostka without relating them to group characters. Their definition in relation to group characters was first introduced by Schur [17, 24]. Indeed, Schur functions are the characters of the irreducible polynomial representations of $Gl_n(\mathbb{C})$.

Characters of representations are fundamental tools in the study of group theory and representation theory. Given a representation of a group, which is a homomorphism from the group to the group of invertible matrices over a field, the character of the representation is a function that assigns to each group element the trace of the corresponding matrix. This trace function encapsulates essential information about the representation in a way that is both concise and powerful.

The expression of Schur functions in terms of the power sum basis has as coefficients the irreducible characters of the symmetric group. Furthermore, the monomial expansion of a Schur function is a generating function for combinatorial objects known as column strict tableaux [19, 25]. This illustrates the important role played by Schur functions in the study of symmetric functions, with deep connections to representation theory, algebraic geometry, and combinatorics. As such, it is undoubtedly of interest to find out properties of these functions including how to multiply and compose them with all symmetric functions including themselves.

In particular, the composition of two Schur functions s_λ and s_μ known as plethysm, and written $s_\lambda[s_\mu]$, represents the composition of irreducible representations. This composition of irreducible representations is not irreducible, and its decomposition in irreducible representations boils down to expressing the plethysm $s_\lambda[s_\mu]$ as a sum of Schur functions [10, 27]. Plethysm is a composition-like operation on symmetric functions [17], which despite its significance, remains one of the more enigmatic areas within symmetric function theory. The notion of plethysm was introduced by Dudley E. Littlewood [1,19].

1.2 Literature Review

The literature on symmetric functions and Schur functions is vast, with key contributions from researchers like Littlewood, Macdonald, and Stanley [19,25,26]. However, the specific area of plethysm coefficients is less developed, with many results being either conjectural or computationally verified for small cases. Some of these results can be found in [27,30], [1] and [10].

Amongst the plethora of papers written on the subject, a good number of them propose algorithms for computing plethysms and also coefficients of plethysms [1–3, 9,18,30]. Some authors give more specific results for particular shapes of partitions [5, 6, 10, 15, 16]. For $n \in \mathbb{N}$, a few specific references for the computation of the plethysms $s_\lambda[s_\mu]$ with $\mu = (n)$ are [17], [28] and [13,15] for $\lambda = (2), (3)$ and $\lambda = (4)$ respectively. Formulas for $s_2[s_\lambda]$ can be found in [7,29] and for $s_2[s_b[s_a]]$ and $s_c[s_2[s_a]]$ in [14]. Also, for computations of plethysms from the angle of representation theory we can refer to [4,11,15,20,21,23].

Despite progress in related areas, the calculation and understanding of plethysm coefficients remain largely unresolved. Some amount of progress has been made specifically for small cases. However, succeeding to get a more efficient and general understanding and handling of these coefficients will lead to a prove (or a counterexample) of Foulkes conjecture. The conjecture states that: if $n \geq m$ then $s_n[s_m] - s_m[s_n]$ is $s-$positive [12]. That is the difference of the plethysms $s_n[s_m]$ and $s_m[s_n]$ has positive coefficients. Hence the need to develop more efficient yet more general formulae to determine these coefficients.

1.3 Objectives

Our main objectives in this work are as follows:

1. To develop new theoretical and computational methods for calculating plethysm coefficients $a^{\gamma}_{\lambda\mu}$ of s_γ in the expression

$$s_\lambda[s_\mu] = \sum_{\gamma} a^{\gamma}_{\lambda\mu} s_\gamma$$

thereby providing new ways to compute these coefficients.

2. To derive explicit formulas for plethysm coefficients in certain special cases, contributing to a better understanding of these coefficients.

3. To investigate the combinatorial and algebraic properties of plethysm coefficients, exploring their implications and potential applications in representation theory and symmetric function theory.

4. To apply the newly developed methods and derived formulas to specific examples, demonstrating their effectiveness and uncovering new insights into the structure of plethysm coefficients.

To this effect, we will be required to understand basic concepts in representation theory, symmetric functions, and combinatorics and be able to build on this background.

This research will bring a plus to the theoretical framework of the theory of symmetric functions by providing new insights into plethysm coefficients. The findings may have significant implications for representation theory and other areas of mathematics, such as algebraic combinatorics and invariant theory.

To achieve this objective, the thesis shall be organised as follows:

We start in chapter two by introducing some basic notions of symmetric functions and their properties. In chapter three we study plethysms with a focus on plethysm of Schur functions and its properties. Chapter four is devoted to the development of new theoretical and computational methods for computing plethysm coefficients as well as explicit formulas for plethysm coefficients and apply these formulas to specific examples.

Chapter TWO

Basic Concepts

This chapter is devoted to the development of basic concepts necessary for the understanding and development of our work. We start by defining partitions in the first section and proceed in section two with the definition of the different types of symmetric functions with a few properties. Since we mainly use the combinatorial definition of Schur functions, it is necessary to first define the implicated combinatorial objects that are the Young tableaux, before giving the definition of Schur functions. The chapter is concluded by giving a formula for Schur function in terms of power sum symmetric function and vice versa and giving some examples on the computation.

2.1 Partitions and Their Orderings

Definition 2.1.1. *A **composition** of an integer n is a representation of n as a sum of positive integers. This sums can simply be written as a sequence of the terms involved. A **weak composition** is a composition in which some terms are allowed to be zero.*

For example, 4 has eight compositions given by:

$$4, 3+1, 2+2, 2+1+1, 1+3, 1+2+1, 1+1+2, 1+1+1+1$$

or written as sequences, we have

$$(4), (3,1), (2,2), (2,1,1), (1,3), (1,2,1), (1,1,2), (1,1,1,1).$$

Definition 2.1.2. *A **partition** of a non-negative integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$, for some non-negative integer $k \geq 1$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\sum_i^k \lambda_i = n$. So, a composition arranged in descending order is a partition.*

For example, $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$ are all partitions of 4.

The set of all partitions of $n \in \mathbb{N}$ is defined as;

$$\mathrm{Par}(n) := \{(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) : \lambda_i \in \mathbb{N} \text{ for } 1 \leqslant i \leqslant k, \lambda_1 + \lambda_2 + \cdots + \lambda_k = n\}.$$

When we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$, $\ell(\lambda)$ is the **length** of the partition and $\ell(\lambda) = i$ if $\lambda_{i+1} = 0$. Also,

$$\mathrm{Par} := \bigcup_{n \geq 0} \mathrm{Par}(n)$$

is the set of all partitions.

Define $m_i = m_i(\lambda)$ as the number of parts of λ equal to i. For example, given $\lambda = (3, 2, 1, 1, 1)$ we get that: $m_1(\lambda) = 3, m_2(\lambda) = 1, m_3(\lambda) = 1$. With this notation, the partition can be written as $\lambda = \langle 1^{m_1} 2^{m_2} \ldots \rangle$ or more concisely as $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \ldots$ and from our example we have $\lambda = 1^3 2^1 3^1$.

Definition 2.1.3. *(First Ordering, $\subseteq$)*
Given $\lambda, \mu \in \mathrm{Par}$, $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i \quad \forall i$. $\subseteq$ is a partial order on Par and $(\mathrm{Par}, \subseteq)$ is called Young's lattice $\mathcal{Y}$.

Definition 2.1.4. *(Second partial Order, $\leqslant$)*
*Second partial order $\leqslant$ defined on $\mathrm{Par}(n)$ for each $n \in \mathbb{N}$. The **dominance** (majorization or natural) order is defined as follows: if $|\lambda| = |\mu|$ then $\mu \leqslant \lambda$ if*

$$\mu_1 + \mu_2 + \cdots + \mu_i \leqslant \lambda_1 + \lambda_2 + \cdots + \lambda_i \qquad \forall i \geq 1.$$

2.2 Symmetric Functions

Symmetric functions play a major role in combinatorics, representation theory and various other mathematical fields. Essentially, a symmetric function is a multi-variable polynomial which is invariant under any permutation of these variables. This property of invariance makes symmetric functions, nice tools for exploring symmetries and associated algebraic structures.

Definition 2.2.1. *Let* $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ *be a sequence of indeterminates. Given a weak composition* $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ *is a* ***monomial of finite degree****.*

Definition 2.2.2. *A formal power series is said to be of homogeneous degree* n *if it is a sum of monomials, each of finite degree* n*, for some fixed* $n \in \mathbb{N}$*. This can be written as*

$$f(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha \mathbf{x}^\alpha$$

where,

1. $|\alpha| = \sum_i \alpha_i$,

2. c_α *are constant coefficients.*

Definition 2.2.3. *A formal power series* f *is a* ***symmetric function*** *if* $f(\mathbf{x}) = f(x_{w(1)}, x_{w(2)}, x_{w(3)}, \ldots)$ *for every permutation* w *of the indices of* $\mathbf{x}$*.*

When there is no ambiguity on the indeterminates, we shall suppress them by writing f in place of $f(\mathbf{x})$. We denote by Λ_R^n the set of homogeneous symmetric functions of degree n and R is the ring over which the functions are defined. If $f, g \in \Lambda_R^n$ and $a, b \in R$, then defining:

$$af(\mathbf{x}) + bg(\mathbf{x}) = \sum_\alpha (ac_\alpha + bd_\alpha)\mathbf{x}^\alpha$$

we have $af + bg \in \Lambda_R^n$. Thus Λ_R^n is a module of R.

We'll be working with $R = \mathbb{Q}$, so $\Lambda_R^n = \Lambda_\mathbb{Q}^n$. Define $\Lambda_\mathbb{Q} = \bigoplus_{n\geq 0} \Lambda_\mathbb{Q}^n$, so that $f \in \Lambda_\mathbb{Q}$ is given by $f = f_0 + f_1 + f_2 + \cdots$, where $f_i \in \Lambda_\mathbb{Q}^i$. We shall suppress $\mathbb{Q}$ when there is no ambiguity on the ring. Clearly, Λ is a $\mathbb{Q}$-vector space.

Definition 2.2.4. *Monomial Symmetric Functions*
Given $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \vdash n$*, we define a monomial symmetric function* m_λ *by:*

$$m_\lambda(\mathbf{x}) = \sum_\alpha \mathbf{x}^\alpha$$

where the sum is taken over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ *of* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$.

Example 2.2.5. $\lambda = (1,1)$ *then any permutation of* λ *will still give* $(1,1)$ *so that*

$$m_{11} = \sum_{\alpha} \mathbf{x}^{\alpha} = \sum_{i \neq j} x_i x_j$$

while for $\lambda = 2$,

$$m_2(\mathbf{x}) = \sum_i x_i^2$$

for $\lambda = (2,1)$,

$$m_{2,1}(\mathbf{x}) = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2$$

Remark 2.2.6. *[26]. We observe that, if* $f = \sum_{\alpha} c_\alpha x^\alpha \in \Lambda^n$ *then* $f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$*.This implies that the set* $\{m_\lambda : \lambda \vdash n\}$ *is a basis for* Λ^n*, implying that dim*$\Lambda^n = p(n)$*, the number of partitions of* n*. Moreover,* $\{m_\lambda : \lambda \in \mathrm{Par}\}$ *is a basis for* Λ.

Definition 2.2.7. *(elementary symmetric functions)*

For a non-negative integer n*, we define an elementary symmetric function* e_n *as*

$$e_n = m_{1^n} = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad n \geq 1$$

with

$$e_0 = m_\emptyset = 1$$

and for a partition λ,

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots, \textit{if } \lambda = (\lambda_1, \lambda_2, \ldots)$$

Example 2.2.8. *for* $\lambda = (2,1,1)$ *we have:*

$$e_\lambda = e_2 e_1 e_1 \quad \textit{with}$$

$$e_2 = \sum_{i<j} x_i x_j, \quad e_1 = m_1 = \sum_i x_i$$

Proposition 7.4.1 of [25] gives a more general expression for e_λ as

$$e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$$

with a clear indication on how to determine the coefficient of m_μ.

