Proceedings of The Canadian Society for Mechanical Engineering International Congress 2018
CSME International Congress 2018
May 27-30, 2018, Toronto, On, Canada

A HIGHER ORDER THEORY OF BEAMS AND ITS APPLICATION TO THE MEMS/NEMS ANALYSIS AND SIMULATIONS

A. Czenkanski
Department of Mechanical Engineering
York University
Toronto, Canada
alex.czekanski@lassonde.yorku.ca

V.V. Zozulya
Composite Materials Department
Centro de Investigacion Cientifica de Yucatan A.C.
Merida, Mexico
zozulya@cicy.mx

Abstract—A higher order theory for beams based on expansion of the two dimensional (2-D) equations of elasticity into Legendre's polynomials series has been developed. The 2-D equations of elasticity have been expanded into Legendre's polynomials series in terms of a thickness coordinate.. Cases of the first and second approximations have been considered in details. For obtained boundary-value problems, a finite element method (FEM) has been used and numerical calculations have been done with MATLAB. Developed theory has been applied for study stress-strain state of the electrostatically and thermally actuated MEMS/NEM.

Keywords- higher order beam theory; MEMS; NEMS; actuators

I. INTRODUCTION

Micro-Electro-Mechanical Systems (MEMS) are microscale devices or miniature embedded systems involving one or more components that enable higher-level functionality. MEMS have a characteristic length scale between 1 mm and 1 µm. Similarly Nano-Electro-Mechanical Systems (NEMS) nanoscale devices. NEMS have a characteristic length scale between 1 nm and 100 µm [3]. The MEMS and NEMS are widely used in engineering industries, communications, defense systems, health care, information technology, environmental monitoring, etc. [3]. Fabrication of the MEMS and NEMS rapidly increase from year to year. Therefore modeling, simulation and mathematical analysis are very important for optimizing process of their fabrication and further safety exploration.

Many MEMS and NEMS structures and devices can be considered as thin-walled structures that are exposed to mechanical loading, high temperature and electromagnetic fields and are in mechanical and thermal contact with other structural elements and massive bases through thin heat conducting layers. Often in the analysis and simulation of such devices classic models based in Euler-Bernoulli or Timoshenko hypothesis give an inaccurate result [1]. For accurate analysis and simulation of such structures and devices higher order theories may be more preferable. These theories are based on expansion of the stress-strain and temperature field components into polynomial Legendre's series in term of thickness [5]. Such an approach has significant advantages because of Legendre's polynomials are orthogonal and as result obtained equations are simpler. This approach was extended and applied to thermally actuated MEMS and NEMS in [6, 7].

In this presentation high order, models of beams based on nonlocal theory of elasticity have been developed and applied for MEMS and NEMS modeling and computer simulations.

II. 2-D NONLOCAL THEORY OF ELASTICITY

We consider a curved elastic rod in a 2-D Euclidian space, which occupies the domain $V=\Omega\times[-h,h]$ with a smooth boundary ∂V . Here $2h$ is thickness, $\Omega=[-l,l]$ is the middle line of the rod and $2l$ is its length. The boundary of the rod ∂V can be presented in the form $\partial V=S\cup\Omega^{+}\cup\Omega^{-}$, where Ω^{+} and Ω^{-} are the upper and lower sides and S denotes lateral sides.

According to the theory of nonlocal elasticity [2], the basic equations for linear, homogeneous, isotropic, nonlocal elastic solid are given in the form of the following system of differential equations.

The equations of motion have the form

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{b}=\rho\ddot{\mathbf{u}} \tag{1}$$

Here $\mathbf{b}$ is a vector of body forces, ρ is a density of material, $\ddot{\mathbf{u}}$ is the acceleration vector.

Kinematic relations simplify and have the form

$$\boldsymbol{\varepsilon}=\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{T}\right) \tag{2}$$

Here $\mathbf{u}$ is a displacements vector and $\boldsymbol{\varepsilon}$ is a tensor of small deformations.

