THEORETICAL AND COMPUTATIONAL ANALYSIS OF CREDIT AND LIQUIDITY RISKS WITH MULTIPLE DEFAULTS

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Abstract

Since the 2008 global financial crisis, regulators have been paying considerable attention to the credit and liquidity risks. Two such concepts (related to credit and liquidity risks) that have been repeatedly mentioned in the regulatory announcements are the credit value adjustment (CVA) and the Incremental Risk Charge (IRC).

The CVA is an adjustment to the previous trade price when the counterparty risk has been added. The IRC is a new type of risk charge defined in Basel II which covers the major exposures of the counterparty and liquidity risk in the trading book.

The current models on CVA and IRC have specific shortcomings. The CVA is currently calculated on a one-period model with restriction on the number of defaults. The IRC is computed using the time consuming Monte Carlo simulations.

In this dissertation, we have made significant contributions to risk analysis by solving CVA in both two-default and full model without the restriction on the number of defaults as well as providing an analytical method for calculating IRC. Our research can be considered as a major step forward in expanding the current credit and liquidity risks models.

Compared to the current one-default CVA calculations, our two-default and full calculations offer the distinct advantages of more accurate and practical CVA results. On top of that, our PDE method provides the speed and accuracy which

allows us to finish a thorough risk exposure analysis and identify the conditions when the first default CVA overestimates or underestimates the counterparty risk.

As opposed to the current numerical approach of calculating IRC, we offer an analytical method which provides an approximation of VaR on the two-period model and exact value of VaR on the infinite-default model. This is the first analytic solution in the literature on the multi-period capital model and may impact the view of current measure of risk controls in the banks. Thus credit risk control can be greatly improved if this new analytic solution can be applied in financial industry.

Combined together, the work in this dissertation makes significant improvements in credit risk analysis in the multi-period credit and liquidity risks models.

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Abbreviations

ASRF: Asymptotic Single Risk Factor

BCBS: Basel Committee on Banking Supervision

CCIRS: Credit Contingent Interest Rate Swap

CDO: Collateralized Debt Obligation

CLO: Collateralized Loan Obligation

CVA: Credit Valuation Adjustment

GA: Granularity Adjustment

IDR: Incremental Default Risk

IRB: Internal Rating-based Approach

IRC: Incremental Risk Charge

LIBOR: London Interbank Offered Rate

MC: Monte Carlo Simulation Model

OSFI: the Office of the Superintendent of Financial Institution

P&L: Profit and Loss

VaR: Value-at-Risk

I Introduction

The 2008 global financial crisis may be the worst financial crisis in the past 70 years. It resulted in the plummet of stock markets around the world, the collapse and subsequent bailout of large financial institute by government, prolonged high unemployment rate, and the subsequent European sovereign-debt crisis. Even to this day, many economies have not fully recovered from the crisis.

The crisis was the result of the false assumption that the housing prices would continue to rise. Before the crisis, thanks to poor regulations, home buyers with even poor credit could easily apply for a mortgage and later refinancing their mortgages. This process led to formation of a complicated loan system that exposed banks to many defaults. As a result, banks decided to reduce their risk to default by packing and selling these similar loans mortgages as Collateralized Debt Obligation (CDO) or Collateralized Loan Obligation (CLO). The CDOs or CLOs issues several tranches according to the default risk level. A most common example of tranches are senior, junior and equity tranches. Banks kept the most risky equity tranche and sold investors the less risky junior and senior tranches. The market ignored the large number of subprime mortgages that were included in the CDOs and CLOs and underestimated the default risk of the junior and senior tranches. In addition, banks charged investors higher spread for these over valued tranches and earned a greater profit. When the housing prices kept rising, borrowers with highest default risk could simply keep refinancing to pay back their mortgage. The underlying default risks were neglected. Clearly, a proper risk control may forbid the banks issuing this large amount of CDOs and CLOs. However, banks were reluctant to execute proper risk controls because of the considerable profit they have made.

Despite the unhealthy practice of issuing subprime mortgages, the stability of the market, due to the consistently increasing house prices, enabled the banks to neglect the long term drawbacks of such mortgages. In reality, the major risk factor of the sub-prime loans is the default of the borrowers.

When the housing bubble burst, the values of all securities related to the U.S. real estate began to plunge. Even the highest classes of CDS with zero default possibility and AAA rating lost value due to thousands of defaults. Rapid loss of value in a very wide range of securities combined with the inability of the risk models to properly assess the risk factors spread crippling fear in the markets. Banks magnified the counterparty risk in an effort to address the massive number of defaults and significantly reduced the amount of money they were lending. Consequently, markets went through a period of harsh illiquidity in the trading book which resulted in tremendous losses for all the involved parties.

With the crisis growing, the counterparty risk emerged as a major threat to all global financial institutions leading to a global freeze of lending practices bringing billions of dollars in losses. Finance researchers, economists and policy makers began to investigate the cause of this crisis. That was the time when the importance of credit and liquidity risk controls were realized by bankers and regulators.

As reported in the Financial Crisis Inquiry Commission Report (2011) [18] lack of financial regulation and supervision was claimed as one of the most important factors contributing to this crisis. Realizing the lack of financial regulation and the importance of counterparty credit risk, regulators established a list of procedures as a precaution aimed at effective prevention of such a crisis. As an immediate response to the financial crisis, the Basel Committee on Banking Supervision (BCBS) announced a series of changes to the Basel II framework in April 2008. These enhancements, referred as Basel 2.5 or Basel II Enhanced included revisions to the Basel II market risk framework and guidelines for computing capital for incremental risk in the trading book. In 2010, the BCBS announced Basel III to regulate bank capital adequacy, stress testing and market liquidity risk. Basel III raised the quality, consistency, and transparency of the capital base, emphasized the liquidity risk, strengthened the risk coverage of the capital framework in which the Credit Valuation Adjustment (CVA) risk was added. These announcements brought various changes to the financial markets all over the world and changed the bank credit risk analysis system.

To fulfil the new requirements set in Basel III, all banks faced pressures from regulators to improve their credit risk management and implement new risk models with specific requirements.

In Canada, the Office of the Superintendent of Financial Institutions (OSFI) encouraged the banks to develop their own models to calculate Incremental Risk Charge (IRC). These models needed to be fully vetted inside each bank and sent to OSFI for approval. If the model of a bank was not approved by OSFI, the bank would be charged a "standard" capital requirement. The "standard" capital requirement is a fixed capital charge set by OSFI. It is normally much higher than the model-based capital charge. From the perspective of the banks, a higher capital charge means less risky but also less profitable investments. Banks have a strong huge motive to develop their own capital models that meet the requirement of the regulations.

Among all important changes of the regulations, the counterparty credit and liquidity risks in the trading books were frequently addressed. For credit risks, CVA and IRC are the two most important concepts. In this dissertation, we will discuss and analyze CVA, IRC and related issues..

CVA is defined as the change of price when counterparty risk is added. In recent years, counterparty risks has started to play an increasingly important role in evaluation of credit risks. When a counterparty defaults, a replacement contract is established and there is a probability that the cost of the replacement contract is significantly higher than that of the original one. The difference between these two prices is called the credit value adjustment (CVA). In some of the earlier pricing literature for credit swaps, counterparty and investor are considered to be default free as in Duffie (1999) [14] and Hull, et al.(2000a) [28]. Counterparty default risk is considered in some studies (as in Hull, et al.(2000b) [29]) while the volatility of the credit spread is neglected (the hazard rate is assumed to be a constant). In other studies, volatility of the credit spread is included but the interest rate of the underlying is assumed to be a constant as in Brigo, et al. (2005) [8] and Sorensen (1994) [38]. In more recent work Brigo, et al. (2008a) [6], Brigo, et al. (2010) [7] and Brigo, et al. (2008b) [9], both stochastic interest rate and hazard rate models are used. In Brigo, et al. (2008a) [6] and Brigo, et al. (2008b) [9], both investor and counterparty defaults, or "bilateral counterparty risk", are included in the models. However, possible correlation between credit spread volatility and interest rate is not considered. In Assefa, et al.(2011) [1], Crepey (2012a) [12] and Crepey (2012b) [13], applications of bilateral counterparty risk have been discussed and analyzed.

In this dissertation, we considered the pricing problem of a new product which is called credit contingent interest rate swap (CCIRS). Following recent literature, we assumed that both hazard rate and interest rate are stochastic with a possible correlation. Our main objective is to investigate the effect of a possible second default and subsequent defaults, which have been neglected in all the existing literature. The basic question we addressed was whether it is justified to ignore the cost associated with one or more possible defaults of the replacement contract of the original CCIRS under normal market conditions. To do so, we first solved the pricing problem of CCIRS without the possibility of a second default. We also find the price of a CVA by allowing a second default and then without the restriction on the number of defaults, and compare the prices using reasonable parameter values for the interest rate and hazard rate.

In the second part of this dissertation, we focus on calculation of the IRC. In response to the financial crisis, the Basel Committee introduced the concept of unsecuritised credit product in trading book (see Basel (2013b) [3]). IRC is measured as the one-year Value-at-Risk (VaR) of unsecuritised credit products in trading book at 99.9 percentile confidence level. Currently, most proposed IRC models are based on a multi-factor multi-period Monte Carlo Simulation model (MC) as in CreditMetrics (2011) [11], Finger (2011) [19], Wilkens, et al.(2013) [45] and Yavin, et al.(2010) [46]. In the latest Basel proposal, IRC would be replaced by Incremental Default Risk (IDR), in which a two-factor modeling framework is suggested (as in Basel (2013a) [2]) and only default losses are considered.

In the model based IRC rules, a few new concepts in the credit risk measurements are introduced by Basel. Among these new concepts, liquidity horizon and constant level of risk over one-year capital horizon are particularly interesting. The liquidity horizon concept allows the banks to model the differences in the underlying liquidity of the trading book position. It represents the time required to sell the position or to hedge all material risk covered by the IRC model in a stressed market. Currently the standard VaR measure is defined as 99-percentile/10-day

VaR, embedded with 10 day liquidity horizon assumption. As witnessed during the financial crisis, banks experienced significant illiquidity in a wide range of credit products held in the trading book.

Another important concept introduced in IRC is the constant level of risk over one-year capital horizon, which allows the banks to model portfolio rebalance of their trading positions in a manner that maintains the initial risk level. Based on my practical observations as an analysist in the bank for four years, proper portfolio re-balancing assumptions are important concepts that makes the IRC model more risk-sensible and relevant to the actual trading portfolio behavior. For more liquidity and highly rated positions, the portfolio rebalancing provides a benefit relative to assuming the same position throughout the capital horizon. On the other hand, in order to hold the initial risk level, one would have to replace the position that defaults within the one-year time horizon and the replaced positions would then carry the same high default risk.

The credit risk capital model in Basel II is the internal rating-based approach (IRB), which is based on Asymptotic Single Risk Factor (ASRF) model. It has been the standard capital charge model for the banking book and is often required as a benchmark by regulators for the trading book. The principle of ASRF model is to model a large credit portfolio via a one risk factor model, which is straightforward and analytically trackable. It assumes that the portfolio is infinitely fine grained driven by one systematic risk factor with the idiosyncratic risk fully diversified away. Since ASRF was introduced, the pros and cons of the IRB approach have been the topic of extensive research in Hibbeln (2010) [27] and Lütkebohmert (2009) [31].

From the credit portfolio management perspective, the major weakness of ASRF model is its inability to capture the concentration risk. Credit concentrations, including both name concentrations and sector concentrations, are probably the single most important cause of major credit problems, which are behind most of the major banking disasters including the most recent financial crisis as mentioned in Basel (2000) [4] and Basel (2006) [5]. Within the IRB approach, the undiversified idiosyncratic risk can be approximated analytically via a granularity adjustment(GA) approximation. GA was first introduced in 2000 by Gordy (2003) [20]. The model was improved and re-established on a more rigorous foundation by Martin, et al.(2002) [32], Wilde (2001) [44], Giyrueriyx, et al.(2000) [26]. A survey of these developments and a primer on the mathematical derivation is presented in Gordy (2004) [21] and a rigorous proof of GA has been done recently by Fermanian (2013) [17]. In recent years, the concept of GA has also been expanded in both the application to model other risks (as in Gordy, et al. (2010) [22]) and credit portfolio risk measures other than capital charges as in Düllmann, et al. (2006) [15], Pykhtin (2004a) [34], Pykhtin (2004b) [35] and Voropaev (2011) [41].

The ASRF measure and its GA are almost fully developed and researched. However, the IRC calculation relies on the time consuming Monte Carlo simulations. So the banks used more efficient ASRF (and its GA) as important measures for effective capital management for banks. In order to achieve a risk sensible comparison, in this dissertation we present a general framework of two-period conditional VaR model in the context of IRC modeling framework in which the liquidity horizon and constant level of risk are considered. Given any time horizon, a two-period adjustment term is derived on top of the standard ASRF model. At the end of the first period, the portfolio is rebalanced to ensure a constant level risk as measured by the credit rating. The analytical approach is then compared with IRC MC models with and without portfolio rebalancing, ASRF with and without standard one-period GA to show how concentration risk, liquidity, and constant level of risk are captured in the new analytical approach. From the IRC (and IDR) modeling perspective, the analytical model can be readily applied to case, in which the portfolio rebalanced at six months with constant level of risk of the portfolio being rating/exposure-at-default based. Then finally, we considered one important question remaining about the multi-period VaR model, which is how to choose a proper liquidity horizon. We presented an exact analytical VaR solution for the infinite-period model, which provides the boundary of VaR with respect to different liquidity horizons.

By solving CVA calculation and analytical approach of VaR calculation by allowing two defaults and without the restriction on the number of defaults, we have made significant and valuable progress in expanding the credit risk research to different dimensions. While the existing research is still completing the current framework of one-period model, our research not only opened a separate door for the future research on CVA and IRC, but also provided guidelines for other potential multi-period credit risk research.

Rest of this thesis is arranged as follows. Chapter II studies the CVA model, applies PDE in a specific CVA pricing problem and extends it to a two-default case and then an no-restriction default case. We also have achieved a thorough risk exposure analysis. Chapter III studies the basic capital requirement model, extends it to a two-period model and solves it analytically by borrowing the logic of GA technique. Then we present an exact analytical solution for infinite-period model. The Chapter IV concludes this dissertation.