Theorem 2.2.9. *[26] Let $\lambda, \mu \vdash n$. Then $M_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda'$, while $M_{\lambda\lambda'} = 1$. Hence the set $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ^n and $\{e_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ. Equivalently, $e_1, e_2, \ldots$ are algebraically independent and generate Λ as a $\mathbb{Q}$-algebra, which we write as*

$$\Lambda = \mathbb{Q}[e_1, e_2, \ldots].$$

Definition 2.2.10. *(complete symmetric functions)*
For a non-negative integer n, we define a complete (homogeneous) symmetric function as

$$h_n = \sum_{\lambda \vdash n} m_\lambda \qquad \textit{with} \qquad h_0 = m_\emptyset = 1$$

and for $\lambda \in \mathrm{Par}$,

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots \qquad \textit{for} \qquad \lambda = (\lambda_1, \lambda_2, \ldots)$$

Example 2.2.11. $\lambda = (3, 2, 1)$,

$$\begin{aligned} h_\lambda &= h_3 h_2 h_1 \\ &= \left(\sum_{\alpha \vdash 3} m_\alpha\right)\left(\sum_{\alpha \vdash 2} m_\alpha\right)(m_1) \\ &= (m_3 + m_{2,1} + m_{1,1,1})(m_2 + m_{1,1})(m_1) \end{aligned}$$

Definition 2.2.12. *(power sum symmetric functions)*
The power sum symmetric function for $n \in \mathbb{N}$ is defined by:

$$p_n = m_n = \sum_i x_i^n, \quad n \geq 1$$

with

$$p_0 = m_\emptyset = 1$$

and for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathrm{Par}$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

2.2.1 Young Tableaux

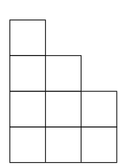
Young tableaux are combinatorial objects that play a fundamental role in the study of symmetric functions, representation theory, and algebraic combinatorics. They provide a visual and structured way to represent partitions and permutations, which are key in various mathematical theories. Young tableaux provide a graphical tool to understand the representations of the symmetric group and general linear groups. In the theory of symmetric functions, Schur functions, which are indexed by partitions, have a natural interpretation in terms of semistandard Young tableaux.

Definition 2.2.13. *(Ferrers Diagram) [22]*
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$. *The* ***Ferrers diagram*** *of shape* λ *is an array of* n *dots having* k *left-justified rows with row* i *containing* λ_i *dots for* $1 \leqslant i \leqslant k$. *In the French convention which we shall use in this work, the rows in Ferrers diagram are counted from bottom to top.*

For example, Ferrers diagram for $\lambda = (3, 3, 2, 1)$ is given by:

$$\begin{array}{lll} \bullet & & \\ \bullet & \bullet & \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array}$$

Squares can be used in place of dots respectively as follows;



Definition 2.2.14. *([22])*

Suppose $\lambda \vdash n$. A ***Young tableau of shape*** *λ is an array T obtained by replacing the dots of the Ferrers diagram of λ with the numbers $1, 2, \ldots, n$ bijectively.*

A Young tableau of shape λ is also called a λ-tableau denoted by T^{λ}. We also write $\mathrm{sh}T = \lambda$ to mean the tableau T is of shape λ.

Remark 2.2.15. *For a partition $\lambda \vdash n$ there are $n!$ Young tableaux.*

For example, given $\lambda = (2, 1) \vdash 3$ the Young tableaux are:

$$\begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 3 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 3 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 1 \\ \hline \end{array}$$

Definition 2.2.16. *A* ***generalized Young tableau*** *of shape λ is an array T obtained by replacing the nodes of λ with positive integers, repetitions allowed. The* ***type****,* ***content*** *or* ***weight****of T is the composition $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$, where μ_i is the number of times i appears in T. By the* ***length*** *of T, denoted $\ell(T)$, we mean the number of rows that constitute T.*

An example is:

$$T = \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array},$$

The first tableau is of shape $\lambda = (2, 1) \vdash 3$ and of content $\mu = (1, 2)$. The second is of shape $\lambda = (3, 1) \vdash 4$ and of content $\mu = (1, 1, 2)$. Both tableaux are of length $\ell(T) = 2$.

Definition 2.2.17. *A tableau T is said to be standard if the entries increase row-wise and column-wise and are only constituted of the integers 1,2, $\ldots$, n where $\lambda \vdash n$.*

As example, the standard Young tableaux for $\lambda = (2, 1)$ are:

$$T = \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

Definition 2.2.18. *A generalized tableau is said to be semistandard if its rows weakly increase and its columns strictly increase.*

An example is given by:

$$T = \begin{array}{|c|c|c}\hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

with content $\mu = (0, 1, 3, 1)$ and $\mathrm{sh}(T) = (3, 2) \vdash 5$.

2.2.2 Schur Function

Definition 2.2.19. *For any generalized tableau T of shape λ, we define $\mathbf{x}^T$ as*

$$\mathbf{x}^T = \prod_{(i,j)\in\lambda} x_{T(i,j)} = \mathbf{x}^\mu$$

where (i, j) indicates the position in the tableau, $T(i, j)$ represents the entry in position (i, j), μ is the content of T and $\mathbf{x}^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m}$ for $\mu = (\mu_1, \mu_2, \dots, \mu_m)$.

Example 2.2.20. *For*

$$T = \begin{array}{|c|c|c}\hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

we have that $\mu = (0, 1, 3, 1)$. So,

$$\mathbf{x}^T = x_2 x_3 x_3 x_3 x_4 = x_2 x_3^3 x_4$$

Definition 2.2.21. *Given a partition λ, the associated* ***Schur function*** *is defined by:*

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$$

where the sum is over all semi-standard λ-tableaux T.

Example 2.2.22. *1) Let's compute $s_8(\mathbf{x})$*

$$s_8(\mathbf{x}) = \sum_T \mathbf{x}^T$$

$\lambda = (8)$, then its family of SSYTs (with alphabet $\{1, 2, 3, 4, 5, 6, 7, 8\}$) is:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline a & b & c & d & e & f & g & h \\ \hline \end{array}$$

with different combinations of integers $1, 2, \ldots, 8$ *in the boxes.*

Then the Schur function is $s_8 = \sum_T \mathbf{x}^T$ *where the different terms will arise with respect to the differences in the content of* T*. For instance, if*

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&4&5&5&6&7\\\hline\end{array}$$

$$\mathbf{x}^T = x_1x_2^2x_4x_5^2x_6x_7$$

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&3&3&4&5&6&6\\\hline\end{array}$$

$$\mathbf{x}^T = x_1x_2x_3^2x_4x_5x_6^2$$

And so on, and s_8 *is the sum of all these.*

2) Next, we compute s_3*. Suppose the largest integer in the box is 3, then for* $\lambda = (3)$*,*

$$\Big\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array},$$
$$\begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&3&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array} \Big\}$$

is the set of SSYTs with contents (in order)

$$\{(111), (12), (102), (21), (201), (3), (03), (021), (012), (003)\},$$

so that

$$s_3 = x_1x_2x_3 + x_1x_2^2 + x_1x_3^2 + x_1^2x_2 + x_1^2x_3 + x_1^3 + x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3$$

Theorem 2.2.23. *(Theorem 7.16.1, [26]) Let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *and* $\mu = (\mu_1, \ldots, \mu_n) \subseteq \lambda$*. Then*

$$s_{\lambda/\mu} = det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^n$$

where we set $h_0 = 1$ *and* $h_k = 0$ *for* $k < 0$*.*

This is actually the Jacobi-Trudi identity, expressing Schur function as a determinant whose entries are h_i's and we deduce from this identity that for $\lambda = (n)$

$$s_n = s_\lambda = h_n.$$

2.2.3 Schur function in terms of Power sum function

The Schur function and the power sum symmetric function can be defined in terms of each other as follows.

Corollary 2.2.24. *([19, 22, 26])*

$$s_\lambda = \sum_{\gamma \vdash |\lambda|} \chi^\lambda(\gamma) p_\gamma / z_\gamma$$

$$p_\gamma = \sum_{\lambda \vdash |\gamma|} \chi^\lambda(\gamma) s_\lambda$$

where $\chi^\lambda(\gamma)$ is the irreducible character of the symmetric group indexed by the partition λ evaluated at a permutation of cycle structure γ, [26]. For $\gamma = (1^{a_1}, 2^{a_2}, \dots, l^{a_l})$, we define $z_\gamma = 1^{a_1} a_1! 2^{a_2} a_2! \cdots , l^{a_l} a_l!$

Example 2.2.25. *1) Lets compute $s_{(3)}$ in terms of power sum symmetric functions*

For $\lambda = (3)$, we consider the partitions γ of 3: $(3), (2,1), (1,1,1)$

The character values $\chi^{(3)}(\gamma)$, the corresponding centralizer orders z_γ and the power sum symmetric functions are:

– *For $\gamma = (3)$:*

$$\chi^{(3)}((3)) = 1, \quad z_{(3)} = 1^0 \cdot 0! \cdot 2^0 \cdot 0! \cdot 3^1 \cdot 1! = 3, \quad p_{(3)} = p_3$$

– *For $\gamma = (2,1)$:*

$$\chi^{(3)}((2,1)) = 1, \quad z_{(2,1)} = 1^1 \cdot 1! \cdot 2^1 \cdot 1! \cdot 3^0 \cdot 0! = 2, \quad p_{(2,1)} = p_2 p_1 = p_{21}$$

– *For* $\gamma = (1,1,1)$*:*

$$\chi^{(3)}((1,1,1)) = 1, \quad z_{(1,1,1)} = 1^3 \cdot 3! \cdot 2^0 \cdot 0! \cdot 3^0 \cdot 0! = 6, \quad p_{(1,1,1)} = p_1^3 = p_{111}$$

Details on the computation of the irreducible characters can be found in [19,25] or [22]. Expression of Power Sum Symmetric Functions

Substituting this in the given expression for Schur function we get:

$$s_{(3)} = \frac{p_3}{3} + \frac{p_2 p_1}{2} + \frac{p_1^3}{6}$$

2) Lets express of p_λ *in terms of Schur functions, for* $\lambda \vdash 3$.

– *For* $\lambda = (3)$:

$$p_3 = \sum_{\lambda \vdash 3} \chi^\lambda((3)) s_\lambda$$

Character values for (3)*:*

$$\chi^{(3)}((3)) = 1, \quad \chi^{(21)}((3)) = -1, \quad \chi^{(111)}((3)) = 1$$

$$p_3 = 1 \cdot s_3 + (-1) \cdot s_{(21)} + 1 \cdot s_{(111)} = s_{(3)} - s_{21} + s_{(111)}$$

– *For* $\lambda = (21)$:

$$p_{(21)} = \sum_{\lambda \vdash 3} \chi^\lambda((21)) s_\lambda$$

Character values for (21)*:*

$$\chi^{(3)}((21)) = 1, \quad \chi^{(21)}((21)) = 0, \quad \chi^{(111)}((21)) = -1$$

$$p_{(21)} = 1 \cdot s_{(3)} + 0 \cdot s_{(21)} + (-1) \cdot s_{(111)} = s_3 - s_{(111)}$$

– *For* $\lambda = (111)$:

$$p_{(111)} = \sum_{\lambda \vdash 3} \chi^\lambda((111)) s_\lambda$$

Character values for (111)*:*

$$\chi^{(3)}((111)) = 1, \quad \chi^{(21)}((111)) = 2, \quad \chi^{(111)}((111)) = 1$$

$$p_{(111)} = 1 \cdot s_{(3)} + 2 \cdot s_{(21)} + 1 \cdot s_{(111)} = s_{(3)} + 2s_{(21)} + s_{(111)}$$

Hence,

$$p_3 = s_{(3)} - s_{21} + s_{(111)}; \quad p_{(21)} = s_3 - s_{(111)}; \quad p_{(111)} = s_{(3)} + 2s_{(21)} + s_{(111)}.$$

Chapter THREE

Plethysm

The five symmetric functions defined in the previous chapter can all be proven to be generators for the ring Λ of symmetric functions [22, 25]. Having defined the different symmetric functions, it is natural to be interested in how these functions interact with each other by addition, multiplication or composition. In this chapter, we discuss the structure of composition of symmetric functions with a particular interest on the compositions of Schur functions, $s_\lambda \circ s_\mu$. This composition of symmetric functions is called **plethysm**. To stay kin to the notation used in the article [10], we shall use square brackets to represent these compositions.

We start this section by giving a rule for the multiplication of Schur functions known as Pieri's rule, which is a special case of the Littlewood-Richardson rule. We then proceed to give a more formal definition of plethysms of symmetric functions.

