

On PCF Polynomials

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Abstract

The author of [27] proves that the set of post-critically finite (PCF) polynomials of given degree is a set of bounded height, up to PGL_2 -conjugacy. This result is extended to show that the set of monic polynomials $g(z) \in \bar{\mathbb{Q}}[z]$ of given degree such that there exists a $d \geq 2$ such that $g(z^d)$ is PCF, is also a set of bounded height. Note that by fixing the degree of a polynomial and algebraic degree of its coefficients, the set of such PCF polynomials is in fact finite, and computable. Bounds on the coefficients for quartic PCF polynomials in $\mathbb{Q}[z]$ are computed, and a search of the resulting space yields 16 distinct conjugacy classes. Infinite families of PCF polynomials containing each of these distinct conjugacy classes are found, giving a lower bound on the number of such conjugacy classes in terms of degree d .

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Chapter 1

Introduction

The dynamics of a rational map over \mathbb{C} are known to be largely determined by the orbits of its critical (ramification) points (points x for which $f'(x) = 0$). For instance:

Theorem 1.1 (see [37], Theorem 8.6). *If f is a rational map of degree $d \geq 2$, the immediate basin of every attracting periodic orbit contains at least one critical point.*

By the Riemann-Hurwitz formula, the number of critical points is bounded above by $2d - 2$: so the number of attracting periodic orbits is as well.

Another example of the importance of critical points' orbits, due to Fatou follows:

Theorem 1.2 (see [37], Theorem 9.5). *The Julia set (see Definition 2.11) of a polynomial f of degree $d \geq 2$ is connected if and only if the orbits of its critical points are bounded.*

In light of the above, we are interested in when the orbits of the critical points are bounded. One case of this is when their orbits are finite:

Definition 1.3. *A rational map f of degree $d \geq 2$ is called postcritically finite (PCF) if the orbits of each of its critical points is finite.*

The Mandelbrot set is the set of c such that the orbit of 0 under iteration by $f(z) = z^2 + c$ is bounded. The PCF quadratic polynomials $z^2 + c$ are those for which this same orbit is finite.

There are infinitely many such maps. However, if we consider the set of *conjugacy classes* of rational maps (under Möbius transformations), we are looking at the moduli space \mathcal{M}_d (see Definition 2.28). Now, over \mathbb{Q} , there are finitely many such points in \mathcal{M}_d : indeed, we have a complete list of the PGL_2 -conjugacy classes of quadratic PCF polynomials with

rational coefficients.

The idea above motivates the definition of height functions, which unfortunately have no formal general definition. The general principles are that they should measure arithmetic complexity, and that they can be used to prove results of the flavour “there are finitely many points of bounded height in some set S ” (for example, see Northcott’s Theorem that there are finitely many points in $\mathbb{P}^N(\bar{\mathbb{Q}})$ of bounded height and algebraic degree).

See Chapter 2 for the construction of the canonical height $\hat{h}_f(P)$ (Definition 2.25) used in the following definition:

Definition 1.4 (see [44], Definition 6.24). *Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over $\bar{\mathbb{Q}}$, $\text{Crit}(f)$ be the set of critical points of f , and define the critical height of f as*

$$\hat{h}_{\text{crit}}(f) = \sum_{P \in \text{Crit}(f)} \hat{h}_f(P) = \sum_{P \in \mathbb{P}^1(\bar{\mathbb{Q}})} (e_P(f) - 1) \hat{h}_f(P).$$

The critical height is of particular importance in studying PCF maps, as the only contributions to this height come from its critical points. \hat{h}_{crit} is well-defined on \mathcal{M}_d modulo change of variables ($(\phi^{-1} \circ f \circ \phi)'(P) = 0 \iff f'(\phi(P)) = 0$, and $\hat{h}_{\phi^{-1} \circ f \circ \phi}(P) = \hat{h}_f(\phi(P))$), and has the following useful properties:

- $\hat{h}_{\text{crit}}(f^n) = n \hat{h}_{\text{crit}}(f)$
- $\hat{h}_{\text{crit}}(f) = 0 \iff f$ is PCF

Definition 1.5 ([45]). *Given an embedding of an algebraic variety $\phi : X \hookrightarrow \mathbb{P}^N$, define an ample Weil height h_ϕ by*

$$h_\phi : X(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}, \quad h_\phi(P) = h(\phi(P))$$

where h is the height on \mathbb{P}^N as in Definition 2.24.

For the following, Lattès maps $f_{E,d}$ are maps induced on $\mathbb{P}^1(\mathbb{C})$ by the multiplication-by- d map on an elliptic curve $E(\mathbb{C})$. The set of equivalence classes in \mathcal{M}_d are points on an algebraic variety ([44], Theorem 2.24). Conjectured by Silverman (see [44], Conjecture 6.29), the following was proven in stages:

Theorem 1.6 (Ingram 2012 [27], Benedetto-Ingram-Jones-Levy 2014 [5], Ingram 2018 [29]). *For any ample Weil height $h_{\mathcal{M}_d}$ on \mathcal{M}_d ,*

$$h_{\mathcal{M}_d} \asymp \hat{h}_{\text{crit}}$$

for non-Lattès maps.

Theorem 1.7 (Thurston 1980s, see [15]). *Any algebraic family of PCF rational maps is either isotrivial or a (flexible) Lattès example.*

So, the non-Lattès PCF maps are a set of bounded height. Note that if there were a non-isotrivial, non-Lattès family of PCF maps f_t , this would contradict Theorem 1.6, as $\hat{h}_{\text{crit}}(f_t) = 0$ but $h_{\mathcal{M}_d}(f_t) \rightarrow \infty$.

Let \mathcal{P}_d denote the moduli space of polynomials of degree d , up to affine conjugacy.

Definition 1.8 ([27]). *Any polynomial of degree d with coefficients in $\bar{\mathbb{Q}}$ is affine conjugate to a map f of the form $z^d + a_{d-2}z^{d-2} + \dots + a_0$. Define the monic centered height on \mathcal{P}_d by*

$$h_{mc}(f) = h(a_{d-2}, \dots, a_0).$$

Theorem 1.9 (Ingram 2012 [27]). *There exist effectively computable constants C_1 and C_2 , both depending just on d , such that*

$$\frac{1}{2d-1}h_{mc}(f) - C_1 \leq \hat{h}_{crit}(f) \leq 4h_{mc}(f) + C_2$$

for all $f \in \mathcal{P}_d(\bar{\mathbb{Q}})$.

So, the set of PCF polynomials of a given degree d is a set of algebraic points of bounded height. In particular, there are finitely many conjugacy classes of PCF polynomials of bounded degree d whose coefficients have bounded algebraic degree B . This result was extended in [5] to rational maps. We extend this result in Chapter 3, to show that the set of polynomials $g(z)$ of degree d over a given number field such that there exists an n with $g(z^n)$ PCF is finite.

As the aforementioned sets are finite, some work has been done to find all such maps for small values of d over \mathbb{Q} . A natural choice for conjugation of a quadratic polynomial is $z^2 + c$, moving its lone critical point to 0. So these act as representatives of the finitely many affine-conjugacy classes of PCF polynomials: there are 3 such for $d = 2$, $B = 1$, ($f(z) = z^2 + c$):

$$c \in \{0, -1, -2\}.$$

The choice becomes more complicated at higher degrees. A finiteness result on the specific case of unicritical polynomials is given in [7]. In [27], the Branner-Hubbard normal form is used, which does not preserve the field of definition of cubic polynomials. As such, in [2], the conjugation is chosen so that the critical points are moved to 0 and 1 (if there are two distinct critical points), and the list of 15 total conjugacy classes of cubic PCF polynomials over \mathbb{Q} is found. In [36], the list of conjugacy classes of quadratic PCF rational maps are

found. Things are further complicated for quartic polynomials, as in the tricritical case it is difficult to find a ‘natural’ conjugation to move the critical points. If all the critical points are rational, they can be moved to 0, 1, and a parameter t , and if they lie in a quadratic extension, we can move them to \sqrt{D} , $-\sqrt{D}$, and a parameter t for some squarefree integer D . But if the critical points lie in an extension of degree 3 or 6, there are less convenient choices.

In the following work, rather than being concerned with the location of the critical points, we conjugate to eliminate the cubic term, and instead look at a resultant (Definition 2.12), which encodes information about the branch points. In Chapter 4, we use this strategy to find the conjugacy classes of quartic PCF polynomials defined over \mathbb{Q} .

Another motivation for the study of PCF maps comes from their connection to arboreal Galois representations. Consider the tree T_∞ formed by the inverse images under $f \in K(x)$ of some base point a . The absolute Galois group of K acts on T_∞ by tree automorphisms, giving a subgroup $G_\infty(f)$ of the group $\text{Aut}(T_\infty)$ of all tree automorphisms [33]. One question asked is the size of the index $[\text{Aut}(T_\infty) : G_\infty(f)]$. It was shown in [40] that for a generic polynomial f , $G_\infty(f)$ is equal to the full automorphism group:

Theorem 1.10 (Odoni 1985 [40]). *Let K be a field of characteristic zero, t_0, \dots, t_{d-1} be algebraically independent over K , and $F(x) = x^d + t_{d-1}x^{d-1} + \dots + t_1x + t_0$. So T_∞ resides in the algebraic closure of $K(t_0, \dots, t_{d-1})$, and*

$$G_\infty(F) = \text{Aut}(T_\infty).$$

Generally, the index of the image of the Galois group in the full automorphism group is expected to be finite, but f being PCF is one of the exceptions. In [32] and [26] are theorems giving that the index is finite under certain conditions: in particular, that f is not PCF. In [10], f not being PCF is one of the requirements for cubic polynomials f giving $[\text{Aut}(T_\infty) : G_\infty(f)]$ finite, and in [6] the arboreal Galois representation of a cubic PCF polynomial is explored. In [33] it is shown that if the field of definition of f PCF of degree d has characteristic 0 or $> d$, then the index is infinite, and a similar result including the converse is proven in [18].

PCF maps often prove to be exceptional cases. In [9], f PCF is shown to imply that the preimage field of K -rational α is finitely ramified over K . In [25], up to three exceptions, is shown that for quadratic PCF polynomials f of a particular form with an iterate reducible modulo every prime, that all iterates are irreducible over \mathbb{Q} . In [42], the main stumbling block in a proof of a case of Thurston’s rigidity theorem is cubic PCF polynomials.

In the following we work over $\mathbb{P}^1(\mathbb{C})$; PCF maps in higher dimension are the subject of ongoing research: over \mathbb{P}^2 ([19], [34], [38], [31]) or higher ([8], [28], [3], [35]). Conjectured and proven for tail length ≤ 2 in [30], it was shown unconditionally in [20] that PCF maps are not Zariski dense in the space End_d^k of endomorphisms of degree d on \mathbb{P}^k for $k \geq 2$ (in contrast to being Zariski dense on \mathbb{P}^1):

Theorem 1.11 (Gauthier-Taflin-Vigny 2023 [20]). *Fix two integers $k, d \geq 2$. There exists a strict subvariety $V_d^k \subsetneq \text{End}_d^k$ such that any PCF endomorphism f is contained in V_d^k .*

Much current research on PCF maps is on the topic of unlikely intersections. A dynamical analogue of the André-Oort conjecture given in [4] and [14] aims to characterize the irreducible subvarieties of \mathcal{M}_d containing a Zariski dense collection of PCF points:

Conjecture 1.12 (Baker-De Marco 2013 [4]). *Suppose $\{f_t : t \in V\}$ is an N -dimensional algebraic family of critically-marked rational maps of degree $d \geq 2$, with V irreducible. Then f_t is post-critically finite for a Zariski-dense subset of $t \in V$ if and only if there are at most N dynamically independent critical points on V .*

This question is answered in the case of cubic polynomials in [17] and [24], for unicritical polynomials in [23] and [11], and in higher dimension for quadratic polynomials in [13]. The distribution of PCF polynomials is explored in the series ([16], [21], [22]).

In Chapter 5, we find some infinite families of PCF polynomials, which contain all of the quartics found in Chapter 4, and further imply that as degree increases, so too does the number of PCF polynomial conjugacy classes.

Chapter 2

Background

In this chapter we provide a brief introduction to arithmetic dynamics (which the interested reader can find a more thorough treatment of in [43], from where many of the definitions in this chapter are pulled). Our principal objects of study are iterated maps:

Definition 2.1. *A dynamical system is a set S together with a map $f : S \rightarrow S$.*

We denote by

$$f^n = f \circ f \circ \dots \circ f$$

the n^{th} iterate of f .

Definition 2.2. *The forward orbit of a point $\alpha \in S$ (under f) is the set*

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \geq 0\}.$$

Our goal is to study the orbits of points in S .

Definition 2.3. *The sets of periodic and preperiodic points of f in S are denoted*

$$\text{Per}(f, S) = \{\alpha \in S : f^n(\alpha) = \alpha \text{ for some } n \geq 1\},$$

$$\text{PrePer}(f, S) = \{\alpha \in S : f^{m+n}(\alpha) = f^m(\alpha) \text{ for some } n \geq 1, m \geq 0\}$$

$$= \{\alpha \in S : \mathcal{O}_f(\alpha) \text{ is finite}\}$$

respectively.

Observe that $\text{Per}(f, S) \subseteq \text{PrePer}(f, S)$ (take $m = 0$, f^0 denotes the identity map on S). Consider the following example of the orbit of a point α under a specific function f :

$$\begin{aligned} \mathcal{O}_f(\alpha) &= \{\alpha, f(\alpha), f^2(\alpha), f^3(\alpha), f^4(\alpha), f^5(\alpha)\} \\ &\subseteq \text{PrePer}(f, S) \end{aligned}$$

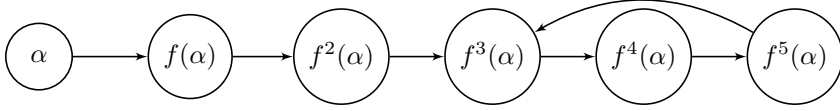


Figure 2.1: Example of a preperiodic point

$$\{f^3(\alpha), f^4(\alpha), f^5(\alpha)\} \subseteq \text{Per}(f, S)$$

In particular, we will be interested in the case where α is a critical (ramification) point of f .

Generally the set S on which we study iterated maps will be the complex projective line (Riemann sphere) $\mathbb{P}^1(\mathbb{C})$, and the maps f will be either polynomial or rational maps:

Definition 2.4. A rational function $f(z) \in \mathbb{C}(z)$ is a quotient

$$f(z) = \frac{F(z)}{G(z)} = \frac{a_0 + a_1z + \dots + a_dz^d}{b_0 + b_1z + \dots + b_dz^d}$$

with no common factors, and $\deg(f) = \max\{\deg(F), \deg(G)\}$.

Definition 2.5. A linear fractional transformation (or Möbius transformation) is a map of the form

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with } ad - bc \neq 0.$$

These maps under composition form the group $\text{PGL}_2(\mathbb{C})$, which is in fact precisely the group of automorphisms of the Riemann sphere, $\text{Aut}(\mathbb{P}^1(\mathbb{C}))$.

Definition 2.6. The linear conjugate (not to be confused with iteration) of a rational map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by a $\phi \in \text{PGL}_2(\mathbb{C})$ is the map

$$f^\phi = \phi^{-1} \circ f \circ \phi.$$

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{f^\phi} & \mathbb{P}^1 \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

Figure 2.2: Conjugation/iteration commutative diagram

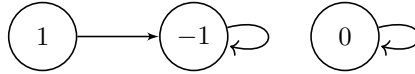
Two rational maps f and g are *linearly conjugate* if $\exists \phi \in \text{PGL}_2(\mathbb{C})$ such that $f = g^\phi$, and this defines an equivalence relation on the set of rational maps.

Usefully, linear conjugation agrees with iteration:

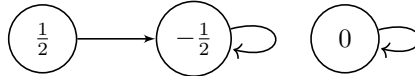
$$(f^\phi)^n = \phi^{-1} \circ f^n \circ \phi = (f^n)^\phi$$

so the dynamics of a map f depends only on its PGL_2 -conjugacy class $[f]$.

Example 2.7. Consider the polynomial $f(z) = z^4 - 2z^2$, a tricritical polynomial with the following critical portrait:



If we conjugate by $\phi(z) = 2z$, we get $f^\phi(z) = (\phi^{-1} \circ f \circ \phi)(z) = 8z^4 - 4z^2$, which has an identical critical portrait, only with the critical points moved:



If α is a critical point for f a PCF rational map, its *tail length* is the minimal n such that $f^n(\alpha)$ is periodic, and its *cycle length* is the minimal period of this periodic cycle.

For example, z^d is PCF for all d ($0 \mapsto 0, \infty \mapsto \infty$)

Some examples of PCF maps (and their corresponding critical portraits) are found in Table 2.1 (we omit ∞ for simplicity):

Definition 2.8. The *ramification index* of f at α is defined as the order of vanishing of α on $(f(z) - f(\alpha))$:

$$e_\alpha(f) = \text{ord}_\alpha(f(z) - f(\alpha)).$$

- f is $e_\alpha(f)$ -to-1 near α
- f is ramified at $\alpha \iff e_\alpha(f) \geq 2$
- $e_\alpha(f) \leq \deg(f)$
- If $e_\alpha(f) = \deg(f)$, we say f is totally ramified at α

| $f(z)$ | $f'(z) = 0$ | Critical Portrait |
|------------------------------|-------------|-------------------|
| $z^2 - 1$ | 0 | |
| $-2z^3 + 3z^2 + \frac{1}{2}$ | 0, 1 | |

Table 2.1: PCF map examples

Theorem 2.9 (Riemann-Hurwitz, 1857 (see [43], Theorem 1.1)). *Given $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ a map of degree d ,*

$$2d - 2 = \sum_{\alpha \in \mathbb{P}^1} (e_\alpha(f) - 1).$$

A consequence of this relation is that a rational map of degree d has exactly $2d - 2$ critical points (counted with multiplicity).

Definition 2.10. *If $\alpha \neq \infty$ is a point of exact period n for f , the multiplier of f at α is*

$$\lambda_\alpha(f) = (f^n)'(\alpha),$$

and the multiplier of f at ∞ (a fixed point) is defined as

$$\lambda_\infty(f) = (f^\phi)'(\phi^{-1}(\infty)) \quad \forall \phi \in PGL_2(\mathbb{C}).$$

The multiplier describes what happens to a small neighbourhood of α under iteration of f . α is called:

$$\begin{aligned} \textit{superattracting} & \quad \text{if } \lambda_\alpha(f) = 0, \\ \textit{attracting} & \quad \text{if } |\lambda_\alpha(f)| < 1, \\ \textit{neutral} & \quad \text{if } |\lambda_\alpha(f)| = 1, \\ \textit{repelling} & \quad \text{if } |\lambda_\alpha(f)| > 1. \end{aligned}$$

Definition 2.11. *Given a map f from a metric space to itself, its Fatou set, $\mathcal{F}(f)$, is the maximal open set on which the iterates of f are equicontinuous, and its Julia set, $\mathcal{J}(f)$, is the complement of the Fatou set.*

Under iteration of f :

- Points in the Fatou set tend to stay close to each other

- Points in the Julia set tend to wander away from each other
- $\mathcal{F}(f)$, $\mathcal{J}(f)$, and $\partial\mathcal{J}(f)$ are completely invariant (that is, $f^{-1}(\mathcal{F}(f)) = \mathcal{F}(f) = f(\mathcal{F}(f))$, etc.)

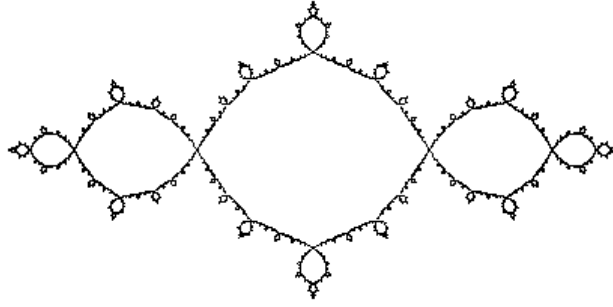


Figure 2.3: Example: Julia set of $z^2 - 1$

We will need the following definition later:

Definition 2.12. *The resultant of two polynomials $A(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_1 z + a_0$, $B(z) = b_e z^e + b_{e-1} z^{e-1} + \dots + b_1 z + b_0$, where $a_d, b_e \neq 0$, is defined as*

$$\text{res}_z(A(z), B(z)) = a_d^e b_e^d \prod_{i=1}^d \prod_{j=1}^e (\alpha_i - \beta_j)$$

where the α_i and β_j are the roots of $A(z)$ and $B(z)$ respectively.

Note that the resultant is equal to zero if and only if the polynomials have a common root. Usefully, the resultant can also be computed as the determinant of their Sylvester matrix:

$$\text{res}_z(A(z), B(z)) = \begin{vmatrix} a_d & 0 & \dots & 0 & b_e & 0 & \dots & 0 \\ a_{d-1} & a_d & \dots & \vdots & b_{e-1} & b_e & \dots & \vdots \\ \vdots & a_{d-1} & \ddots & 0 & \vdots & b_{e-1} & \ddots & 0 \\ a_1 & \vdots & \ddots & a_d & b_1 & \vdots & \ddots & b_e \\ a_0 & a_1 & \ddots & a_{d-1} & b_0 & b_1 & \ddots & b_{e-1} \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & 0 & \ddots & a_1 & \vdots & 0 & \ddots & b_1 \\ 0 & 0 & \dots & a_0 & 0 & 0 & \dots & b_0 \end{vmatrix}$$

We further have, if the coefficients are in an integral domain,

$$\text{res}_z(A(z), B(z)) = (-1)^{de} b_e^d \prod_{j=1}^e A(\beta_j)$$

where the β_j are the roots of B . Note that if we take the resultant of a polynomial f and its derivative f' , we get a polynomial in the coefficients of f which encodes information about its branch points – that is, the images of the critical points, $f(\beta_j)$ where the β_j are the critical points of f .

Definition 2.13. *An absolute value on a field K is a map*

$$|\cdot| : K \rightarrow \mathbb{R}$$

such that:

- $|\alpha| \geq 0$
- $|\alpha| = 0 \iff \alpha = 0$
- $|\alpha\beta| = |\alpha| \cdot |\beta|$ for all $\alpha, \beta \in K$
- $|\alpha + \beta| \leq |\alpha| + |\beta|$ for all $\alpha, \beta \in K$

The rationals \mathbb{Q} have the usual real absolute value (archimedean):

$$|\alpha|_\infty = \max\{\alpha, -\alpha\}$$

and for each prime p , a p -adic absolute value (non-archimedean):

$$|\alpha|_p = p^{-e_p(\alpha)}$$

where $e_p(\alpha)$ is the exponent of p in the unique factorization of α into powers of primes.

A non-archimedean absolute value satisfies the ultrametric inequality:

$$|\alpha + \beta| \leq \max\{|\alpha|_v, |\beta|_v\}$$

with equality if $|\alpha|_v \neq |\beta|_v$.

Definition 2.14. *Two absolute values $|\cdot|_a$ and $|\cdot|_b$ on a field K are called equivalent if there exists a $c \in \mathbb{R}$ such that $\forall x \in K, |x|_a^c = |x|_b$.*

Theorem 2.15 (Ostrowski 1916, [41]). *Every non-trivial absolute value on \mathbb{Q} is equivalent to either the usual real absolute value, or a p -adic absolute value.*

This motivates the following definition:

Definition 2.16. *The set of standard absolute values on \mathbb{Q} , denoted $M_{\mathbb{Q}}$, consists precisely of the usual real absolute value and the p -adic absolute values.*

Example 2.17. *Observe that the product $\prod |\alpha|_v$ over all absolute values v is equal to 1:*

$$\left| \frac{10}{9} \right| \cdot \left| \frac{10}{9} \right|_2 \cdot \left| \frac{10}{9} \right|_3 \cdot \left| \frac{10}{9} \right|_5 = \frac{10}{9} \cdot \frac{1}{2} \cdot 9 \cdot \frac{1}{5} = 1$$

Given an $\alpha \in \mathbb{Q}$ written in lowest terms $\alpha = \frac{a}{b}$, define its height to be $H(\alpha) = \max\{|a|, |b|\}$.

Given a $P \in \mathbb{P}^N(\mathbb{Q})$ (in all of the following take N, M to be fixed positive integers), we can find homogeneous coordinates

$$P = [x_0, \dots, x_N] \text{ with } x_0, \dots, x_N \in \mathbb{Z} \text{ and } \gcd(x_0, \dots, x_N) = 1$$

by multiplying through by an integer to clear the denominators, and cancelling any common factors. Then define

$$H(P) = \max\{|x_0|, \dots, |x_N|\}.$$

Observe that a set of points of bounded height is finite:

$$|\{P \in \mathbb{P}^N(\mathbb{Q}) \mid H(P) \leq B\}| \leq (2B + 1)^{N+1}.$$

Denote by M_K the *set of standard absolute values on K* : the set of all absolute values on K whose restriction to \mathbb{Q} is in $M_{\mathbb{Q}}$.

For any $v \in M_K$, let K_v denote the completion of K at v . The *local degree of v* is then defined as $n_v = [K_v : \mathbb{Q}_v]$.

Definition 2.18. *Let K/\mathbb{Q} be a number field, and let $P \in \mathbb{P}^N(K)$ be a point with homogeneous coordinates $P = [x_0, \dots, x_N]$, all in K . The height of P (relative to K) is defined as*

$$H_K(P) = \prod_{v \in M_K} \max\{|x_0|_v, \dots, |x_N|_v\}^{n_v}.$$

Proposition 2.19 (see [43], Prop 3.4). *Let K/\mathbb{Q} and $P \in \mathbb{P}^N(K)$ be as above.*

- (a) *This height $H_K(P)$ is independent of choice of homogeneous coordinates for P*
- (b) *$H_K(P) \geq 1$*
- (c) *If L/K is a finite extension, then $H_L(P) = H_K(P)^{[L:K]}$*

Proof. (a) Any other choice of coordinates for P has form $P = [\alpha x_0, \dots, \alpha x_N]$.

$$\prod_{v \in M_K} \max_i \{|\alpha x_i|_v\}^{n_v} = \prod_{v \in M_K} |\alpha|_v^{n_v} \max_i \{|x_i|_v\}^{n_v} = \prod_{v \in M_K} \max_i \{|x_i|_v\}^{n_v}$$

since

$$\prod_{v \in M_K} |\alpha|_v^{n_v} = 1 \quad \forall \alpha \in K^* \quad (\text{see [43], Prop 3.3}).$$

(b) Dividing the homogeneous coordinates by a nonzero x_j yields a homogeneous coordinate which is at least 1, so every factor of the product in Definition 2.18 is at least 1.

$$\begin{aligned} \text{(c) } H_L(P) &= \prod_{w \in M_L} \max_i \{|x_i|_w\}^{n_w} = \prod_{v \in M_K} \prod_{\substack{w \in M_L \\ w|v}} \max_i \{|x_i|_w\}^{n_w} \\ &= \prod_{v \in M_K} \max_i \{|x_i|_v\}^{[L:K]n_v} = H_K(P)^{[L:K]} \end{aligned}$$

(by [43], Prop 3.2). □

Definition 2.20. Let $P \in \mathbb{P}^N(\bar{\mathbb{Q}})$ be a point whose coordinates are algebraic numbers. The (absolute) height of P , $H(P)$, is defined by choosing any number field K such that $P \in \mathbb{P}^N(K)$, and setting

$$H(P) = H_K(P)^{1/[K:\mathbb{Q}]}.$$

Theorem 2.21 (Northcott 1950, [39]). Let K/\mathbb{Q} be a number field, and B be any constant. Then the set

$$\{P \in \mathbb{P}^N(K) : H_K(P) \leq B\}$$

is finite. More generally, for any constants B and D , the set

$$\{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) : H(P) \leq B \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\}$$

is finite.