II A Multiple Defaults CVA Problem: Credit Contingent Interest Rate Swap Pricing—What Happens after the Default?

An interest-rate swap is a contract between two parties where one party (e.g. the bank) receives a fixed amount periodically in exchange for the LIBOR linked floating payments to the counterparty. When a counterparty defaults, a replacement contract is established and there is a probability that the cost of the replacement contract is significantly higher than that of the original one. The difference between these two prices is called the credit value adjustment (CVA). From the risk management point of view, it is important for financial institutions such as banks to understand the risk of counterparty defaults and estimate CVA of their portfolios. Currently, many banks operate under the assumption that CVA is the one time replacement cost of an existing contract. Completely ignoring the fact that the counterparty of the replacement contract could also default. Therefore, they either overestimate or underestimate the true CVA. In this chapter, we systematically investigate the risk involved in the current practice. Using Credit Contingent Interest Rate Swap (CCIRS) as an example, we present a detailed analysis of the CVA and strong evidences that the risk involved by ignoring the possibility of subsequent default of counterparties could be significant, especially for relatively long contracts with low credit ratings.

Our results are useful for financial institutions and financial regulators as they can serve as guidelines for estimating the true value of counterparty risk.

II.1 CVA and CCIRS

Its credit value adjustment (CVA) is the expected cost due to interest rate changes as well as the replacement costs in the cases of defaults of both parties. In Brigo, et al. (2008a) [6] and Brigo, et al. (2008b) [9], a general formula for pricing CVA was introduced using the following notations:

$$
\begin{aligned}
&\tau_I : \text{default} \quad \text{time} \quad \text{of} \quad \text{investor}, \\
&\tau_C : \text{default} \quad \text{time} \quad \text{of} \quad \text{counterparty}, \\
&\tau_U : \text{default} \quad \text{time} \quad \text{of} \quad \text{underlying.} \\
&T : \text{maturity} \quad \text{of} \quad \text{the} \quad \text{underlying}, \\
&A = \{\tau_I \le \tau_C \le T\}, \qquad B = \{\tau_I \le T \le \tau_C\}, \\
&C = \{\tau_C \le \tau_I \le T\}, \qquad D = \{\tau_C \le T \le \tau_I\}, \\
&E = \{T \le \tau_I \le \tau_C\}, \qquad F = \{T \le \tau_C \le \tau_I\}.
\end{aligned}
$$

The price of a CVA under these notations is given as

$$
\begin{aligned}
E\{\Pi^D(t,T)|\mathfrak{F}_t\} = {} & E\{\Pi(t,T)|\mathfrak{F}_t\} \\
& + E\{\text{LGD}_I \cdot I(A \cup B) \cdot P(t,\tau_I) \cdot [-\text{NPV}(\tau_I)]^+|\mathfrak{F}_t\} \\
& - E\{\text{LGD}_C \cdot I(C \cup D) \cdot P(t,\tau_C) \cdot [\text{NPV}(\tau_C)]^+|\mathfrak{F}_t\},
\end{aligned}
$$

where $E\{\Pi(t,T)|\mathfrak{F}_t\}$ is the price under the assumption that both the investor and counterparty are default-free, and the second and third terms are the replacement costs. $\mathfrak{F}_t$ contains the full information before time t, $\text{LGD} = (1-\text{RecoverRate})$ is the loss given default, $\text{NPV}(t)$ is net present value of the residual payoff for the investor until maturity from time t, $P(t_1, t_2)$ is the price at t_1 of a zero coupon bond matured at time t_2, i.e. the discount rate from time t_1 to t_2. If only counterparty risk is considered from the viewpoint of the bank (investor), the cost due to counterparty default is

$$E\left\{\Pi^D(t,T)|\mathfrak{F}_t\right\} = E\left\{\Pi(t,T)|\mathfrak{F}_t\right\} \\ - E\left\{\mathrm{LGD}_C \cdot I(\tau_C < T) \cdot P(t,\tau_C) \cdot [\mathrm{NPV}(\tau_C)]^+|\mathfrak{F}_t\right\} \ . \quad \text{(II.1.1)}$$

Notice that the CVA is always non-negative when only the counterparty risk is taken in to account. But if the bilateral counterparty risk exists, CVA also can be negative. More importantly, the above formulas are correct only when the swap expires at the defaults, or the counterparty of the replacement contract is default free.

CCIRS is a contract which can cover the loss due to the counterparty default in interest rate swap. Suppose the bank enters an interest-rate swap with a counterparty so that the bank receives from the counterparty a fixed amount periodically in exchange for the LIBOR linked floating payment from the bank. If the counterparty defaults, the bank needs to enter another swap agreement. However, the fixed rate will likely be different from the original one since the interest rate environment and number of remaining payments have changed. Thus, the bank bears the risk of making higher payment due to the possibility of default of the counterparty. There is also the possibility that in case of a default, the new rate is lower, but this scenario is of no concern to the bank from a risk management point of view. The purchase of a CCIRS eliminates that risk, and the fair price of CCIRS should be the expectation of the possible loss at the time when CCIRS is issued. Therefore, the pricing problem of CCIRS is equivalent to that of a CVA problem for interest rate swap when only counterparty risk is considered as in formula (II.1.1), under the assumption that the replacement contract is default free. When the counterparty of the replacement contract is not default free, the pricing formulae (II.1.1) underestimates the risk.

II.2 CCIRS with Default-Free Replacement Contracts

To price a CCIRS, we first describe how an interest rate swap works and the relationship between the fixed and floating legs of the swap. A swap is a derivative contract in finance in which two counterparties enter an agreement to exchange certain benefits of one party's financial instrument another. The benefits in question depend on the type of financial instruments involved. Specifically, if the two counter-parties agree to exchange one stream of interest rate payments against another stream of payments, the derivative is an interest rate swap. If the two counter-parties sign an interest rate swap contract, then one counterparty agrees to make fixed payments at specified times. Normally the payment is the product of the notional value, the time interval between payments and the agreed fixed rate , i.e. $Nol \times \Delta t \times R_{\text{fixed}}$. In return, it will receive a stream of payments based on the floating rate. Similarly, the payment is normally the product of notional value Nol, the time between payments Δt and the current floating rate $R_{\text{floating}}(t)$, which is usually an indexed reference rate (such as LIBOR) with a fixed spread S_p (can be 0). i.e. $Nol \times \Delta t \times (R_{\text{floating}}(t) + S_p)$. For example, a company signs an interest rate swap contract with a bank. The swap requires the company to pay a fixed rate at 5% in each payment time and the company receives a payment at the LIBOR rate in return. The notional value is \$1,000,000. The maturity of the swap is five years and payment is made semi-annually. Every half a year, company pays $1,000,000 \times 0.5 \times 5\% = \$25,000$ and receives $1,000,000 \times 0.5 \times LIBOR(t)$, where t is the time when payment is made.

II.2.1 CCIRS Pricing

In an interest rate swap, one party is required to make payments during each Δt period, from t_1 to t_n. Let t be the current time, the (random) default time for the counterparty is τ, the next payment time is t_k, the last payment time is t_n and T is the expiry time for the swap. If the default does not occur, the present value of the remaining payments at time τ is $Nol \cdot A_\tau(\tau, T)K$, where $A_\tau(\tau, T) = \Delta t_i \sum_{i=k_\tau}^{n} P(\tau, t_i)$ is the remaining annuity after time τ and $P(\tau, t_i)$ is the t_i-maturity zero coupon bond price at time τ. When the counterparty defaults at time τ, the payment of the replacement contract is $Nol \cdot \Delta t R_\tau(\tau, T)$, where $R_\tau(\tau, T)$ is the new fixed swap-rate calculated at time τ. The present value of the remaining payments (assuming no additional defaults) at time τ is $Nol \cdot A_\tau(\tau, T)R_\tau(\tau, T)$. Normally a fraction of the present value Rec can be recovered at default. Therefore, only the portion $1 - Rec$ needs to be covered by CCIRS.

When the counterparty of the replacement contract is default-free, we can now write down the cost of replacing the swap. For the counterparty paying the fixed rate, the possible loss when $\tau < T$ is

$$\begin{aligned} v(\tau) &= (1 - Rec)(Nol \cdot A_\tau(\tau, T)R_\tau(\tau, T) - Nol \cdot A_\tau(\tau, T)K)^+ \\ &= Nol \cdot (1 - Rec)A_\tau(\tau, T)(R_\tau(\tau, T) - K)^+; \end{aligned}$$

while for the counterparty receiving the fixed rate, the possible loss is

$$\begin{aligned} v(\tau) &= (1 - Rec)(Nol \cdot A_\tau(\tau, T)K - Nol \cdot A_\tau(\tau, T)R_\tau(\tau, T))^+ \\ &= Nol \cdot (1 - Rec)A_\tau(\tau, T)(K - R_\tau(\tau, T))^+. \end{aligned}$$

The derivative price at time t is simply the discounted expected value of $v(\tau)$ at time t under the risk-neutral measure, i.e., $v(t) = E\left[I_{\tau<T}\exp(-\int_t^\tau r(s)ds)v(\tau)|\mathfrak{F}_t\right]$.

In our case, we only considered the price of CCIRS when the investor is paying the fixed rate. The price at default time τ is

$$v(\tau) = Nol \cdot (1 - Rec)A_\tau(\tau, T)(R_\tau(\tau, T) - K)^+. \qquad \text{(II.2.1)}$$

The derivative price at time t is

$$v(t) = Nol \cdot (1 - Rec) \cdot E\left[I_{\tau<T}\exp(-\int_t^\tau r(s)ds)A_\tau(\tau, T)(R_\tau(\tau, T) - K)^+ \middle| \mathfrak{F}_t \right]. \qquad \text{(II.2.2)}$$

Since both Nol and Rec are constants, we only need to compute the scaled price

$$E\left[I_{\tau<T}\exp(-\int_t^\tau r(s)ds)A_\tau(\tau, T)(R_\tau(\tau, T) - K)^+ \middle| \mathfrak{F}_t \right]. \qquad \text{(II.2.3)}$$

The final price can be obtained by multiplying $Nol \times (1 - Rec)$.

II.2.2 Model Selection

Since the default time τ involves hazard rate process, a proper model of this process needs to be chosen. On top of that, we need to choose a proper model for interest rate as well. In our research, we assume the hazard rate process is the same for all counter-parties with a same credit rating. Then we assume that both the interest rate and hazard rate follow the mean reverting Cox-Ingersoll-Ross (CIR) model (introduced in Cox et al. (1985) [10]), a widely used model in industry. CIR model is given by

$$dr = a_1(b_1 - r(t))dt + \sigma_1\sqrt{r(t)}dB_t^1 \qquad \text{(II.2.4)}$$

$$d\lambda = a_2(b_2 - \lambda(t))dt + \sigma_2\sqrt{\lambda(t)}dB_t^2 \qquad \text{(II.2.5)}$$

where $d[B_t^1, B_t^2] = \rho dt$, i.e., the hazard and interest rates are correlated with coefficient ρ. When $2a_1b_1 > \sigma_1^2$ and $2a_2b_2 > \sigma_2^2$, this model ensures the interest rate and hazard rate are always positive and will never touch zero.

II.2.3 Fixed Rate for an Interest Rate Swap

In this subsection, we derive the formula for the fixed rate for an interest rate swap $R_t(t,T)$, signed at time t, maturing at time T, with n payments at time $t_1, t_2, \ldots, t_n$. Suppose the time t for issuing the new swap is between t_{k-1} and t_k, which means the next payment time is t_k.

For the fixed leg, the present value of all the fixed payment at time t is

$$Nol \cdot R_t \Delta t \sum_{i=k}^{n} P(t,t_i),$$

where Nol is the notional amount, R_t (short for $R_t(t,T)$) is the fixed rate, $P(t,t_i)$ is the zero-coupon bond price at time t and matures at time t_i. For the floating leg, one can use no-arbitrage argument to show that the present value of the payments is equivalent to an investment in bonds that mature at t_i ($i = k, ..., n$), which is given by $Nol \cdot [1 - P(t,t_n) + S_p \Delta t \sum_{i=k}^{n} P(t,t_i)]$, where S_p is the fixed spread added on floating index (i.e. LIBOR, etc), t_n is the last payment date.

We can now compute the fixed rate of swap by equating the value of the fixed leg and floating leg since the swap contract itself has no value at the time of signing the contract.

$$R_t(t,T) = \frac{1 - P(t,t_n)}{\Delta t \sum_{i=k}^{n} P(t,t_i)} + S_p \tag{II.2.6}$$

In fact $\Delta t \sum_{i=k}^{n} P(t,t_i) = A_t(t,T)$, $A_t(t,T)$ is the into-forward annuity from time t to maturity T observed at time t. From (II.2.3), it can be seen that S_p can be absorbed into the fixed rate K in the valuation. Without loss of generality, we assume $S_p = 0$ in rest of this dissertation.

II.2.4 Discount Factor under Stochastic Short Rate

Denoting the discount factor from time t_1 to time t_2 is $P(t_1, t_2)$, we expect

$$P(t_1, t_2) = E[\exp(-\int_{t_1}^{t_2} r(s)ds)|\mathfrak{F}_{t_1}],$$

where $r(s)$ is short rate at time s and it follows:

$$dr = a_1(b_1 - r(t))dt + \sigma_1\sqrt{r(t)}dB_t^1.$$

Since $\exp(-\int_r^{t_1} r(s)ds)P(t_1, t_2)$ is a martingale, $P(t_1, t_2)$ satisfies the partial differential equation (PDE)

$$\partial_t P + a_1(b_1 - r)\partial_r P + \frac{1}{2}\sigma_1^2 r\partial_{rr}P = rP \tag{II.2.7}$$

with terminal condition $P(t_2, t_2) = 1$. This PDE is solved analytically as

$$P(t_1, t_2) = \Lambda(t_1, t_2)\exp(-B(t_1, t_2)r(t)), \tag{II.2.8}$$

where

$$\begin{aligned}
\Lambda(t_1, t_2) &= \left\{\frac{2\gamma\exp[\frac{1}{2}(a_1 + \gamma)(t_2 - t_1)]}{(\gamma + a_1)\{\exp[\gamma(t_2 - t_1)] - 1\} + 2\gamma}\right\}^{\frac{2a_1b_1}{\sigma_1^2}}, \\
B(t_1, t_2) &= \frac{2\{\exp[\gamma(t_2 - t_1)] - 1\}}{(\gamma + a_1)\{\exp[\gamma(t_2 - t_1)] - 1\} + 2\gamma}
\end{aligned}$$

with $\gamma = \sqrt{a_1^2 + 2\sigma_1^2}$.