Theorem 3.0.1. *Pieri's Rule [9]. For a partition λ and a non negative integer n, we have that*

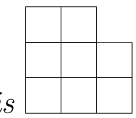
$$s_\lambda s_n = \sum_{\nu} s_\nu$$

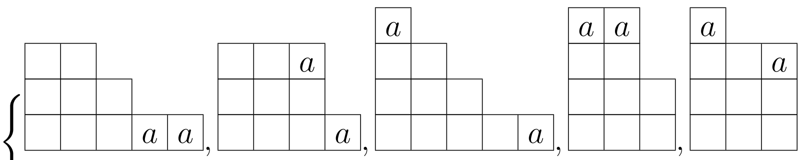
where the sum is taken over all partitions $\nu \vdash |\lambda| + n$ such that ν/λ is a horizontal strip of size n.

By ν/λ a horizontal strip of size n we mean a Young tableau of skew shape ν/λ. This is obtained when the squares that constitute the tableau of shape λ are removed from the tableau of shape ν. Hence, the number of terms in the product depends on the number of ways in which n squares can be added to the tableau of shape λ without adding squares on the same column.

Example 3.0.2. *Lets compute $s_2 s_1$ and $s_{332} s_2$.*

1) For $\lambda = (2)$ the corresponding tableaux is $\begin{array}{|c|c|}\hline & \\ \hline\end{array}$ and to obtain a strip ν/λ of size 1, we require $\nu \in \left\{ \begin{array}{|c|c|c|}\hline & & a \\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} a & \multicolumn{1}{c}{} \\ \hline & \\ \hline\end{array} \right\}$, where the added squares are the ones *labelled with a. Hence, we obtain the product* $s_2 s_1 = s_3 + s_{21}$.

2) Similarly, for $\lambda = (332)$ *the corresponding tableau is*  *and we require*

$\nu \in \Bigg\{$  $\Bigg\}$

and as such we get $s_{332} s_2 = s_{532} + s_{433} + s_{4321} + s_{3331} + s_{3322}$.

Definition 3.0.3. *[25] Consider symmetric functions* $f, g \in \Lambda$ *with* $f := \sum_{i \geq 1} x^{a^{(i)}}$ *a sum of monomials, where* $a^{(i)}$ *are weak compositions and* $x^{a^{(i)}}$ *is as in definition 2.2.1. By the* ***plethysm*** $g[f]$ *we mean the composition of both functions given by*

$$g[f] = g(x^{a^{(1)}}, x^{a^{(2)}}, x^{a^{(3)}}, \ldots)$$

In terms of power sum basis, suppose $f = \sum_\lambda c_\lambda p_\lambda$ with $c_\lambda \in \mathbb{Q}$, then for any positive integer l, and any partition μ,

$$p_l[f] = \sum_\lambda c_\lambda p_{l\lambda} = \sum_\lambda c_\lambda p_{l\lambda_1} p_{l\lambda_2} p_{l\lambda_3} \cdots p_{l\lambda_{\ell(\lambda)}}$$

and

$$p_\mu[f] = p_{\mu_1}[f] p_{\mu_2}[f] p_{\mu_3}[f] \cdots p_{\mu_{\ell(\mu)}}[f].$$

And for $g = \sum_\mu d_\mu p_\mu$,

$$g[f] = \sum_\mu d_\mu p_\mu[f].$$

Remark 3.0.4. *For* $f, g, h \in \Lambda$ *and* $a, b \in \mathbb{Q}$,

$$(af + bg)[h] = af[h] + bg[h]$$

$$(fg)[h] = f[h]g[h]$$

Observe that for a power sum symmetric functions p_m, p_n $(m, n \in \mathbb{N})$, we have:

$$f[p_n] = f(x_1^n, x_2^n, \ldots) = \sum_{i \geq 1} x^{a(i)n} = p_n[f].$$

$$p_m[p_n] = p_m(x_1^n, x_2^n, \ldots) = \sum_{i \geq 1} x_i^{mn} = p_{mn}.$$

Similarly, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$

$$p_\lambda[p_n] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[p_n] = \prod_{i=1}^{\ell(\lambda)} p_{n\lambda_i}$$

Now, in the case of Schur functions, by definition of plethysm we have that $s_\mu[p_d] = p_d[s_\mu]$ for any $d \in \mathbb{N}$ and from Corollary 2.2.24,

$$s_\lambda[s_\mu] = \sum_{\gamma \vdash |\lambda|} \frac{\chi^\lambda(\gamma)}{z_\gamma} p_\gamma[s_\mu]$$

But

$$\begin{aligned} p_\gamma[s_\mu] &= \prod_{i=1}^{\ell(\gamma)} p_{\gamma_i}[s_\mu] \\ &= \prod_{i=1}^{\ell(\gamma)} s_\mu[p_{\gamma_i}] \end{aligned}$$

Hence,

$$s_\lambda[s_\mu] = \sum_{\gamma \vdash |\lambda|} \frac{\chi^\lambda(\gamma)}{z_\gamma} \prod_{i=1}^{\ell(\gamma)} s_\mu[p_{\gamma_i}].$$

Example 3.0.5. *Let's look at some examples of computations.*

1) $s_3[s_1] = \sum_{\gamma \vdash 3} \frac{\chi^3(\gamma)}{z_\gamma} p_\gamma[s_1]$ *But from example 2.2.25 in the previous chapter, we get that*

$$s_3[s_1] = \frac{p_3}{3}[s_1] + \frac{p_{21}}{2}[s_1] + \frac{p_{111}}{6}[s_1] = \frac{p_3}{3} + \frac{p_2 p_1}{2} + \frac{p_{111}}{6}$$

since $s_1 = x_1 + x_2 + x_3$. *And thus*

$$s_3[s_1] = s_3.$$

Indeed, for any symmetric function f, $f[s_1] = f$.

2) Similarly, we have that $s_2 = \frac{1}{2}p_{11} + \frac{1}{2}p_2$ *so that,*

$$\begin{aligned}
s_3[s_2] &= \frac{1}{3}p_3[s_2] + \frac{1}{2}p_{21}[s_2] + \frac{1}{6}p_{111}[s_2] \\
&= \frac{1}{3}p_3[s_2] + \frac{1}{2}p_2[s_2]p_1[s_2] + \frac{1}{6}p_1[s_2]p_1[s_2]p_1[s_2] \\
&= \frac{1}{6}p_3[p_{11} + p_2] + \frac{1}{4}p_{21}[p_{11} + p_2] + \frac{1}{12}p_{111}[p_{11} + p_2] \\
&= \frac{1}{3}(\frac{1}{2}p_6 + \frac{1}{2}p_{33}) + \frac{1}{2}(\frac{1}{2}p_4 + \frac{1}{2}p_{22})(\frac{1}{2}p_2 + \frac{1}{2}p_{11}) + \frac{1}{6}\big(\frac{1}{2}p_2 + \frac{1}{2}p_{11}\big)^3 \\
&= s_6 + s_{42} + s_{222}
\end{aligned}$$

3) Also,

$$\begin{aligned}
s_2[s_3] &= \frac{1}{2}\,(p_{11} + p_2)\,[\frac{1}{3}p_3 + \frac{1}{2}p_{21} + \frac{1}{6}p_{111}] \\
&= \frac{1}{2}\left(p_{11}[\frac{1}{3}p_3 + \frac{1}{2}p_{21} + \frac{1}{6}p_{111}] + p_2[\frac{1}{3}p_3 + \frac{1}{2}p_{21} + \frac{1}{6}p_{111}]\right) \\
&= \frac{1}{2}\left(\frac{1}{3}p_{11}[p_3] + \frac{1}{2}p_{11}[p_{21}] + \frac{1}{6}p_{11}[p_{111}] + \frac{1}{3}p_2[p_3] + \frac{1}{2}p_2[p_{21}] + \frac{1}{6}p_2[p_{111}]\right) \\
&= \frac{1}{6}p_{33} + \frac{1}{4}(p_{21})^2 + \frac{1}{12}(p_1)^6 + \frac{1}{6}p_6 + \frac{1}{4}p_{42} + \frac{1}{12}(p_2)^3 \\
&= s_6 + s_{42}.
\end{aligned}$$

3.1 Plethysms $s_\lambda[s_k]$ with $\lambda \vdash 3$

Now that we have given the definition for plethysm, we observe that the plethysm of Schur functions in particular can be significantly difficult to compute in terms of Schur basis owing to the necessity to transit from Schur basis to power sum basis and back for each computation. Also, in some cases we might need to know only the coefficients of the terms, for example if we know a coefficient will be zero, this can help speed up the computation since there will be no need computing these terms again. A number of authors including [8, 9] have worked on algorithms for specific cases of partitions of small integers so far.

Our aim here will be to explore in details the theorem 5.3 from [10] which focuses on the case of $s_\lambda[s_k]$ for $\lambda \vdash 3$ and any integer $k \geq 1$. This theorem uses a combinatorial approach to expressing the plethysms and gives a faster and more efficient method to compute them.

Henceforth in this work, by $\mathrm{SSYT}_{(k,k,k)}$ we will mean semi-standard Young tableaux with content k 1's, k 2's and k 3's, for $k \in \mathbb{N}$. Another notion on tableaux which will be useful in this section and subsequent ones is the **transpose** of a tableau, denoted T^t for any tableau T. It is an involution (that is $(T^t)^t = T$) operation on tableaux obtained by reflecting it about the non-main diagonal line. More simply, with respect to the next definition we have that;

- $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ implies $T = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$

- $T = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array}$ implies $T = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline\end{array}$.

In the next definition we give a different definition to the type of a tableau and we only refer to this definition throughout the rest of the work when we mention type of a tableau. To refer to the type as given in definition 2.2.16, we use the term content which was then given as its synonym as we have been doing throughout.

Definition 3.1.1. *For $S \in SSYT_{(k,k,k)}$, let N_r represent the number of cells with the label r in the second row of S for $1 \leqslant r \leqslant 3$.*

1 If S is standard and $k = 1$, then let type$(S) = S$.

2) If S has one or two rows, then we define:

a) type$(S) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ if N_2 is even, $N_3 \geq 2N_2$, but $N_3 \neq 2N_2 + 1$.

b) *type*$(S) = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$ *if* N_2 *is odd,* $N_3 \geq 2N_2$*, but* $N_3 \neq 2N_2 + 1$.

c) *type*$(S) = \begin{array}{|c|c}\cline{1-1} 3 & \\\hline 1 & \multicolumn{1}{c|}{2}\\\hline\end{array}$ *if* N_2 *is even, and* $N_3 < 2N_2$ *or* $N_3 = 2N_2 + 1$.

d) *type*$(S) = \begin{array}{|c|c}\cline{1-1} 2 & \\\hline 1 & \multicolumn{1}{c|}{3}\\\hline\end{array}$ *if* N_2 *is odd, and* $N_3 < 2N_2$ *or* $N_3 = 2N_2 + 1$.

3) *If* S *has three rows and* $\overline{S}$ *is* S *with the first column removed, then type*$(S) =$ *type*$(\overline{S})^t$.

Next, if we consider T a standard tableau of size 3, define the set

$$\mathrm{Tab}_{T,k} = \{S \in \mathrm{SSYT}_{(k,k,k)} \mid \mathrm{type}(S) = T\}$$

Also, define a linear and a bilinear operator on symmetric functions respectively as follows:

- $$s_\mu \downarrow_k = \begin{cases} s_\mu & \text{if } \ell(\mu) \leqslant k, \\ 0 & \text{else.} \end{cases}$$
- $s_\lambda \odot s_\mu = s_{\lambda+\mu}$ where the addition is done component-wise filling in parts with zero if the lengths of the partitions do not coincide. That is, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$ with $\ell(\lambda) > \ell(\mu)$ without loss of generality, $\lambda + \mu = \lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_{\ell(\mu)} + \mu_{\ell(\mu)}, \lambda_{k+1} + 0, \ldots, \lambda_{\ell(\lambda)} + 0$.

We shall use the following theorem which is a Schur functions version of Theorem 2.1 from [27] to prove Lemma 3.1.3 .

Theorem 3.1.2. *The plethysm* $s_3[s_n]$ *can be described as:*

1) $s_3[s_0] = 1 \qquad s_3[s_1] = s_3$

2) $(s_3[s_n]) \downarrow_2 = s_{66} \odot (s_3[s_{n-4}]) \downarrow_2 + \sum_{k=2}^{n} s_{3n-k,k} + s_{3n}$

3) $s_3[s_n] = (s_3[s_n]) \downarrow_2 + s_{222} \odot (s_3[s_{n-2}]) + s_{441} \odot ((s_3[s_{n-3}]) \downarrow_2)$.