Proof. (Sketch) Divide the homogeneous coordinates of P by a nonzero coordinate so one of them equals 1. Then

$$K = \mathbb{Q}(P) = \mathbb{Q}(x_0, \dots, x_N) \supset \mathbb{Q}(x_i) \quad \forall 0 \leq i \leq N$$

so $[\mathbb{Q}(x_i) : \mathbb{Q}] \leq D$. Let $K = \mathbb{Q}(P)$ and $d = [K : \mathbb{Q}]$, and estimate the heights of individual coordinates by

$$B \geq \left(\prod_{v \in M_K} \max\{|x_0|_v, \dots, |x_N|_v\}^{n_v} \right)^{\frac{1}{d}} \geq \prod_{v \in M_K} \left((\max\{|x_0|_v, 1\}^{n_v} \cdots \max\{|x_N|_v, 1\}^{n_v})^{\frac{1}{N}} \right)^{\frac{1}{d}}$$

$$= (H(x_0) \cdots H(x_N))^{\frac{1}{N}} \geq H(x_i)^{\frac{1}{N}} \quad \forall 0 \leq i \leq N.$$

So each coordinate of P lies in a field of bounded degree and has bounded height, so it suffices to prove

$$\{\alpha \in \bar{\mathbb{Q}} : H(\alpha) \leq B \text{ and } [\mathbb{Q}(\alpha) : \mathbb{Q}] = d\} \quad (2.1)$$

is finite. Factor the minimal polynomial of α as

$$F_\alpha(X) = X^d + a_1 X^{d-1} + \dots + a_d = (X - \alpha_1) \cdots (X - \alpha_d).$$

Then estimate the absolute values of the coefficients a_1, \dots, a_d in terms of the absolute values of the roots $\alpha_1, \dots, \alpha_d$ for each $v \in M_{\mathbb{Q}(\alpha)}$, by using the fact that the coefficients are the elementary symmetric polynomials of the roots together with the triangle inequality. We find

$$H([1, a_1, \dots, a_d]) \leq 2^d H(\alpha_1) \cdots H(\alpha_d)$$

and since the $H(\alpha_i) = H(\alpha) \leq B$ for every i we get

$$H([1, a_1, \dots, a_d]) \leq (2B)^d.$$

This height bound implies there are finitely many possibilities for the minimal polynomial of α , and since each has d roots, there are finitely many possibilities for α , and so 2.1 is finite. \square

Theorem 2.21 says that there are finitely many points in $\mathbb{P}^N(\bar{\mathbb{Q}})$ of bounded height and degree.

The height of a point P measures its arithmetic complexity. Its image under a map of degree d has height approximately the d^{th} power of $H(P)$:

Definition 2.22 (see [43], pg. 89). *A rational map of degree d between projective spaces, $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^M$ defined by*

$$\phi(P) = [f_0(P), \dots, f_M(P)],$$

where $f_0, \dots, f_M \in \bar{K}[X_0, \dots, X_N]$ are homogeneous polynomials of degree d with no common factors, is called a morphism if it is defined at every point of $\mathbb{P}^N(\bar{K})$: that is, if the only solution to the simultaneous equations

$$f_0(X_0, \dots, X_N) = \dots = f_M(X_0, \dots, X_N) = 0$$

is the trivial solution $X_0 = \dots = X_N = 0$.

Theorem 2.23 (see [43], Theorem 3.11). *If $\phi : \mathbb{P}^N(\bar{K}) \rightarrow \mathbb{P}^M(\bar{K})$ is a morphism of degree d , there exist constants $C_1, C_2 > 0$, depending on ϕ , such that*

$$C_1 H(P)^d \leq H(\phi(P)) \leq C_2 H(P)^d \quad \forall P \in \mathbb{P}^N(\bar{K}).$$

Proof. (Sketch) Define the absolute value of a polynomial f to be the maximum of its coefficients' absolute values, and the absolute value of a collection of polynomials $\phi = [f_0, \dots, f_M]$ to be the maximum of the absolute values of the f_i .

Then using the triangle inequality, bound the absolute value of each $f_i(P)$ in terms of the absolute values of f_i and P . Then taking the maximum over i , bound the absolute value of $\phi(P)$ in terms of the absolute values of ϕ and P . Use this to similarly bound the height $H(\phi(P))$ from above, and the constant C_2 is given explicitly by

$$C_2 = \binom{N+d}{d} H(\phi).$$

The lower bound uses the Nullstellensatz to say there is an exponent e such that

$$X_i^e = \sum_j^M g_{i,j} f_j \quad \forall 1 \leq i \leq N$$

for homogeneous polynomials $g_{i,j}$ of degree $e-d$. For each $P \in \mathbb{P}^N(K)$, estimate for p -adic v

$$|P|_v^e = \max_{0 \leq i \leq N} |x_i|_v^e \leq C_{v,M,N,d,e} \left(\max_{\substack{0 \leq i \leq N \\ 0 \leq j \leq M}} |g_{i,j}|_v \right) |P|_v^{e-d} |\phi(P)|_v$$

for some constant $C_{v,M,N,d,e}$. Letting $|g|_v = \max_{i,j} |g_{i,j}|_v$, this yields the estimate

$$H(P)^d \leq C_{M,N,d,e} H(g) H(\phi(P))$$

where $H(g)$, the height determined by the coefficients of the $g_{i,j}$, is not dependent on P . So we have a lower bound $H(P)^d \leq CH(\phi(P))$ where C is not dependent on P . \square

Definition 2.24. *Define the logarithmic height of P (relative to K) as*

$$h_K : \mathbb{P}^N(K) \rightarrow \mathbb{R}, \quad h_K(P) = \log H_K(P)$$

and its absolute logarithmic height as

$$h : \mathbb{P}^N(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}, \quad h(P) = \log H(P).$$

So, we have that $h(f(P))$ and $dh(P)$ differ by a bounded amount (by Theorem 2.23):

$$h(f(P)) = dh(P) + O(1) \quad \forall P \in \mathbb{P}^N(\bar{\mathbb{Q}}).$$

Given functions $f : S \rightarrow S$ and $h : S \rightarrow \mathbb{R}$ satisfying the above property, there exists a unique function (see [43], Theorem 3.20), which we call the canonical height for each morphism f :

Definition 2.25 (see [12], Theorem 1.1). *Let $f : \mathbb{P}^N \rightarrow \mathbb{P}^N$ be a morphism of degree $d \geq 2$. The canonical height function (associated to f) is the unique function*

$$\hat{h}_f : \mathbb{P}^N(\bar{\mathbb{Q}}) \rightarrow \mathbb{R}$$

satisfying

$$\hat{h}_f(P) = h(P) + O(1) \quad \text{and} \quad \hat{h}_f(f(P)) = d\hat{h}_f(P).$$

Theorem 2.26 (Call-Silverman 1993, [12]). *Let $f : \mathbb{P}^N \rightarrow \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over $\bar{\mathbb{Q}}$ and let $P \in \mathbb{P}^N(\bar{\mathbb{Q}})$. Then*

$$P \in \text{PrePer}(f) \iff \hat{h}_f(P) = 0.$$

Proof. If P is preperiodic, then $d^{-n}h(f^n(P)) \rightarrow 0$ as $n \rightarrow \infty$.

If $\hat{h}_f(P) = 0$, then letting K be such that $P \in \mathbb{P}^N(K)$ and f is defined over K , we have

$$h(f^n(P)) = \hat{h}_f(f^n(P)) + O(1) = d^n \hat{h}_f(P) + O(1) = O(1) \quad \forall n \geq 0.$$

So the orbit

$$\mathcal{O}_f(P) = \{P, f(P), f^2(P), \dots\} \subset \mathbb{P}^N(K)$$

is a set of bounded height, and so it is finite by Theorem 2.21. So P is preperiodic. \square

So, the canonical heights of the critical points of a PCF map must all be equal to zero. This is what motivates the definition of $\hat{h}_{\text{crit}}(f)$, Definition 1.4.

A rational map of degree d , $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, is specified by homogeneous polynomials F and G :

$$f = [F, G] = [a_0X^d + a_1X^{d-1}Y + \dots + a_dY^d, b_dX^d + b_1X^{d-1}Y + \dots + b_dY^d]$$

such that F and G have no common factors ($\text{res}(F, G) \neq 0$).

So, f is determined by the $2d + 2$ coefficients. Denote by \mathbf{a} the $(d + 1)$ -tuple (a_0, \dots, a_d) , and $F_{\mathbf{a}}$ the associated polynomial.

Definition 2.27. *The set of rational functions of degree d , Rat_d is identified with an open subset of \mathbb{P}^{2d+1} via*

$$\{[\mathbf{a}, \mathbf{b}] \in \mathbb{P}^{2d+1} : \text{res}(F_{\mathbf{a}}, F_{\mathbf{b}}) \neq 0\} \xrightarrow{\sim} \text{Rat}_d[\mathbf{a}, \mathbf{b}] \mapsto [F_{\mathbf{a}}, F_{\mathbf{b}}].$$

Rat_d is an affine variety defined over \mathbb{Q} . There is a natural map

$$\begin{aligned} f : \mathbb{P}^1 \times \text{Rat}_d &\rightarrow \mathbb{P}^1 \times \text{Rat}_d, \\ ([X, Y], [\mathbf{a}, \mathbf{b}]) &\mapsto ([F_{\mathbf{a}}(X, Y), F_{\mathbf{b}}(X, Y)], [\mathbf{a}, \mathbf{b}]). \end{aligned}$$

Any algebraic family of degree d rational maps factors through Rat_d .

For example, if we are not in characteristic 2, any quadratic polynomial $f(z) = Ax^2 + Bz + C$ can be conjugated to the form $z^2 + c$: if $\phi(z) = (2z - B)/(2A)$, then

$$f^\phi(z) = (\phi^{-1} \circ f \circ \phi)(z) = z^2 + \left(AC - \frac{1}{4}B^2 + \frac{1}{2}B \right).$$

Each c value represents a different conjugacy class, so the space of these conjugacy classes can be identified with \mathbb{A}^1 .

The map

$$\text{PGL}_2 \times \text{Rat}_d \rightarrow \text{Rat}_d, \quad (\phi, f) \mapsto f^\phi = \phi^{-1} \circ f \circ \phi$$

is an algebraic group action of PGL_2 on Rat_d defined over \mathbb{Q} .

Definition 2.28. *The moduli space of rational maps of degree d on \mathbb{P}^1 is the quotient space*

$$\mathcal{M}_d = \text{Rat}_d / \text{PGL}_2$$

where PGL_2 acts on Rat_d via conjugation as above.

Points in \mathcal{M}_d are conjugacy classes of degree d rational maps. For example, denote by B the set of all degree 2 polynomials. We have seen that the moduli space of degree 2 polynomials is \mathbb{A}^1 . So we have a commutative diagram

$$\begin{array}{ccc}
PGL_2 \times B & \longrightarrow & B \\
\downarrow & & \downarrow \langle \cdot \rangle \\
B & \xrightarrow{\langle \cdot \rangle} & \mathbb{A}^1
\end{array}
\qquad
\begin{array}{ccc}
(\phi, f) & \longmapsto & f^\phi \\
\downarrow & & \downarrow \\
f & \longmapsto & c
\end{array}$$

Figure 2.4: \mathcal{P}_2 commutative diagram

where $\langle \cdot \rangle$ maps each $f(z) = Az^2 + Bz + C$ to the point $(AC - \frac{1}{4}B^2 + \frac{1}{2}B) \in \mathbb{A}^1$ (that is, to the unique value of c when f is conjugated to the form $f_c(z) = z^2 + c$).

Chapter 3

Composition With a Power Map

3.1 Introduction

It is easy to show that, for a fixed n , there are only finitely many c in any given number field such that $z^n + c$ is PCF. This is just by a basic argument bounding $h(c)$. But in fact the bound is better than this. If $z^n + c$ is PCF, then c is an algebraic integer satisfying $|c| < |2|^{1/(n-1)}$, and so $h(c) \leq \frac{1}{n-1} \log 2$. In other words, there exist only finitely many c in a given number field such that there exists a $n \geq 2$ with $z^n + c$ PCF. This can be generalized.

Theorem 3.1. *Over a number field K there are only finitely many monic polynomials $g(z)$ of fixed degree m such that there exists an $n \geq 2$ with $g(z^n)$ PCF. Moreover,*

$$nh_{crit}(g(z^n)) + \zeta \geq h(g),$$

where ζ is a constant depending on m , but independent of n .

This lets us find a finite list of polynomials of fixed degree m such that if there exists an n with $g(z^n)$ PCF, then g is in this list (note that if we don't let n vary and fix $n = 1$, then this is the result from [27]). After proving some intermediate lemmas, the proof of this theorem is given on page 30.

3.2 Local estimates

Here we take a local field, K , complete with respect to some absolute value (that is, complete as a metric space via the metric $d(x, y) = |x - y|$) and algebraically closed. In general we will drop subscripts related to the absolute value, as it is now fixed. We have $n \geq 2$ fixed,

$m = \deg(g)$, and $d = mn = \deg(f)$ for $f(z) = g(z^n)$. To fix notation, we will also set

$$g(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$$

and

$$g'(z) = m \prod_{i=1}^{m-1} (z - c_i),$$

noting that this imposes a relation between the a_i and the c_j (see below).

Lemma 3.2. *For $g(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ with critical points c_j , $j \in \{1, m-1\}$,*

$$\log \|\mathbf{c}\| := \log \max |c_i| \leq \log \max_{i \neq 0} |a_i|^{1/(m-i)} + \log m.$$

Moreover, $\log \max_{i \neq 0} |a_i|^{1/(m-i)} \leq \log \|\mathbf{c}\| + \varepsilon_v$, for some nonnegative real number ε_v , where $\varepsilon_v = 0$ for all but finitely many places $v \in M_K$.

Proof. In one direction, note that σ_k , the basic symmetric polynomial of degree k in $m-1$ variables, has $\binom{m-1}{k} \leq 2^{m-1}$ terms.

Vieta's formulas relate the coefficients of a polynomial to sums and products of its roots. Where the roots of a general polynomial $p(x) = b_{m-1}x^{m-1} + \dots + b_0$ are c_1, \dots, c_{m-1} , we have the elementary symmetric polynomials of its roots:

$$\begin{aligned} \sigma_1(c_1, \dots, c_{m-1}) &= c_1 + \dots + c_{m-1} = -\frac{b_{m-2}}{b_{m-1}}, \\ \sigma_2(c_1, \dots, c_{m-1}) &= (c_1c_2 + c_1c_3 + \dots + c_1c_{m-1}) + \dots + c_{m-2}c_{m-1} = \frac{b_{m-3}}{b_{m-1}}, \\ &\vdots \\ \sigma_{m-1}(c_1, \dots, c_{m-1}) &= c_1c_2 \cdots c_{m-1} = \frac{b_0}{b_{m-1}}. \end{aligned}$$

Recall that $g'(z) = m \prod_{i=1}^{m-1} (z - c_i)$. Put it in the form $g'(z) = b_{m-1}z^{m-1} + \dots + b_0$.

Then for every $j \in [1, m-1]$, $ja_j = b_{j-1} = \pm b_{m-1} \cdot \sigma_{m-1-(j-1)}(c_1, \dots, c_{m-1}) = \pm m \cdot \sigma_{m-j}(c_1, \dots, c_{m-1})$

$$\begin{aligned}
|a_j|^{1/(m-j)} &\leq \left| \frac{m}{j} \right|^{1/(m-j)} \cdot |\sigma_{m-j}(\mathbf{c})|^{1/(m-j)} \\
&\leq \left| \frac{m}{j} \right|^{1/(m-j)} \cdot (2^{m-1})_v \cdot \|\mathbf{c}\|^{m-j}^{1/(m-j)} \\
&= \left| \frac{m}{j} \right|^{1/(m-j)} \cdot (2^{m-1})_v \cdot \|\mathbf{c}\|.
\end{aligned}$$

This second inequality follows by using the triangle/ultrametric inequality. If v is archimedean, clearly we should have a product (since the $(m-j)$ th symmetric polynomial has at most 2^{m-1} monomials, and each term is a product of $m-j$ terms). On the other hand, if v is nonarchimedean, then by the ultrametric inequality, the sum of these monomials is at most the maximum of them, and so we can have $(2^{m-1})_v = 1$.

So if v restricts on \mathbb{Q} to a p -adic absolute value with $p > m$, then $|m/j| = 1$ and $(2^{m-1})_v = 1$, so we have $\log \max_{j \neq 0} |a_j|^{1/(m-j)} \leq \log \|\mathbf{c}\|$ for all but finitely many places (that is, $\varepsilon_v = 0$). For the remaining nonarchimedean places, we have

$$\log |a_j|^{1/(m-j)} \leq \log \|\mathbf{c}\| + \frac{1}{m-j} \log |m/j|,$$

and so we can take

$$\varepsilon_v = \max \left\{ \frac{1}{m-j} \log |m/j|, 0 \right\},$$

where, since $j \leq m-1$, it can be easily seen that

$$\varepsilon_v \leq \log(m-1). \tag{3.1}$$

In the archimedean case we have

$$\log |a_j|^{1/(m-j)} \leq \log \|\mathbf{c}\| + \frac{1}{m-j} \log |m/j| + (m-1) \log^+ |2|,$$

and so we can take

$$\varepsilon_v = \max \left\{ \frac{1}{m-j} \log |m/j|, 0 \right\} + (m-1) \log^+ |2|$$

where similarly we can see that

$$\varepsilon_v \leq \log m + (m-1) \log^+ |2|. \tag{3.2}$$

In the other direction, note that for all $i \in [1, m-1]$,

$$g'(c_i) = mc_i^{m-1} + (m-1)a_{m-1}c_i^{m-2} + \cdots + a_1 = 0,$$

since the c_i 's are roots of $g'(z)$.

In a non-archimedean absolute value we would have a contradiction if the lead term is larger than all others ($|mc_i^{m-1}| > |ja_jc_i^{j-1}|$ for all $j \leq m-1$), because then by the ultrametric inequality, this sum would equal the value of its maximum term, and hence not be zero. So there must be some j with $|mc_i^{m-1}| \leq |ja_jc_i^{j-1}|$. So, $\forall i \in [0, m-1], \exists j \in [1, m-1]$ such that if $c_i \neq 0$:

$$\begin{aligned}
& |mc_i^{m-1}| \leq |ja_jc_i^{j-1}| \\
\implies |c_i|^{m-j} & \leq \left| \frac{j}{m} a_j \right| \\
\implies |c_i| & \leq \left| \frac{j}{m} a_j^{\frac{1}{m-j}} \right| \\
& \leq \max_{0 \leq i \leq m-1} \left| \frac{i}{m} a_i^{\frac{1}{m-i}} \right| \\
& \leq \max_{i \neq 0} \left| a_i^{\frac{1}{m-i}} \right| \cdot \max_{i \neq 0} \left| \frac{i}{m} \right| \\
& = \max_{i \neq 0} \left| a_i^{\frac{1}{m-i}} \right| \cdot \max \left\{ \left| \frac{1}{m} \right|, \left| \frac{2}{m} \right|, \dots, \left| \frac{m-2}{m} \right|, \left| \frac{m-1}{m} \right| \right\} \\
\implies \log \|\mathbf{c}\| & \leq \log \max_{i \neq 0} |a_i|^{1/(m-i)} + \log m.
\end{aligned}$$

In the archimedean absolute value we would have a contradiction if the lead term is larger than all others times $m-1$: (that is, $|mc_i^{m-1}| > (m-1)|ja_jc_i^{j-1}|$ for all $j \leq m-1$), since then it would be larger than the sum of all other terms combined, and hence adding it to them would not yield zero. So there must be some j with $|mc_i^{m-1}| \leq (m-1)|ja_jc_i^{j-1}|$. So, $\forall i \in [0, m-1], \exists j \in [1, m-1]$ such that if $c_i \neq 0$:

$$\begin{aligned}
|mc_i^{m-1}| &\leq (m-1)|ja_jc_i^{j-1}| \\
\implies |c_i|^{m-j} &\leq \left| \frac{j(m-1)}{m} a_j \right| \\
\implies |c_i| &\leq \left| \frac{j(m-1)^{\frac{1}{(m-j)}}}{m} a_j^{\frac{1}{(m-j)}} \right| \\
&\leq \max_{0 \leq i \leq m-1} \left| \frac{i(m-1)^{\frac{1}{(m-i)}}}{m} a_i^{\frac{1}{(m-i)}} \right| \\
&\leq \max_{i \neq 0} \left| a_i^{\frac{1}{(m-i)}} \right| \cdot \max_{i \neq 0} \left| \frac{i(m-1)^{\frac{1}{(m-i)}}}{m} \right| \\
&\leq \max_{i \neq 0} \left| a_i^{\frac{1}{(m-i)}} \right| \cdot \max_{i \neq 0} \left| \left(m-2 + \frac{1}{m} \right)^{\frac{1}{(m-i)}} \right| \\
\implies \log \|\mathbf{c}\| &\leq \log \max_{i \neq 0} |a_i|^{1/(m-i)} + \log m.
\end{aligned}$$

□

Let

$$G_f(z) = \lim_{k \rightarrow \infty} \frac{\log^+ |f^k(z)|}{\deg(f)^k}$$

where $f(z) = g(z^n)$. Also, let

$$(2m)_v = \begin{cases} 2m, & \text{if } v \text{ is archimedean} \\ 1, & \text{otherwise} \end{cases}$$

Lemma 3.3. *Let*

$$C_g = \log \max\{1, |a_i|^{1/(m-i)}\}.$$

Then for $n \log |z| > C_g + \log(2m)_v$ we have

$$G_{f,v}(z) \geq \log^+ |z|_v + \frac{1}{d-1} \log \frac{1}{2},$$

and in the non-archimedean case we have

$$G_{f,v}(z) = \log^+ |z|.$$

Proof. Observe, for every $i \leq m-1$:

$$\begin{aligned}
C_g + \log(2m)_v &< n \log |z| \\
\implies \left| a_i^{\frac{1}{(m-i)}} \right| &< |z|^n \\
\implies |a_i| &< |z|^{n(m-i)} \\
\implies |a_i z^{ni}| &< |z|^{nm}.
\end{aligned}$$

So, in the non-archimedean absolute values, the condition $n \log |z| > C_g + \log(2m)_v$ just ensures that the lead term of $f(z) = g(z^n) = z^{nm} + a_{m-1}z^{n(m-1)} + \dots + a_1z^n + a_0$ dominates.

So,

$$\log^+ |g(z^n)| = \log^+ |z^{nm}| = \log^+ |z^d| = d \log^+ |z|,$$

and by induction

$$\log^+ |f^k(z)| = d^k \log^+ |z|.$$

Then,

$$G_f(z) = \lim_{k \rightarrow \infty} \frac{\log^+ |f^k(z)|}{\deg(f)^k} = \log^+ |z|.$$

In the archimedean absolute value, we have, for every $i \leq m-1$

$$\begin{aligned}
\log(2m)_v + C_g &< n \log |z| \\
\implies 2m|a_i|^{1/(m-i)} &< |z|^n \\
\implies (2m)^{m-i}|a_i| &< |z|^{n(m-i)} \\
\implies 2m|a_i z^{ni}| &< |z|^{nm} \\
\implies |a_i z^{ni}| &< \frac{1}{2m}|z|^{nm}.
\end{aligned}$$

Then,

$$\begin{aligned}
|g(z^n)| &= |z^{nm} + a_{m-1}z^{n(m-1)} + \dots + a_1z^n + a_0| \\
&\geq |z^{nm}| - |a_{m-1}z^{n(m-1)}| - \dots - |a_1z^n| - |a_0| \\
&\geq |z^{nm}| - \frac{1}{2m}|z^{nm}| - \dots - \frac{1}{2m}|z^{nm}| - \frac{1}{2m}|z^{nm}| \\
&= |z^{nm}| - \frac{1}{2}|z^{nm}| \\
&= \frac{1}{2}|z^d|.
\end{aligned}$$

Then

$$\frac{1}{2}|z^{nm}| \leq |g(z^n)| \leq \frac{3}{2}|z^{nm}|,$$

(note that the right-hand inequality follows by doing a similar argument to above, only

$|g(z^n)| \leq |z^{nm}| + |a_{m-1}z^{n(m-1)}| + \dots + |a_1z^n| + |a_0| \leq \dots \leq \frac{3}{2}|z^d|$, and by induction,

$$\left(\frac{1}{2}\right)^{\frac{d^k-1}{d-1}} |z^{nm^k}| \leq |f^k(z)| \leq \left(\frac{3}{2}\right)^{\frac{d^k-1}{d-1}} |z^{nm^k}|.$$

Proof. Proof of the left-hand inequality by induction. The right-hand inequality would have a similar argument. First, if we assume that $m \geq 2$ (as we can since the introduction paragraph takes care of the case $m = 1$):

$$\begin{aligned} n \log |z| &> C_g + \log(2m)_v \geq \log(2m)_v \\ \implies |z|^n &> 2m \\ \implies |z|^{nm-1} &> 4 \quad (\text{since } nm - 1 > n \text{ and } m \geq 2) \\ \implies |g(z^n)| &\geq \frac{1}{2}|z^{nm}| > 2|z| > |z|. \end{aligned}$$

That is, if z meets the hypothesis of Lemma 3.3, then so does $g(z^n)$.

$$\begin{aligned}
|f^{k+1}(z)| &= |f(f^k(z))| \\
&= |(f^k(z))^{nm} + a_{m-1}(f^k(z))^{n(m-1)} + \dots + a_1(f^k(z))^n + a_0| \\
&\geq |(f^k(z))^{nm}| - \frac{1}{2m}|(f^k(z))^{nm}| - \dots - \frac{1}{2m}|(f^k(z))^{nm}| - \frac{1}{2m}|(f^k(z))^{nm}| \\
&\geq \left| \left(\left(\frac{1}{2} \right)^{\frac{d^k-1}{d-1}} |z^{d^k}| \right)^d \right| - \frac{1}{2} \left| \left(\left(\frac{3}{2} \right)^{\frac{d^k-1}{d-1}} |z^{d^k}| \right)^d \right| \\
&= \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-d}{d-1}} |z^{d^{k+1}}| \right| - \frac{1}{2} \left| \left(\frac{3}{2} \right)^{\frac{d^{k+1}-d}{d-1}} |z^{d^{k+1}}| \right| \\
&= \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-d}{d-1}} - \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{d^{k+1}-d}{d-1}} \right| |z^{d^{k+1}}| \\
&= \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-d}{d-1}} \left(1 - \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{d^{k+1}-d}{d-1}} \right) \right| |z^{d^{k+1}}| \\
&= \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-1}{d-1}} \left(\frac{1}{2} \right)^{\frac{-(d-1)}{d-1}} \left(1 - \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{d^{k+1}-d}{d-1}} \right) \right| |z^{d^{k+1}}| \\
&= \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-1}{d-1}} \cdot 2 \left(1 - \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{d^{k+1}-d}{d-1}} \right) \right| |z^{d^{k+1}}| \\
&\geq \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-1}{d-1}} \cdot 2 \left(1 - \frac{1}{2} \cdot 3 \right) \right| |z^{d^{k+1}}| \quad (\text{since clearly } \frac{d^{k+1}-d}{d-1} \geq 1) \\
&= \left| \left(\frac{1}{2} \right)^{\frac{d^{k+1}-1}{d-1}} \cdot (-1) \right| |z^{d^{k+1}}| \\
&= \left(\frac{1}{2} \right)^{\frac{d^{k+1}-1}{d-1}} |z^{d^{k+1}}|.
\end{aligned}$$

□

Then, taking logarithms and limits yields

$$\frac{1}{d-1} \log \frac{1}{2} \leq \lim_{k \rightarrow \infty} \frac{\log |f^k(z)|}{\deg(f)^k} - \log |z| \leq \frac{1}{d-1} \log \frac{3}{2},$$

so

$$G_f(z) = \lim_{k \rightarrow \infty} \frac{\log^+ |f^k(z)|}{\deg(f)^k} \geq \log^+ |z| + \frac{1}{d-1} \log \frac{1}{2}.$$

□

Lemma 3.4. For each $1 \leq i \leq m-1$, let $G_i(c_1, \dots, c_{m-1}) = g(c_i) - a_0$, and let $G(z) =$

$g(z) - a_0$. Then the polynomials $G_i \in \mathbb{Q}[c_1, \dots, c_{m-1}]$ are homogeneous forms of degree m , with no common root over $\bar{\mathbb{Q}}$ other than the trivial root $c_i = 0$ for all i .

Proof. See Lemma 7 of [27]. □

Observe that

$$\begin{aligned} C_g &= \log \max\{1, |a_i|^{1/(m-i)}\} \\ &\leq \max\{\log \max_{i \neq 0} |a_i|^{1/(m-i)}, \log |a_0|^{1/m}\} \\ &\leq \max\{\log \|\mathbf{c}\| + \varepsilon_v, \log |a_0|^{1/m}\} \quad (\text{by Lemma 3.2}) \\ &= \max\{\log \|\mathbf{c}\| + \varepsilon_v, \log |g(0)|^{1/m}\} \quad (\text{since } a_0 \text{ is the constant term of } g). \end{aligned}$$

Note that if 1 dominates in the above, then we have $C_g \leq 0$, and so Lemma 3.6 follows trivially.

Recall that the critical points of f are the solutions $f'(c) = 0$, and the branch points are the $f(c)$.

Lemma 3.5. *There is a branch point β of $f(z) = g(z^n)$ and a real number $\delta_v \leq 0$ with*

$$\log |\beta|_v \geq \deg(g)C_g + \delta_v,$$

for every $v \in M_{\mathbb{Q}}$, where $\delta_v = 0$ for all but finitely many v .