II.2.5 Solution Methodologies

II.2.5.1 Monte Carlo Method

To use the Monte-Carlo method (MC), we generate M realizations for $r(t)$ and $\lambda(t)$ based on equations (II.2.4) and (II.2.5) . The default time τ is generated using the process of $\lambda(t)$. We compute $A_\tau(\tau, T)$ and $R_\tau(\tau, T)$ using (II.2.6) and (II.2.8) given in the previous sections. The value of a CCIRS is computed using equation (II.2.1) at the default time τ. In each realization where $r(t)$ becomes a constant process, we find the discounted value of $v(\tau)$ at the initial time ($t = 0$) with a discount rate $\exp\left(-\int_t^\tau r(s)ds\right)$. This discounted value is the initial price of CCIRS for each path. The expected CCIRS price is obtained by taking the average value of all the M realizations of Monte Carlo simulated paths. When M is large, the computational speed of Monte Carlo simulations decreases quickly which limits the implementation of Monte Carlo simulations.

II.2.5.2 PDE Approach

Although Monte Carlo method is easy to implement, the PDE based approach provides a more efficient alternative compared to time consuming Monte Carlo simulations. In this subsection, we derive the partial differential equations that are needed for pricing a CCIRS.

Lemma II.2.1 *τ is the first jump time of a Poisson Process with parameter λ.*

$$V(t) = E\left[\exp(-\int_t^\tau r(s)ds)f(\tau)\middle|\mathfrak{F}_t\right]$$

where $\mathfrak{F}_t = \mathfrak{G}_t \cup \sigma(I_{t>\tau}, 0 \le t \le T)$. $\mathfrak{F}_t$ contains the full information before time t, $\sigma(I_{t>\tau}, 0 \le t \le T)$ contains the information that whether there has been a default before time t. $\mathfrak{G}_t$ contains the full information before time t except the information contained in $\sigma(I_{t>\tau}, 0 \le t \le T)$. Use $E_t[\bullet]$ to represent $E[\bullet|\mathfrak{G}_t]$ for short. Then we have:

$$V(t) = I_{t<\tau}E_t\left[\int_t^T f(s)\lambda(s)\exp(-\int_t^s (r(u)+\lambda(u))du)ds\right]$$

Proof See [30], Prop 3.1. ∎

The value of CCIRS at time t can be represented as

$$v(t) = E\left[I_{\tau<T}\exp(-\int_t^{\tau} r(s)ds)A_{\tau}(\tau,T)(R_{\tau}(\tau,T)-K)^{+} \middle| \mathfrak{F}_t \right].$$

Actually the PDE of $v(t)$ is the direct result of Feynman-Kac theorem. We decided to elaborate the proof here for the completeness of our thesis.

Theorem II.2.2

$$F(t) = v(t)\exp(-\int_0^t (r(u)+\lambda(u))du) + \int_0^t f(s)\lambda(s)\exp(-\int_0^s (r(u)+\lambda(u))du)ds$$

is a martingale, where

$$f(t) = I_{t<T}A_t(t,T)(R_t(t,T)-K)^{+}.$$

Proof Use Lemma II.2.1, we have

$$v(t) = E_t\left[\int_t^T f(s)\lambda(s)\exp(-\int_t^s (r(u)+\lambda(u))du)ds \right], \tag{II.2.9}$$

which gives

$$\begin{aligned} F(t) &= v(t)\exp(-\int_0^t (r(u)+\lambda(u))du) + \int_0^t f(s)\lambda(s)\exp(-\int_0^s (r(u)+\lambda(u))du)ds \\ &= E_t\left[\int_0^T f(s)\lambda(s)\exp(-\int_0^s (r(u)+\lambda(u))du)ds \right]. \end{aligned}$$

Notice the last expression above is a martingale since the expectation does not contain t. Denote the function inside the expectation as H. This means $F(t) = E_t(H)$. Given the definition of a martingale, we only need to show

1. $E_t(|H|) < \infty$;

2. $E_t[F(s)] = F(t)$, $(s > t)$.

The first inequality is true evidently. Since

$$E_t\left[F(s)\right] = E_t\left[E_s(H)\right] = E_t\left(H\right) = F(t),$$

$F(t)$ is a martingale. ∎

Theorem II.2.3 *The PDE satisfied by $v(t, r, \lambda)$ is*

$$(\partial_t + \mathcal{L})v + \lambda(f - v) - rv = 0 \tag{II.2.10}$$

with terminal condition $v(T, r, \lambda) = 0$, where

$$\mathcal{L} = a_1(b_1 - r)\partial_r + \frac{1}{2}\sigma_1^2 r\partial_{rr} + a_2(b_2 - \lambda)\partial_\lambda + \frac{1}{2}\sigma_2^2\lambda\partial_{\lambda\lambda} + \rho\sigma_1\sigma_2\sqrt{r\lambda}\partial_{r\lambda},$$
$$f = A_t(t, T)(R_t(t, T) - K)^+$$

Proof To simplify notation, we denote

$$\hat{D}(t) = \exp(-\int_0^t (r(u)+\lambda(u))du), \quad M(t) = \int_0^t f(s)\lambda(s)\exp(-\int_0^s (r(u)+\lambda(u))du)ds$$

and $F(t) = v(t)\hat{D}(t) + M(t)$. Recall the models for r and λ

$$dr = a_1(b_1 - r(t))dt + \sigma_1\sqrt{r(t)}dB_t^1, \quad d\lambda = a_2(b_2 - \lambda(t))dt + \sigma_2\sqrt{\lambda(t)}dB_t^2.$$

Applying Ito's lemma

$$d\hat{D}(t) = -\hat{D}(t)(r(t) + \lambda(t))dt, \quad dM(t) = f(t)\lambda(t)\hat{D}(t)dt,$$

and

$$\begin{aligned} dv &= v_t dt + v_r dr + v_\lambda d\lambda + \frac{1}{2}v_{rr}drdr + v_{r\lambda}drd\lambda + \frac{1}{2}v_{\lambda\lambda}d\lambda d\lambda, \\ &= v_t dt + v_r(a_1(b_1 - r)dt + \sigma_1\sqrt{r}dB_t^1) + v_\lambda(a_2(b_2 - \lambda)dt + \sigma_2\sqrt{\lambda}dB_t^2) \\ &+ \frac{1}{2}v_{rr}\sigma_1^2 r dt + \rho\sigma_1\sigma_2\sqrt{r\lambda}v_{r\lambda}dt + \frac{1}{2}v_{\lambda\lambda}\sigma_2^2 dt \end{aligned}$$

$$= (\partial_t + \mathcal{L})v dt + v_r \sigma_1 \sqrt{r} dB_t^1 + v_\lambda \sigma_2 \sqrt{\lambda} dB_t^2.$$

This leads to

$$\begin{aligned} dF(t) &= v(t) d\hat{D}(t) + \hat{D}(t) dv + dM(t) \\ &= -v(t)\hat{D}(t)(r(t) + \lambda(t))dt \\ &+ \hat{D}(t) \left((\partial_t + \mathcal{L})v dt + v_r \sigma_1 \sqrt{r} dB_t^1 + v_\lambda \sigma_2 \sqrt{\lambda} dB_t^2 \right) + f(t)\lambda(t)\hat{D}(t)dt \\ &= \hat{D}(t) \left[(\partial_t + \mathcal{L})v - v(r + \lambda) + f\lambda \right] dt + \hat{D}(t) v_r \sigma_1 \sqrt{r} dB_t^1 + \hat{D}(t) v_\lambda \sigma_2 \sqrt{\lambda} dB_t^2. \end{aligned}$$

From Theorem II.2.2, we know $F(t)$ is a martingale. Therefore the coefficient of the dt term in $dF(t)$ must vanish, which gives

$$(\partial_t + \mathcal{L})v - v(r + \lambda) + f\lambda = 0,$$

which can be rearranged to

$$(\partial_t + \mathcal{L})v + \lambda(f - v) - rv = 0.$$

∎

The remaining task is to find the terminal and boundary condition for it. If the counterparty defaults at or after the maturity T, there is no need to replace the original swap. In this case, the price of CCIRS is zero. This gives us the terminal condition as

$$v(T, \lambda, r) = 0. \tag{II.2.11}$$

For the boundary condition, we can look at the characteristic function of r and λ. For $r = 0$, the $\frac{dr}{dt} = a_1 b_1 > 0$. And for this PDE, we have a terminal condition which means we need to solve backward. And from the direction of the characteristic function of r on $r = 0$, we can clearly see the value goes towards the boundary line. For $r = \infty$, the diffusion term can be ignored because its order is $\sqrt{r}$. So $\frac{dr}{dt} = a_1(b_1 - r) < 0$, this means the value goes towards the boundary line as well. The behavior of the value respect to λ is the same. This means for boundary condition, we can simply replace the derivative on the boundary with the derivative of the inside point beside the boundary.

The PDE with the terminal and boundary conditions is solved using finite difference method, second order in time and alternating-direct-implicit in time.

II.2.6 Numerical Results

II.2.6.1 Parameter Values

In this section, we will do a rough estimation by using real data to calculate the parameters of our CIR Model. This estimation is used here to give us a reasonable sense about how these parameters should be picked up.

For interest rate, it is widely accepted that the risk-free rate curve is the best approximation of short rate. And LIBOR rate curve is always used when a risk-free curve is needed. Theoretically, if all LIBOR curves are risk-free, they should be identical if compounded to an annual rate. In our paper, we can reasonably assume the 12 Month LIBOR Rate is the best approximation of risk-free rate. So we used the 5-year data of 12 Month LIBOR Rate starting from May 1st, 2009 to April 30, 2014.

Partial data of 12 Month LIBOR rate has been put in Table 1. We used the rate 0.5490% on April 30, 2014 as the initial rate. The mean is used as the long term average, i.e.,

$$b_1 = mean(\text{data}) = 0.909\%$$

and

$$mean(\text{data}) = \frac{\sum_{i=1}^{N} r_i}{N},$$

where N is number of data in five years and r_i is the 12 Month LIBOR in ith day.

And we know the conditional variance of the interest rate at any time t is

$$Variance[r_t|r_0] = r_0\frac{\sigma_1^2}{a_1}(e^{-a_1 t} - e^{-2a_1 t}) + \frac{b_1\sigma_1^2}{2a_1}(1 - e^{-a_1 t})^2. \qquad \text{(II.2.12)}$$

For sufficient large t, this variance will turn into a long term variance and r_0 has almost no effect. Then we let $t = 100$, and then we can reasonably assume the long term variance equals to the variance of the 12 month LIBOR rate. Then we can have

$$\sigma_1 = \sqrt{\frac{Variance(\text{data})}{\frac{b_1}{2a_1}(1 - e^{-100a_1})^2}}$$

and

$$Variance(\text{data}) = \frac{\sum_{i=1}^{N}(r_i - \bar{r})^2}{N-1},$$

where $\bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i$. Assuming $a_1 = 1$, we have $\sigma_1 = 0.038060013$. These estimated parameters of b_1, σ_1 and a_1 have been put in Table 2.

For hazard rate, a commonly model applied in industry is the CDS spread approach. This model assumes

$$\lambda(t) = \frac{\text{spread}(t)}{1 - R}, \qquad \text{(II.2.13)}$$

where R is recovery rate and normally assumed to be 0.4.

We used the USD Financial sector 12 Month CDS spread data from May 1st, 2009 to April 30, 2014. Partial data of these CDS spreads has been put in Table 1 as well. Applying the model in (II.2.13), we used the rate on April 30, 2014 to approximate the initial hazard rate. The mean is used to estimate the long term average, i.e.

$$b_2 = \frac{mean(\text{data})}{1 - R}$$

and

$$mean(\text{data}) = \frac{\sum_{i=1}^{N} spread_i}{N}.$$

And we have the same formula for conditional variance of the hazard rate at any time t as (II.2.12),

$$Variance[\lambda_t|\lambda_0] = \lambda_0 \frac{\sigma_2^2}{a_2}(e^{-a_2 t} - e^{-2a_2 t}) + \frac{b_2 \sigma_2^2}{2a_2}(1 - e^{-a_2 t})^2. \qquad \text{(II.2.14)}$$

Since the relationship between the hazard rate and spread in (II.2.13) exists, it is reasonable to assume the variance of spread equal to the variance of the hazard rate. Then we have

$$\sigma_2 = \sqrt{\frac{Variance(\text{data})}{\frac{b_2}{2a_2}(1 - e^{-100a_2})^2}}$$

and

$$Variance(\text{data}) = \frac{\sum_{i=1}^{N}(spread_i - \overline{spread})^2}{N-1},$$

where $\overline{spread} = \frac{1}{N}\sum_{i=1}^{N} spread_i$. Assume $a_2 = 1$, we can then get σ_2. All these parameters for different ratings are shown in Table 3.

The rating A is normally used as a testing grade. So we used the parameters for rating A here, i.e., $\lambda_0 = 0.64683\%, b_2 = 1.1736\%, \sigma_2 = 0.035502957, a_2 = 1$.

The other parameters are set as follows, maturity T is set to be 5 years. The fixed rate K for original swap is 0.909%, ρ=0.2 and the notional value is $250,000,000.

II.2.6.2 Monte-Carlo Results

In the Monte Carlo simulations, we set the number of realizations M as 1,000,000 and divide the time to maturity (T) into 2,000 equal parts. The computational time is 648 seconds and the price is $2,204.58.

II.2.6.3 PDE Results

We use a 100×100 grid for the interest rate and hazard rate and the number of time steps is 600 over a 5 year period. The price for CCIRS is $2,236.22, which is close to the result got by the Monte Carlo simulation, within 1.9 seconds. When we increase the number of time steps to 2000, the same numbers used in the Monte-Carlo simulations, the total computational time increases to 3.62 seconds and the price is $2,235.02, which suggests that the time step is sufficiently acceptable for the spatial grid chosen. We have also obtained the result by assuming constant hazard rate, which is obtained by solving the reduced PDE, with a 100 grid points in r and 600 time steps over 5 years. The price for a constant hazard rate is $1,201.53, quite different from the one with a stochastic hazard rate given in the table.