According to Eringen's nonlocal elasticity theory [1] the stress at a point $\mathbf{x}$ in a body is functional of the strain field at every point of the body. In differential form nonlocal constitutive relations are presented as

 Copyright © 2018 by CSME

$$(1-(e_0 a)^2 \nabla^2)\boldsymbol{\sigma} = \bar{\lambda}(tr\boldsymbol{\varepsilon})\mathbf{I}\delta_{ij} + 2\mu\boldsymbol{\varepsilon} \tag{3}$$

Here $\bar{\lambda} = \lambda \dfrac{2\mu}{\lambda + 2\mu}$, λ and μ are Lamè constants of classical elasticity, e_0 is the constant of material, a is a parameter describes internal characteristic length.

The differential equations of motion in the form of displacements can be represented as the following

$$\mathbf{L}_u \cdot \mathbf{u} + (1-(e_0 a)^2 \nabla^2)(\mathbf{b} - \rho \mathbf{I}_u \cdot \ddot{\mathbf{u}}) = 0 \tag{4}$$

where $\mathbf{I}_u$ is the matrix operator that has on the main diagonal elements equal to one and

$$\mathbf{L}_u \cdot \mathbf{u} = \mu \nabla^2 \mathbf{u} + (\bar{\lambda} + \mu)\nabla(\nabla \cdot \mathbf{u}) \tag{5}$$

Differential operator of Hamilton and its combina-tions in orthogonal system of coordinates related to the middle line of the beam are presented in [5].

III. 1-D FORMULATION

In order to reduce the 2-D problem for the couple stress theory of elastic curved beams to a 1-D one, we expand the physical parameters, that describe the stress-strain state of the beam into the Legendre polynomials series along the coordinate x_2. Then any continuous function $f(p)$ can be represented by Legendre's polynomial series according to formula

$$f(x_2) = \sum_{k=0}^{\infty} a_k P_k(\varpi),\ a_n = \frac{2k+1}{2} \int_{-h}^{h} f(\varpi) P_k(\varpi) dx_2 \tag{6}$$

where $\varpi = x_2 / h$.

By expanding all functions contained in the equations (1)-(3) in the Legendre's polynomial series one can obtain differential equations of motion for Legendre's polynomial series coefficients of the displacements in the form (4).

For the first order approximation theory we have

$$\begin{aligned} u_\alpha(x_1, x_2) &= u_\alpha^0(x_1) P_0(\omega) + u_\alpha^1(x_1) P_1(\omega), \\ \varepsilon_{\alpha\beta}(x_1, x_2) &= \varepsilon_{\alpha\beta}^0(x_1) P_0(\omega) + \varepsilon_{\alpha\beta}^1(x_1) P_1(\omega) \\ \sigma_{\alpha\beta}(x_1, x_2) &= \sigma_{\alpha\beta}^0(x_1) P_0(\omega) + \sigma_{\alpha\beta}^1(x_1) P_1(\omega) \end{aligned} \tag{7}$$

The matrix differential operator $\mathbf{L}_u$ vectors of displacements and body forces have the form

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^{00} & L_{12}^{00} & L_{11}^{01} & L_{12}^{01} \\ L_{21}^{00} & L_{22}^{00} & L_{12}^{01} & L_{22}^{01} \\ L_{11}^{10} & L_{12}^{01} & L_{11}^{11} & L_{12}^{11} \\ L_{21}^{10} & L_{22}^{10} & L_{21}^{11} & L_{22}^{11} \end{vmatrix},\ \mathbf{u} = \begin{vmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{vmatrix},\ \mathbf{b} = \begin{vmatrix} b_1^0 \\ b_2^0 \\ b_1^1 \\ b_2^1 \end{vmatrix} \tag{8}$$