Proof. Let $G_i(c_1, \dots, c_{m-1}) = G(c_i) \in \mathbb{Q}[c_1, \dots, c_{m-1}]$ be the homogeneous polynomials defined in Lemma 3.4. By that lemma, these polynomials have no common root other than the trivial one, and so as in (Lemma 8, [27]), by Hilbert's Nullstellensatz,

$$\deg(g) \log \|\mathbf{c}\| - B_v = m \log \max\{|c_1|_v, \dots, |c_{m-1}|_v\} - B_v \leq \log \max\{|G(c_i)|_v\}$$

where $B_v \geq 0$ is 0 for all but finitely many $v \in M_K$.

Note that the branch points of $f(z) = g(z^n)$ are $f(0) = a_0 = g(0)$ (with multiplicity $n - 1$) and the $m - 1$ many branch points of g , $f(c_i^{1/n}) = g(c_i)$ (each with multiplicity n):

$$f'(z) = g'(z^n) \cdot n z^{n-1} \quad g'(z^n) = m \prod_{i=1}^{m-1} (z^n - c_i).$$

So $\log \max\{|g(c_i)|_v, |g(0)|_v\}$ gives \log of a branch point. For non-archimedean v , there exists a c_i such that:

$$\begin{aligned}
\log \max\{|g(c_i)|_v, |g(0)|_v\} &\geq \log |g(c_i) - g(0)|_v && \text{(by the ultrametric inequatlity)} \\
&= \log |G(c_i)|_v \\
&\geq \deg(g) \log \|\mathbf{c}\| - B_v && \text{(by the above inequality)}.
\end{aligned}$$

$$\underline{\text{Case 1:}} \log |g(0)|^{1/m} < \log \|\mathbf{c}\| + \varepsilon_v \implies C_g \leq \log \|\mathbf{c}\| + \varepsilon_v \implies C_g - \varepsilon_v \leq \log \|\mathbf{c}\|$$

$$\implies \log \max\{|g(c_i)|_v, |g(0)|_v\} \geq \deg(g)C_g - \deg(g)\varepsilon_v - B_v,$$

so choose $\delta_v = -(\deg(g)\varepsilon_v + B_v)$.

Case 2: $\log |g(0)|^{1/m} \geq \log \|\mathbf{c}\| + \varepsilon_v \implies C_g \leq \frac{1}{m} \log |g(0)|$, and so the above inequality clearly holds ($\log \max\{|g(c_i)|_v, |g(0)|_v\} \geq \log |g(0)|_v \geq \deg(g)C_g$).

For archimedean v , we'd like to do something similar to the above, this time using the triangle inequality...

$$\begin{aligned}
\log(2 \max\{|g(c_i)|, |g(0)|\}) &\geq \log |g(c_i) - g(0)| && \text{(by the triangle inequatlity)} \\
&= \log |G(c_i)|_v \\
&\geq \deg(g) \log \|\mathbf{c}\| - B_v && \text{(by the above inequality)}.
\end{aligned}$$

$$\underline{\text{Case 1:}} \log |g(0)|^{1/m} < \log \|\mathbf{c}\| + \varepsilon_v \implies C_g \leq \log \|\mathbf{c}\| + \varepsilon_v.$$

$$\implies \log \max\{|g(c_i)|, |g(0)|\} \geq \deg(g)C_g - \deg(g)\varepsilon_v - B_v - \log(2),$$

so choose $\delta_v = -(\deg(g)\varepsilon_v + B_v + \log(2))$.

$$\underline{\text{Case 2:}} \log |g(0)|^{1/m} \geq \log \|\mathbf{c}\| + \varepsilon_v \implies C_g \leq \frac{1}{m} \log |g(0)| \implies \log |g(0)| \geq \deg(g)C_g,$$

and so take $\beta = g(0)$ and we get $\log |\beta| \geq \deg(g)C_g$

Finally, note that since both ε_v and B_v are zero for all but finitely many $v \in M_{\mathbb{Q}}$, the same holds for δ_v . \square

Lemma 3.6. *For any $v \in M_{\mathbb{Q}}$, there is a $\zeta_v \leq 0$ such that*

$$\frac{1}{n}(C_g + \zeta_v) < \lambda_{crit,v}(f)$$

for all $\mathbf{c} \in \mathbb{A}^{d-1}(\bar{\mathbb{Q}})$. Furthermore, $\zeta_v = 0$ for all but finitely many $v \in M_{\mathbb{Q}}$.

Proof. By Lemma 3.5., there exists a branch point β of $f(z) = g(z^n)$ with $\log |\beta|_v \geq \deg(g)C_g + \delta_v$. So, $n \log |\beta|_v \geq nmC_g + n\delta_v$, and if $nmC_g + n\delta_v > (2m)_v C_g$, then β satisfies the hypothesis of Lemma 3.3. For all but finitely many v , $\delta_v = 0$, and clearly $nm > (2m)_v$ since $n \geq 2$, so this would be true. For the remaining v ...

Case 1 ($C_g > \frac{-n\delta_v}{nm-(2m)_v}$):

If $C_g > \frac{-n\delta_v}{nm-(2m)_v} = \frac{n(\deg(g)\varepsilon_v + B_v)}{nm-(2m)_v}$, we have $n \log |\beta|_v > (2m)_v C_g$. So in this case, we also have β satisfying the hypothesis of Lemma 3.3.

Case 2a (v is archimedean, $C_g \leq \frac{-n\delta_v}{nm-(2m)_v}$):

$$\begin{aligned}
C_g &\leq \frac{n(\deg(g)\varepsilon_v + B_v + \log 2)}{nm - (2m)_v} \\
&= \frac{n(m\varepsilon_v + B_v + \log 2)}{nm - \log 2m} && \text{(by definition of } (2m)_v) \\
&\leq \frac{n(m(\log m + (m-1)\log 2) + B_v + \log 2)}{nm - \log 2m} && \text{(by 3.2, from Lemma 3.2)} \\
&\leq \frac{m(\log m + (m-1)\log 2) + B_v + \log 2}{m - \frac{\log 2m}{n}} \\
&\leq \frac{m(\log m + (m-1)\log 2) + B_v + \log 2}{m - \frac{\log 2m}{2}} && \text{(since } n \geq 2).
\end{aligned}$$

Case 2b (v is non-archimedean, $C_g \leq \frac{-n\delta_v}{nm-(2m)_v}$):

$$\begin{aligned}
C_g &\leq \frac{n(\deg(g)\varepsilon_v + B_v)}{nm - (2m)_v} \\
&= \frac{n(m\varepsilon_v + B_v)}{nm - 1} && \text{(by definition of } (2m)_v) \\
&\leq \frac{n(m \log(m-1) + B_v)}{nm - 1} && \text{(by 3.1, from Lemma 3.2)} \\
&\leq \frac{m \log(m-1) + B_v}{m - \frac{1}{n}} \\
&\leq \frac{m \log(m-1) + B_v}{m - \frac{1}{2}} && \text{(since } n \geq 2).
\end{aligned}$$

So by Lemma 3.3, for all but finitely many v we have $G_{f,v}(\beta) \geq \log^+ |\beta|_v + \frac{1}{d-1} \log \frac{1}{2}$ for some branch point β .

$$\begin{aligned}
&\implies mC_g + \delta_v + \frac{1}{d-1} \log \frac{1}{2} < \log^+ |\beta|_v + \frac{1}{d-1} \log \frac{1}{2} \leq \max\{G_{f,v}(\beta) : \text{branch } \beta\} \\
&\implies mC_g + \delta_v + \frac{1}{d-1} \log \frac{1}{2} < mn\lambda_{crit,v}(f) \\
&\implies \frac{1}{n} \left(C_g + \frac{1}{m} \left(\delta_v + \frac{1}{d-1} \log \frac{1}{2} \right) \right) < \lambda_{crit,v}(f) \\
&\implies \frac{1}{n} C_g + \zeta_v < \lambda_{crit,v}(f)
\end{aligned}$$

where $\zeta_v = \frac{1}{m} \left(\delta_v + \frac{1}{d-1} \log \frac{1}{2} \right)$ is clearly non-positive. Note δ_v is zero at all but finitely many places, and $\zeta_v = \frac{1}{m} \delta_v$ at the non-archimedean places, so we have $\zeta_v = 0$ for all but finitely many v .

For the remaining finitely many v in Case 2, we have C_g bounded above by a constant κ not dependent on n . Since $\lambda_{crit,v}(f) \geq 0$, we can simply choose $\zeta_v = -\frac{\kappa}{n}$, and the result follows.

□

We now proceed to prove Theorem 3.1,

$$n\hat{h}_{crit}(f) + \zeta \geq h(g).$$

Proof. Defining $\hat{h}_{crit}(f)$ in terms of local contributions...

$$\begin{aligned}
\hat{h}_{crit}(f) &= \sum_{v \in M_K} n_v(\lambda_{crit,v}(f)) \quad \left(n_v = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \right) \\
&> \sum_{v \in M_K} n_v \frac{1}{n} (C_g + \zeta_v) \quad (\text{by Lemma 3.6}) \\
&= \frac{1}{n} \left(h(g) + \sum_{v \in M_K} \zeta_v \right)
\end{aligned}$$

where the constant $\sum_{v \in M_K} \zeta_v$ does not depend on n .

□

3.3 Explicit computations

In this section we will compute all of the possibilities for monic quadratics whose composition with a power map could be PCF.

Proposition 3.7. *Let $g(z) = z^2 + az + b \in \mathbb{Q}[z]$. The set of (a, b) such that there exists $n \geq 2$ where $g(z^n)$ is PCF is a set of bounded height.*

Proof. We begin with the archimedean case, and omit subscripts on the absolute value. Note $g(z^n) = z^{2n} + az^n + b$. Let $|z|^n > \max\{4|a|, 16|b|^{\frac{1}{2}}\}$. Then $\frac{1}{4}|z|^n > |a|$ and $\frac{1}{256}|z|^{2n} > |b|$.

$$\begin{aligned} |g(z^n)| &\geq |z|^{2n} - |a||z|^n - |b| \\ &> |z|^{2n} - \frac{1}{4}|z|^n|z|^n - \frac{1}{256}|z|^{2n} \\ &= \frac{191}{256}|z|^{2n} \\ &\geq |z| \quad \text{if } |z|^n \geq \left(\frac{256}{191}|z|\right)^{\frac{1}{2}}. \end{aligned}$$

So, z is a wandering point if $|z|^n > \max\{4|a|, 16|b|^{\frac{1}{2}}\}$ and $|z|^n \geq \left(\frac{256}{191}|z|\right)^{\frac{1}{2}}$.

So, if z is preperiodic then $|z|^n \leq \max\{4|a|, 16|b|^{\frac{1}{2}}, \left(\frac{256}{191}|z|\right)^{\frac{1}{2}}\}$.

If $g(z^n)$ is PCF, then its critical points are preperiodic.

$$\frac{d}{dz}(g(z^n)) = nz^{n-1}g'(z^n) = nz^{n-1}(2(z^n) + a),$$

so the critical points are 0 and solutions to $z^n = -\frac{a}{2}$.

Case 1: $16|b|^{\frac{1}{2}} \geq 4|a|$.

$g(z^n)$ is PCF $\implies b$ is preperiodic (since the critical point $0 \mapsto b$)

$$\implies |b|^n \leq \max \left\{ 4|a|, 16|b|^{\frac{1}{2}}, \left(\frac{256}{191}|b|\right)^{\frac{1}{2}} \right\} = 16|b|^{\frac{1}{2}}$$

$$\implies |b|^{\frac{2n-1}{2}} \leq 16$$

$$\implies |b| \leq 16^{\frac{2}{2n-1}}.$$

Case 2: $16|b|^{\frac{1}{2}} < 4|a|$ ($\implies 256|b| < 16|a|^2 \implies |b| < \frac{1}{16}|a|^2$).

$$\begin{aligned}
g(z^n) \text{ is PCF} &\implies \left(-\frac{a}{2}\right)^2 - \frac{a^2}{2} + b \text{ is preperiodic (since the critical point} \\
&\quad z^n = -\frac{a}{2} \text{ maps to it)} \\
&\implies -\frac{a^2}{4} + b \text{ is preperiodic} \\
&\implies \left|-\frac{a^2}{4} + b\right|^n \leq \max\left\{4|a|, 16|b|^{\frac{1}{2}}, \left(2\left|-\frac{a^2}{4} + b\right|\right)^{\frac{1}{2}}\right\} = 4|a| \\
&\implies \left|\frac{3a^2}{16}\right|^n \leq 4|a| \quad (\text{since } |b| < \frac{1}{16}|a|^2 \implies \left|\frac{3a^2}{16}\right|^n \leq \left|-\frac{a^2}{4} + b\right|^n) \\
&\implies |a|^{2n-1} \leq \frac{4}{(3/16)^n} \\
&\implies |a| \leq \left(\frac{4}{(3/16)^n}\right)^{\frac{1}{2n-1}}.
\end{aligned}$$

A simple argument can bound a and b p -adically. Note that $|g(z^n)|_p \leq \max\{|z^{2n}|_p, |az^n|_p, |b|_p\}$, with equality if one of the terms exceeds the others.

Supposing $|a|_p > 1$ yields that the critical point $z^n = -\frac{a}{2}$ wanders unless $\left|(-\frac{a}{2})^{4n}\right|_p \leq |b|_p$. But in this case the branch point b wanders. So $|a|_p \leq 1$, and then if $|b|_p > 1$, b wanders, so we also have $|b|_p \leq 1$.

So $a, b \in \mathbb{Z}$. By the archimedean bounds above, we then have that $|a| \leq 4$ and $|b| \leq 6$.

So by the above bounds, $a \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ and $b \in \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$, so there are $9 \cdot 13 = 117$ possibilities. □

We could then search to find, say, all monic quadratics whose composition with z^3 is PCF, by searching within the above 117. This turns up the following:

| $f(z)$ | Classification |
|------------------|--|
| z^6 | Uncritical |
| $z^6 - 1$ | Uncritical |
| $z^6 + 2z^3$ | See Prop 5.1, $f(z)$, $a = 2$, $m = 1$, $n = 3$ |
| $z^6 - 2z^3 + 1$ | See Prop 5.1, $h(z)$, $a = 2$, $m = 1$, $n = 3$ |

Table 3.1: PCF polynomials of form $z^6 + az^3 + b$ for $a, b \in \mathbb{Q}$

Note that the 117 provide a complete list of quadratics whose composition with z^n could possibly be PCF, for any n . Determining for which of the 117 there exists an n with $g(z^n)$ PCF (and if so, for how many such n) remains an open problem.

Chapter 4

Quartic PCF Polynomials $\in \mathbb{Q}[z]$

In this chapter, we want to find the conjugacy classes of quartic PCF polynomials $g(z) \in \mathbb{Q}[z]$. We take a different approach to that of [2], as there is not a natural conjugation to move three distinct critical points. Instead, we conjugate to simplify the coefficients as much as possible, and make use of the resultant's relationship with the branch points.

We first conjugate to eliminate the cubic term (if $h(z) = a_1z^4 + a_2z^3 + a_3z^2 + a_4z + a_5$, where $\psi(z) = z + -\frac{a_2}{4a_1}$, the conjugation $\psi^{-1} \circ h \circ \psi$ eliminates the cubic term), so let $g(z) = az^4 + bz^2 + cz + d$. Note that the conjugation $f(z) = g^\phi(z) = (\phi^{-1} \circ g \circ \phi)(z) \in \mathbb{Q}[z]$ where $\phi(z) = dz$ ensures that we can move the constant term to 1 (if it is nonzero). So we can assume throughout the work that $d \in \{0, 1\}$. Then

$$T - f(z) = -az^4 - bz^2 - cz + (T - d), \quad f'(z) = 4az^3 + 2bz + c$$

and

$$\text{res}_z(T - f(z), f'(z)) = (-1)^{12}(4a)^4 \prod_{j=1}^3 (T - f(\zeta_j)) = 256a^4 \prod_{j=1}^3 (T - f(\zeta_j))$$

where the ζ_j are the 3 critical points of f .

Now we can also compute the resultant as the following determinant:

$$\operatorname{res}_z(T - f(z), f'(z)) = \begin{vmatrix} T-d & 0 & 0 & c & 0 & 0 & 0 \\ -c & T-d & 0 & 2b & c & 0 & 0 \\ -b & -c & T-d & 0 & 2b & c & 0 \\ 0 & -b & -c & 4a & 0 & 2b & c \\ -a & 0 & -b & 0 & 4a & 0 & 2b \\ 0 & -a & 0 & 0 & 0 & 4a & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 4a \end{vmatrix}$$

$$\begin{aligned} &= a^2(256a^2T^3 + (128ab^2 - 768a^2d)T^2 + (768a^2d^2 - 256ab^2d + 144abc^2 + 16b^4)T \\ &\quad + (128ab^2d^2 - 256a^2d^3 - 144abc^2d + 27ac^4 - 16b^4d + 4b^3c^2)) \\ &= 256a^4(T^3 + \left(\frac{b^2}{2a} - 3d\right)T^2 + \left(3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2}\right)T \\ &\quad + \left(\frac{b^2d^2}{2a} - d^3 - \frac{9bc^2d}{16a} + \frac{27c^4}{256a} - \frac{b^4d}{16a^2} + \frac{b^3c^2}{64a^2}\right)). \end{aligned}$$

So, since $f(\zeta_j)$ are the 3 branch points of f , we have that

$$\sigma_1 = |f(\zeta_1) + f(\zeta_2) + f(\zeta_3)|_v = \left| \frac{b^2}{2a} - 3d \right|_v,$$

$$\sigma_2 = |f(\zeta_1)f(\zeta_2) + f(\zeta_1)f(\zeta_3) + f(\zeta_2)f(\zeta_3)|_v = \left| 3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2} \right|_v,$$

$$\sigma_3 = |f(\zeta_1)f(\zeta_2)f(\zeta_3)|_v = \left| \frac{b^2d^2}{2a} - d^3 - \frac{9bc^2d}{16a} + \frac{27c^4}{256a} - \frac{b^4d}{16a^2} + \frac{b^3c^2}{64a^2} \right|_v,$$

where v is either the usual archimedean absolute value $|\cdot|_\infty$ or one of the non-archimedean absolute values $|\cdot|_p$ for any prime p .

4.1 Bounds

Let

$$C_{f,v} = (8)_v \max \left\{ \left| \frac{b}{a} \right|_v^{\frac{1}{2}}, \left| \frac{c}{a} \right|_v^{\frac{1}{3}}, \left| \frac{d}{a} \right|_v^{\frac{1}{4}}, \left| \frac{1}{a} \right|_v^{\frac{1}{3}} \right\}$$

where, as in Chapter 3,

$$(8)_v = \begin{cases} 8, & \text{if } v \text{ is archimedean} \\ 1, & \text{otherwise} \end{cases}.$$

For non-archimedean absolute values, for any prime p , by the ultrametric inequality if

$$|f(\zeta_j)|_p \leq C_{f,p} \quad \forall j$$

then

$$\sigma_i \leq \max_j \{|f(\zeta_j)|_p\}^i \leq C_{f,p}^i \quad \forall i.$$

So if f is PCF, we must have

$$\max \{\sigma_1, \sqrt{\sigma_2}, \sqrt[3]{\sigma_3}\} \leq C_{f,p}.$$

We use this fact to obtain bounds on each of the coefficients.

Throughout the following lemmas, we will split into cases which generally result in a dominant term in the sums comprising the σ 's, and then we have strict equality in the ultrametric inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

Lemma 4.1. *If f is PCF, then there exists a conjugate of f which maintains the field of definition such that $d \in \{0, 1\}$, and $|a|_p, |b|_p, |c|_p \leq 1$ for all $p \neq 2, 3$.*

Proof. In each case, we start by supposing that f is PCF and $|a|_p > 1$. We obtain a contradiction, and then show that $|a|_p \leq 1 \implies |b|_p, |c|_p \leq 1$, to bound all three coefficients.

If $|c^2|_p > \left|\frac{b^3}{a}\right|_p$, then in σ_3 , $\left|\frac{27c^4}{256a}\right|_p > \left|\frac{b^6}{a^3}\right|_p$, so if $\left|\frac{b^2}{a}\right|_p \geq 1$, then this term dominates, and we have $\sigma_3 = \left|\frac{c^4}{a}\right|_p$. This fact is used in Cases 1.1 and 1.2 below.

Case 1: $d = 1$

So $\left|\frac{d}{a}\right|_p^{\frac{1}{4}} = \left|\frac{1}{a}\right|_p^{\frac{1}{4}} \geq \left|\frac{1}{a}\right|_p^{\frac{1}{3}}$, so $C_{f,p} = \max \left\{ \left|\frac{b}{a}\right|_p^{\frac{1}{2}}, \left|\frac{c}{a}\right|_p^{\frac{1}{3}}, \left|\frac{1}{a}\right|_p^{\frac{1}{4}} \right\}$.

Case 1.1: $\left|\frac{b^2}{a}\right|_p > 1 = |d|_p$

Under these hypotheses, we have

$$\sigma_1 = \left| \frac{b^2}{2a} - 3d \right|_p = \max \left\{ \left| \frac{b^2}{a} \right|_p, |3|_p \right\} = \left| \frac{b^2}{a} \right|_p.$$

First, suppose $|c^2|_p \leq \left|\frac{b^3}{a}\right|_p$, and so $C_{f,p} = \max \left\{ \left|\frac{b}{a}\right|_p^{\frac{1}{2}}, \left|\frac{1}{a}\right|_p^{\frac{1}{4}} \right\}$. Note that $\left|\frac{b^2}{a}\right|_p > 1 \implies |b|_p^2 > |a|_p \implies |b|_p > |a|_p^{\frac{1}{2}}$, so $\sigma_1 = \left|\frac{b^2}{a}\right|_p > \left|\frac{ba^{\frac{1}{2}}}{a}\right|_p = \left|\frac{b}{a^{\frac{1}{2}}}\right|_p > \left|\frac{b}{a}\right|_p^{\frac{1}{2}}$, since if $\left|\frac{b^2}{a}\right|_p > 1$ and $|a|_p \geq 1$, then $|b|_p > 1$. Also $|b|_p^2 > |a|_p \implies |b|_p^{\frac{3}{2}} > |a|_p^{\frac{3}{4}}$, so

$\sigma_1 = \left| \frac{b^2}{a} \right|_p > \left| \frac{b^{\frac{1}{2}} a^{\frac{3}{4}}}{a} \right|_p = \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_p > \left| \frac{1}{a} \right|_p^{\frac{1}{4}}$, so $\sigma_1 > C_{f,p}$ in this case.

Now suppose $|c^2|_p > \left| \frac{b^3}{a} \right|_p$, and so $\sigma_3 = \left| \frac{c^4}{a} \right|_p$ and $C_{f,p} = \max \left\{ \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{4}} \right\}$. First note that $|c^2|_p > \left| \frac{b^3}{a} \right|_p \implies |c|_p > \left| \frac{b^{\frac{3}{2}}}{a^{\frac{1}{2}}} \right|_p$, and $|b^2|_p > |a|_p \implies |b|_p > |a|_p^{\frac{1}{2}}$. Putting it together,

$$\sqrt[3]{\sigma_3} = \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p > \left| \frac{b^{\frac{3}{2}} c^{\frac{1}{3}}}{a^{\frac{5}{6}}} \right|_p > \left| \frac{b^{\frac{1}{2}} a^{\frac{1}{2}} c^{\frac{1}{3}}}{a^{\frac{5}{6}}} \right|_p = \left| \frac{b^{\frac{1}{2}} c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_p > \left| \frac{c}{a} \right|_p^{\frac{1}{3}}$$

(again since $|b|_p > 1$). Similarly,

$$\sqrt[3]{\sigma_3} = \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p > \left| \frac{b^2}{a} \right|_p > \left| \frac{b^{\frac{1}{2}} a^{\frac{3}{4}}}{a} \right|_p = \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_p > \left| \frac{1}{a} \right|_p^{\frac{1}{4}}.$$

So we have $\sqrt[3]{\sigma_3} > C_{f,p}$ in this case as well, and so we must have $|a|_p \leq 1$.

Case 1.2: $\left| \frac{b^2}{a} \right|_p = |d|_p = 1$

Case 1.2.1: $|c^2|_p > \left| \frac{b^3}{a} \right|_p = |b|_p$

Under these hypotheses, we have $\sigma_3 = \left| \frac{c^4}{a} \right|_p$.

So, this case looks identical to Case 1.1, since we still have $|c^2|_p > \left| \frac{b^3}{a} \right|_p$ and $|b|_p^2 \geq |a|_p$ (only they're equal now). So $|a|_p \leq 1$.

Case 1.2.2: $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p = |b|_p$

Since $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p \implies \left| \frac{c}{a} \right|_p^{\frac{1}{3}} \leq \left| \frac{b}{a} \right|_p^{\frac{1}{2}} = \left| \frac{1}{a} \right|_p^{\frac{1}{4}}$, we have $C_{f,p} = \left| \frac{1}{a} \right|_p^{\frac{1}{4}} < 1$.

By way of contradiction, suppose that

$$\left| \frac{b^2}{2a} - 3d \right|_p < 1, \quad \left| 3d^2 - \frac{b^2 d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2} \right|_p < 1, \text{ and}$$

$$\left| \frac{b^2 d^2}{2a} - d^3 - \frac{9bc^2 d}{16a} + \frac{27c^4}{256a} - \frac{b^4 d}{16a^2} + \frac{b^3 c^2}{64a^2} \right|_p < 1.$$

$$\left| \frac{b^2}{2a} - 3d \right|_p = \left| \frac{b^2}{2a} - 3 \right|_p < 1 \implies \frac{b^2}{2a} = 3 + \varepsilon \text{ and } \frac{b}{2a} = \frac{3 + \varepsilon}{b} \text{ where } |\varepsilon|_p < 1$$

$$\begin{aligned}
&\implies \left| 3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2} \right|_p = \left| 3 - 2(3+\varepsilon) + \frac{9bc^2}{16a} + \frac{(3+\varepsilon)^2}{4} \right|_p < 1 \\
&\implies \left| -\frac{3}{4} + \frac{9bc^2}{16a} \right|_p < 1 \text{ and } \left| -\frac{3}{4} + \frac{9(3+\varepsilon)c^2}{8b} \right|_p < 1 \\
&\implies \frac{9bc^2}{16a} = \delta + \frac{3}{4} \text{ and } c^2 = \left(\frac{8b}{9(3+\varepsilon)} \right) \left(\delta + \frac{3}{4} \right) \text{ where } |\delta|_p < 1.
\end{aligned}$$

$$\begin{aligned}
\sigma_3 &= \left| \frac{b^2d^2}{2a} - d^3 - \frac{9bc^2d}{16a} + \frac{27c^4}{256a} - \frac{b^4d}{16a^2} + \frac{b^3c^2}{64a^2} \right|_p \\
&= \left| (3+\varepsilon) - 1 - \frac{9bc^2}{16a} + \frac{27c^4}{256a} - \frac{(3+\varepsilon)^2}{4} + \frac{(3+\varepsilon)bc^2}{32a} \right|_p \\
&= \left| (3+\varepsilon) - 1 - \delta - \frac{3}{4} + \left(\frac{1}{6(3+\varepsilon)} \right) \left(\delta + \frac{3}{4} \right)^2 - \frac{(3+\varepsilon)^2}{4} + \left(\frac{3+\varepsilon}{18} \right) \left(\delta + \frac{3}{4} \right) \right|_p \\
&= \left| -1 + \varepsilon - \delta + \left(\frac{1}{6(3+\varepsilon)} \right) \left(\delta + \frac{3}{4} \right)^2 - \frac{3\varepsilon}{2} - \frac{\varepsilon^2}{4} + \frac{\delta}{6} + \frac{1}{8} + \frac{\varepsilon\delta}{18} + \frac{\varepsilon}{18} \right|_p < 1 \\
&\implies \left| -\frac{7}{8} + \frac{3}{32(3+\varepsilon)} \right|_p < 1
\end{aligned}$$

since clearly every other term in the sum has p -adic absolute value less than 1, and these 2 are the only terms with p -adic absolute value equal to 1.

$$\begin{aligned}
&\implies \frac{-7}{8} + \frac{3}{96 + 32pk} = pl \text{ for some } k, l \in \mathbb{Z} \\
&\implies p((32kl)p + (96l + 28k)) = -81 \\
&\implies p \mid 81 \implies p = 3
\end{aligned}$$

which is a contradiction, since $p \neq 2, 3$. So $\sigma_j = 1 > \left| \frac{1}{a} \right|_p^{\frac{1}{4}} = C_{f,p}$ for some j .