We are not going to provide a proof of this theorem here, the details of this calculation can be found in [27]. The proof uses a result of Thrall [28] and an analysis of the recursive structure of plethysm coefficients.

Lemma 3.1.3. *(Lemma 5.5, [10]) For $k \geq 1$,*

$$s_3[s_k]\downarrow_2 = \sum_S s_{shape(S)},$$

where the sum is over all $S \in Tab_{\boxed{1\,|\,2\,|\,3},k}$ with $\ell(S) \leqslant 2$.

Proof. We aim to show that each term in the expression of $s_3[s_k]\downarrow_2$ has one corresponding tableau of length less than or equal to 2. From the item 2 of Theorem 3.1.2, we have that

$$s_3[s_n]\downarrow_2 = s_{66} \odot (s_3[s_{n-4}])\downarrow_2 + \sum_{k=2}^{n} s_{3n-k,k} + s_{3n}.$$

For $S \in \mathrm{Tab}_{\boxed{1\,|\,2\,|\,3},k}$ with $\ell(S) \leqslant 2$, we get that the term $s_{66} \odot (s_3[s_{n-4}])\downarrow_2$ encompasses the functions corresponding to those tableaux that contain at least two 2s and at least four 3s (since $N_3 \geqslant 2N_2$) in the second row. That is, these tableaux have $\boxed{1\,|\,1\,|\,1\,|\,1\,|\,2\,|\,2}$ in the first row and $\boxed{2\,|\,2\,|\,3\,|\,3\,|\,3\,|\,3}$ in the second row which are of type $\boxed{1\,|\,2\,|\,3}$. Actually, observing carefully we see that the tableau $\begin{array}{|c|c|c|c|c|c|}\hline 2&2&3&3&3&3\\ \hline 1&1&1&1&2&2\\ \hline\end{array}$ corresponding to the function s_{66} is of type $\boxed{1\,|\,2\,|\,3}$ and in computations, any difference in the final function (with corresponding different tableau) for this term is justified by the linear $\odot$ operation with the recursive part of this term, $(s_3[s_{n-4}])\downarrow_2$.

On the other hand, the tableau with a single row is catered for by the term s_{3n}. Observe that for these tableau, $N_2 = N_3 = 0$. That is N_2 is even and $N_3 \neq 2N_2 + 1$. Hence this tableau is of type $\boxed{1\,|\,2\,|\,3}$.

The remaining tableaux, those with k 3's in the second row for $2 \leqslant k \leqslant n$ are enumerated by the middle term. Here also, $N_2 = 0$ is even, $N_3 > 2N_2$ and $k = N_3 \neq 2N_2 + 1 = 1$. Thus these tableaux too are of type $\boxed{1\,|\,2\,|\,3}$. $\square$

Remark 3.1.4. *The operator $\odot$ being bilinear, we have that:*

1) If $n = 4$, *then* $s_3[s_{n-4}] = s_3[s_0] = 1$ *and* $s_{66} \odot (s_3[s_{n-4}]) \downarrow_2 = s_{66}$.

2) If $n < 4$, *then* $s_3[s_{n-4}] = 0$ *and* $s_{66} \odot (s_3[s_{n-4}]) \downarrow_2 = 0$.

Example 3.1.5. *Let's compute* $s_3[s_4] \downarrow_2$. *The list of tableaux* $S \in Tab_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},4}$ *with* $\ell(S) \leqslant 2$ *is*

$$\left\{ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&2&2&2&3&3&3&3\\\hline\end{array}, \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-2} 3&3&\multicolumn{8}{c}{}\\\hline 1&1&1&1&2&2&2&2&3&3\\\hline\end{array}, \right.$$

$$\left. \begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{1-3} 3&3&3&\multicolumn{6}{c}{}\\\hline 1&1&1&1&2&2&2&2&3\\\hline\end{array}, \begin{array}{|c|c|c|c|c|c|c|c|}\cline{1-4} 3&3&3&3&\multicolumn{4}{c}{}\\\hline 1&1&1&1&2&2&2&2\\\hline\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 2&2&3&3&3&3\\\hline 1&1&1&1&2&2\\\hline\end{array} \right\} \textit{ and}$$

$s_3[s_4] \downarrow_2 = s_{66} + s_{102} + s_{93} + s_{84} + s_{12}$.

Lemma 3.1.6. *(Lemma 5.6, [10]) For* $k \geq 1$,

$$\sum_{S \in Tab_{\begin{array}{|c|c|}\cline{1-1} 3&\multicolumn{1}{c}{}\\\hline 1&2\\\hline\end{array},k}} s_{shape(S)} = \sum_{S \in Tab_{\begin{array}{|c|c|}\cline{1-1} 2&\multicolumn{1}{c}{}\\\hline 1&3\\\hline\end{array},k}} s_{shape(S)}.$$

Proof. To proof this equality, we show that the following map

$$\phi : \mathrm{Tab}_{\begin{array}{|c|c|}\cline{1-1} 3&\multicolumn{1}{c}{}\\\hline 1&2\\\hline\end{array},k} \to \mathrm{Tab}_{\begin{array}{|c|c|}\cline{1-1} 2&\multicolumn{1}{c}{}\\\hline 1&3\\\hline\end{array},k}$$

is bijective. For $S \in \mathrm{Tab}_{\begin{array}{|c|c|}\cline{1-1} 3&\multicolumn{1}{c}{}\\\hline 1&2\\\hline\end{array},k}$ the map is defined as follows:

Case 1. If the length of the second row of S is odd, then let $\phi(S) = S'$ be the tableau where one takes a 3 from the second row of S and exchanges it with a 2 in the first row.

Case 2. If the length of the second row of S is even, then let $\phi(S) = S'$ be the tableau such that one takes a 3 from the first row of S and exchanges it with a 2 in the second row.

Case 3. If the length of S is 3, then let $\overline{S} \in SSYT_{(k,k,k)}$ be the tableau S with the first column removed. Define $\phi(S)$ to be $\phi^{-1}(\overline{S})$ to which a column of length 3 is added.

We justify that in each of the 3 cases, $\phi(S) \in \mathrm{Tab}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k}$ and this image is unique. Firstly, observe that $S \in \mathrm{Tab}_{\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}, k}$ implies that N_2 is even, and either $N_3 < 2N_2$ or $N_3 = 2N_2 + 1$.

Let $N_2^{S'}$ and $N_3^{S'}$ represent the number of 2's and 3's in the second row of S'.

For case 1, we have that $N_2^{S'} = N_2 + 1$ and $N_3^{S'} = N_3 - 1$. So, $N_2^{S'}$ is odd and:

- if $N_3 < 2N_2$, then $N_3 - 1 < N_3 < 2N_2 < 2N_2 + 2 = 2(N_2 + 1)$. That is, $N_3^{S'} < 2N_2^{S'}$.

- if $N_3 = 2N_2 + 1$, then $N_3 - 1 = 2N_2 < 2(N_2 + 1)$. That is, $N_3^{S'} < 2N_2^{S'}$.

Hence in both cases we obtain that $S' = \phi(S) \in \mathrm{Tab}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k}$.

For case 2, we have that $N_2^{S'} = N_2 - 1$ and $N_3^{S'} = N_3 + 1$. So, $N_2^{S'}$ is odd and we start by observing that for all tableaux S in this case $N_3 + 3 < 2N_2$ or $N_3 + 2 = 2N_2$ and we decompose the cases as follows:

- if $N_3 + 3 < 2N_2$, then $N_3 + 1 < 2N_2 - 2 = 2(N_2 - 1)$. That is, $N_3^{S'} < 2N_2^{S'}$.

- if $N_3 + 2 = 2N_2$, then $N_3 + 1 = 2N_2 - 1 = 2(N_2 - 1) + 1$. That is, $N_3^{S'} = 2N_2^{S'} + 1$.

Hence in both cases we obtain that $S' = \phi(S) \in \mathrm{Tab}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k}$.

For case 3, observe that, if $S \in \mathrm{Tab}_{\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}, k}$ with $\ell(S) = 3$, then $\mathrm{type}(\overline{S}) = \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$. That is, $\overline{S} \in \mathrm{Tab}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k}$. Let $T = \phi^{-1}(\overline{S})$, then $\mathrm{type}(T) = \mathrm{type}(\phi^{-1}(\overline{S})) = \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$. Also, let T' be the tableau T to which a column of length 3 is added, then we observe that $\mathrm{type}(T') = \mathrm{type}(T)^t = \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$, implying that $T' \in \mathrm{Tab}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k}$. But $\mathrm{type}(T') = \mathrm{type}(\phi(S))$. Hence $S' = \phi(S) \in \mathrm{Tab}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k}$.

Furthermore, it is clear by the definition of ϕ that for $S, T \in \mathrm{Tab}_{\ytableaushort{3,12},k}$, if $\phi(S) = \phi(T)$, then $S = T$. Also, for all $S' \in \mathrm{Tab}_{\ytableaushort{2,13},k}$ there exist $S \in \mathrm{Tab}_{\ytableaushort{3,12},k}$ such that $S' = \phi(S)$. Hence ϕ is a bijection and thus

$$\sum_{S \in \mathrm{Tab}_{\ytableaushort{3,12},k}} s_{\mathrm{shape}(S)} = \sum_{S \in \mathrm{Tab}_{\ytableaushort{2,13},k}} s_{\mathrm{shape}(S)}.$$

□

One more useful notion before to be considered before we delve in proving the theorem of interest is the notion of the $s-$perp trick. We first define the action of $s-$perp operator denoted $s^{\perp}$.

Definition 3.1.7. *[10] Let λ be a partition and $f \in \Lambda$. The action of $s_{\lambda}^{\perp}$ on f is defined by*

$$s_{\lambda}^{\perp} f = \sum_{\mu} \langle f, s_{\lambda} s_{\mu} \rangle s_{\mu}.$$

Note that the scalar product involved is the **Hall inner product** defined by

$$\langle s_{\lambda}, s_{\mu} \rangle = \langle h_{\lambda}, m_{\mu} \rangle = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise .} \end{cases}$$

Proposition 3.1.8. *The s-perp trick [10] Let f and g be two symmetric functions of homogeneous degree d. If*

$$s_r^{\perp} f = s_r^{\perp} g \quad \textit{for all } 1 \leqslant r \leqslant d,$$

then $f = g$. The same statement holds true if $s_r^{\perp}$ is replaced by $s_{1^r}^{\perp}$, mutatis mutandis.

Theorem 3.1.9. *(Theorem 5.3, [10])*

Let $T \in \left\{ \ytableaushort{123}, \ytableaushort{2,13}, \ytableaushort{3,12}, \ytableaushort{3,2,1} \right\}$ be a standard tableau with shape a partition of

3. Then for any $k \geq 1$,

$$s_{shape(T)}[s_k] = \sum_{\substack{S \in SSYT_{(k,k,k)} \\ type(S)=T}} s_{shape(S)}$$

where the type(S) for $S \in SSYT_{(k,k,k)}$ is defined in Definition 3.1.1.

Proof. The proof is by induction on k. The base step taken at $k = 1$, we aim to show that if $T \in \left\{ \ytableaushort{123}, \ytableaushort{2,13}, \ytableaushort{3,12}, \ytableaushort{3,2,1} \right\}$ is a standard tableau with shape a partition of 3, then,

$$s_{\text{shape}(T)}[s_1] = \sum_{\substack{S \in \text{SSYT}_{(1,1,1)} \\ \text{type}(S)=T}} s_{\text{shape}(S)}$$

By definition and (1) of example 3.0.5 we have that;

$$s_3[s_1] = s_3$$

$$s_{21}[s_1] = s_{21}$$

and

$$s_{111}[s_1] = s_{(111)}.$$

Now, lets check that we have the same results using the theorem. First note that $SSYT_{(111)} = \left\{ \ytableaushort{123}, \ytableaushort{2,13}, \ytableaushort{3,12}, \ytableaushort{3,2,1} \right\}$ where $SSYT_\lambda$ is the set of SSYTs with content λ, we obtain the following;

- For $T = \ytableaushort{123}$, $shape(T) = (3)$, $S \in SSYT_{(111)}$ and $type(S) = T$ implies that $S = \ytableaushort{123}$ so $s_{shape(S)} = s_3$. Hence $s_3[s_1] = s_3$
- For $T = \ytableaushort{3,2,1}$, $shape(T) = (111)$, $S \in SSYT_{(111)}$ and $type(S) = T$ implies that

$S = \ytableaushort{3,2,1}$ so $s_{shape(S)} = s_{(111)}$. Hence $s_{(111)}[s_1] = s_{(111)}$

- Observe that by definition 3.1.1, $type\left(\ytableaushort{3,12}\right) = \ytableaushort{3,12}$ and $type\left(\ytableaushort{2,13}\right) = \ytableaushort{2,13}$. So, proceeding as in the previous cases, for $S = \ytableaushort{3,12}$ or $S = \ytableaushort{2,13}$ we get $s_{shape(S)} = s_{(21)}$. Hence $s_{(21)}[s_1] = s_{(21)}$.