For the second order approximation theory we have

$$\begin{aligned} u_\alpha(x_1, x_2) &= u_\alpha^0(x_1) P_0(\omega) + u_\alpha^1(x_1) P_1(\omega) + u_\alpha^2(x_1) P_2(\omega), \\ \varepsilon_{\alpha\beta}(x_1, x_2) &= \varepsilon_{\alpha\beta}^0(x_1) P_0(\omega) + \varepsilon_{\alpha\beta}^1(x_1) P_1(\omega) + \varepsilon_{\alpha\beta}^2(x_1) P_2(\omega), \\ \sigma_{\alpha\beta}(x_1, x_2) &= \sigma_{\alpha\beta}^0(x_1) P_0(\omega) + \sigma_{\alpha\beta}^1(x_1) P_1(\omega) + \sigma_{\alpha\beta}^2(x_1) P_2(\omega). \end{aligned} \tag{9}$$

The matrix differential operator $\mathbf{L}_u$ vectors of displacements and body forces have the form

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^{00} & L_{12}^{00} & L_{11}^{01} & L_{12}^{01} & L_{11}^{02} & L_{12}^{02} \\ L_{21}^{00} & L_{22}^{00} & L_{21}^{01} & L_{22}^{01} & L_{21}^{02} & L_{22}^{02} \\ L_{11}^{10} & L_{12}^{10} & L_{11}^{11} & L_{12}^{11} & L_{11}^{12} & L_{12}^{12} \\ L_{21}^{10} & L_{22}^{10} & L_{21}^{11} & L_{22}^{11} & L_{12}^{12} & L_{22}^{12} \\ L_{11}^{20} & L_{12}^{20} & L_{11}^{21} & L_{12}^{21} & L_{11}^{22} & L_{12}^{22} \\ L_{21}^{20} & L_{22}^{20} & L_{21}^{21} & L_{21}^{12} & L_{21}^{22} & L_{22}^{22} \end{vmatrix},\ \mathbf{u} = \cdot \begin{vmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{vmatrix},\ \mathbf{f} = \cdot \begin{vmatrix} f_1^0 \\ f_2^0 \\ f_1^1 \\ f_2^1 \\ f_1^2 \\ f_2^2 \end{vmatrix} \tag{10}$$

Explicit expressions for differential operators in the matrix differential operators (8) for the beams are presented in [3].

The equations presented here can be used for modeling and stress-strain calculations of the beams by considering nonlocal effects at micro- and nanoscales

IV. APPLICATION TO MODELING OF THE ELECTROSTATICALLY ACTUATED MICRO AND NANO BEAMS

Let us consider an elastic beam of the length l, width b and thickness $2h$ which is suspended above the rigid foundation with an initial gap h_0 in the electromagnetic field. The beam and foundation are perfect conductors and separated by a dielectric medium of permittivity $\varepsilon_0 \varepsilon_r$, where $\varepsilon_0 = 8.854 \cdot 10^{-12} F/m$ is the vacuum permittivity and ε_r is the relative permittivity. A positive potential difference V between the two conductors causes the beam to electrostatically deflect downwards. We assume that the gap h_0 is commensurable with the beam displacements which are assumed to be small. There is an electro conducting medium in the gap between the foundation and the beam, which does not resist the beam deformation.

From an electrical point of view, the system behaves as a variable gap capacitor. We do not consider fringing fields in the present work. Therefore a distributed force on the deformable beam due to the electric field depends on the potential difference between the two conductors and on their geometries. Because of we consider that $h_0 / l \ll 1$, it is reasonable to assume that at every point the electrostatic force per unit length depends only on the local deflection and equals the force per unit length acting on an infinitely long straight beam separated by a distance $h_0 + u_2(x_1)$ from a ground plane. Then the magnitude of the electrostatic force F_e acting on the deformable electrode along its normal is given by

$$F_e = \frac{\varepsilon_0 V^2}{2(h_0 - u_2)^2} \tag{11}$$

Thus the expression for the electrostatic force at a point on the plate depends only on the local gap h_0 and the validity of the analysis is limited to those variable gap capacitors which are locally parallel to each other.