So $|a|_p \leq 1$. Then $|c^2|_p \leq |b|_p$ and $|b|_p^2 = |a|_p \implies |b|_p, |c|_p \leq 1$.

Case 1.3: $\left| \frac{b^2}{a} \right|_p < |d|_p = 1$

Under these hypotheses, we have $\sigma_1 = 1$.

First, suppose $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p$, and so $C_{f,p} = \max \left\{ \left| \frac{b}{a} \right|_p^{\frac{1}{2}}, \left| \frac{1}{a} \right|_p^{\frac{1}{4}} \right\}$. Note that $\left| \frac{b^2}{a} \right|_p < 1 \implies |b|_p^{\frac{1}{2}} < |a|_p^{\frac{1}{4}} \implies \left| \frac{b}{a} \right|_p^{\frac{1}{2}} < \left| \frac{a^{\frac{1}{4}}}{a^{\frac{1}{2}}} \right|_p = \left| \frac{1}{a^{\frac{1}{4}}} \right|_p < 1 = \sigma_1$. Also clearly $\left| \frac{1}{a} \right|_p^{\frac{1}{4}} < 1 = \sigma_1$ since $|a|_p > 1$, so $\sigma_1 > C_{f,p}$ in this case.

Now suppose $|c^2|_p > \left| \frac{b^3}{a} \right|_p$, and so $C_{f,p} = \max \left\{ \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{4}} \right\}$. First note that if $|c|_p < |a|_p$, then $\left| \frac{c}{a} \right|_p^{\frac{1}{3}} < 1 = \sigma_1$, so we have $\sigma_1 > C_{f,p}$. So we can assume that $|c|_p \geq |a|_p > 1$. Then we have that $\sigma_3 = \left| \frac{c^4}{a} \right|_p$, since under these hypotheses it dominates all other terms in the sum comprising σ_3 . Then $\sqrt[3]{\sigma_3} = \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p > \left| \frac{c}{a} \right|_p^{\frac{1}{3}}$ since $|c|_p > 1$, and $\left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p \geq |c|_p > \left| \frac{1}{a} \right|_p^{\frac{1}{4}}$ since $|a|_p > 1$, so $\sqrt[3]{\sigma_3} > C_{f,p}$ in this case as well, and so $|a|_p \leq 1$.

If $|a|_p \leq 1$ and $\left| \frac{b^2}{a} \right|_p < 1$, then $|b|_p < 1$. Then $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p < |b|_p \implies |c|_p < 1$. Finally, suppose $|c^2|_p > \left| \frac{b^3}{a} \right|_p$ and $|c|_p > 1$. Then $\sigma_3 = 256 \left| \frac{c^4}{a} \right|_p$, so $\sqrt[3]{\sigma_3} = \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p > \left| \frac{c}{a} \right|_p^{\frac{1}{3}} > \left| \frac{1}{a} \right|_p^{\frac{1}{4}}$, so we have $\sqrt[3]{\sigma_3} > C_{f,p}$. So $|c|_p \leq 1$.

Case 2: $d = 0$

So, $\left| \frac{d}{a} \right|_p^{\frac{1}{4}} = 0$, so $C_{f,p} = \max \left\{ \left| \frac{b}{a} \right|_p^{\frac{1}{2}}, \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}$, and $\sigma_1 = \left| \frac{b^2}{a} \right|_p$.

In this case, we can conjugate to clear the denominator of a , so $|a|_p \leq 1$.

Finally, we prove that $|b|_p, |c|_p \leq 1$ in those cases where it has not already been shown. In Cases 1.1 and 2, we have $\sigma_1 = \left| \frac{b^2}{a} \right|_p$, so:

First, suppose $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p$, and so $C_{f,p} = \max \left\{ \left| \frac{b}{a} \right|_p^{\frac{1}{2}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}$.

$$\begin{aligned} \sigma_1 = \left| \frac{b^2}{a} \right|_p > C_{f,p} &\iff \left| \frac{b^2}{a} \right|_p > \left| \frac{b}{a} \right|_p^{\frac{1}{2}} \text{ and } \left| \frac{b^2}{a} \right|_p > \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \\ &\iff |b|_p^3 > |a|_p. \end{aligned}$$

So, f PCF $\implies |a|_p \geq |b|_p^3 \implies |b|_p \leq 1$. So $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p \leq 1 \implies |c|_p \leq 1$.

In Cases 1.1, 1.21, and 2, we have $|c^2|_p > \left| \frac{b^3}{a} \right|_p \implies \sigma_3 = \left| \frac{c^4}{a} \right|_p$ (all have been shown above except Case 2, which is clear since $d = 0 \implies \sigma_3 = \max \left\{ \left| \frac{27c^4}{256a} \right|_p, \left| \frac{b^3c^2}{64a^2} \right|_p \right\}$) and $C_{f,p} = \max \left\{ \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}$. First note that if $|c|_p > 1$, then $\sqrt[3]{\sigma_3} = \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p > \left| \frac{c}{a} \right|_p^{\frac{1}{3}} > \left| \frac{1}{a} \right|_p^{\frac{1}{3}}$, so we have $\sqrt[3]{\sigma_3} > C_{f,p}$. So we can assume that $|c|_p \leq 1$, so $C_{f,p} = \left| \frac{1}{a} \right|_p^{\frac{1}{3}}$. Then in Cases 1.1 and 2 we have $\sigma_1 = \left| \frac{b^2}{a} \right|_p \leq C_{f,p} \iff |b|_p^3 \leq |a|_p \implies |b|_p \leq 1$. In Case 1.21, $|c^2|_p > |b|_p \implies |b|_p < 1$.

So we have shown that after a change of variables, in all cases $|a|_p, |b|_p, |c|_p \leq 1$. □

Lemma 4.2. *If f is PCF, and $p \neq 2, 3$, then $|a|_p < 1 \implies |b|_p < 1$.*

Proof. First, note that since our coefficients are in \mathbb{Q} , $|b|_p < 1 \implies |b|_p \leq \frac{1}{p}$. We will write this proof based around the cases of Lemma 4.1. Note that we already have $|b|_p \leq 1$. Suppose $|a|_p < 1$ and $|b|_p = 1$.

In Case 1.1, we have

$$C_{f,p} = \left\{ \left| \frac{b}{a} \right|_p^{\frac{1}{2}}, \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}, \quad \sigma_1 = \left| \frac{b^2}{a} \right|_p.$$

If $|c|_p < \frac{1}{|a|_p^2}$, then $\sigma_1 = \left| \frac{b^2}{a} \right|_p = \left| \frac{1}{a} \right|_p > \left| \frac{b}{a} \right|_p^{\frac{1}{2}} > \left| \frac{1}{a} \right|_p^{\frac{1}{3}}$. Then, $|c|_p < \frac{1}{|a|_p^2} \implies \left| \frac{c}{a} \right|_p^{\frac{1}{3}} < \left| \frac{1}{a} \right|_p^{\frac{1}{3}}$. So $\sigma_1 > C_{f,p}$.

If $|c|_p \geq \frac{1}{|a|_p^2}$, then $|c|_p > 1 \geq |b|_p$, and

$$\sigma_2 = \max \left\{ |3d^2|, \left| \frac{b^2 d}{a} \right|, \left| \frac{9bc^2}{16a} \right|, \left| \frac{b^4}{16a^2} \right| \right\} = \max \left\{ 1, \left| \frac{1}{a} \right|, \left| \frac{c^2}{a} \right|, \left| \frac{1}{a^2} \right| \right\}.$$

So, $\sqrt{\sigma_2} = \sqrt{\left| \frac{c^2}{a} \right|_p} = \left| \frac{c}{a^{\frac{1}{2}}} \right|_p \geq \left| \frac{c^{\frac{1}{3}}}{a^{\frac{1}{6}}} \right|_p > \left| \frac{c}{a} \right|_p^{\frac{1}{3}}$. We also have $\sqrt{\left| \frac{c^2}{a} \right|_p} = \left| \frac{c}{a^{\frac{1}{2}}} \right|_p > \left| \frac{b}{a} \right|_p^{\frac{1}{2}} = \left| \frac{1}{a} \right|_p^{\frac{1}{2}} > \left| \frac{1}{a} \right|_p^{\frac{1}{3}}$. So $\sqrt{\sigma_2} > C_{f,p}$.

So in this case we have $|a|_p < 1 \implies |b|_p \leq \frac{1}{p}$.

In Cases 1.2 and 1.3, we have $\left| \frac{b^2}{a} \right|_p \leq 1$, so to satisfy the condition $|a|_p \geq |b|_p^2$ with $|a|_p \leq \frac{1}{p}$ requires at least $|b|_p \leq \frac{1}{p}$.

In Case 2, $d = 0$. If $\left| \frac{b^2}{a} \right|_p > 1$, then we use the same proof from Case 1.1.

If $\left| \frac{b^2}{a} \right|_p \leq 1$, we have $|b|_p \leq \frac{1}{p}$. □

Lemma 4.3. *If f is PCF, then there exists a conjugate of f which maintains the field of definition such that $d \in \{0, 1\}$, $|a|_2 \leq 1$, $|b|_2 \leq \frac{1}{2}$, and $|c|_2 \leq \frac{1}{4}$.*

Proof. In each case, we start by supposing that f is PCF and $|a|_2 > 1$. We obtain a contradiction, and then show that $|a|_2 \leq 1 \implies |b|_2 \leq \frac{1}{2}$ and $|c|_2 \leq \frac{1}{4}$, to bound all three coefficients.

If $|c^2|_2 \geq \left| \frac{b^3}{a} \right|_2$, then in σ_3 , $\left| \frac{27c^4}{256a} \right|_2 > 256 \left| \frac{b^6}{a^3} \right|_2$, so if $\left| \frac{b^2}{a} \right|_2 \geq \frac{1}{2}$, then this term dominates, and we have $\sigma_3 = \left| \frac{c^4}{a} \right|_2$. This fact is used in Cases 1.1 and 1.2 below.

Case 1: $d = 1$

So, $\left| \frac{d}{a} \right|_2^{\frac{1}{4}} = \left| \frac{1}{a} \right|_2^{\frac{1}{4}} \geq \left| \frac{1}{a} \right|_2^{\frac{1}{3}}$, so $C_{f,2} = \max \left\{ \left| \frac{b}{a} \right|_2^{\frac{1}{2}}, \left| \frac{c}{a} \right|_2^{\frac{1}{3}}, \left| \frac{1}{a} \right|_2^{\frac{1}{4}} \right\}$.

Case 1.1: $2 \left| \frac{b^2}{a} \right|_2 > 1 = |d|_2$

Under these hypotheses, we have

$$\sigma_1 = \left| \frac{b^2}{2a} - 3d \right|_2 = \max \left\{ 2 \left| \frac{b^2}{a} \right|_2, |3|_2 \right\} = 2 \left| \frac{b^2}{a} \right|_2.$$

First, suppose $|c^2|_2 < \left| \frac{b^3}{a} \right|_2$, and so $C_{f,2} = \max \left\{ \left| \frac{b}{a} \right|_2^{\frac{1}{2}}, \left| \frac{1}{a} \right|_2^{\frac{1}{4}} \right\}$. Note that $\left| \frac{b^2}{a} \right|_2 > \frac{1}{2} \implies |b|_2^2 > \frac{1}{2}|a|_2 \implies |b|_2 > \frac{1}{\sqrt{2}}|a|_2^{\frac{1}{2}}$, so $\sigma_1 = \left| \frac{b^2}{2a} \right|_2 > \sqrt{2} \left| \frac{ba^{\frac{1}{2}}}{a} \right|_2 = \sqrt{2} \left| \frac{b}{a^{\frac{1}{2}}} \right|_2 > \left| \frac{b}{a} \right|_2^{\frac{1}{2}}$, since if $\left| \frac{b^2}{a} \right|_2 \geq 1$ and $|a|_2 \geq 1$, then $|b|_2 \geq 1$. Also $|b|_2^2 > \frac{1}{2}|a|_2 \implies |b|_2^{\frac{3}{2}} > \frac{1}{2^{\frac{3}{4}}}|a|_2^{\frac{3}{4}}$, so $\sigma_1 = 2 \left| \frac{b^2}{a} \right|_2 > 2^{\frac{1}{4}} \left| \frac{b^{\frac{1}{2}}a^{\frac{3}{4}}}{a} \right|_2 = 2^{\frac{1}{4}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_2 > \left| \frac{1}{a} \right|_2^{\frac{1}{4}}$, so $\sigma_1 > C_{f,2}$ in this case.

Now suppose $|c^2|_2 \geq \left| \frac{b^3}{a} \right|_2$, and so $\sigma_3 = 256 \left| \frac{c^4}{a} \right|_2$ and $C_{f,2} = \max \left\{ \left| \frac{c}{a} \right|_2^{\frac{1}{3}}, \left| \frac{1}{a} \right|_2^{\frac{1}{4}} \right\}$. First note that $|c^2|_2 \geq \left| \frac{b^3}{a} \right|_2 \implies |c|_2 \geq \left| \frac{b^{\frac{3}{2}}}{a^{\frac{1}{2}}} \right|_2$, and $|b^2|_2 > \frac{1}{2}|a|_2 \implies |b|_2 > \frac{1}{\sqrt{2}}|a|_2^{\frac{1}{2}}$. Putting it together, $\sqrt[3]{\sigma_3} = 2^{\frac{8}{3}} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_2 \geq 2^{\frac{8}{3}} \left| \frac{b^{\frac{3}{2}}c^{\frac{1}{3}}}{a^{\frac{5}{6}}} \right|_2 > 2^{\frac{13}{6}} \left| \frac{b^{\frac{1}{2}}a^{\frac{1}{2}}c^{\frac{1}{3}}}{a^{\frac{5}{6}}} \right|_2 = 2^{\frac{13}{6}} \left| \frac{b^{\frac{1}{2}}c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_2 > \left| \frac{c}{a} \right|_2^{\frac{1}{3}}$ (again since $|b|_2 \geq 1$). Similarly, $\sqrt[3]{\sigma_3} = 2^{\frac{8}{3}} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_2 \geq 2^{\frac{8}{3}} \left| \frac{b^2}{a} \right|_2 > 2^{\frac{23}{12}} \left| \frac{b^{\frac{1}{2}}a^{\frac{3}{4}}}{a} \right|_2 = 2^{\frac{23}{12}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_2 > \left| \frac{1}{a} \right|_2^{\frac{1}{4}}$. So we have $\sqrt[3]{\sigma_3} > C_{f,2}$ in this case as well, and so we must have $|a|_2 \leq 1$.

Case 1.2: $2 \left| \frac{b^2}{a} \right|_2 = |d|_2 = 1$

Case 1.2.1: $|c^2|_2 \geq \left| \frac{b^3}{a} \right|_2 = \frac{1}{2}|b|_2$

Under these hypotheses, we have $\sigma_3 = 256 \left| \frac{c^4}{a} \right|_2$.

So, this case looks identical to Case 1.1, since we still have $|c^2|_2 \geq \left|\frac{b^3}{a}\right|_2$, $|b|_2^2 \geq \frac{1}{2}|a|_2$ (only they're equal now), and $\sigma_3 = 256 \left|\frac{c^4}{a}\right|_2$, and if $\left|\frac{b^2}{a}\right|_2 = \frac{1}{2}$ and $|a|_2 \geq 1$, then $|b|_2^2 \geq \frac{1}{2} \implies |b|_2 \geq \frac{1}{\sqrt{2}} \implies |b|_2 \geq 1$ (since $b \in \mathbb{Q}$) $\implies |a|_2 \leq 1$ (by the argument of Case 1.1).

So since $\left|\frac{b^2}{a}\right|_2 = \frac{1}{2}$, $|b|_2^2 \leq \frac{1}{2} \implies |b|_2 \leq \frac{1}{\sqrt{2}}$. It remains to bound $|c|_2$.

Case 1.2.2: $|c^2|_2 < \left|\frac{b^3}{a}\right|_2 = \frac{1}{2}|b|_2$

Under these hypotheses, we have

$$\sigma_2 = \left|3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2}\right|_2 = \max\left\{1, \frac{1}{2}, 16\left|\frac{bc^2}{a}\right|_2, 16\left|\frac{b^4}{a^2}\right|_2\right\} = 16\left|\frac{b^4}{a^2}\right|_2 = 4.$$

So $C_{f,2} = \max\left\{\left|\frac{b}{a}\right|_2^{\frac{1}{2}}, \left|\frac{1}{a}\right|_2^{\frac{1}{4}}\right\}$. As in case 1.2.1, we have $|b|_2 \geq 1$ since $|a|_2 \geq 1$. $\sqrt{\sigma_2} = 2^2 \left|\frac{b^2}{a}\right|_2 = 2^{\frac{3}{2}} \left|\frac{ba^{\frac{1}{2}}}{a}\right|_2 = 2^{\frac{3}{2}} \left|\frac{b}{a^{\frac{1}{2}}}\right|_2 > \left|\frac{b}{a}\right|_2^{\frac{1}{2}}$, and $\sqrt{\sigma_2} = 2^2 \left|\frac{b^2}{a}\right|_2 = 2^{\frac{5}{4}} \left|\frac{b^{\frac{1}{2}}a^{\frac{3}{4}}}{a}\right|_2 = 2^{\frac{5}{4}} \left|\frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}}\right|_2 > \left|\frac{1}{a}\right|_2^{\frac{1}{4}}$, and so $\sqrt{\sigma_2} > C_{f,2}$.

So $|a|_2 \leq 1$, and so $|b|_2 \leq \frac{1}{2}$, as in Case 1.2.1. It remains to bound $|c|_2$.

Case 1.3: $2\left|\frac{b^2}{a}\right|_2 < |d|_2 = 1$

Under these hypotheses, we have

$$\sigma_1 = \left|\frac{b^2}{2a} - 3d\right|_2 = \max\left\{2\left|\frac{b^2}{a}\right|_2, |3|_2\right\} = 1.$$

First, suppose $|c^2|_2 \leq \left|\frac{b^3}{a}\right|_2$, and so $C_{f,2} = \max\left\{\left|\frac{b}{a}\right|_2^{\frac{1}{2}}, \left|\frac{1}{a}\right|_2^{\frac{1}{4}}\right\}$. Note that $\left|\frac{b^2}{a}\right|_2 < \frac{1}{2} \implies |b|_2^{\frac{1}{2}} < \frac{1}{2^{\frac{1}{4}}}|a|_2^{\frac{1}{4}} \implies \left|\frac{b}{a}\right|_2^{\frac{1}{2}} < \frac{1}{2^{\frac{1}{4}}}\left|\frac{a^{\frac{1}{4}}}{a^{\frac{1}{2}}}\right|_2 = \frac{1}{2^{\frac{1}{4}}}\left|\frac{1}{a^{\frac{1}{4}}}\right|_2 < 1 = \sigma_1$ (since $|a|_2 > 1$). Also clearly $\left|\frac{1}{a}\right|_2^{\frac{1}{4}} < 1 = \sigma_1$ since $|a|_2 > 1$, so $\sigma_1 > C_{f,2}$ in this case.

Now suppose $|c^2|_2 > \left|\frac{b^3}{a}\right|_2$, and so $C_{f,2} = \max\left\{\left|\frac{c}{a}\right|_2^{\frac{1}{3}}, \left|\frac{1}{a}\right|_2^{\frac{1}{4}}\right\}$. First note that if $|c|_2 < |a|_2$, then $\left|\frac{c}{a}\right|_2^{\frac{1}{3}} < 1 = \sigma_1$, so we have $\sigma_1 > C_{f,2}$. So we can assume that $|c|_2 \geq |a|_2 > 1$. Then we have that $\sigma_3 = 256 \left|\frac{c^4}{a}\right|_2$, since it exceeds all other terms in the sum comprising σ_3 (for instance, $256 \left|\frac{c^4}{a}\right|_2 > 256 \left|\frac{bc^{\frac{10}{3}}}{a^{\frac{8}{3}}}\right|_2 \geq 256 \left|\frac{ba^{\frac{2}{3}}c^{\frac{8}{3}}}{a^{\frac{8}{3}}}\right|_2 = 256 \left|\frac{bc^{\frac{8}{3}}}{a}\right|_2 > 16 \left|\frac{bc^2}{a}\right|_2$). Then $\sqrt[3]{\sigma_3} = 2^{\frac{8}{3}} \left|\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}\right|_2 > \left|\frac{c}{a}\right|_2^{\frac{1}{3}}$ since $|c|_2 > 2$, and $2^{\frac{8}{3}} \left|\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}\right|_2 \geq |c|_2 > \left|\frac{1}{a}\right|_2^{\frac{1}{4}}$ since $|a|_2 > 1$, so

$\sqrt[3]{\sigma_3} > C_{f,2}$ in this case as well, and so $|a|_2 \leq 1$.

If $|a|_2 \leq 1$ and $\left|\frac{b^2}{a}\right|_2 < \frac{1}{2}$, then $|b|_2 \leq \frac{1}{2}$. Then $|c^2|_2 \leq \left|\frac{b^3}{a}\right|_2 < \frac{1}{2}|b|_2 \implies |c^2|_2 < \frac{1}{4} \implies |c|_2 \leq \frac{1}{4}$. It remains to bound $|c|_2$ when $|c^2|_2 > \left|\frac{b^3}{a}\right|_2$.

Case 2: $d = 0$

So, $\left|\frac{d}{a}\right|_2^{\frac{1}{4}} = 0$, so $C_{f,2} = \max\left\{\left|\frac{b}{a}\right|_2^{\frac{1}{2}}, \left|\frac{c}{a}\right|_2^{\frac{1}{3}}, \left|\frac{1}{a}\right|_2^{\frac{1}{3}}\right\}$, and $\sigma_1 = \left|\frac{b^2}{a}\right|_2$.

In this case, we can conjugate to clear the denominator of a , so $|a|_2 \leq 1$.

Finally, we prove that $|b|_2, |c|_2 \leq 1$ in those cases where it has not already been shown. In Cases 1.1 and 2 if $|c^2|_2 < \left|\frac{b^3}{a}\right|_2$, then

$$\sigma_2 = \max\left\{|3d^2|_2, \left|\frac{b^2d}{a}\right|_2, 16\left|\frac{bc^2}{a}\right|_2, 16\left|\frac{b^4}{a^2}\right|_2\right\} = 16\left|\frac{b^4}{a^2}\right|_2$$

and $C_{f,2} = \max\left\{\left|\frac{b}{a}\right|_2^{\frac{1}{2}}, \left|\frac{1}{a}\right|_2^{\frac{1}{3}}\right\}$. So

$$\begin{aligned} \sqrt{\sigma_2} = 4\left|\frac{b^2}{a}\right|_2 > C_{f,2} &\iff 4\left|\frac{b^2}{a}\right|_2 > \left|\frac{b}{a}\right|_2^{\frac{1}{2}} \text{ and } 4\left|\frac{b^2}{a}\right|_2 > \left|\frac{1}{a}\right|_2^{\frac{1}{3}} \\ &\iff 16|b|_2^3 > |a|_2 \text{ and } 8|b|_2^3 > |a|_2 \\ &\iff 8|b|_2^3 > |a|_2. \end{aligned}$$

So, f PCF $\implies |a|_2 \geq 8|b|_2^3 \implies |b|_2^3 \leq \frac{1}{8} \implies |b|_2 \leq \frac{1}{2}$. Now in all cases, $|c^2|_2 \leq \left|\frac{b^3}{a}\right|_2 \leq \frac{1}{8} \implies |c|_2 \leq \frac{1}{2^{\frac{3}{2}}} \implies |c|_2 \leq \frac{1}{4}$.

When $|c^2|_2 \geq \left|\frac{b^3}{a}\right|_2$, we have $C_{f,2} = \max\left\{\left|\frac{c}{a}\right|_2^{\frac{1}{3}}, \left|\frac{1}{a}\right|_2^{\frac{1}{3}}\right\}$ and if $|c|_2 > 1$ and $|a|_2 \leq 1$, we have $\sigma_3 = 256\left|\frac{c^4}{a}\right|_2$. First note that if $|c|_2 \geq \frac{1}{4}$, then $\sqrt[3]{\sigma_3} = 2^{\frac{8}{3}}\left|\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}\right|_2 \geq 2^{\frac{2}{3}}\left|\frac{c}{a}\right|_2^{\frac{1}{3}} > \left|\frac{1}{a}\right|_2^{\frac{1}{3}}$, so we have $\sqrt[3]{\sigma_3} > C_{f,2}$. So we can assume that $|c|_2 \leq \frac{1}{8}$, so $C_{f,2} = \left|\frac{1}{a}\right|_2^{\frac{1}{3}}$. Then $\sigma_1 = 2\left|\frac{b^2}{a}\right|_2 \leq C_{f,2} \iff 2^{\frac{3}{2}}|b|_2^3 \leq |a|_2 \implies |b|_2 \leq \frac{1}{2}$.

So in all cases we have shown $|a|_2 \leq 1$, $|b|_2 \leq \frac{1}{2}$, and $|c|_2 \leq \frac{1}{4}$.

□

Lemma 4.4. *If f is PCF, then $|a|_2 \leq \frac{1}{2} \implies |b|_2 \leq \frac{1}{4}$.*

Proof. This proof follows a similar strategy to Lemma 4.2, using the cases from Lemma 4.3

here.

Suppose that $|a|_2 \leq \frac{1}{2}$ and $|b|_2 = \frac{1}{2}$ (since we already have $|b|_2 \leq \frac{1}{2}$, so we only need to rule out equality, and then we conclude $|b|_2 \leq \frac{1}{4}$).

Note that $2 \left| \frac{b^2}{a} \right|_2 \geq 1$, and so Case 1.3 does not occur.

In Case 1.1 $\left(2 \left| \frac{b^2}{a} \right|_2 > 1 \right)$, we have $|a|_2 \leq \frac{1}{4}$. $\sigma_1 = 2 \left| \frac{b^2}{a} \right|_2 = \frac{1}{2} \left| \frac{1}{a} \right|_2 > C_{f,2} = \max \left\{ \left| \frac{b}{a} \right|_2^{\frac{1}{2}}, \left| \frac{1}{a} \right|_2^{\frac{1}{4}} \right\}$, so f is not PCF, a contradiction.

For Case 1.2 $\left(2 \left| \frac{b^2}{a} \right|_2 = 1 \right)$, we have $|a|_2 = \frac{1}{2}$. Case 1.2.1 never occurs, since $|c|_2 \leq \frac{1}{4} \implies \frac{1}{16} \geq |c^2|_2 \geq \left| \frac{b^3}{a} \right|_2 = \frac{1}{2} |b|_2 = \frac{1}{4}$, a contradiction. In Case 1.2.2, we have $C_{f,2} = 2^{\frac{1}{3}} < 2 = 4 \left| \frac{b^2}{a} \right|_2 = \sqrt{\sigma_2}$.

In Case 2, when $|c^2|_2 < \left| \frac{b^3}{a} \right|_2$, we have $|a|_2 \geq 8|b|_2^3 \implies |b|_2^3 \leq \frac{1}{16} \implies |b|_2 \leq \frac{1}{4}$. We never have that $|c^2|_2 \geq \left| \frac{b^3}{a} \right|_2$, since $\left| \frac{b^3}{a} \right|_2 \geq \frac{1}{4} > \frac{1}{16} \geq |c^2|_2$.

□

Lemma 4.5. *If f is PCF, then there exists a conjugate of f which maintains the field of definition such that $d \in \{0, 1\}$, and either $|a|_3 = 3^9$, $|b|_3 = 3^4$, and $|c|_3 = 3^3$, or $|a|_3 \leq 3^5$, $|b|_3 \leq 3^2$, and $|c|_3 \leq 3^2$.*

Proof. In the following, we start by supposing that f is PCF and $|a|_3 > 3^5$. We obtain a contradiction in most cases, and then show that $|a|_3 \leq 3^5$ implies the desired bounds on $|b|_3$ and $|c|_3$. In one narrow case (1.2.3 below), we get the unfortunate bounds $|a|_3 \leq 3^9 \implies |b|_3 \leq 3^4$ and $|c|_3 \leq 3^3$.

Case 1: $d = 1$

So, $\left| \frac{d}{a} \right|_3^{\frac{1}{4}} = \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \geq \left| \frac{1}{a} \right|_3^{\frac{1}{3}}$, so $C_{f,3} = \max \left\{ \left| \frac{b}{a} \right|_3^{\frac{1}{2}}, \left| \frac{c}{a} \right|_3^{\frac{1}{3}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$.

