BLOCK SYSTEMS OF RANKS 3 AND 4 TOROIDAL HYPERTOPES

ERIC JAMES LOEPP ENS

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Abstract

This dissertation deals with abstract combinatorial structure of toroidal polytopes and toroidal hypertopes. Abstract polytopes are objects satisfying the main combinatorial properties of a classical (geometric) polytope. A regular toroidal polytope is an abstract polytope which can be constructed from the string affine Coxeter groups. A hypertope is a generalization of an abstract polytope, and a regular toroidal hypertope is a hypertope which can be constructed from any affine Coxeter group. In this thesis we classify the rank 4 regular toroidal hypertopes. We also seek to find all block systems on a set of (hyper)faces of toroidal polytopes and hypertopes of ranks 3 and 4 as well as the regular and chiral toroidal polytopes of ranks 3. A block system of a set X under the action of a group G is a partition of X which is invariant under the action of G.

Keywords: Regularity, Chirality, Toroids, Thin Geometries, Hypermaps, Abstract Polytopes, Block Systems

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I also want to thank my family for their encouragement and support without which I would not have been able to spend the last 6 years thinking about fun math problems.

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1 Introduction

The study of polytopes begins with some of the ancient origins of formalized mathematics. Pythagoras set up his Pythagorean school to study mathematics and philosophy where geometric shapes held special, mystical meanings. This mystic view amongst the ancient Greeks persisted for centuries and we still refer to the five regular convex polyhedra as *Platonic solids* (see Figure 1.1). From a mathematical point of view, Theatetus (himself a contemporary of Plato) gave a mathematical description of all five Platonic solids and may have been responsible for the first known proof that no other convex regular polyhedra exist.

Figure 1.1: The five Platonic solids. Models built and photographed by Dr. David Gunderson

Another set of highly symmetric polyhedra were said to have been described by Archimedes and are still referred to as the *Archimedean solids*.

The draw of these geometric shapes held sway thousands of years after which Johannes Kepler was convinced the orbits of the planets were determined by the Platonic solids (see Figure 1.2). He held this view until careful observation led him to the discovery that planetary orbits were elliptical. Kepler would also make important discoveries in geometry itself with the discovery of a set of "star" shaped polyhedra.

The modern concept and study of polytopes began with initial work by H.S.M. Coxeter in his book *Regular Polytopes* [11] and by Branko Grünbaum in *Convex Polytopes* [19]. The concept of an abstract polytope was formalized by Egon Schulte in his dissertation [30] and then later with L. Danzer in a series of papers [13], [14]. It was later collected and expanded in the book *Abstract Regular Polytopes* [26] by Peter McMullen and Egon Schulte.

The concept of a polytope is extremely broad and there is no single definition widely used. Intuitively the term polytope generalizes into any dimension the idea of a polygon or polyhedron. Some definitions may include polytopes with an infinite number of faces called *apeirotopes* or polytopes whose faces intersect each other called *star polytopes*. We will be focused on primarily two definitions, *convex polytopes* and *abstract polytopes*, with a heavy focus on the latter.

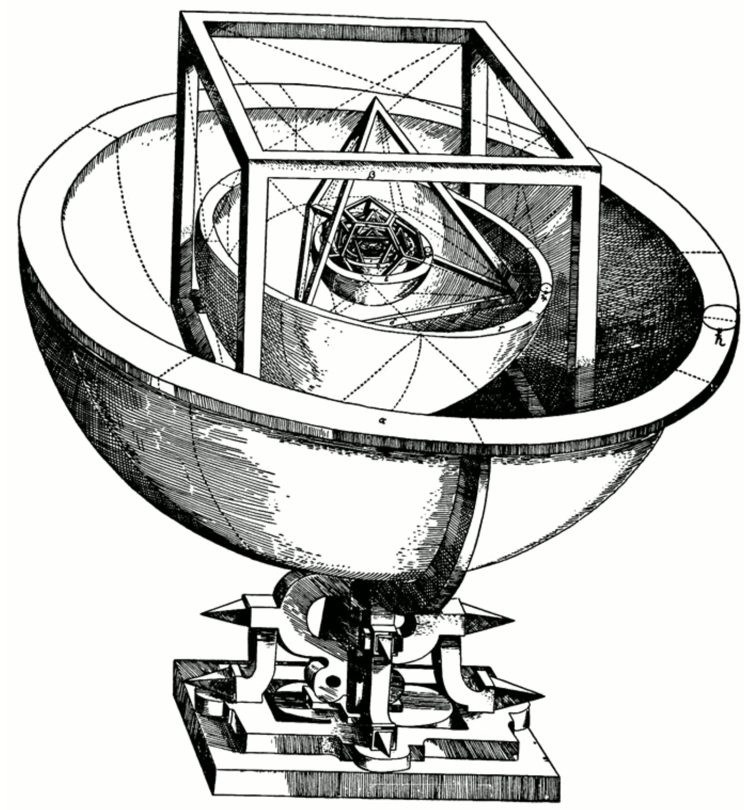

Figure 1.2: Kepler's platonic solids model for the solar system.

A convex polytope in n dimensions is the convex hull of a finite number of points in $\mathbb{E}^n$[1]. Because a convex polytope exists in space it can be described by usual Euclidean metrics like distances. This is contrasted by the next way to describe a polytope, that of an abstract polytope.

When describing an abstract polytope we look to use the field of combinatorics. To do this an abstract polytope is described as a set whose elements are the vertices,

[1] Think of the shape obtained by shrink-wrapping a set of points

edges and other similar analogues in as high a rank as needed as well as specific relationships between these element. These relationships are called *incidences* and describe inclusion. For example, a vertex is incident with an edge when it lies on that edge. In this context, that means that an abstract polytope is described by how many vertices there are, how many edges each face has, etc as well as their incidences. This concept differs from a traditional polytope because it doesn't include measurements like angles or area. As a result there is no differentiation between a square and a rectangle or a skew 4-gon. All have four edges and four vertices, with each vertex connecting two edges.

A more traditionally defined polytope like a convex polytope can be associated with an abstract polytope by using the convex polytopes' incidences to define a partial order.

An abstract polytope is described as a partially ordered set. This partial order locates edges above vertices and faces above edges, etc. There are two other important elements that must be included, a unique smallest element representing an empty set, and a unique largest element representing the whole polytope. Then we can connect various elements in the set by the incidence relationship, ie an edge is connected to the two vertices that it connects. The graph whose vertices are elements of an abstract polytope and whose edges are the incidence relationships is called the incidence graph. A more useful graph is called the *Hasse diagram*.

This is constructed by arranging the vertices of incidence graph in horizontal layers according to their rank with the smallest element at the bottom, followed by vertices, edges, etc. Then we preserve only those edges from the incidence graph which connect one "level" of elements with the level directly above or below. See Figure 1.3 as an example of a square bottomed pyramid together with its Hasse diagram. U is the unique largest element which represents the whole polytope, the uppercase letters P through T are the faces, the lower case letters the edges and the upper case letters A through E are the vertices and the empty set is the unique lowest element.

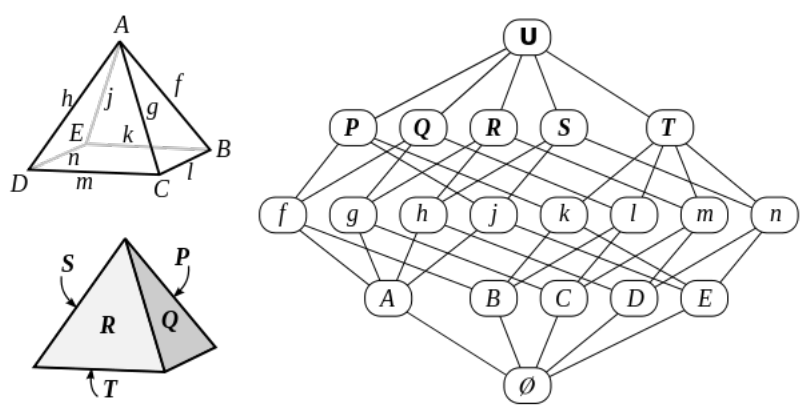

Figure 1.3: A square pyramid and accompanying Hasse diagram [20].

One of the most interesting concepts in connection with a classical polytope is that of symmetry. For convex polytopes, which exists in space, we define its symmetries as *isometries* of space which fix the convex polytope. From the perspective of abstract polytopes, a symmetry is replaced by an automorphism of a polytope that is a bijection which preserves its partial order. Both symmetries and automorphisms form a group by composition. For a convex polytope $\mathcal{P}$, we call this group its *Symmetry group*, denoted by $\mathrm{Sym}(\mathcal{P})$. For an abstract polytope we call this group its *Automorphism group* denoted by $\mathrm{Aut}(\mathcal{P})$ or more commonly $\Gamma(\mathcal{P})$. Around these automorphisms arises the study of *Coxeter groups*, where the regular polytopes' automorphism groups are quotients of Coxeter groups.

A maximal chain from the bottom of a polytope's Hasse diagram to the top consists of one element of each rank, from the smallest element to a vertex to an edge and so on. Such a chain is referred to as a *flag*. In the classical theory, a polytope is said to be *regular* if for all ranks all its faces of that rank are alike, and the faces themselves are all regular all the way down to the vertices which must also all be alike. Additionally, the vertex-figures (the face you get when you "slice" off a vertex) must also be regular. Note that the faces must then all be regular polygons, with each side length and angle measure the same. In the case of abstract polytopes, we say a polytope is regular when each flag can be mapped to each other flag by an automorphism.

An important class of polytope is a *tessellation* (its higher dimensional versions sometimes called *honeycombs*). When we talk about a tessellation we mean, informally, a locally finite collection of polytopes which covers a Euclidean, spherical, or hyperbolic space in a face-to-face manner. See Figure 1.4 for an example of a portion of the infinite tiling of the Euclidean plane by squares. Even though it arises from 2 dimensions, it is a rank 3 polytope, since its *facets*, or proper faces of highest dimension are rank 2 polytopes.

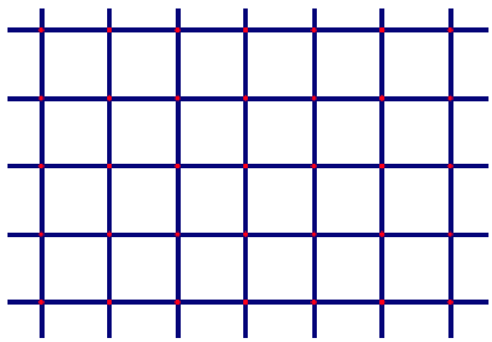

Figure 1.4: The tiling of the Euclidean plane by squares.

The next key concept is that of a *toroidal polytope*. A toroidal polytope is an abstract polytope that can be seen as a tessellation on a torus. A toroidal polytope can then be defined as a "quotient" of a Euclidean tessellation by linearly independent translations. For a precise definition of a toroidal polytope see [23]. Highly symmetric types of these polytopes are well known and understood, in particular the regular and chiral toroidal polytopes have been classified for rank 3 by Coxeter in 1948 [10] (see also [12]) and for any rank by McMullen and Schulte [25]. Regular toroidal polytopes (and also regular toroidal hypertopes, which we define below) are strongly related to a special class of Coxeter groups, the infinite irreducible Coxeter groups of Euclidean type which are also known as affine Coxeter groups (see, for example page 73 in [26]). The symmetry groups of regular tessellations of Euclidean space are precisely the affine Coxeter groups with string diagrams (see theorem 3B5 in [26]).

Informally, a 2-dimensional torus is a doughnut shaped surface, but more specifically we can form a toroidal polyhedron by starting with rectangular portion of a tessellation, then glueing two of the opposite sides to form a tube, then glueing two ends of the tube to form a torus. This process can be scaled up to higher dimensions as well, though it gets much more difficult to visualize. In 4 dimensions, this involves glueing opposite faces of a 3 dimensional parallelepiped. Figure 1.5 shows which sides of a rectangle get glued to each other as well as a toroidal polytope whose faces are squares.

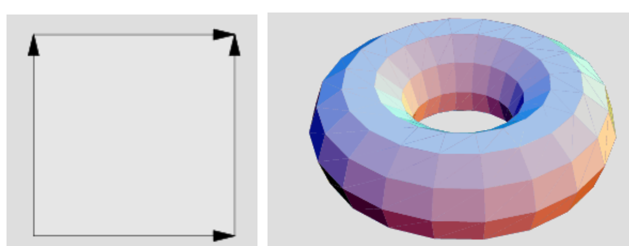

Figure 1.5: A torus represented in 2 and 3 dimensions [34].

A block system, or in this case, a block system on the faces of a polytope can be illustrated using colourings. A block system on the faces of a polytope can be seen as a way to colour the faces such that the colouring is not necessarily fixed by the group of automorphisms, but is respected by it. By way of example (see Figure

1.6), the faces of a cube are coloured by three colours, each face the same colour as the opposite face, this colouring does not get fixed by the group of symmetries of the cube as, after a rotation, red faces may be where green faces were prior, but each face is still the same colour as the opposite face. The rule for colouring the cube is still satisfied.

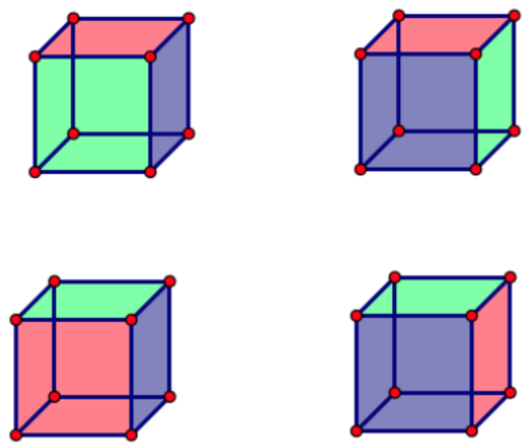

Figure 1.6: A block system on a cube and several rotations.

On the other hand, consider the colouring where all the faces of a cube are coloured blue except the top face which is coloured red. Then if the cube is rotated by 90 degrees so that the red face moves to face the reader (see Figure 1.7), the rule we used to colour the cube is no longer satisfied.

One of the two main results that will be examined in this thesis is the finding of all the block systems on various important types of toroidal polytopes of rank 3

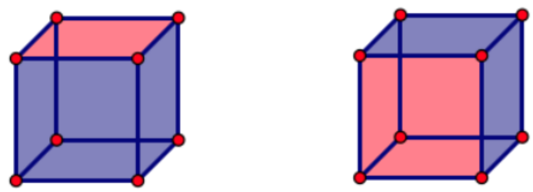

Figure 1.7: A colouring which is not a block system

and 4. To do this, I use the various symmetries of the initial tessellations as well as the techniques used by McMullen and Schulte in [26] to construct the toroids. This can help to examine a relationship between toroids and constructing block systems.

The other main area of investigation involves hypertopes. The concept of a hypertope has recently been introduced by Fernandes, Leemans and Weiss (see [18]). A hypertope can be seen as a generalization of a polytope. Or, from another perspective, as a generalization of a hypermap. For more infomation on hypermaps see [9]. Hypertopes resemble polytopes in many ways, however some of the more intuitive structure of polytope may not exist in a hypertope. An example is the rule that, in a polytope, an edge of a rank 3 polytope must be incident with exactly 2 faces and exactly two vertices, this need not be the case for hypertopes. Also there is no partial order on the set of elements in a hypertope like there is for a polytope. Like polytopes, we can find infinite hypertopes that we can then turn into toroidal hypertopes. Figure 1.8 is an example of a representation of an infinite hypertope.

In this case, the grey triangles with red centres represent "hyperedges" which connect three green vertices and three, white "hyperfaces". A notable property of this particular hypertope is that there is no properties which distinguish a hyperface from a hyperedge from a hypervertex. Though we illustrate each with distinguishing features to help contextualize the object in comparison to a polytope.

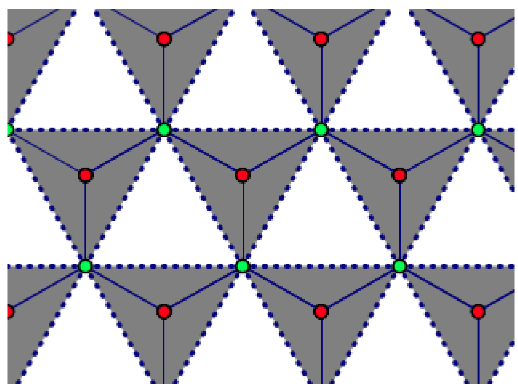

Figure 1.8: A "tiling" by hypertopes.

A *regular toroidal hypertope* (see Chapter 4 for a precise definition) can be seen as a quotient of a "tessellation" by hypertopes by a normal subgroup of translations, denoted by Λ_T.

The result I present in this thesis in regards to hypertopes is the classification of all regular rank 4 toroidal hypertopes, this has been previously published in [16]. The work extends the results of Schulte and McMullen in [25] on the classification of regular toroidal polytopes. For each of the two non-string affine Coxeter groups I showed there are three families of such toroids. I also include a formulation of a definition for a regular toroidal hypertope starting at Proposition 4.2.1 as well as the classification of the toroids described in sections 4.4, 4.5 and 4.6. Each affine Coxeter group in rank 4 (which are usually denoted by $\widetilde{C_3}$, $\widetilde{B}_3$ and $\widetilde{A}_3$), as we shall see, can be associated with the group $\widetilde{C_3} = [4,3,4]$, the symmetry group of the cubic tessellation of $\mathbb{E}^3$. The *Coxeter complex*, denoted by $\mathcal{C}$, of $\widetilde{C_3}$ can be seen as the simplicial complex obtained by the barycentric subdivision of the cubic tessellation $\{4,3,4\}$. The Coxeter complex for the other two rank 4 affine Coxeter groups can be obtained by doubling the rank 3 simplicies for $\widetilde{B}_3$ and quadrupling them for $\widetilde{A}_3$.

Block Systems are a concept without a great deal of literature describing them and without any work done connecting them to polytopes (much less hypertopes). The context in which block systems most often appear is in the study of permutation groups. In particular when a permutation group action has only trivial block systems it is a primitive permutation group and can be used to find primitive Galois groups[4][15][29]. Thus, the literature around block systems is primarily concerned with avoiding them. The work I undertake moves away from permutation groups and into Geometry in order to treat block systems as a subject themselves. Furthermore, I tie the new concept of hypertopes into the problem of finding block systems, I have found all the block systems of particular "hyperfaces" of these toroidal hypertopes under the action of their automorphism groups. This is done throughout chapters 5 and 6. Of particular interest to the reader are the cases discussed in sections 5.4 and 6.2 as these cases have block systems which are more difficult to find and require more than a simple lemma.

2 Abstract Polytopes

2.1 Abstract Polytopes

In this section we will define abstract polytopes and investigate their "symmetries." Much of this chapter is collected in Abstract Regular Polytopes by McMullen and Schülte[26]. Traditionally, polytopes are thought of as geometric objects defined in space using measurements like angles and lengths and volumes. In the study of abstract polytopes however, we will define them as combinatorial objects, in which case we use *rank* in an analogy to dimension in a geometric space. Instead of the inclusion relationship between elements of a polytope like vertices or faces we describe a combinatorial relationship by defining a polytope as a partially ordered set together with a rank function and a number of conditions. These conditions ensure that the poset behaves in a way that it satisfies the main properties of geometric polytopes.

The analogue of the dimension of a geometric polytope when viewed as an abstract polytope is the *rank* of the abstract polytope. We call elements of a polytope

$\mathcal{P}$ *faces*. We also call rank 0 faces *vertices* and rank 1 faces *edges* as usual. If $\mathcal{P}$ is *rank* n then the rank $(n-1)$ faces we call *facets*. We also describe a unique rank n face representing the whole polytope and a unique rank -1 face representing the empty set.

Definition 2.1.1. An *abstract polytope* of rank n is a (possibly infinite) partially ordered set $\mathcal{P}$ with a strictly monotone rank function with range $\{-1, 0, \ldots, n\}$ that satisfies the following properties:

- **$\mathcal{P}$ has a unique minimal element and a unique maximal element.** We call these *improper faces* and denote the smallest element by $\emptyset$ and the largest element by $\mathcal{P}_n$. Elements of $\mathcal{P}$ of rank j are called *j-faces*.

- **All maximal chains have $n+2$ elements.** These subsets are called *flags*. The *j-face* of a flag Φ is denoted $(\Phi)_j$ and the set of all flags of $\mathcal{P}$ is denoted $\mathcal{F}(\mathcal{P})$.

- **Given faces $F < G$ with $rank(F) = i-1$ and $rank(G) = i+1$ there are exactly two *i-faces* H_1 and H_2 where $F < H_1, H_2 < G$.** This is referred to as the *diamond condition*.

- **Given a pair of flags Φ and Ψ, there exists a sequence of flags $\Phi_0, \ldots, \Phi_k$ such that $\Phi_0 = \Phi$, $\Phi_k = \Psi$, $|\Phi_i \cap \Phi_{i+1}| = n+1$ and $\Phi \cap \Psi \subseteq \Phi_i$**

for all i**.** This condition is referred to as *strongly flag-connected*. The condition that $|\Phi_i \cap \Phi_{i+1}| = n + 1$ says that each flag in the sequence differs from the subsequent flag by only one element.

Figure 2.3 shows the partially ordered set of a tetrahedra where one can observe its various flags. We also note Figure 2.1 as an example of an object which is not a polytope because it does not satisfy strong flag-connectivity. We note the pinched section of the object and the two flags highlighted in yellow share a vertex, but any sequence of i-adjacent flags from one to the other would require moving away from that vertex.

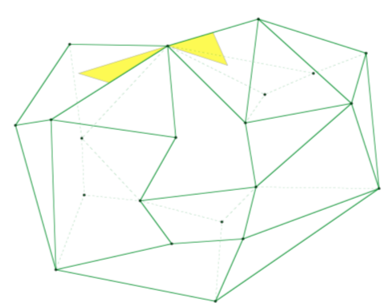

Figure 2.1: A geometric object which does not satisfy strong flag connectivity. From Isabel Hubard's doctoral dissertation [22].

One key difference between defining a polytope in this way, as opposed to the more classical convex polytope, is that all p-gons (for example all 5-sided polygons) are equivalent, see Figure 2.2.

Figure 2.2: Both the pentagon and the pentagram have 5 vertices and 5 edges with 2 edges at each vertex.

We note that a convex polytope can be described as an abstract polytope by describing its incidences. Furthermore we can similarly describe a flag for a convex polytope.

From this point on, we will refer to an abstract polytope as simply a polytope, since that is the scope of this thesis. When needed, we will differentiate by specifying a polytope as a convex polytope.

Remark 1. From the diamond condition we can see that for each i in the range $\{0, n-1\}$, every flag Φ has unique flag Φ^i differing from Φ only at the i-face and thus $|\Phi \cap \Phi^i| = n+1$. We say that Φ^i is *i-adjacent* to Φ. We also note that $(\Phi^i)^i = \Phi$ and that if $|i-j| \geq 2$ then $\Phi^{i,j} = \Phi^{j,i}$.

Given a property like strong flag-connectivity it is natural to wonder if there is a weaker version of this property. There is, though it is not much use to us in this thesis. We define this weaker property by dropping the condition that $\Phi \cap \Psi \subseteq \Phi_i$.

We say $\mathcal{P}$ is *flag-connected* if, given a pair of flags Φ and Ψ, there exists a sequence of flags $\Phi_0, \dots, \Phi_k$ such that $\Phi_0 = \Phi$, $\Phi_k = \Psi$, $|\Phi_i \cap \Phi_{i+1}| = n+1$ for all i.

Definition 2.1.2. Given two faces F and G of a polytope $\mathcal{P}$, a *section* is a subset $G/F := \{H : F \leq H \leq G\}$.

Given a vertex F_0, the section $\mathcal{P}/F_0$ is the *vertex figure* at F_0. A facet F_{n-1} for example, can also be seen as the section $F_{n-1}/\emptyset$. We can also see that all sections of a polytope are also polytopes with $rank(G/F) = rank(G) - rank(F) - 1$.

The *incidence graph* of a poset X is defined as the graph with the vertex set X where the element x_i is adjacent to x_j if and only if $x_i \leq x_j$ or $x_j \leq x_i$. A diagram often used to represent posets is an oriented graph called the *Hasse diagram* which describes each i-face's incidences with its adjacent $i+1$ and $i-1$ faces with rank oriented vertically. Figure 2.3 shows the Hasse diagram of a tetrahedron and the earlier Figure 1.3 shows the Hasse diagram for a square bottomed pyramid.

We now introduce some theorems which are equivalent to the definition of an abstract polytope which can be used to find more meaningful results later.

Definition 2.1.3. A poset X with maximum element F and minimum element $\emptyset$ is *connected* if the incidence graph of $X \setminus \{\emptyset, F\}$ is connected.

Theorem 2.1.4 (Proposition 2A1 in [26])**.** *A poset $\mathcal{P}$ satisfying the first three conditions in definition 2.1.1 is strongly flag-connected if and only if every section*

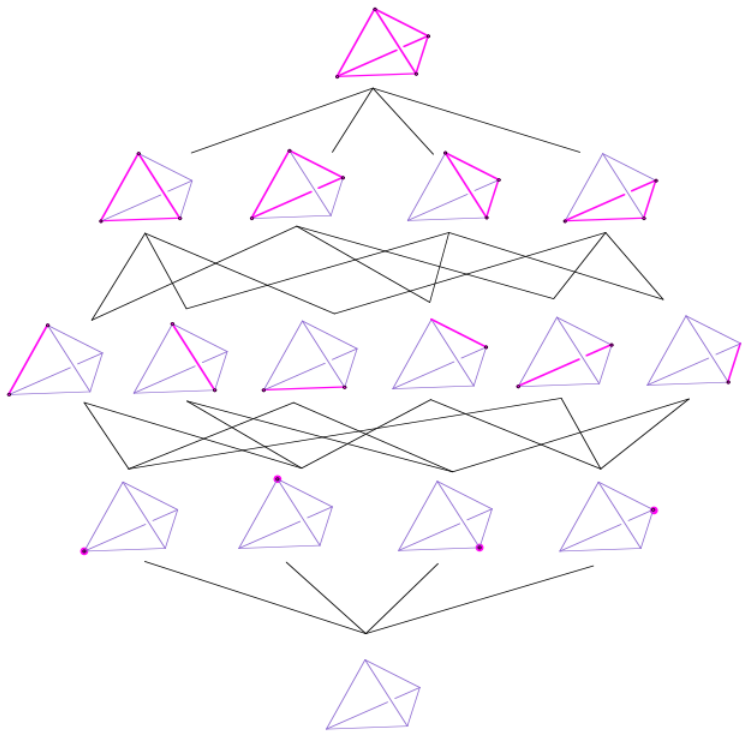

Figure 2.3: The Hasse diagram of a tetrahedron. From Isabel Hubard's doctoral dissertation [22].

of $\mathcal{P}$ of rank at least 2 is connected.

Proposition 2.1.5. *If $\mathcal{P}$ is a poset with rank $n \geq 3$ satisfying the first two properties in definition 2.1.1 and all facets and vertex figures of $\mathcal{P}$ are polytopes, then $\mathcal{P}$ is a polytope if and only if it is flag-connected.*

Proof. The diamond condition holds since every diamond is contained in a face or vertex figure. Meanwhile, $\mathcal{P}$ is strongly flag-connected since all sections are connected. ■

Earlier we mentioned that, when seen an abstract rank 2 polytope, all p-gons are equivalent. So it becomes necessary to find a way to compare abstract polytopes.

Definition 2.1.6. An *isomorphism* from a polytope $\mathcal{P}$ to a polytope $\mathcal{Q}$ is a bijection from $\mathcal{P}$ to $\mathcal{Q}$ that preserves the partial order. If such an isomorphism exists, $\mathcal{P}$ is said to be *isomorphic* to $\mathcal{Q}$.

We now introduce a special class of polytopes which are called equivelar. These are polytopes which can be described with a *type* which will be explained below. Recall that given a flag Φ of a polytope, the $i-face$ of that flag is denote by $(\Phi)_i$.