Copyright © 2018 by CSME

For convenience we introduce the nondimensional parameters of the beam deformation $\hat{u}_\alpha^k = u_\alpha^k / h_0$ and the nondimensional coordinate $\hat{x}_1 = x_1 / l$. Then the electrostatic force (11) can be presented in the form

$$F_e = \frac{\alpha V^2}{(1-\hat{u}_2)^2}, \quad \alpha = \frac{3\varepsilon_0 l^2}{4\mu b h^2 h_0^2} \tag{12}$$

An applied direct current (DC) voltage V between the beam and foundation results in the deflection of the beam and a consequent change in the system capacitance. The applied voltage has an upper limit beyond which the electrostatic force is not balanced by the elastic restoring force in the deformable conductor. Beyond this critical voltage the deformable conductor snaps and touches the lower rigid foundation. As it was mentioned in introduction, this phenomenon is called pull-in instability. We study static response of beam till pull-in instability using high order models developed here for fixed-fixed and simply supported beams.

For solution of the system of differential equation (4) with operators (8) we used simple iterative algorithm as it have been proposed in [4]. Algorithm consists in the following. We start from the initial value of $\mathbf{u}^0(x_1)$, for example and calculate next value form the equation

$$\mathbf{u}^{n+1} = \mathbf{T} \cdot \mathbf{u}^n \tag{13}$$

where is the operator that act in the Banach space, which properties depend on used model. In our case it is the operator inverse to (8).

Algorithm (13) is not always convergent, and for its convergence operator $\mathbf{T}$ has to be a contraction in corresponding Banach space. In that case $\mathbf{T}$ has a fixed point which is a solution of the system of differential equations (4) and can be obtained from the initial value $\mathbf{u}^0(x_1)$ using iterative algorithm (13), as the limit of the iterative sequence. We do not study here conditions for convergence of the algorithm, it can be done mathematically using methods of functional analysis.

This approach we explore for investigation of the pull-in instability of the beams using first order and second order theories developed here. On each step of iteration we calculate displacements using differential equations (4) with differential operators (8)for first and second order theories respectively. Corresponding differential equations we solve numerically using finite element method (FEM). All calculations and post processing analysis have been done using commercial software MATLAB. We performed finite element analysis with MATLAB. We used PDE mode with coefficient form impute module. The differential equations (4)are presented in the form convenient for MATLAB input. In 1-D PDE coefficient module are used finite elements of Lagrange type from linear to quantic order. Also in MATLAB there is option for mesh refinement. For details we refer to the corresponding software manuals. Our numerical experiments show that for the problem under consideration FEM has good convergence and use quadratic elements with one mesh refinement gives accurate results.

We calculated dimensionless vertical displacements of the beam using first order and second order and Euler-Bernoulli theories. Calculations have been done using iteration algorithm (13) defined above. The algorithm demonstrates good convergence for $V < V_{in}$, where V_{in} is the critical value of the applied voltage V, that corresponds to pull-in instability of the beam and called pull-in voltage. In the vicinity of V_{in} convergence of the algorithm becomes worse and worse and finally is broken down for value of V close to V_{in}. We define approximate value of V_{in} using that convergence property of the algorithm. For fixed-fixed beam all considered theories give the same result $V_{in} \approx 81\,\mathrm{V}$. Results of calculation of the displacements at the middle point $x_1 = l/2$ for various values of the applied voltage V, up to point of pull-in instability are presented. The results obtained using different theories are in a good correspondence.

V. Application to modeling of the thermally actuated micro and nano beams

Let us consider an elastic beam of the length l, width b and thickness $2h$, which is settled above the rigid foundation with an initial gap h_0 in the thermal field. There is a heat-conducting medium in the gap between the foundation and the beam. The medium does not resist the beam deformation, and heat exchange between the foundation and the beam is due to the thermal conductivity of the medium. We assume that the gap h_0 is commensurable with the beam displacements which are assumed to be small.

With taking into account temperature field the 1-D differential equations of thermoelasticity and head conductivity of the beam can be presented in the form

$$\begin{aligned} &\mathbf{L}_u \cdot \mathbf{u} + \mathbf{L}_\theta \cdot \boldsymbol{\theta} + \mathbf{f} = 0, \\ &\mathbf{L}_{\theta\theta} \cdot \boldsymbol{\theta} + \mathbf{Q} + \boldsymbol{\chi} = 0 \end{aligned} \tag{14}$$

The matrix differential operator $\mathbf{L}_u$ and vectors $\mathbf{u}$ and $\mathbf{f}$ have the forms (8) and (10) for first and second order approximations, respectively. The differential operators $\mathbf{L}_{\theta\theta}$ and $\mathbf{L}_\theta$ and vectors $\boldsymbol{\theta}$, $\mathbf{Q}$, and $\boldsymbol{\chi}$ can be presented in the following forms.