Case 1.1: $\left| \frac{b^2}{a} \right|_3 > \frac{1}{3} = |3d|_3$

Under these hypotheses, we have

$$\sigma_1 = \left| \frac{b^2}{2a} - 3d \right|_2 = \max \left\{ \left| \frac{b^2}{a} \right|_2, \frac{1}{3} \right\} = \left| \frac{b^2}{a} \right|_2.$$

First, suppose $|c^2|_3 \leq \left| \frac{b^3}{a} \right|_3$, and so $C_{f,3} = \max \left\{ \left| \frac{b}{a} \right|_3^{\frac{1}{2}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$. Note that $\left| \frac{b^2}{a} \right|_3 > \frac{1}{3} \implies |b|_3^2 > \frac{1}{3}|a|_3 \implies |b|_3^2 \geq |a|_3$, so $\sigma_1 = \left| \frac{b^2}{a} \right|_3 \geq \left| \frac{ba^{\frac{1}{2}}}{a} \right|_3 = \left| \frac{b}{a^{\frac{1}{2}}} \right|_3 \geq \sqrt{3} \left| \frac{b}{a} \right|_3^{\frac{1}{2}} > \left| \frac{b}{a} \right|_3^{\frac{1}{2}}$, since if $\left| \frac{b^2}{a} \right|_3 \geq 1$ and $|a|_3 > 1$, then $|b|_3 \geq 3$. Also $|b|_3^2 \geq |a|_3 \implies |b|_3^{\frac{3}{2}} \geq |a|_3^{\frac{3}{4}}$, so $\sigma_1 = \left| \frac{b^2}{a} \right|_3 \geq \left| \frac{b^{\frac{1}{2}} a^{\frac{3}{4}}}{a} \right|_3 = \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_3 \geq \sqrt{3} \left| \frac{1}{a} \right|_3^{\frac{1}{4}} > \left| \frac{1}{a} \right|_3^{\frac{1}{4}}$, so $\sigma_1 > C_{f,3}$ in this case.

Next suppose $3 \left| \frac{b^3}{a} \right|_3 \leq |c^2|_3 \leq 9 \left| \frac{b^3}{a} \right|_3$, and so under these hypotheses

$$\sigma_3 = \max \left\{ \left| \frac{b^2 d^2}{2a} \right|_3, |d^3|_3, \left| \frac{9bc^2 d}{16a} \right|_3, \left| \frac{27c^4}{256a} \right|_3, \left| \frac{b^4 d}{16a^2} \right|_3, \left| \frac{b^3 c^2}{64a^2} \right|_3 \right\} = \left| \frac{b^3 c^2}{a^2} \right|_3.$$

$C_{f,3} = \max \left\{ \left| \frac{c}{a} \right|_3^{\frac{1}{3}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$. We have $\sqrt[3]{\sigma_3} = \left| \frac{bc^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right|_3 \geq 3^{\frac{1}{6}} \left| \frac{b^{\frac{2}{3}} c^{\frac{1}{3}}}{a^{\frac{2}{3}}} \right|_3 \geq 3^{\frac{1}{6}} \left| \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{3}}}{a^{\frac{2}{3}}} \right|_3 = 3^{\frac{1}{6}} \left| \frac{b^{\frac{1}{2}} c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_3 \geq 3^{\frac{1}{3}} \left| \frac{c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}}$. Also, $\sqrt[3]{\sigma_3} = \left| \frac{bc^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right|_3 \geq 3^{\frac{1}{3}} \left| \frac{b^2}{a} \right|_3 \geq 3^{\frac{1}{3}} \left| \frac{a^{\frac{3}{4}} b^{\frac{1}{2}}}{a} \right|_3 = 3^{\frac{1}{3}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_3 > \left| \frac{1}{a} \right|_3^{\frac{1}{4}}$ (since if $\left| \frac{b^2}{a} \right|_3 > \frac{1}{3}$ and $|a|_3 > 1$, then $|b|_3 > 1$), and so $\sqrt[3]{\sigma_3} > C_{f,3}$.

Now suppose $|c^2|_3 > 9 \left| \frac{b^3}{a} \right|_3$, and so under these hypotheses

$$\sigma_2 = \max \left\{ |3d^2|_3, \left| \frac{b^2 d}{a} \right|_3, \left| \frac{9bc^2}{16a} \right|_3, \left| \frac{b^4}{16a^2} \right|_3 \right\} = \frac{1}{9} \left| \frac{bc^2}{a} \right|_3.$$

$C_{f,3} = \max \left\{ \left| \frac{c}{a} \right|_3^{\frac{1}{3}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$. We have $\sqrt{\sigma_2} = \frac{1}{3} \left| \frac{b^{\frac{1}{2}} c}{a^{\frac{1}{2}}} \right|_3 \geq 3^{-\frac{1}{3}} \left| \frac{b^{\frac{2}{3}} c^{\frac{1}{3}}}{a^{\frac{2}{3}}} \right|_3 \geq 3^{-\frac{1}{3}} \left| \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{3}}}{a^{\frac{2}{3}}} \right|_3 = 3^{-\frac{1}{3}} \left| \frac{b^{\frac{1}{2}} c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_3 \geq 3^{\frac{1}{6}} \left| \frac{c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}}$. Also, $\sqrt{\sigma_2} = \frac{1}{3} \left| \frac{b^{\frac{1}{2}} c}{a^{\frac{1}{2}}} \right|_3 > \left| \frac{b^2}{a} \right|_3 \geq \left| \frac{a^{\frac{3}{4}} b^{\frac{1}{2}}}{a} \right|_3 = \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_3 > \left| \frac{1}{a} \right|_3^{\frac{1}{4}}$ (since if $\left| \frac{b^2}{a} \right|_3 > \frac{1}{3}$ and $|a|_3 > 1$, then $|b|_3 > 1$), and so $\sqrt{\sigma_2} > C_{f,3}$.

So $|a|_3 \leq 1$. So $\left| \frac{1}{a} \right|_3^{\frac{1}{3}} \geq \left| \frac{1}{a} \right|_3^{\frac{1}{4}}$, changing $C_{f,3}$ in the following argument. So, when $|c^2|_3 \leq \left| \frac{b^3}{a} \right|_3$, we have

$$\sigma_1 = \left| \frac{b^2}{a} \right|_2 > C_{f,3} \iff \left| \frac{b^2}{a} \right|_3 > \left| \frac{b}{a} \right|_3^{\frac{1}{2}} \text{ and } \left| \frac{b^2}{a} \right|_3 > \left| \frac{1}{a} \right|_3^{\frac{1}{3}} \iff |b|_3^3 > |a|_3.$$

So f PCF $\implies |a|_3 \geq |b|_3^3 \implies |b|_3 \leq 1$, and $|c^2|_3 \leq \left| \frac{b^3}{a} \right|_3 \leq 1 \implies |c|_3 \leq 1$.

If $3 \left| \frac{b^3}{a} \right|_3 \leq |c^2|_3 \leq 9 \left| \frac{b^3}{a} \right|_3$, then first note if $|c|_3 \geq 3$, then $\sqrt[3]{\sigma_3} = \left| \frac{bc^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right|_3 \geq \frac{1}{3^{\frac{2}{3}}} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_3 \geq 3^{\frac{1}{3}} \left| \frac{c}{a} \right|_3^{\frac{1}{3}} > \left| \frac{c}{a} \right|_3^{\frac{1}{3}} \geq \left| \frac{1}{a} \right|_3^{\frac{1}{3}}$, so we must have $|c|_3 \leq 1$, and so $|b|_3 \leq \frac{1}{3}$.

Finally, if $|c^2|_3 \geq 27 \left| \frac{b^3}{a} \right|_3$, then if $|b|_3 \geq 3$, $\sqrt{\sigma_2} = \frac{1}{3} \left| \frac{b^{\frac{1}{2}}c}{a^{\frac{1}{2}}} \right|_3 \geq \left| \frac{b^{\frac{3}{2}}c^{\frac{1}{3}}}{a^{\frac{5}{6}}} \right|_3 \geq \left| \frac{b^{\frac{1}{2}}c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}}$, and also $\sqrt{\sigma_2} = \frac{1}{3} \left| \frac{b^{\frac{1}{2}}c}{a^{\frac{1}{2}}} \right|_3 \geq \sqrt{3} \left| \frac{b^2}{a} \right|_3 \geq \sqrt{3} \left| \frac{b^{\frac{2}{3}}}{a^{\frac{1}{3}}} \right|_3 > \left| \frac{1}{a} \right|_3^{\frac{1}{3}}$, so we must have $|b|_3 \leq 1$. Then if $|c|_3 > 3^{\frac{3}{2}} |a|_3^{-\frac{1}{8}} \implies |c|_3^{\frac{2}{3}} > 3 |a|_3^{-\frac{1}{12}}$ and $|c|_3 > 3$, $\sqrt{\sigma_2} = \frac{1}{3} \left| \frac{b^{\frac{1}{2}}c}{a^{\frac{1}{2}}} \right|_3 \geq \frac{1}{3} \left| \frac{c}{a^{\frac{1}{4}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}} > \left| \frac{1}{a} \right|_3^{\frac{1}{3}}$, so we must have $|c|_3 \leq 3^{\frac{3}{2}} |a|_3^{-\frac{1}{8}} \leq 3^3$, since by the lemma below, $|a| \leq 655211 \implies |a|_3 \geq 3^{-12}$. So $|c|_3 \leq 27$ (although this only occurs at $|a| = 531441$: otherwise we have $|a|_3 \geq 3^{-11}$, and so $|c|_3 \leq 3^{\frac{23}{8}} \implies |c|_3 \leq 9$).

Case 1.2: $\left| \frac{b^2}{a} \right|_3 = |3d|_3 = \frac{1}{3}$

Under these hypotheses,

$$|c^2|_3 \leq 9 \left| \frac{b^3}{a} \right|_3 \implies \sigma_3 = 1 \quad |c^2|_3 \geq 81 \left| \frac{b^3}{a} \right|_3 \implies \sigma_3 = \frac{1}{27} \left| \frac{c^4}{a} \right|_3.$$

Case 1.2.1: $|c^2|_3 \leq 9 \left| \frac{b^3}{a} \right|_3 = 3|b|_3$

If $|c^2|_3 \leq \left| \frac{b^3}{a} \right|_3$, then $C_{f,3} = \max \left\{ \left| \frac{b}{a} \right|_3^{\frac{1}{2}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$. $\sqrt[3]{\sigma_3} = 1 = 3^{\frac{1}{4}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_3 > \left| \frac{b}{a} \right|_3^{\frac{1}{2}}$ (since $|a|_3 > 1$), and clearly also exceeds $\left| \frac{1}{a} \right|_3^{\frac{1}{4}}$ (since $|b|_3^2 = \frac{1}{3}|a|_3 \implies |b|_3^2 \geq 1 \implies |b|_3 \geq 1$), so $\sqrt[3]{\sigma_3} > C_{f,3}$ in this case.

If $3 \left| \frac{b^3}{a} \right|_3 \leq |c^2|_3 \leq 9 \left| \frac{b^3}{a} \right|_3$, then $C_{f,3} = \max \left\{ \left| \frac{c}{a} \right|_3^{\frac{1}{3}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$. $\sqrt[3]{\sigma_3} = 1 = 3^{\frac{1}{4}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_3 \geq 3^{\frac{1}{12}} \left| \frac{c^{\frac{1}{3}}}{a^{\frac{1}{12}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}}$ (since $|c^{\frac{1}{3}}|_3 \leq 3^{\frac{1}{6}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{6}}} \right|_3$, and $|a|_3 > 1$), and also exceeds $\left| \frac{1}{a} \right|_3^{\frac{1}{4}}$ by the same argument as above, so $\sqrt[3]{\sigma_3} > C_{f,3}$ in this case as well.

So $|a|_3 \leq 1$, and so $\left| \frac{b^2}{a} \right|_3 = \frac{1}{3} \implies |b|_3 \leq \frac{1}{3}$. Then $|c^2|_3 \leq 3|b|_3 \implies |c|_3 \leq 1$.

Case 1.2.2: $|c^2|_3 \geq 81 \left| \frac{b^3}{a} \right|_3 = 27|b|_3$

So $C_{f,3} = \max \left\{ \left| \frac{c}{a} \right|_3^{\frac{1}{3}}, \left| \frac{1}{a} \right|_3^{\frac{1}{4}} \right\}$. Then $\sqrt[3]{\sigma_3} = \frac{1}{3} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_3 \geq \sqrt{3} \left| \frac{b^{\frac{1}{2}}c^{\frac{1}{3}}}{a^{\frac{1}{3}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}}$ (since $|b|_3 \geq 1$), and also $\sqrt[3]{\sigma_3} = \frac{1}{3} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_3 \geq 3 \left| \frac{b^{\frac{2}{3}}}{a^{\frac{1}{3}}} \right|_3 = 3^{\frac{11}{12}} \left| \frac{b^{\frac{1}{2}}}{a^{\frac{1}{4}}} \right|_3 > \left| \frac{1}{a} \right|_3^{\frac{1}{4}}$, so we have $\sqrt[3]{\sigma_3} > C_{f,3}$ in this case.

So $|a|_3 \leq 1$, and so $\left| \frac{b^2}{a} \right|_3 = \frac{1}{3} \implies |b|_3 \leq \frac{1}{3}$. Then if $|c|_3 > 3$, we have $\sqrt[3]{\sigma_3} = \frac{1}{3} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_3 > \left| \frac{c}{a} \right|_3^{\frac{1}{3}} > \left| \frac{1}{a} \right|_3^{\frac{1}{3}}$, so we must have $|c|_3 \leq 3$.

Case 1.2.3: $|c^2|_3 = 27 \left| \frac{b^3}{a} \right|_3 = 9|b|_3$

In this case there is no dominant term in the sums describing the σ , so by way of contradiction we suppose that the sums of the dominant *terms* are all strictly less than their maxima. That is, suppose

$$\begin{aligned}
& \left| \frac{b^2}{2a} - 3d \right|_3 < \frac{1}{9}, \quad \left| 3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2} \right|_3 < \frac{1}{3}, \text{ and} \\
& \left| \frac{b^2d^2}{2a} - d^3 - \frac{9bc^2d}{16a} + \frac{27c^4}{256a} - \frac{b^4d}{16a^2} + \frac{b^3c^2}{64a^2} \right|_3 < 1. \\
& \left| \frac{b^2}{2a} - 3d \right|_3 = \left| \frac{b^2}{2a} - 3 \right|_3 < \frac{1}{9} \implies \frac{b^2}{2a} = 3 + \varepsilon \text{ and } \frac{b}{2a} < \frac{3 + \varepsilon}{b} \text{ where } |\varepsilon|_3 < \frac{1}{9} \\
& \implies \left| 3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2} \right|_3 = \left| 3 - 2(3 + \varepsilon) + \frac{9bc^2}{16a} + \frac{(3 + \varepsilon)^2}{4} \right|_3 < \frac{1}{3} \\
& \implies \left| -\frac{3}{4} + \frac{9bc^2}{16a} \right|_3 < \frac{1}{3} \text{ and } \left| -\frac{3}{4} + \frac{9(3 + \varepsilon)c^2}{8b} \right|_3 < \frac{1}{3} \\
& \implies \frac{9bc^2}{16a} = \delta + \frac{3}{4} \text{ and } c^2 = \left(\frac{8b}{9(3 + \varepsilon)} \right) \left(\delta + \frac{3}{4} \right) \text{ where } |\delta|_3 < \frac{1}{3}. \\
\sigma_3 &= \left| \frac{b^2d^2}{2a} - d^3 - \frac{9bc^2d}{16a} + \frac{27c^4}{256a} - \frac{b^4d}{16a^2} + \frac{b^3c^2}{64a^2} \right|_3 \\
&= \left| (3 + \varepsilon) - 1 - \frac{9bc^2}{16a} + \frac{27c^4}{256a} - \frac{(3 + \varepsilon)^2}{4} + \frac{(3 + \varepsilon)bc^2}{32a} \right|_3 \\
&= \left| (3 + \varepsilon) - 1 - \delta - \frac{3}{4} + \left(\frac{1}{6(3 + \varepsilon)} \right) \left(\delta + \frac{3}{4} \right)^2 - \frac{(3 + \varepsilon)^2}{4} + \frac{3 + \varepsilon}{18} \left(\delta + \frac{3}{4} \right) \right|_3 \\
&= \left| -1 + \varepsilon - \delta + \left(\frac{1}{6(3 + \varepsilon)} \right) \left(\delta + \frac{3}{4} \right)^2 - \frac{6\varepsilon}{4} - \frac{\varepsilon^2}{4} + \frac{3\delta}{18} + \frac{9}{72} + \frac{\varepsilon\delta}{18} + \frac{3\varepsilon}{72} \right|_3 < 1 \\
&\implies \left| -\frac{7}{8} + \frac{3}{32(3 + \varepsilon)} \right|_3 < 1
\end{aligned}$$

since clearly every other term in the sum has 3-adic absolute value less than 1, and these 2 are the only terms with 3-adic absolute value equal to 1.

$$\begin{aligned}
& \implies \frac{-7}{8} + \frac{3}{96 + 864k} = 3l \text{ for some } k, l \in \mathbb{Z} \\
& \implies 288l(9k + 1) + 756k = -81 \\
& \implies 32l(9k + 1) + 84k = -9
\end{aligned}$$

which is a diophantine equation with no solutions.

So $\max\{\sigma_1, \sqrt{\sigma_2}, \sqrt[3]{\sigma_3}\} \geq \frac{1}{9}$. We have $C_{f,3} = \max\left\{\left|\frac{c}{a}\right|_3^{\frac{1}{3}}, \left|\frac{1}{a}\right|_3^{\frac{1}{4}}\right\}$, so $|a|_3 > 3^9 \implies \left|\frac{1}{a}\right|_3^{\frac{1}{4}} < \frac{1}{3^{\frac{9}{4}}}$. Also, $\left|\frac{c}{a}\right|_3^{\frac{1}{3}} = 3^{\frac{1}{6}} \left|\frac{b}{a}\right|_3^{\frac{1}{2}} = 3^{-\frac{1}{12}} \left|\frac{1}{a}\right|_3^{\frac{1}{4}} < \frac{1}{9}$. So $|a|_3 > 3^9 \implies \max\{\sigma_1, \sqrt{\sigma_2}, \sqrt[3]{\sigma_3}\} > C_{f,3}$.

So we have $|a|_3 \leq 3^9$ in this case. Then $\frac{1}{\sqrt{3}}|a|_3^{\frac{1}{2}} = |b|_3 \implies |b|_3 \leq 3^4$. Finally, $3|b|_3^{\frac{1}{2}} = |c|_3 \implies |c|_3 \leq 3^3$.

Note that the relations $\frac{1}{\sqrt{3}}|a|_3^{\frac{1}{2}} = |b|_3$ and $3|b|_3^{\frac{1}{2}} = |c|_3$ prohibit $|a|_3 \in \{3^6, 3^7, 3^8\}$ (since our coefficients are in \mathbb{Q}), so if $|a|_3 \neq 3^9$, then $|a|_3 \leq 3^5$ and $|b|_3, |c|_3 \leq 3^2$.

Case 1.3: $\left|\frac{b^2}{a}\right|_3 < 3|d|_3 = \frac{1}{3}$

Under these hypotheses, we have

$$\sigma_1 = \frac{1}{3}, \text{ and if } |c^2|_3 \leq 81 \left|\frac{b^3}{a}\right|_3, \text{ then } \sigma_3 = 1.$$

First, suppose $|c^2|_3 \leq \left|\frac{b^3}{a}\right|_3$, and so $C_{f,3} = \max\left\{\left|\frac{b}{a}\right|_3^{\frac{1}{2}}, \left|\frac{1}{a}\right|_3^{\frac{1}{4}}\right\}$, and $\sqrt[3]{\sigma_3} = 1$. Note that $\left|\frac{b^2}{a}\right|_3 < \frac{1}{3} \implies |b|_3^{\frac{1}{2}} < \frac{1}{3^{\frac{1}{4}}}|a|_3^{\frac{1}{4}} \implies \left|\frac{b}{a}\right|_3^{\frac{1}{2}} < \frac{1}{3^{\frac{1}{4}}}\left|\frac{a^{\frac{1}{4}}}{a^{\frac{1}{2}}}\right|_3 = \frac{1}{3^{\frac{1}{4}}}\left|\frac{1}{a^{\frac{1}{4}}}\right|_3 < \sqrt[3]{\sigma_3}$ (since $|a|_3 > 1$). Also clearly $\left|\frac{1}{a}\right|_3^{\frac{1}{4}} < 1 = \sqrt[3]{\sigma_3}$ since $|a|_3 > 1$, so $\sqrt[3]{\sigma_3} > C_{f,3}$ in this case.

Now suppose $|c^2|_3 > \left|\frac{b^3}{a}\right|_3$, and so $C_{f,3} = \max\left\{\left|\frac{c}{a}\right|_3^{\frac{1}{3}}, \left|\frac{1}{a}\right|_3^{\frac{1}{4}}\right\}$. First note that if $3\left|\frac{b^3}{a}\right|_3 \leq |c^2|_3 \leq 81\left|\frac{b^3}{a}\right|_3 \leq 9|b|_3$, then $\left|\frac{c}{a}\right|_3^{\frac{1}{3}} \leq 3^{\frac{1}{3}}\left|\frac{b^{\frac{1}{6}}}{a^{\frac{1}{3}}}\right|_3 \leq 3^{\frac{1}{6}}\left|\frac{1}{a^{\frac{1}{4}}}\right|_3 < 1 = \sqrt[3]{\sigma_3}$ (since $|a|_3 \geq 3 \implies \left|\frac{1}{a^{\frac{1}{4}}}\right|_3 \leq 3^{-\frac{1}{4}}$), and like above we also have $\left|\frac{1}{a}\right|_3^{\frac{1}{4}} < 1 = \sqrt[3]{\sigma_3}$, so $\sqrt[3]{\sigma_3} > C_{f,3}$ in this case.

For the last case, suppose $|a|_3 > 3^5$ and $|c^2|_3 > 81\left|\frac{b^3}{a}\right|_3$, and so $C_{f,3} = \max\left\{\left|\frac{c}{a}\right|_3^{\frac{1}{3}}, \left|\frac{1}{a}\right|_3^{\frac{1}{4}}\right\}$. First note that if $|c|_3 < \frac{1}{3^{\frac{1}{3}}}|a|_3$, then $\left|\frac{c}{a}\right|_3^{\frac{1}{3}} < \frac{1}{3} = \sigma_1$, and $|a|_3 > 3^5 \implies \left|\frac{1}{a}\right|_3^{\frac{1}{4}} < \frac{1}{3} = \sigma_1$, so we have $\sigma_1 > C_{f,3}$. So we can assume that $|c|_3 \geq \frac{1}{3^{\frac{1}{3}}}|a|_3 \geq 3^3$. Then we have that $\sigma_3 = \frac{1}{27}\left|\frac{c^4}{a}\right|_3$ because it exceeds all other terms in the sum comprising σ_3 (for instance, $\frac{1}{27}\left|\frac{c^4}{a}\right|_3 > \frac{1}{3^{\frac{5}{3}}}\left|\frac{bc^{\frac{10}{3}}}{a^{\frac{4}{3}}}\right|_3 \geq \frac{1}{3^{\frac{8}{3}}}\left|\frac{ba^{\frac{1}{3}}c^3}{a^{\frac{4}{3}}}\right|_3 = \frac{1}{3^{\frac{7}{3}}}\left|\frac{bc^3}{a}\right|_3 \geq 3^{\frac{1}{3}}\left|\frac{bc^2}{a}\right|_3 > \frac{1}{9}\left|\frac{bc^2}{a}\right|_3$). Then $\sqrt[3]{\sigma_3} = \frac{1}{3}\left|\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}\right|_3 > \left|\frac{c}{a}\right|_3^{\frac{1}{3}}$ since $|c|_3 > 3$, and $\sqrt[3]{\sigma_3} = \frac{1}{3}\left|\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}\right|_2 \geq \frac{1}{9}|c|_3 \geq 3 > \left|\frac{1}{a}\right|_3^{\frac{1}{4}}$ since $|a|_3 > 1$, so $\sqrt[3]{\sigma_3} > C_{f,3}$ in this case as well, and so $|a|_3 \leq 3^5$.

If $|a|_3 \leq 3^5$ and $\left|\frac{b^2}{a}\right|_3 \leq \frac{1}{9}$, then $|b|_3 \leq 3$. Then $|c^2|_3 \leq 81 \left|\frac{b^3}{a}\right|_3 \leq 9|b|_3 \implies |c^2|_3 \leq 27 \implies |c|_3 \leq 3$, and if $|c^2|_3 > 81 \left|\frac{b^3}{a}\right|_3$, then if $|c|_3 > 9$ we have $\sqrt[3]{\sigma_3} = \frac{1}{3} \left|\frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}\right|_3 > \left|\frac{c}{a}\right|_3^{\frac{1}{3}} > \left|\frac{1}{a}\right|_3^{\frac{1}{3}}$, so we must have $|c|_3 \leq 9$.

Case 2: $d = 0$

This argument bounding $|b|_3$ and $|c|_3$ largely follows that of Case 1.1, since in this case, we can conjugate to clear the denominator of a , so take $|a|_3 \leq 1$.

If $|c^2|_3 \leq \left|\frac{b^3}{a}\right|_3$, then σ_1 is the same, so the argument is unchanged. If $3 \left|\frac{b^3}{a}\right|_3 \leq |c^2|_3 \leq 9 \left|\frac{b^3}{a}\right|_3$, then $\sqrt[3]{\sigma_3}$ is the same as far as showing that $|c|_3 \leq 1$. Then $3 \left|\frac{b^3}{a}\right|_3 \leq |c^2|_3 \leq 1 \implies 3|b|_3^3 \leq 1 \implies |b|_3 \leq \frac{1}{3}$. If $81 \left|\frac{b^3}{a}\right|_3 \leq |c^2|_3$, then the same argument shows $|c|_3 \leq 3$, and so $81 \left|\frac{b^3}{a}\right|_3 \leq |c^2|_3 \leq 3 \implies 81|b|_3^3 \leq 3 \implies |b|_3 \leq \frac{1}{3}$.

Finally, suppose $|c^2|_3 = 27 \left|\frac{b^3}{a}\right|_3$. So $\sigma_2 = \left|\frac{9bc^2}{16a}\right|_3 = 3 \left|\frac{b^4}{a^2}\right|_3$, and so if $\left|\frac{b^2}{a}\right|_3 > \frac{1}{3}$ then $|b|_3 \leq 1$ and $|c|_3 \leq 3^{\frac{3}{2}}|a|_3^{-\frac{1}{8}}$. If on the other hand we have $\left|\frac{b^2}{a}\right|_3 \leq \frac{1}{3}$, then $|a|_3 \leq 1 \implies |b|_3^2 \leq \frac{1}{3} \implies |b|_3 \leq \frac{1}{3}$. Finally, $|c^2|_3 = 27 \left|\frac{b^3}{a}\right|_3 \leq 27|b|_3 \implies |c^2|_3 \leq 9 \implies |c|_3 \leq 3$.

□

We next tackle the archimedean bounds. By the triangle inequality, note that there must exist branch points $f(\zeta_j), f(\zeta_k), f(\zeta_l)$, not necessarily distinct, such that

$$|f(\zeta_j)|_\infty \geq \frac{\sigma_1}{3} \quad \text{and} \quad |f(\zeta_k)|_\infty \geq \sqrt{\frac{\sigma_2}{3}} \quad \text{and} \quad |f(\zeta_l)|_\infty \geq \sqrt[3]{\sigma_3}$$

since the σ are symmetric polynomials of the branch points. We can use these quantities with the constant $C_{f,\infty}$ to find an archimedean bound. Alternatively (as we do in the below lemma), we can find a bound on a coefficient above which we determine a point to wander. That is, we show that $|a| > M$ and $|\alpha| > N \implies |f(\alpha)| > |\alpha|$, where $M \in \mathbb{R}^+$ and N is some function of the coefficients. We then show that one of the above σ_i expressions exceeds N , and so conclude that $|a| \leq M$. We can suppose that a and d have the same sign, since this is the worse case and leads to the worse bounds (since if a and d have the same sign, then $\frac{\sigma_1}{3} = \left|\frac{b^2}{6a} - d\right| = \left|\frac{b^2}{6a}\right| + 1$).