Definition 2.1.7. We say that an abstract polytope $\mathcal{P}$ is *equivelar* if, for every flag Φ of $\mathcal{P}$ and every $i \in \{1, \ldots, n-1\}$, there exists an integer p_i such that the sections $(\Phi)_{i+1}/(\Phi)_{i-2}$ are all isomorphic p_i-gons.

An equivelar polytope can be assigned a *Schläfli type* $\{p_1, \ldots, p_{n-1}\}$. The Schläfli type of a p-gon is simply $\{p\}$ and the Schläfli type of an equivelar polyhedron is $\{p, q\}$, where it has p-gons for facets and q-gons for vertex-figures. By way of example, the Schläfli type of a cube is $\{4, 3\}$ telling us it has squares for facets and triangles as vertex-figures. When seen as an abstract polytope, the tiling of the Euclidean plane by squares has Schläfli type $\{4, 4\}$. A square bottomed pyramid however, is not equvaler since the vertex figure of the top is different from the other vertex figures. An additional side note is that all the Platonic solids are equivelar as are all polygons.

2.2 Automorphisms of Polytopes

One of the key concepts in the study of polytopes is that of "symmetry". In the previous section we introduced the concept of an isomorphism between polytopes. As mentioned in the introduction, the symmetry group of a convex polytope is the group of isometries preserving the polytope. This group also acts on the flags of the convex polytope. We use that to develop a concept of "symmetry" for an abstract polytope. The following concepts are collected in [26].

Definition 2.2.1. An *automorphism* of a polytope $\mathcal{P}$ is an isomorphism from $\mathcal{P}$ to itself.

Now we can show a property of automorphisms relating to flag adjacency.

Proposition 2.2.2. *For an abstract polytope $\mathcal{P}$, if γ is an automorphism of $\mathcal{P}$ and Φ a flag, then $(\Phi\gamma)^i = \Phi^i\gamma$.*

Proof. The j-faces of $(\Phi\gamma)^i$ and $\Phi^i\gamma$ are the same when $i \neq j$. Let F and G be the two i-faces between $i+1$ and $i-1$ faces of Φ labeled in Figure 2.4 as H^{i+1} and H^{i-1}.

Since γ is an automorphism $H^{i-1}\gamma \leq F\gamma$, $G\gamma \leq H^{i+1}\gamma$ and $F\gamma \neq G\gamma$ (since otherwise γ is a bijection). Therefore the i-face of $(\Phi\gamma)^i$ is $G\gamma$ and this is also the

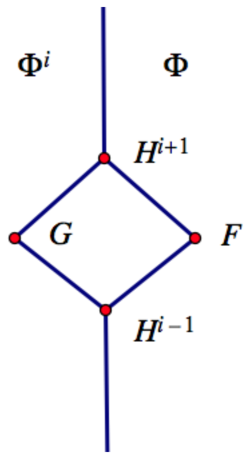

Figure 2.4: Flags F and G

i-face of $\Phi^i\gamma$. ■

Corollary 2.2.3. $\Phi^{i_1,\ldots,i_k}\gamma = (\Phi\gamma)^{i_1,\ldots,i_k}$

Proposition 2.2.4. *A bijection of the flags of a polytope $\mathcal{P}$ that preservers i-adjacency for every $i \in \{-1,\ldots,n\}$ induces an automorphism of $\mathcal{P}$.*

Proof. Suppose φ is such a bijection, then we define ψ from $\mathcal{P}$ to $\mathcal{P}$ by $F\psi = G$ whenever $rank(F) = rank(G)$ and if $F \in \Phi$ then $G \in \Phi\varphi$.

If $F \in \Phi$ and $F \in \Phi'$ then there exists $i_1,\ldots,i_k$ such that $\Phi' = \Phi^{i_1,\ldots,i_k}$ where $i_j \neq rank(F)$. Since ψ preserves $rank(F)$-adjacencies, $\Phi'\psi = (\Phi^{i_1,\ldots,i_k})\psi = (\Phi\psi)^{i_1,\ldots,i_k}$ and the $rank(F)$-face must coincide. Thus ψ is well-defined.

Now we show ψ preserves incidence. If $F_1 \leq F_2 \in \Phi$ then $F_1\psi, F_2\psi \in \Phi\varphi$, and ψ is a bijection since if G is a face of $\mathcal{P}$, then G belongs to a flag Φ and it is the image of the face in $\Phi\varphi^{-1}$ of the same rank.

Finally we show that ψ is a bijection. Let $F_1, F_2 \in \mathcal{P}$ with $rank(F_1) = rank(F_2) = k$. If $F_1\psi = F_2\psi$ then there exists flags Φ_1, Φ_2 with $F_i \in \Phi_i$ such that the face of rank k in $\Phi_1\varphi$ and $\Phi_2\varphi$ are the same. Then $\Phi_2\varphi = (\Phi_1\varphi)^{i_1,\ldots,i_l}$ with $i_j \neq k$ and $\Phi_2 = \Phi_1^{i_1,\ldots,i_l}$ so they have the same k-face and $F_1 = F_2$. ■

The set of automorphisms of a polytope $\mathcal{P}$ forms a group under composition, which is denoted $\Gamma(\mathcal{P})$. Let $\gamma \in \Gamma(\mathcal{P})$ be an automorphism that fixes some flag Φ, let Ψ be another flag of $\mathcal{P}$. Then we can find $i_1, \ldots, i_k$ where $\Psi = \Phi^{i_1,\ldots,i_k}$ and $\Psi\gamma = \Phi^{i_1,\ldots,i_k}\gamma = (\Phi\gamma)^{i_1,\ldots,i_k} = \Phi^{i_1,\ldots,i_k} = \Psi$ and so γ is the identity. $\Gamma(\mathcal{P})$ then acts freely on the set of flags. Consequently, the orbits of all flags under $\Gamma(\mathcal{P})$ have the same size which is the order of $\Gamma(\mathcal{P})$.

We will utilize orbits to define the meaning of a regular polytope in the context of abstract polytopes. A polytope whose automorphism group induces k orbits on the set of flags is called a *k-orbit* polytope.

Definition 2.2.5. A polytope is *(combinatorially) regular* if it is a 1-orbit polytope.

This definition is equivalent to saying that a polytope $\mathcal{P}$ is regular if $\Gamma(\mathcal{P})$ acts transitively on the set of flags $\mathcal{F}(\mathcal{P})$. Since regularity for a convex polytope can be defined as requiring its symmetry group to act transitively on its flags, a regular convex polytope when seen as an abstract polytope retains its regularity as combinatorial regularity (Theorem 2B1 in [26]). As such Platonic solids are all combinatorially regular.

Let us fix a flag Φ of a regular polytope $\mathcal{P}$ and call this flag a *base flag* of $\mathcal{P}$. Then we denote by ρ_i the unique [2] automorphism of $\mathcal{P}$ mapping Φ to Φ^i. We call these automorphisms *distinguished generators* of $\mathcal{P}$ with respect to the base flag Φ.

We denote the identity automorphism by *id*. Note that if a polytope is regular, it is also equivelar (though the converse is not necessarily true). This arises from the fact that given two flags Φ and Ψ the sections Φ_{i+1}/Φ_{i-2} and Ψ_{i+1}/Ψ_{i-2} are isomorphic. If a polytope $\mathcal{P}$ is a regular polytope with Schläfli type $\{p_1, \ldots, p_{n-1}\}$ then the order $o(\rho_{i-1}\rho_i) = p_i$ for every i. In particular,

$$(\rho_{i-1}\rho_i)^{p_i} = id. \tag{2.1}$$

The following proposition shows the relationship between regularity and distinguished generators.

Proposition 2.2.6. *Given an n-polytope $\mathcal{P}$ and a base flag Φ, $\mathcal{P}$ is regular if and only if distinguished generators ρ_i exists for all $i \in \{0, 1, \ldots, n-1\}$.*

Proof. The forward implication follows directly from the definition of a regular polytope, so we will prove the backwards implication.

---

[2] Uniqueness follows from the fact that $\Gamma(\mathcal{P})$ acts freely on the flags of $\mathcal{P}$ and there is only one orbit of flags, thus $|\Gamma(\mathcal{P})| = |\mathcal{F}(\mathcal{P})|$

Let Φ be the base flag of $\mathcal{P}$ and Ψ be another flag. Then there exists $i_1, \dots, i_k$ such that $\Psi = \Phi^{i_1,\dots,i_k} = (\Phi\rho_{i_1})^{i_2,\dots,i_k} = \Phi^{i_2,\dots,i_k}\rho_{i_1} = \dots = \Phi\rho_{i_k}\dots\rho_{i_1}$ (by induction). ■

Following from the previous proposition we can note that if a polytope $\mathcal{P}$ is regular then

$$\Gamma(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle, \tag{2.2}$$

where each ρ_i is distinguished generator from a base flag of $\mathcal{P}$. Important properties of such generators are $\rho_i^2 = id$ and if $|i - j| \geq 2$ then

$$\rho_i \rho_j = \rho_j \rho_i. \tag{2.3}$$

The last equation is simply a restatement of the property that $\Phi^{jk} = \Phi^{kj}$.

Given a base flag Φ of a regular polytope $\mathcal{P}$ and its associated distinguished generators, then

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_j : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle \tag{2.4}$$

for any $I, J \subseteq \{0, \dots, n-1\}$. A group generated by list of generators is said to satisfy the *intersection condition* when it satisfies equation 2.4. The proof that the automorphism group of a regular polytope satisfies the intersection condition can be found in Theorem 2B10 in [26].

Now we look at a class of polytopes whose automorphism groups do not act transitively on flags. In particular we are interested in cases where polytopes are highly symmetric though not quite regular. More specifically we look for polytopes having no symmetry by reflections but the polytopes have a great deal of other symmetry. An example of such a polytope is the snub cube in Figure 2.5. However, we will look for polytopes with more symmetry than the snub cube, in fact, we will require that these polytopes have all possible symmetries by rotation.

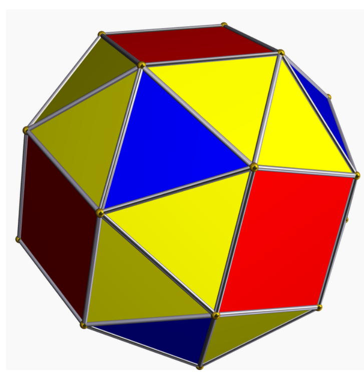

Figure 2.5: A snub cube which has the same rotations as a cube but no reflections. Image was made using Robb Webb's Stella Software [33]

When dealing with abstract polytopes, we will tighten up the conditions and say that an abstract polytope is *chiral* when it has precisely two flag orbits and each flag is in the opposite orbit as all of its adjacent flag.

Though, as we shall see, the snub cube does not satisfy our formal definition for chirality since the snub cube does not have two flag orbits, but instead has many more. From the perspective of abstract polytopes however we must be more strict.

Since a chiral polytope has no automorphisms which act like reflections, we can find an isomorphic "mirror image." We call this the *enantiomorphic form* much like a left and right hand. Figure 2.6 shows a chiral polytope and its enantiomorphic form. These polytopes are formed by identifying opposite sides of its boundaries to form a torus. These chiral toroidal polytopes are discussed in more depth later in Section 3.4.

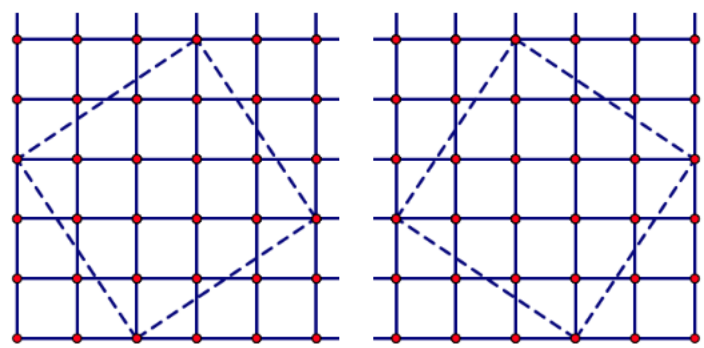

Figure 2.6: A chiral toroid and its enantiomorphic form.

Let $\mathcal{P}$ be a polytope, then if Φ and Ψ are flags in the same flag-orbit and $\gamma \in \Gamma(\mathcal{P})$ then $\Phi\gamma$ and $\Psi\gamma$ are in the same flag-orbit.

Definition 2.2.7. For $I \subset \{0, \ldots, n-1\}$ we say that a 2-orbit polytope $\mathcal{P}$ is in class 2_I if for any flag Φ, Φ^i and Φ are in the same flag orbit if and only if $i \in I$.

An example of a polytope in class $2_{\{0,1\}}$ is the cuboctahedron, seen in Figure

2.7.

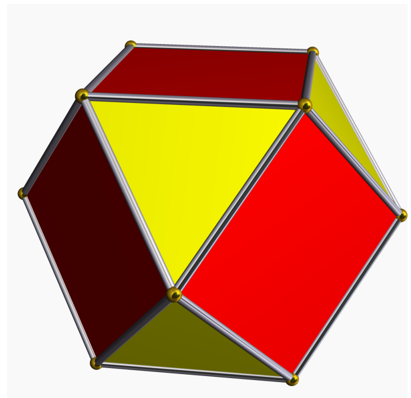

Figure 2.7: A cuboctahedron which is in class $2_{\{0,1\}}$. Image also made with Robb Webb's Stella software[33].

An n-polytope is *chiral* if it is in class $2_\emptyset$, more simply denoted by class 2. This says that for every flag of a chiral polytope all adjacent flags are in the opposite flag-orbit. Now we look to describe the automorphism group of a chiral polytope.

Proposition 2.2.8. *If $\mathcal{P}$ is a chiral n-polytope and Φ a fixed base flag, then $\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ where σ_i is defined by $\Phi\sigma_i = \Phi^{i,i-1}$.*

Proof. Define the flag $\Psi := \Phi^{i_1,\ldots,i_k}$ with k even. Every flag in the same orbit as Φ can be expressed as such a string and all flags in the other flag orbit are of the form $(\Phi\gamma)^j$ for some automorphism $\gamma \in \Gamma(\mathcal{P})$, with $0 \leq j \leq n-1$. So we prove this theorem by induction on k.

For $k = 2$, if $\Psi = \Phi^{i_1,i_2}$ with $i_1 > i_2$, then $\Phi^{i_1,i_2} = \Phi^{i_1,i_1-1,i_1-1,i_1-2,i_1-2,\ldots,i_2+1,i_2} =$

$(\Phi\sigma_{i_1})^{i_1-1,\dots i_2+1,i_2} = \dots = \Phi\sigma_{i_2+1}\dots\sigma_{i_1-1}\sigma_{i_1}$. Now, if $i_1 < i_2$, $\Phi^{i_1,i_2} = \Phi^{i_1,i_1+1,i_1+1,\dots,i_2-1,i_2} = \Phi\sigma_{i_2}^{-1}\dots\sigma_{i_1+1}^{-1}$, since $\Phi\sigma_{i+1}^{-1} = \Phi^{i,i+1}$.

So, there exists a $\gamma \in \langle\sigma_1,\dots,\sigma_{n-1}\rangle$ where $\Phi\gamma = \Psi$.

Inductively, for $k > 2$, $\Phi^{i_1,i_2,\dots,i_k} = (\Phi\gamma)^{i_3,\dots,i_k}$ with $\gamma \in \langle\sigma_1,\dots,\sigma_{n-1}\rangle = (\Phi^{i_3,\dots,i_k})\gamma = \Phi\gamma'$. $\blacksquare$

Proposition 2.2.9. *The facets of a chiral polytope are either regular or chiral [31].*

An important subgroup of the automorphism group of a regular polytope $\mathcal{P}$ is the *rotation subgroup* $\Gamma^+(\mathcal{P})$ which is generated by $\sigma_i = \rho_{i-1}\rho_i$ for $i = 1,\dots,n-1$. From these we have the following two identities,

$$\sigma_i^{p_i} = id,\ i = 1,\dots,n-1, \tag{2.5}$$

$$(\sigma_i\sigma_{i+1}\dots\sigma_j)^2 = id, j > i. \tag{2.6}$$

We can also note from the above two propositions that if $\mathcal{P}$ is chiral, then the rotation subgroup $\Gamma^+(\mathcal{P}) = \Gamma(\mathcal{P})$.

We now introduce the concept of a dual polytope. With a polytope $\mathcal{P}$, we can associate the *dual polytope* $\mathcal{P}^\delta$. For convex polyhedra (or maps), we identify a vertex at the centre of each face and connect them with edges when two of the original faces are joined by an edge. Old vertices become faces, old faces become vertices and

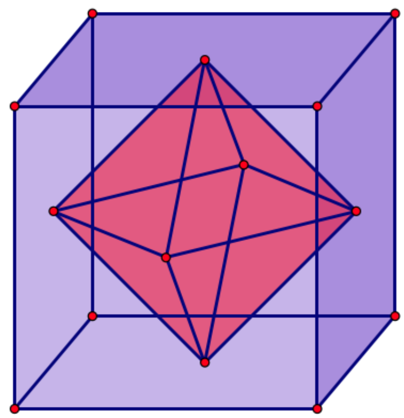

Figure 2.8: A cube with its dual, an octahedron, inscribed inside.

old edges become new edges. This can be easily understood by way of an example. The dual of a cube is the octahedron (see Figure 2.8).

Definition 2.2.10. Given an abstract polytope $\mathcal{P}$, the *dual* $\mathcal{P}^\delta$ of $\mathcal{P}$ has the same elements of the partially ordered set, but the partial order is reversed.

Proposition 2.2.11 (Page 28 of [26])**.** *$\mathcal{P}^\delta$ is a polytope.*

Remark 2. The dual of a flag is a flag and if Ψ is the dual flag of Φ, the Ψ^{n-1-i} is the dual flag of Φ^i.

Proposition 2.2.12. *If $\mathcal{P}$ is a 2-orbit polytope in class 2_I and $J = \{n-1-i : i \in I\}$, then $\mathcal{P}^\delta$ is in class 2_J.*

Proof. Consider a flag Ψ of $\mathcal{P}^\delta$ which has the opposite order of some flag Φ of $\mathcal{P}$.

Since $\mathcal{P}$ is in class 2_I we know that for all $i \in I$, Φ^i is in the same orbit as Φ. Then the flag Ψ^{n-1-i} is in the same flag orbit as Ψ for all $i \in I$.

If $i \notin I$, then Ψ^{n-1-i} and Ψ are in different orbits. $\blacksquare$

Some special polytopes are isomorphic to their own duals, we call these *self-dual.* An example of this is a tetrahedron. We call the isomorphism that maps a self-dual polytope $\mathcal{P}$ to its dual $\mathcal{P}^\delta$ a *duality* and note that every self-dual polytope has the same number of dualities as automorphisms.

Proposition 2.2.13. *Every self-dual polytope $\mathcal{P}$ has $|\Gamma(\mathcal{P})|$ dualities.*

Proof. Let δ be a duality then $\delta\gamma$ is a duality for every $\gamma \in \Gamma(\mathcal{P})$, and if $\gamma_1 \neq \gamma_2$, then $\delta\gamma_1 \neq \delta\gamma_2$. Hence there are at least $|\Gamma(\mathcal{P})|$ dualities.

Now, if δ_1 and δ_2 are dualities then $\delta_1 \circ \delta_2 \in \Gamma(\mathcal{P})$, so $\delta_2 = \delta_1^{-1}\gamma$. $\blacksquare$

2.3 Coxeter Groups

At the end of the last section we began to outline a few properties of automorphism groups of polytopes, in this section we will describe in detail the types of groups that are automorphism groups of abstract regular polytopes.

Definition 2.3.1. A *C-group* of *rank* n is a group generated by involutions $\rho_0, \ldots, \rho_{n-1}$ satisfying

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_j : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle$$

for all $I, J \subseteq \{0, \ldots, n-1\}$. If $\rho_i \rho_j = \rho_j \rho_i$ when $|i-j| \geq 2$, then it is called a *string C-group* of *rank n*.

In the future we will use the notation $\Gamma_I = \langle \rho_i : i \in I \rangle$ and $\Gamma_i = \langle \rho_j : i \neq j \rangle$. From the last section we can see then, that if $\mathcal{P}$ is a regular polytope of rank n then $\Gamma(\mathcal{P})$ is a string C-group of rank n. The relationship is deeper than that however, so we arrive at the next theorem.

Theorem 2.3.2 (Theorem 2E11 from [26]). *Given* $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ *a rank* n *string C-group, then the partially ordered set*

$$\mathcal{P}(\Gamma) = \{\Gamma_i \gamma : \gamma \in \Gamma, i \in \{0, \ldots, n-1\}\} \cup \{F_{-1}\} \cup \{F_n\}$$

where $F_{-1} \leq \Gamma_i \gamma \leq F_n$ *for all* $\gamma \in \Gamma$ *and* $i \in \{0, \ldots, n-1\}$*, and* $\Gamma_{i_1} \gamma_i \leq \Gamma_{i_2} \gamma_j$ *if and only if* $i_1 \leq i_2$ *and* $\Gamma_{i_1} \gamma_i \cap \Gamma_{i_2} \gamma_j \neq \emptyset$ *is a regular polytope satisfying* $\Gamma(\mathcal{P}(\Gamma)) \cong \Gamma$.

A generalization of the above theorem will appear in proposition 4.1.5.

We can also describe the relationship between abstract polytopes and other groups (specifically, other C-groups).

Definition 2.3.3. An abstract n-polytope $\mathcal{P}$ admits a *flag action* from a group $T = \langle t_0, \ldots, t_{n-1} \rangle$ if $t_i^2 = id$ for every i and it acts on the flags of $\mathcal{P}$ by mapping every flag Φ to $\Phi^i = \Phi t_i$.

Now we introduce a specific type C-group in order to use Theorem 2.3.2 to construct particular polytopes.

Definition 2.3.4. A *Coxeter group of rank* n is a group generated by $r_0, \ldots, r_{n-1}$ with presentation $\langle r_0, \ldots, r_{n-1} : (r_i r_j)^{p_{i,j}} = id \rangle$ where $p_{i,i} = 1$, for $i = 0, \ldots, n-1$ and $p_{i,j} = p_{j,i} \geq 2$ for $i \neq j$.

A Coxeter group G, can be represented by a diagram $\mathcal{D}(G)$ called the *Coxeter diagram* which is a graph where each involution r_i is represented by a vertex, and two vertices r_i and r_j are connected by an edge if $p_{i,j} = 3$ and an edge labelled with $p_{i,j}$ when $p_{i,j} \geq 4$. A Coxeter group is called *irreducible* or *reducible* when its diagram is connected or disconnected, respectively. See Figure 2.9 for an example of a Coxeter diagram. We will be concentrating entirely on irreducible Coxeter groups.

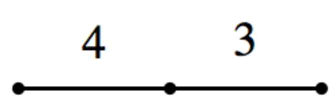

Figure 2.9: The Coxeter diagram for the group $[4, 3]$, the automorphism group of the cube.

Of particular interest in this thesis are Coxeter groups isomorphic to groups generated by reflections in Euclidean spaces which contain a normal subgroup of translations. These Coxeter groups are called *affine* Coxeter groups whose representation as a group of linear transformations of a vector space is called the *canonical representation*. Details on this representation can be found in chapter 3 of [26].

G is a called a *string* Coxeter group when it has a string diagram, which is to say that $p_{i,j} = 2$ whenever $|i - j| \geq 2$. G is called the *universal Coxeter group* with string diagram when $p_{i,i+1} = \infty$ for all i. We can observe that the universal Coxeter group with string diagram of rank n is admitted as a flag action on every n-polytope.

If G is a rank n Coxeter group with generators r_i, $i \in \{0, \ldots, n-1\}$ and $J \subseteq \{0, \ldots, n-1\}$, then we call the subgroup $G_J = \langle r_j : j \in J \rangle$ a *parabolic subgroup*. Additionally, we write $G_i = \langle r_j : j \neq i \rangle$.

Theorem 2.3.5 ([32]). *Every Coxeter group G satisfies the intersection condition, that is*

$$G_I \cap G_J = G_{I \cap J}.$$

Thus, every Coxeter group is also a C-group, and every string Coxeter group is a string C-group. For any Coxeter group with a string diagram, we define $p_i = p_{i-1,i}$ for $i = 1, \ldots, n-1$ and denote this group by $[p_1, p_2, \ldots, p_{n-1}]$. The corresponding polytope $\mathcal{P} = \mathcal{P}([p_1, \ldots, p_{n-1}])$ is denoted by $\{p_1, p_2, \ldots, p_{n-1}\}$.

In the next chapter we look at a class of polytopes that we construct from tessellations of $\mathbb{E}^n$ called toroidal polytopes.

3 Toroidal Polytopes

An important class of polytopes, and the subject of this thesis, are *toroidal polytopes*. Whereas polytopes like the Platonic Solids can be seen as spherical tilings, toroidal polytopes can be seen as tilings of a torus. To find these we need to develop a way to derive them from tesselations.

Definition 3.0.1. [26] Given a polytope $\mathcal{P}$, we say $\mathcal{Q}$ is a *quotient by group action* of $\mathcal{P}$ if $\mathcal{Q}$ is obtained as $\mathcal{P}/N$ for $N \leq \Gamma(\mathcal{P})$ with the mapping $\pi : \mathcal{P} \to \mathcal{P}/N$ given by $f\pi := fN$ where fN is the orbit of the face f under the subgroup N.

In order to define toroidal polytopes, we begin by introducing polytopes called tessellations. These are the polytopes that we will take quotients of to form toroidal polytopes.

3.1 Tessellations of Euclidean space

Informally, a tessellation is a tiling of the Euclidean space by convex polytopes (other non-Euclidean spaces can also be tiled, but we will focus on the Euclidean case). For example, in figure 3.1 we see a portion of the tiling of $\mathbb{E}^2$ by squares, which we note is itself a rank 3 polytope with an infinite number of faces.

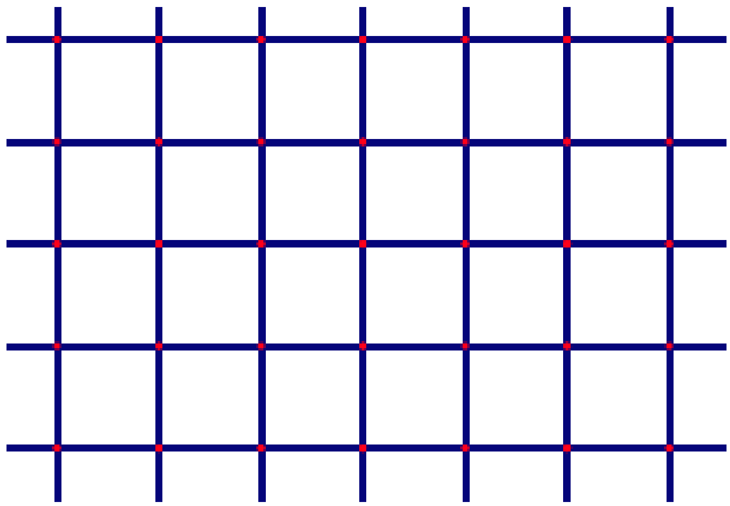

Figure 3.1: A portion of the tiling of the Euclidean plane by squares.

Definition 3.1.1. A *tessellation* on Euclidean n-space $\mathbb{E}^n$ is a locally finite collection $\mathcal{T}$ of rank n convex polytopes, called *cells*, which cover $\mathbb{E}^n$ and where any two cells with a non-empty intersection have disjoint interiors and meet at a common i-face for some i.

Remark 3. Typically the name *tessellation* is reserved for tilings on the Euclidean plane $\mathbb{E}^2$, while the name *honeycomb* is an "old fashioned" name used to describe the same idea in higher ranks. In this thesis we will use primarily the first term to describe any rank.

A *flag* of a tessellation is defined as a flag on any of its cells. Two flags of a tessellation $\mathcal{T}$ are i-adjacent if they differ only in a face of rank i. From this we have, for a flag Ψ, a unique i-adjacent flag Ψ^i. By the virtue of each of the cells being a polytope, a tessellation is also an abstract polytope of rank $n+1$. A tessellation $\mathcal{T}$ is then regular if its symmetry group $\mathrm{Sym}(\mathcal{T})$ is transitive on its flags. It follows that if a tessellation is regular, then all of its cells must be isomorphic regular polytopes. Furthermore, the symmetry group of a regular tessellation $\mathcal{T}$ is a Coxeter group and $\mathcal{T}$ is equivelar, that is it has an associated Schläfli type.