Table 4 shows the comparison of results obtained by using the PDE method and Monte Carlo simulation. The differences are small but PDE approach is much faster.

In addition to the savings in computational time, the PDE approach also generates the price of CCIRS for the entire range of interest and hazard rates, as shown by Figures 1. It can be seen that the price is in general an increasing function of the interest and hazard rates, since higher hazard rates mean higher probability of default.

II.3 CCIRS with Defaultable Replacement Contract

In the previous section, when the counterparty defaults and a new replacement swap contract is signed, it was assumed that the counterparty of the new contract is default-free. Therefore, the CCIRS price obtained in the previous section is only an approximation, which may underestimate the real price. This is justified for a counterparty with a high credit rating when the time to maturity is short. In practice, however, the time to maturity of these contracts is relatively long (e.g., 10 years). Therefore, it will be of practical interest to investigate the effect of the default-free assumption, which is the focus of this section.

II.3.1 PDE for the Second Default Problem

We assume that the replacement contract could also default but its replacement is default free. In the rest of the thesis, this is called "two-default" problem, which is a more accurate approximation of the time cost than the default free replacement contract model, or the "one-default" problem discussed previously. The "second" default problem is actually conditional on the happening of the first default. Let τ_1 and τ_2 ($\tau_2 > \tau_1$) be the default times of the original and replacement counterparties, respectively. They are the first and the second jumps time of the Cox process with hazard rate λ given by (II.2.5). Recall that the price of CCIRS with a default-free replacement contractor is given by (II.2.3) as

$$V(t) = Nol \cdot (1 - Rec)E\left[I_{\tau<T} \exp(-\int_t^{\tau_1} r(s)ds) A_{\tau_1}(\tau_1, T)(R_{\tau_1}(\tau_1, T) - K)^+ \middle| \mathfrak{F}_t \right].$$

We can rewrite this equation as:

$$V(t) = Nol \cdot (1 - Rec)E\left[D(t, \tau_1) f(\tau_1) \middle| \mathfrak{F}_t \right] , \qquad \text{(II.3.1)}$$

where

$$f(\tau_1) = \begin{cases} A_{\tau_1}(\tau_1, T)(R_{\tau_1}(\tau_1, T) - K)^+, & \tau_1 < T \ ; \\ 0, & \tau_1 \geq T \ , \end{cases}$$

and

$$D(t, \tau_1) = \exp\left[-\int_t^{\tau_1} r(s)ds \right] .$$

Again,$Nol \cdot (1 - Rec)$ is a constant and we will drop it in the following discussion knowing that the final price can be obtained by multiplying our numerical solution with it. When the counterparty of the replacement contractor is allowed to default, there are three scenarios.

(i). Only one default occurs before maturity. Based on equation (II.2.1), the loss at the first default τ_1 is

$$v(\tau_1) = A_{\tau_1}(\tau_1, T)(R_{\tau_1}(\tau_1, T) - K)^+$$

(ii). Both defaults occur before maturity. The fixed rate for a new swap at τ_2 is $R_{\tau_2}(\tau_2, T)$. The fixed rate payment of the replacement swap between τ_1 and τ_2 is $\Delta t R_{\tau_1}(\tau_1, T)$ and the fixed rate payment of the second replacement swap between τ_2 and T is $\Delta t R_{\tau_2}(\tau_2, T)$. The discounted value of all the payments between τ_1 and τ_2 is $A_{\tau_1}(\tau_1, \tau_2) R_{\tau_1}(\tau_1, T)$. The discounted value of all the payments between τ_2 and T is $D(\tau_1, \tau_2) N A_{\tau_2}(\tau_2, T) R_{\tau_2}(\tau_2, T)$. The value of CCIRS at time τ_1 is the sum

$$v(\tau_1) = A_{\tau_1}(\tau_1, \tau_2)(R_{\tau_1}(\tau_1, T) - K)^+ + A_{\tau_2}(\tau_2, T)(R_{\tau_2}(\tau_2, T) - K)^+ D(\tau_1, \tau_2).$$

(iii). The first default happens after maturity. There is no cost and the value of CCIRS is zero.

Considering all cases above, the CCIRS price is given by

$$W(t) = E\left[D(t, \tau_1) f(\tau_1, \tau_2) | \mathfrak{F}_t\right]$$

where

$$f(\tau_1, \tau_2) = \begin{cases} 0 & \tau_2 > \tau_1 > T; \\ A_{\tau_1}(\tau_1, T)(R_{\tau_1}(\tau_1, T) - K)^+ & \tau_2 > T > \tau_1; \\ A_{\tau_1}(\tau_1, \tau_2)(R_{\tau_1}(\tau_1, T) - K)^+ & \\ \qquad + A_{\tau_2}(\tau_2, T)(R_{\tau_2}(\tau_2, T) - K)^+ D(\tau_1, \tau_2) & T > \tau_2 > \tau_1. \end{cases}$$

To simplify notation, let

$$\widetilde{A}(t_1, t_2) = \begin{cases} A_{t_1}(t_1, t_2) & T > t_2 > t_1; \\ A_{t_1}(t_1, T) & t_2 > T > t_1; \\ 0 & \text{otherwise}. \end{cases} \tag{II.3.2}$$

We rewrite $f(\tau_1, \tau_2)$ as

$$\widetilde{A}(\tau_1, \tau_2)(R_{\tau_1}(\tau_1, T) - K)^+ + \widetilde{A}(\tau_2, T)(R_{\tau_2}(\tau_2, T) - K)^+ D(\tau_1, \tau_2)$$

and $W(t)$ can be written as

$$W(t) = E\Big\{D(t,\tau_1)[\widetilde{A}(\tau_1,\tau_2)(R_{\tau_1}(\tau_1,T)-K)^+ \\ + \widetilde{A}(\tau_2,T)(R_{\tau_2}(\tau_2,T)-K)^+ D(\tau_1,\tau_2)]\Big|\mathfrak{F}_t\Big\}. \qquad \text{(II.3.3)}$$

To derive the PDE for $W(t)$, we need the following theorems and corollaries.

Corollary II.3.1 *For any $T > 0$ and $\tau > t$, let Z_t be a $\mathfrak{G}_t$-adapted stochastic process and $Z_t \not\equiv 0$ when $t \geq T$, then:*

$$E\left[D(t,\tau)Z_\tau\,|\mathfrak{F}_t\right] = E_t\left[\int_t^{+\infty} Z_s\lambda_s\hat{D}(t,s)ds\right].$$

Proof Denote $t_i = i\Delta t$, $i = 0, 1, ...$, we have $Z_s^{(i)} = Z_s I_{t_i \leq s < t_{i+1}}$ and $Z_s = \sum_{i=0}^{\infty} Z_s^{(i)}$. It follows that

$$\begin{aligned} E\left[D(t,\tau)Z_\tau\,|\mathfrak{F}_t\right] &= E\left[D(t,\tau)\sum_{i=0}^{\infty} Z_\tau^{(i)}\middle|\mathfrak{F}_t\right] \\ &\quad (\text{Since } D(t,\tau)Z_\tau^{(i)} > 0,\ \text{by Tonelli's Theorem}) \\ &= \sum_{i=0}^{\infty} E\left[D(t,\tau)Z_\tau^{(i)}\middle|\mathfrak{F}_t\right]. \end{aligned}$$

Since each $Z_\tau^{(i)} = 0$ when $\tau \geq t_{i+1}$, Theorem II.2.2 applies and

$$\begin{aligned} \sum_{i=0}^{\infty} E\left[D(t,\tau)Z_\tau^{(i)}\middle|\mathfrak{F}_t\right] &= \sum_{i=0}^{\infty} I_{\tau\geq t}E_t\left[\int_t^{t_{i+1}} Z_s^{(i)}\lambda_s\hat{D}(t,s)ds\right] \\ &= \sum_{i=0}^{\infty} I_{\tau\geq t}E_t\left[\int_t^{t_{i+1}} Z_s I(t_i \leq s < t_{i+1})\lambda_s\hat{D}(t,s)ds\right] \\ &= I_{\tau\geq t}\sum_{i=1}^{\infty} E_t\left[\int_{t_i}^{t_{i+1}} Z_s\lambda_s\hat{D}(t,s)ds\right] \\ &\quad \left(\text{Since } \int_{t_i}^{t_{i+1}} Z_s\lambda_s\hat{D}(t,s)ds > 0,\ \text{by Tonelli's Theorem}\right) \\ &= I_{\tau\geq t}E_t\left[\sum_{i=1}^{\infty}\int_{t_i}^{t_{i+1}} Z_s\lambda_s\hat{D}(t,s)ds\right] \end{aligned}$$

$$= I_{\tau \geq t} E_t \left[\int_t^{+\infty} Z_s \lambda_s \hat{D}(t,s) ds \right]. \tag{II.3.4}$$

This ends the proof. ∎

Corollary II.3.2 *From Corollary II.3.1, let $r_t \equiv 0$ and $Z(t) \equiv 1$, for any $\tau > t$ we have*

$$1 = E_t \left[\int_t^{+\infty} \lambda_s \exp \left(- \int_t^s \lambda_u du \right) ds \right].$$

Theorem II.3.3 *(This is a stronger result than Corollary II.3.2.)*

$$\int_t^{\infty} \lambda_s \exp \left(- \int_t^s \lambda_u du \right) ds = 1.$$

Proof From the proof of Proposition 3.1 in [30], the density of the default time for $s > t$ is given by

$$\frac{\partial}{\partial s} \mathrm{P}(\tau \leq s | \tau > t, \mathfrak{G}_T) = \lambda_s \exp \left(- \int_t^s \lambda_u du \right).$$

We know the integration of density function is 1, which proves Theorem II.3.3. ∎

Theorem II.3.4

$$A_u(u,s) = A_u(u,T) - E[D(u,s) A_s(s,T) | \mathfrak{F}_u].$$

Proof First, we have

$$A_u(u,T) - A_u(u,s) = \quad \Delta t \sum_{i=k_u}^{n} P(u,t_i) - \Delta t \sum_{i=k_u}^{j_s} P(u,t_i), = \Delta t \sum_{i=j_s+1}^{n} P(u,t_i).$$

where k_u is the next payment time after time u, and j_s is the closest payment time which is before or equal to s.

Since

$$P(u,t_i) = E[D(u,s) D(s,t_i) | \mathfrak{F}_u],$$

we have

$$\begin{aligned}
\Delta t \sum_{i=j_s+1}^{n} P(u,t_i) &= E\left[\Delta t \sum_{i=j_s+1}^{n} D(u,s)D(s,t_i)\middle| \mathfrak{F}_u\right] \\
&= E\left[D(u,s)\Delta t E_s\left[\sum_{i=j_s+1}^{n} D(s,t_i)\right]\middle| \mathfrak{F}_u\right] \\
&= E\left[D(u,s)\Delta t \sum_{i=j_s+1}^{n} P(s,t_i)\middle| \mathfrak{F}_u\right] \\
&= E\left[D(u,s)A_s(s,T)\middle| \mathfrak{F}_u\right].
\end{aligned}$$

∎

Corollary II.3.5

$$\widetilde{A}(u,s) = \widetilde{A}(u,T) - E[D(u,s)\widetilde{A}(s,T)|\mathfrak{F}_u] \qquad (s > u).$$

Proof For $s > T$, the left-hand-side equals to $\widetilde{A}(u,T)$, and the right-hand-side equals to $\widetilde{A}(u,T) - 0$. Therefore the Corollary is true. For $u > T$, both sides of the equation equal to zero. Finally, for $s < T$, the left-hand-side equals to $A(u,s)$ and the right-hand-side equals to $A_u(u,T) - E[D(u,s)A_s(s,T)|\mathfrak{F}_u]$. Applying Theorem II.3.4 proves the Corollary. ∎

With these preparations, we are now in the position to derive the PDE for $W(t)$. We note that

$$\begin{aligned}
W(t) = E\Big\{& D(t,\tau_1)[\widetilde{A}(\tau_1,\tau_2)(R_{\tau_1}(\tau_1,T) - K)^+ \\
&+ \widetilde{A}(\tau_2,T)(R_{\tau_2}(\tau_2,T) - K)^+ D(\tau_1,\tau_2)]\Big|\mathfrak{F}_t\Big\} \\
= E\Big\{& D(t,\tau_1)E\big[\widetilde{A}(\tau_1,\tau_2)(R_{\tau_1}(\tau_1,T) - K)^+ \\
&+ \widetilde{A}(\tau_2,T)(R_{\tau_2}(\tau_2,T) - K)^+ D(\tau_1,\tau_2)\big|\mathfrak{F}_{\tau_1}\big]\Big|\mathfrak{F}_t\Big\} \\
= E\Big\{& D(t,\tau_1)E\big[\widetilde{A}(\tau_1,\tau_2)(R_{\tau_1}(\tau_1,T) - K)^+\big|\mathfrak{F}_{\tau_1}\big]\Big|\mathfrak{F}_t\Big\}
\end{aligned}$$

$$+ E\Big\{D(t,\tau_1)E\big[\widetilde{A}(\tau_2,T)(R_{\tau_2}(\tau_2,T)-K)^+D(\tau_1,\tau_2)\big|\mathfrak{F}_{\tau_1}\big]\Big|\mathfrak{F}_t\Big\},$$

which can be separated into two parts as

$$W_A(t) = E\Big\{D(t,\tau_1)E\big[\widetilde{A}(\tau_1,\tau_2)(R_{\tau_1}(\tau_1,T)-K)^+\big|\mathfrak{F}_{\tau_1}\big]\Big|\mathfrak{F}_t\Big\}, \tag{II.3.5}$$

and

$$W_B(t) = E\Big\{D(t,\tau_1)E\big[\widetilde{A}(\tau_2,T)(R_{\tau_2}(\tau_2,T)-K)^+D(\tau_1,\tau_2)\big|\mathfrak{F}_{\tau_1}\big]\Big|\mathfrak{F}_t\Big\}. \tag{II.3.6}$$