These results all coincide with the result obtained using the definition. Hence this completes the base step of the proof.

Suppose the theorem holds true for arbitrary k.

Assuming that the theorem holds for a fixed k and for all tableaux $T \in \left\{\ytableaushort{123}, \ytableaushort{2,13}, \ytableaushort{3,12}, \ytableaushort{3,2,1}\right\}$, we aim to prove that the theorem holds for $k+1$.

By Lemma 3.1.3 we have that $s_3[s_{k+1}] \downarrow_2$ gives all the terms involving tableaux $S \in \mathrm{Tab}_{\ytableaushort{123},k+1}$ with length $\ell(S) \leqslant 2$. For the terms with length 3 we use the following formula which is a special case of equation (5.2) from (Proposition 5.2, [10]) making use of the s-perp trick:

$$s_{1^r}^{\perp} s_h[s_w] = s_{h-r}[s_w] s_{1^r}[s_{w-1}] \tag{3.1.1}$$

with $r = h = 3$, and $w = k+1$ resulting to $s_{111}^{\perp} s_3[s_{k+1}] = s_0[s_{k+1}] s_{111}[s_k] = s_{111}[s_k]$. We thus deduce inductively that the terms of length 3 are given by those $S \in \mathrm{Tab}_{\ytableaushort{123},k+1}$ with $\ell(S) = 3$ and hence,

$$s_3[s_{k+1}] = \sum_{S \in \mathrm{Tab}_{\ytableaushort{123},k+1}} s_{\mathrm{shape}(S)}.$$

Furthermore, we make use of equations (1) and (2) in Example 9 from Section I.8 of [19] to which we associate the Jacobi-Trudi identity to derive the last remaining cases.

- For $s_{21}[s_{k+1}]$, we have that;

$$s_2[s_{k+1}] = \sum_{S \in \mathrm{SSYT}_{(k+1,k+1)}} s_{\mathrm{shape}(S)},$$

where the sum is over $S \in \mathrm{SSYT}_{(k+1,k+1)}$ with N_2 even. This in turn implies that,

$$\begin{aligned} s_{21}[s_{k+1}] + s_3[s_{k+1}] &= s_2[s_{k+1}]s_1[s_{k+1}] \qquad \text{(by Pieri's rule)} \\ &= s_2[s_{k+1}]s_{k+1} \\ &= \sum_{S \in \mathrm{Tab}_{\ytableau{3 \\ 1 & 2}\,,k+1}} s_{\mathrm{shape}(S)} + \sum_{S \in \mathrm{Tab}_{\ytableau{1 & 2 & 3}\,,k+1}} s_{\mathrm{shape}(S)}. \end{aligned}$$

Note that the last equality holds because $\mathrm{Tab}_{\ytableau{1 & 2 & 3}\,,k+1} \cup \mathrm{Tab}_{\ytableau{3 \\ 1 & 2}\,,k+1}$ is the collection of all tableaux $S \in \mathrm{SSYT}_{(k+1,k+1,k+1)}$ with N_2 even. This said, subtracting the expression obtained earlier for $s_3[s_{k+1}]$ from the above equation we get

$$s_{21}[s_{k+1}] = \sum_{S \in \mathrm{Tab}_{\ytableau{3 \\ 1 & 2}\,,k+1}} s_{\mathrm{shape}(S)}$$

and by Lemma 3.1.6, we get

$$\sum_{S \in \mathrm{Tab}_{\ytableau{2 \\ 1 & 3}\,,k+1}} s_{\mathrm{shape}(S)} = s_{21}[s_{k+1}] = \sum_{S \in \mathrm{Tab}_{\ytableau{3 \\ 1 & 2}\,,k+1}} s_{\mathrm{shape}(S)}$$

- We use the same technique for $s_{111}[s_{k+1}]$, starting from the fact that;

$$s_{11}[s_{k+1}] = \sum_{S} s_{\mathrm{shape}(S)}.$$

where the sum is taken over $S \in \mathrm{SSYT}_{(k+1,k+1)}$ with N_2 odd. This implies that

$$\begin{aligned} s_{111}[s_{k+1}] + s_{21}[s_{k+1}] &= s_{11}[s_{k+1}]s_1[s_{k+1}] \qquad \text{(by Pieri's rule)} \\ &= s_{11}[s_{k+1}]s_{k+1} \\ &= \sum_{S \in \mathrm{Tab}_{\ytableaushort{3,2,1},k+1}} s_{\mathrm{shape}(S)} + \sum_{S \in \mathrm{Tab}_{\ytableaushort{2,13},k+1}} s_{\mathrm{shape}(S)} \end{aligned}$$

Here too we note that the last equality holds because $\mathrm{Tab}_{\ytableaushort{3,2,1},k+1} \cup \mathrm{Tab}_{\ytableaushort{2,13},k+1}$ is the collection of all tableaux $S \in \mathrm{SSYT}(k+1, k+1, k+1)$ with N_2 odd. Then, using the expression obtained for $s_{21}[s_{k+1}]$ and by subtracting from both sides of the equation we get,

$$s_{111}[s_{k+1}] = \sum_{S \in \mathrm{Tab}_{\ytableaushort{3,2,1},k+1}} s_{\mathrm{shape}(S)} \tag{3.1.2}$$

This ends the proof. $\square$

Lets have some examples of the plethysms computed with the aid of this theorem.

Example 3.1.10. *Let* $T \in \left\{ \ytableaushort{123}, \ytableaushort{2,13}, \ytableaushort{3,12}, \ytableaushort{3,2,1} \right\}$ *be a standard tableau with shape a partition of 3.*

1) For $k = 2$, *we have*

$$SSYT_{(2,2,2)} = \left\{ \ytableaushort{112233}, \ytableaushort{3,11223}, \ytableaushort{2,11233}, \ytableaushort{33,1122}, \ytableaushort{22,1133}, \ytableaushort{23,1123}, \ytableaushort{233,112}, \ytableaushort{3,2,1123}, \ytableaushort{3,23,112}, \ytableaushort{3,22,113}, \ytableaushort{33,22,11} \right\}$$

These tableaux fall in the following four categories. In each category we give detailed steps for analysing and classifying one of the tableaux and give the full expression for the respective plethysms with the tableau corresponding to each term.

– *for* $S = \begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}$ $N_2 = 0$ *is even,* $N_3 = 2 \geq 2N_2$, *but* $N_3 \neq 2N_2 + 1$. *So* *type*$(S) = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{42}.

$$s_3[s_2] = s_6 + s_{42} + s_{222}$$

with corresponding tableaux $\begin{ytableau} 1 & 1 & 2 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}$, *and* $\begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{ytableau}$ *respectively.*

– *for* $S = \begin{ytableau} 2 \\ 1 & 1 & 2 & 3 & 3 \end{ytableau}$ $N_2 = 1$ *is odd, and* $N_3 = 0 < N_2$. *So type*$(S) = \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{51}.

$$s_{21}[s_2] = s_{51} + s_{42} + s_{321}$$

with corresponding tableaux $\begin{ytableau} 2 \\ 1 & 1 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 3 \\ 1 & 1 & 2 & 3 \end{ytableau}$, *and* $\begin{ytableau} 3 \\ 2 & 3 \\ 1 & 1 & 2 \end{ytableau}$ *respectively.*

– *for* $S = \begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}$ $N_2 = 2$ *is even, and* $N_3 = 0 < N_2$. *So type*$(S) = \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{42}.

$$s_{21}[s_2] = s_{51} + s_{42} + s_{321}$$

with corresponding tableaux $\begin{ytableau} 3 \\ 1 & 1 & 2 & 2 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}$, *and* $\begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 & 3 \end{ytableau}$ *respectively.*

– $S = \begin{ytableau} 3 \\ 2 \\ 1 & 1 & 2 & 3 \end{ytableau}$ *has three rows and* $\overline{S} = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$, *with type*$(\overline{S}) = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$. *So type*$(S) = (\begin{ytableau} 1 & 2 & 3 \end{ytableau})^t = \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}$ *and the corresponding Schur function to* *this tableau is* s_{411}

$$s_{111}[s_2] = s_{33} + s_{411}$$

with corresponding tableaux $\begin{ytableau} 2 & 3 & 3 \\ 1 & 1 & 2 \end{ytableau}$, *and* $\begin{ytableau} 3 \\ 2 \\ 1 & 1 & 2 & 3 \end{ytableau}$ *respectively.*

The results in the next examples were computed using the SageMathCell with the commands given in the appendix A.

2) For $k = 3$, *we have*

– *For tableaux of type* $\begin{ytableau} 1 & 2 & 3 \end{ytableau}$,

$$s_3[s_3] = s_{441} + s_{63} + s_9 + s_{522} + s_{72}$$

with corresponding tableaux $\begin{ytableau} 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \end{ytableau}$, $\begin{ytableau} 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{ytableau}$, $\begin{ytableau} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 & 1 & 1 & 2 & 3 \end{ytableau}$ *and* $\begin{ytableau} 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 \end{ytableau}$ *respectively.*

– *For tableaux of type* $\begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$,

$$s_{21}[s_3] = s_{54} + s_{531} + s_{63} + s_{72} + s_{81} + s_{432} + s_{621}$$

with corresponding tableaux $\begin{ytableau} 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 \end{ytableau}$, $\begin{ytableau} 3 \\ 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 3 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 3 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{ytableau}$, *and* $\begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 & 1 & 2 & 3 & 3 \end{ytableau}$ *respectively.*

– *For tableaux of type* $\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$,

$$s_{21}[s_3] = s_{54} + s_{531} + s_{63} + s_{72} + s_{81} + s_{432} + s_{621}$$

with corresponding tableaux $\begin{array}{ccccc} 2&3&3&3& \\ 1&1&1&2&2 \end{array}$, $\begin{array}{ccccc} 3&&&& \\ 2&2&2&& \\ 1&1&1&3&3 \end{array}$, $\begin{array}{cccccc} 2&2&2&&& \\ 1&1&1&3&3&3 \end{array}$, $\begin{array}{ccccccc} 2&3&&&&& \\ 1&1&1&2&2&3&3 \end{array}$, $\begin{array}{cccccccc} 2&&&&&&& \\ 1&1&1&2&2&3&3&3 \end{array}$, $\begin{array}{cccc} 3&3&& \\ 2&2&2& \\ 1&1&1&3 \end{array}$ *and* $\begin{array}{cccccc} 3&&&&& \\ 2&3&&&& \\ 1&1&1&2&2&3 \end{array}$ *respectively.*

– *For tableaux of type* $\begin{array}{c} 3 \\ 2 \\ 1 \end{array}$,

$$s_{111}[s_3] = s_{63} + s_{531} + s_{711} + s_{333}$$

with corresponding tableaux $\begin{array}{cccccc} 2&3&3&&& \\ 1&1&1&2&2&3 \end{array}$, $\begin{array}{ccccc} 3&&&& \\ 2&3&3&& \\ 1&1&1&2&2 \end{array}$, $\begin{array}{cccccccc} 3&&&&&&& \\ 2&&&&&&& \\ 1&1&1&2&2&2&3&3 \end{array}$, *and* $\begin{array}{ccc} 3&3&3 \\ 2&2&2 \\ 1&1&1 \end{array}$ *respectively.*

3) $s_3[s_4] = s_{12} + s_{102} + s_{93} + s_{84} + s_{822} + s_{741} + s_{66} + s_{642} + s_{444}$ *with corresponding tableaux* $\begin{array}{ccccccccccccc} 1&1&1&1&2&2&2&2&2&3&3&3&3 \end{array}$, $\begin{array}{cccccccccc} 3&3&&&&&&&& \\ 1&1&1&1&2&2&2&2&3&3 \end{array}$, $\begin{array}{ccccccccc} 3&3&3&&&&&& \\ 1&1&1&1&2&2&2&2&3 \end{array}$, $\begin{array}{cccccccc} 3&3&3&3&&&& \\ 1&1&1&1&2&2&2&2 \end{array}$, $\begin{array}{cccccccc} 3&3&&&&&& \\ 2&2&&&&&& \\ 1&1&1&1&2&2&3&3 \end{array}$, $\begin{array}{ccccccc} 3&&&&&& \\ 2&2&3&3&&& \\ 1&1&1&1&2&2&3 \end{array}$, $\begin{array}{cccccc} 2&2&3&3&3&3 \\ 1&1&1&1&2&2 \end{array}$, $\begin{array}{cccccc} 3&3&&&& \\ 2&2&3&3&& \\ 1&1&1&1&2&2 \end{array}$ *and* $\begin{array}{cccc} 3&3&3&3 \\ 2&2&2&2 \\ 1&1&1&1 \end{array}$ *respectively.*

Chapter FOUR

An Alternative Approach to Type Classification

In this chapter, we present an alternative to definition 3.1.1 which gives a different yet equivalent classification for tableaux $S \in \mathrm{SSYT}_{(k,k,k)}$, $k \geq 1$. It is named **type'**. With this new classification, we examine the validity of the results established in Section 3.1 of the previous chapter.