For the first order approximation theory

$$\begin{aligned} \theta_i(\mathbf{x}) &= \theta_i^0(x_1) P_0(\omega) + \theta_i^1(x_1) P_1(\omega), \\ \chi_i(\mathbf{x}) &= \chi_i^0(x_1) P_0(\omega) + \chi_i^1(x_1) P_1(\omega) \end{aligned} \tag{15}$$

Copyright © 2018 by CSME

$$\mathbf{L}_\theta = \mathbf{E}\cdot\begin{vmatrix} L_1^0 \\ L_2^0 \\ L_1^1 \\ L_2^1 \end{vmatrix}, \boldsymbol{\theta} = \begin{vmatrix} \theta_1^0 \\ \theta_2^0 \\ \theta_1^1 \\ \theta_2^1 \end{vmatrix}, \ \mathbf{L}_{\theta\theta} = \begin{vmatrix} L^0 & 0 \\ 0 & L^1 \end{vmatrix}, \mathbf{Q} = \begin{vmatrix} Q_2^0 \\ Q_2^1 \end{vmatrix}, \boldsymbol{\chi} = \begin{vmatrix} \overline{\chi}^0 \\ \overline{\chi}^1 \end{vmatrix} \quad (16)$$

For the second order approximation theory

$$\begin{aligned} \theta_i\left(\boldsymbol{x}\right) &= \theta_i^0\left(x_1\right)P_0\left(\omega\right) + \theta_i^1\left(x_1\right)P_1\left(\omega\right) + \theta_i^2\left(x_1\right)P_2\left(\omega\right), \\ \chi_i\left(\boldsymbol{x}\right) &= \chi_i^0\left(x_1\right)P_0\left(\omega\right) + \chi_i^1\left(x_1\right)P_1\left(\omega\right) + \chi_i^2\left(x_1\right)P_2\left(\omega\right) \end{aligned} \quad (17)$$

$$\mathbf{L}_\theta = \mathbf{E}\cdot\begin{vmatrix} L_1^0 \\ L_2^0 \\ L_1^1 \\ L_2^1 \\ L_1^2 \\ L_2^2 \end{vmatrix}, \boldsymbol{\theta} = \begin{vmatrix} \theta_1^0 \\ \theta_2^0 \\ \theta_1^1 \\ \theta_2^1 \\ \theta_1^2 \\ \theta_2^2 \end{vmatrix}, \ \mathbf{L}_{\theta\theta} = \begin{vmatrix} L^0 & 0 & 0 \\ 0 & L^1 & 0 \\ 0 & 0 & L^2 \end{vmatrix}, \mathbf{Q} = \begin{vmatrix} Q_2^0 \\ Q_2^1 \\ Q_2^2 \end{vmatrix}, \boldsymbol{\chi} = \begin{vmatrix} \overline{\chi}^0 \\ \overline{\chi}^1 \\ \overline{\chi}^2 \end{vmatrix} \quad (18)$$

In some cases under action of mechanical load and temperature filed lower side Ω^- of the beam can be in unilateral mechanical contact with the rigid foundation. In this case the area of close mechanical contact ∂V_e and contact forces are not known in advance. Therefore unilateral mechanical contact conditions have the form of inequalities

$$u_n = \ \geq h_0 \ , q_n \geq 0 \ , \left(u_n - h_0\right)q_n = 0, \quad \forall \mathbf{x} \in \partial V_e \quad (19)$$

For the case if temperature set on upper side of the beam $\theta^+(x_1)$ and on the foundation $\theta^-(x_1)$ respectively, then contact conditions through heat conduction layer we have

$$T_k = \frac{\lambda_T\left(h_0 - u_2\right)\left(3\theta^+ + 6\theta^0 - 10\theta^1\right) + \lambda_T^* h\theta^-}{9\lambda_T\left(h_0 - u_2\right) + \lambda_T^* h} \quad (20)$$

Here $u_2(x_1)$ is calculated using representation (7) and (9) for the first and second order approximation theories, respectively.