II.3.1.1 PDE for $W_A(t)$

From the definition of $\widetilde{A}(t_1,t_2)$ in (II.3.2), when $\tau_2 > T$, $\widetilde{A}(\tau_1,\tau_2) = A_{\tau_1}(\tau_1,T) \not\equiv 0$. Using Corollary II.3.1 yields

$$\begin{aligned} E\big[\widetilde{A}(\tau_1,\tau_2)\big|\mathfrak{F}_{\tau_1}\big] &= I_{\tau_2>\tau_1}E_{\tau_1}\left[\int_{\tau_1}^{\infty}\widetilde{A}(\tau_1,s)\lambda(s)\hat{D}(\tau_1,s)ds\right] \\ &= E_{\tau_1}\left[\int_{\tau_1}^{\infty}\widetilde{A}(\tau_1,s)\lambda(s)\hat{D}(\tau_1,s)ds\right] \end{aligned}$$

since $\tau_2 > \tau_1$. Denote

$$l(u) = E_u\left[\int_u^{\infty}\widetilde{A}(u,s)\lambda(s)\exp\left(-\int_u^s\lambda_k dk\right)ds\right],$$

and note that $l(u) = 0$ when $u \geq T$, due to $\widetilde{A}(T,s) \equiv 0$ by the definition of $\widetilde{A}(t_1,t_2)$. With this new notation, we have

$$\begin{aligned} W_A(t) &= E\Big\{D(t,\tau_1)(R_{\tau_1}(\tau_1,T)-K)^+E\big[\widetilde{A}(\tau_1,\tau_2)|\mathfrak{F}_{\tau_1}\big]\Big|\mathfrak{F}_t\Big\} \\ &= E\Big[D(t,\tau_1)(R_{\tau_1}(\tau_1,T)-K)^+l(\tau_1)\Big|\mathfrak{F}_t\Big] \\ &= E_t\left[\int_t^T D(t,s)\lambda(s)(R_s(s,T)-K)^+l(s)ds\right]. \end{aligned}$$

Since $D(0,t)v_1(t) + \int_0^t D(t,s)\lambda(s)(R_s(s,T)-K)^+l(s)ds$ is a martingale. we obtain the PDE for $W_A(t)$ as

$$(\partial_t + \mathcal{L})W_A + \lambda(f - W_A) - rW_A = 0 \tag{II.3.7}$$

with $W_A(T, r, \lambda) = 0$, where $f = l(t)(R_t(t, T) - K)^+$ and

$$\mathcal{L} = a_1(b_1 - r)\partial_r + \frac{1}{2}\sigma_1^2 r\partial_{rr} + a_2(b_2 - \lambda)\partial_\lambda + \frac{1}{2}\sigma_2^2\lambda\partial_{\lambda\lambda} + \rho\sigma_1\sigma_2\sqrt{r\lambda}\partial_{r\lambda}.$$

By Corollary II.3.5, we have

$$\begin{aligned}
l(u) &= E_u\left[\int_u^\infty (\widetilde{A}(u,T) - D(u,s)\widetilde{A}(s,T))\lambda(s)\exp\left(-\int_u^s \lambda_k dk\right) ds\right] \\
&= E_u\left[\int_u^\infty \widetilde{A}(u,T)\lambda(s)\exp\left(-\int_u^s \lambda_k dk\right) ds\right] \\
&\quad - E_u\left[\int_u^\infty \widetilde{A}(s,T)\lambda(s)D(u,s)\exp\left(-\int_u^s \lambda_k dk\right) ds\right] \\
&= \widetilde{A}(u,T)E_u\left[\int_u^\infty \lambda(s)\exp\left(-\int_u^s \lambda_k dk\right) ds\right] \\
&\quad - E_u\left[\int_u^\infty \widetilde{A}(s,T)\lambda(s)\hat{D}(u,s)ds\right].
\end{aligned}$$

By Theorem II.3.3, we have $E_u\left[\int_u^\infty \lambda(s)\exp\left(-\int_u^s \lambda_k dk\right) ds\right] = 1$, then

$$l(u) = \widetilde{A}(u,T) - E_u\left[\int_u^\infty \widetilde{A}(s,T)\lambda(s)\hat{D}(u,s)ds\right].$$

Denote

$$h(u) = E_u\left[\int_u^\infty \widetilde{A}(s,T)\lambda(s)\hat{D}(u,s)ds\right].$$

Since $\widetilde{A}(s,T) = 0$ for $s > T$, and $\widetilde{A}(s,T) = A_s(s,T)$ for $s < T$, we have

$$h(u) = E_u\left[\int_u^\infty A_s(s,T)\lambda(s)\hat{D}(u,s)ds\right].$$

It can be verified that

$$\hat{D}(0,u)h(u) + \int_0^u \hat{D}(0,s)\lambda(s)A_s(s,T)ds$$

is a martingale, which yields the PDE for $h(u)$ as

$$(\partial_t + \mathcal{L})h + \lambda(f - h) - rh = 0 \tag{II.3.8}$$

with $h(T, r, \lambda) = 0$, where $f = A_t(t, T)$ and

$$\mathcal{L} = a_1(b_1 - r)\partial_r + \frac{1}{2}\sigma_1^2 r\partial_{rr} + a_2(b_2 - \lambda)\partial_\lambda + \frac{1}{2}\sigma_2^2\lambda\partial_{\lambda\lambda} + \rho\sigma_1\sigma_2\sqrt{r\lambda}\partial_{r\lambda}.$$

After we obtain $h(t)$, we can find $l(t)$ using $l(t) = \widetilde{A}(t, T) - h(t)$ and solve the PDE for $W_A(t)$.

II.3.1.2 PDE for of $W_B(t)$

From (II.3.6), we have

$$\begin{aligned} W_B(t) &= E\left[D(t, \tau_2)\widetilde{A}(\tau_2, T)(R_{\tau_2}(\tau_2, T) - K)^+ \middle| \mathfrak{F}_t\right] \\ &= E\left[D(t, \tau_1)E\left[\widetilde{A}(\tau_2, T)(R_{\tau_2}(\tau_2, T) - K)^+ P(\tau_1, \tau_2)\middle|\mathfrak{F}_{\tau_1}\right] \middle|\mathfrak{F}_t\right] \\ &= E\left[D(t, \tau_1)E_{\tau_1}\left[\int_{\tau_1}^T \widetilde{A}(s, T)(R_s(s, T) - K)^+\lambda(s)D(\tau_1, s)ds\right] \middle|\mathfrak{F}_t\right]. \end{aligned}$$

Here we have used Lemma II.2.1. Let

$$p(u) = E_u\left[\int_u^T \widetilde{A}(s, T)(R_s(s, T) - K)^+\lambda(s)D(u, s)ds\right].$$

When $s \leq T$, since $\widetilde{A}(s, T) = A_s(s, T)$ by definition, we can rewrite $p(u)$ as

$$p(u) = E_u\left[\int_u^T A_s(s, T)(R_s(s, T) - K)^+\lambda(s)D(u, s)ds\right].$$

Thus,

$$\begin{aligned} W_B(t) &= E\left[D(t, \tau_1)p(\tau_1)\middle|\mathfrak{F}_t\right] \\ &\quad \text{(by Lemma II.2.1)} \\ &= E_t\left[\int_t^T D(t, s)\lambda(s)p(s)ds\right] \end{aligned}$$

It can be verified that $D(0,t)g(t)+\int_0^t D(t,s)\lambda(s)p(s)ds$ is a martingale, from which we obtain the PDE for $W_B(t)$ as

$$(\partial_t + \mathcal{L})W_B + \lambda(p - W_B) - rW_B = 0 \tag{II.3.9}$$

with $W_B(T, r, \lambda) = 0$, where

$$\mathcal{L} = a_1(b_1 - r)\partial_r + \frac{1}{2}\sigma_1^2 r\partial_{rr} + a_2(b_2 - \lambda)\partial_\lambda + \frac{1}{2}\sigma_2^2\lambda\partial_{\lambda\lambda} + \rho\sigma_1\sigma_2\sqrt{r\lambda}\partial_{r\lambda}.$$

Since $p(u) = E_u\left[\int_u^T A_s(s,T)(R_s(s,T) - K)^+\lambda(s)D(u,s)ds\right]$ is defined similarly as the value in equation (II.2.9), we can derive the PDE for $p(u)$ in a similar way, which is given by

$$(\partial_t + \mathcal{L})p + \lambda(f - p) - rp = 0 \tag{II.3.10}$$

with $p(T, r, \lambda) = 0$, where

$$\mathcal{L} = a_1(b_1 - r)\partial_r + \frac{1}{2}\sigma_1^2 r\partial_{rr} + a_2(b_2 - \lambda)\partial_\lambda + \frac{1}{2}\sigma_2^2\lambda\partial_{\lambda\lambda} + \rho\sigma_1\sigma_2\sqrt{r\lambda}\partial_{r\lambda}\ ,$$

$$f = A_t(t,T)(R_t(t,T) - K)^+.$$

II.3.1.3 CCIRS Price $W(t)$

We solve $W_A(t, r, \lambda)$ using two PDEs (II.3.7)-(II.3.8) and $W_B(t, r, \lambda)$ using (II.3.9)-(II.3.10) numerically with the ADI finite difference method. We can then obtain the final CCIRS price using $W(t, r, \lambda) = W_A(t, r, \lambda) + W_B(t, r, \lambda)$.

II.3.2 Results

II.3.2.1 Verifications

We used the same parameters given in the previous section. In the Monte-Carlo simulations, we run 1,000,000 realizations and partition the 5 year to maturity into 2,000 equal parts. The computational time is 660 seconds and the price is $2,223.51.

The reliability of PDE usually lies on its convergence test. To test the stability and convergence of our PDE, we have done the convergence test in Appendix A. The results in this Appendix give us confidence that our numerical results are reliable. After this convergence test, we cover the time, interest rate, hazard rate with a $600 \times 100 \times 100$ grid and solve this PDE. The computational time is 4.07 seconds, due to the fact that we need to solve four PDEs. The CCIRS price is \$2,263.50. We can see these results are consistent to the ones from the Monte-Carlo simulation. The comparison is given in Table 5.

The PDE technique can also provide the solution on any point of the grid. We have chosen a simulated path for r and λ. Using the same time for calculating the price of CCIRS with 2-default, we quickly got the price on each annual node. The results are shown in Table 6.

II.4 CCIRS: Full Problem

In the previous section we showed that the model based on one default underestimate CCIRS price, by comparing the additional cost to the one-default price. In this section, we consider the full problem where no restriction on the number of defaults is imposed.

II.4.1 A New PDE

Without any restriction on the number of defaults, we have developed a more accurate model that gives the precise price of CCIRS. The CCIRS value at t is the expected value of all the losses after the first default $\tau_1(\tau_1 > t)$.

Case1, if $\tau_1 \geq T$, the value is 0.

Case2, if $\tau_1 < T$, at time τ_1, the new CCIRS should cover all the losses starting from τ_2, but it does not cover the loss between τ_1 and τ_2.

Now in case 2, we can discuss 2 other cases. Case2-1, if $\tau_2 \geq T$, the the loss between τ_1 and τ_2 is $f(\tau_1)$ (i.e. $A_{\tau_1}(\tau_1, T)(R_{\tau_1}(\tau_1, T) - K)^+$).

Case2-2, if $\tau_2 < T$, the the loss between τ_1 and τ_2 is $A_{\tau_1}(\tau_1, \tau_2)(R_{\tau_1}(\tau_1, T) - K)^+$. So by combining two cases, we can write

$$W(t) = E\left[D(t, \tau_1)\widetilde{W}(\tau_1, \tau_2)\middle|\mathfrak{F}_t\right] \tag{II.4.1}$$

where

$$\widetilde{W}(t_1, t_2) = \begin{cases} 0 & t_1 \geq T \ ; \\ W(t_1) + \widetilde{f}(t_1, t_2), & t_1 < T \ , \end{cases} \tag{II.4.2}$$

and

$$\widetilde{f}(t_1, t_2) = \begin{cases} A_{t_1}(t_1, T)(R_{t_1}(t_1, T) - K)^+, & t_1 < T \leq t_2 \ ; \\ A_{t_1}(t_1, t_2)(R_{t_1}(t_1, T) - K)^+ & t_1 < t_2 < T \ . \end{cases} \tag{II.4.3}$$

Or simply rewrite the formula of $\widetilde{W}(t_1, t_2)$ using the notation in equation (II.3.2),

$$\widetilde{W}(t_1, t_2) = \begin{cases} 0 & t_1 \geq T \ ; \\ W(t_1) + \widetilde{A}(t_1, t_2)(R_{t_1}(t_1, T) - K)^+ & t_1 < T \ . \end{cases} \tag{II.4.4}$$

So the way to get the PDE should be the same complexity as the 2-default case following the similar procedures.

$$(\partial_t + \mathcal{L})W + \lambda f - rW = 0 \tag{II.4.5}$$

with $W(T) = 0$, where $f = l(t)(R_t(t, T) - K)^+$. $l(t)$ is defined in previous section as:

$$l(u) = E_u\left[\int_u^\infty \widetilde{A}(u, s)\lambda(s)\exp\left(-\int_u^s \lambda_k dk\right) ds\right].$$

Solving $l(u)$ has been done as well. $l(t) = \widetilde{A}(t, T) - h(t)$ and PDE for $h(t)$ is

$$(\partial_t + \mathcal{L})h + \lambda(f - h) - rh = 0 \tag{II.4.6}$$

with $h(T, r, \lambda) = 0$, where $f = A_t(t, T)$.

II.4.2 Numerics

To compare the prices of the one-default, two-default and full model (ie., the model without restriction on the default numbers), the parameters have been chosen the same as previous tests. The results are shown in the Table 7. The PDE solution of the difference of full model price and two-default prices for different initial interest rate and hazard rate is in Figure 2 and Figure 3. The PDE solution of the difference of two-default and one-default prices for different initial interest rate and hazard rate is in Figure 3. On top of that, we have also shown the difference of these prices with respect to different correlations between the interest rate and hazard rate in Table 8.

These comparisons provide us the first impression that the second and more defaults can have non-trivial effect on the CCIRS price. In the next section, we will more clear knowledge about the second and more default impact by applying a thorough risk exposure analysis.