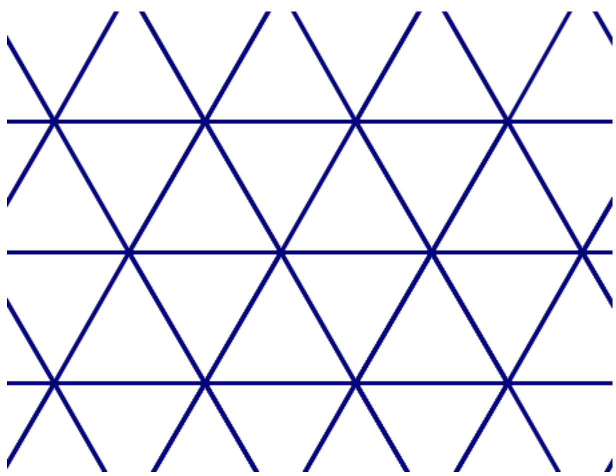

Figure 3.2: The triangular tiling $\{3, 6\}$.

The rank 3 regular tessellations of $\mathbb{E}^2$ are tilings by triangles, squares and hexagons which have Schläfli types $\{3, 6\}, \{4, 4\}, \{6, 3\}$ respectively (see Figure 3.2 for a portion of $\{3, 6\}$ and Figure 3.3 for a portion of $\{6, 3\}$). We also note that the dual of the triangular tiling is the hexagonal tiling while the square tiling is self-dual. A simple geometric argument can be made to verify that these are the only regular rank 3 tessellations. The only regular tiling of $\mathbb{E}^3$ is the tiling by cubes. In fact, for any rank $n \geq 3$ there is a regular tessellation of $\mathbb{E}^n$ by n-cubes. Meanwhile, in $\mathbb{E}^4$, we also have regular tessellations with types $\{3, 3, 4, 3\}$ and $\{3, 4, 3, 3\}$. However, only tessellations up to rank 4 are within the scope of this thesis.

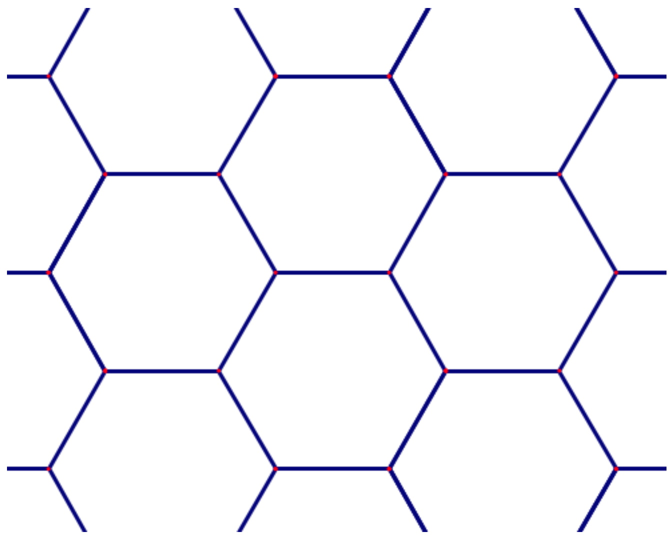

Figure 3.3: The hexagonal tiling $\{6, 3\}$.

These tessellations can be recovered from the string affine Coxeter groups using the construction from Theorem 2.3.2 which are isomorphic to the tessellations (see Corollary 2E14 from [26]). As the standard representation of affine Coxeter groups are the only Coxeter groups that contain Euclidean translations, the only regular tessellations of rank 3 and 4 are $\{3, 6\}, \{4, 4\}, \{6, 3\}$ and $\{4, 3, 4\}$, as the groups $[3, 6], [4, 4], [6, 3]$ and $[4, 3, 4]$ are the only string affine Coxeter groups of ranks 3 and 4.

Since a tessellation $\mathcal{T}$ covers $\mathbb{E}^n$ we can describe some of its symmetries by translations along vectors. When $\mathcal{T}$ is $\{3,6\}$, $\{4,4\}$ or $\{4,3,4\}$, we start by fixing a base flag, Φ and declaring the vertex in that flag to be the origin of the underlying Euclidean space (for $\mathcal{T} = \{6,3\}$ we take the centre of the face of the base flag to be the origin). Then, the reflection that maps Φ to Φ^0 fixes a hyperplane H_0 with normal vector α_0 from the origin to the vertex in Φ^0. The orbit of α_0 under the stabilizer of the origin forms a *root system*. Figure 3.4 shows an example for $\{4,4\}$. This root system generates an integer lattice which can be seen as the translation subgroup $\mathbf{T} \leq \mathrm{Sym}(\mathcal{T})$.

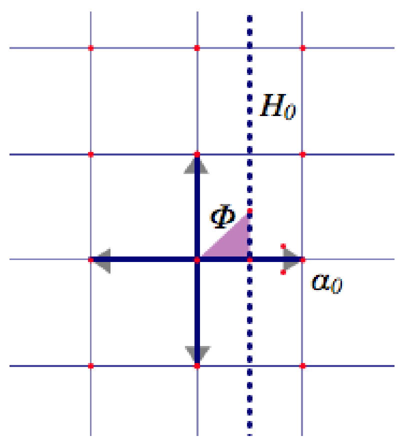

Figure 3.4: The root system for $\{4,4\}$.

If we denote by $\mathbf{S}$ the stabilizer of the origin, then $\mathrm{Sym}(\mathcal{T}) = \mathbf{T} \rtimes \mathbf{S}$. Now that we have a description of translations we can briefly mention how they interact with other isometries. If we have a translation t by a vector u and some isometry σ fixing the origin then $\sigma^{-1}t\sigma = t_{u\sigma}$ where $t_{u\sigma}$ is a translation by the image of u under σ.

3.2 Regular Toroidal Polytopes

Now we have a class of polytopes and a means of finding a subgroup to take a quotient with, so we can now build toroidal polytopes.

Definition 3.2.1. A *toroid* of rank $n+1$ or an $(n+1)$-*toroid* is a quotient of a tessellation $\mathcal{T}$ of $\mathbb{E}^n$ by a rank n subgroup $\Lambda \leq \mathbf{T}$ of its translations (or the corresponding integer lattice). We denote this toroid by $\mathcal{T}/\Lambda$.

Each subgroup Λ is generated by n linearly independent translations $\{u_1, \ldots, u_n\}$ which yield a lattice $\boldsymbol{\Lambda} := o\Lambda$ whose corresponding vectors u_i determine a fundamental region of the lattice. This region is the parallelepiped whose edges are determined by the vectors u_i. If the generating vector is too small, then the fundamental region for the toroid is not large enough to contain enough faces and edges to satisfy the definition of a polytope and the result of a quotient will not be a polytope. Thus, if the fundamental region is sufficiently large, $\mathcal{T}/\Lambda$ is a polytope and its faces and flags are the orbits of the faces and flags of $\mathcal{T}$ under Λ. If $\mathcal{T}$ is equivelar of type $\{p_1, \ldots, p_k\}$ then $\mathcal{T}/\Lambda$ is also equivelar of the same type.

Such a toroidal polytope is regular when its automorphism group is transitive on the flags. To be able to characterize when a toroid is regular we look at the action of symmetries on toroids. A symmetry $\gamma \in \mathrm{Sym}(\mathcal{T})$ induces an automorphism of $\mathcal{T}/\Lambda$ if and only if $\mathbf{\Lambda}$ is invariant under γ, that is, when $\gamma^{-1}\Lambda\gamma = \Lambda$. So, given a regular tessellation $\mathcal{T}$ and translation subgroup Λ, the toroid $\mathcal{T}/\Lambda$ is regular when $\gamma^{-1}\Lambda\gamma = \Lambda$ for all $\gamma \in \mathrm{Sym}(\mathcal{T})$ or equivalently, for all γ in the stabilizer of the origin. [23]

Next we will characterize the regular toroids of rank 3 and 4. We begin with the tessellation $\mathcal{T} = \{4,4\}$ of $\mathbb{E}^2$ by squares and obtain the toroid $\{4,4\}/\Lambda$ where Λ is a rank 2 translation subgroup. The automorphism group of $\{4,4\}$ is $[4,4]$ generated by reflections ρ_0, ρ_1 and ρ_2 defined next. Pick a base flag Φ, then taking the 0-face of Φ to be the origin, the 0-face of Φ^0 to be the point $(1,0)$ and the 0-face of $\Phi^{1,0}$ to be $(0,1)$. Then ρ_0, ρ_1 and ρ_2 are reflections in the lines $x = 1/2$, $y = x$ and $y = 0$ respectively. Furthermore, the unit translation $(1,0)$ can be expressed by the automorphism $\rho_1\rho_2\rho_1\rho_0$.

If we then want to generate a certain lattice $\mathbf{\Lambda}$ invariant under the stabilizer of the origin with two linearly independent vectors, one of which is the vector (s,t), then, by rotation clockwise by $\pi/2$, the other generator of Λ must be $(t,-s)$. However, by the reflection ρ_0 the vector (t,s) must also be in Λ, so we arrive at the necessary identity $st(s-t) = 0$ (see page 18 of [26]). In such a case we denote the resulting toroid by $\{4,4\}/\Lambda_{(s,t)}$ or more simply, by $\{4,4\}_{(s,t)}$ (where, as noted, s or t are 0 or $s = t$). Figure 3.5 gives two examples of these toroids.

Figure 3.5: The regular toroid where $s = 4$ and $t = 0$ as well as the regular toroid where $s = t = 2$. The latter is determined by the dotted lines in the figure on the right.

Remark 4. For this section as well as all subsequent sections and chapters, while s and t can be any integer we will assume they are non-negative without loss of generality. When we get to the chapter of block systems it will be especially useful to take these values as non-negative.

As mentioned earlier, if the generating vector is too small, then the quotient is not a polytope. This is the case for the vectors $(1, 0), (0, 1)$ or $(1, 1)$.

Since $\{3, 6\}$ and $\{6, 3\}$ are duals, it suffices to treat only one of them. The vertices of $\{3, 6\}$ and the faces of $\{6, 3\}$ form the integer lattice we use to describe the translation subgroup. In this case we describe the coordinate axis along edges of the tessellation and in particular along the translation vectors $(1, 0)$ and $(0, 1)$ shown in Figure 3.6. As before, the generators ρ_0, ρ_1 and ρ_2 of $[3, 6]$ are reflections in the lines $y = -2x + 1$, $y = x$ and $y = 0$ respectively. This time the translation $(1, 0)$ is the automorphism $\rho_1\rho_2\rho_1\rho_2\rho_1\rho_0$. The two translation vectors $(1, 0)$ and $(0, 1)$ as shown in Figure 3.6 form the canonical basis for the translation subgroup.

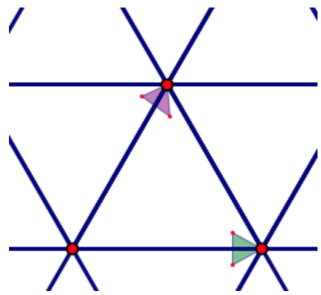

Figure 3.6: The green arrow indicates the translation along the vector $(1, 0)$ and the purple one along $(0, 1)$.

So then a translation subgroup $\Lambda_{(s,t)}$ is generated by (s, t) and its images under the stabilizer of the origin. The conditions for regularity are the same as before, namely $\{3, 6\}_{(s,t)}$ (and the dual, $\{6, 3\}_{(s,t)}$) is regular when $st(s - t) = 0$ that is, s or t is 0 or $s = t$. We give two examples below in Figure 3.7.

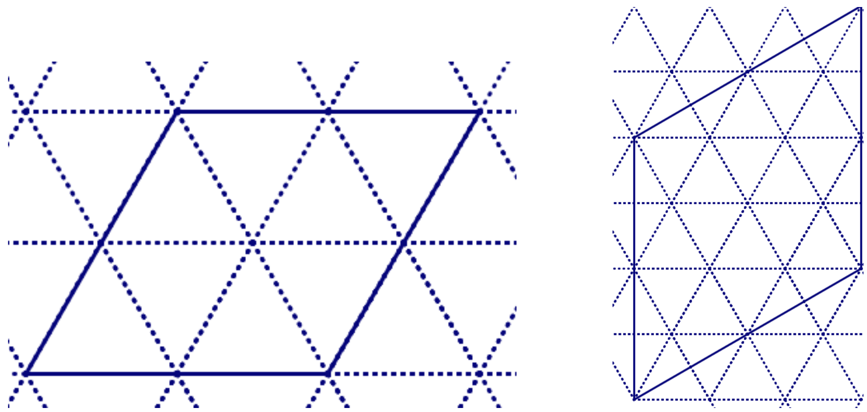

Figure 3.7: Two regular toroids $\{3,6\}_{(2,0)}$ and $\{3,6\}_{(2,2)}$.

3.3 Regular Cubic Toroids

Now we seek to characterize the rank 4 regular toroids. These are quotients of the cubic tessellation $\mathcal{T} = \{4,3,4\}$ whose group of symmetries is isomorphic to the string Coxeter group $[4,3,4]$. As generators of $[4,3,4]$ we take ρ_1, ρ_2, ρ_3 to be with reflections in the hyperplanes with normal vectors $(1,-1,0), (0,1,-1), (0,0,1)$ respectively, and the generator ρ_0 with the reflection in the plane through $(1/2,0,0)$ with normal vector $(1,0,0)$. Then,

$$\begin{aligned}(x,y,z)\rho_0 &= (1-x,y,z),\\ (x,y,z)\rho_1 &= (y,x,z),\\ (x,y,z)\rho_2 &= (x,z,y),\\ (x,y,z)\rho_3 &= (x,y,-z).\end{aligned} \tag{3.1}$$

As described earlier, we can recover the cubic tessellation from the standard representation of the group $[4, 3, 4]$ using Theorem 2.3.2 and the translation subgroup $\mathbf{T} \leq [4, 3, 4]$ generated by translations along the usual vectors so we can write $\mathbf{T} = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1) \rangle$.

The regular polytope which results from factoring the regular tessellation $\mathcal{T} = \{4, 3, 4\}$ by a subgroup Λ of $\mathbf{T}$ in $[4, 3, 4]$, is denoted by $\mathcal{T}/\Lambda$ (as above).

We let $\Lambda_{\mathbf{s}}$ be the translation subgroup generated by the vector $\mathbf{s}$ and its images under the stabilizer of the origin in $[4, 3, 4]$ and hence under permutations and changes of sign of its coordinates. The regular polytope $\mathcal{T}/\Lambda_{\mathbf{s}}$ is denoted by $\{4, 3, 4\}_{\mathbf{s}} := \{4, 3, 4\}/\Lambda_{\mathbf{s}}$ and the corresponding group $[4, 3, 4]/\Lambda_{\mathbf{s}}$ is written as $[4, 3, 4]_{\mathbf{s}}$. The following lemma lists all possible such subgroups of $\mathbf{T}$.

Lemma 3.3.1. *Let Λ be a subgroup of $\boldsymbol{T}$, and if for every $\boldsymbol{a} \in \Lambda$, the image of $\boldsymbol{a}$ under all changes of sign and permutations of coordinates (which is conjugation of $\boldsymbol{a}$ by the stabilization of the origin in $[4, 3, 4]$) is also in Λ, then $\Lambda =$*

$$
\begin{gathered}
\langle (x, 0, 0), (0, x, 0), (0, 0, x) \rangle, \\
\langle (x, x, 0), (-x, x, 0), (0, -x, x) \rangle \text{ or} \\
\langle (x, x, x), (2x, 0, 0), (0, 2x, 0) \rangle.
\end{gathered}
\tag{3.2}
$$

Proof. As adapted from page 165 from [26].

Let s be the smallest positive integer from all coordinates of vectors in Λ, then we can assume that $(s, s_2, s_3) \in \Lambda$. Then $(-s, s_2, s_3) \in \Lambda$ and thus $2se_1 \in \Lambda$ and so too are each $2se_i$. By adding and subtracting multiples of these we can find a vector all of whose coordinates are values in $\{-s, 0, s\}$. It then follows that Λ is generated by all permutations of $(s^k, 0^{3-k})$ with all changes of sign for some $k \in \{1, 2, 3\}$.

If $k = 1$ then we have the first basis mentioned in the lemma, the second if $k = 2$ and the third when $k = 3$. ■

It follows from the previous lemma that $\Lambda_{\mathbf{s}} = s\Lambda_{(1^k, 0^{n-l})}$, and thus, as can be seen in [26] Theorem 6D1, we have the following theorem.

Theorem 3.3.2. *[Theorem 6D1 in [26]] The only regular toroidal polytopes whose automorphism groups are quotients of* $[4, 3, 4]$ *are* $\{4, 3, 4\}_{\boldsymbol{s}}$ *where* $\boldsymbol{s} = (s, 0, 0), (s, s, 0)$ *or* (s, s, s) *and* $s \geq 2$.

Proof. If $s = 1$ the resulting tessellation of the torus is not strongly connected. Conjugation of vectors in Λ by ρ_1, ρ_2 and ρ_3 are precisely all permutations of coordinates and changes of sign, this theorem follows directly from lemma 3.3.1. ■

The following theorem also appears in [26] along with its proof. This theorem describes the group of each toroid. To arrive at the following result (and subsequent related results in Sections 4.4 and 4.5) we note that the mirror of reflection ρ_0 is $x = 1/2$ while the mirrors for ρ_1, ρ_2 are $x = y$ and $y = z$ respectively and the mirror for ρ_3 is $z = 0$.

Theorem 3.3.3 (Theorem 6D4 in [26]). *Let* $\boldsymbol{s} = (s^k, 0^{3-k})$*, with* $s \geq 2$ *and* $k = 1, 2, 3$*. Then the group* $[4,3,4]_{\boldsymbol{s}}$ *is a quotient of the Coxeter group* $[4,3,4] = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$*, where the generators are specified in* (3.1)*, factored out by a single extra relation which is*

$$(\rho_1\rho_2\rho_3\rho_2\rho_1\rho_0)^s = id, \ \text{if } k = 1,$$

$$(\rho_2\rho_3\rho_2\rho_1\rho_0)^{2s} = id, \ \text{if } k = 2,$$

$$(\rho_3\rho_2\rho_1\rho_0)^{3s} = id, \ \text{if } k = 3.$$

As explained in [26], a geometric argument can be used to verify the intersection condition for these groups when $s \geq 2$. However, note that $[4,3,4]_{\mathbf{s}}$ does not satisfy the intersection condition when $s \leq 1$ and thus is not a C-Group. We show the breakdown of the intersection condition for $s = 1$ by way of example for $k = 1$ where cases for $k = 2, 3$ follow similar arguments [26].

When $s = 1$, the identity $\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1 = id$ tells us that $\rho_0 \in \langle \rho_1, \rho_2, \rho_3 \rangle$ so G does not satisfy the intersection condition.

3.4 Chiral Toroids

In section 2.2 we briefly defined chiral polytopes. In this section we will characterize the chiral toroidal polytopes of rank 3.

We recall that the automorphism group of a chiral polytope has two orbits on the flags and that any two adjacent flags are in different flag orbits.

Constructing a chiral toroid from the tessellation $\{4,4\}$ is much like constructing a regular toroid. If we begin with a translation vector (s,t) which will generate $\Lambda_{(s,t)}$, then $\Lambda_{(s,t)}$ must be invariant under rotations, that is, under the group $[4,4]^+$. Since the rotation subgroup $[4,4]^+$ is an index 2 subgroup of $\{4,4\}$, the subgroup fixing the origin is then $\langle \rho_1\rho_2 \rangle$. Thus, we can take $(t,-s)$ as the other generator so we have the condition that the toroid $\{4,4\}_{(s,t)}$ is chiral when $st(s-t) \neq 0$, if it were equal to zero then it would be regular. Figure 3.8 shows a chiral toroid and its enantiomorphic form.

For $\{3,6\}$ (and its dual $\{6,3\}$) the process is similar and we arrive at the same conditions that $\{3,6\}_{(s,t)}$ and $\{6,3\}_{(s,t)}$ are chiral when $st(s-t) \neq 0$.

The non-existence of chiral toroids of rank 4 will be dealt with in section 4.6 together with toroidal hypertopes of that rank.

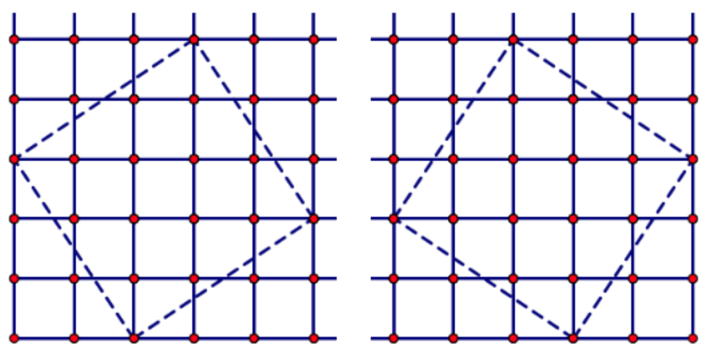

Figure 3.8: A chiral toroid based on the vector $(3, 2)$ and its enantiomorphic form based on the vector $(2, 3)$.

4 Hypertopes and Coxeter Complexes

The relationship described in previous chapters between string Coxeter groups and abstract polytopes is deep. If we seek to generalize the concept of an abstract polytope we may do so by looking at the Coxeter groups. In particular, we can look at Coxeter groups which are not string groups. By removing the string condition we no longer have the hierarchy of vertices, edges, faces etc. In other words, the structure no longer is a partial order. In this chapter we describe such a generalization and call it a hypertope.

The concept of a hypertope has recently been introduced by Fernandes, Leemans and Weiss (see [18]). A hypertope can be seen as a generalization of a polytope. Or, from another perspective, as a generalization of a hypermap. For more infomation on hypermaps see [9].

Since the focus of this thesis is on toroidal structures we will work towards the concept of toroidal hypertopes. A *regular toroidal hypertope* (see the following section for a precise definition) can be seen as a quotient $\mathcal{C}/\Lambda_R$ of a structure called a *Coxeter complex* of an affine Coxeter group by a normal subgroup of translations, denoted Λ_R where R represents a generating set identifying the normal subgroup. In this chapter we will develop formally the concepts of hypertopes and Coxeter complexes. Following that we will review rank 3 regular (and also chiral) toroidal hypertopes within the definition framework developed and then characterize the two families of regular toroidal hypertopes of rank 4 which are not polytopes. These are hypertopes associated with the affine Coxeter groups $\widetilde{B}_3$ and $\widetilde{A}_3$. These results I have published in [16]. Together with the regular toroidal polytopes formed from cubic tilings these make up all regular toroidal hypertopes of rank 4 since the groups $[4, 3, 4]$, $\widetilde{B}_3$ and $\widetilde{A}_3$ are the only affine Coxeter groups of rank 4.

4.1 Incidence Systems and Hypertopes

Details of the concepts we review here are given in [18]. Before we begin with the set of definitions which lead to a hypertope, we give an example of an infinite hypertope which is *not* a polytope. This is the same example given in the introduction (and also explained in detail later in section 4.3) and can be interpreted as a hypertope "tiling". We can see a visualization in Figure 4.1. Here we can describe a set of "hypervertices" (the green vertices), "hyperedges" (the grey triangles centred with a red vertex) and "hyperfaces" (the white faces). We can notice that the hyperedges connect three vertices where an edge of a polytope must connect precisely two vertices.

More difficult to see from a visualization is the incidence relationship between pairs of "hyperelements[3]." In this hypertope, there is nothing which determines an ordering of the types of elements which gives a partial order on the set elements. Each element of each type exhibits triangular symmetry as each element of each type is incident to exactly three elements of each other type. The illustration, therefore, could just as easily represented the hypervertices as triangles or the hyperfaces as hypervertices.

As we introduce definitions the reader can refer to this example to help contextualize them.

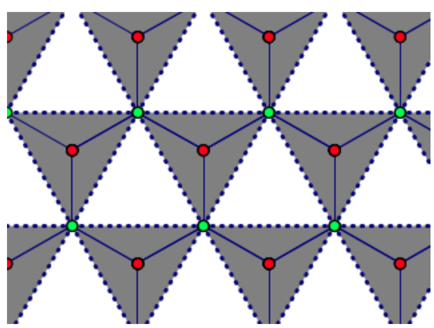

Figure 4.1: A hypertope which is not a polytope.

We begin by building up types of combinatorial structures and adding conditions on them until we name one a hypertope. We will also note that abstract polytopes

[3]The reader may choose to forgive this abuse of notation as this term will not appear again and is used only for rhetorical purposes.

are hypertopes.

Definition 4.1.1. An *incidence system* $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called *elements* of Γ;
- I is a set whose elements are called *types* of Γ;
- $t : X \to I$ is a *type function* that associates to each element $x \in X$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation of X called *incidence*, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The *incidence graph* of Γ is the graph whose vertices are the elements of X and two vertices are joined by an edge if they are incident in Γ. A *flag* is a set of pairwise incident elements of Γ and the *type* of a flag F is $\{t(x) : x \in F\}$. A *chamber* is a flag of type I. An element x is said to be *incident* to a flag F when x is incident to all elements of F and we write $x * F$. The *rank* of Γ is the cardinality of I.

Referring back to our example, the hypervertices, hyperedges and hyperfaces are the three types and no element of one type is incident to an element of that same type.

Definition 4.1.2. An *incidence geometry* is an incidence system Γ where every flag is contained in a chamber.

Let $\Gamma := (X, *, t, I)$ be an incidence system and F a flag of Γ. The *residue* of F in Γ is the incidence system $\Gamma_F := \{X_F, *_F, t_F, I_F\}$ where

- $X_F := \{x \in X : x * F, x \notin F\}$;
- $I_F := I \backslash t(F)$;
- t_F and $*_F$ are the restrictions of t and $*$ to X_F and I_F.

If each residue of rank at least 2 of Γ has a connected incidence graph then Γ is said to be *residually connected*. Γ is *thin* when every residue of rank 1 contains exactly 2 elements. From our example, a residue of rank 1 would be derived from a flag which contains 2 incident elements. Choose, for example, a flag consisting of a green hypervertex and white hyperface. The resulting residue is all elements of the hypertope which are the hyperedges which are incident to that flag, of which there are precisely two.

If we pick a flag from our example consisting of one hypervertex and one hyperedge which are incident then the rank 1 residue of that flag is precisely the two hyperfaces which are incident to that flag. Other rank 1 residues can similarly be seen to have exactly two elements.

Furthermore, Γ is *chamber-connected* when for each pair of chambers C and C', there exists a sequence of chambers $C =: C_0, C_1, \dots C_n := C'$ such that $|C_i \cap C_{i+1}| = |I| - 1$. An incidence system is *strongly chamber-connected* when all of its residues of rank at least 2 are chamber-connected. Chamber-connectivity here fulfils the same purpose as that of flag-connectivity in an abstract polytope.

Proposition 4.1.3 (Proposition 2.1 in [18]). *Let Γ be a thin incidence geometry. Then Γ is residually connected if and only if Γ is strongly chamber-connected.*

A *hypertope* is a strongly chamber-connected thin incidence geometry. To reinforce the relationship between polytopes and hypertopes we will sometimes refer to the *elements* of a hypertope Γ as *hyperfaces* of Γ, and *elements of type i* as *hyperfaces of type i*.

We can now see that the example described above is, in fact, a hypertope.

Let $\Gamma := (X, *, t, I)$ be an incidence system. An *automorphism* of Γ is a mapping $\alpha : (X, I) \to (X, I)$ where $(x, t(x)) \mapsto (\alpha(x), t(\alpha(x)))$ where

- α is a bijection on X (inducing a bijection on I);
- for each $x, y \in X, x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x, y \in X, t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism α is *type-preserving* when, for each $x \in X, t(\alpha(x)) = t(x)$. We denote by $Aut(\Gamma)$ the group of automorphisms of Γ and by $Aut_I(\Gamma)$ the group of type-preserving automorphisms of Γ.