4.1 Alternative classification

In this section, we start by giving the definition of type' for the respective tableaux and subsequently proceed to investigate and establish results pertaining to this definition. In particular, we try to restate and prove the lemma 3.1.3, lemma 3.1.6 and theorem 3.1.9 using this classification. We point out that the first difference between the two classifications is observed for the tableaux $\begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}$ and $\begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}$ and thus, these definitions are developed with the assumption that $\begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}$ is of type $\begin{ytableau} 1 & 2 & 3 \end{ytableau}$ as given by definition 3.1.1 and $\begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}$ is of type' $\begin{ytableau} 1 & 2 & 3 \end{ytableau}$ as given by the following definition.

Definition 4.1.1. *For $S \in SSYT_{(k,k,k)}$, let N_r represent the number of cells with the label r in the second row of S for $1 \leqslant r \leqslant 3$.*

1 If S is standard and $k = 1$, then let type'$(S) = S$.

2) If S has one or two rows, then we define:

a) type'$(S) = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$ if N_2 is even, $N_3 \leqslant 2N_2$, and $N_3 \equiv 0, 1 \pmod 4$.

b) type'$(S) = \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}$ if N_2 is odd, $N_3 \leqslant 2N_2$, and $N_3 \equiv 2, 3 \pmod 4$.

c) type'$(S) = \ytableaushort{3,12}$ *if* N_2 *is even, and* $N_3 > 2N_2$ *or* $N_3 \equiv 2, 3 \pmod 4$.

d) type'$(S) = \ytableaushort{2,13}$ *if* N_2 *is odd, and* $N_3 > 2N_2$ *or* $N_3 \equiv 0, 1 \pmod 4$.

3) If S *has three rows and* $\overline{S}$ *is* S *with the first column removed, then type'*$(S) =$ *type'*$(\overline{S})^t$.

Next, if we consider T a standard tableau of size 3, define the set

$$\overline{\text{Tab}}_{T,k} = \{S \in \text{SSYT}_{(k,k,k)} \mid \text{type'}(S) = T\}$$

We state and proof results homologous to Lemma 3.1.3, Lemma 3.1.6 and Theorem 3.1.9 based on Definition 4.1.1. The proofs are done analogously to their counterparts given in the previous chapter.

Lemma 4.1.2. *For* $k \geq 1$,

$$s_3[s_k] \downarrow_2 = \sum_S s_{shape(S)},$$

where the sum is over all $S \in \overline{Tab}_{\ytableaushort{123},k}$ *with* $\ell(S) \leqslant 2$.

Proof. We aim to show that each term in the expression of $s_3[s_k] \downarrow_2$ has one corresponding tableau of length less than or equal to 2. From the item 2 of Theorem 3.1.2, we have that

$$s_3[s_n] \downarrow_2 = s_{66} \odot (s_3[s_{n-4}]) \downarrow_2 + \sum_{k=2}^{n} s_{3n-k,k} + s_{3n}.$$

For $S \in \overline{\text{Tab}}_{\ytableaushort{123},k}$ with $\ell(S) \leqslant 2$, the only possible tableau corresponding to s_{66} is $\ytableaushort{223333,111122}$. Hence, for $n \geq 4$, the term $s_{66} \odot (s_3[s_{n-4}]) \downarrow_2$ encompasses the functions corresponding to those tableaux that contain at least four 3s and at least two 2s (since $N_3 \leqslant 2N_2$) in the second row. If $n < 4$, then the term is 0.

On the other hand, the tableau with a single row is catered for by the term s_{3n}. Observe that for these tableau, $N_2 = N_3 = 0$. That is N_2 is even, $N_3 \leqslant 2N_2$ and $N_3 \equiv 0 \pmod 4$. Hence this tableau is of type' $\ytableaushort{123}$.

The remaining tableaux, are enumerated by the middle term. Each term correspond to exactly one tableau of shape $(3n-k, k)$ with $2 \leqslant k \leqslant n$. Here, these tableaux have the particularity that $N_2 = k$ if k is even (that is, $N_3 = 0 \equiv 0 \pmod 4$) and, $N_2 = k-1$ when k is odd (that is, $N_3 = 1 \equiv 1 \pmod 4$). Thus these tableaux too are of type' $\ytableaushort{123}$. $\square$

Example 4.1.3. *1) Let's compute $s_3[s_4]\downarrow_2$. The list of tableaux $S \in \overline{Tab}_{\ytableaushort{123},4}$ with $\ell(S) \leqslant 2$ is*

$$\left\{ \ytableaushort{223333,111122}, \ytableaushort{22,1111223333}, \ytableaushort{223,111122333}, \ytableaushort{2222,11113333}, \ytableaushort{1111222223333} \right\}$$

and

$$s_3[s_4]\downarrow_2 = s_{66} + s_{102} + s_{93} + s_{84} + s_{12}.$$

2) For $k = 5$, the list of tableaux $S \in \overline{Tab}_{\ytableaushort{123},5}$ with $\ell(S) \leqslant 2$ is

$$\left\{ \ytableaushort{223333,111112223}, \ytableaushort{22,1111122233333}, \ytableaushort{223,111112223333}, \ytableaushort{2222,11111233333}, \ytableaushort{22223,1111123333}, \ytableaushort{111112222233333} \right\}.$$

We thus get

$$s_3[s_5]\downarrow_2 = s_{96} + s_{132} + s_{123} + s_{114} + s_{105} + s_{15}.$$

Conjecture 4.1.4. *For $k \geq 1$,*

$$\sum_{S \in \overline{Tab}_{\ytableaushort{3,12},k}} s_{shape(S)} = \sum_{S \in \overline{Tab}_{\ytableaushort{2,13},k}} s_{shape(S)}.$$

Proof. To proof this equality, we show that the following map

$$\phi : \overline{\mathrm{Tab}}_{\begin{smallmatrix}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{smallmatrix},k} \to \overline{\mathrm{Tab}}_{\begin{smallmatrix}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{smallmatrix},k}$$

such that $\phi(S) = S'$, is a shape preserving bijection. If $S \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{smallmatrix},k}$ is of length 2, let N_2 and N_3 represent respectively the number of 2's and 3's in the second row of S. Similarly, let $N_2^{S'}$ and $N_3^{S'}$ represent respectively the number of 2's and 3's in the second row of S'. For $S \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{smallmatrix},k}$ the map is defined as follows:

Case 1. If $N_3 > 2N_2$, then $\phi(S) = S'$ is the tableau where an odd number of 3's from the second row are exchanged with 2's from the first row such that $N_3^{S'} \equiv 0, 1 \pmod 4$.

Case 2. If $N_3 \equiv 2, 3 \pmod 4$ with $N_3 \leqslant 2N_2$, then $\phi(S) = S'$ is the tableau where an odd number of 3's from the first row are exchanged with 2's from the second row such that $N_3^{S'} > 2N_2^{S'}$ modulo 4.

Case 3. If the length of S is 3, then let $\overline{S} \in SSYT_{(k,k,k)}$ be the tableau S with the first column removed. Define $\phi(S)$ to be $\phi^{-1}(\overline{S})$ to which a column of length 3 is added.

We justify that in each of the 3 cases, $\phi(S) \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{smallmatrix},k}$ and this image is unique. Firstly, observe that $S \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}\boxed{3} & \\ \boxed{1} & \boxed{2}\end{smallmatrix},k}$ implies that N_2 is even, and either $N_3 > 2N_2$ or $N_3 \equiv 2, 3 \pmod 4$.

For either case 1 or case 2, we have that $N_2^{S'} = N_2 + d$ where d is an odd integer (positive in case 1 and negative in case 2) representing the number of 3's exchanged with 2's. So we easily see that $N_2^{S'}$ is odd for both cases. It is unclear how the choice of the integer d can be made with precision. Clarifying this step of the proof will enable to have this conjecture as a lemma.

Additionally, from the above definition of ϕ, we easily conclude that $S' = \phi(S) \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix},k}$ for both case 1 and case 2.

For case 3, observe that, if $S \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix},k}$ with $\ell(S) = 3$, then type'$(\overline{S}) = \begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}$. That is, $\overline{S} \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix},k}$. Let $T = \phi^{-1}(\overline{S})$, then type'$(T) =$ type'$(\phi^{-1}(\overline{S})) = \begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix}$. Also, let T' be the tableau T to which a column of length 3 is added, then we observe that type'$(T') =$ (type'$(T))^t = \begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}$, implying that $T' \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix},k}$. But type'$(T') =$ type'$(\phi(S))$. Hence $S' = \phi(S) \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix},k}$.

Furthermore, it is clear by the definition of ϕ that for $S, T \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix},k}$, if $\phi(S) = \phi(T)$, then $S = T$. Also, for all $S' \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix},k}$ there exists $S \in \mathrm{Tab}_{\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix},k}$ such that $S' = \phi(S)$. Hence ϕ is a bijection and thus

$$\sum_{S \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix},k}} s_{\mathrm{shape}(S)} = \sum_{S \in \overline{\mathrm{Tab}}_{\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix},k}} s_{\mathrm{shape}(S)}.$$

□

Since the last two results are analogues of lemma 3.1.3 and lemma 3.1.6 respectively with a different classification of tableaux, we can also deduce that theorem 3.1.9 holds true with this new classification of tableaux. We however state it here as a conjecture since the last result which is used in its proof has not been fully proven as a lemma.

Conjecture 4.1.5. *Let* $T \in \left\{ \begin{smallmatrix}1 & 2 & 3\end{smallmatrix}, \begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}, \begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix}, \begin{smallmatrix}3 \\ 2 \\ 1\end{smallmatrix} \right\}$ *be a standard tableau with* *shape a partition of 3. Then for any $k \geq 1$,*

$$s_{shape(T)}[s_k] = \sum_{\substack{S \in SSYT_{(k,k,k)} \\ type'(S)=T}} s_{shape(S)}$$

where the type'(S) for $S \in SSYT_{(k,k,k)}$ is defined in Definition 4.1.1.

Proof. Just as in Theorem 3.1.9, the proof is by induction on k. The base step taken at $k = 1$, we aim to show that if $T \in \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1&3\\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1&2\\ \hline\end{array}, \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} \right\}$ is a standard tableau with shape a partition of 3, then,

$$s_{\text{shape}(T)}[s_1] = \sum_{\substack{S \in \text{SSYT}_{(1,1,1)} \\ \text{type'}(S)=T}} s_{\text{shape}(S)}$$

By definition 4.1.1 and (1) of example 3.0.5 we have that;

$$s_3[s_1] = s_3$$

$$s_{21}[s_1] = s_{21}$$

and

$$s_{111}[s_1] = s_{(111)}.$$

Checking that we have the same results using the theorem, we first note that $SSYT_{(111)} = \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1&3\\ \hline\end{array}, \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1&2\\ \hline\end{array}, \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} \right\}$. We then obtain the following;

- For $T = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}$, $shape(T) = (3)$, $S \in SSYT_{(111)}$ and type'$(S) = T$ implies that $S = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}$ so $s_{shape(S)} = s_3$. Hence $s_3[s_1] = s_3$
- For $T = \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array}$, $shape(T) = (111)$, $S \in SSYT_{(111)}$ and type'$(S) = T$ implies that

$S = \ytableaushort{3,2,1}$ so $s_{shape(S)} = s_{(111)}$. Hence $s_{(111)}[s_1] = s_{(111)}$

- Observe that by definition 4.1.1, type' $\left(\ytableaushort{3,12}\right) = \ytableaushort{3,12}$ and type' $\left(\ytableaushort{2,13}\right) = \ytableaushort{2,13}$. So, proceeding as in the previous cases, for $S = \ytableaushort{3,12}$ or $S = \ytableaushort{2,13}$ we get $s_{shape(S)} = s_{(21)}$. Hence $s_{(21)}[s_1] = s_{(21)}$.