II.5 Risk Exposure Analysis

In actual markets, most exposures of the credit products come from the credit risk exposure, notional exposure and maturity exposure. Certainly, the interest rate exposure should be included in CCIRS pricing model. Since the notion is a constant and proportional to the price of CCIRS, only the other three kinds of mentioned exposures will be examined by comparing prices under two different models in this section.

From the simulation results, we found Monte Carlo method can only provide a large range of results. This instability may have impact on the risk exposure analysis and leads to a wrong conclusion. In this section, only PDEs technique has been applied and achieved a great results.

First we have weighed the impact of hazard rate model parameters of different ratings on one, two defaults and the full models. We used the parameters in Table 2 and Table 3 to achieve more reasonable results. Maturity is now 10 years, $\rho = 0.2$ and $K = 1\%$. The values of three different models are shown in Table 9. We can see if the counterparty has a high credit ratings (AAA or AA), these three prices in Table 9 are very close. If the counterparty has a medium credit ratings (A or BBB), the difference between these three prices are small but not noticeable. If the counterparty has a low credit ratings (BB or B), the difference between these three prices are huge and cannot be ignored. We have a first insight of these three models and how they behaved with respect to different ratings.

The second test we have done is the impact of the long term average hazard rate on the prices of three models, under the assumption that the initial hazard rate equals the long term average hazard rate. The results can be can be observed clearly in Figure 4. We found when the long term average hazard rate is larger, the value is larger. This can be easily understood because when the hazard rate is higher, the two-default model captures more exposures and the full model without the restriction on the default numbers captures the real exposures. More risk exposures means higher prices.

From the previous two test, since the difference of three values is generally very small for high rating counterparties, it is hard to test the impact of other parameters on the comparison ratios if we choose AAA and AA as the testing grade. Among the other four credit ratings, A is always used as a standard testing rating and the testing of B rating will generate the most visible impact on the difference. So in the rest of the tests, we used the parameters close to the rating A and B to see how the parameters can affect the values of three models and their comparison ratios.

In the next step, the impact on CCIRS price of reverting speed and volatility of hazard rate on three model prices have been tested as well. The prices comparison from rating A and B are shown in Table 10 and Table 12. The ratios comparison of two-default model price/one-default model price and the full model price/one-default model price from rating A and B are shown in Figure 5 and Figure 7. The parameters for interest rate are from Table 2. The results show the values and the ratios are not that sensitive to the hazard rate volatility and reverting speed. Combined this new results and the results from the first and second test above, we can say that the value of hazard rate itself plays a major role in affecting the ratio of the two default model price and the full model price.

In the second groups of tests, the impact of the interest rate on the three model prices has been assessed. In the first test, the initial and long term average interest rate is ranging from 0.4% to 2%, a_1 is chosen to be 0.5, 1, 1.5 and 2 respectively. The other parameters are chosen as rating A and B parameters in Table 3. The notional value is \$250,000,000, $\rho = 0.2$, Maturity is 5 years and $K = 0.5\%$. The different model prices with different chosen parameters have been put in Table 14$\sim$21. From these results, we find when the interest rate is larger, all three values increase quickly. This behavior can be easily explained. The CCIRS price mainly relies on the excessive amount of the new fixed rate over the old one. From equation (II.2.6), it is easy to prove that if the interest rate is constant, the reasonable fixed rate of the swap is very close to the interest rate. So when the long term average interest rate is larger, it has higher probability to sign a new contract with a higher fixed rate, i.e. the value of CCIRS is higher.

The comparison of ratios of two-default value/one-default price and the full value/one-default price are shown in Figure 9$\sim$16. The observations from these results are more complicated. Actually there are three factors affecting the ratios.

These three factors are number of defaults, reverting speed of interest rate and the trend of interest rate change. Because of the complexity of these effects from the three factors, we discuss them in two cases.

In the first case when the long term average is smaller than K, the number of defaults and the reverting speed dominates the ratios. The reason for this domination is intuitive. When the interest rate is low, it is less likely to sign a new fixed rate higher than K. So on the one hand more defaults actually provides more chances of signing a higher fixed rate. So in this case more defaults mean higher price of CCIRS. On the other hand when the reverting speed of interest rate is larger, the interest rate will concentrate more around the average. So when the average is smaller than K, lower reverting speed can provide more randomness of the interest rate to be higher than K. Then again, more defaults means higher price.

In the second case when the long term average is above K, the effect of number of defaults becomes less important. Because when average is already higher than K, the fixed rate signed at the first default time is likely to be more than K, so more defaults did not provide much more chances of signing higher fixed rate. Especially when the reverting speed is large, it is very obvious that all values become close to each other when the average rate is around and above K.

The other observation we found is for very large average rate, the two-default value can be even smaller and the full value is the smallest. To better understand this phenomenon, we have run two new tests.

We wanted to test the three model prices when the interest rate will gradually increase or gradually decrease. So in the first new test, we let the initial interest rate be 0.549% and long term average rate ranges from 0.6% to 5%. In the second new test, we let the initial interest rate be 4% and long term average rate ranges from 0.4% to 2%. The reverting speed of interest rate is 0.1. The parameters of hazard rate model is chosen as the parameters for rating B. $K = 1\%$, maturity is 10 years and $\rho = 0.2$. The results are shown in Figure 18 and Figure 20. In test 1 where the interest rate gradually increases, the newer fixed rate after each default will likely be larger. So intuitively the two-default value is larger and the full value is the largest, as proved by the results in Figure 18. However, when the interest rate gradually decreases, the newer fixed rate after each default will likely be lower. In this case, the CCIRS holder can actually take benefit from more defaults. Because the two-default price can be lower and full price is the lowest.

In the next test, the effect of the volatility of interest rate is tested as well and the three model prices are in Table 22 and Table 23. The ratios comparison of two-default model price/one-default model price and full model price/one-default model price from rating A and B are shown in Figure 21 and Figure 22. The other parameters are chosen as in Table 2 and Table 3. $K = 1\%$, $\rho = 0.2$ and maturity is 10 years. The results reflect that volatility provides higher chance of signing the new fixed rate higher than K and pushes the value higher. And in this case when the average rate is close to K, more possible defaults actually brings more chances of signing higher fixed rate as well. The combination of these two effects leads to the higher ratio of two defaults value and full model value to one default value when the volatility is higher.

The last test studies the maturity exposure in three different default model. The maturity is chosen from 5 years to 15 years. The initial and long term average interest rate are 4% and the initial and long term average hazard rate are 10%. The results shown in Table 24 and Table 25. The ratios comparison of the two-default model price/one-default model price and the full model price/one-default model price from rating A and B are shown in Figure 23 and Figure 24. The results follow the intuition as well. More possible defaults will happen with longer maturity. When the average interest rate is around the K, more defaults lead to greater risk of getting higher fixed rate than one default.

After these analyses of risk exposures, it is reasonable to conclude that the effect of the possible second default or more defaults in CCIRS pricing model can not be neglected, especially when the risk of default is not low and the interest rate floats around the original fixed swap rate, in which case the sensitive impact of its randomness greatly affects the two-default price. The two-default model might be good enough for the pricing of rating A default risk. But for counterparties with higher default risks, the full model captures the correct risk.

II.6 Chapter Conclusion

In this chapter, we investigated the importance of additional defaults in the pricing of CCIRS. We compared the results using both the Monte-Carlo simulation and PDEs based methods. As the PDE approach is computationally more efficient, it allows us to carry out extensive risk exposure analysis. Our results indicate that the risk due to the default of the replacement contract in an interest rate swap is significant. Therefore, the assumption of a default-free replacement contract may overestimate or underestimate CCIRS price and the risk exposure due to counterparty default. The CVA of the second and subsequent defaults can only be ignored when the credit risk is very low, i.e. when the counterparty has a high credit rating with short maturity. In addition, our results also show that the interest rate environment and maturity of the contract play important roles in the risk composition of the final CCIRS price.

One shortcoming of the two-default model is that the effect of subsequent defaults is assumed to be small and consequently are not modelled. To overcome this shortcoming, we have considered the pricing problem with no restriction on the number of defaults. The price computed using this full model captures the real or true cost of the counterparty risk and can be used as a benchmark when comparing the prices computed using the one-default and two-default model, both are approximations.

III An Analytical Value at Risk (VaR) Approach for Credit Portfolio with Liquidity Horizon and Portfolio Rebalancing

Current Incremental Risk Charge (IRC) calculation relies on time consuming Monte Carlo simulation (MC), so the banks still use more efficient Asymptotic Single Risk Factor (ASRF) and its Granularity Adjustment (GA) as important measures for the effective capital management for banks. In order to achieve a risk sensible comparison, we provide a general framework of two-period conditional VaR model in the context of IRC modeling framework in which the liquidity horizon and constant level of risk are considered. Compared to the original ASRF model, two-period conditional VaR model is more practical from the risk point of view when liquidity risk is added. So finding a proper analytic solution is a very valuable research work. By borrowing the GA technique, we will then present an analytic approach to the two-period conditional VaR in this chapter. On top of that, we will also provide an exact solution to VaR in the infinite-period model.

III.1 General Framework of two period conditional VaR model

III.1.1 ASRF model for Credit Portfolio and the GA approximation

The general framework of ASRF and its GA are presented in this section. For a credit portfolio, the loss function within a one-factor modeling framework is defined as

$$L_N = \sum_{i=1}^{N} u_i \mathbf{I}_{\{X_i > U_i\}} \ , \tag{III.1.1}$$

$$X_i = \rho_i S + \sqrt{1-\rho_i^2}\xi_i \ , \tag{III.1.2}$$

where L_N is the portfolio loss, u_i is the loss given default of the ith asset, ξ_i is the idiosyncratic factor, ρ_i is the positive correlation between asset factor X_i and systematic factor S, and U_i is the threshold to determine if the default of the ith trade will happen. S and all the ξ_i are assumed to be i.i.d Gaussian variables $N(0,1)$.

Denote $\alpha_q(\cdot)$ as the q percentile value of random variable. i.e.,

$$P(X \le \alpha_q(X)) = q \ . \tag{III.1.3}$$

Then the q percentile VaR of this portfolio is denoted $\alpha_q(L_N)$, which can be found by MC.

Although there is no direct analytic solution for the q percentile VaR, its approximation can be calculated in ASRF. In ASRF, the most important assumption is that when the portfolio is large enough, the individual risk of each trade will be diversified away. With this assumption and by the law of large numbers, Gordy

(2003) [20] demonstrated:

$$L_N \to E(L_N|S), \quad \text{a.s.} \tag{III.1.4}$$

This means $\alpha_q(L_N) \approx \alpha_q[E(L_N|S)]$. If the conditional expectation of loss function $f(s) = E[L_N|S = s]$ is monotonic,which is the assumption of most models, we have $\alpha_q[E(L_N|S)] = E(L_N|\alpha_q(S))$ because of the monotonic property of $\alpha_q(\cdot)$. If the loss function is defined as in (III.1.1), we have

$$\begin{aligned}\alpha_q(L_N) &\approx \alpha_q[E(L_N|S)] \\ &= \alpha_q\left[\sum_{i=1}^{N} u_i\left(1-\Phi\left(\frac{U_i-\rho_i S}{\sqrt{1-\rho_i^2}}\right)\right)\right] \\ &= \sum_{i=1}^{N} u_i\left(1-\Phi\left(\frac{U_i-\rho_i\alpha_q(S)}{\sqrt{1-\rho_i^2}}\right)\right)\end{aligned} \tag{III.1.5}$$

This is how the capital requirement calculation (IRB approach) is implemented based on the ASRF assumption. In practice, however, the infinitely fine grained portfolio does not exist, so there is a difference between VaR (i.e. $\alpha_q(L_N)$) and $E(L_N|\alpha_q(S))$. The summation of $E(L_N|\alpha_q(S))$ and the difference of $\alpha_q(L_N)$ and $\alpha_q[E(L_N|S)]$ is considered to be the new VaR. This calculation of the difference is the main goal of GA.

The key method of GA proposed by Gordy (2003) [20] was the second order Taylor expansion. We know

$$\alpha_q(L_N) = \alpha_q[E(L_N|S) + \varepsilon(L_N - E(L_N|S))]|_{\varepsilon=1} \ .$$

Let $z(\varepsilon) = \alpha_q[E(L_N|S) + \varepsilon(L_N - E(L_N|S))]$; applying the second order Taylor expansion on $\varepsilon = 0$, we have,

$$z(\varepsilon) \approx z(0) + z'(0)\varepsilon + z''(0)\frac{\varepsilon^2}{2} \ . \tag{III.1.6}$$

$$
\begin{aligned}
&\alpha_q[E(L_N|S) + \varepsilon(L_N - E(L_N|S))]|_{\varepsilon=1} \\
=&z(1) \\
\approx&z(0) + z'(0) \cdot 1 + z''(0) \cdot \frac{1^2}{2} \\
=&\alpha_q[E(L_N|S)] + \frac{\partial \alpha_q}{\partial \varepsilon}[E(L_N|S) + \varepsilon(L_N - E(L_N|S))]|_{\varepsilon=0} \\
+&\frac{1}{2}\frac{\partial^2 \alpha_q}{\partial \varepsilon^2}[E(L_N|S) + \varepsilon(L_N - E(L_N|S))]|_{\varepsilon=0} \ .
\end{aligned} \tag{III.1.7}
$$

Then the value of GA, the difference between $\alpha_q(L_N)$ and $\alpha_q[E(L_N|S)]$, is approximately the sum of the first and the second derivatives. Wilde (2001) [44] proved that GA can be expressed as

$$
\mathrm{GA}_N = -\frac{1}{2h(\alpha_q(S))}\frac{d}{dx}\left(\frac{\sigma^2(L_N|S=x)h(x))}{\frac{dE(L_N|S=x)}{dx}}\right)\Bigg|_{x=\alpha_q(S)} , \tag{III.1.8}
$$

where h is the density function of the systematic risk factor S. The detailed expression of $h(x), E(L_N|S=x)$ and $\sigma^2(L_N|S=x)$ depend on the chosen model.