An incidence system Γ is *flag-transitive* if $Aut_I(\Gamma)$ is transitive on all flags of type J for each $J \subseteq I$. It is *chamber-transitive* if $Aut_I(\Gamma)$ is transitive on all chambers of Γ. Furthermore, it is *regular* if the action of $Aut_I(\Gamma)$ on the flags of each type is semiregular and transitive.

Proposition 4.1.4 (Proposition 2.2 in [18])**.** *Let* Γ *be an incidence geometry.* Γ *is chamber-transitive if and only if it is flag-transtive.*

A *regular hypertope* is a flag-transitive hypertope. We note that every abstract regular polytope can be viewed as a regular hypertope where I is the set of ranks, t identifies each element as a vertex, etc and $*$ is the usual incidence relationship. In chapter 2 we described how we can both define a group of the automorphisms of a regular abstract polytope as well as define a polytope from a quotient of a string Coxeter group satisfying the intersection condition. We can develop a similar relationship with regular hypertopes.

For a hypertope Γ, if the action of $Aut_I(\Gamma)$ is semiregular but results in two chamber orbits, where adjacent chambers are in different orbits, we say the hypertope is *chiral*.

Given an incidence system Γ and a chamber C of Γ, we denote by G_i the stabilizer of the element of type i of C in $G := Aut_I(\Gamma)$. When Γ is a regular hypertope, the subgroups $\cap_{j\in I\setminus i} G_j$ are cyclic groups of order 2 (see [18]) whose generators we denote by ρ_i. The set $\{\rho_i : i \in I\}$ of involutions generates $Aut_I(\Gamma)$ and is called the set of *distinguished generators* of $Aut_I(\Gamma)$.

The following two propositions show how to construct a regular hypertope, that is starting from a particular group with some of its subgroups we can construct a regular hypertope whose automorphism group is the group we started with.

Proposition 4.1.5 ([32])**.** *Let n be a positive integer and I a finite set. Let G be a group together with a family of subgroups $(G_i)_{i\in I}$, X the set consisting of all cosets $G_ig, g \in G, i \in I$ and $t : X \to I$ defined by $t(G_ig) = i$. Define an incidence relation $*$ on $X \times X$ by :*

$$G_ig_2 * G_jg_2 \text{ if and only if } G_ig_1 \cap G_jg_2 \text{ is non-empty in } G.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having at least one chamber. Moreover, the group G acts by right multiplication as a group of type-preserving automorphisms of Γ. Finally, the group G is transitive on the flags of rank less than 3.

Whenever Γ is constructed as in the above proposition it is written as $\Gamma(G; (G_i)_{i\in I})$ and if it is an incidence geometry it is called a *coset geometry*. If G acts transitively on all chambers of Γ (thus also flags of any type) we say that G is *flag-transitive* on Γ or that Γ is flag-transitive.

Now we note that we can construct a coset geometry $\Gamma(G; (G_i)_{i\in I})$ using a C-group $G = \langle r_i : i \in I\rangle$ of rank p by setting $G_i := \langle r_j : j \in I\backslash\{i\}\rangle$ for all $i \in I$.

We introduce the following proposition which lets us know when the constructions we use produce regular hypertopes.

Proposition 4.1.6 (4.6 in [18]). *Let $G = \langle r_i : i \in I \rangle$ be a C-group of rank p and let $\Gamma := \Gamma(G; (G_i)_{i \in I})$ with $G_i = \langle r_j : j \in I \backslash \{i\} \rangle$ for all $i \in I := \{0, \ldots, p-1\}$. If G is flag-transitive on Γ, then Γ is a regular hypertope.*

4.2 Coxeter Complexes

The concept of a *Coxeter complex* (and a more general *order complex*) is very broad and applies to may different types of Coxeter groups. These details are described chapter 2 of [26]. Here we will restrict our considerations to affine Coxeter groups of rank 3 and 4. Recall that an affine Coxeter group is an infinite Coxeter group admitting a faithful representation as isometries of a Euclidean space.

Let G be a rank $n = 3$ or 4 affine Coxeter group with generators r_i (with $i \in \{0, \ldots, n-1\}$) where each r_i is a reflection through an associated affine hyperplane, H_i. These hyperplanes determine an n-simplex in $\mathbb{E}^n$ which we call the *fundamental domain* or the *fundamental simplex* of G and denote it by ε. Let F_i be the vertex of the fundamental simplex not on H_i. The specific construction of these hyperplanes is the key element of the canonical representation of G and the subsequent fundamental simplex is explained in detail in [24] sections 1.15 and 6.5.

We define the *Coxeter complex* to be the collection $\mathcal{C}$ of all εg ($g \in G$), or the orbit of ε under G. We note that $\mathcal{C}$ covers $\mathbb{E}^n$ where for any εx and εy with non-empty intersection, the intersection is contained in a hyperplane. When G is the group of symmetries of a tessellation of $\mathbb{E}^n$, the Coxeter complex forms the *barycentric subdivision* of that tessellation.

Proposition 4.2.1. *The Coxeter complex* $\mathcal{C}$ *of an affine Coxeter group* G *of rank* $n \leq 4$ *is isomorphic to* $\Gamma(G; (G_i)_{i \in I})$.

Proof. We note that the fundamental simplex of $\mathcal{C}$ is an n-simplex where each vertex F_i of the fundamental simplex can also be associated with a subgroup G_i of G, where G_i is the stabilizer of F_i. So we choose the fundamental simplex to be the base chamber $\{G_i id\}_{i \in I}$. Since the Coxeter complex $\mathcal{C}$ is defined as the orbit of the fundamental simplex, G acts transitively on all the simplices making up $\mathcal{C}$.

So we can find a bijection between the set of vertices of all simplices in $\mathcal{C}$ and elements in $\Gamma(G; (G_i)_{i \in I})$. If we define incidence on these vertices by whether or not they lie on a common simplex we can see that this bijection preserves incidence. And since each vertex is the image of a unique F_i, the bijection preserves type ■

Given that $\mathcal{C}$ is isomorphic to $\Gamma(G; (G_i)_{i \in I})$ we can call $\mathcal{C}$ a hypertope.

Corollary 4.2.2. *The Coxeter complex* $\mathcal{C}$ *of an affine Coxeter group* G *is a regular hypertope.*

Proof. $\mathcal{C}$ is clearly chamber-transitive and thus flag-transitive, and so by Proposition 4.1.6, it is a regular hypertope. ■

Remark 5. The example we begin this chapter with is the Coxeter complex for the group $\widetilde{A_3}$. Together with the regular tessellations $\{3, 6\}, \{6, 3\}$ and $\{4, 4\}$ are the only rank 3 hypertopes constructed from the Coxeter complex of an affine Coxeter group. We also note that this need not result in a polytope since an affine Coxeter group need not be a string group and so may not inherit a partial order.

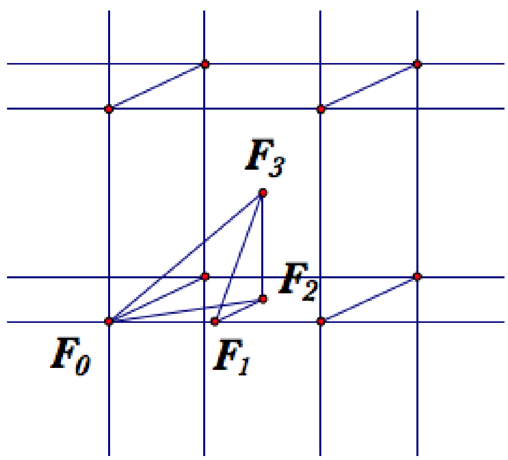

Figure 4.2: Fundamental simplex of $[4, 3, 4]$

As we noted in section 3.3, abstract polytopes are also hypertopes (though the converse is not always true). So we look at the cubic tessellation $\{4, 3, 4\}$ as an example. Figure 4.2 shows the fundamental simplex for $[4, 3, 4]$. One vertex of the fundamental simplex coincides with a vertex of the tessellation, one with the midpoint of an edge, centre of a face and centre of a cell.

We can use a similar method as that in Chapter 3 to develop an integer lattice and translation subgroup. Let G be an affine Coxeter group of rank $n = 3$ or 4 with generators r_i. Here r_i with $1 \leq i \leq n-1$ will stabilize a point which, without loss of generality, can be assumed to be the origin o in $\mathbb{E}^2$ or $\mathbb{E}^3$ (with the exception of the group $\{6,3\}$ which was dealt with in chapter 3). Then the maximal parabolic subgroup G_0 is a finite crystallographic subgroup, which is a group that leaves the origin fixed. For details, see [8] page 108-109. The normal vectors to the reflection planes of the generators of G_0 are called the $fundamental\ roots$. The images of the fundamental roots under G_0 form a *root system* for G_0.

The lattice Λ, generated by the root system is called the *root lattice*, and the fundamental roots form the integral basis for Λ. The parallelotope[4] determined by the fundamental roots is called the *fundamental region* (not to be confused with a fundamental domain or chamber). This lattice gives us (and can be identified with) the translation subgroup $T \leq G$ generated by the root lattice of G_0, note that $G = G_0 \rtimes T$ [24]. For convenience we identify the translations with its vectors and in addition a lattice also corresponds with the roots of the Coxeter group G.

If K is a set of linearly independent translations in T, and $T_K \leq T$ the subgroup generated by K, then the sublattice $\Lambda_K \leq \Lambda$ is the lattice induced by oT_K, the orbit of the origin under T_K. We note that $\mathcal{C}$ as defined above is a regular hypertope and each maximal simplex in $\mathcal{C}$ represents a chamber where each vertex of the simplex is an element of a different type. In rank $n \leq 4$, when the quotient $\mathcal{C}/\Lambda_K$ is a

[4] A generalization of a parallelepiped in n-dimensions.

regular (chiral) hypertope, we say it is a *regular* (*chiral*) *toroidal hypertope* of rank $n \leq 4$. $\mathcal{C}/\Lambda_K$ is a regular hypertope when Λ_K is large enough to ensure the corresponding group satisfies the intersection condition and when Λ_K is invariant under G_0, ie $r_i \Lambda_K r_i = \Lambda_K$ for $i \leq 4$. Similarly, Λ_K must be sufficiently large in order for $\mathcal{C}/\Lambda_K$ to be a chiral toroidal hypertope. It is important to note that, in addition to denoting a lattice, Λ_K also denotes a set of vectors as well as a translation subgroup of G along those vectors. If Λ_K is generated by a vector $\mathbf{s}$ and all permutations and changes in sign of the coordinates $\mathbf{s}$ then we will write $\Lambda_{\mathbf{s}}$ in place of Λ_K.

4.3 Toroidal Hypertopes of rank 3

The affine Coxeter groups of rank 3 are the affine Coxeter groups with a string diagram mentioned in section 3.2 as well as the group $\widetilde{A}_2$. Since we have already found the regular toroids for the string groups (which are polytopes) it suffices to find the regular toroidal hypertope constructed from $\widetilde{A}_2$. $\widetilde{A}_2$ is generated by the three generators $\widetilde{\rho_0}, \rho_1, \rho_2$ defined below, and has the Coxeter diagram depicted in figure 4.3

If we let $\{\rho_0, \rho_1, \rho_2\}$ be the set of generators of the group $[6, 3]$ and ε the fundamental region for the hexagonal tiling $\{6, 3\}$. Then we can double the fundamental region by replacing ρ_0 with $\widetilde{\rho_0} = \rho_0 \rho_1 \rho_0$ seen in Figure 4.4. We note that the result-

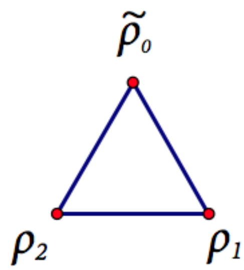

Figure 4.3: The group $\widetilde{A}_2$.

ing change means that what was the set of vertices in $\{6, 3\}$ has been divided into two sets which we can call the set of hypervertices associated with $\widetilde{\rho}_0$ and hyperedges associated with ρ_1. The remaining set which were the faces of $[6, 3]$ are now called hyperfaces. However, it is important to note that when seen as a hypertope the set of hypervertices, hyperedges and hyperfaces are mutually isomorphic and there is nothing to differentiate them except arbitrary names.

The translation subgroup of $\widetilde{A}_2$ remains the same as that of $[6, 3]$, though translation by the vector $(1, 0)$ must now be expressed with the new generators $(\widetilde{\rho}_0\rho_1\rho_2\rho_1)$. Thus, the subgroup $\Lambda_{(s,t)} \leq \mathbf{T}$ is invariant under conjugation by the $\widetilde{A}_3$ when $st(s - t) = 0$ making $\mathcal{C}/\Lambda$ a regular hypertope and chiral otherwise. For more details on these conclusions see [18].

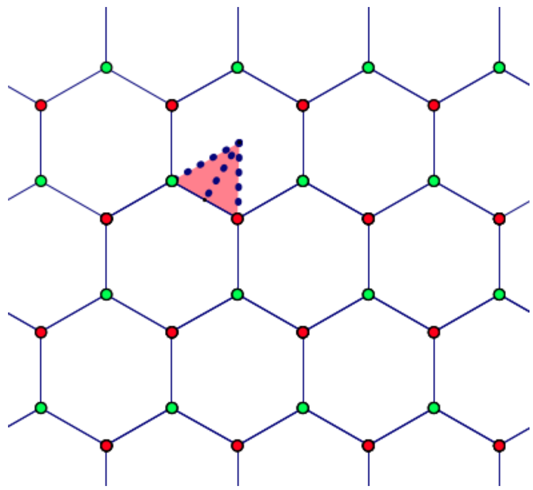

Figure 4.4: The infinite hypertope built from $\widetilde{A}_2$. The red and green vertices and the hexagonal faces are the elements of each type and the shaded area showing the doubling of the fundamental region.

4.4 Toroidal Hypertopes whose automorphism group is $\widetilde{B}_3$

We now discuss the regular toroidal hypertopes of rank 4. Let $\{\rho_0, \rho_1, \rho_2, \rho_3\}$ be the set of generators of $[4, 3, 4]$ as in Section 3.3 and ε the corresponding fundamental simplex. We can double this fundamental simplex by replacing the generator ρ_0 with $\widetilde{\rho_0} = \rho_0\rho_1\rho_0$. Then $\widetilde{\rho_0}$ is a reflection through the hyperplane through the point $(1, 0, 0)$ with normal vector $(1, 1, 0)$. The transformation of a general vector by $\widetilde{\rho_0}$ is

$$(x, y, z)\widetilde{\rho_0} = (1 - y, 1 - x, z). \tag{4.1}$$

Then $\{\widetilde{\rho_0}, \rho_1, \rho_2, \rho_3\}$ generates $\widetilde{B}_3$, a subgroup of $[4, 3, 4]$ of index 2. The Coxeter diagram for this group is the non-linear diagram in figure 4.5. In this section we let $G(=\widetilde{B}_3) := \langle\widetilde{\rho_0}, \rho_1, \rho_2, \rho_3\rangle$ and let $\mathcal{C}(\widetilde{B}_3)$ be the Coxeter complex of $G = \widetilde{B}_3$.

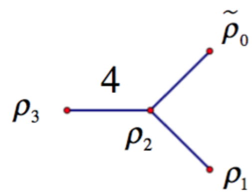

Figure 4.5: Coxeter diagram for $\widetilde{B}_3$

The fundamental simplex of $\widetilde{B}_3$ is the simplex in Figure 4.6 bounded by the planes H_1, H_2, H_3 (fixed by ρ_1, ρ_2, ρ_3 respectively) and H_0 (now fixed by $\widetilde{\rho_0}$). Let, as above, F_i be the vertex of the fundamental simplex opposite to H_i. The orbit of each vertex, F_j of the fundamental simplex of $\widetilde{B}_3$ represents the set of hyperfaces of type j. Since this fundamental simplex shares vertices F_0, F_2 and F_3 with the fundamental simplex of $\langle\rho_0, \rho_1, \rho_2, \rho_3\rangle$ we will use the same names for hyperfaces as the names in section 3, namely, vertices, faces and facets. Though the orbit or F_1 (which is isomorphic to the orbit of F_0 since conjugating by the reflection $\widetilde{\rho_0}$ preserves ρ_2 and ρ_3 (as isometries), while interchanges ρ_1 and $\widetilde{\rho_0}$) will be called hyperedges.

Now the translation subgroup of G is different from the translation subgroup in the previous section since the set of vertices of $\{4, 3, 4\}$ now represent vertices and hyperedges (hyperfaces of type 0 and 1 respectively). The translation subgroup associated with this fundamental simplex is $T = \langle (1,1,0), (-1,1,0), (0,-1,1) \rangle$.

We then note that the translation by vector $(1,1,0)$ is the transformation (by right multiplication) $w_1 = \rho_2\rho_3\rho_2\rho_1\rho_2\rho_3\rho_2\widetilde{\rho}_0$, by vector $(-1,1,0)$ is $w_2 = \rho_2\rho_3\rho_2\widetilde{\rho}_0\rho_2\rho_3\rho_2\rho_1$ and by vector $(0,-1,1)$ is $w_3 = \rho_1\rho_3\rho_2\widetilde{\rho}_0\rho_1\rho_2\rho_1\rho_2\rho_3\rho_2$.

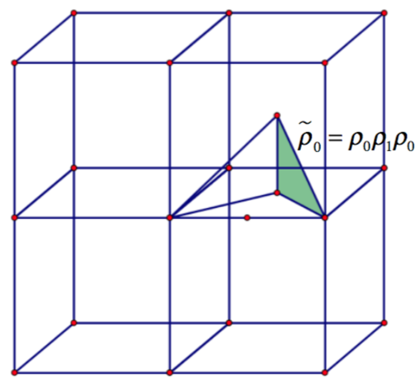

Figure 4.6: Fundamental simplex of $\widetilde{B}_3$

Now, to form a root lattice Λ we have the freedom to choose the crystollographic subgroup G_0 by fixing either a vertex or a hyperedge (see [8] page 108-109). We choose to leave out $\widetilde{\rho}_0$ since this reflection does not fix F_0. Doing so leaves $[3,4]$ as the subgroup we are conjugating with, which is the same as was for $[4,3,4]$. We also note that if we chose ρ_1 rather than $\widetilde{\rho}_0$ then the result is functionally the same since we are still conjugating by $[3,4] = \langle \widetilde{\rho}_0, \rho_2, \rho_3 \rangle$.

We now note that although the same conditions are satisfied as in Lemma 3.3.1, T is now a different set. So instead we have the following Lemma.

Lemma 4.4.1. *If* $T = \langle (1,1,0), (-1,1,0), (0,-1,1)\rangle$*,* $\Lambda \leq T$ *a subgroup and if for every* $\boldsymbol{a} \in \Lambda$*, the image of* $\boldsymbol{a}$ *under changes of sign and permutations of coordinates is also in* Λ*, then* $\Lambda = \langle (2x,0,0), (0,2x,0), (0,0,2x)\rangle$*,* $\langle (x,x,0), (-x,x,0), (0,-x,x)\rangle$ *or* $\langle (2x,2x,2x), (4x,0,0), (0,4x,0)\rangle$ *for some positive integer* x*.*

Proof. We will only modify the proof of Lemma 3.3.1. In that proof we arrive at a generating set $(s^k, 0^{3-k})$ for each $k \in \{1,2,3\}$. However, this proof will require adjustment since T is different than the translation subgroup of Section 3.

Similar arguments to the ones used in the proof to Lemma 3.3.1 can now be used to show that for $k = 1$ or $k = 3$, s is even. For $k = 2$, Λ is generated by permutations and changes of sign of $(s,s,0)$. This needs no further examination since it is clearly in T.

■

As in section 3.3, we describe the groups that will be used to construct each of the toroids. We denote by $G_{\mathbf{s}}$ the quotient $\widetilde{B}_3/\Lambda_{\mathbf{s}}$. We denoted w_1 as the translation $(1,1,0)$ while $(\rho_1\rho_2\rho_3\rho_2\widetilde{\rho_0})^2$ is the translation $(2,0,0)$ and $(\rho_3\rho_2\rho_1\rho_3\rho_2\widetilde{\rho_0})^3$ is the translation $(2,2,2)$. And now that the the mirror for $\widetilde{\rho_0}$ is $y = 1 - x$ we have the following theorem.

Theorem 4.4.2. *Let* $\boldsymbol{s} = (2s,0,0), (s,s,0)$ *with* $s \geq 2$ *or* $(2s,2s,2s)$ *with* $s \geq 1$*. Then the group* $G_{\boldsymbol{s}} = \widetilde{B}_3/\Lambda_{\boldsymbol{s}}$ *is the Coxeter group* $\widetilde{B}_3 = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \rho_3\rangle$ *with Coxeter* *diagram in figure 4.5, factored out by the single extra relation which is*

$$(\rho_1\rho_2\rho_3\rho_2\widetilde{\rho_0})^{2s} = id \text{ if } \boldsymbol{s} = (2s, 0, 0),$$

$$(\rho_2\rho_3\rho_2\rho_1\rho_2\rho_3\rho_2\widetilde{\rho_0})^{s} = id \text{ if } \boldsymbol{s} = (s, s, 0),$$

$$(\rho_3\rho_2\rho_1\rho_3\rho_2\widetilde{\rho_0})^{3s} = id \text{ if } \boldsymbol{s} = (2s, 2s, 2s).$$

Here, as in section 3, we have that $G_{\mathbf{s}}$ fails the intersection property for small enough s. However, because the fundamental simplex is doubled, this time when $\mathbf{s} = (2s, 2s, 2s)$, $G_{\mathbf{s}}$ satisfies the intersection condition for $s \geq 1$ while $s \geq 2$ is still necessary for the other two vectors. Verifying that $G_{\mathbf{s}}$ fails the intersection condition for $s = 1$ when $\mathbf{s} = (2s, 0, 0)$ and $(s, s, 0)$ follows similar calculations as those done in section 3. When $s = 1$ we start with each of the factored relations for the first and second vectors. From the first we arrive at the identity $\widetilde{\rho_0}\rho_2\rho_3\rho_2\widetilde{\rho_0} = \rho_1\rho_2\rho_3\rho_2\rho_1$ and for the second vector we have the identity $\widetilde{\rho_0}\rho_2\rho_3\rho_2 = \rho_2\rho_3\rho_2\rho_1$. These identities show us $\widetilde{\rho_0}$ can be found in the maximal parabolic subgroup $G_{\widetilde{0}} := \langle \rho_1, \rho_2, \rho_3 \rangle$. In the first case the element $\widetilde{\rho_0}\rho_2\rho_3\rho_2\widetilde{\rho_0} = \rho_1\rho_2\rho_3\rho_2\rho_1$ is in $G_{\widetilde{0}} \cap G_1$ but not in $\langle \rho_2, \rho_3 \rangle$ and in the second case the intersection of $G_{\widetilde{0}}$ and $\langle \widetilde{\rho_0} \rangle$ is non-empty. Both of these cases violates the intersection condition.

MAGMA[3] can be used to verify that $G_{\mathbf{s}}$ satisfies the intersection condition when $\mathbf{s} = (4,0,0), (2,2,0)$ or $(2,2,2)$. To see that it also satisfies the intersection condition for greater values of s can be seen with a geometric argument.

The orbit of the base chamber under each parabolic subgroup of $G_{\mathbf{s}}$ can be seen as a collection of chambers making up the orbit under the corresponding parabolic subgroup of $\widetilde{B}_3$ and then that same collection duplicated by the lattice $\Lambda_{\mathbf{s}}$. So these orbits can be interpreted as a number of repeats of one *initial collection*. This initial collection is the orbit of the base chamber under a particular parabolic subgroup of $\widetilde{B}_3$. For instance, the orbit of the base chamber under the subgroup $\langle \rho_1, \rho_2, \rho_3 \rangle$ consists of the initial octahedron made up of 48 chambers surrounding the origin as well as copies of that octahedron repeated around each vertex making up the integer lattice of Λ_{s}.

Given the collection of chambers from two such subgroups, there will always be some intersection between the two initial collections of each subgroup (sometimes it is just the base chamber itself). However, If $G_{\mathbf{s}}$ fails the intersection condition, then there will be an intersection with the chambers from the initial collection from one subgroup with the chambers from a repeated collection from the other subgroup.

So, by starting with $G_{\mathbf{s}}$ for some s, as s is increased, each of the repeated collections (associated with the lattice $\Lambda_{\mathbf{s}}$) get further and further apart making the types of intersections which demonstrate the failure of the intersection condition more unlikely. So if $G_{\mathbf{s}}$ satisfies the intersection condition for some s then $G_{\mathbf{s}'}$ will satisfy the intersection condition for any $s' \geq s$.

Adopting a similar notation as in the previous section and using $\Lambda_{\mathbf{s}}$ as defined in Section 4.2, we now have the following theorem.

Theorem 4.4.3. *The regular toroidal hypertopes of rank 4 constructed from* $G(= \widetilde{B}_3) = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \rho_3 \rangle$, *where the generators are specified in* (3.1) *and* (4.1), *are* $\mathcal{C}(\widetilde{B}_3)/\Lambda_{\mathbf{s}}$ *where* $\mathcal{C}(\widetilde{B}_3)$ *is the Coxeter complex of* $\widetilde{B}_3$ *and* $\mathbf{s} = (2s, 0, 0), (s, s, 0)$ *with* $s \geq 2$ *or* $(2s, 2s, 2s)$ *with* $s \geq 1$.

Proof. To begin we suppose $\mathcal{C}(\widetilde{B}_3)/\Lambda_{\mathbf{s}}$ is such a regular toroidal hypertope. Then $\mathbf{s}$ must be such that $\Lambda_{\mathbf{s}}$ is invariant under conjugation by a subgroup of G which is the symmetry group of "vertex"-figure (by vertex we mean, the element at the origin). In this case our subgroup ends up being $[3, 4]$ as was described before Lemma 4.4.1.

Now, since we are conjugating by $[3, 4] = \langle \rho_1, \rho_2, \rho_3 \rangle$, $\Lambda_{\mathbf{s}}$ must contain all permutations and changes of sign of any vector in $\Lambda_{\mathbf{s}}$ (which we discovered in the proof of Theorem 3.3.2 which is also on page 165 of [26]). Thus, by Lemma 4.4.1, $\mathbf{s} = (2s, 0, 0), (s, s, 0)$ or $(2s, 2s, 2s)$. However, we still do not know if this construction yields a regular hypertope. To do this, we start by noting that the Coxeter complex $\mathcal{C}(\widetilde{B}_3)$ formed from G is precisely the hypertope $\Gamma(G; (G_i)_{i \in I})$ (the construction of which follows from [18]).

So we need to show that $\mathcal{C}(\widetilde{B}_3)$ is flag-transitive (or, equivalently, chamber-transitive). To do so we will note the rank 3 residue $\Gamma_{\widetilde{0}} := \Gamma(G_{\widetilde{0}}; (G_{\{\widetilde{0}, i\}})_{i \in \{1,2,3\}})$.

This is isomorphic to the cube, a regular polyhedron, which is flag-transitive.

So we pick two chambers in $\Gamma(G;(G_i)_{i\in I}) = \mathcal{C}(\widetilde{B}_3)$, C_1 as a chamber containing the element $G_{\widetilde{0}}g_0$ of type $\widetilde{0}$ and C_2 as a chamber containing the element $G_{\widetilde{0}}h_0$ of type $\widetilde{0}$. Then, since $G = G_{\widetilde{0}} \rtimes T$ and T acts transitively on elements of type $\widetilde{0}$ there is a translation $t \in G$ such that $C_1 t = \{G_{\widetilde{0}}h_0, X, Y, Z\}$ which is some chamber that shares the same element of type $\widetilde{0}$ as C_2. Then the chambers $C_1 t$ and C_2 are both in some rank 3 residue which is isomorphic to $\Gamma_{\widetilde{0}}$. Since this residue is flag-transitive, there is some element, $g \in G$ such that $C_1 tg = C_2$. Thus $\mathcal{C}(\widetilde{B}_3)$ is chamber-transitive and thus flag-transitive. So, by proposition 4.6 from [18] this is a regular hypertope.