Lemma 4.6. *If $f(z) = az^4 + bz^2 + cz + d \in \mathbb{Q}[z]$ is PCF, then $|a| < 816916$.*

Proof. Let $|\alpha| \geq \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$ for some $x, y, z > 0$. Suppose that

$$|a| > \frac{1}{z^3 \left(1 - \left| \frac{1}{x^2} \right| - \left| \frac{1}{y^3} \right| \right)}. \quad (4.1)$$

Then

$$\begin{aligned} \frac{1}{z^3} &< |a| \left(1 - \left| \frac{1}{x^2} \right| - \left| \frac{1}{y^3} \right| \right) \\ \implies \frac{1}{|\alpha|^3} &< |a| \left(1 - \left| \frac{1}{x^2} \right| - \left| \frac{1}{y^3} \right| \right) \\ \iff |\alpha| &< |\alpha| \cdot |\alpha|^3 |a| \cdot \left(1 - \left| \frac{1}{x^2} \right| - \left| \frac{1}{y^3} \right| \right) \\ &= |\alpha| \left(|a\alpha^3| - \left| \frac{a}{x^2} \alpha^3 \right| - \left| \frac{a}{y^3} \alpha^3 \right| \right) \\ &\leq |a\alpha^4 + b\alpha^2 + c\alpha + d| \\ &= |f(\alpha)|. \end{aligned}$$

That is, if 4.1 holds, then α is a wandering point.

First, consider the case that $d = 1$. Throughout the proof, we will use the above argument, showing that a σ_i exceeds either $x \left| \frac{b}{a} \right|^{\frac{1}{2}}$ or $y \left| \frac{c}{a} \right|^{\frac{1}{3}}$ for sufficiently large $|a|$, and that it exceeds a constant z , so we get another bound on $|a|$ from equation 4.1, and take the worst of the two.

Case 1: Suppose $|c|^2 \leq \frac{x^6}{y^6} \left| \frac{b^3}{a} \right|$, so $x \left| \frac{b}{a} \right|^{\frac{1}{2}} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}$:

Suppose $|b| \geq n|a|^{\frac{1}{2}}$ for some $n > 0$. Suppose that

$$|a| \geq \left(\frac{x}{n^{\frac{1}{2}} \left(\frac{n}{6} - \frac{1}{n} \right)} \right)^4 \quad z = n \left(\frac{n}{6} - \frac{1}{n} \right) \quad (4.2)$$

where $\left(\frac{n}{6} - \frac{1}{n} \right) > 0$, so we must have $n > \sqrt{6}$. Then $\frac{\sigma_1}{3} \geq \left(\frac{n}{6} - \frac{1}{n} \right) \left| \frac{b}{a^{\frac{1}{2}}} \right| \geq n \left(\frac{n}{6} - \frac{1}{n} \right) = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \left(\frac{x}{n^{\frac{1}{2}} \left(\frac{n}{6} - \frac{1}{n} \right)} \right)^4 \\
\Rightarrow & \left(\frac{n}{6} - \frac{1}{n} \right) (n^{\frac{1}{2}} |a|^{\frac{1}{4}}) \geq x \\
& \Rightarrow \left(\frac{n}{6} - \frac{1}{n} \right) |b|^{\frac{1}{2}} \geq x \\
\Leftrightarrow & \left(\frac{n}{6} - \frac{1}{n} \right) \left| \frac{b}{a^{\frac{1}{2}}} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \Rightarrow \frac{n}{6} \left| \frac{b}{a^{\frac{1}{2}}} \right| - 1 \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \Rightarrow \left| \frac{b^2}{6a} \right| - 1 \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \Rightarrow \frac{\sigma_1}{3} \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}}.
\end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

Next suppose $|b| \leq m|a|^{\frac{1}{2}}$ for some $0 < m < n$. Suppose that

$$|a| \geq \left(\frac{x\sqrt{m}}{1 - \frac{m^2}{6}} \right)^4 \quad z = 1 - \frac{m^2}{6} \quad (4.3)$$

where $1 - \frac{m^2}{6} > 0$, so we must have $m < \sqrt{6}$. Then $\frac{\sigma_1}{3} \geq 1 - \frac{m^2}{6} = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \left(\frac{x\sqrt{m}}{1 - \frac{m^2}{6}} \right)^4 \\
\Leftrightarrow & 1 - \frac{m^2}{6} \geq x\sqrt{m} \left| \frac{1}{a} \right|^{\frac{1}{4}} \\
& \Rightarrow 1 - \frac{m^2}{6} \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \Rightarrow 1 - \left| \frac{b^2}{6a} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \Rightarrow \frac{\sigma_1}{3} \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}}.
\end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

We can narrow the interval (m, n) containing $\sqrt{6}$ in the above at the cost of getting a worse bound on $|a|$. However, it will be necessary to narrow this interval in order to deal

with the case that $m|a|^{\frac{1}{2}} < |b| < n|a|^{\frac{1}{2}}$ using either $\sqrt{\frac{\sigma_2}{3}}$ or $\sqrt[3]{\sigma_3}$. We will work to optimize these choices to make the worst bound on $|a|$ across all cases as low as possible.

Suppose $m|a|^{\frac{1}{2}} < |b| < n|a|^{\frac{1}{2}}$. Suppose that

$$|a| \geq \left(\frac{x}{m^{\frac{3}{2}}\beta} \right)^4 \quad z = m^{\frac{3}{2}}\beta \quad (4.4)$$

where $\sqrt{\left(\frac{1}{3n^2} - \frac{3x^6}{16y^6} - \frac{1}{m^4} - \frac{1}{48}\right)} = \beta$. Then $\sqrt{\frac{\sigma_2}{3}} \geq m\beta \left| \frac{b}{a^{\frac{1}{2}}} \right| \geq m^{\frac{3}{2}}\beta = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned} & |a| \geq \left(\frac{x}{m^{\frac{3}{2}}\beta} \right)^4 \\ \Leftrightarrow & m^{\frac{3}{2}}\beta |a|^{\frac{1}{4}} \geq x \\ \Rightarrow & m\beta |b|^{\frac{1}{2}} \geq x \\ \Leftrightarrow & m\beta \left| \frac{b}{a^{\frac{1}{2}}} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\ \Rightarrow & \sqrt{\left(\frac{1}{3n^2} - \frac{3x^6}{16y^6} - \frac{1}{m^4} - \frac{1}{48}\right)} \left| \frac{b^2}{a} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\ \Rightarrow & \sqrt{\left(\frac{1}{3n^2} - \frac{3x^6}{16y^6} - \frac{1}{m^4} - \frac{1}{48}\right)} \left| \frac{b^4}{a^2} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\ \Rightarrow & \sqrt{\left| \frac{3bc^2}{16a} \right| + 1 + \left| \frac{b^4}{48a^2} \right| - \left| \frac{b^2d}{3a} \right|} \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\ \Rightarrow & \sqrt{\frac{\sigma_2}{3}} \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}}. \end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

Note that in the above, we require $\beta > 0$. As m, y get larger and n, x get smaller we can increase the quantity under the root to satisfy this requirement.

We optimize Case 1 by equating bounds 4.2, 4.3, 4.4. If all of these conditions are met, then we have a critical point which wanders. So, if f is PCF then $|a| \leq$ the maximum of these bounds. When $x = 1.09, y = 3.37, m = 2.38$, and $n = 2.52$, we find an approximate optimal bound $|a| \leq 816916$.

Case 2: $|c|^2 > \frac{x^6}{y^6} \left| \frac{b^3}{a} \right|$, so $x \left| \frac{b}{a} \right|^{\frac{1}{2}} < y \left| \frac{c}{a} \right|^{\frac{1}{3}} = 3.72 \left| \frac{c}{a} \right|^{\frac{1}{3}}$:

Case 2.1: a and b have the same sign:

Suppose $|b| \geq n|a|^{\frac{1}{2}}$ for some $n > 0$. Suppose that

$$|a| \geq \left(\frac{y^3}{n^{\frac{3}{2}}x^2\beta} \right)^4 \quad z = \frac{x^3n^2}{y^3}\beta \quad (4.5)$$

where $\sqrt{\left(\frac{3}{16} - \frac{y^6}{3n^2x^6}\right)} = \beta$. Then $\sqrt{\frac{\sigma_2}{3}} \geq \sqrt{n}\beta \left| \frac{c}{a^{\frac{1}{4}}} \right| \geq \frac{x^3n^2}{y^3}\beta = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned} |a| &\geq \left(\frac{y^3}{n^{\frac{3}{2}}x^2\beta} \right)^4 \\ &\iff \frac{x^2n^{\frac{3}{2}}}{y^2}\beta \geq y \left| \frac{1}{a} \right|^{\frac{1}{4}} \\ &\iff \frac{x^2n^{\frac{3}{2}}}{y^2}\beta \left| \frac{c^{\frac{1}{3}}}{a^{\frac{1}{12}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt{n}\beta \left| \frac{c}{a^{\frac{1}{4}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt{\left(\frac{3}{16} - \frac{y^6}{3n^2x^6}\right)} \left| \frac{b^{\frac{1}{2}}c}{a^{\frac{1}{2}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt{\left(\frac{3}{16} - \frac{y^6}{3n^2x^6}\right)} \left| \frac{bc^2}{a} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt{\left| \frac{3bc^2}{16a} \right| + 1 + \left| \frac{b^4}{48a^2} \right| - \left| \frac{b^2d}{3a} \right|} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt{\frac{\sigma_2}{3}} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}. \end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$. We can see that the smaller y is relative to x , the smaller we can make n .

Next, suppose $|b| < n|a|^{\frac{1}{2}}$ and $|c|^2 \geq \frac{x^6}{y^6}pn|a|^{\frac{1}{2}} > \frac{x^6}{y^6}p|b|$. Suppose that

$$|a| \geq \frac{y^{16}}{p^2n^2x^{12}\delta^4} \quad z = (pn)^{\frac{2}{3}}\frac{x^4}{y^4}\delta \quad (4.6)$$

where $\sqrt[3]{\left(\frac{27}{256} - \frac{9y^6}{16px^6} - \frac{n^2y^{12}}{16p^2x^{12}} - \frac{y^{12}}{p^2n^2x^{12}}\right)} = \delta$. Then $\sqrt[3]{\sigma_3} \geq \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}} \right| \geq (pn)^{\frac{2}{3}}\frac{x^4}{y^4}\delta = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \frac{y^{16}}{p^2 n^2 x^{12} \delta^4} \\
& \iff \delta p^{\frac{1}{2}} n^{\frac{1}{2}} \frac{x^3}{y^3} |a|^{\frac{1}{4}} \geq y \\
& \implies \delta |c| \geq y \\
& \iff \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt[3]{\left(\frac{27}{256} - \frac{9y^6}{16px^6} - \frac{n^2 y^{12}}{16p^2 x^{12}} - \frac{y^{12}}{p^2 n^2 x^{12}} \right) \left| \frac{c^4}{a} \right|} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt[3]{\left| \frac{27c^4}{256a} + \left| \frac{b^2 d^2}{2a} \right| + \left| \frac{b^3 c^2}{64a^2} \right| - \left| \frac{9bc^2 d}{16a} \right| - \left| \frac{b^4 d}{16a^2} \right| - 1} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt[3]{\sigma_3} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}.
\end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$. Note that the smaller n is, the smaller we can make p .

Next, suppose $|c|^2 < \frac{x^6}{y^6} pn |a^{\frac{1}{2}}|$ and $|b| \leq l |a|^{\frac{1}{2}}$ or $m |a|^{\frac{1}{2}} \leq |b| \leq n |a|^{\frac{1}{2}}$, where the interval (l, m) is centered on $\sqrt{6}$. Suppose that

$$|a| \geq \left(\frac{x(pn)^{\frac{1}{6}}}{1 - \frac{l^2}{6}} \right)^4 \quad z = 1 - \frac{l^2}{6} \quad (4.7)$$

where $1 - \frac{l^2}{6} > 0$, so we must have $l < \sqrt{6}$. So $\frac{\sigma_1}{3} \geq 1 - \frac{l^2}{6} = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \left(\frac{x(pn)^{\frac{1}{6}}}{1 - \frac{l^2}{6}} \right)^4 \\
& \iff 1 - \frac{l^2}{6} \geq x(pn)^{\frac{1}{6}} \left| \frac{1}{a} \right|^{\frac{1}{4}} \\
& \implies 1 - \frac{l^2}{6} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \left| \frac{b^2}{6a} - 1 \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \frac{\sigma_1}{3} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}.
\end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

We have x and y fixed by Case 1, so we optimize over 4.5, 4.6, 4.7, varying n , p , and l . We get a worse bound on $|a|$ the more we constrict the interval (l, m) around $\sqrt{6}$. So

to exclude as small an interval as possible, we optimize a bound close to but less than the $|a| \leq 816916$ obtained in Case 1, achieved when $n = 39.5, p = 108700, l = 1.79$.

So the next case to consider is $1.78|a|^{\frac{1}{2}} < |b| < 3.11|a|^{\frac{1}{2}}$ where $|c|^2 < \frac{x^6}{y^6}pn|a|^{\frac{1}{2}} \implies |c| < 4916|a|^{\frac{1}{4}}$. We next use $\sqrt{\frac{\sigma_2}{3}}$ to eliminate part of this case. Specifically, suppose $s|a|^{\frac{1}{2}} \leq |b| \leq t|a|^{\frac{1}{2}}$ and $q|a|^{\frac{1}{4}} \leq |c| \leq r|a|^{\frac{1}{4}}$. Then we find that

$$\begin{aligned} \sqrt{\frac{\sigma_2}{3}} &\geq \max \left\{ \sqrt{\left| \frac{3bc^2}{16a} \right| + 1 + \left| \frac{b^4}{48a^2} \right| - \left| \frac{b^2d}{3a} \right|}, \sqrt{\left| \frac{b^2d}{3a} \right| - \left| \frac{3bc^2}{16a} \right| - 1 - \left| \frac{b^4}{48a^2} \right|} \right\} \\ &\geq \max \left\{ \sqrt{\left(\frac{3sq^2}{16} + 1 + \frac{s^4}{48} - \frac{t^2}{3} \right)}, \sqrt{\left(\frac{s^2}{3} - \frac{3tr^2}{16} - 1 - \frac{t^4}{48} \right)} \right\} = \delta \\ &\geq 3.37 \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\iff |a| \geq \left(\frac{3.37r^{\frac{1}{3}}}{\delta} \right)^4 \iff |a| \geq 816916 \end{aligned}$$

except in $1.78|a|^{\frac{1}{2}} < |b| < 3.11|a|^{\frac{1}{2}}$ and $|c| < 1.02|a|^{\frac{1}{4}}$.

We use $\sqrt[3]{\sigma_3}$ for some of the remaining cases. We computationally check small subintervals in $1.78|a|^{\frac{1}{2}} < |b| < 3.11|a|^{\frac{1}{2}}$ and $0.15|a|^{\frac{1}{4}} \leq |c| \leq 1.02|a|^{\frac{1}{4}}$ or $|c| < 0.15|a|^{\frac{1}{4}}$ and $2.18|a|^{\frac{1}{2}} < |b| < 3.26|a|^{\frac{1}{2}}$.

Specifically, suppose $s|a|^{\frac{1}{2}} \leq |b| \leq t|a|^{\frac{1}{2}}$ and $q|a|^{\frac{1}{4}} \leq |c| \leq r|a|^{\frac{1}{4}}$. Then we find that

$$\begin{aligned} \sqrt[3]{\sigma_3} &\geq \max \left\{ \sqrt[3]{\left| \frac{9bc^2d}{16a} \right| + \left| \frac{b^4d}{16a^2} \right| + 1 - \left| \frac{27c^4}{256a} \right| - \left| \frac{b^2d^2}{2a} \right| - \left| \frac{b^3c^2}{64a^2} \right|}, \right. \\ &\quad \left. \sqrt[3]{\left| \frac{27c^4}{256a} \right| + \left| \frac{b^2d^2}{2a} \right| + \left| \frac{b^3c^2}{64a^2} \right| - \left| \frac{9bc^2d}{16a} \right| - \left| \frac{b^4d}{16a^2} \right| - 1} \right\} \\ &\geq \max \left\{ \sqrt[3]{\left(\frac{9sq^2}{16} + \frac{s^4}{16} + 1 - \frac{t^3r^2}{64} - \frac{27r^4}{256} - \frac{t^2}{2} \right)}, \right. \\ &\quad \left. \sqrt[3]{\left(\frac{27q^4}{256} + \frac{s^2}{2} + \frac{s^3q^2}{64} - \frac{9tr^2}{16} - \frac{r^4}{16} - 1 \right)} \right\} = \delta \geq 3.37r^{\frac{1}{3}}|a|^{-\frac{1}{4}} \\ &\iff |a| \geq \left(\frac{3.37r^{\frac{1}{3}}}{\delta} \right)^4 \iff |a| \geq 816916 \end{aligned}$$

for all choices of s, t, r, q in the above intervals.

So the last subcase is $|c| < 0.15|a|^{\frac{1}{4}}$ and $1.78|a|^{\frac{1}{2}} < |b| < 2.1|a|^{\frac{1}{2}}$.

$$\frac{\sigma_1}{3} = \left| \frac{b^2}{6a} - 1 \right| \geq 1 - \frac{t^2}{6} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \iff |a| \geq \frac{3.37(0.15)^{\frac{1}{3}}}{1 - \frac{2.18^2}{6}} \iff |a| \geq 10$$

which rules out the last case.

Case 2.2: a and b have the opposite sign:

Similar to Case 2.1. Suppose $|c|^2 \geq \frac{x^6}{y^6} p \left| \frac{b^3}{a} \right|$ and $|b| \geq n|a|^{\frac{1}{2}}$. Suppose that

$$|a| \geq \frac{y^{16}}{p^2 n^6 x^{12} \delta^4} \quad z = p^{\frac{2}{3}} n^2 \frac{x^4}{y^4} \delta \quad (4.8)$$

where $\sqrt[3]{\left(\frac{27}{256} - \frac{y^6}{64px^6} - \frac{9y^6}{16pn^2x^6} - \frac{y^{12}}{16p^2n^2x^{12}} - \frac{y^{12}}{p^2n^3x^{12}} \right)} = \delta$. Then $\sqrt[3]{\sigma_3} \geq \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right| \geq p^{\frac{2}{3}} n^2 \frac{x^4}{y^4} \delta = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned} |a| &\geq \frac{y^{16}}{p^2 n^6 x^{12} \delta^4} \\ &\iff \delta p^{\frac{1}{2}} n^{\frac{3}{2}} \frac{x^3}{y^3} |a|^{\frac{1}{4}} \geq y \\ &\implies \delta |c| \geq y \\ &\iff \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt[3]{\left(\frac{27}{256} - \frac{y^6}{64px^6} - \frac{9y^6}{16pn^2x^6} - \frac{y^{12}}{16p^2n^2x^{12}} - \frac{y^{12}}{p^2n^3x^{12}} \right)} \left| \frac{c^4}{a} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt[3]{\left| \frac{27c^4}{256a} + \left| \frac{b^2d^2}{2a} \right| - \left| \frac{b^3c^2}{64a^2} \right| - \left| \frac{9bc^2d}{16a} \right| - \left| \frac{b^4d}{16a^2} \right| - 1} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\implies \sqrt[3]{\sigma_3} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}. \end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

Similarly, suppose $|c|^2 \geq \frac{x^6}{y^6} p \left| \frac{b^3}{a} \right|$, $|c|^2 \geq \frac{x^6}{y^6} p |a|^{\frac{1}{2}}$ and $|b| < n|a|^{\frac{1}{2}}$. Then as above, suppose that

$$|a| \geq \frac{y^{16}}{p^2 x^{12} \delta^4} \quad z = p^{\frac{2}{3}} \frac{x^4}{y^4} \delta \quad (4.9)$$

where $\sqrt[3]{\left(\frac{27}{256} - \frac{y^6}{64x^6p} - \frac{9ny^6}{16px^6} - \frac{ny^{12}}{16p^2x^{12}} - \frac{y^{12}}{p^2x^{12}} \right)} = \delta$. Then $\sqrt[3]{\sigma_3} \geq \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right| \geq p^{\frac{2}{3}} \frac{x^4}{y^4} \delta = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \frac{y^{16}}{p^2 x^{12} \delta^4} \\
& \iff \delta p^{\frac{1}{2}} \frac{x^3}{y^3} |a|^{\frac{1}{4}} \geq y \\
& \implies \delta |c| \geq y \\
& \iff \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt[3]{\left(\frac{27}{256} - \frac{y^6}{64x^6 p} - \frac{9ny^6}{16px^6} - \frac{ny^{12}}{16p^2 x^{12}} - \frac{y^{12}}{p^2 x^{12}} \right) \left| \frac{c^4}{a} \right|} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt[3]{\left| \frac{27c^4}{256a} + \frac{b^2 d^2}{2a} - \frac{b^3 c^2}{64a^2} - \frac{9bc^2 d}{16a} - \frac{b^4 d}{16a^2} - 1 \right|} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt[3]{\sigma_3} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}.
\end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

Next, suppose $|c|^2 \geq \frac{x^6}{y^6} p \left| \frac{b^3}{a} \right|$, $|c|^2 < \frac{x^6}{y^6} p |a|^{\frac{1}{2}}$ and $|b| \leq l |a|^{\frac{1}{2}}$ or $m |a|^{\frac{1}{2}} \leq |b| \leq n |a|^{\frac{1}{2}}$, where the interval (l, m) is centered on $\sqrt{6}$. Then suppose

$$|a| \geq \left(\frac{x p^{\frac{1}{6}}}{1 - \frac{l^2}{6}} \right)^4 \quad z = 1 - \frac{l^2}{6} \quad (4.10)$$

where $1 - \frac{l^2}{6} > 0$, so we must have $l < \sqrt{6}$. Then $\frac{\sigma_1}{3} \geq 1 - \frac{l^2}{6} = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \left(\frac{x p^{\frac{1}{6}}}{1 - \frac{l^2}{6}} \right)^4 \\
& \iff 1 - \frac{l^2}{6} \geq x p^{\frac{1}{6}} \left| \frac{1}{a} \right|^{\frac{1}{4}} \\
& \implies 1 - \frac{l^2}{6} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \left| \frac{b^2}{6a} - 1 \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \frac{\sigma_1}{3} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}.
\end{aligned}$$

So we have a critical point α such that $|\alpha| > \max\{x \left| \frac{b}{a} \right|^{\frac{1}{2}}, y \left| \frac{c}{a} \right|^{\frac{1}{3}}, z\}$.

We have x and y fixed by Case 1, so we optimize over 4.8, 4.9, 4.10, varying n , p , and l . We get a worse bound on $|a|$ the more we constrict the interval (l, m) around $\sqrt{6}$ and the

smaller we have p . So to exclude as small an interval and make p as small as possible, we aim to get a bound smaller than but close to the $|a| \leq 816916$ obtained in Case 1, achieved when $n = 1, p = 7000, l = 2.21$.

The next case to consider is $|c|^2 \geq \frac{x^6}{y^6} 7000 \left| \frac{b^3}{a} \right|$, $|c|^2 < \frac{x^6}{y^6} 7000 |a|^{\frac{1}{2}}$ and $2.21|a|^{\frac{1}{2}} \leq |b| \leq 2.69|a|^{\frac{1}{2}}$.

$$|c|^2 < \frac{x^6}{y^6} 7000 |a|^{\frac{1}{2}} \implies |c|^2 < 8.02|a|^{\frac{1}{2}}, \text{ and}$$

$$|c|^2 \geq \frac{x^6}{y^6} 7000 \left| \frac{b^3}{a} \right| \wedge 2.21|a|^{\frac{1}{2}} \leq |b| \implies |c|^2 \geq 8.02(2.21)^3 |a|^{\frac{1}{2}},$$

a contradiction, so this case is empty. Our remaining case to consider is

$$|c|^2 < 8.02|a|^{\frac{1}{2}} \implies y \left| \frac{c}{a} \right|^{\frac{1}{3}} < 4.77|a|^{-\frac{1}{4}}.$$

Once again, we can use $\frac{\sigma_1}{3}$ and $|b| \leq l|a|^{\frac{1}{2}}$ or $m|a|^{\frac{1}{2}} \leq |b|$, where the interval (l, m) is centered on $\sqrt{6}$.

$$\begin{aligned} \frac{\sigma_1}{3} &\geq \left| \frac{b^2}{6a} - 1 \right| \geq 1 - \frac{l^2}{6} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\iff 1 - \frac{l^2}{6} \geq 4.77|a|^{-\frac{1}{4}} \\ &\iff |a| \geq \left(\frac{4.77}{1 - \frac{l^2}{6}} \right)^4. \end{aligned}$$

So once again, constricting the interval (l, m) of unaccounted for cases, we achieve the same bound $|a| \leq 816916$ except in $2.24|a|^{\frac{1}{2}} \leq |b| \leq 2.66|a|^{\frac{1}{2}}$.

So our final case (I promise. Well... as long as $d = 1$) is, since $|c|^2 < 8.02|a|^{\frac{1}{2}}$,

$$|c| < 2.84|a|^{\frac{1}{4}} \implies y \left| \frac{c}{a} \right|^{\frac{1}{3}} < 4.77|a|^{-\frac{1}{4}} \text{ and } 2.24|a|^{\frac{1}{2}} \leq |b| \leq 2.66|a|^{\frac{1}{2}}.$$

Now, as at the end of Case 1, we can confirm that $\sqrt{\frac{\sigma_2}{3}} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}$ by checking small subintervals in $|c| < 2.73|a|^{\frac{1}{4}}$ and $2.24|a|^{\frac{1}{2}} \leq |b| \leq 2.66|a|^{\frac{1}{2}}$.

Specifically, suppose $s|a|^{\frac{1}{2}} \leq |b| \leq t|a|^{\frac{1}{2}}$ and $q|a|^{\frac{1}{4}} \leq |c| \leq r|a|^{\frac{1}{4}}$. Then we find that

$$\begin{aligned}
\sqrt{\frac{\sigma_2}{3}} &\geq \min \left\{ \sqrt{\left| \frac{3bc^2}{16a} \right| + \left| \frac{b^2d}{3a} \right| - 1 - \left| \frac{b^4}{48a^2} \right|}, \sqrt{\left| \frac{b^2d}{3a} \right| + 1 + \left| \frac{b^4}{48a^2} \right| - \left| \frac{3bc^2}{16a} \right|} \right\} \\
&\geq \min \left\{ \sqrt{\left(\frac{3sq^2}{16} + \frac{s^2}{3} - 1 - \frac{t^4}{48} \right)}, \sqrt{\left(\frac{s^2}{3} + 1 + \frac{s^4}{48} - \frac{3tr^2}{16} \right)} \right\} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
&\iff |a| \geq \left(\frac{3.37r^{\frac{1}{3}}}{\min \left\{ \sqrt{\left(\frac{3sq^2}{16} + \frac{s^2}{3} - 1 - \frac{t^4}{48} \right)}, \sqrt{\left(\frac{s^2}{3} + 1 + \frac{s^4}{48} - \frac{3tr^2}{16} \right)} \right\}} \right)^4 \\
&\iff |a| \geq 816916
\end{aligned}$$

for all choices of s, t, r, q in the above intervals.

Our remaining cases are $2.24|a|^{\frac{1}{2}} \leq |b| \leq 2.66|a|^{\frac{1}{2}}$ and $2.73|a|^{\frac{1}{4}} \leq |c| \leq 2.84|a|^{\frac{1}{4}}$.

First, note that the remaining cases are easily handled by $\frac{\sigma_1}{3}$ if a is negative (since then $\frac{\sigma_1}{3} = \left| \frac{b^2}{6a} - 1 \right| = \left| \frac{b^2}{6a} \right| + 1$). So we can assume that a is positive and b is negative. So

$$\begin{aligned}
\sqrt[3]{\sigma_3} &\geq \max \left\{ \sqrt[3]{\left| \frac{9bc^2d}{16a} \right| + \left| \frac{27c^4}{256a} \right| + \left| \frac{b^2d^2}{2a} \right| - \left| \frac{b^4d}{16a^2} \right| - 1 - \left| \frac{b^3c^2}{64a^2} \right|}, \right. \\
&\quad \left. \sqrt[3]{\left| \frac{b^3c^2}{64a^2} \right| + \left| \frac{b^4d}{16a^2} \right| + 1 - \left| \frac{9bc^2d}{16a} \right| - \left| \frac{27c^4}{256a} \right| - \left| \frac{b^2d^2}{2a} \right|} \right\} \\
&\geq \max \left\{ \sqrt[3]{\left(\frac{9sq^2}{16} + \frac{27q^4}{256} + \frac{s^2}{2} - \frac{t^4}{16} - 1 - \frac{t^3r^2}{64} \right)}, \right. \\
&\quad \left. \sqrt[3]{\left(\frac{s^3q^2}{64} + \frac{s^4}{16} + 1 - \frac{9tr^2}{16} - \frac{27r^4}{256} - \frac{t^2}{2} \right)} \right\} = \delta \geq 3.37r^{\frac{1}{3}}|a|^{-\frac{1}{4}} \\
&\iff |a| \geq \left(\frac{3.37r^{\frac{1}{3}}}{\delta} \right)^4 \iff |a| \geq 816916
\end{aligned}$$

in all of the remaining intervals, which rules out the last case.