III.1.2 Analytical VaR with Liquidity Horizon and Portfolio Rebalancing

III.1.2.1 Two-period Credit Portfolio VaR measure and its ASRF and GA terms

The two-period credit portfolio valuation and loss model will be described in this section. In order to model the portfolio rebalancing within one year time horizon, we divide the one year horizon into two half-year periods. In the first half, the credit portfolio follows standard factor model as outlined in III.1.1. Then at the end of the first period, the portfolio can be rebalanced according to what happens in the first period. For example, if one asset defaults in the first period, we choose to replace it with a similar asset of the same LGD and rating (i.e. same default probability and asset correlation). Then we can say that at the end of six month, the portfolio is replenished such that it maintains a constant level of risk. For some assets, we can also assume that no action is needed. From the default risk perspective, this has the embedded assumption that this asset has a liquidity horizon of one year. In this chapter, we assume that all assets have a six month liquidity horizon. Therefore, we need to model the losses aggregated in two periods with the losses in the second period conditional on the portfolio rebalancing assumptions.

Let S_1, S_2 be the realizations of the systematic factor in the end of the first and second period. They are assumed to be independent. Similar to one-period default model (III.1.1), the two-step one-factor default model is:

$$L_N = \sum_{i=1}^{N} \left[u_i \mathbf{I}_{\{T_1^{(i)} > U_i\}} + u_i \mathbf{I}_{\{T_2^{(i)} > U_i\}} \right], \tag{III.1.9}$$

where:

$$T_1^{(i)} = \rho_i S_1 + \sqrt{1 - \rho_i^2} \xi_i \ , \tag{III.1.10}$$

$$T_2^{(i)} = \rho_i S_2 + \sqrt{1 - \rho_i^2} \xi_i' \ , \tag{III.1.11}$$

u_i is the loss given default of asset i, all the ξ_i and ξ_i' are the idiosyncratic factors which are independent across each other and across each systematic factor S_1, S_2 and ρ_i is the positive correlation between asset factor $T_1^{(i)}$ and systematic factor S_1. It is the same as the correlation between asset factor $T_2^{(i)}$ and systematic factor S_2 since the trade has the same behavior as the trade in the first period no matter it defaults or not. U_i is the threshold to determine if the default of the ith trade will happen. S_1, S_2 and all the ξ_i and ξ_i' are assumed to be Gaussian distributed variables $N(0, 1)$.

Similar to the standard GA, the second order Taylor expansion can be applied.

Rewrite $L_N = \alpha_q[E(L_N|S_1, S_2)] + \varepsilon[L_N - E(L_N|S_1, S_2)]|_{\varepsilon=1}$.

Use Taylor expansion and proceed as (III.1.7):

$$\begin{aligned}\alpha_q(L_N) &= \alpha_q[E(L_N|S_1, S_2)] + \varepsilon[L_N - E(L_N|S_1, S_2)]|_{\varepsilon=1} \\ &\approx \alpha_q[E(L_N|S_1, S_2)] + \frac{\partial \alpha_q}{\partial \varepsilon}[E(L_N|S_1, S_2) + \varepsilon(L_N - E(L_N|S_1, S_2))]|_{\varepsilon=0} \\ &+ \frac{1}{2}\frac{\partial^2 \alpha_q[E(L_N|S_1, S_2) + \varepsilon(L_N - E(L_N|S_1, S_2))]}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} \quad . \end{aligned} \tag{III.1.12}$$

If we can calculate the values of the three components of the sum in (III.1.12), we will know the VaR of this portfolio. The summation of the first and the second derivative is the value of GA.

III.1.2.2 Two-period "ASRF" Term in Equation (III.1.12)

With the formula of L_N in (III.1.9), $E(L_N|S_1, S_2)$ is calculated as:

$$\begin{aligned}E(L_N|S_1, S_2) &= \sum_{i=1}^{N}\left\{u_i E[\mathbf{I}_{\{T_1^{(i)}>U_i\}}|S_1, S_2] + u_i E[\mathbf{I}_{\{T_2^{(i)}>U_i\}}|S_1, S_2]\right\} \\ &= \sum_{i=1}^{N}\left\{u_i[1 - \Phi(\frac{U_i - \rho_i S_1}{\sqrt{1-\rho_i^2}})]\right\} \\ &+ \sum_{i=1}^{N}\left\{u_i[1 - \Phi(\frac{U_i - \rho_i S_2}{\sqrt{1-\rho_i^2}})]\right\} \quad , \end{aligned} \tag{III.1.13}$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distributed variable. Denote

$$X = E(L_N|S_1, S_2) \ , \tag{III.1.14}$$

$$l(s) = \sum_{i=1}^{N}\left\{u_i[1 - \Phi(\frac{U_i - \rho_i s}{\sqrt{1-\rho_i^2}})]\right\} \quad . \tag{III.1.15}$$

So (III.1.13) can be simply written as

$$X = l(S_1) + l(S_2) \ . \tag{III.1.16}$$

Since ρ_i is always positive, it is obvious that $l(s)$ is a strictly monotonically increasing function and

$$\lim_{s \to -\infty} l(s) = 0 \ , \tag{III.1.17}$$

$$\lim_{s \to +\infty} l(s) = \sum_{i=1}^{N} u_i \ . \tag{III.1.18}$$

To simplify the rest narrations in this chapter, the limit of any function $f(\cdot)$ will be rewritten as:

$$f(\pm\infty) = \lim_{x \to \pm\infty} f(x) \ . \tag{III.1.19}$$

So the limits of function l are rewritten as:

$$l(-\infty) = 0 \ , \tag{III.1.20}$$

$$l(+\infty) = \sum_{i=1}^{N} u_i \ . \tag{III.1.21}$$

To calculate $\alpha_q(X)$, the cumulative distribution function of X needs to be calculated, which is

$$F_X(t) = P(X \leq t) \ . \tag{III.1.22}$$

From (III.1.17) and (III.1.18), function $l(s)$ is bounded between 0 and $\sum_{i=1}^{N} u_i$. So

$$F_X(t) = \begin{cases} 0, & \text{if } t \leq 0 \ ; \\ 1, & \text{if } t \geq 2\sum_{i=1}^{N} u_i \ . \end{cases} \tag{III.1.23}$$

For any $0 < t < 2\sum_{i=1}^{N} u_i$,

$$
\begin{aligned}
& X \le t \\
\Leftrightarrow\ & l(S_1) + l(S_2) \le t \\
\Leftrightarrow\ & \begin{cases} \{S_2 \le l^{-1}(t - l(S_1)), S_1 > l^{-1}(t - \sum_{i=1}^{N} u_i)\} & \\ \qquad\qquad \bigcup \{S_2 \in \mathbb{R}, S_1 \le l^{-1}(t - \sum_{i=1}^{N} u_i)\}, & t \ge \sum_{i=1}^{N} u_i\ ; \\ \{S_2 \le l^{-1}(t - l(S_1)), S_1 < l^{-1}(t)\}, & t < \sum_{i=1}^{N} u_i\ . \end{cases}
\end{aligned}
\tag{III.1.24}
$$

The $F_X(t)$ in both cases can be calculated given the systematic factor S_1 and S_2 are normally distributed and uncorrelated. The density function of X will be calculated here for future use.

(i) In the first case, i.e. $2\sum_{i=1}^{N} u_i > t \ge \sum_{i=1}^{N} u_i$,

$$
F_X(t) = \int_{l^{-1}(t - \sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\Phi(l^{-1}(t - l(s_1)))ds_1 + \Phi(l^{-1}(t - \sum_{i=1}^{N} u_i))\ , \tag{III.1.25}
$$

where $\phi(\cdot)$ is the density function of the standard normal random variable. Note when $t = \sum_{i=1}^{N} u_i$, $F_X(t) = \int_{-\infty}^{+\infty} \phi(s_1)\Phi(l^{-1}(t - l(s_1)))ds_1$. This does not violate the formula of $F_X(t)$ in (III.1.25) with the notation (III.1.19), i.e. $\Phi(-\infty) = \lim_{x\to\infty} \Phi(x) = 0$.

To calculate $f_X(t)$, the following propositions are required.

Proposition III.1.1 *let* $F(t) = \int_{b(t)}^{a(t)} f(t,x)dx$, *where* $f(t,x) \in C(D)$, $\frac{\partial f(t,x)}{\partial t} \in C(D)$, $D = \{(x,t)|x \in [\alpha,\beta], t \in [m,n]\}$, $C(D)$ *is the set of all continuous functions* *on* D. $a'(t)$ *and* $b'(t)$ *exist when* $t \in [m, n]$. *And* $\alpha \le a(t) \le \beta, \alpha \le b(t) \le \beta$. *then*

$$F'(t) = f(t, a(t))a'(t) - f(t, b(t))b'(t) + \int_{b(t)}^{a(t)} \frac{\partial f(t, x)}{\partial t} dx \ . \tag{III.1.26}$$

Proposition III.1.2

$$\frac{dl^{-1}(t)}{dt} = \frac{1}{l'(l^{-1}(t))} \ . \tag{III.1.27}$$

Then use Proposition III.1.1 and Proposition III.1.2 in the calculation of $f_X(t)$.

When $t \neq \sum_{i=1}^{N} u_i$,

$$\begin{aligned}
f_X(t) &= F'_X(t) \\
&= -\phi(l^{-1}(t - \sum_{i=1}^{N} u_i))\Phi(l^{-1}(\sum_{i=1}^{N} u_i))\frac{1}{l'(l^{-1}(t - \sum_{i=1}^{N} u_i))} \\
&+ \int_{l^{-1}(t - \sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1)))\frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 \\
&+ \phi(l^{-1}(t - \sum_{i=1}^{N} u_i))\frac{1}{l'(l^{-1}(t - \sum_{i=1}^{N} u_i))} \\
&(\text{Since } \Phi(l^{-1}(\sum_{i=1}^{N} u_i)) = \Phi(+\infty) = 1) \\
&= -\phi(l^{-1}(t - \sum_{i=1}^{N} u_i))\frac{1}{l'(l^{-1}(t - \sum_{i=1}^{N} u_i))} \\
&+ \int_{l^{-1}(t - \sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1)))\frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 \\
&+ \phi(l^{-1}(t - \sum_{i=1}^{N} u_i))\frac{1}{l'(l^{-1}(t - \sum_{i=1}^{N} u_i))} \\
&= \int_{l^{-1}(t - \sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1)))\frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 \ .
\end{aligned} \tag{III.1.28}$$

When $t = \sum_{i=1}^{N} u_i$,

$$\begin{aligned} f_X(t) &= F'_X(t) \\ &= \int_{-\infty}^{+\infty} \phi(s_1)\phi(l^{-1}(t-l(s_1)))\frac{1}{l'(l^{-1}(t-l(s_1)))}ds_1 \\ &\text{(since } l^{-1}(0) = -\infty) \\ &= \int_{l^{-1}(t-\sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\phi(l^{-1}(t-l(s_1)))\frac{1}{l'(l^{-1}(t-l(s_1)))}ds_1 \ . \end{aligned} \tag{III.1.29}$$

So in case (i),

$$f_X(t) = \int_{l^{-1}(t-\sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\phi(l^{-1}(t-l(s_1)))\frac{1}{l'(l^{-1}(t-l(s_1)))}ds_1 \ . \tag{III.1.30}$$

(ii) In the second case, i.e. $0 < t < \sum_{i=1}^{N} u_i$,

$$F_X(t) = \int_{-\infty}^{l^{-1}(t)} \phi(s_1)\Phi(l^{-1}(t-l(s_1)))ds_1 \ , \tag{III.1.31}$$

Similarly, use Proposition III.1.1 and Proposition III.1.2 to calculate $f_X(t)$.

$$\begin{aligned} f_X(t) &= \phi(l^{-1}(t))\Phi(l^{-1}(t-t))\frac{1}{l'(l^{-1}(t))} \\ &+ \int_{-\infty}^{l^{-1}(t)} \phi(s_1)\phi(l^{-1}(t-l(s_1)))\frac{1}{l'(l^{-1}(t-l(s_1)))}ds_1 \\ &\text{(Since } \Phi(l^{-1}(t-t)) = \Phi(-\infty) = 0) \\ &= \int_{-\infty}^{l^{-1}(t)} \phi(s_1)\phi(l^{-1}(t-l(s_1)))\frac{1}{l'(l^{-1}(t-l(s_1)))}ds_1 \ . \end{aligned} \tag{III.1.32}$$

Following are the values of $f_X(t)$ on two boundaries in both cases. In case (i) when $t = \sum_{i=1}^{N} u_i$,

$$f(t) = \int_{-\infty}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1))) \frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 \ , \tag{III.1.33}$$

and in case (ii) when $t \to \sum_{i=1}^{N} u_i$,

$$\lim_{t \to \sum_{i=1}^{N} u_i} f(t) = \int_{-\infty}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1))) \frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 \ . \tag{III.1.34}$$

In case (i) when $t \to 2\sum_{i=1}^{N} u_i$,

$$\lim_{t \to 2\sum_{i=1}^{N} u_i} f_X(t) = \int_{+\infty}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1))) \frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 = 0 \ , \tag{III.1.35}$$

and in case (ii) when $t \to 0$,

$$\lim_{t \to 0} f_X(t) = \int_{-\infty}^{-\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1))) \frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1 = 0 \ . \tag{III.1.36}$$

The two conditions $(III.1.33) = (III.1.34)$ and $(III.1.35) = (III.1.36) = 0$ give the continuity of the density function for all $t \in \mathbb{R}$.

Then

$$f_X(t) = \begin{cases} \int_{l^{-1}(t - \sum_{i=1}^{N} u_i)}^{+\infty} \phi(s_1)\phi(l^{-1}(t - l(s_1))) \frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1, & \text{if } 2\sum_{i=1}^{N} u_i > t \geq \sum_{i=1}^{N} u_i \ ; \\ \int_{-\infty}^{l^{-1}(t)} \phi(s_1)\phi(l^{-1}(t - l(s_1))) \frac{1}{l'(l^{-1}(t - l(s_1)))} ds_1, & \text{if } \sum_{i=1}^{N} u_i > t \geq 0 \ ; \\ 0, & \text{otherwise} \ . \end{cases} \tag{III.1.37}$$

To simplify the expression of $f_X(t)$, define a function $g_t(s_1)$ on

$$\left\{ (t, s_1) \middle| (t, s_1) \in \left[\sum_{i=1}^{N} u_i, 2\sum_{i=1}^{N} u_i \right) \times \left(l^{-1}(t - \sum_{i=1}^{N} u_i), +\infty \right) \right.$$

$$\bigcup \left[0, \sum_{i=1}^{N} u_i\right) \times \left(-\infty, l^{-1}(t)\right)\Bigg\} \tag{III.1.38}$$

as

$$g_t(s_1) = l^{-1}(t - l(s_1)). \tag{III.1.39}$$

Since $l(x)$ is a strictly monotonically increasing function, $g_t(s_1)$ is a strictly monotonically decreasing function w.r.t s_1.