So now we want to know if $\Gamma(G';(G'_i)_{i\in I})$ is a regular hypertope where G' is the group $G/\Lambda_{\mathbf{s}}$ where s satisfies the starting conditions (since otherwise G' fails the intersection condition and the resulting construction fails to be thin). Just as before, we take two chambers Φ and Ψ from $\Gamma(G';(G'_i)_{i\in I})$. Then to each of these chambers we can associate a family of chambers Φ' and Ψ' in $\mathcal{C}(\widetilde{B}_3)$. Since $\mathcal{C}(\widetilde{B}_3)$ is chamber-transitive there exist chambers $\Phi_1 \in \Phi'$ and $\Psi_1 \in \Psi'$ in $\mathcal{C}(\widetilde{B}_3)$ where, since $\Lambda_{\mathbf{s}}$ is invariant under G, $\Phi_1\psi = \Psi_1$ and $\psi \in G$. Then $\Phi(\Lambda_{\mathbf{s}}\psi) = \Psi$. We can see this by noting that Φ_1 and Ψ_1 are the members of their respective families which lie inside the fundamental region of $\Lambda_{\mathbf{s}}$ and so ψ is in the stabilizer of the origin.

Thus $\Gamma(G';(G'_i)_{i\in I})$ is chamber-transitive and thus flag transitive, so is also a regular hypertope by Proposition 4.1.6.

■

4.5 Toroidal Hypertopes whose automorphism group is $\widetilde{A}_3$

We can show that this group is, yet again a subgroup of $[4, 3, 4]$ by doubling the fundamental region a second time and now defining $\widetilde{\rho}_3 = \rho_3\rho_2\rho_3$ which is a reflection in the plane through the origin with normal vector $(0, 1, 1)$. Transformation of a general vector by $\widetilde{\rho}_3$ is

$$(x, y, z)\widetilde{\rho}_3 = (x, -z, -y). \tag{4.2}$$

Now we let $G(= \widetilde{A}_3) := \langle \widetilde{\rho}_0, \rho_1, \rho_2, \widetilde{\rho}_3 \rangle$ and $\mathcal{C}(\widetilde{A}_3)$ be the Coxeter complex of G. The defining relations for G are implicit in the Coxeter diagram in figure 4.7.

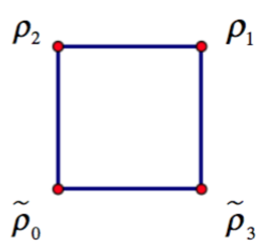

Figure 4.7: Coxeter diagram for $\widetilde{A}_3$

Here the fundamental region of $\widetilde{A}_3$ is a tetrahedron bounded by the planes H_i. This fundamental region shares the planes fixed by $\widetilde{\rho}_0, \rho_1, \rho_2$ with the fundamental region of $\widetilde{B}_3$ as well as the vertices F_0, F_1 and F_3. The stabilizers of each vertex of the fundamental region are also isomorphic since all maximal parabolic subgroups of $\widetilde{A}_3$ are pairwise isomorphic to the automorphism group of the tetrahedron. This implies that the set of hyperfaces of types i and j are isomorphic for each $i, j \in \{0, 1, 2, 3\}$.

This fundamental region gives us the same translation subgroup as we had in the previous section. Though now we must use the new generators to find the translations. We define $w_1 = \rho_2\rho_1\widetilde{\rho}_3\rho_1\rho_2\widetilde{\rho}_0 = (1, 1, 0)$, $w_2 = \rho_2\widetilde{\rho}_3\widetilde{\rho}_0\widetilde{\rho}_3\rho_2\rho_1 = (-1, 1, 0)$ and $w_3 = \rho_1\widetilde{\rho}_0\widetilde{\rho}_3\widetilde{\rho}_0\rho_1\rho_2 = (0, -1, 1)$.

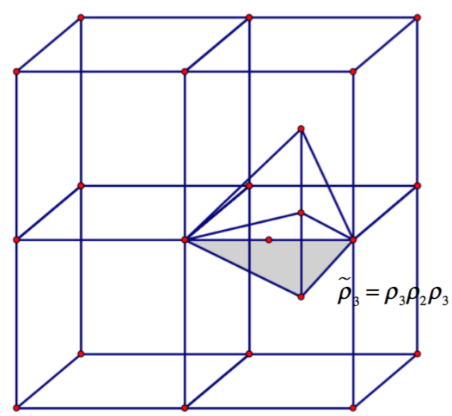

Figure 4.8: Fundamental region of $\widetilde{A}_3$

Using these translations, for an arbitrary translation $(a, b, c) \in \Lambda$, we have that $\rho_1(a, b, c)\rho_1 = (b, a, c)$. Similarly, conjugating by ρ_2 yields (a, c, b) and conjugating by $\widetilde{\rho}_3$ yields$(a, -c, -b)$. So if we conjugate by $\rho_1\rho_2\rho_1$ then we get (c, b, a) and so Λ must be closed under permutation of coordinates. Now, we know Λ must also contain $(a, -b, -c)$ and adding this to (a, b, c) gives $(2a, 0, 0)$, which then subtracted from (a, b, c) is $(-a, b, c)$ and so by permuting before finding the change in sign, and then permuting back, we have all changes in sign.

We leave out $\widetilde{\rho_0}$ to form the crystollographic subgroup G_0. A property of this group is that we can use any generator of $\widetilde{B}_3$ to form a crystollographic subgroup G_i and still finish with the same objects. This leaves ρ_1, ρ_2 and $\widetilde{\rho_3}$ with which to conjugate Λ. As in the regular case, ρ_1 and ρ_2 show us that Λ must consist of all permutations of the coordinates of vectors.

As in the previous section, we describe the groups of each of the toroids. Earlier we noted w_1 as the translation $(1, 1, 0)$ while $(\rho_1\widetilde{\rho_3}\rho_2\widetilde{\rho_0})^2$ is the translation $(2, 0, 0)$ and $(\widetilde{\rho_3}\rho_1\rho_2\widetilde{\rho_0})^3$ is the translation $(2, 2, 2)$. And now that the the mirror for $\widetilde{\rho_0}$ is $y = 1 - x$ while the mirror for $\widetilde{\rho_3}$ is $y = -z$ we have the following theorem.

Theorem 4.5.1. *Let $\boldsymbol{s} = (2s, 0, 0), (s, s, 0)$ with $s \geq 2$ or $(2s, 2s, 2s)$ with $s \geq 1$. Then the group $G_{\boldsymbol{s}} = \widetilde{A}_3/\Lambda_{\boldsymbol{s}}$ is the Coxeter group $\widetilde{A}_3 = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \widetilde{\rho_3} \rangle$ (with Coxeter group specifed in figure 4.7), factored out by the single extra relation which is*

$$(\rho_1\widetilde{\rho_3}\rho_2\widetilde{\rho_0})^{2s} = id \text{ if } \boldsymbol{s} = (2s, 0, 0),$$

$$(\rho_2\rho_1\widetilde{\rho_3}\rho_1\rho_2\widetilde{\rho_0})^{s} = id \text{ if } \boldsymbol{s} = (s, s, 0),$$

$$(\widetilde{\rho_3}\rho_1\rho_2\widetilde{\rho_0})^{3s} = id \text{ if } \boldsymbol{s} = (2s, 2s, 2s).$$

For the same reasons as in Section 4.4, the intersection condition is satisfied for $\mathbf{s} = (2s, 2s, 2s)$ when $s \geq 1$.

Theorem 4.5.2. *The regular toroidal hypertopes of rank 4 induced by* $G(= \widetilde{A}_3) = \langle \widetilde{\rho_0}, \rho_1, \rho_2, \widetilde{\rho_3} \rangle$ *(where the generators are specified in* (3.1)*,* (4.1) *and* (4.2)*) are* $\mathcal{C}(\widetilde{A}_3)/\Lambda_{\boldsymbol{s}}$ *where* $\mathcal{C}(\widetilde{A}_3)$ *is the Coxeter complex of* $\widetilde{A}_3$ *and* $\boldsymbol{s} = (2s, 0, 0), (s, s, 0)$ *for* $s \geq 2$ *or* $(2s, 2s, 2s)$ *with* $s \geq 1$*.*

Proof. Since the translation subgroup is the same as in the previous Section and conjugating Λ by $\rho_1, \rho_2, \widetilde{\rho_3}$ gives all permutations and changes in sign of a general vector in Λ, the same arguments for lemma 4.4.1 and theorem 4.4.3 will prove this theorem. ■

4.6 Non-existence of rank 4 Toroidal Chiral Hypertopes

Here we recall that for an abstract polytope to be chiral its automorphism group must have two orbits when acting on flags and that adjacent flags are in different orbits. Chiral polytopes of any rank are examined in depth in [31]. The existence of these objects in any rank was proved in [27]. There is also a notion of chirality in hypermaps as well, see for example, [2]. Similarly we say that a hypertope is chiral if its automorphism group action has two chamber orbits and adjacent chambers are in different orbits [18].

As in section 2, given an affine Coxeter group G and associated Coxeter complex $\mathcal{C}$, we define a subgroup $G_0 \leq G$ as the maximal parabolic subgroup fixing the origin. Given a set K of linearly independent translations in G and T_K, the translation subgroup generated by K, we denote by Λ_K the lattice induced by the orbit of the origin under T_K. When Λ_K is invariant under the rotation subgroup ${G_0}^+$ but there is no element of $G_0 \setminus {G_0}^+$ preserving Λ_K, then in rank 4 we say that the quotient $\mathcal{C}/\Lambda_K$ is a *chiral toroidal hypertope* of rank 4.

The proof that there are no chiral toroids of rank 4 for the group $[4, 3, 4]$ comes from page 178 from [26] and the same proof can be adapted for the other two rank 4 affine Coxeter groups. In fact that proof (as we shall see) needs only minor changes to be adapted to all affine cases. The basic idea for the proof is by contradiction and says that since $\mathcal{C}/\Lambda$ is chiral, Λ is invariant under the rotation group $[3, 4]^+$, so Λ is invariant under compositions of an even number of even permutations with an even number of sign changes and by compositions of an odd number of odd permutations with an odd number of sign changes. It then goes to show that if $(a, b, c) \in \Lambda$ then $(b, a, c) \in \Lambda$, which is the image of (a, b, c) under an odd permutation, which is a contradiction. Therefore no such Λ can exist.

We will use the same method to show the same is true for the other two groups.

Theorem 4.6.1. *There are no rank 4 chiral toroidal hypertopes.*

Proof. In [26] it was shown that there are no rank 4 chiral polytopes which are quotients of $\{4,3,4\}$, so we extend the same result for constructions of hypertopes from $\widetilde{B}_3$ and $\widetilde{A}_3$. In previous sections we found that if Λ is a subgroup of the translations that is invariant under conjugation by the stabilizer of the origin in $\widetilde{B}_3$ and $\widetilde{A}_3$ with the translation $(a,b,c) \in \Lambda$, then Λ also contains all permutations and changes of sign of (a,b,c), just as it did with the stabilizer in $[4,3,4]$.

Thus conjugation of Λ by the stabilizer of the rotation subgroup of each of these groups results in all compositions of even permutations with an even number of sign changes or all compositions of odd permutations with an odd number of sign changes, just as for $[4,3,4]$. So the same arguments and calculations from page 178 in [26] still hold and show that $(b,a,c) \in \Lambda$ and we develop the same contradiction.

■

5 Block Systems of Polytopes

Block systems come from the theory of permutation groups where they are primarily used to build the concept of a primitive group action (a subject not relevant to this thesis). A group action is called primitive if the only block systems are trivial block systems. The paper [4] and text [15] give details on blocks in the context of permutation groups. In particular, one can use block systems as a computational means of finding out if a group is primitive [1]. An important note is that block systems go by different names in different texts. The paper [4] and text [29] refer to block systems as "systems of imprimitivity" which reflects the importance in permutation groups where the action has only trivial block systems.

Another technique uses block systems as a tool to "force" a primitive group action when the usual group action is imprimitive. When a group acts on a set imprimitively, there will be a block system where the action of the group on those blocks is primitive [5]. This technique is used by Leemans and co-authors in [6] and

[5]Such a block system is sometimes called *maximal.*

[17] as tool to deal with cases where a group action is transitive but not primitive in order to find bounds on the rank for polytopes whose automorphism groups are the alternating or symmetric groups.

Block systems are also relevant in the study of combinatorially designs by examining designs called *block-transitive* designs. This has been done by Cameron and Praeger in, for example, [5] and [28].

This chapter and the next will then develop existence and classification theorems for block systems in relation to polytopes and hypertopes. The methods used in these chapters are geometric, which contrasts the more algebraic methods in the above mentioned literature.

We begin this chapter by introducing the concept of a *block system* of facets of a polytope under the group action of its automorphism group. Then we characterize all block systems on the regular tessellations $\{4,4\}$ and $\{4,3,4\}$ respectively as well as on the regular and chiral toroids constructed from those tessellations. Finally we deal with block systems of the facets of polytopes resulting from $\{3,6\}$ and $\{6,3\}$ which, as we shall see, are considerably more complex. These results will likely only require minor adjustments in order to extend to some regular toroidal polytopes of ranks higher than 4. However, the polytopes $\{3,6\}$ and $\{6,3\}$ will show us that higher rank results are not guaranteed to be direct extensions. In the next chapter we will extend these ideas to hypertopes.

5.1 Block Systems of Facets of Polytopes

Some interesting problems are motivated by colouring the facets of a polytope $\mathcal{P}$ and observing the results under the group action of $\Gamma(\mathcal{P})$ on the facets. In particular we can seek to colour the facets of $\mathcal{P}$ in such a way that a colouring of facets of $\mathcal{P}$ is *invariant* under automorphisms of $\mathcal{P}$.

An example of such a colouring is the colouring of the faces of a cube (see Figure 5.1). We use three colours to colour the faces of the cube where each face is the same colour as its opposite face. Under any automorphism γ of the cube, opposite faces are still the same colour, though the specific colours may be in different places.

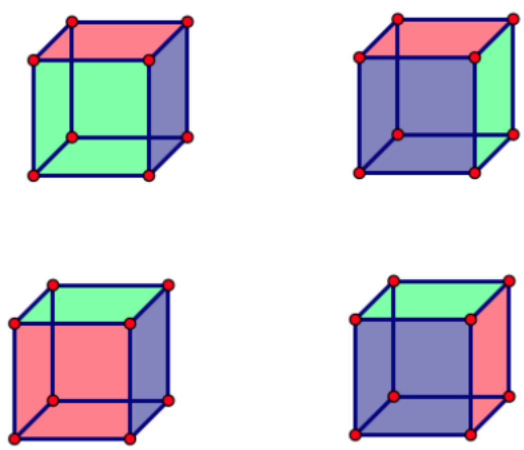

Figure 5.1: A block system on a cube and several rotations.

We can interpret the set of facets of a single colour as a *block* which we will now describe in detail.

Definition 5.1.1. Given a group G acting on a set $\mathcal{F}$ a *block* is a subset $F \subseteq \mathcal{F}$ where for any $g \in G$ either

- $Fg = F$, or
- $Fg \cap F = \emptyset$.

A block B is said to be trivial if $B = \emptyset, \mathcal{F}$ or a singleton set.

Theorem 5.1.2. *[29] Given a group G acting transitively on a set $\mathcal{F}$, and let B be a non-trivial block of $\mathcal{F}$.*

(i) If $g \in G$, then Bg is a block.

(ii) There are elements $g_1, \ldots, g_m \in G$ such that $\{Bg_1, \ldots, Bg_m\}$ is a partition of $\mathcal{F}$.

Definition 5.1.3. Given a group G acting transitively on a set $\mathcal{F}$ and a block B, a *block system* on $\mathcal{F}$ under the action of G is a partition of $\mathcal{F}$ into blocks $\{Bg_1, \ldots, Bg_m\}$.

In this thesis we will only deal with block systems which have a finite number of blocks.

With each block we will associate a colour as colourings are useful ways to visualize partitions of facets. In this chapter we will discuss only block systems of facets of a polytope and so we will sometimes refer to a block system on $\mathcal{P}$ rather than on the facets of $\mathcal{P}$. Additionally, the group action will be assumed to be $\Gamma(\mathcal{P})$ acting on the set of facets of $\mathcal{P}$, though when discussing chiral polytopes we will be more specific about defining the group action.

Given two block systems A and B on a set $\mathcal{F}$ we adapt the language of set theory and say that A is a *refinement* of B if every block $\alpha \in A$ is a subset of some block $\beta \in B$. In this case we say that A is *finer* than B and that B is *coarser* than A.

Given a polytope $\mathcal{P}$ of rank k we let $\mathcal{F}_{k-1}$ be the set of facets of $\mathcal{P}$. Then we can see that under the group action of $\Gamma(\mathcal{P})$ we can find block systems of $\mathcal{F}_{k-1}$ by partitioning it into singleton sets or into the trivial partition $\{\mathcal{F}_{k-1}\}$. This is akin to colouring every facet of a polytope with a different colour, or with the same colour. Here the singleton block system is finer than the trivial block system. The set $\mathcal{F}_{k-1}$ of facets of $\mathcal{P}$ is *primitive* if those are the only block systems. If $\mathcal{F}_{k-1}$ is not primitive, then we know we can find another block system which is a refinement of the trivial block system and which can then be refined further to the singleton block system.

Lemma 5.1.4. *Suppose x and y are facets of a polytope $\mathcal{P}$ and χ_1 and χ_2 are blocks in the same block system of the facets of $\mathcal{P}$. For any $x, y \in \chi_1$, and any $\gamma \in \Gamma(\mathcal{P})$, if $x\gamma \in \chi_2$, then $y\gamma \in \chi_2$.*

Remark 6. For the proof of this lemma it helps to note that a direct consequence of the definition of a block is that if the image of a subset under the group action is partially contained within itself and partially somewhere else, that subset cannot be a block.

Proof. First suppose that $\chi_1 = \chi_2$, which we will simply call χ. Then, by the definition of a block, $\chi\gamma = \chi$ or $\chi\gamma \cap \chi = \emptyset$ and since $x\gamma \in \chi\gamma \cap \chi$ we know that $\chi\gamma = \chi$ so $y\gamma \in \chi$.

Now suppose that $\chi_1 \neq \chi_2$. Since block systems are partitions we know that $\chi_1 \cap \chi_2 = \emptyset$ and also that $\chi_1\gamma \cap \chi_1 = \emptyset$ from the definition of a block and our choice of γ. Suppose now the contrary, which is to say that there exists a $\gamma \in \Gamma(\mathcal{P})$ where $x\gamma \in \chi_2$ and $y\gamma \notin \chi_2$.

Since χ_1 and χ_2 are in the same block system there is a $g \in \Gamma(\mathcal{P})$ such that $\chi_2 g = \chi_1$. Then we note that $x\gamma g \in \chi_1$ but $y\gamma g \notin \chi_1$ since g is a bijection, $|\chi_1| = |\chi_2|$ and $y\gamma \notin \chi_2$. Thus χ_1 is not a block, which is a contradiction since γg splits χ_1 in the manner described in the above remark. ■

We can also add another lemma giving us a method for finding individual blocks.

Lemma 5.1.5. *Let $\mathcal{P}$ be a k-polytope where $\Gamma(\mathcal{P})$ acts transitively on its facets, and let $\{\chi_i : i = 1, \dots, d\}$ be a block system on $\mathcal{F}_{k-1}$ under the action of $\Gamma(\mathcal{P})$. Then $|\chi_i| = |\chi_j|$ for every $i, j \leq d$ and if $|\mathcal{F}_{k-1}|$ is finite, $|\chi_i| \mid |\mathcal{F}_{k-1}|$.*

Proof. If $\chi_i = \chi_j$ we are done, otherwise we note that $\chi_i \cap \chi_j = \emptyset$. Since $\mathcal{P}$ is facet transitive, for each $x_i \in \chi_i$ and each $x_j \in \chi_j$, there exists $\gamma \in \Gamma(\mathcal{P})$ such that, $x_i\gamma = x_j$. Let $y \in \chi_i$ where $y \neq x_i$. By Lemma 5.1.4, $y\gamma \in \chi_j$ and also $y\gamma \neq x_j$ (since γ is a bijection). Therefore $\chi_i\gamma \subseteq \chi_j$ and $|\chi_i| \leq |\chi_j|$. Similarly $|\chi_j| \leq |\chi_i|$.

■

The preceding two lemmas can be used to find many block systems on the faces of the Platonic solids. In addition to trivial and singleton block systems, we can extend the example we used for the cube earlier. Block systems can be found for each Platonic solid besides the tetrahedron by defining each block to consist of a face and its antipodal face. The octahedron has an additional block system with two blocks where adjacent faces are in different blocks (see Figure 5.2).

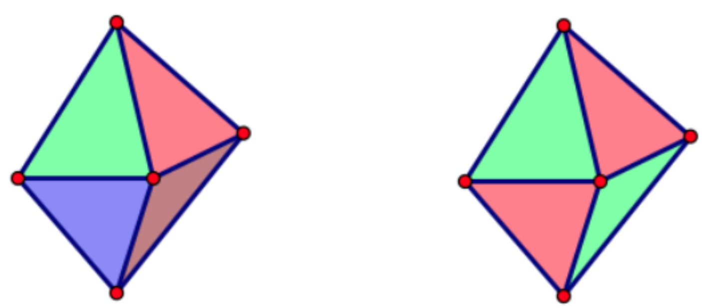

Figure 5.2: Two non-trivial block systems on the octahedron

We now introduce a theorem that will enable us to find block systems on facets of polytopes using normal subgroups.

Lemma 5.1.6 (Lemma 9.16(ii) in [29])**.** *Let $\mathcal{P}$ be a facet-transitive polytope. If $N \trianglelefteq \Gamma(\mathcal{P})$, then the orbits on the facets of $\mathcal{P}$ under N form a block system of the facets of $\mathcal{P}$ under the action of $\Gamma(\mathcal{P})$.*

Proof. Let x be a facet of $\mathcal{P}$ and B_x be the N-orbit of x, that is $B_x = \{xn : n \in N\}$. Further, suppose $B_x g \cap B_x \neq \emptyset$ where $g \in \Gamma(\mathcal{P})$. Then, since $N \trianglelefteq \Gamma(\mathcal{P})$ we know that $B_x g = B_{xg}$, $B_x g \cap B_x = B_{xg} \cap B_x$. Then we can find $n, n' \in N$ where $xgn = xn'$, and so $xg = xn'n^{-1} \in B_x$ thus $g \in N$ and so $B_x g = B_x$ and the orbits of N form a block system. ■

We can also shoehorn the definition of a block system to the Archimedean solids. These polyhedra are not face-transitive, but we can find block systems for each orbit of faces (often with faces in different orbits being different polygons) and then combining them. By way of example the symmetry group of a truncated cube acts on its octagonal faces in the same way as the action on the faces of a cube, while it acts on the triangular faces in the same way as it acts the action on the faces of an octahedron. So a block system on the faces of a truncated cube is a combination of the block systems from a cube with those of an octahedron. In fact, if one wants to find block systems on the facets of a polytope which is not facet transitive, this is the way one can do it. Though this is outside the scope of this thesis.

Remark 7. In the next sections we will be finding block systems on a number of regular toroidal polytopes and hypertopes. The general strategy for this is to define a partition as a "candidate" for a block system and then prove that it is, indeed, a block system. As such, the elements of the candidate block system will be referred to simply as 'subsets' before they can be proved to be blocks.

5.2 Block Systems on the faces of {4,4} and resulting toroids

The polytope $\{4, 4\}$ is the infinite tiling of the Euclidean plane with squares (see Figure 5.3). We follow section 3.2 in setting up the coordinate system as well as in choosing the generators ρ_0, ρ_1 and ρ_2 for the Coxeter group $[4, 4]$. We will then label the faces of {4,4} as (x, y) by identifying the face with the cartesian coordinate of its lower lefthand corner. Occasionally this will also be used to identify vertices of {4,4} (it will be made clear which this refers to, to avoid confusion).

Theorem 5.2.1. *If two adjacent faces of* $\{4, 4\}$ *are in the same block of faces under the action of* $[4, 4]$*, then all faces of* $\{4, 4\}$ *are in the same block and the block system is trivial.*

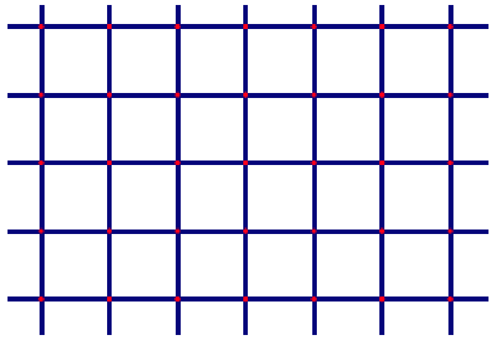

Figure 5.3: A portion of the tiling of the Euclidean plane by squares.

Proof. Suppose, without loss of generality, that the two adjacent faces, F and G in the same block B, are along the same row. Now let Z be the image of F under the automorphism γ where γ is the reflection in a vertical line through the centre of G.

Then, by Lemma 5.1.4, Z must also be in B. Inductively, every face along that same row must be in B.

If δ is the reflection in a diagonal line through the centre of G, δ fixes G but maps the row to the column through G. Then using Lemma 5.1.4 we see this column is also in B. Then, using the same process which showed the entire row was in B, we can carry the column through G across the entire plane to show that all the faces of $\{4, 4\}$ are in the same block.

■

To find block systems (in this and also subsequent sections), we will start by defining partitions. These partitions are "candidate block systems," so we will proceed to prove that these partitions are block systems. Finally we will prove that there are no other block systems for the given polytope.

Lemma 5.1.6 gives us the tools to find two classes of block systems for $\{4,4\}$. For each positive integer a we can associate two distinct block systems, one in each class. These are each based on the normal subgroups of $[4,4]$ which yield the regular toroids $\{4,4\}_{(a,0)}$ and $\{4,4\}_{(a,a)}$.

The first of these classes are the partitions X_a of the faces of $\{4,4\}$ (for a positive integer a). We define these as the partition of the faces of $\{4,4\}$ into subsets $\chi_1 \ldots \chi_d$, where each subset is unique and a translation of the subset $\chi_1 := \{(ma, na) : m, n \in \mathbb{Z}\}$ (an example is in Figure 5.4). So, if a face (x, y) is in some block χ_i, then every other face in $\chi_i, i \in \{1, \ldots, d\}$ has the form $(x + ma, y + na)$ for some integers m and n. There are a^2 translates of χ_1 so X_a partitions the faces of $\{4,4\}$ into $d = a^2$ subsets. Alternatively we can view this as colouring the faces of $\{4,4\}$ with a^2 colours using X_a.

The second class consists of the partitions Y_a of the faces of $\{4,4\}$ (for a positive integer a). These partitions are defined as being all the translates of the subset $\chi := \{(na - ma, na + ma) : n, m \in \mathbb{Z}\}$. As before, if a face (x, y) is in some subset χ', then every other face in χ' has the form $(x + na - ma, y + na + ma)$ for some integers n and m. There are $2a^2$ translates of χ so Y_a partitions the faces of $\{4,4\}$

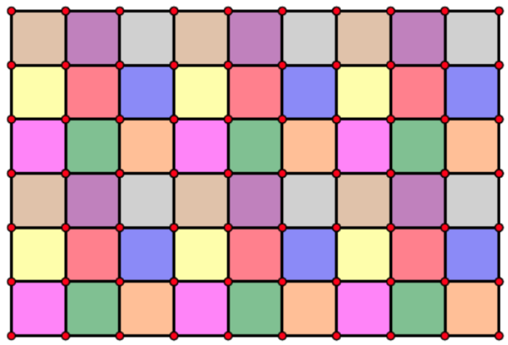

Figure 5.4: The partition X_3 with blocks indicated by colours.

into $2a^2$ subsets.

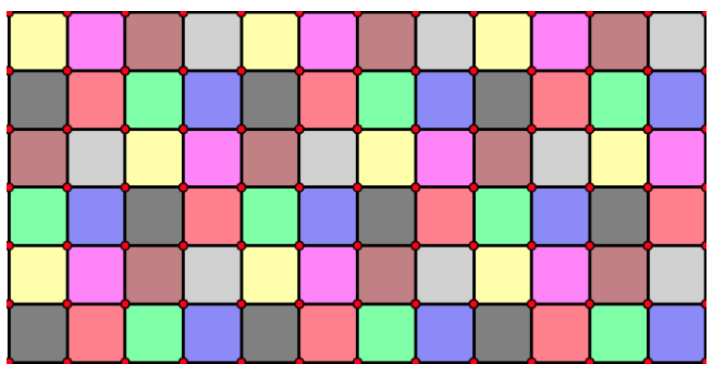

Figure 5.5: The partition Y_2 with blocks indicated by colours.