Thus here too the results all coincide with the result obtained using the definition and this completes the base step of the proof.

Suppose the theorem holds true for arbitrary k.

Assuming that the theorem holds for a fixed k and for all tableaux $T \in \left\{ \ytableaushort{123}, \ytableaushort{2,13}, \ytableaushort{3,12}, \ytableaushort{3,2,1} \right\}$, we aim to prove that the theorem holds for $k+1$.

By Lemma 4.1.2 we have that $s_3[s_{k+1}] \downarrow_2$ gives all the terms involving tableaux $S \in \overline{\mathrm{Tab}}_{\ytableaushort{123},k+1}$ with length $\ell(S) \leqslant 2$. For the terms with length 3 we again make use of the s-perp trick formula below which is a special case of equation (5.2) from Proposition 5.2, [10]:

$$s_{1^r}^{\perp} s_h[s_w] = s_{h-r}[s_w] s_{1^r}[s_{w-1}] \tag{4.1.1}$$

with $r = h = 3$, and $w = k+1$ resulting to $s_{111}^{\perp} s_3[s_{k+1}] = s_0[s_{k+1}] s_{111}[s_k] = s_{111}[s_k]$. We thus deduce inductively that the terms of length 3 are given by those $S \in \overline{\mathrm{Tab}}_{\ytableaushort{123},k+1}$ with $\ell(S) = 3$ and hence,

$$s_3[s_{k+1}] = \sum_{S \in \overline{\mathrm{Tab}}_{\ytableaushort{123},k+1}} s_{\mathrm{shape}(S)}.$$

Furthermore, we make use of equations (1) and (2) in Example 9 from Section I.8 of [19] to which we associate the Jacobi-Trudi identity to derive the last remaining cases.

- For $s_{21}[s_{k+1}]$, we have that;

$$s_2[s_{k+1}] = \sum_{S \in \mathrm{SSYT}_{(k+1,k+1)}} s_{\mathrm{shape}(S)},$$

where the sum is over $S \in \mathrm{SSYT}_{(k+1,k+1)}$ with N_2 even. This in turn implies that,

$$\begin{aligned} s_{21}[s_{k+1}] + s_3[s_{k+1}] &= s_2[s_{k+1}]s_1[s_{k+1}] \qquad \text{(by Pieri's rule)} \\ &= s_2[s_{k+1}]s_{k+1} \\ &= \sum_{S \in \overline{\mathrm{Tab}}_{\ytableaushort{3,12},k+1}} s_{\mathrm{shape}(S)} + \sum_{S \in \overline{\mathrm{Tab}}_{\ytableaushort{123},k+1}} s_{\mathrm{shape}(S)}. \end{aligned}$$

Note that the last equality holds because $\overline{\mathrm{Tab}}_{\ytableaushort{123},k+1} \cup \overline{\mathrm{Tab}}_{\ytableaushort{3,12},k+1}$ is the collection of all tableaux $S \in \mathrm{SSYT}_{(k+1,k+1,k+1)}$ with N_2 even. This said, subtracting the expression obtained earlier for $s_3[s_{k+1}]$ from the above equation we get

$$s_{21}[s_{k+1}] = \sum_{S \in \overline{\mathrm{Tab}}_{\ytableaushort{3,12},k+1}} s_{\mathrm{shape}(S)}$$

and by Conjecture 4.1.4, we get

$$\sum_{S \in \overline{\mathrm{Tab}}_{\ytableaushort{2,13},k+1}} s_{\mathrm{shape}(S)} = s_{21}[s_{k+1}] = \sum_{S \in \overline{\mathrm{Tab}}_{\ytableaushort{3,12},k+1}} s_{\mathrm{shape}(S)}$$

- We use the same technique for $s_{111}[s_{k+1}]$, starting from the fact that;

$$s_{11}[s_{k+1}] = \sum_S s_{\mathrm{shape}(S)}.$$

where the sum is taken over $S \in \mathrm{SSYT}_{(k+1,k+1)}$ with N_2 odd. This implies that

$$\begin{aligned} s_{111}[s_{k+1}] + s_{21}[s_{k+1}] &= s_{11}[s_{k+1}]s_1[s_{k+1}] \qquad \text{(by Pieri's rule)} \\ &= s_{11}[s_{k+1}]s_{k+1} \\ &= \sum_{S \in \overline{\mathrm{Tab}}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}, k+1}} s_{\mathrm{shape}(S)} + \sum_{S \in \overline{\mathrm{Tab}}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k+1}} s_{\mathrm{shape}(S)} \end{aligned}$$

Here too we note that the last equality holds because $\overline{\mathrm{Tab}}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}, k+1} \cup \overline{\mathrm{Tab}}_{\begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, k+1}$ is the collection of all tableaux $S \in \mathrm{SSYT}(k+1, k+1, k+1)$ with N_2 odd. Then, using the expression obtained for $s_{21}[s_{k+1}]$ and by subtracting from both sides of the equation we get,

$$s_{111}[s_{k+1}] = \sum_{S \in \overline{\mathrm{Tab}}_{\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}, k+1}} s_{\mathrm{shape}(S)}$$

This completes the proof. $\square$

Let's revisit the examples given previously using the classification type'.

Example 4.1.6. *Let* $T \in \left\{ \begin{ytableau} 1 & 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau} \right\}$ *be a standard tableau with shape a partition of 3.*

1) For $k = 2$*, we have*

$$SSYT_{(2,2,2)} = \left\{ \begin{ytableau} 1 & 1 & 2 & 2 & 3 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 1 & 2 & 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 1 & 2 & 3 & 3 \end{ytableau}, \begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}, \right.$$

$$\left. \begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}, \begin{ytableau} 2 & 3 \\ 1 & 1 & 2 & 3 \end{ytableau}, \begin{ytableau} 2 & 3 & 3 \\ 1 & 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 2 \\ 1 & 1 & 2 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 3 \\ 1 & 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 & 3 \end{ytableau}, \begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{ytableau} \right\}$$

These tableaux fall in the following four categories. In each category we give detailed steps for analysing and classifying one of the tableaux and give the full expression for the respective plethysms with the tableau corresponding to each term.

– *for* $S = \begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}$ $N_2 = 2$ *is even,* $N_3 = 0 < 2N_2$ *and* $N_3 \equiv 0 \pmod 4$. *So type'*$(S) = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{42}.

$$s_3[s_2] = s_6 + s_{42} + s_{222}$$

with corresponding tableaux $\begin{ytableau} 1 & 1 & 2 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau}$, *and* $\begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{ytableau}$ *respectively.*

– *for* $S = \begin{ytableau} 2 \\ 1 & 1 & 2 & 3 & 3 \end{ytableau}$ $N_2 = 1$ *is odd, and* $N_3 = 0 \equiv 0 \pmod 4$. *So type'*$(S) = \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{51}.

$$s_{21}[s_2] = s_{51} + s_{42} + s_{321}$$

with corresponding tableaux $\begin{ytableau} 2 \\ 1 & 1 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 3 \\ 1 & 1 & 2 & 3 \end{ytableau}$, *and* $\begin{ytableau} 3 \\ 2 & 3 \\ 1 & 1 & 2 \end{ytableau}$ *respectively.*

– *for* $S = \begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}$ $N_2 = 0$ *is even, and* $N_3 = 2 > 2N_2$. *So type'*$(S) = \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{42}.

$$s_{21}[s_2] = s_{51} + s_{42} + s_{321}$$

with corresponding tableaux $\begin{ytableau} 3 \\ 1 & 1 & 2 & 2 & 3 \end{ytableau}$, $\begin{ytableau} 3 & 3 \\ 1 & 1 & 2 & 2 \end{ytableau}$, *and* $\begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 & 3 \end{ytableau}$ *respectively.*

– $S = \begin{ytableau} 3 \\ 2 \\ 1 & 1 & 2 & 3 \end{ytableau}$ *has three rows and* $\overline{S} = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$, *with type'*$(\overline{S}) = \begin{ytableau} 1 & 2 & 3 \end{ytableau}$.

So type'(S) $= (\begin{ytableau}1&2&3\end{ytableau})^t = \begin{ytableau}3\\2\\1\end{ytableau}$ *and the corresponding Schur function to this tableau is* s_{411}

$$s_{111}[s_2] = s_{33} + s_{411}$$

with corresponding tableaux $\begin{ytableau}2&3&3\\1&1&2\end{ytableau}$, *and* $\begin{ytableau}3\\2\\1&1&2&3\end{ytableau}$ *respectively.*

2) For $k = 3$, *we computed using the SageMathCell with the commands given in the appendix B;*

– *For tableaux of type* $\begin{ytableau}1&2&3\end{ytableau}$,

$$s_3[s_3] = s_{441} + s_{63} + s_9 + s_{522} + s_{72}$$

with corresponding tableaux $\begin{ytableau}3\\2&2&3&3\\1&1&1&2\end{ytableau}$, $\begin{ytableau}2&2&3\\1&1&1&2&3&3\end{ytableau}$, $\begin{ytableau}1&1&1&2&2&2&3&3&3\end{ytableau}$, $\begin{ytableau}3&3\\2&2\\1&1&1&2&3\end{ytableau}$ *and* $\begin{ytableau}2&2\\1&1&1&2&3&3&3\end{ytableau}$ *respectively.*

– *For tableaux of type* $\begin{ytableau}3\\1&2\end{ytableau}$,

$$s_{21}[s_3] = s_{54} + s_{531} + s_{63} + s_{72} + s_{81} + s_{432} + s_{621}$$

with corresponding tableaux $\begin{ytableau}2&2&3&3\\1&1&1&2&3\end{ytableau}$, $\begin{ytableau}3\\2&2&3\\1&1&1&2&3\end{ytableau}$, $\begin{ytableau}3&3&3\\1&1&1&2&2&2\end{ytableau}$, $\begin{ytableau}3&3\\1&1&1&2&2&2&3\end{ytableau}$, $\begin{ytableau}3\\1&1&1&2&2&2&3&3\end{ytableau}$, $\begin{ytableau}3&3\\2&2&3\\1&1&1&2\end{ytableau}$, *and* $\begin{ytableau}3\\2&2\\1&1&1&2&3&3\end{ytableau}$ *respectively.*

– *For tableaux of type* $\begin{ytableau}2\\1&3\end{ytableau}$,

$$s_{21}[s_3] = s_{54} + s_{531} + s_{63} + s_{72} + s_{81} + s_{432} + s_{621}$$

with corresponding tableaux $\begin{ytableau} 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 \end{ytableau}$, $\begin{ytableau} 3 \\ 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 \end{ytableau}$, $\begin{ytableau} 2 & 2 & 2 \\ 1 & 1 & 1 & 3 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{ytableau}$ *and* $\begin{ytableau} 3 \\ 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 \end{ytableau}$ *respectively.*

– *For tableaux of type* $\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau}$,

$$s_{111}[s_3] = s_{63} + s_{531} + s_{711} + s_{333}$$

with corresponding tableaux $\begin{ytableau} 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 \end{ytableau}$, $\begin{ytableau} 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 & 3 & 3 \end{ytableau}$, $\begin{ytableau} 3 \\ 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{ytableau}$, *and* $\begin{ytableau} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{ytableau}$ *respectively.*

Conclusion

Our objectives in this work were; to develop new theoretical and computational methods for calculating plethysm coefficients; to derive explicit formulas for plethysm coefficients in certain special cases; to investigate the combinatorial and algebraic properties of plethysm coefficients; and finally to apply the newly developed methods and derived formulas to specific examples, demonstrating their effectiveness and uncovering new insights into the structure of plethysm coefficients.