Next consider the case that $d = 0$. We start by proving a relationship between the p -adic absolute values of a and b . Conjugate so that a is a cube-free integer (note that if $a = p_1^{e_1} \cdots p_k^{e_k}$, then if $\phi(z) = \left(p_1^{-\lfloor \frac{e_1}{3} \rfloor} \cdots p_k^{-\lfloor \frac{e_k}{3} \rfloor} \right) z$, the conjugation $\phi^{-1} \circ f \circ \phi$ results in $a \in \mathbb{Z}$ cube-free). Then

$$C_{f,p} = \max \left\{ \left| \frac{b}{a} \right|_p^{\frac{1}{2}}, \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}.$$

If $|c^2|_p \leq \left| \frac{b^3}{a} \right|_p$, then $C_{f,p} = \max \left\{ \left| \frac{b}{a} \right|_p^{\frac{1}{2}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}$, and

$$|b|_p^3 > |a|_p \text{ (if } p \neq 2), \text{ or } 2^{\frac{3}{2}}|b|_2^3 > |a|_2 \implies \sigma_1 = \left| \frac{b^2}{2a} \right|_p > C_{f,p}.$$

Since $|a|_p \in \left\{ 1, \frac{1}{p}, \frac{1}{p^2} \right\}$, we have

$$|b|_p^3 \leq |a|_p \implies |b|_p^2 \leq |a|_p, \text{ and } 2^{\frac{3}{2}}|b|_2^3 \leq |a|_2 \implies |b|_2^2 \leq |a|_2.$$

If $|c^2|_p > \left| \frac{b^3}{a} \right|_p$, then $C_{f,p} = \max \left\{ \left| \frac{c}{a} \right|_p^{\frac{1}{3}}, \left| \frac{1}{a} \right|_p^{\frac{1}{3}} \right\}$, and if $p \neq 2, 3$

$$|b|_p^3 > |a|_p \implies \sqrt{\sigma_2} = \left| \frac{b^{\frac{1}{2}c}}{a^{\frac{1}{2}}} \right|_p > C_{f,p}.$$

For $p = 3$, we split into cases. If $|c^2|_3 > 9 \left| \frac{b^3}{a} \right|_3$, then

$$3^{-\frac{2}{3}}|b|_3^3 > |a|_3 \implies \sqrt{\sigma_2} = \frac{1}{3} \left| \frac{b^{\frac{1}{2}c}}{a^{\frac{1}{2}}} \right|_3 > C_{f,3}$$

and if $3 \left| \frac{b^3}{a} \right|_3 \leq |c^2|_3 \leq 9 \left| \frac{b^3}{a} \right|_3$, then

$$3^{-\frac{2}{3}}|b|_3^3 > |a|_3 \implies \sigma_1 = \left| \frac{b^2}{a} \right|_3 > C_{f,3}.$$

Finally,

$$|b|_2^3 > |a|_2 \implies \sqrt[3]{\sigma_3} = 2^{\frac{8}{3}} \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_2 > C_{f,2}.$$

Since $|a|_p \in \left\{ 1, \frac{1}{p}, \frac{1}{p^2} \right\}$, we have

$$|b|_p^3 \leq |a|_p \implies |b|_p^2 \leq |a|_p \text{ and } 3^{-\frac{2}{3}}|b|_3^3 \leq |a|_3 \implies |b|_3^2 \leq |a|_3 \text{ and}$$

$$|b|_2^3 \leq |a|_2 \implies |b|_2^2 \leq |a|_2.$$

So we have, unless $b = 0$ (which is a case we will have to deal with separately),

$$|b|_p^2 \leq |a|_p \text{ for every prime } p \implies |b|_\infty^2 \geq |a|_\infty.$$

First, suppose $|c|^2 \leq \frac{x^6}{y^6} \left| \frac{b^3}{a} \right|$, so $x \left| \frac{b}{a} \right|^{\frac{1}{2}} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}$. Then suppose

$$|a| \geq (6x)^4 \quad z = \frac{1}{6}. \tag{4.11}$$

Then $\frac{\sigma_1}{3} = \left| \frac{b^2}{6a} \right| \geq \frac{1}{6} = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq (6x)^4 \\
& \iff |a|^{\frac{1}{4}} \geq 6x \\
& \implies \frac{|b|^{\frac{1}{2}}}{6} \geq x \\
& \implies \frac{1}{6} \left| \frac{b}{a^{\frac{1}{2}}} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \implies \left| \frac{b^2}{6a} \right| \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}} \\
& \implies \frac{\sigma_1}{3} \geq x \left| \frac{b}{a} \right|^{\frac{1}{2}}.
\end{aligned}$$

Next suppose $|c|^2 > \frac{x^6}{y^6} \left| \frac{b^3}{a} \right|$, so $x \left| \frac{b}{a} \right|^{\frac{1}{2}} < y \left| \frac{c}{a} \right|^{\frac{1}{3}}$. Then suppose

$$|a| \geq \frac{y^{12}}{x^8 \beta^4} \quad z = \beta \frac{x^3}{y^3} \quad (4.12)$$

where $\sqrt{\frac{3}{16} - \frac{y^6}{48x^6}} = \beta$. Then $\sqrt{\frac{\sigma_2}{3}} \geq \beta \left| \frac{b^{\frac{1}{2}} c}{a^{\frac{1}{2}}} \right| \geq \beta \frac{x^3}{y^3} \left| \frac{b^2}{a} \right| \geq \beta \frac{x^3}{y^3} = z$ (that is, we have a critical point α such that $|\alpha| \geq z$). Also,

$$\begin{aligned}
& |a| \geq \frac{y^{12}}{x^8 \beta^4} \\
& \iff \beta \frac{x^2}{y^2} |a|^{\frac{1}{4}} \geq y \\
& \implies \beta \frac{x^2}{y^2} |b|^{\frac{1}{2}} \geq y \\
& \implies \beta \frac{x^2}{y^2} \left| \frac{b^{\frac{3}{2}}}{a^{\frac{1}{2}}} \right| \geq y \\
& \implies \sqrt{\frac{3}{16} - \frac{y^6}{48x^6}} \left| \frac{b^{\frac{1}{2}} c^{\frac{2}{3}}}{a^{\frac{1}{6}}} \right| \geq y \\
& \implies \sqrt{\frac{3}{16} - \frac{y^6}{48x^6}} \left| \frac{b^{\frac{1}{2}} c}{a^{\frac{1}{2}}} \right| \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}} \\
& \implies \sqrt{\frac{\sigma_2}{3}} \geq y \left| \frac{c}{a} \right|^{\frac{1}{3}}.
\end{aligned}$$

Optimizing with equations 4.11 and 4.12, we find $|a| \leq 4979$ when $x = 1.4, y = 1.752$.

Finally, we must address the case that $b = d = 0$. Note that if $f(z) = az^4 + cz$, then $f'(z) = 4az^3 + c$, so the critical points have absolute value $|\alpha| = \frac{1}{2^{\frac{1}{3}}} \left| \frac{c}{a} \right|^{\frac{1}{3}}$.

Note that

$$|\sqrt[3]{\sigma_3}|_p = \left| \sqrt[3]{\frac{27}{256}} \right|_p \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right|_p, \quad C_{f,p} = \max \left\{ \left| \frac{c}{a} \right|^{\frac{1}{3}}, \left| \frac{1}{a} \right|^{\frac{1}{3}} \right\}$$

so it is simple to show that if f is PCF, $|c|_2 \leq \frac{1}{4}$, $|c|_3 \leq 3$, and $|c|_p \leq 1$ for $p \neq 2, 3$. So $|c| \geq \frac{4}{3}$. It turns out that if $|c| = \frac{4}{3}$ then for all a , the critical points of f are fixed, and so f is PCF. For larger $|c|$ (which by the above, $|c| \geq \frac{8}{3}$), we can show that a critical point wanders.

Since we have a critical point $|\alpha| = \frac{1}{2^{\frac{2}{3}}} \left| \frac{c}{a} \right|^{\frac{1}{3}}$,

$$\begin{aligned} |f(\alpha)| &= |\alpha| (|a\alpha^3| + |c|) \\ &\geq |\alpha| \left(-\frac{1}{4}|c| + |c| \right) \\ &\geq |\alpha| \left(\frac{3}{4} \cdot \frac{8}{3} \right) \\ &> |\alpha| \end{aligned}$$

so a critical point wanders, and f is not PCF. □

Lemma 4.7. *If $f(z) = az^4 + bz^2 + cz + d \in \mathbb{Q}[z]$ is PCF, then $|b| < 3318$.*

Proof.

$$C_{f,\infty} = 8 \cdot \max \left\{ \left| \frac{b}{a} \right|^{\frac{1}{2}}, \left| \frac{c}{a} \right|^{\frac{1}{3}}, \left| \frac{1}{a} \right|^{\frac{1}{3}}, \left| \frac{1}{a} \right|^{\frac{1}{4}} \right\}$$

Clearly our worst bounds will occur for the largest values of a , since we have larger powers of a in the denominators of the σ . $\left| \frac{1}{a} \right|^{\frac{1}{3}}$ and $\left| \frac{1}{a} \right|^{\frac{1}{4}}$ will be dominated in $C_{f,\infty}$ unless b, c are quite small. First, suppose $|c|^2 \leq \left| \frac{b^3}{a} \right|$, so $8 \left| \frac{b}{a} \right|^{\frac{1}{2}} \geq 8 \left| \frac{c}{a} \right|^{\frac{1}{3}}$. Suppose $|b| > n|a|^{\frac{1}{2}} = \sqrt{816916n}$.

$$\begin{aligned} \frac{\sigma_1}{3} &\geq \left| \frac{b^2}{6a} \right| - 1 > \frac{n}{6} \left| \frac{b}{a^{\frac{1}{2}}} \right| - 1 > \left(\frac{n}{6} - \frac{1}{n} \right) \left| \frac{b}{a^{\frac{1}{2}}} \right| > 8 \left| \frac{b}{a} \right|^{\frac{1}{2}} \\ &\iff \left(\frac{n}{6} - \frac{1}{n} \right) |b|^{\frac{1}{2}} > 8 \\ &\iff \sqrt{n} \left(\frac{n}{6} - \frac{1}{n} \right) |a|^{\frac{1}{4}} > 8 \\ &\iff \sqrt{n} \left(\frac{n}{6} - \frac{1}{n} \right) > \frac{8}{816916^{\frac{1}{4}}} \end{aligned}$$

$$\iff n \geq 2.96.$$

Next, suppose $|c|^2 > \left| \frac{b^3}{a} \right|$, so $8 \left| \frac{b}{a} \right|^{\frac{1}{2}} < 8 \left| \frac{c}{a} \right|^{\frac{1}{3}}$.

$$\begin{aligned} \sqrt{\frac{\sigma_2}{3}} &\geq \sqrt{\left| \frac{3bc^2}{16a} - 1 - \left| \frac{b^4}{48a^2} \right| - \left| \frac{b^2d}{3a} \right| \right|} \\ &\geq \sqrt{\left(\frac{3}{16} - \frac{1}{n^2} - \frac{1}{48} - \frac{1}{3n} \right) \left| \frac{bc^2}{a} \right|} \\ &\geq \sqrt{\left(\frac{3}{16} - \frac{1}{n^2} - \frac{1}{48} - \frac{1}{3n} \right) \left| \frac{b^{\frac{1}{2}}c}{a^{\frac{1}{2}}} \right|} \\ &\geq \sqrt{n}\beta \left| \frac{c}{a^{\frac{1}{4}}} \right| \quad \left(\text{Where } \sqrt{\left(\frac{3}{16} - \frac{1}{n^2} - \frac{1}{48} - \frac{1}{3n} \right)} = \beta \right) \\ &\geq n^{\frac{3}{2}}\beta \left| \frac{c^{\frac{1}{3}}}{a^{\frac{1}{12}}} \right| \geq 8 \left| \frac{c}{a} \right|^{\frac{1}{3}} \\ &\iff n^{\frac{3}{2}}\beta \geq 8 \left| \frac{1}{a} \right|^{\frac{1}{4}} \\ &\iff n^{\frac{3}{2}}\beta \geq \frac{8}{816916^{\frac{1}{4}}} \\ &\iff n \geq 3.67. \end{aligned}$$

So we find $|b| \leq \sqrt{816916} \cdot \max\{2.96, 3.67\} < 3318$. \square

Lemma 4.8. *If $f(z) = az^4 + bz^2 + cz + d \in \mathbb{Q}[z]$ is PCF, then $|c| < 153.4$.*

Proof. As above, we may assume the largest value of $|a|$, and that $\left| \frac{1}{a} \right|^{\frac{1}{3}}$ and $\left| \frac{1}{a} \right|^{\frac{1}{4}}$ will be dominated in $C_{f,\infty}$. First, suppose $|c|^2 \leq \left| \frac{b^3}{a} \right|$, so $8 \left| \frac{b}{a} \right|^{\frac{1}{2}} \geq 8 \left| \frac{c}{a} \right|^{\frac{1}{3}}$. From the above lemma, we then have that $|b| \leq 2.99|a|^{\frac{1}{2}}$, and so $8 \left| \frac{b}{a} \right|^{\frac{1}{2}} \leq 8\sqrt{2.99} \left| \frac{1}{a} \right|^{\frac{1}{4}} = \frac{8\sqrt{2.96}}{816916^{\frac{1}{4}}}$. Suppose $|c| > n|a|^{\frac{1}{4}}$.

$$\begin{aligned} \sqrt[3]{\sigma_3} &\geq \sqrt[3]{\left| \frac{27c^4}{256a} + \left| \frac{b^2d^2}{2a} \right| - \left| \frac{b^3c^2}{64a^2} \right| - \left| \frac{9bc^2d}{16a} \right| - \left| \frac{b^4d}{16a^2} \right| - 1 \right|} \\ &\geq \sqrt[3]{\frac{27n^4}{256} - \frac{2.96^6}{64} - \frac{9(2.96)^4}{16} - \frac{2.96^4}{16} - 1} \\ &\geq \sqrt[3]{\frac{27n^4}{256} - 38.5} > \frac{8\sqrt{2.96}}{816916^{\frac{1}{4}}} \\ &\iff n > 4.38. \end{aligned}$$

Next, suppose $|c|^2 > \left| \frac{b^3}{a} \right|$, so $8 \left| \frac{b}{a} \right|^{\frac{1}{2}} < 8 \left| \frac{c}{a} \right|^{\frac{1}{3}}$, and by the above lemma we have that $|b| \leq 3.67|a|^{\frac{1}{2}}$.

$$\begin{aligned}
\sqrt[3]{\sigma_3} &\geq \sqrt[3]{\left| \frac{27c^4}{256a} + \left| \frac{b^2d^2}{2a} \right| - \left| \frac{b^3c^2}{64a^2} \right| - \left| \frac{9bc^2d}{16a} \right| - \left| \frac{b^4d}{16a^2} \right| - 1 \right|} \\
&\geq \sqrt[3]{\left(\frac{27}{256} - \frac{1}{64} - \frac{9(3.67)}{16n^2} - \frac{3.67}{16n^2} - \frac{1}{n^4} \right) \left| \frac{c^4}{a} \right|} \\
&\geq \delta \left| \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}} \right| \quad \left(\text{Where } \sqrt[3]{\left(\frac{23}{256} - \frac{9(3.67)}{16n^2} - \frac{3.67}{16n^2} - \frac{1}{n^4} \right)} = \delta \right) \\
&> 8 \left| \frac{c}{a} \right|^{\frac{1}{3}} \iff \delta|c| > 8 \\
&\iff n\delta > \frac{8}{816916^{\frac{1}{4}}} \\
&\iff n > 5.1.
\end{aligned}$$

So we find $|c| \leq 816916^{\frac{1}{4}} \cdot \max\{4.38, 5.1\} < 153.4$. □

Repeating the arguments of the above two lemmas when $d = 0$ results in the improved bounds $|b| < 226$ and $|c| < 32$.

4.2 Search

Combining the results from the previous section, our search space is $a \in [-816916, 816916]$, a is an integer multiple of $\frac{1}{3^9}$, $b \in [-3318, 3318]$, b is an integer multiple of $\frac{2}{3^4}$, $c \in [-154, 154]$, c is an integer multiple of $\frac{4}{3^3}$, and $d \in \{0, 1\}$. So the size of our search space is approximately 3.6×10^{19} , which gives a long runtime to any search of the full space.

We can however rule out broad swaths of this space which violate the archimedean bound. We apply a series of filters to the search space to make it more manageable. For $d = 1$:

1. First, we check the corners of unit boxes formed by the intervals $[a, a + 1]$, $[b, b + 1]$, to see whether $\frac{\sigma_1}{3} > C_{f,\infty}$ at all four corners (assuming the worst case, with the largest possible value of $|c|$). Since the function describing σ_1 is monotone except where $a = 0$ or $b = 0$, it is monotone in both variables within each of these boxes, and so if the above condition holds, we can rule out such boxes. This eliminates most possible pairs (a, b) : those that remain are displayed in the below plot:

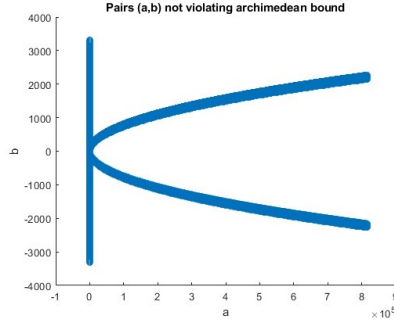


Figure 4.1: Improved bounds on a, b , where $d = 1$

2. For each $a \geq 10$, we search the above results to compute bounds

$$n_{1,a} \leq \left| \frac{b}{a^{\frac{1}{2}}} \right| \leq n_{2,a}.$$

The smaller this interval, the more useful this information is in the following argument bounding $|c|$. Below is displayed $n_{1,a}$ and $n_{2,a}$ for $a \geq 5000$ (where the interval is sufficiently small to help).

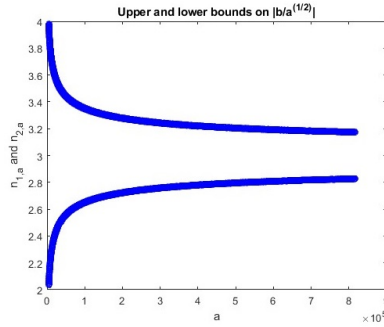


Figure 4.2: Bounds $n_{1,a} \leq \left| \frac{b}{a^{\frac{1}{2}}} \right| \leq n_{2,a}$

3. Use the above relationship to bound $|c|$ for each $a \geq 10$. We have that $n_{2,a}^{\frac{1}{2}} \left| \frac{1}{a} \right|^{\frac{1}{4}} \geq \left| \frac{b}{a} \right|^{\frac{1}{2}}$. We can suppose $|c| \geq 1$, and so $\left| \frac{c}{a} \right|^{\frac{1}{3}} \geq \left| \frac{1}{a} \right|^{\frac{1}{3}}$ and $n_{2,a}^{\frac{1}{2}} > 1$, so $n_{2,a}^{\frac{1}{2}} \left| \frac{1}{a} \right|^{\frac{1}{4}} \geq \left| \frac{1}{a} \right|^{\frac{1}{4}}$. So,

$$C_{f,\infty} \leq \max \left\{ n_{2,a}^{\frac{1}{2}} \left| \frac{1}{a} \right|^{\frac{1}{4}}, \left| \frac{c}{a} \right|^{\frac{1}{3}} \right\}.$$

Note that at $\left|\frac{b}{a^{\frac{1}{2}}}\right| = 2\sqrt{2}$, we have the minimum value of the below

$$1 - \frac{b^2}{3a} + \frac{b^4}{48a^2} = 1 - \frac{(2\sqrt{2})^2}{3} + \frac{(2\sqrt{2})^4}{48} = -\frac{1}{3}$$

and the value becomes a larger positive at $\left|\frac{b}{a^{\frac{1}{2}}}\right| = n_{2,a}$ (from Step 3). So, where

$$m = \max \left\{ \frac{1}{3}, \left| 1 - \frac{n_{2,a}^2}{3} + \frac{n_{2,a}^4}{48} \right| \right\},$$

for each a , search over some possible values of x , letting

$$|c| > \frac{m}{x} |a|^{\frac{1}{4}} \implies x^2 \left| \frac{c^2}{a^{\frac{1}{2}}} \right| > m.$$

So we have

$$\sqrt{\frac{\sigma_2}{3}} \geq \sqrt{\frac{n_{1,a}(3c^2)}{16a^{\frac{1}{2}}} - m} = \sqrt{\frac{3n_{1,a} - 16x^2}{16}} \left| \frac{c}{a^{\frac{1}{4}}} \right|.$$

So, if we also have

$$|c| > \left(\frac{3n_{1,a} - 16x^2}{16n_{2,a}} \right)^{-\frac{1}{2}} \quad \text{and} \quad |c| > \left(\frac{3n_{1,a} - 16x^2}{16} \right)^{-\frac{3}{4}} |a|^{\frac{1}{4}},$$

then $\sqrt{\frac{\sigma_2}{3}} > C_{f,\infty}$, and we have a wandering critical point. So, we have an upper bound on $|c|$ equal to the maximum of the above 3 quantities. This results in improved bounds on $|c|$ for each value of a (for instance, $|c| < 15$ for every $a \geq 20200$).

4. For the remaining steps, we now check each individual value of c . Taking partial derivatives, we can show that the function describing

$$\sigma_2 = \left| 1 - \frac{b^2}{3a} + \frac{b^4}{48a^2} + \frac{3bc^2}{16a} \right|$$

is monotonic in each variable when a , b , and c are nonzero and provided

$$|c| \neq \left| \frac{\sqrt{2}\sqrt{8ab - b^3}}{3} \right| \quad \text{and} \quad |c| \neq \left| \frac{\sqrt{b}\sqrt{3a - 4b^2}}{3\sqrt{a}} \right|.$$

So, if the above two expressions are either strictly less or strictly greater than $|c|$ at all 4 corners of unit boxes as in Step 1, and are monotonic within these boxes, then we can conclude that σ_2 is monotone.

Once again taking partial derivatives of the above expressions, we have that they are

monotone provided

$$|b| \neq 2\sqrt{\frac{2}{3}}\sqrt{a}, \quad |b| \neq 2\sqrt{2}\sqrt{a}, \quad |b| \neq \frac{1}{2}\sqrt{a}, \quad \text{and } b \neq 0.$$

Thankfully, the above are monotone. So, if each of

$$\frac{2\sqrt{\frac{2}{3}}\sqrt{a}}{b}, \quad \frac{2\sqrt{2}\sqrt{a}}{b}, \quad \text{and } \frac{\frac{1}{2}\sqrt{a}}{b}$$

lie within one of $(-\infty, -1)$, $(-1, 1)$, or $(1, \infty)$ at all 4 corners of boxes, then we have the original desired monotonicity.

Now filtering boxes once again, using the improved bounds together with our tests for monotonicity of σ_1 and σ_2 , we eliminate more regions, as in Step 1 above. This eliminates all $a > 11570$.

5. Taking partial derivatives we can show that the function describing

$$\sigma_3 = \left| \frac{b^2 d^2}{2a} - d^3 - \frac{9bc^2 d}{16a} + \frac{27c^4}{256a} - \frac{b^4 d}{16a^2} + \frac{b^3 c^2}{64a^2} \right|$$

is monotonic in each variable when a , b , and c are nonzero and provided

$$|c| \neq \frac{2\sqrt{\frac{\pm\sqrt{b^2(108^2+18ab^2+b^4)+18ab-b^3}}{a}}}{3\sqrt{3}}, \quad |c| \neq 2\sqrt{b},$$

$$|c| \neq \frac{4\sqrt{-b(b^2-4a)}}{\sqrt{3}\sqrt{12a-b^2}}, \quad \text{and } |c| \neq \frac{\sqrt{\frac{2}{3}}\sqrt{b(36a-b^2)}}{3\sqrt{a}}.$$

So, as before, we check that each of the above expressions lie on one side of $|c|$ for all 4 corners of boxes, as in Step 4. But then we need to ensure that the expressions themselves are monotone, so taking partial derivatives again, we find that we need

$$\sqrt{108a^2b^2 + 18ab^4 + b^6} + 9ab + b^3 \neq 0,$$

which holds, as the equality has no real solutions,

$$\pm \frac{3b(6a+b^2)^2}{\sqrt{108a^2b^2 + 18ab^4 + b^6}} + 18a - 3b^2 \neq 0,$$

$$2(48a^2 - 32ab^2 + b^4) \neq 0, \quad \text{and } b^2 - 12a \neq 0.$$

So, we check that at all 4 corners each of the above 4 expressions are either negative or positive, and then check that they are themselves monotone. Once again taking

partial derivatives, we finally require

$$|b| \neq 2\sqrt{4a \pm \sqrt{13a}}, \quad (36\sqrt{183}a^3 + 487a^3)^{\frac{2}{3}} - a^2 \neq 0,$$

$$\text{and } |b| \neq \sqrt{\frac{2}{3}} \sqrt{\left(\sqrt[3]{487 + 36\sqrt{183}} + \frac{1}{\sqrt[3]{487 + 36\sqrt{183}}} - 8 \right) a}.$$

So, we check that the first and third of the above expressions are either strictly less or strictly greater than $|b|$, and that the second inequality holds, and we have the original desired monotonicity.

6. Finally, we put boxes through the above σ_3 filter, and check individual points in those that pass it. For this, we check all possible triples (a, b, c) , filtering through all 3 σ 's. This results in the improved bounds $a \in [-2, 10]$, $b \in [-10, 9]$, and $|c| \leq \frac{84}{27}$. The test for individual points involves iterating critical points 4 times: if they don't escape any of the archimedean or p -adic bounds, then they are potentially PCF, to be verified by hand.

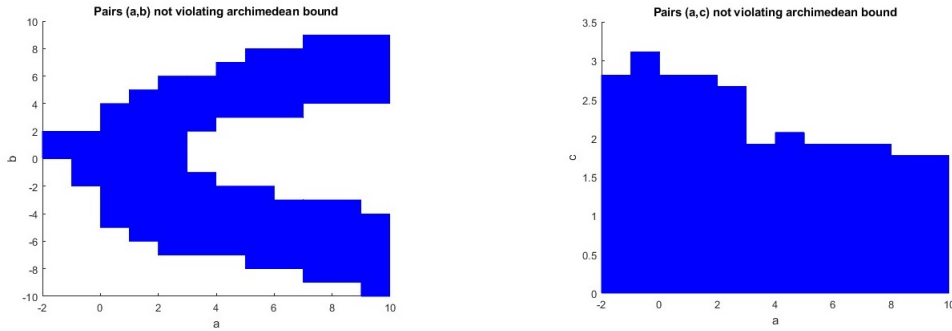


Figure 4.3: Improved bounds on a, b, c , where $d = 1$

Note that in the case $d = 0$ we have that $a, b \in \mathbb{Z}$, $|b|_2 \leq \frac{1}{2}$, $|c|_2 \leq \frac{1}{4}$, and $|c|_3 \leq 3$. So, a, b , and c are integer multiples of 1, 2, and $\frac{4}{3}$ respectively, and the search space is already small enough to be tractable. With the requirement that $|b|^2 \geq |a|$ (as was shown in the proof of Lemma 4.6), we filter through all 3 σ 's, leaving a relatively small set to check, displayed below ($|c| \leq \frac{72}{27}$):

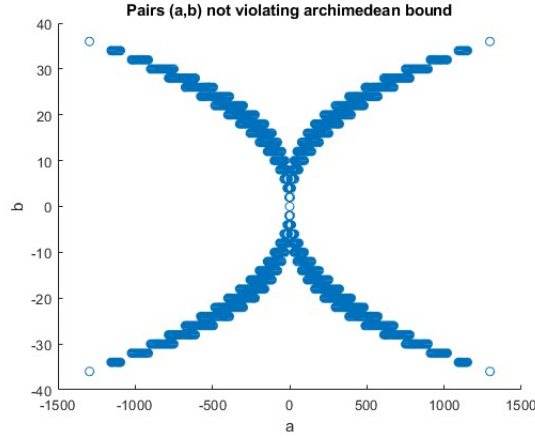


Figure 4.4: Improved bounds on a, b , where $d = 0$

Finally, the points which passed the above archimedean filters are tested p -adically. Lemma 4.2 and Lemma 4.4 help by reducing the number of valid b for each a (especially as a grows larger, and hence is increasingly likely to contain large prime factors). See Appendix A for the code used in the search algorithm.