Theorem 5.2.2. *Given any positive integer a, X_a and Y_a are block systems of the faces of $\{4,4\}$ under the group action of $[4,4]$.*

Proof. These two partitions divide the set of faces into orbits under the normal subgroups $\Lambda_{(a,0)}$ and $\Lambda_{(a,a)}$ and thus by Lemma 5.1.6, are block systems.

■

Remark 8. Notice that X_a is a refinement of X_b when $b \mid a$.

Remark 9. Notice that Y_a is a refinement of Y_b when $b \mid a$. It is important to note that X_{2a} is also a refinement of the block system Y_a.

Theorem 5.2.3. *The only non-trivial, non-singleton block systems for the faces of* $\{4,4\}$ *under the group action of* $[4,4]$ *are* X_a *and* Y_a *for all positive integers* a.

Proof. Suppose Z is a block system of the faces of {4,4}. Since Z is a block system and {4,4} is regular, all blocks from Z are translates of each other (this follows from Lemma 5.1.4 and the fact that there is a translation between any two faces of $\{4,4\}$), so pick the block of faces $\theta \in Z$ containing $(0,0)$. Within this block there must be two faces with a minimum distance between them (where this distance is the Euclidean distance between lower lefthand corners of the faces). Without loss of generality we can assume one of these faces is the face at the origin. Then let (x,y) be the other face in θ with that minimum distance from $(0,0)$. We can also assume that $x \leq y$ (since otherwise we can reflect by ρ_1 and use those coordinates).

The simplest case is when this distance is 0, that is when $x^2 + y^2 = 0$. In this case this is the trivial block system where all faces are their own singleton block.

We again take note of the definitions from Section 3.2 for $[4,4]$. Reflecting the face (x,y) by ρ_0 gives us the face $(-x,y)$ and reflecting in ρ_1 gives the face (y,x). Since both ρ_0 and ρ_1 fix the face $(0,0)$, and since $(x,y) \in \theta$, $(-x,y)$ and (y,x) are also in θ (see Figure 5.6). The squares of the distances between these two faces

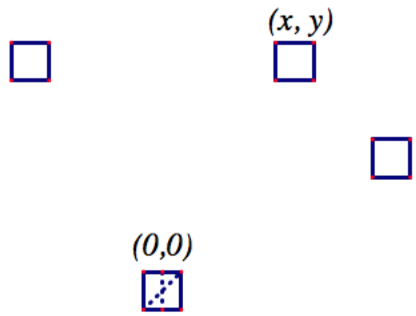

Figure 5.6: The images of (x, y) under ρ_1 and ρ_2

and (x, y) are $4x^2$ and $2(y-x)^2$, respectively. Since the distance between (x, y) and $(0, 0)$ is minimal we know that $4x^2$ and $2(x-y)^2$ must both be larger than x^2+y^2 or one of them must be equal to 0.

Since $x <\leq y$, we know that $x^2+y^2 \leq 4x^2$. So we suppose $x^2+y^2 \leq 2(y-x)^2$, then $4xy \leq x^2+y^2$. Since $x \leq y$, $4x^2 \leq 4xy \leq x^2+y^2$. Then $4x^2 = 0$ or $4x^2 = x^2+y^2$. If $4x^2 = 0$, then $x = -x = 0$, otherwise, $4xy \leq 4x^2$, but since $x \leq y$, $x = y$.

Lemma 5.1.4 and translations by integer combinations of the vectors (x, y), $(-x, y)$ and $y, x)$ will show that Θ must then contain a block from X_a or Y_b for some positive integers a or b. We can then also note that Θ *must* be such a block, since otherwise it would contain some face not in the block from X_a or Y_b which would violate the minimum distance between pairs of faces. Thus, Z is either X_a or Y_b for some positive integers a or b. ■

5.2.1 Block Systems of the faces of $\{4,4\}_{(p,0)}$ and $\{4,4\}_{(p,p)}$

Recall that each regular toroid of Schläfli type $\{4,4\}$ can be described as $\{4,4\}_{(p,0)}$ or $\{4,4\}_{(p,p)}$ where p is a positive integer. Block systems of the infinite polytope $\{4,4\}$ provide a starting point for block systems of the faces of these toroidal maps. Suppose we begin with a block system of the faces of $\{4,4\}_{(p,0)}$. We can extend this block system to a partition of the faces of $\{4,4\}$ by the translation subgroup $\Lambda_{(p,0)}$. This is itself a block system since if a part in this partition where not a block, then its counterpart in $\{4,4\}_{(p,0)}$ would not be a block either. So, any block system of $\{4,4\}_{(p,0)}$, when extended to $\{4,4\}$ as a partition, must also be a block system.

If we start with the singleton block system of the faces of the toroid $\{4,4\}_{(p,0)}$ we have p^2 blocks (recalling that $\{4,4\}_{(p,0)}$ has p^2 faces). We can extend this block system to $\{4,4\}$ by the translation subgroup $\Lambda_{(p,0)}$ this becomes the block system X_p. So if we want to find all block systems of the faces of $\{4,4\}_{(p,0)}$ it suffices to find all block systems of the faces of $\{4,4\}$ that are coarser than X_p. This means that any block system coarser than X_p must be of the form X_a or Y_b where $a \mid p$ and $2b \mid p$.

Theorem 5.2.4. *Let a and p be positive integers such that $a \mid p$. Then the partition X_a induces a block system of the faces of $\{4,4\}_{(p,0)}$ and if $2a \mid p$ then the partition Y_a induces a block system of the faces of $\{4,4\}_{(p,0)}$.*

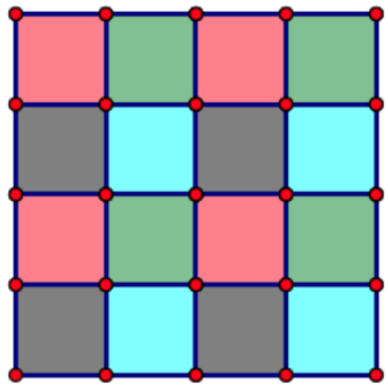

Figure 5.7: The block system X_2 on the faces of $\{4, 4\}_{(4,0)}$.

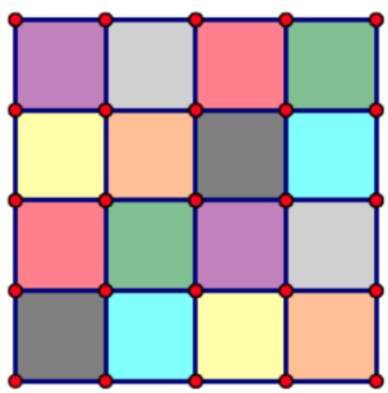

Figure 5.8: The block system Y_2 on the faces of $\{4, 4\}_{(4,0)}$.

Proof. As mentioned above it suffices to find block systems, X_a and Y_a that are coarser than X_p.

Let χ be a subset such that $\chi \in X_p$, then if X_a is coarser than X_p, there exists a subset $\theta \in X_a$ where $\chi \subset \theta$. Since each element of X_a (and also X_p) are translates of each other, it suffices to work at the level of χ. We can assume that all faces in θ are of the form (ma, na) so then to show X_a is coarser than X_p it suffices to show that every element of χ is of the form (ma, na). Since every element of χ is of the form (xp, yp) for some x, y, X_a is coarser than X_p if and only if $a \mid p$.

Thus X_a is a block system for $\{4,4\}_{(p,0)}$.

Similarly we can find a subset η of Y_a and assume the faces in η are of the form $((s+t)a, (s-t)a)$ for some s and t. Then if $2a \mid p$, $(xp, yp) = (2xp'a, 2yp'a) = ((s+t)a, (s-t)a)$, where $p = 2ap'$. Thus $\chi \subset \eta$ and thus Y_a is coarser than X_p. So we have that Y_a is a block system of $\{4,4\}_{(p,0)}$. ∎

Now we turn our attention to $\{4,4\}_{(p,p)}$. As before finding block systems of $\{4,4\}_{(p,p)}$ amounts to finding block systems, X_a or Y_a that are coarser than Y_p.

Theorem 5.2.5. *Let a and p be positive integers such that $a \mid p$. Then the partitions X_a and Y_a induce block systems of the faces of $\{4,4\}_{(p,p)}$.*

Proof. This proof follows the same form as that of Theorem 5.2.4 except that χ_i is a subset of Y_p and as such faces in χ_i are of the form $((x+y)p, (x-y)p)$. ∎

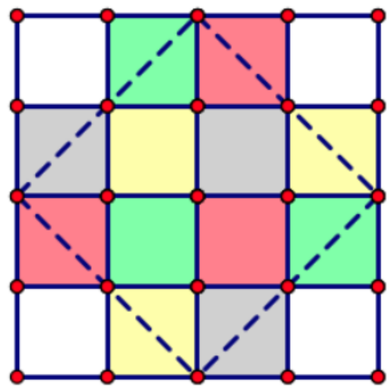

Figure 5.9: The block system X_2 on the faces of $\{4,4\}_{(2,2)}$.

Since we proved that X_a and Y_b are the only block systems on the faces of $\{4,4\}$ under the action of $[4,4]$ then we can see that the only block systems on the faces of the regular toroidal maps of type $\{4,4\}$ are induced by X_a and Y_b under the conditions in Theorems 5.2.4 and 5.2.5.

5.2.2 Block Systems on the faces of the chiral polytope $\{4,4\}_{(p,q)}$

Now we move to the chiral toroid $\{4,4\}_{(p,q)}$. As we shall see, we can no longer simply find block systems that are coarser than those for $\{4,4\}$. But since $\Gamma(\{4,4\}_{(p,q)})$ is a quotient of the group $[4,4]^+$, which is generated by rotations we can start by looking for block systems of the faces of $\{4,4\}$ under the group action of $[4,4]^+$.

Given positive integers a, b, we will start by defining a partition, $Z_{a,b}$ by translates of the subset χ where a face $F \in \chi$ if F is of the form $(ma - nb, mb + na)$ for

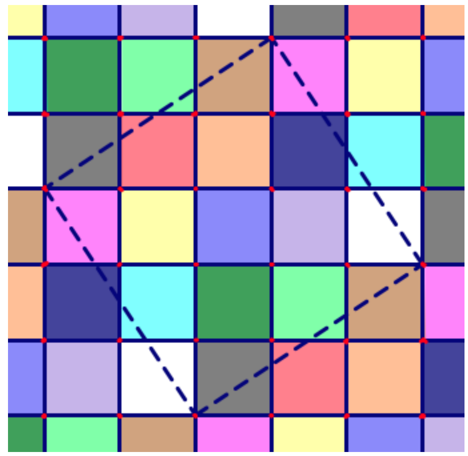

Figure 5.10: The block system $Z_{3,2}$.

some m, n (see Figure 5.10).

Theorem 5.2.6. *A partition of the faces of* $\{4, 4\}$ *is a block system under the group action of* $[4, 4]^+$ *if and only if it is of the form* $Z_{a,b}$ *for some non-negative integers* a, b.

Proof. We will once again use Lemma 5.1.6. The partition $Z_{a,b}$ of the faces of $\{4, 4\}$ partitions the faces into the orbits of the subgroup $\Lambda_{(a,b)}$ which is a normal subgroup of $[4, 4]^+ = \langle \sigma_1, \sigma_2 \rangle$, and so it is a block system of the faces of $\{4, 4\}$ under the action of $[4, 4]^+$. These rotation groups were defined in Section 2.2.

Now suppose X is a block system on the faces of $\{4, 4\}$ under the group action of $[4, 4]^+$ that is not of the form $Z_{a,b}$. Since $[4, 4]^+$ contains the same translations as $[4, 4]$, each pair of block in X must be translates of each other. Now, for some x and y, X must be a coarser $Z_{x,y}$. We can see this by first choosing an arbitrary block in X, which can be the block containing the origin, and picking any other face in that same block which can have coordinates (x, y) (since x and y are arbitrary). We can then find other faces in this block by finding the images of (x, y) under rotation around the centre of the origin, then continuing this process for each new face. Each of these new faces must all be in the same block of X but they also make up a block in the block system $Z_{x,y}$. Since all blocks or $Z_{x,y}$ are translates of each other and all blocks of X are translates of each other, each block in $Z_{x,y}$ must be contained in a block of X, and thus X is coarser than $Z_{x,y}$. Note that we can find many such (x, y) in order to find several $Z_{x,y}$'s which are finer than X.

So we pick x and y such that X is coarser than $Z_{x,y}$, and $Z_{x,y}$ is the coarsest such block system of that form. In other words, the minimum Euclidean distance $d = \sqrt{x^2 + y^2}$ (where the distance is the same as defined earlier) between faces in a block of $Z_{x,y}$ is minimal while still being finer than X (at some point, if d gets too small, $Z_{x,y}$ will become coarser than X).

Pick a block χ of X. Then there exists some block θ of $Z_{x,y}$ such that $\theta \subset \chi$. But since X is not of the form $Z_{x,y}$, there is a face $F \in \chi$ where $F \notin \theta$. There is also $G \in \theta \subset \chi$ where the Euclidean distance between G and F is smaller than d (since all faces in θ are at least d distance apart).

Then take the vector (r, s) from F to G whose magnitude is smaller than d. We can use (r, s) to form a block of $Z_{r,s}$ which is itself a subset of χ. By the same argument showing X was coarser $Z_{x,y}$, we can see that X is coarser than $Z_{r,s}$. However, $\sqrt{r^2+s^2} < d$ which is a contradiction. So there cannot be block systems of the faces of $\{4,4\}$ under the group action of $[4,4]^+$ that are not of the form $Z_{a,b}$.

■

Next we use these block systems to find block systems of $\{4,4\}_{(p,q)}$. Just like in the previous section, any block system of $\{4,4\}_{(p,q)}$ must induce a block system of $\{4,4\}$ under the group action of $\Gamma^+(\{4,4\})$ when extended to a partition of the faces of $\{4,4\}$. So any block system of $\{4,4\}_{(p,q)}$ must be induced by a block system coarser than $Z_{p,q}$ under the group action of $[4,4]^+$.

Lemma 5.2.7. *If $x(a^2+b^2) = p^2+q^2$ then x is a sum of two squares (though perhaps trivially as z^2+0).*

Proof. A generalization of Fermat's theorem on the sum of two squares states that a number is the sum of two squares if and only if its prime decomposition contains no primes congruent to 3 mod 4 raised to an odd power[6].

So consider the prime decomposition of $p^2+q^2 = p_0^{\alpha_0}\dots p_k^{\alpha_k}$. Then for each p_i which is congruent to 3 mod 4, α^i is even. Then, since a^2+b^2 is a divisor of p^2+q^2, its prime factorization is a subset of the prime factorization of p^2+q^2 and for all *its* primes congruent to 3 mod 4, their exponents are also even. So now we look at

[6]This generalization is often called *The sum of two squares theorem* and is a commonly used result.

the prime factorization of x. It must also be a subset of the prime factorization of p^2+q^2, and looking specifically at its primes which are congruent to 3 mod 4, they must have an even power, since it will be the difference of two evens. And so by Fermat's theorem, it must also be a sum of two squares. ■

Theorem 5.2.8. *Given non-negative integers a, b, p, q, if there exist integers m, n such that $p = ma - nb$ and $q = mb + na$ then the partition, $Z_{a,b}$, of the faces of $\{4,4\}$ induces a block system of the faces of $\{4,4\}_{(p,q)}$ under the group action of $[4,4]^+$.*

Proof. We want to show that $Z_{a,b}$ is a coarser block system of $\{4,4\}$ (under the group action of $[4,4]^+$) than $Z_{p,q}$ when $a^2+b^2 \mid p^2+q^2$. To show this it is sufficient to show for blocks χ of $Z_{a,b}$ and θ of $Z_{p,q}$ such that where $(0,0) \in \chi, \theta$, $\theta \subset \chi$. To do this, we only need to show that $(0,0), (p,q) \in \chi$ since all faces of θ can be obtained from $(0,0)$ and (p,q) by using automorphisms which fix one and move the other to locate more faces in that block and iterating. We already know $(0,0) \in \chi$ by construction.

We know that any face is an element of χ if it is of the form $(na - mb, ma + nb)$ for some n, m. So to show that the face (p,q) is in χ we only need that $p = xa - yb$ and $q = ya + xb$ and so (p,q) as a face in the block χ.

■

5.3 Block systems of the cells of $\{4,3,4\}$, $\{4,3,4\}_{(p,0,0)}$, $\{4,3,4\}_{(p,p,0)}$ and $\{4,3,4\}_{(p,p,p)}$

As in Section 5.2, we will begin by finding block systems of the facets of $\{4,3,4\}$ then find coarser block systems. We want to pay close attention to the techniques in that section since they will often apply in rank 4.

The first step is to show that the partitions of the faces of $\{4,3,4\}$ which are block systems are precisely those which get identified under a lattice which generates a regular toroidal polytope. To describe translations and lattices we take a base flag and identify the vertex in that flag with the origin and then the cell of that flag is the associated cell with the origin. The positive x-axis is in the direction of the reflection of the origin by ρ_0, the positive y-axis is the reflection of the x-axis by ρ_1 and the positive z-axis is the reflection of the y-axis by ρ_2.

So we will define three partitions. For a positive integer a, X'_a partitions the facets of $\{4,3,4\}$ into translations of the set $\{(ma,na,oa) : m,n,o \in \mathbb{Z}\}$, while Y'_a partitions the facets of $\{4,3,4\}$ into translations of set $\{(ma-na, ma+na-oa, oa : m,n,o) \in \mathbb{Z}\}$ and Z'_a partitions the facets of $\{4,3,4\}$ into translations of the set $\{(ma+2na, ma+2oa, ma) : m,n,o \in \mathbb{Z}\}$.

Theorem 5.3.1. *The partitions X'_a, Y'_a and Z'_a for all positive integers a of the facets of $\{4,3,4\}$ are the only block systems on the facets of $\{4,3,4\}$ under the* *group action of* $[4,3,4]$.

Proof. Since X'_a, Y'_a and Z'_a are all based on orbits of normal subgroups of translations (the ones which determine regular toroids) we know they are block systems. So all we need to show is that they are the only block systems for $\{4,3,4\}$. An argument similar to the one used in the proof of Theorem 5.2.3 would suffice, however, we will use another argument that is more concise and illustrates the connection between regular toroids more closely.

Recall Section 3.3 and the association of $\{4,3,4\}$ with cartesian coordinates. Suppose Ω is a block system on the facets of $\{4,3,4\}$, then similarly to the earlier section, all blocks are translates of each other. So we pick the block containing the facet labelled $(0,0,0)$, then this block will form an integer lattice with translations from $(0,0,0)$ to the other facets in the block. Let (x,y,z) be another cell in the same block with the associated translation vector. Then ρ_0, ρ_1 and ρ_2 all fix the cell $(0,0,0)$ and so the image of (x,y,z) under each is $(-x,y,z),(y,x,z)$ and (x,z,y), respectively. Thus, for a vector in the integer lattice generated by the block, the lattice also contains vectors with all permutations of coordinates and all changes of sign.

Thus, by Lemma 3.3.1, the lattice is generated by translations to $(a,0,0)$, $(a,a,0)$ or (a,a,a), which are the block systems X'_a, Y'_a and Z'_a. ∎

So then to find block systems on the facets of $\{4,3,4\}_{(p,0,0)}$, $\{4,3,4\}_{(p,p,0)}$ or $\{4,3,4\}_{(p,p,p)}$ we will look for block systems which are coarser than X'_p, Y'_p and Z'_p. The following theorems are similar to those in section 5.2.1. We recall Remark 9 and note that we have a similar result where X'_a is a refinement of Y'_a or Z'_a.

Theorem 5.3.2. *Let a and p be positive integers such that $a|p$, then the partition X'_a induces a block system on the facets of $\{4,3,4\}_{(p,0,0)}$. If $2a|p$, then the partitions Y'_a and Z'_a induce block systems on the facets of $\{4,3,4\}_{(p,0,0)}$.*

Proof. Since $\{4,3,4\}_{(p,0,0)}$ determines the block system X'_p on $\{4,3,4\}$, it suffices to show that any block in X'_p is a subset of a block in X'_a when $a|p$ and Y'_a and Z'_a when $2a|p$. Since all blocks in these block systems are translates of the block containing the origin, we need only use that block.

Suppose $a|p$, and (xp, yp, zp) is any face in a block from X'_p, since $a|p$, $p = ka$ for some k. So then any face in that block is of the form (xka, yka, zka), which is in the block from X'_a containing the origin.

Now, suppose $2a|p$, that is, that $p = 2ak$ for some k. Then $(xp, yp, zp) = (2xa, 2ya, 2za) = ((x+y+z)a-(y+z-x)a, (x+y+z)a+(y+z-x)a-(2z)a, 2za)$ which is in the block containing the origin from Y'_a.

Finally, we suppose $2a|p$ and $p = 2ak$. Then $(xp, yz, zp) = (2xa, 2ya, 2za) = (2za + 2(x-z)a, 2za + 2(y-z)a, 2za)$ which is in the block containing the origin from Z'_a. ∎

The proofs for the following two theorems are much the same as the above proof.

Theorem 5.3.3. *Let a and p be positive integers such that $a|p$, then the partitions X'_a and Y'_a induce block systems on the facets of $\{4,3,4\}_{(p,p,0)}$. If $2a|p$, then the partition Z'_a induces a block system on the facets of $\{4,3,4\}_{(p,p,0)}$*

Theorem 5.3.4. *Let a and p be positive integers such that $a|p$, then the partitions X'_a and Z'_a induce block systems on the facets of $\{4,3,4\}_{(p,p,p)}$. If $2a|p$, then the partition Y'_a induces a block system on the facets of $\{4,3,4\}_{(p,p,p)}$.*

Since there are no chiral polytopes $\{4,3,4\}_{(a,b,c)}$, we can not find block systems for them.

5.4 Block Systems of $\{3,6\}$, $\{6,3\}$ and resulting toroids

The regular tiling $\{3,6\}$ has the interesting property that the translation subgroup does not act transitively on the faces. As a result, finding block systems of the faces is considerably more complex so it will be given its own section, along with its dual $\{6,3\}$.

5.4.1 Block Systems of the faces of $\{3,6\}$

The previous section might imply that the the block systems of the faces of a regular tessellation are precisely those which form regular toroids, but as we shall see in this section, there are sometimes other block systems.

The infinite tiling $\{3,6\}$ of the Euclidean plane with triangles has Schläfli type $\{3,6\}$. This tiling provides a different scenario since the translation subgroup of $[3,6]$ does not act transitively on the set of faces, unlike $\{4,4\}$. Visually this can be seen as some of the triangular faces pointing up, while others point down.

To label our faces we will use the usual cartesian coordinates, where a chosen base flag determines the origin by its vertex while the x and y axes which lie along the edges of the tessellation. This is expanded on in Section 3.2. However, we will also need to identify two separate faces per coordinate, we will label faces $(x,y)^+$ and $(x,y)^-$ (see Figure 5.11).

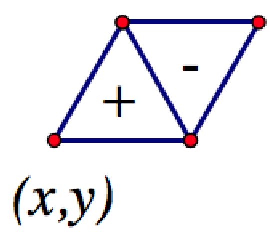

Figure 5.11: The two different triangles in (x,y).

Before beginning finding block systems we need to mention an important geometric property of $\{3,6\}$.

Lemma 5.4.1. *For an equilateral triangle, T, with its edges along those of $\{3,6\}$ and vertices coinciding with vertices of $\{3,6\}$, denote by $l(T)$ the length of the edges*

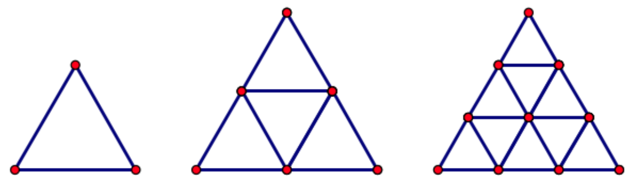

Figure 5.12: $l = 1, 2, 3$.

of T. Then T has a centre[7] *at:*

(a) a vertex of $\{3, 6\}$ if $l(T) \equiv 0 \bmod 3$,

(b) a face of $\{3, 6\}$ pointing in the same direction as T if $l(T) \equiv 1 \bmod 3$,

(c) a face of $\{3, 6\}$ pointing in the opposite direction as T if $l(T) \equiv 2 \bmod 3$.

Proof. We can assume, without loss of generality, that T is pointed upwards. We proceed by induction on the side length, l, of T. The lemma is clear from Figure 5.12 for $l = 1, 2, 3$.

Now we assume the theorem is true for $l = k$. If $k \equiv 0 \bmod 3$ then by the induction hypothesis its centre is a vertex. A triangle with $l = k+1$ has a base with one more horizontal edge, one more edge on the left side (in the direction of the 'y-axis') and with the new right side being formed by connecting the new endpoints.

[7] An interesting problem is how to define the centre of triangle, with different definitions yielding different centres. However, with an equilateral triangle, they coincide. To be more precise, we could specify the centroid, though it is not necessary.

So its centre becomes a face point in the same direction whose left vertex is the centre of the triangle with $l = k$.

If $k \equiv 1 \bmod 3$ then its centre is a face pointing in the same direction. As before, by increasing the side length of T by 1 means its centre increases its base by one and its left edge by one, becoming a triangle with 4 faces. This triangle has a centre as a downward pointing face.

If $k \equiv 2 \bmod 3$ then its centre is a face pointing in the opposite direction. So the centre of the larger triangle means adding edges in the y direction and horizontal edges on the right side making a hexagon, whose centre is a vertex. Figure 5.13 demonstrates how the induction step changes the centre.

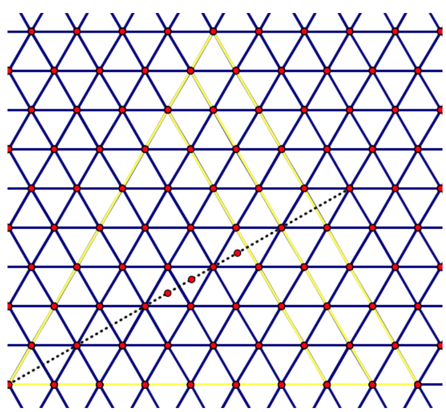

Figure 5.13: As the side length of the triangle increases, the centre of rotation processes along the dotted line.

■

With this in mind we will consider two types of block systems of the faces of $\{3,6\}$. The first of these are based on the orbits of normal subgroups. In these block systems all faces within any block point in the same direction. The other type of block systems are not necessarily based on the orbits of normal subgroups. These block systems may have blocks containing both upward and downward pointing faces.

We define a partition Θ_a as partitioning the faces of $\{3,6\}$ into all translations of the subset $\theta^+ := \{(na,ma)^+ : m,n \in \mathbb{Z}\}$ as well as the reflections of these subsets (which contain downward pointing faces). This means that the other subsets in the partition Θ_a are the translations of the subset $\theta^- := \{(na,ma)^- : m,n \in \mathbb{Z}\}$. Figure 5.14 colours the faces of a particular subset from Θ_3. Θ_a then partitions the faces of $\{3,6\}$ into the orbits of the normal subgroup $\Lambda_{(a,0)}$ of the automorphism group of $\{3,6\}$ which is the Coxeter Group $[3,6]$ whose generators we name ρ_0, ρ_1 and ρ_2 as the canonical generators of $[3,6]$ as defined in Section 3.2.

Theorem 5.4.2. *Let a be a positive integer, then the partition Θ_a of the faces of $\{3,6\}$ is a block system of the faces of $\{3,6\}$ under the group action of $[3,6]$.*

Proof. This follows from Lemma 5.1.6.