To achieve these objectives, we started by understanding the basic notions of partitions and symmetric functions, followed by notions on plethysms, specifically the case of Schur functions. The objectives were then achieved through theorem 3.1.9 and conjecture 4.1.5. Precisely, in theorem 3.1.9 we essentially provide details for the proof of theorem 5.3 from the referenced article and provide examples. This theorem tells us that the plethysm of any Schur functions s_λ and s_k where λ is a partition of 3 and $k \geqslant 1$ is an integer, can expressed as the sum of specific Schur functions s_μ with coefficients 1 where μ is a partition of $3k$. In chapter 4, a different classification is defined and through the proof of conjecture 4.1.5 it is shown that theorem 3.1.9 still holds true with this new definition. Hence giving an alternative way to compute the plethysms $s_\lambda[s_k]$.

For future work, it will be of interest to either establish how the integer d in conjecture 4.1.4 can be accurately determined or provide a different approach to proving the conjecture. An immediate interesting follow up to this work will be to investigate the case of such plethysms where λ is a partition of 4 then a higher integer, since a greater goal will be to be able to give a general approach to computing these plethysms for any partition of any positive integer. Also, it will be interesting for a future work to investigate the case of plethysms $s_\lambda[s_\mu]$ with μ an arbitrary partition.

Appendix A

A.1. SageMath code for classification of tableaux using definition 3.1.1

```
#Sage Code for $s_{\lambda}[s_k]$ based on definition of type
SymmetricFunctions(QQ).inject_shorthands()
Tableaux.options(convention='French')
def has_type_123(T):
    if len(T)==1:
        return True
    if len(T)==2:
        N2 = T[1].count(2)
        N3 = T[1].count(3)
        return is_even(N2) and N3>=2*N2 and N3!=2*N2+1
    if len(T)==3:
        Trem = [T[0][1:], T[1][1:], T[2][1:]]
        if len(Trem[2])==0:
            Trem = Trem[:2]
        if len(Trem[1])==0:
            Trem = Trem[:1]
        if len(Trem[0])==0:
            Trem = []
        return has_type_321(Tableau(Trem))
    return False
def has_type_321(T):
    if T.shape()==[1,1,1]:
        return True
    if len(T)==2:
```

```
        N2 = T[1].count(2)
        N3 = T[1].count(3)
        return is_odd(N2) and N3>=2*N2 and N3!=2*N2+1
    if len(T)==3:
        Trem = [T[0][1:], T[1][1:], T[2][1:]]
        if len(Trem[2])==0:
            Trem = Trem[:2]
        if len(Trem[1])==0:
            Trem = Trem[:1]
        if len(Trem[0])==0:
            Trem = []
        return has_type_123(Tableau(Trem))
    return False
def has_type_312(T):
    return not has_type_123(T) and \
     not has_type_321(T) and \
     is_even(T[1].count(2))
def has_type_213(T):
    return not has_type_123(T) and \
     not has_type_321(T) and \
     is_odd(T[1].count(2))


Tab123 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_123(T):
        Tab123 = Tab123.union(Set([T]))
print("\n The types 123 are:")
print(Tab123)
```

```
Tab321 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_321(T):
        Tab321 = Tab321.union(Set([T]))
print("\n The types 321 are:")
print(Tab321)


Tab213 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_213(T):
        Tab213 = Tab213.union(Set([T]))
print("\n The types 213 are:")
print(Tab213)


Tab312 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_312(T):
        Tab312 = Tab312.union(Set([T]))
print("\n The types 312 are:")
print(Tab312)
```

A.2. Outputs for $k = 4$

```
 The types 123 are:
{[[1, 1, 1, 1, 2, 2, 2, 2], [3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2], [2, 2, 3, 3, 3, 3]],
[[1, 1, 1, 1], [2, 2, 2, 2], [3, 3, 3, 3]],
```

[[1, 1, 1, 1, 2, 2, 2, 2, 3], [3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2], [2, 2, 3, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 3, 3], [2, 2], [3, 3]],
[[1, 1, 1, 1, 2, 2, 3], [2, 2, 3, 3], [3]]}

The types 321 are:
{[[1, 1, 1, 1, 2, 2, 2, 3], [2, 3, 3], [3]],
[[1, 1, 1, 1, 2], [2, 2, 2, 3, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3], [2, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3, 3], [2], [3]],
[[1, 1, 1, 1, 2, 3], [2, 2, 2], [3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2], [2, 3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2], [2, 3, 3, 3], [3]]}

The types 213 are:
{[[1, 1, 1, 1, 2, 3, 3], [2, 2, 2], [3, 3]],
[[1, 1, 1, 1, 2, 3], [2, 2, 2, 3, 3], [3]],
[[1, 1, 1, 1, 2], [2, 2, 2, 3], [3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3], [2, 3], [3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3, 3], [2, 3]],
[[1, 1, 1, 1, 2, 3, 3, 3, 3], [2, 2, 2]],
[[1, 1, 1, 1, 2, 3], [2, 2, 2, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3], [2]],
[[1, 1, 1, 1, 2, 3, 3], [2, 2, 2, 3], [3]],
[[1, 1, 1, 1, 2, 2, 2, 3], [2, 3, 3, 3]],
[[1, 1, 1, 1, 2, 3, 3], [2, 2, 2, 3, 3]],
[[1, 1, 1, 1, 2, 3, 3, 3], [2, 2, 2], [3]],
[[1, 1, 1, 1, 2, 3, 3, 3], [2, 2, 2, 3]]}

```
The types 312 are:
{[[1, 1, 1, 1, 2, 2, 3, 3], [2, 2, 3, 3]],
[[1, 1, 1, 1, 2, 2, 3, 3, 3], [2, 2, 3]],
[[1, 1, 1, 1, 2, 2, 3], [2, 2, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2], [2, 2, 3, 3, 3], [3]],
[[1, 1, 1, 1, 2, 2, 3, 3], [2, 2, 3], [3]],
[[1, 1, 1, 1, 3], [2, 2, 2, 2], [3, 3, 3]],
[[1, 1, 1, 1, 3, 3, 3], [2, 2, 2, 2], [3]],
[[1, 1, 1, 1, 2, 2, 3, 3, 3, 3], [2, 2]],
[[1, 1, 1, 1, 2, 2, 3], [2, 2, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3], [3]],
[[1, 1, 1, 1, 2, 2, 3, 3, 3], [2, 2], [3]],
[[1, 1, 1, 1, 3, 3, 3, 3], [2, 2, 2, 2]],
[[1, 1, 1, 1, 3, 3], [2, 2, 2, 2], [3, 3]]}
```

Appendix B

B.1. SageMath code for classification of tableaux using definition 4.1.1

```
#Sage Code for $s_{\lambda}[s_k]$ based on definition of type'
SymmetricFunctions(QQ).inject_shorthands()
Tableaux.options(convention='French')

def has_typep_123(T):
    if len(T) == 1:
        return True
    if len(T) == 2:
        N2 = T[1].count(2)
        N3 = T[1].count(3)
        return (N2 % 2 == 0) and (N3 <= 2 * N2) and \
         (N3 % 4 == 0 or N3 % 4 == 1)
    if len(T) == 3:
        Trem = [T[0][1:], T[1][1:], T[2][1:]]
        Trem = [row for row in Trem if row]  # Remove empty rows
        return has_typep_321(Tableau(Trem))
    return False

def has_typep_321(T):
    if T.shape() == [1, 1, 1]:
        return True
    if len(T) == 2:
        N2 = T[1].count(2)
        N3 = T[1].count(3)
```

```
        return (N2 % 2 == 1) and (N3 <= 2 * N2) and \
         (N3 % 4 == 2 or N3 % 4 == 3)
    if len(T) == 3:
        Trem = [T[0][1:], T[1][1:], T[2][1:]]
        Trem = [row for row in Trem if row]  # Remove empty rows
        return has_typep_123(Tableau(Trem))
    return False

def has_typep_312(T):
    if len(T) == 1:
        return False
    if len(T) == 2:
        N2 = T[1].count(2)
        N3 = T[1].count(3)
        return (N2 % 2 == 0) and ((N3 > 2 * N2) or \
         (N3 % 4 == 2 or N3 % 4 == 3))
    if len(T) == 3:
        Trem = [T[0][1:], T[1][1:], T[2][1:]]
        Trem = [row for row in Trem if row]  # Remove empty rows
        return has_typep_213(Tableau(Trem))

def has_typep_213(T):
    if len(T) == 1:
        return False
    if len(T) == 2:
        N2 = T[1].count(2)
        N3 = T[1].count(3)
        return (N2 % 2 == 1) and ((N3 > 2 * N2) or \
         (N3 % 4 == 0 or N3 % 4 == 1))
    if len(T) == 3:
```

```
        Trem = [T[0][1:], T[1][1:], T[2][1:]]
        Trem = [row for row in Trem if row]  # Remove empty rows
        return has_typep_312(Tableau(Trem))

Tab123 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_123(T):
        Tab123 = Tab123.union(Set([T]))
print("\n The tableaux with typep 123 are:")
print(Tab123)



Tab321 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_321(T):
        Tab321 = Tab321.union(Set([T]))
print("\n The tableaux with typep 321 are:")
print(Tab321)



Tab213 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
    if has_type_213(T):
        Tab213 = Tab213.union(Set([T]))
print("\n The tableaux with typep 213 are:")
print(Tab213)



Tab312 = Set()
for T in SemistandardTableaux(eval=[k,k,k]):
```

```
    if has_type_312(T):
        Tab312 = Tab312.union(Set([T]))
print("\n The tableaux with typep 312 are:")
print(Tab312)
```

B.2. Outputs for $k = 4$

```
 The tableaux with typep 123 are:
{[[1, 1, 1, 1, 2, 2, 3, 3, 3], [2, 2, 3]],
[[1, 1, 1, 1, 2, 2], [2, 2, 3, 3, 3, 3]],
[[1, 1, 1, 1], [2, 2, 2, 2], [3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 3, 3], [2, 2], [3, 3]],
[[1, 1, 1, 1, 2, 2, 3, 3, 3, 3], [2, 2]],
[[1, 1, 1, 1, 2, 2, 3], [2, 2, 3, 3], [3]],
[[1, 1, 1, 1, 3, 3, 3, 3], [2, 2, 2, 2]],
[[1, 1, 1, 1, 3, 3], [2, 2, 2, 2], [3, 3]]}

 The tableaux with typep 321 are:
{[[1, 1, 1, 1, 2], [2, 2, 2, 3, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3], [2, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3, 3], [2], [3]],
[[1, 1, 1, 1, 2, 3, 3], [2, 2, 2, 3], [3]],
[[1, 1, 1, 1, 2, 3, 3], [2, 2, 2, 3, 3]],
[[1, 1, 1, 1, 2, 3, 3, 3], [2, 2, 2], [3]],
[[1, 1, 1, 1, 2, 3], [2, 2, 2], [3, 3, 3]]}

 The tableaux with typep 213 are:
{[[1, 1, 1, 1, 2, 3], [2, 2, 2, 3, 3], [3]],
```

```
[[1, 1, 1, 1, 2], [2, 2, 2, 3], [3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3], [2, 3], [3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3, 3], [2, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3], [2, 3, 3], [3]],
[[1, 1, 1, 1, 2, 3, 3, 3, 3], [2, 2, 2]],
[[1, 1, 1, 1, 2, 2, 2], [2, 3, 3, 3], [3]],
[[1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3], [2]],
[[1, 1, 1, 1, 2, 3], [2, 2, 2, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 3], [2, 3, 3, 3]],
[[1, 1, 1, 1, 2, 3, 3], [2, 2, 2], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2], [2, 3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 3, 3, 3], [2, 2, 2, 3]]}

 The tableaux with typep 312 are:
{[[1, 1, 1, 1, 2, 2, 2, 2], [3, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 3, 3], [2, 2, 3, 3]],
[[1, 1, 1, 1, 2, 2, 3], [2, 2, 3, 3, 3]],
[[1, 1, 1, 1, 2, 2], [2, 2, 3, 3, 3], [3]],
[[1, 1, 1, 1, 2, 2, 3, 3], [2, 2, 3], [3]],
[[1, 1, 1, 1, 3], [2, 2, 2, 2], [3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3], [3, 3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3, 3], [3, 3]],
[[1, 1, 1, 1, 3, 3, 3], [2, 2, 2, 2], [3]],
[[1, 1, 1, 1, 2, 2], [2, 2, 3, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 3], [2, 2, 3], [3, 3]],
[[1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3], [3]],
[[1, 1, 1, 1, 2, 2, 3, 3, 3], [2, 2], [3]]}
```

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