Theorem 4.9. *There are exactly 16 conjugacy classes of quartic PCF polynomials defined over \mathbb{Q} , representatives of which are given in Table 4.1.*

| $f(z)$ | $f'(z) = 0$ | Critical Portrait |
|---|-------------------------------|-------------------|
| z^4 | 0 | |
| $-z^4 + 1$ | 0 | |
| $-2z^4 + 1$ | 0 | |
| $\frac{8}{3^5}z^4 - \frac{4}{3^2}z^2 + \frac{8}{3^2}z + 1$ | $-3, \frac{3}{2}$ | |
| $\frac{125}{3^5}z^4 - \frac{10}{3^2}z^2 - \frac{8}{3^2}z + 1$ | $-\frac{3}{5}, \frac{6}{5}$ | |
| $\frac{17576}{3^9}z^4 - \frac{52}{3^4}z^2 - \frac{8}{3^3}z + 1$ | $-\frac{9}{26}, \frac{9}{13}$ | |

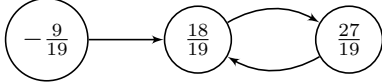
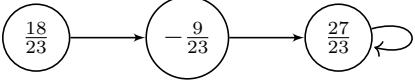
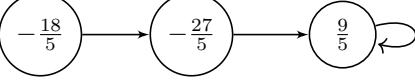
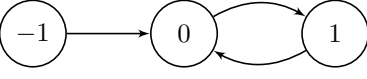
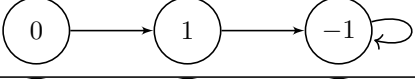
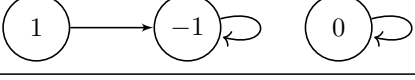

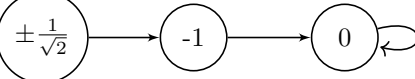
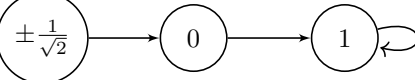
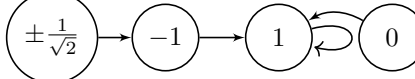
| | | |
|---|--|--|
| $-\frac{6859}{3^9}z^4 + \frac{38}{3^4}z^2 + \frac{8}{3^3}z + 1$ | $-\frac{9}{19}, \frac{18}{19}$ |  |
| $\frac{48668}{3^9}z^4 - \frac{184}{3^4}z^2 - \frac{32}{3^3}z + 1$ | $-\frac{9}{23}, \frac{18}{23}$ |  |
| $\frac{500}{3^9}z^4 - \frac{40}{3^4}z^2 + \frac{32}{3^3}z + 1$ | $-\frac{18}{5}, \frac{9}{5}$ |  |
| $z^4 - 2z^2 + 1$ | $-1, 0, 1$ |  |
| $2z^4 - 4z^2 + 1$ | $-1, 0, 1$ |  |
| $z^4 - 2z^2$ | $-1, 0, 1$ |  |
| $\frac{1}{3}z^4 + \frac{4}{3}z$ | $\zeta_k = e^{(i\pi+2k)/3}, k \in \{0, 1, 2\}$ |  |
| $4z^4 - 4z^2$ | $0, \pm \frac{1}{\sqrt{2}}$ |  |
| $4z^4 - 4z^2 + 1$ | $0, \pm \frac{1}{\sqrt{2}}$ |  |
| $8z^4 - 8z^2 + 1$ | $0, \pm \frac{1}{\sqrt{2}}$ |  |

Table 4.1: Quartic PCF polynomials

The first 3 conjugacy classes are unicritical, the next 6 are bicritical, and the next 7 are tricritical, which can be divided into two groups dependent on whether the critical points are rational. The last polynomial is the Chebyshev polynomial $T_4(z)$.

Note that the bicritical conjugacy classes could be made less messy by removing the requirement that the representative be of the form $az^4 + bz^2 + cz + d$. They are conjugate to those given in Table 4.2, obtained by moving the critical points to 0 and 1.

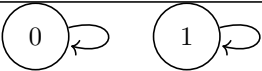

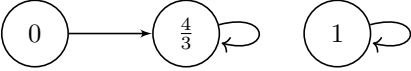
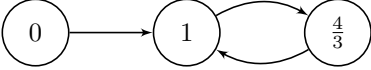
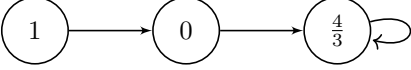
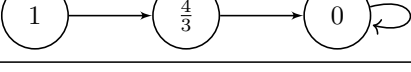
| $f(z)$ | $f'(z) = 0$ | Critical Portrait |
|--|-------------|--|
| $-3z^4 + 4z^3$ | 0, 1 |  |
| $3z^4 - 4z^3 + 1$ | 0, 1 |  |
| $z^4 - \frac{4}{3}z^3 + \frac{4}{3}$ | 0, 1 |  |
| $-z^4 + \frac{4}{3}z^3 + 1$ | 0, 1 |  |
| $4z^4 - \frac{16}{3}z^3 + \frac{4}{3}$ | 0, 1 |  |
| $-4z^4 + \frac{16}{3}z^3$ | 0, 1 |  |

Table 4.2: Bicritical Quartic PCF polynomials

Chapter 5

Families of PCF polynomials

Note that there were no quartic examples found where the algebraic degree of the critical points over \mathbb{Q} is greater than 2. Further, the total number of conjugacy classes found is close to that found in the cubic case in [2]. This might be explained by the fact that as the polynomial degree d increases, there is a tradeoff between increased degrees of freedom, and a more stringent requirement of needing more critical points to be preperiodic.

Consider: what happens to the number of conjugacy classes as $d \rightarrow \infty$, if the number field K remains fixed? We have some additional infinite families to help us explore this question.

In this chapter, a number of infinite families of conjugacy classes of PCF polynomials are described. Many were found by matching polynomials of different degree with similar critical portraits. Some were found by attempting to construct PCF polynomials of higher degree with similar critical portraits to known PCF polynomials of lower degree.

First, let unicritical, bicritical, tricritical, denote polynomials with 1, 2, 3, ... distinct critical points respectively. Observe in Table 5.1 that there are 6 infinite families of conjugacy classes for bicritical polynomials, for any degree $d \geq 3$:

| $f(z)$ | $f'(z) = 0$ | Critical Portrait |
|---------------------------|-------------|-------------------|
| $-(d-1)z^d + dz^{d-1}$ | 0, 1 | |
| $(d-1)z^d - dz^{d-1} + 1$ | 0, 1 | |

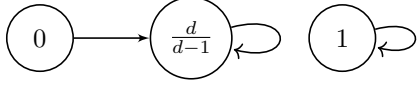
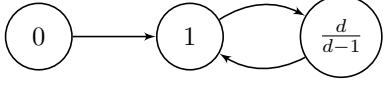
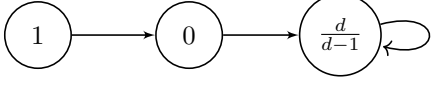
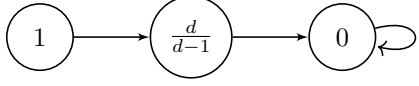
| | | |
|---|------|--|
| $z^d - \frac{d}{d-1}z^{d-1} + \frac{d}{d-1}$ | 0, 1 |  |
| $-z^d + \frac{d}{d-1}z^{d-1} + 1$ | 0, 1 |  |
| $dz^d - \frac{d^2}{d-1}z^{d-1} + \frac{d}{d-1}$ | 0, 1 |  |
| $-dz^d + \frac{d^2}{d-1}z^{d-1}$ | 0, 1 |  |

Table 5.1: Bicritical PCF polynomials

Similarly, there are 2 infinite families of conjugacy classes for tricritical polynomials given in Table 5.2, for any degree $d \geq 4$ (where 0 has multiplicity $d - 3$, and ± 1 have multiplicity 1). Where d is even, they are conjugate to one another.

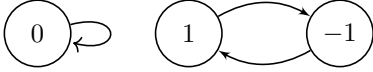
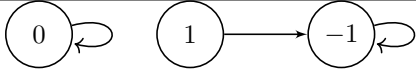
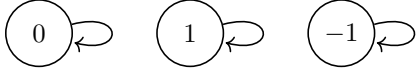
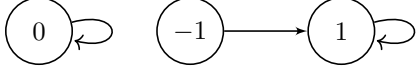
| $f(z)$ | Critical Portrait |
|---|---|
| $\frac{d-2}{2}z^d - \frac{d}{2}z^{d-2}$ |  where d is odd |
| |  where d is even |
| $-\left(\frac{d-2}{2}z^d - \frac{d}{2}z^{d-2}\right)$ |  where d is odd |
| |  where d is even |

Table 5.2: Tricritical PCF polynomials

We also have the following $(d - 1)$ -critical polynomials in Table 5.3, with distinct critical points $\zeta_1, \dots, \zeta_{d-1}$:


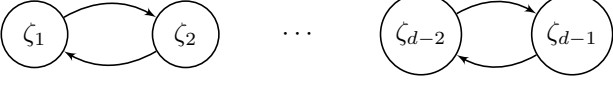

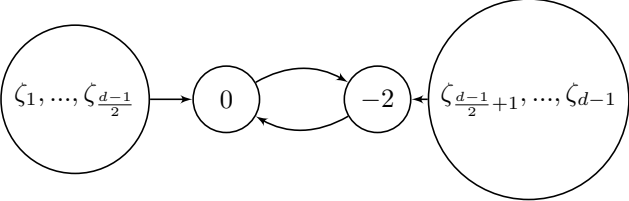
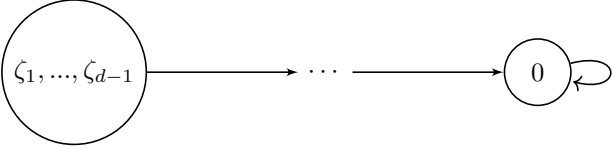
| $f(z)$ | Critical Portrait |
|------------------------|--|
| $z^d + \frac{d}{d-1}z$ |  $\forall i$ |
| $z^d - \frac{d}{d-1}z$ |  where d is odd |

Table 5.3: $(d - 1)$ -critical PCF polynomials

Note that these in Table 5.3 coincide with two of the bicritical families from Table 5.1 in the cubic case. Next, we have families of polynomials given in Table 5.4 which are found by making slight changes to Chebyshev polynomials conjugated to be monic (dots are used in rows 1 and 3 because these critical portraits vary: their only commonality is that each critical point ends at the same fixed point under iteration):

| $f(z)$ | Critical Portrait |
|---|---|
| Where $T_d(z) = z^d + \dots + a_2 z^2 \pm 2$, take $(\frac{1}{2})(z^d + \dots + a_2 z^2) \pm 2$ |  where $d \pmod{4} \equiv 0$ |
| |  where $d \pmod{4} \equiv 2$ |
| Where $T_d(z) = z^d + \dots + a_2 z^2 \pm 2$, take $(\frac{1}{2})(z^d + \dots + a_2 z^2)$ |  where $d \pmod{4} \equiv 0$ |

| | |
|---|---|
| | <p style="text-align: center;">where $d \pmod{4} \equiv 2$</p> |
| <p>Where $T_d(z) \sim z^d - a_{d-2}z^{d-2} \dots + a_1z$, take $z^d + a_{d-2} z^{d-2} \dots + a_1 z$</p> | <p style="text-align: center;">where $d \pmod{4} \equiv 1$</p> |
| | <p style="text-align: center;">where $d \pmod{4} \equiv 3$</p> |

Table 5.4: Chebyshev-like PCF polynomials

For Table 5.5 and Table 5.6, let $m \in \mathbb{Z}$, and define $T(z) = \binom{m}{d} z^d - \binom{m}{d-1} z^{d-1} + \dots - \binom{m}{m+1} z^{m+1}$. Where d is odd, let $m = \frac{d-1}{2}$. We have bicritical families in Table 5.5:

| $f(z)$ | Critical Portrait |
|----------------------|-------------------|
| $T(1)^{-1}T(z)$ | |
| $-T(1)^{-1}T(z) + 1$ | |

Table 5.5: d odd, bicritical PCF polynomials

Where $d \pmod{6} \equiv 4$, let $m = \frac{d-1}{3}$. We have tricritical families in Table 5.6:

| $f(z)$ | Critical Portrait |
|-----------------------|-------------------|
| $T(1)^{-1}T(z)$ | |
| $-T(1)^{-1}T(z) + 1$ | |
| $-2T(1)^{-1}T(z) + 1$ | |

Table 5.6: $d \pmod 6 \equiv 4$, tricritical PCF polynomials

Between the 3 unicriticals, $T_d(z)$, and families from the preceding tables, all quartics found in Theorem 4.9 are accounted for by some infinite family, classified in Table 5.7:

| $f(z)$ | Classification |
|--|------------------|
| z^4 | Unicritical |
| $-z^4 + 1$ | Unicritical |
| $-2z^4 + 1$ | Unicritical |
| $-3z^4 + 4z^3$ | Table 5.1, Row 1 |
| $3z^4 - 4z^3 + 1$ | Table 5.1, Row 2 |
| $z^4 - \frac{4}{3}z^3 + \frac{4}{3}$ | Table 5.1, Row 3 |
| $-z^4 + \frac{4}{3}z^3 + 1$ | Table 5.1, Row 4 |
| $4z^4 - \frac{16}{3}z^3 + \frac{4}{3}$ | Table 5.1, Row 5 |
| $-4z^4 + \frac{16}{3}z^3$ | Table 5.1, Row 6 |
| $z^4 - 2z^2 + 1$ | Table 5.6, Row 2 |
| $2z^4 - 4z^2 + 1$ | Table 5.6, Row 3 |
| $z^4 - 2z^2$ | Table 5.6, Row 1 |
| $z^4 + \frac{4}{3}z$ | Table 5.3, Row 1 |
| $4z^4 - 4z^2$ | Table 5.4, Row 3 |
| $4z^4 - 4z^2 + 1$ | Table 5.4, Row 1 |
| $8z^4 - 8z^2 + 1$ | $T_4(z)$ |

Table 5.7: Quartic PCF polynomial classification

Note every quartic PCF polynomial with rational coefficients is conjugate to one of these, but not necessarily via a conjugation over \mathbb{Q} .

Proposition 5.1. *For any $a \in \mathbb{Z}_{\geq 0}$, $m, n \in \mathbb{N}$, if $d = mn + a + 1$, and $T(z) = \left(\frac{\binom{m}{0}}{d}\right) z^d - \left(\frac{\binom{m}{1}}{d-n}\right) z^{d-n} + \left(\frac{\binom{m}{2}}{d-2n}\right) z^{d-2n} - \dots + \left(\frac{\binom{m}{a+1}}{a+1}\right) z^{a+1}$, then $f(z) = T(1)^{-1}T(z)$ is PCF. If we further have that n and a are even, then $g(z) = -T(1)^{-1}T(z)$ is PCF. If $a \pmod n \equiv -1$, then $h(z) = -T(1)^{-1}T(z) + 1$ is PCF.*

Proof. Note that $f'(z) = T(1)^{-1}z^a(z^n - 1)^m$. So, our critical points are the n^{th} roots of unity, each of multiplicity m , and 0, of multiplicity a . We clearly have that 0 is a fixed point. The n^{th} roots of unity are closed under f :

$$f(e^{2\pi ik/n}) = T(1)^{-1} \left(\left(e^{2\pi ik/n} \right)^d T(1) \right) = \left(e^{2\pi ik/n} \right)^d,$$

thus f is PCF. Note that if n is even, we have an even number of critical points which are roots of unity. So, their negatives are also roots of unity, and we thus have that $g = -f$ is PCF. Further, if a is also even, then d is odd, and g is not conjugate to f , since the lead term of f has odd degree.

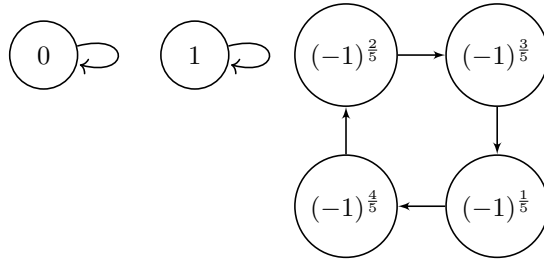
Since $a \pmod n \equiv -1 \implies d \pmod n \equiv 0$, roots of unity are mapped by f to 1. So they are mapped by h to 0, and we have $h(0) = 1, h(1) = 0$, so h is PCF. \square

Note the above explains Table 5.2 and Table 5.5. We can also explain the first row of Table 5.6: since $d \pmod 6 \equiv 4 \implies d - 1 = m(2(1) + 1)$ (that is, where there are two roots of unity and zero as the critical points, all of multiplicity m). The second and third rows of Table 5.6 are explained by the fact that roots of unity are mapped to 1 (as in h), and the fact that those roots of unity are precisely -1 and 1, respectively. It also explains the first two rows of Table 5.1 (taking $a = d - 2, m = 1$, and $n = 1$).

In [1], a lower bound of $\lfloor \frac{d-1}{2} \rfloor$ is found on the number of PGL_2 conjugacy classes of degree $d \geq 3$ conservative polynomials in $\mathbb{Q}[z]$. Conservative polynomials are those whose critical points are fixed, a special case of PCF polynomials. The families in Proposition 5.1 provide more examples in the less restrictive PCF case (close to $(d - 1)^2$ many).

Example 5.2. Where degree $d = 13, a = 2, n = 5$, and $m = 2$. by Proposition 5.1, where $T(z) = \frac{1}{13}z^{13} - \frac{2}{8}z^8 + \frac{1}{3}z^3$, the polynomial $f(z) = T(1)^{-1}T(z) = \frac{156}{25}T(z) = \frac{12}{25}z^{13} - \frac{39}{25}z^8 + \frac{52}{25}z^3$ is PCF, with 6 distinct critical points, each of multiplicity 2.

Note that $f'(z) = \frac{156}{25}z^2(z^5 - 1)^2$, and we have the critical portrait:



Note that we can construct similar rational functions $h(z)$ with the property that the

critical points are roots of unity, which are closed under h .

Example 5.3. Let $T(z)$ be as in Proposition 5.1, and let $h(z) = \frac{T(1)}{T(z)}$. Then we have $h'(z) = -\frac{T(1)T'(z)}{(T(z))^2}$, with critical points 0 and the $(n-1)^{\text{th}}$ roots of unity, with 0 mapped to ∞ and roots of unity closed under h .

Table 5.8 gives the cubic PCF polynomials from [2], and the corresponding infinite families they belong to. The rows with a ‘?’ are those which did not belong to any of the infinite families described in this chapter.

| $f(z)$ | Classification |
|---|-------------------------------------|
| z^3 | Unicritical |
| $-z^3 + 1$ | Unicritical |
| $-2z^3 + 3z^2 + \frac{1}{2}$ | ? |
| $-z^3 + \frac{3}{2}z^2 + 1$ | Table 5.1, Row 4 |
| $-2z^3 + 3z^2$ | Table 5.1, Row 1 & Table 5.3, Row 1 |
| $-3z^3 + \frac{9}{2}z^2$ | Table 5.1, Row 6 |
| $-z^3 + \frac{3}{2}z^2 - 1$ | ? |
| $-4z^3 + 6z^2 - \frac{1}{2}$ | Table 5.4, Row 6 |
| $2z^3 - 3z^2 + 1$ | Table 5.1, Row 2 & Table 5.3, Row 2 |
| $4z^3 - 6z^2 + \frac{3}{2}$ | $T_3(z)$ |
| $2z^3 - 3z^2 + \frac{1}{2}$ | ? |
| $3z^3 - \frac{9}{2}z^2 + 1$ | Table 5.1, Row 5 |
| $z^3 - \frac{3}{2}z^2$ | Table 5.1, Row 3 |
| $-\frac{1}{4}z^3 + \frac{3}{2}z + 2$ | ? |
| $-\frac{1}{28}z^3 - \frac{3}{4}z + \frac{7}{2}$ | ? |

Table 5.8: Cubic PCF polynomial classification

The cubics in rows 3, 7, 11, 14, 15 appear to be exceptional, and do not fit into any of the infinite families found thus far in the above. It is straightforward to check whether there can exist bicritical polynomials with $d > 3$ that have isomorphic critical portraits to these. For instance, if there exists a bicritical quintic similar to $-2z^3 + 3z^2 + \frac{1}{2}$, then it must be of either of the forms

$$\begin{aligned}
 & f'(z) = xz^3(z-1) \text{ or } f'(z) = xz^2(z-1)^2 \\
 \implies & f(z) = x \left(\frac{z^5}{5} - \frac{z^4}{4} \right) + C \text{ or } f(z) = x \left(\frac{z^5}{5} - \frac{2z^4}{4} + \frac{z^3}{3} \right) + C \\
 \implies & f(0) = C \text{ and } f(1) = -\frac{x}{20} + C \text{ or } f(1) = \frac{x}{30} + C
 \end{aligned}$$

$$f(C) = 1 \text{ and } f^2(C) = C$$

$$\begin{aligned} \implies x \left(\frac{C^5}{5} - \frac{C^4}{4} \right) + C = 1 \text{ and } x \left(\frac{(-\frac{x}{20} + C)^5}{5} - \frac{(-\frac{x}{20} + C)^4}{4} \right) + C = C \\ \text{or } f(C) = x \left(\frac{C^5}{5} - \frac{2C^4}{4} + \frac{C^3}{3} \right) + C = 1 \\ \text{and } x \left(\frac{(\frac{x}{30} + C)^5}{5} - \frac{3(\frac{x}{30} + C)^4}{4} + (\frac{x}{30} + C)^3 - \frac{(\frac{x}{30} + C)^2}{2} \right) + C = C, \end{aligned}$$

systems which have no solutions over the rationals.

A cursory search for quintic PCF polynomials with rational coefficients yields only polynomials belonging to one of the infinite families described in this chapter (the expected quintics are given in Table 5.9). Specifically, the region searched was (for quintics conjugated to the form $az^5 + bz^3 + cz^2 + dz + k$) $|a| \leq 100$, $|b| \leq 50$, $|c| \leq 20$, $|d| \leq 10$, $k \in \{0, 1\}$, with $|a|_p, |b|_p, |c|_p, |d|_p \leq 1 \forall p \neq 2$, and $|a|_2, |b|_2, |c|_2, |d|_2 \leq 16$. The quintics not found by this search were the bicriticals from Table 5.1, whose 2-adic absolute values are much higher when conjugated to this form.

| $f(z)$ | Classification |
|--|--|
| z^5 | Uncritical |
| $-z^5 + 1$ | Uncritical |
| $z^5 - \frac{5}{4}z^4 + \frac{5}{4}$ | Table 5.1, Row 3 |
| $-z^5 + \frac{5}{4}z^4 + 1$ | Table 5.1, Row 4 |
| $5z^5 - \frac{25}{4}z^4 + \frac{5}{4}$ | Table 5.1, Row 5 |
| $-5z^5 + \frac{25}{4}z^4$ | Table 5.1, Row 6 |
| $6z^5 - 5z^3 + \frac{15}{8}z$ | Prop 5.1, $f(z)$, $a = 0$, $m = 2$, $n = 2$ |
| $-6z^5 + 5z^3 - \frac{15}{8}z$ | Prop 5.1, $g(z)$ $a = 0$, $m = 2$, $n = 2$ |
| $-4z^5 + 15z^4 - 20z^3 + 10z^2$ | Prop 5.1, $f(z)$ $a = 1$, $m = 3$, $n = 1$ |
| $4z^5 - 15z^4 + 20z^3 - 10z^2 + 1$ | Prop 5.1, $h(z)$ $a = 1$, $m = 3$, $n = 1$ |
| $6z^5 - 15z^4 + 10z^3$ | Prop 5.1, $f(z)$, $a = 2$, $m = 2$, $n = 1$ |
| $-6z^5 + 15z^4 - 10z^3 + 1$ | Prop 5.1, $h(z)$ $a = 2$, $m = 2$, $n = 1$ |
| $-4z^5 + 5z^4$ | Prop 5.1, $f(z)$ $a = 3$, $m = 1$, $n = 1$ |
| $4z^5 - 5z^4 + 1$ | Prop 5.1, $h(z)$ $a = 3$, $m = 1$, $n = 1$ |
| $\frac{3}{2}z^5 - \frac{5}{2}z^3$ | Prop 5.1, $f(z)$, $a = 2$, $m = 1$, $n = 2$ |
| $-\frac{3}{2}z^5 + \frac{5}{2}z^3$ | Prop 5.1, $g(z)$, $a = 2$, $m = 1$, $n = 2$ |
| $-\frac{2}{3}z^5 + \frac{5}{3}z^2$ | Prop 5.1, $f(z)$, $a = 1$, $m = 1$, $n = 3$ |
| $z^5 + \frac{5}{4}z$ | Table 5.3, Row 1 |
| $z^5 - \frac{5}{4}z$ | Table 5.3, Row 2 |
| $z^5 - 5z^3 + 5z$ | $T_5(z)$ |

Table 5.9: Quintic PCF polynomials from aforementioned infinite families

Conjecture 5.4. *Every conjugacy class of PCF polynomials with rational coefficients belongs to one of the infinite families described in this chapter, aside from the 5 exceptional*

cubics in Table 5.8. Namely:

- *Uncritical polynomials*
- *Chebyshev polynomials*
- *Families from Table 5.1, Table 5.3 , Table 5.4 , or Table 5.6*
- *Families generated by Proposition 5.1*

Appendix A: Sage Pseudocode

Below is pseudocode for finding the triples (a, b, c) to be checked (where $|a|_3 = 3^9$, $d = 1$), within the intervals $[a_0, a_1]$, $[b_0, b_1]$, and $[c_0, c_1]$:

```

for i in range(39(a0), 39(a1)):
    # Skip multiples of 3, which would give |a|3 ≤ 38
    if mod(i,3) != 0:
        I = list(factor(i))
        p = 1
        for l in range(0,len(I)):
            # |a|p < 1 ⇒ |b|p < 1, Lemmas 4.2, 4.4
            if I[l][0] != 3:
                p = p * I[l][0]
        a =  $\frac{i}{3^9}$ 
        for j in range(34(b0), 34(b1), 2p):
            # Skip multiples of 3, which would give |b|3 ≤ 33
            if mod(j,3) != 0:
                b =  $\frac{j}{3^4}$ 
                for k in range(33(c0), 33(c1), 4):
                    # Skip multiples of 3, which would give |c|3 ≤ 32
                    if mod(k,3) != 0:
                        c =  $\frac{k}{3^3}$ 

```

Each triple (a, b, c) is run through the following filters to eliminate polynomials which are not PCF:

```

potential = true
C = 8 * max { | $\frac{b}{a}$ |1/2, | $\frac{c}{a}$ |1/3, | $\frac{d}{a}$ |1/4, | $\frac{1}{a}$ |1/3 }

```

```


$$\sigma_1 = \frac{b^2}{2a} - 3d$$


$$\sigma_2 = 3d^2 - \frac{b^2d}{a} + \frac{9bc^2}{16a} + \frac{b^4}{16a^2}$$


$$\sigma_3 = \frac{b^2d^2}{2a} - d^3 - \frac{9bc^2d}{16a} + \frac{27c^4}{256a} - \frac{b^4d}{16a^2} + \frac{b^3c^2}{64a^2}$$

if max  $\left\{ \left| \frac{\sigma_1}{3} \right|, \left| \frac{\sigma_2}{3} \right|^{\frac{1}{2}}, \left| \sigma_3 \right|^{\frac{1}{3}} \right\} \leq C$ :
    primes = Primes()
    for l in range(M):
        p = primes.unrank(l)
        pa = QQ.valuation(p)
         $C_p = \max \left\{ (p^{-pa(b/a)})^{1/2}, (p^{-pa(c/a)})^{1/3}, (p^{-pa(d/a)})^{1/4}, (p^{-pa(1/a)})^{1/3} \right\}$ 
        if max  $\left\{ (p^{-pa(\sigma_1)}), (p^{-pa(\sigma_2)})^{\frac{1}{2}}, (p^{-pa(\sigma_3)})^{\frac{1}{3}} \right\} > C_p$ :
            potential = false
if potential == true:
    x = PolynomialRing(ComplexField(), 'x').gen()
    derivative =  $4ax^3 + 2bx + c$ 
    crit = derivative.roots()
    potential = false
    for k in range(0, len(crit)):
         $\mathcal{O}_k = \text{crit}[k][0]$ 
        for n in range(5):
             $\mathcal{O}_k = a * \mathcal{O}_k^4 + b * \mathcal{O}_k^2 + c * \mathcal{O}_k + d$ 
            if max  $\{ |\mathcal{O}_k| \} > C$ :
                break
        if n == 4:
            potential = true

```

That is, if none of the sigmas exceed either the archimedean constant or the p -adic constant for the first M primes, and none of the critical points exceed the archimedean constant after 4 iterations, we have a potentially PCF polynomial, to be checked manually.

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