■

Next we define the partition H_a. For a positive integer a, H_a partitions the faces

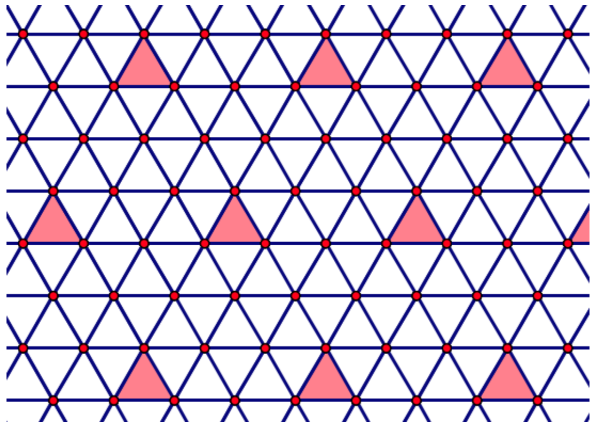

Figure 5.14: One subset of faces in the partition Θ_3.

of $\{3,6\}$ into translations of the subset $\eta^+ := \{(na - ma, na + 2ma)^+ : m, n \in \mathbb{Z}\}$ as well as translations of the subset $\eta^- := \{(na - ma, na + 2ma)^- : m, n \in \mathbb{Z}\}$ (see Figure 5.15 for a subset from H_2). H_a partitions the faces of $\{3,6\}$ into the orbits of the normal subgroup $\Lambda_{(a,a)}$.

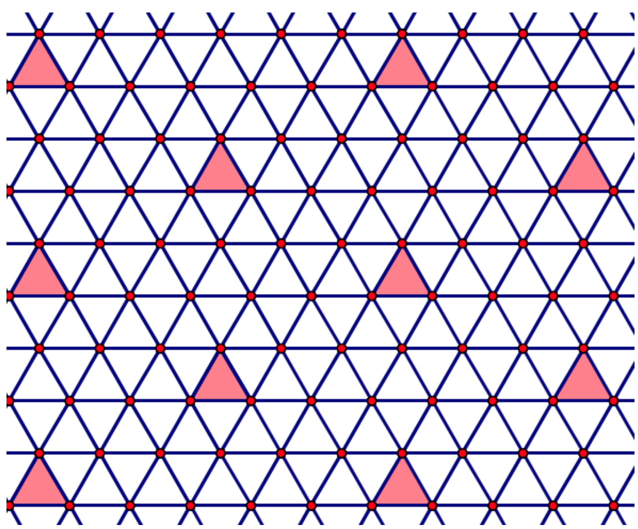

Figure 5.15: One subset of faces in the partition H_2.

Theorem 5.4.3. *Let a be a positive integer, then the partition H_a of the faces of $\{3,6\}$ is a block system of the faces of $\{3,6\}$ under the group action of $[3,6]$.*

Now we define partitions which are not necessarily derived from normal subgroups. For a positive integer a let M_a partition the faces of $\{3,6\}$ into all the translations of the subset $\mu := \{(m(3a+2), n(3a+2))^+ : m, n \in \mathbb{Z}\} \cup \{(-(a+1) + m(3a+2), -(a+1) + n(3a+2))^- : m, n \in \mathbb{Z}\}$ (Figure 5.16 shows a subset from M_0).

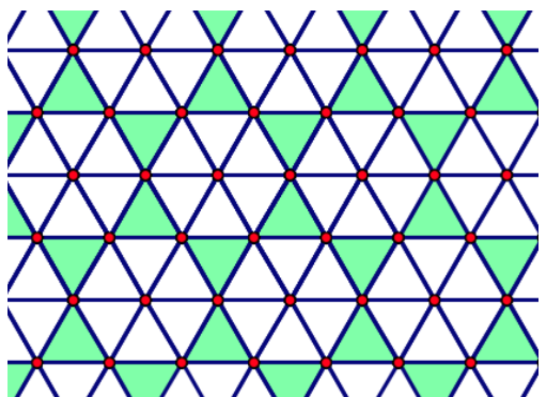

Figure 5.16: One subset of faces in the partition M_0.

To see if this set is the orbit of a normal subgroup we note the transformation that maps an upward pointing face to a downward pointing face. Without loss of generality we can assume that these faces are the $(0,0)^+$ and the face $(-a-1, -a-1)^-$. One transformation mapping $(0,0)^+$ to $(-a-1, -a-1)^-$ is the reflection in the line $y = -x - a$ (which is the bisector of the line connecting the centres of the triangular faces) and another is a rotation of π radians about the intersection of the line $y = -x - a$ and the line $y = x$[8]. If we conjugate either of these transformations by ρ_2 the images of $(0,0)^+$ are the faces $(-2a-1, a)^-$ or $(-4a-1, 2a-1)^-$, respectively. The latter of which is only in the same subset as $(-a-1, -a-1)^-$ (and hence the orbit of a normal subgroup) if $a = 0$ and the former never. Thus this partition is *not* based on the orbits of normal subgroups, unless $a = 0$.

Theorem 5.4.4. *For a positive integer a, the partition M_a of the faces of $\{3,6\}$ is a block system of the faces of $\{3,6\}$ under the group action of $[3,6]$.*

Proof. We first note that M_a is a coarser partition of the faces of $\{3,6\}$ than Θ_{3a+2} where for each i there exists a j and k such that $\mu_i = \theta_j^- \cup \theta_k^+$, each of which are blocks.

So we will prove this directly from the definition. Since all subsets in M_a are translates of each other, it suffices to show for the subset F containing the origin, $Fg = F$ or $Fg \cap F = \emptyset$ for each g that generates $[3,6]$.

Now, since ρ_0 and ρ_1 both map upward pointing faces to upward pointing faces and downward pointing faces to downward pointing faces then, by our initial observation, since ρ_0 and ρ_1 both fix the face at the origin and since Θ_{3a+2} is a block system we have that $F\rho_i = F$ for $i = 0, 1$.

For ρ_2, then the subset $F\rho_2$ consists of faces $f = (m(3a+2), n(3a+2))^+\rho_2$ and

[8] There are, in fact, 4 other such transformations. However, they consist of the two mentioned transformations followed by rotations about the centre of the face, fixing it. So it suffices to treat only the two mentioned

$g = (-(a+1) + m(3a+2), -(a+1) + n(3a+2))^{-}\rho_2$. That is,

$$f = ((m+n)(3a+2), -n(3a+2) - 1)^{-} \text{ and}$$

$$g = ((m+n)(3a+2) - 2(a+1), -n(3a+2) + a)^{+}.$$

The second coordinate shows that neither f nor g are in F, and so $F\rho_2 \cap F = \emptyset$.

■

Now we will define one more partition containing faces pointing in both directions which is very similar to M_a. We define the partition M'_a to partition the faces of $\{3,6\}$ into translates of the subset $\mu := \{(m(3a+1), n(3a+1))^{+} : m, n \in \mathbb{Z}\} \cup \{(a + n(3a+1), a + n(3a+1))^{-} : m, n \in \mathbb{Z}\}$.

Theorem 5.4.5. *For a positive integer a, the partition M'_a of the faces of $\{3,6\}$ is a block system of the faces of $\{3,6\}$ under the group action of $[3,6]$.*

Proof. This proof follows the same argument as the proof for theorem 5.4.4. ■

Theorem 5.4.6. *For a positive integer a, the block systems Θ_a, H_a, M_a and M'_a of the faces of $\{3,6\}$ are the only block systems of the faces of $\{3,6\}$ under the group action of $[3,6]$.*

Proof. The argument that Θ_a and H_a are the only block systems of the faces of $\{3,6\}$ with subsets of faces pointing in the same direction is the same argument as the proof of Theorem 5.2.3. That is the argument that assumes that there must then be some face in a block that is not of the usual form, and that its distance from the origin must be minimal, and then using reflections to find a contradiction. So we will show that M_a and M'_a are the only block system of the faces of $\{3,6\}$ with subsets containing faces pointing in both direction.

We will do this by constructing a block χ of some block system X with faces pointing in both directions, and show that X must be either M_a, M'_a or is the trivial block system.

Without loss of generality assume $(0,0)^+$ is in χ and let Φ be the associated base flag. Then we pick an upward and downward pointing face in χ whose straight line distances (measured between their centres) between them is minimal. The upward pointing face can be assumed to be the origin and then the downward pointing face is $(a,b)^-$.

Then by reflecting by ρ_0 we find that $(-a-b-1,b)^-$ is also in χ. These two faces show X must be coarser than Θ_x where $x = |a-(-a-b-1)|$, and so then the faces $(a,b-x)^-, (a,b+x)^- \in \chi$ which lies above or below $(0,0)^+$ (see Figure 5.17).

The convex hull of the faces $(a,b)^-$, $(-a-b-1,b)^-$ and $(a,b-x)^-$ (or $(a,b+x)^-$ for $b<0$) form an equilateral triangle (more or less) with side lengths along the edges of $\{3,6\}$ and side length $x+1$. We also know that the centre of this triangle

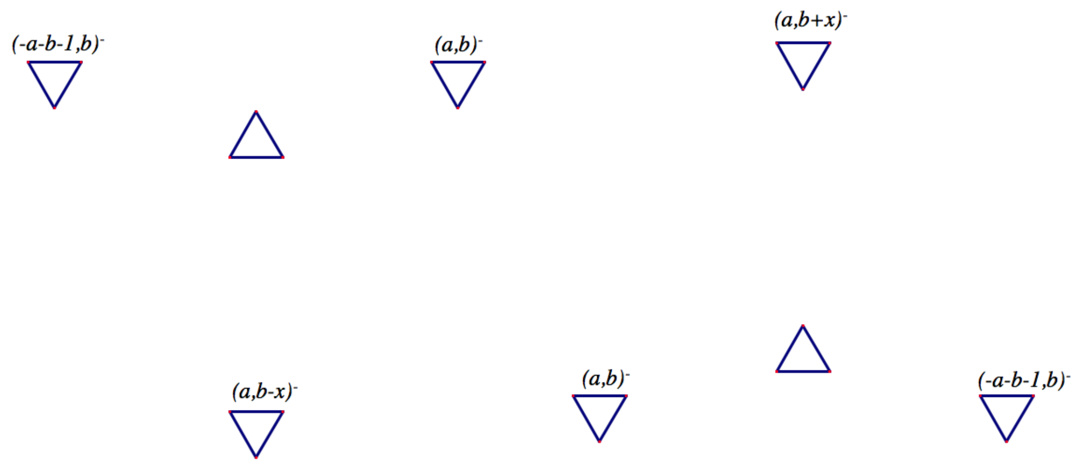

Figure 5.17: The images of (a, b) if $b > 0$ or $b < 0$.

is below $(0,0)^+$ if $b > 0$ since otherwise $(a, b - x)^-$ is closer to $(0,0)^+$ than (a, b). Similarly, if $b < 0$ the centre of rotation is above $(0,0)^+$.

Furthermore, if we rotate by $2\pi/3$ radians around the centre of that triangle it will fix χ, so that means the images of $(0,0)^+$ under those rotations are also in χ. But then, the images of $(0,0)^+$ under those rotations are also upward facing triangles inside the triangle formed by $(a,b)^-, (-a-b-1,b)^-$ and $(a, b-x)^-$ (see Figure 5.18). The convex hull of the face $(0,0)^+$ and its images under these rotations also form an equilateral triangle (more or less), we define the side length of this triangle as $y + 1$. The case with $b < 0$ and $(a, b + x)^-$ is similar.

Then X is also coarser than Θ_y where $y \leq x/2$. Furthermore, the distance between the origin and $(a, b)^-$ is less than or equal to y as well. Since X is coarser than Θ_y, $(a, b)^-$ must then lie inside a triangle formed by orbit of the origin in Θ_y,

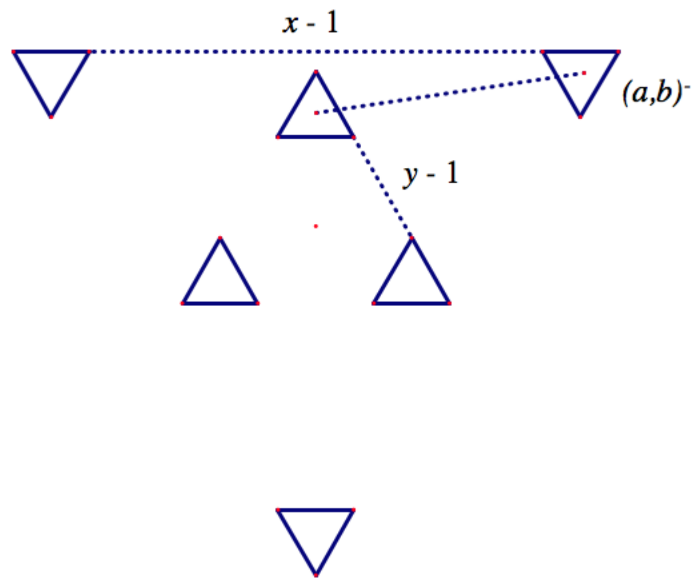

Figure 5.18: The images of $(0,0)^+$ after rotating. Note that the doted images are -1 as x is that distance plus 1. Also note the maximum value for y is $x/2$ and the minimum distance between the origin and (a,b) is roughly $x/2$. Since otherwise that puts $(0,0)^+$ outside the triangle.

which means that the distance between $(a,b)^-$ and any of the (upward pointing) corners of this new triangle is less than the distance between the origin and $(a,b)^-$.

This is a contradiction unless $y=0$ and the centre of the triangle formed by $(a,b)^-, (-a-b-1,b)^-$ and $(a,b-x)^-$ (or $(a,b+x)^-$) is the centre of the triangular face $(0,0)^+$ and thus $a=b$. So the downward pointing faces are $(a,a)^-, (-2a-1,a)^-$ and $(a,-2a-1)^-$.

Lemma 5.4.1 part (c) tells us $x-1 \equiv 2 \bmod 3$ or $a \equiv 2 \bmod 3$ when $a>0$ and $a \equiv 1 \bmod 3$ when $a<0$. And thus X is either M_a or M'_a. ■

Remark 10. Note that all block systems of {3,6} are coarser than Θ_a for some a.

5.4.2 Block Systems of the faces of $\{3,6\}_{(p,0)}$ and $\{3,6\}_{(p,p)}$

Just as in Section 5.2.1, finding block systems of $\{3,6\}_{(p,0)}$ and $\{3,6\}_{(p,p)}$ amounts to finding block systems that are coarser than Θ_p and H_p. We omit the proofs in this section as the method is identical to that of Section 5.2.1, which involves finding when one block system is a refinement of another.

Theorem 5.4.7. *For positive integers a and p, the following induce block systems of the faces of $\{3,6\}_{(p,0)}$ under the group action of $[3,6]$:*

(i) the partition Θ_a if $a \mid p$,

(ii) the partition H_a if $3a \mid p$,

(iii) the partition M_a *if* $3a + 2 \mid p$ *and*

(iv) the partition M'_a *if* $3a + 1 \mid p$.

Theorem 5.4.8. *For positive integers* a *and* p*, the following induce block systems of the faces of* $\{3, 6\}_{(p,p)}$ *under the group action of* $[3, 6]$*:*

(i) the partition Θ_a *if* $a \mid p$*,*

(ii) the partition H_{3a} *if* $3a \mid p$*,*

(iii) the partition M_{3a} *if* $3a + 2 \mid p$ *and*

(iv) the partition M'_{3a}*, if* $3a + 1 \mid p$.

By the same arguments in Section 5.4.1 these block systems are the only block systems of the faces of $\{3, 6\}_{(p,0)}$ and $\{3, 6\}_{(p,p)}$ under the group action of $[3, 6]$.

5.4.3 Block Systems of the faces of the chiral polytope $\{3, 6\}_{(p,q)}$

We proceed in the same manner that we did with $\{4, 4\}_{(p,q)}$, by starting with block systems of $\{3, 6\}$ under the group action of $[3, 6]^+$ and then finding coarser block systems. And like the previous two sections we will look at partitions in two categories. One where all faces in a given subset point in the same direction and the other where a subset can contain faces pointing in both directions.

We will start with partitions where all faces in each part face in the same direction. If we look at the two faces $(x, y)^+$ and $(x, y)^-$ as one face together, many of the same arguments from Section 5.2.2 carry over. Then it will suffice to split up the two faces and create separate parts for the downward pointing faces.

As before, we define the partition $\Sigma_{a,b}$ for positive integers a, b as being all the translations of the subset $\chi^+ := \{(ma - nb, na + (m + n)b)^+ : m, n \in \mathbb{Z}\}$ and all translations of the subset $\chi^- := \{(ma - nb, na + (m + n)b)^- : m, n \in \mathbb{Z}\}$ (see Figure **??** which shows a particular subset from $\Sigma_{2,1}$).

Theorem 5.4.9. *For positive integers a, b, the partition $\Sigma_{a,b}$ of the faces of $\{3, 6\}$ is a block system under the group action of $[3, 6]^+$.*

Proof. Once again, we note that the translation subgroup generated by the vectors (a, b) and $(-b, a + b)$ is a normal subgroup of $[3, 6]^+$, so by Lemma 5.1.6 $\Sigma_{a,b}$ is a block system.

■

Theorem 5.4.10. *For non-negative integers a, b, $\Sigma_{a,b}$ is the only block system of the faces of $\{3, 6\}$ under the group action of $[3, 6]^+$ in which all faces in any given block point in the same direction.*

Remark 11. We note an important property that two faces of $\{3, 6\}$ pointing in the same direction will determine a block system $\Sigma_{x,y}$ for some x and y.

Proof. The proof of this theorem is analogous to that of the second part of Theorem 5.2.6. The strategy is once again to suppose there is some other block system, which must have a block containing a face that does not fit the Σ block system paradigm and we can find a face within that paradigm which is closest to the face outside it. We can then use those two faces to form a new Σ which violate a minimum distance. ∎

To discuss block systems of $\{3,6\}$ under the group action of $[3,6]^+$ in which blocks have faces pointing in both directions we will take a slightly different approach. Instead we will construct a partition coarser than $\Sigma_{a,b}$ that is a block system and suits our requirements.

Before we can construct this partition, we need a few lemmas, the following lemma and Lemma 5.4.1.

Lemma 5.4.11. *Given faces $(0,0)^+,(a,b)^+,(-b,a+b)^+$ and $(a-b,a+2b)^+$, the centre of rotations fixing setwise $(0,0)^+,(a,b)^+,(-b,a+b)^+$ and $(a,b)^+,(-b,a)^+,(a-b,a+2b)^+$ are either both centres of faces of $\{3,6\}$ pointing upward or with one being the centre of a downward pointing face and a vertex.*

Proof. Name the three faces $(0,0)^+,(a,b)^+,(-b,a+b)^+$ as X and $(a,b)^+,(-b,a+b)^+,(a-b,a+2b)^+$ as Y, further name $A=(0,0)^+, B=(a,b)^+, C=(-b,a+b)^+$ and $D=(a-b,a+2b)^+$. By similarity through rotations we can assume that

$a \geq b \geq 0$.

To find the centre of X we form a triangle with the same centre as X by extending the left side of A, the base of B and the right side of C. When a,b are chosen as we have above, this triangle always points up. It also has side length $a - b + 1$

To find the centre of Y we form a triangle with the same centre by extending the base of C, right side of B and left side of D. This is always a downward pointing triangle (unless it is just a single vertex) with side length $a - b - 1$.

Then, by Lemma 5.4.1 X has its centre as either a vertex, upward pointing face or downward pointing face. depending on if $a - b + 1 \equiv 0, 1, 2 \mod 3$.

If the centre of X is a vertex, then the centre of Y is a face pointing in the same direction of the triangle formed from edges of Y, namely pointing downward. If the centre of X is an upward point face, then the centre of Y is also an upward pointing face. Finally if the centre of X is a downward pointing face then the centre of Y is a vertex. ■

Now we seek to construct a partition of the faces of $\{3, 6\}$ that is a block system under the group action of $[3, 6]^+$ with subsets containing both upward and downward pointing faces.

First we recall Remark 11. Furthermore, any pair of faces in one subset pointing in opposite directions can be used to find more faces pointing in either direction by rotating through the centre of each. Thus any partition of the faces of $\{3,6\}$ under the group action of $[3,6]^+$ with faces in subsets pointing in both directions must be coarser than some block system $\Sigma_{a,b}$. So to find such a partition we will use the above lemmas.

The block system $\Sigma_{a,b}$ will place the 4 faces satisfying the conditions of lemma 5.4.11 into the same subset (we can assume they all belong to χ_0^+). If $a-b \equiv 1,2 \bmod 3$ we can find that the centres of rotations of the 4 faces are the centre of a downward pointing face, $(x_1,x_2)^-$, and a vertex. Now we note that rotation of $\pi/3$ around the centre of that downward pointing face fixes the subset χ_0^+. It also fixes the block of $\Sigma_{a,b}$ containing $(x_1,x_2)^-$. Then we can place all faces from $\chi_x^- = \{(x_1+ma-nb, x_2+na+(m+n)b)^- : m,n \in \mathbb{Z}\}$ into the same as block χ_0^+.

It is important to note here that all blocks in $\Sigma_{a,b}$ are translates of either χ_0^+ or χ_j^-. This produces a partition $\Pi_{a,b}$ where $(x_1,x_2) = (\frac{a-b-1}{3}, \frac{a+2b-1}{3})^-$ when $a-b \equiv 1 \bmod 3$ and $(x_1,x_2) = (\frac{-a+b-1}{3}, \frac{-a-2b-1}{3})^-$ when $a-b \equiv 2 \bmod 3$.

Theorem 5.4.12. *When a,b are positive integers such that $a-b \equiv 1,2 \bmod 3$, the partition $\Pi_{a,b}$ of the faces of $\{3,6\}$ is a block system under the group action of $[3,6]^+$ and all block systems with upward and downward pointing faces in the same block are of this form.*

Proof. Since all subsets in $\Pi_{a,b}$ are translates of each other, we pick one and show that it satisfies the definition of a block. We also note that the upward and downward pointing components are each blocks in $\Sigma_{a,b}$.

It suffices to show that the subset π of $\Pi_{a,b}$ containing $(0,0)^+$ is a block. This block can be described as $s^+ \cup s^-$ where s^+ is the block containing the origin in $\Sigma_{a,b}$ and s^- is the block of $\Sigma_{a,b}$ containing $(x_1, x_2)^-$.

$[3,6]^+$ is generated by rotations $\sigma_1 = \rho_0\rho_1$ and $\sigma_2 = \rho_0\rho_2$. We start with $\pi\sigma_1$, and since $s^+\sigma_1 = s^+$ we need to show that $s^-\sigma_1 = s^-$[9]. By definition, (x_1, x_2) is the centre of rotation of the "triangle" formed by the convex hull of $(0,0)^+, (a,b)^+$ and $(-b, a+b)^+$ (if $|a-b| \equiv 1 \bmod 3$) or $(a,b)^+, (-b, a+b)^+$ and $(a-b, a+2b)^+$ (if $|a-b| \equiv 2 \bmod 3$). Furthermore, the centre of every face in s^- is the centre of rotation of 3 faces of s^+ by the nature of the translation by vector (a,b).

Thus, every face in s^+ is also the centre of rotation of a triangle formed by the convex hull of three faces in s^-. In particular, $(0,0)^+$ is the centre of rotation of $(x_1, x_2)^-, (-x_1 - x_2, x_1)^-$ and $(x_2, -x_1 - x_2)^-$ if $a - b \equiv 1 \bmod 3$ or the centre of rotation of $(x_1 - a, x_2 - b)^-, (-x_1 + a - x_2 + b, x_1 - a)^-$ and $(x_2 - b, -x_1 + a - x_2 + b)^-$ if $a - b \equiv 2 \bmod 3$. Since these triangles are also equilateral, and $(0,0)^+$ is their centre, $s^-\sigma_1 = s^-$.

Next we consider $\pi\sigma_2$. Since $s^+\sigma_2 \cap s^+ = \emptyset$ we need to show that $s^-\sigma_2 \cap s^+ = \emptyset$. We need not show $s^-\sigma_2 \cap s^- = \emptyset$ since σ_2 interchanges upward and downward pointing faces. First we note that $(x_1, x_2)^-\sigma_2 = (-x_1, -x_2 - 1)^+$ and $(0,0)^+\sigma_2 =$

[9] it would suffice to show that $s^-\sigma_1 \subseteq s^- \cup s^+$, but we know $s^-\sigma_1 \cap s^+ = \emptyset$ since otherwise s^+ would not be a block

$(0,-1)^-$. To do this we note that for any block $s \in \Sigma_{a,b}$ if $s\mathbf{t} = s$ for some translation $\mathbf{t}$, then $\mathbf{t} = (ma-nb, na+(m+n)b)$ for some integers m, n. For any other translation **t'**, s**t'** $\cap s = \emptyset$.

We can note that $(0,-1)^-$ is in a different element of $\Pi_{a,b}$ than $(x_1, x_2)^-$ and $(0,0)$ (unless $\Pi_{a,b}$ is the trivial block system with a single block) and so the translation $\mathbf{p} = (x_1, x_2+1)$ mapping $(0,-1)^-$ to $(x_1, x_2)^-$ is *not* of the form $(ma-nb, na+(m+n)b)$ and so neither is its inverse $\mathbf{p}^{-1} = (-x_1, -x_2-1)$. However, this is precisely the translation mapping $(0,0)^+$ to $(x_1, x_2)^-\sigma_2$ and so $(x_1, x_2)^-\sigma_2$ cannot be in s^+. Thus $(s^- \cup s^+)\sigma_2 \cap (s^- \cup s^+) = \emptyset$ and so $\Pi_{a,b}$ is a block system.

Finally we show that this is the only such block system. We assume there is a block system Ω with upward and downward pointing faces. Since any block will contain upward and downward pointing faces, all blocks must be translates of each other. So choose a block and choose the pair consisting of upward and downward pointing faces which are closest to each other. By rotating around one we find the images of the other ensuring that Ω is coarser than $\Sigma_{x,y}$ for some x, y. Rotating around the other face shows that Ω is coarser than $\Pi_{x,y}$ for the same x, y as before since each face is at the centre of rotation of the faces pointing in the other direction.

To show that Ω is exactly $\Pi_{x,y}$ we must simply note that if it is not, there must be some other face in the above subset not described by $\Pi_{x,y}$. This face would then necessarily lie within one of the regions formed by opposite pointing faces and the translations generated by (x, y). But then that face would lie nearer to an opposite pointing face than the pair of faces we earlier chose. ■

Now we turn our attention to finding block systems of the faces of $\{3,6\}_{(p,q)}$. As before, this simply amounts for finding block systems that are coarser than $\Sigma_{p,q}$.

Theorem 5.4.13. *Given non-negative integers a, b, p, q, if there exist integers m, n such that $p = ma - nb$ and $q = na + (m+n)b$ then the partition $\Sigma_{a,b}$ induces a block system of the faces of $\{3,6\}_{(p,q)}$ under the action of $[3,6]^+$.*

Proof. Since we can express p and q in the form $p = na - mb$ and $q = ma + (m+n)b$, we pick a block χ_i^+ of $\Sigma_{p,q}$, if $(x_1, x_2)^+ \in \chi_i^+$, then any face, F, in χ_i^+ can be written as $(x_1 + n'p - m'q, x_2 + m'p + (n' + m')q)^+$. Using the above identity, this face can instead be written:

$$\begin{aligned} F &= (x_1 + n'p - m'q, x_2 + m'p + (n' + m')q)^+ \\ &= (x_1 + n'(na - mb) - m'(ma + (m+n)b), \\ &\quad x_2 + m'(na - mb) + (n' + m')(ma + (m+n)b))^+ \\ &= (x_1 + (n'n - m'm)a - (n'm + m'm + m'n)b, \\ &\quad x_2 + (n'm + m'm + m'n)a + ((n'n - m'm) + (n'm + m'm + m'n))b)^+ \end{aligned}$$

So we can see that every face in χ_i^+ is also in some block θ_j^+, of $\Sigma_{a,b}$ that contains

$(x_1, x_2)^+$. The same argument shows that every block χ_i^- of $\Sigma_{p,q}$ is contained in some block θ_j^- of $\Sigma_{a,b}$ and so $\Sigma_{a,b}$ is coarser than $\Sigma_{p,q}$. ■

Since block systems $\Pi_{a,b}$ are built from $\Sigma_{a,b}$ the following corollary characterizes when $\Pi_{a,b}$ can be refined into $\Sigma_{a,b}$ by simply splitting each block up.