It is important to mention, that systems of differential equations of thermoelasticity an heat conductivity (14) are coupled. Their connectedness is not the one usually related to dynamical thermoelasticity. We consider stationary problem and here the connectedness of the corresponding equations caused by change of the heat conducting conditions during the microbeam deformations. One can see that in the equations of heat is presented function $u_2(x_1)$ that is deflection of the beam. Presence of the function $u_2(x_1)$ in the equation (20) turns the problem into nonlinear one.

For solution of the problem we use iterative algorism developed in [51, 53]. In the first step of iteration we assume that deflection of the beam $u_2(x_1) = 0$. In this case we have traditional uncoupled problem of thermoelasticity and heat conductivity. For that uncoupled problem any analytical or numerical method can be used and corresponding equations of thermoelasticity and heat conductivity can be solved independently. We refer below to this case as uncoupled or traditional formulation. In the next steps of iterations we substitute in (20) deflection $u_2(x_1)$ obtained from the solution of the problem in previous step of iteration. Our previous and fulfilled here researches show that in the problems under consideration algorithm is convergent and convergence is fast enough.

In this study the differential equations of thermoelasticity and heat conductivity (14) we solve numerically using finite element method (FEM). All calculations and post processing analysis have been done using commercial software MATLAB and COMSOL Multiphysics. We performed finite element analysis with COMSOL Multiphysics. We used PDE mode with coefficient form impute module. The differential equations (14) are presented in the form convenient for MATLAB input. In 1-D PDE coefficient module are used finite elements of Lagrange type from linear to quantic order. Also in MATLAB there is option for mesh refinement. For details we refer to the corresponding software manuals. Our numerical experiments show that for the problem under consideration FEM has good convergence and use quadratic elements with one mesh refinement gives accurate results. Obtained with MATLAB results of the finite element analysis then have been further analyzed and compared with results obtained by Euler-Bernoulli theory.

ACKNOWLEDGMENT

This work was partly supported by the Committee of Science and Technology of Mexico (CONACYT) by the Research Grants (Ciencia Basica, Reference No 256458), which is gratefully acknowledged. The grants provided by Lassonde School of Engineering and NSERC Quanser Chair in Design Engineering are also acknowledged.

REFERENCES

[1] H. Altenbach and V.A. Eremeev *Shell Like Structures. Non Classical Theories and Applications*. - Springer, New York, 2011. – 761 p.

[2] A.C. Eringen Microcontinuum Field Theories I. Foundations and Solids, Springer, New York, 1999, 336 p.

[3] Lyshevski S.E. Nano- and Micro-Electromechanical Systems. Fundamentals of Nano- and Microengineering. 2nd edition, CRC Press LLC, 2005. – 744 p.

[4] J.A. Pelesko and D.H. Bernstein *Modeling of MEMS and NEMS*. Chapman & Hall; 2002 -364 p.

[5] I.N. Vekua *Shell theory, general methods of construction*, Pitman Advanced Pub. Program., Boston, 1986 – 302 p.

[6] V.V. Zozulya and A. Saez A high order theory of a thermo elastic beams and its application to the MEMS/NEMS analysis and simulations. *Archive of Applied Mechanics*, 2016, **86**(7), 1255–1272.

[7] Zozulya V.V., Saez A. A high order theory of a thermo elastic beams and its application to the MEMS/NEMS analysis and simulations. *Archive of Applied Mechanics*, 2015, **86**(7), 1255–1272.

[8] Czekanski A., Meguid S.A., Solution of dynamic frictional contact problems using nondifferentiable optimization, *International Journal of Mechanical Sciences*, 2001, **43**, 1369–1386

Copyright © 2018 by CSME