Corollary 5.4.14. *For positive integers a, b, if the faces of $\{3,6\}_{(p,q)}$ can be partitioned into a block system by $\Sigma_{a,b}$ and $|a-b| \equiv 1, 2 \bmod 3$ then $\Pi_{a,b}$ induces a block system on the faces of $\{3,6\}_{(p,q)}$.*

5.4.4 Block Systems of the faces of $\{6,3\}$, $\{6,3\}_{(p,0)}$ and $\{6,3\}_{(p,p)}$

The third polytope we look at is the infinite tiling of the Euclidean plane with hexagons whose automorphism group is the Coxeter group $[6,3]$ generated by ρ_0, ρ_1 and ρ_2 (described in Section 3.1). In order to identify faces we draw a triangular lattice like that in $\{3,6\}$ with each vertex at the centre of a hexagonal face, then use the same coordinates as with $\{3,6\}$ (see Figure 5.19).

Unlike $\{3,6\}$ we do not have to worry about two different types of faces. In fact the methods of finding block systems of $\{6,3\}$ are much the same as in $\{3,6\}$ without the third block system which had faces pointing in both directions within a subset. The proofs that these partitions are block systems follow the same arguments as well. Because of the similarity in arguments, theorems will be offered

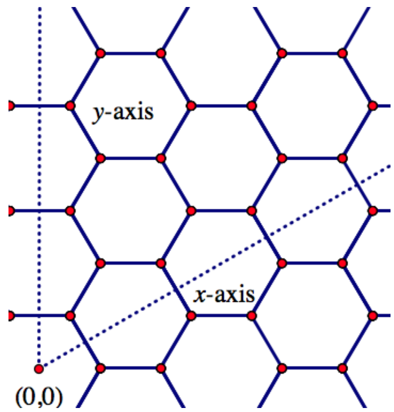

Figure 5.19: The coordinates for $\{6,3\}$.

without proof.

We define the partition Λ_a for a positive integer a as being the translations of $\lambda = \{(mx, nx) : m, n \in \mathbb{Z}\}$ (see figure 5.20), and we define partition K_a for a positive integer a as being the translations of $\kappa = \{(mx-nx, mx+2nx) : m, n \in \mathbb{Z}\}$ (see figure 5.21).

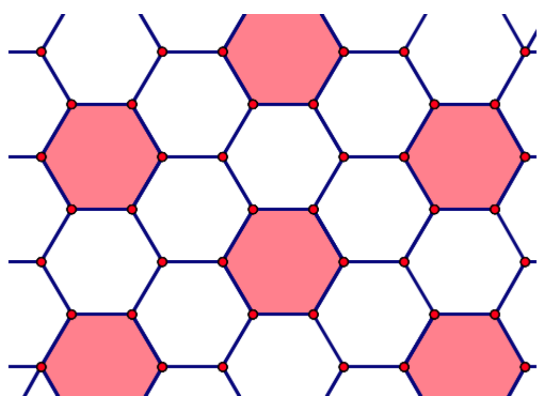

Figure 5.20: One block of faces in the partition Λ_2.

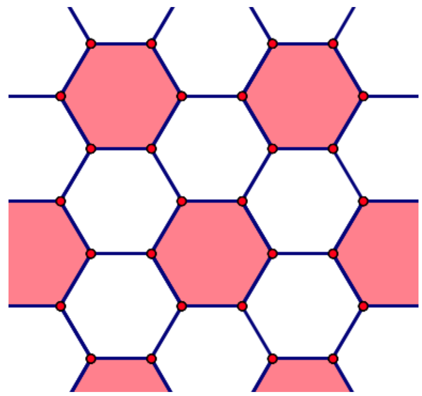

Figure 5.21: One block of faces in the partition K_1.

Theorem 5.4.15. *The partitions Λ_a and K_a of the faces of $\{6,3\}$ are block systems of the faces of $\{6,3\}$ under the action of $[6,3]$ for all positive integers a, and are the only such block systems.*

Theorem 5.4.16. *The partition Λ_a of the faces of $\{6,3\}_{(p,0)}$ is a block system under the action of $[6,3]$ for any positive integer a such that $a \mid p$, and the partition K_a is a block system under the action of $[6,3]$ for all positive integers a such that $3a \mid p$.*

Theorem 5.4.17. *The partition Λ_a of the faces of $\{6,3\}_{(p,p)}$ is a block system under the action of $[6,3]$ for all positive integers a such that $a \mid p$, and the partition K_a is a block system under the action of $[6,3]$ for all positive integers a such that $a \mid p$.*

5.4.5 Block Systems of the faces of the chiral polytope $\{6,3\}_{(p,q)}$.

Just as in the previous section, the results follow very closely with other polytopes and so they will be offered without proof.

For positive integers a, b, we define the partition $I_{a,b}$ of the faces of $\{6, 3\}$ as being all translations of $\iota_0 = \{(ma - nb, na + (m + n)b) : m, n \in \mathbb{Z}\}$.

Theorem 5.4.18. *Let a, b be positive integers, then the partition $I_{a,b}$ of the faces of $\{6, 3\}$ is a block system under the action of $[6, 3]^+$. Any block system of the faces of $\{6, 3\}$ is a block system under the action of $[6, 3]^+$ must be of this form.*

Theorem 5.4.19. *Let a, b, m, n be positive integers such that $p^2 + pq + q^2 = (a^2 + ab + b^2)(m^2 + mn + n^2)$, then the partition $I_{a,b}$ induces a block system of the faces of $\{6, 3\}_{(p,q)}$ under the action of $[6, 3]^+$.*

6 Block Systems of Toroidal Hypertopes

Now we look to find block systems for the toroidal hypertopes described in Chapter 4. The first adjustment we must make is the types of objects we are partitioning. With polytopes we chose facets (polygons and cubes). With the set of types in hypertopes not having a total order, the choice of which set of elements of a single type to partition seems less clear. However, two classes of regular toroidal hypertopes (those constructed from $\widetilde{A}_2$ and $\widetilde{A}_3$) have the property that each set of elements of one type is isomorphic to every other set of elements of another type. The other class (toroidal hypertopes built from $\widetilde{B}_3$) has a clear analog to the cubic tessellation and resulting regular toroids.

As in the previous chapter, most of the machinery is done by finding block systems over the infinite hypertope and its Coxeter complex. Then the block systems for the regular toroidal hypertopes are found by looking for coarser block systems.

6.1 Block Systems on Rank 3 Toroidal Hypertopes

In rank 3, the only regular hypertopes which are not polytopes are those which are constructed from the group $\widetilde{A}_2$.

We start by re-visualizing the regular hypertope $\mathcal{C}(\widetilde{A}_2)$. In Chapter 4.3 we described the group $\widetilde{A}_2$ as being generated by $\widetilde{\rho_0}$, ρ_1 and ρ_2 and visualized it using the hexagonal tiling. Starting with the base chamber we distinguish one element of each type. One interchanged by $\widetilde{\rho_0}$ which is represented by a vertex of the hexagonal tiling, one interchanged by ρ_1 represented by a vertex adjacent to the previous vertex, and one interchanged by ρ_2 represented by a hexagonal face. We will refer to the type of each of these elements as type $\widetilde{0}$, 1 and 2, respectively.

Since both elements of type $\widetilde{0}$ and type 1 are equivalent and both represented by vertices of the hexagonal tiling we note that elements of either type are incident with three elements of opposing type as well as three elements of type 2 and so both exhibit triangular geometry. So we will transform the elements of type 1 into triangles which also transforms the elements of type 2 from hexagons into triangles. The elements of type $\widetilde{0}$ will be the remaining vertices. Figure 6.1 shows elements of type $\widetilde{0}$ as green vertices, elements of type 1 as grey triangles containing red vertices and elements of type 2 as white triangles.

Since each set of elements of a given type are isomorphic to the other sets, we

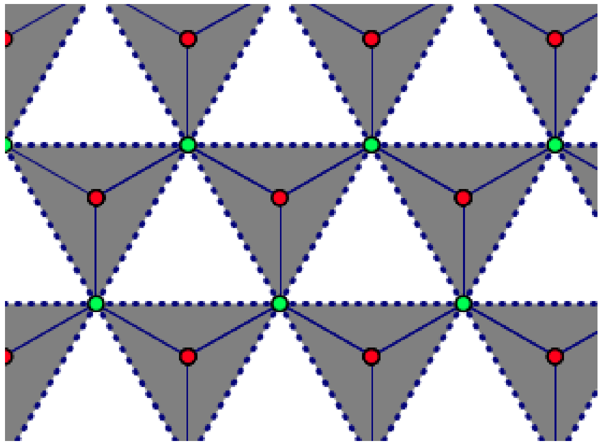

Figure 6.1: A revisualization of $\mathcal{C}(\widetilde{A}_2)$.

can choose to partition the set of elements of type 2. Now we describe the coordinate system for the regular hypertope $\mathcal{C}(\widetilde{A}_2)$. We begin with a base chamber, then we associate the origin with the element of type $\widetilde{0}$, then the image of the origin $(0,0)$ under $\widetilde{\rho_0}$ we label $(1,0)$. The line connecting $(0,0)$ and $(1,0)$ we call the x-axis and its image under ρ_1 is the y-axis.

Now, most importantly, we notice that the action $\widetilde{A}_2$ on the elements of type 2 is the same as the action of the index 2 subgroup of $[3,6]$ preserving the set of upward pointing faces of $\{3,6\}$. Additionally, both $\{3,6\}$ and $\mathcal{C}(\widetilde{A}_2)$ admit the same translations. The result is that the elements of type 2 of $\mathcal{C}(\widetilde{A}_2)$ and the resulting regular toroidal hypertopes can be partitioned into block systems Θ_a and H_a from section 5.4.1, but simply not including those blocks that contain downward pointing faces.

As such, we define a partition Θ'_a as partitioning the elements of type 2 of $\mathcal{C}(\widetilde{A}_2)$ into all translations of the subset $\theta' := \{(na, ma) : m, n \in \mathbb{Z}\}$. We also define the partition H'_a, for a positive integer a, H'_a partitions the elements of type 2 of $\mathcal{C}(\widetilde{A}_2)$ into translations of the subset $\eta' := \{(na - ma, na + 2ma)^+ : m, n \in \mathbb{Z}\}$

Theorem 6.1.1. *Let a be a positive integer, then the partitions Θ'_a and H'_a induce block systems of the elements of type 2 of $\mathcal{C}(\widetilde{A}_2)$ under the group action of $\widetilde{A}_2$.*

Theorem 6.1.2. *For positive integers a and p, if $a \mid p$ then the partition Θ'_a induces a block system of the faces of $\mathcal{C}(\widetilde{A}_2)_{(p,0)}$ and $\mathcal{C}(\widetilde{A}_2)_{(p,p)}$ under the group action of $\widetilde{A}_2$. If $a \mid p$ then then the partition H'_a induces a block system of the faces of $\mathcal{C}(\widetilde{A}_2)_{(p,0)}$ and $\mathcal{C}(\widetilde{A}_2)_{(p,p)}$ under the group action of $\widetilde{A}_2$.*

6.2 Block Systems on the regular toroids $\mathcal{C}(\widetilde{B}_3)_{\mathbf{s}}$

Begin by recalling the presentation of the group $\widetilde{B}_3$, and from that we can label the types from $\mathcal{C}(\widetilde{B}_3)_{\mathbf{s}}$ as types $\widetilde{0}, 1, 2$ and 3. Unlike the previous section (and the next section), the elements of each type are not equivalent. However, the elements of type 3 (seen as points in space) are located precisely at the centres of the facets of $\{4, 3, 4\}$, so we will partition those and name them *hypercells* for simplicity. Additionally we will use the same cartesian labelling convention for the tessellation $\{4, 3, 4\}$. In order to identify each element of type 3 we will use the same method as in Chapter 4. An important note is that the integer lattice, $\mathbb{Z}^3$ has both elements of types $\widetilde{0}$ and of type 1, where (x, y, z) is an element of type $\widetilde{0}$ when $x + y + z$ is even and is of type 1 when $x + y + z$ is odd.

We note that the image of the element of type 3 represented by (x, y, z) under the generator $\widetilde{\rho_o}$ is $(x, y, z)\widetilde{\rho_0} = (1 - y, 1 - x, z)$, while the image of (x, y, z) under the other generators is the same as in section 5.3.

Just as before, it is necessary to begin with finding block systems on the infinite hypertope $\mathcal{C}(\widetilde{B}_3)$. Here we note that because the translation subgroup of $\widetilde{B}_3$ is generated by $(1, 1, 0), (-1, 1, 0)$ and $(0, -1, 1)$ we are unable to translate a element of type 3 to the adjacent element of type 3 (much like upward and downward pointing face of $\{3, 6\}$). To describe these two (disjoint) sets of hypercells, we refer to hypercells as the *original* when they can be obtained by a translation of the hyperface of type labelled $(0, 0, 0)$ and *unoriginal* when not. We note that this partition is itself a block system, though it will be described as a particular instance of a later defined block system.

To find the block systems we will utilize the same strategy as for $\{4, 3, 4\}$, by claiming that the translations which form a regular toroid also identify hypercells in the same block. Though, in this case, the vector from the origin to an unoriginal element of type 3 is *not* a translation, nor will it form a regular toroidal hypertope, but we can still use it to create a block system.

We introduce the following three partitions of the hypercells of $\mathcal{C}(\widetilde{B}_3)$.

- $\mathcal{X}_a$ partitions hypercells into the images of the subset $\{(ma, na, oa) : m, n, o \in \mathbb{Z}\}$ under the action of $\widetilde{B}_3$.
- $\mathcal{Y}_a$ partitions hypercells into the images of the subset $\{(ma - na, ma + na - oa, oa) : m, n, o \in \mathbb{Z}\}$ under the action of $\widetilde{B}_3$.
- $\mathcal{Z}_a$ partitions hypercells into the images of the subset $\{(ma + 2na, ma + 2oa, ma) : m, n, o \in \mathbb{Z}\}$ under the action of $\widetilde{B}_3$.

Theorem 6.2.1. *The partitions $\mathcal{X}_a, \mathcal{Y}_a$ and $\mathcal{Z}_a$ of the hypercells of $\mathcal{C}(\widetilde{B}_3)$ are the only block systems of the hypercells of $\mathcal{C}(\widetilde{B}_3)$ under the group action of $\widetilde{B}_3$.*

Proof. We first show that these partitions are block systems. We start with an element of type 3 which, without loss of generality, can be the element labelled $(0, 0, 0)$, then the other element is either $G = (ma, na, oa), (ma - na, ma + na - oa, oa)$ or $(ma+2na, ma+2oa, ma)$ for some $m, n, o \in \mathbb{Z}$ depending on the partition.

We note that the element of type 3 at the origin is fixed under $\widetilde{\rho}_0, \rho_1$ and ρ_2 so, the image of any other element of type 3 must remain in that same subset. Meanwhile, its image under ρ_3 is $(0, 0, 0)\rho_3 = (0, 0, -1)$.

For $\mathcal{X}_a$, the images of $G = (ma, na, oa)$ are as follows.

$$G\widetilde{\rho}_0 = (-na, -ma, oa)$$

$$G\rho_1 = (na, ma, oa)$$

$$G\rho_2 = (ma, oa, na)$$

$$G\rho_3 = (ma, na, -1 - oa)$$

Thus, $\widetilde{\rho}_0, \rho_1$ and ρ_2 preserve the subset while ρ_3 sends it to another subset. So the images of G under $\widetilde{B}_3$ is always in the same subset as the image of the origin and so $\mathcal{X}_a$ is a block system on the hypercells.

For $\mathcal{Y}_a$, the images of $G = (ma - na, ma + na - oa, oa)$ are as follows.

$$G\widetilde{\rho}_0 = (-ma - na + oa, -ma + na, oa) = (pa - na, pa + na - oa, oa)$$

$$G\rho_1 = (ma + na - oa, ma - na, oa) = (ma - pa, ma + pa - oa, oa)$$

$$G\rho_2 = (ma - na, oa, ma + na - oa) = (ma - na, ma + na - qa, qa)$$

$$G\rho_3 = (ma - na, ma + na - oa, -oa - 1)$$

For $\mathcal{Z}_a$, the images of $G = (ma + 2na, ma + 2oa, ma)$ are as follows.

$$G\widetilde{\rho}_0 = (-ma - 2oa, -ma - 2na, ma) = (ma + 2(-m - o)a, ma + 2(-m - n)a, ma)$$

$$G\rho_1 = (ma + 2oa, ma + 2na, ma)$$

$$G\rho_2 = (ma + 2na, ma, ma + 2oa) = (ra + 2(n - o)a, ra + 2(-o)a, ra)$$

$$G\rho_3 = (ma + 2na, ma + 2oa, -ma - 1)$$

So the images of G under $\widetilde{B}_3$ for G in both $\mathcal{Y}_a$ or $\mathcal{Z}_a$ is always in the same subset as the image of the origin and so $\mathcal{Y}_a$ and $\mathcal{Z}_a$ is a block system on the hypercells.

Now we will show that there are no other block systems. Suppose Ω is a block system of the hypercells of $\mathcal{C}(\widetilde{B}_3)$. Let (p, q, r) be the hypercell in the same block as the hypercell at the origin where the distance between (p, q, r) and the origin is the smallest distance between any two elements in that block. Consider the rotation around the vertical line (parallel to the z-axis) through the centre of (p, q, r). The image of the hypercell at the origin under that rotation is the hypercell labelled $(2p, 2q, 0)$. So $(2p, 2q, 0)$ must also be in this block and the plane $z = 0$ contains more hypercells than just the origin.

Now let $(x, y, 0)$ be the hypercell in the plane $z = 0$ with the shortest Euclidean distance from the hypercell at the origin and without loss of generality, we can assume $x \geq y$. We can then also show that $(y, x, 0)$ and $(-x, y, 0)$ are also hypercells in that same block. If d_0 is the minimal distance among hypercells with $z = 0$ in this block, then $d_0^2 = x^2 + y^2$.

Then, let d_1 be the distance between $(x, y, 0)$ and $(y, x, 0)$ and d_2 be the distance between $(x, y, 0)$ and $(-x, y, 0)$. Then,

$$d_1^2 = 2x^2 + 2y^2 - 4xy$$

$$d_2^2 = 4x^2.$$

By the same argument in the proof of Theorem 5.2.3, either $x = 0$ or $x = y$. Since ρ_1 and ρ_2 give us all permutations of coordinates for hypercells, we can then use the hypercells labelled $(0, y, 0)$ or $(y, y, 0)$ along with the hypercell at the origin with $\widetilde{B}_3$ to locate other elements in this block. In particular we can show that this block contains the blocks of $\mathcal{X}_y$ or $\mathcal{Y}_y$ which contain the element at the origin. Thus, Ω is coarser than one of $\mathcal{X}_y$ or $\mathcal{Y}_y$.

We then return to the hypercell (p, q, r) of type 3, which, without loss of generality, can lie within the fundamental region of the lattice determined by $(x, y, 0)$ and the group $\widetilde{B}_3$. There are two possible such lattices. The first is when $x = 0$ and the lattice is generated by $(y, 0, 0), (0, y, 0)$ and $(0, 0, y)$. The second is when $x = y$ and the lattice is generated by $(y, y, 0), (-y, y, 0)$ and $(0, -y, y)$.

If the hypercell (p, q, r) is, itself, a lattice point then since it is the closest hypercell to the origin Ω is $\mathcal{X}_y$ or $\mathcal{Y}_y$ depending on the value for x. Otherwise, we will perform a series of 180 degree rotations fixing (p, q, r). These are rotations of 180 degrees around the lines through the hypercell (p, q, r) and parallel to the x, y or z axis. These rotations must preserve the lattice, otherwise the image of a hypercell at a lattice point will violate the minimum distance.

If $x = 0$, then the only location for the hypercell (p, q, r) that ensures these rotations preserve the lattice is either (p, p, p), in which case Ω is $\mathcal{Z}_p$, or $(p, p, 0)$ (or $(p, 0, p)$ or $(0, p, p)$), in which case Ω is $\mathcal{Y}_p$.

If $x = y$, then (p, q, r) must be $(0, p, 0)$ (or permutations of) in order for the described rotations to preserve the lattice, in which case Ω is $\mathcal{X}_p$.

Thus, Ω must be one of the three partitions and so, $\mathcal{X}_a, \mathcal{Y}_a$ and $\mathcal{Z}_a$ must be the only block systems of the facets of $\mathcal{C}(\widetilde{B}_3)$ under the group action of $\widetilde{B}_3$. ■

Once again, we have done the majority of the work and what remains when finding block systems on the toroids $\mathcal{C}(\widetilde{B}_3)_{\mathbf{s}}$ is to find block systems which are coarser than $\mathcal{X}_a, \mathcal{Y}_a$ and $\mathcal{Z}_a$, depending on which toroid is being partitioned.

The proofs for the following three theorems are identical to the proofs of theorems 5.3.2, 5.3.3 and 5.3.4.

Theorem 6.2.2. *For all positive integers a, the partition $\mathcal{X}_a$ induces a block system on the hypercells of $\mathcal{C}(\widetilde{B}_3)_{(p,0,0)}$ when $a|p$, the partitions $\mathcal{Y}_a$ and $\mathcal{Z}_a$ induce block* *systems on the hypercells of* $\mathcal{C}(\widetilde{B}_3)_{(p,0,0)}$ *when* $2a|p$.

Theorem 6.2.3. *For all positive integers* a*, the partitions* $\mathcal{X}_a$ *and* $\mathcal{Y}_a$ *induce block systems on the hypercells of* $\mathcal{C}(\widetilde{B}_3)_{(p,p,0)}$ *when* $a|p$*, the partition* $\mathcal{Z}_a$ *induces a block system on the hypercells of* $\mathcal{C}(\widetilde{B}_3)_{(p,p,0)}$ *when* $2a|p$.

Theorem 6.2.4. *For all positive integers* a*, the partitions* $\mathcal{X}_a$ *and* $\mathcal{Z}_a$ *induce block systems on the hypercells of* $\mathcal{C}(\widetilde{B}_3)_{(p,p,p)}$ *when* $a|p$*, the the partition* $\mathcal{Y}_b$ *induce a block system on the hypercells of* $\mathcal{C}(\widetilde{B}_3)_{(p,p,p)}$ *when* $2a|p$.

6.3 Block Systems on the regular toroids $\mathcal{C}(\widetilde{A}_3)_{\mathbf{s}}$

In this section deciding which set of elements to partition is irrelevent as each set is equivalent to the other sets. Furthermore, the group fixing any particular element is the group $[3,3]$, the same as the group fixing the elements of type 3 in the previous section. The other important thing to note is that there exists a translation mapping any element of one type to an element of the same type, so we do not have the issue of original and unoriginal hypercells.

So, the set of elements which we will partition will be the set associated with the generator $\widetilde{\rho_3}$, which we will call elements of type $\widetilde{3}$ or more simply, reuse the name *hypercell.* Using the same coordinate system we have used in this chapter and in previous chapters, we know that these hypercell all have coordinates (x, y, z) where

$x + y + z$ is even.

So now we have that block systems on one set of elements of $\mathcal{C}(\widetilde{A}_3)$ are precisely those where an individual block determines a regular toroid. So we will define the following partitions then prove they are block systems.

- $\mathcal{A}_a$ partitions the hypercells into the translates of subset $\{(2ma, 2na, 2oa) : m, n, o \in \mathbb{Z}\}$ under the action of $\widetilde{A}_3$.
- $\mathcal{B}_a$ partitions the hypercells into the translates of subset $\{(ma - na, ma + na - oa, oa) : m, n, o \in \mathbb{Z}\}$ under the action of $\widetilde{A}_3$.
- $\mathcal{C}_a$ partitions the hypercells into the translates of subset $\{(2ma + 4na, 2ma + 4oa, 2ma) : m, n, o \in \mathbb{Z}\}$ under the action of $\widetilde{A}_3$.

Theorem 6.3.1. *The partitions $\mathcal{A}_a, \mathcal{B}_a$ and $\mathcal{C}_a$ of the hypercells of $\mathcal{C}(\widetilde{A}_3)$ are the only non-singleton and not trivial block systems of the hypercells of $\mathcal{C}(\widetilde{A}_3)$ under the group action of $\widetilde{A}_3$ for all positive integers a.*

Proof. First we note that while this hypertope does not present the same difficulties with "original" and "unoriginal" faces as $\mathcal{C}(\widetilde{B}_3)$, the coordinates of the hypercells of $\mathcal{C}(\widetilde{A}_3)$ are the same coordinates of the "original" hypercells of $\mathcal{C}(\widetilde{B}_3)$. Furthermore, while the translation subgroup of $\widetilde{A}_3$ is transitive on the set of hypercells, it is *not* transitive on the integer lattice. Furthermore, the subgroup of $\mathcal{C}(\widetilde{A}_3)$

which fixes the hypercell labelled $(0,0,0)$ is $\langle \widetilde{\rho_0}, \rho_1, \rho_2 \rangle$, which is the subgroup that fixes the hypercell labelled $(0,0,0)$ of $\mathcal{C}(\widetilde{B}_3)$. And finally, the action of this subgroup on either group of hyperfaces is identical. Thus, the proof of theorem 6.2.1 also proves this theorem. $\blacksquare$

And then, similarly, we finish with the following results.

Theorem 6.3.2. *The partition $\mathcal{A}_a$ induces a block system on the hypercells of $\mathcal{C}(\widetilde{A}_3)_{(p,0,0)}$ when $a|p$, the partitions $\mathcal{B}_a$ and $\mathcal{C}_a$ induce block systems on the hypercells of $\mathcal{C}(\widetilde{A}_3)_{(p,0,0)}$ when $2a|p$, for all positive integers a.*

Theorem 6.3.3. *The partitions $\mathcal{A}_a$ and $\mathcal{B}_a$ induce block systems on the hypercells of $\mathcal{C}(\widetilde{A}_3)_{(p,p,0)}$ when $a|p$, the partition $\mathcal{C}_a$ induces a block system on the hypercells of $\mathcal{C}(\widetilde{A}_3)_{(p,p,0)}$ when $2a|p$ for all positive integers a.*

Theorem 6.3.4. *The partitions $\mathcal{A}_a$ and $\mathcal{C}_a$ induce block systems on the hypercells of $\mathcal{C}(\widetilde{A}_3)_{(p,p,p)}$ when $a|p$, the partition $\mathcal{B}_a$ induces a block system on the hypercells of $\mathcal{C}(\widetilde{A}_3)_{(2p,2p,2p)}$ when $a|p$ for all positive integers a.*

7 Conclusions

In this thesis we have characterized all regular toroidal hypertopes of ranks up to 4. The work grew out of the original characterization of hypertopes by Maria-Elisa Fernandes, Dimitri Leemans and Asia Ivic Weiss in [18] as well as the characterizations of regular toroidal polytopes in [26]. Those results from Sections 4.4 to 4.6 of the non-polytopal regular toroidal hypertopes of rank precisely 4 are original results and were published in [16].

This thesis also found block systems of these toroidal hypertopes under the group action of their automorphism group.

The natural extension of the results in this thesis is to characterize the regular toroidal hypertopes of any rank. A more approachable problem is to extend the results in [23] to hypertopes and characterize the non-regular toroidal hypertopes of rank 4.

A deeper relationship between block systems on facets of polytopes also warrants investigation. In particular, given the automorphism group Γ of a regular hypertope, the relationship between subgroups H such that $\Gamma' < H < \Gamma$ where Γ' is the stabilizer of a facet or hyperface and a block system. It is well established that when H is a normal subgroup it determines a block system. We have also seen that a block system need not come from the orbits of a normal subgroup, in those cases we can also seek to more completely explain the relationship between such block systems and non-normal subgroups which contain Γ'. This relationship may enable us to find easier methods of finding such block systems.

Thus, we can take Chapter 5 as an exhaustive proof of the following corollary.

Corollary 7.0.1. *Given a tessellation $\mathcal{T}$ of rank $n \leq 4$ with an automorphism group G with maximal parabolic subgroup G_{n-1} which stabilizes a facet, every block system of the facets of $\mathcal{T}$ under the action of G are derived from the orbits of a subgroup H where $G_{n-1} < H < G$.*

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