Define an interval $\Omega(t)$ as:

$$\Omega(t) = \begin{cases} (l^{-1}(t - \sum_{i=1}^{N} u_i), +\infty), & \text{Case 1: } \; 2\sum_{i=1}^{N} u_i > t \geq \sum_{i=1}^{N} u_i \; ; \\ (-\infty, l^{-1}(t)), & \text{Case 2: } \; \sum_{i=1}^{N} u_i > t > 0 \; . \end{cases} \tag{III.1.40}$$

Then

$$f_X(t) = \begin{cases} \int_{\Omega(t)} \phi(s_1)\phi(g_t(s_1))(l'(g_t(s_1)))^{-1} ds_1, & \text{if } 2\sum_{i=1}^{N} u_i > t \geq 0 \; ; \\ 0, & \text{otherwise} \; . \end{cases} \tag{III.1.41}$$

Then using (III.1.25) and (III.1.31) to solve the $\alpha_q(X)$ numerically from equation

$$F_X(\alpha_q(X)) = q \; . \tag{III.1.42}$$

III.1.2.3 Calculate the Second Term in Equation (III.1.12)

Define $Y = L_N - E(L_N|S_1, S_2)$, then the first order term can be rewritten as

$$\left.\frac{\partial \alpha_q(X + \varepsilon Y)}{\partial \varepsilon}\right|_{\varepsilon=0} \; . \tag{III.1.43}$$

Here Rau-Bredow(2002) [36] proved the following theorem.

Theorem III.1.3 *Consider two random variables X and Y with a joint density function $f(x,y)$ and $\alpha_q(\cdot)$ is defined in the same way as in this chapter. Then:*

$$\frac{\partial \alpha_q(X+\varepsilon Y)}{\partial \varepsilon} = E\left[Y | X+\varepsilon Y = \alpha_q(X+\varepsilon Y)\right] \ , \tag{III.1.44}$$

$$\frac{\partial^2 \alpha_q(X+\varepsilon Y)}{\partial \varepsilon^2} = \tag{III.1.45}$$

$$- \left[\frac{\partial \sigma^2(Y|X+\varepsilon Y = s)}{\partial s} + \sigma^2(Y|X+\varepsilon Y = s)\frac{\partial ln f_{X+\varepsilon Y}(s)}{\partial s}\right]_{s=\alpha_q(X+\varepsilon Y)}, \tag{III.1.46}$$

where $f_{X+\varepsilon Y}(s)$ is the density function of $X+\varepsilon Y$ and $\sigma^2(Y|X+\varepsilon Y = s)$ is the conditional variance of Y.

Using Theorem III.1.3:

$$\begin{aligned}
\left.\frac{\partial \alpha_q(X+\varepsilon Y)}{\partial \varepsilon}\right|_{\varepsilon=0} &= E\left[Y|X = \alpha_q(X)\right] \\
&= E[L_N - E(L_N|S_1,S_2)|E(L_N|S_1,S_2) = \alpha_q(E(L_N|S_1,S_2))] \\
&= E[L_N|E(L_N|S_1,S_2) = \alpha_q(E(L_N|S_1,S_2))] \\
&- E[E(L_N|S_1,S_2)|E(L_N|S_1,S_2) = \alpha_q(E(L_N|S_1,S_2))] \ .
\end{aligned} \tag{III.1.47}$$

Since $E(L_N|S_1,S_2)$ is $\sigma(S_1,S_2)$-measurable, and

$$\sigma(E(L_N|S_1,S_2) = \alpha_q(E(L_N|S_1,S_2))) \subseteq \sigma(S_1,S_2),$$

by tower property,

$$\begin{aligned}
&E[E(L_N|S_1,S_2)|E(L_N|S_1,S_2) = \alpha_q(E(L_N|S_1,S_2))] \\
&= E[L_N|E(L_N|S_1,S_2) = \alpha_q(E(L_N|S_1,S_2))] \ .
\end{aligned} \tag{III.1.48}$$

So

$$\left.\frac{\partial\alpha_q(X+\varepsilon Y)}{\partial\varepsilon}\right|_{\varepsilon=0} = 0 \ . \tag{III.1.49}$$

It is elaborate that the first derivative of two-period GA, as well as the first derivative of one-period GA, are both 0.

III.1.2.4 Calculate the Third Term in Equation (III.1.12)

Using Theorem III.1.3:

$$\begin{aligned}
&\left.\frac{1}{2}\frac{\partial^2\alpha_q[E(L_N|S_1,S_2)+\varepsilon(L_N-E(L_N|S_1,S_2))]}{\partial\varepsilon^2}\right|_{\varepsilon=0} \\
&= \left.\frac{1}{2}\frac{\partial^2\alpha_q(X+\varepsilon Y)}{\partial\varepsilon^2}\right|_{\varepsilon=0} \\
&= -\frac{1}{2}\left[\frac{\partial\sigma^2(Y|X=s)}{\partial s}+\sigma^2(Y|X=s)\frac{dlnf_X(s)}{ds}\right]_{s=\alpha_q(X)} \ .
\end{aligned} \tag{III.1.50}$$

First, the value of $\sigma^2(Y|X=s)$ is required:

$$\begin{aligned}
\sigma^2(Y|X=s) &= \sigma^2(L_N-X|X=s) \\
&= \sigma^2(L_N-s|X=s) \\
&= \sigma^2(L_N|X=s) \\
&= E(L_N^2|X=s)-E^2(L_N|X=s) \ .
\end{aligned} \tag{III.1.51}$$

Then $E(L_N|X=s)$ and $E(L_N^2|X=s)$ are calculated for the value of $\sigma^2(Y|X=s)$ in section III.1.2.4.1 and III.1.2.4.2.

Second, the derivative of the variance (i.e. $\frac{\partial\sigma^2(Y|X=s)}{\partial s}$) is calculated in section III.1.2.4.3.

Finally, $\frac{dlnf_X(s)}{ds}$ is calculated in section III.1.2.4.4.

III.1.2.4.1 Calculate $E(L_N|X=s)$

Please recall the loss function L_N is defined as

$$L_N = \sum_{i=1}^{N} \left[u_i \mathbf{I}_{\{T_1^{(i)} > U_i\}} + u_i \mathbf{I}_{\{T_2^{(i)} > U_i\}} \right] , \tag{III.1.52}$$

where:

$$T_1^{(i)} = \rho_i S_1 + \sqrt{1 - \rho_i^2} \xi_i , \tag{III.1.53}$$

$$T_2^{(i)} = \rho_i S_2 + \sqrt{1 - \rho_i^2} \xi_i' , \tag{III.1.54}$$

u_i is the loss given default of asset i, all the ξ_i and ξ_i' are the idiosyncratic factors which are independent across each other and across each systematic factor S_1, S_2 and ρ_i is the correlation between asset factor $T_1^{(i)}$ and systematic factor S_1. It is the same as the correlation between asset factor $T_2^{(i)}$ and systematic factor S_2 since the trade has the same behavior as the trade in the first period no matter it defaults or not. U_i is the threshold to determine if the default of the ith trade will happen. S_1, S_2 and all the ξ_i and ξ_i' are assumed to be Gaussian distributed variables $N(0, 1)$.

Then

$$\begin{aligned} E(L_N | X = s) &= E \left\{ \sum_{i=1}^{N} \left[u_i \mathbf{I}_{\{T_1^{(i)} > U_i\}} + u_i \mathbf{I}_{\{T_2^{(i)} > U_i\}} \right] | X = s \right\} \\ &= \sum_{i=1}^{N} \left\{ u_i E[\mathbf{I}_{\{T_1^{(i)} > U_i\}} | X = s] + u_i E[\mathbf{I}_{\{T_2^{(i)} > U_i\}} | X = s] \right\} . \end{aligned} \tag{III.1.55}$$

By symmetry,

$$E[\mathbf{I}_{\{T_1^{(i)} > U_i\}} | X = s] = E[\mathbf{I}_{\{T_2^{(i)} > U_i\}} | X = s] \qquad (\forall i = 1, 2, ..., N) . \tag{III.1.56}$$

So the formulas for $E[\mathbf{I}_{\{T_1^{(i)} > U_i\}} | X = s]$ are enough to give the value of $E(L_N | X = s)$.

Remember $X = l(S_1) + l(S_2)$ is defined in (III.1.15). Proposition B.1 in the Appendix B can be directly applied. To simplify the calculation, the definition in (III.1.41) will be used:

$$\begin{aligned}
&E[\mathbf{I}_{\{T_1^{(i)}>U_i\}}|X=s] \\
&= E[\mathbf{I}_{\{\rho_i S_1+\sqrt{1-\rho_i^2}\xi_i>U_i\}}|X=s] \\
&= \frac{1}{f_X(s)}\int_{-\infty}^{+\infty}\int_{s_1\in\Omega(s)} \mathbf{I}_{\{\rho_i s_1+\sqrt{1-\rho_i^2}x>U_i\}}\phi(x)\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}dxds_1 \\
&= \frac{1}{f_X(s)}\int_{s_1\in\Omega(s)}\int_{\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}}^{+\infty} \phi(x)\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}dxds_1 \\
&= \frac{1}{f_X(s)}\int_{s_1\in\Omega(s)}\left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right]\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1\ .
\end{aligned} \tag{III.1.57}$$

Then all $E[\mathbf{I}_{\{T_1^{(i)}>U_i\}}|X=s]$ are able to be calculated based on formula (III.1.57). Following the equation (III.1.56), all $E[\mathbf{I}_{\{T_2^{(i)}>U_i\}}|X=s]$ are known by symmetry. Then $E(L_N|X=s)$ is calculated by equation (III.1.55).

III.1.2.4.2 Calculate $E(L_N^2|X=s)$

First,

$$\begin{aligned}
&E(L_N^2|X=s) \\
&= E\left\{\left\{\sum_{i=1}^{N}\left[u_i\mathbf{I}_{\{T_1^{(i)}>U_i\}}+u_i\mathbf{I}_{\{T_2^{(i)}>U_i\}}\right]\right\}^2\middle| X=s\right\} \\
&= E\left\{\sum_{i=1}^{N}u_i^2\mathbf{I}_{\{T_1^{(i)}>U_i\}}+\sum_{i,j=1(i\neq j)}^{N}u_iu_j\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_1^{(j)}>U_j\}}+\sum_{i=1}^{N}u_i^2\mathbf{I}_{\{T_2^{(i)}>U_i\}}\right. \\
&\quad+\sum_{i,j=1(i\neq j)}^{N}u_iu_j\mathbf{I}_{\{T_2^{(i)}>U_i\}}\mathbf{I}_{\{T_2^{(j)}>U_j\}}+\sum_{i,j=1}^{N}u_iu_j\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_2^{(j)}>U_j\}}
\end{aligned}$$

$$
+ \sum_{i,j=1}^{N} u_i u_j \mathbf{I}_{\{T_2^{(i)}>U_i\}} \mathbf{I}_{\{T_1^{(j)}>U_j\}} \Bigg| X = s \Bigg\}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} u_i^2 E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} | X = s\right) + \sum_{i,j=1 (i \neq j)}^{N} u_i u_j E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_1^{(j)}>U_j\}} | X = s\right) \\
&+ \sum_{i=1}^{N} u_i^2 E\left(\mathbf{I}_{\{T_2^{(i)}>U_i\}} | X = s\right) + \sum_{i,j=1 (i \neq j)}^{N} u_i u_j E\left(\mathbf{I}_{\{T_2^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right) \\
&+ 2 \sum_{i,j=1}^{N} u_i u_j E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right) \ . \qquad \text{(III.1.58)}
\end{aligned}
$$

A property of indicator function, which is $I^2_{\{\text{event}\}} = I_{\{\text{event}\}}$, has been used in this calculation.

Second, in order to calculate $E(L_N^2 | X = s)$, the values of the following are required:
$E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} | X = s\right)$, $E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_1^{(j)}>U_j\}} | X = s\right)$, $E\left(\mathbf{I}_{\{T_2^{(i)}>U_i\}} | X = s\right)$,
$E\left(\mathbf{I}_{\{T_2^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right)$ and $E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right)$.

Again, by symmetry,

$$
E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} | X = s\right) = E\left(\mathbf{I}_{\{T_2^{(i)}>U_i\}} | X = s\right) \ , \qquad \text{(III.1.59)}
$$

$$
E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_1^{(j)}>U_j\}} | X = s\right) = E\left(\mathbf{I}_{\{T_2^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right) \ , \qquad \text{(III.1.60)}
$$

$$
E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right) = E\left(\mathbf{I}_{\{T_1^{(j)}>U_j\}} \mathbf{I}_{\{T_2^{(i)}>U_i\}} | X = s\right) \ . \qquad \text{(III.1.61)}
$$

The value of $E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} | X = s\right)$ has already been calculated in section III.1.2.4.1.

Similarly, apply Proposition B.1 in the Appendix B to calculate
$E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_1^{(j)}>U_j\}} | X = s\right)$ and $E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_2^{(j)}>U_j\}} | X = s\right)$ as:

$$
E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}} \mathbf{I}_{\{T_1^{(j)}>U_j\}} | X = s\right)
$$

$$
\begin{aligned}
&= E\left[\mathbf{I}_{\{\rho_i S_1+\sqrt{1-\rho_i^2}\xi_i>U_i\}}\mathbf{I}_{\{\rho_j S_1+\sqrt{1-\rho_j^2}\xi_j>U_j\}}|X=s\right] \\
&= \frac{1}{f_X(s)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{s_1\in\Omega(s)}\mathbf{I}_{\{\rho_i s_1+\sqrt{1-\rho_i^2}x_1>U_i\}}\mathbf{I}_{\{\rho_j s_1+\sqrt{1-\rho_j^2}x_2>U_j\}} \\
&\qquad\qquad \cdot\phi(x_1)\phi(x_2)\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}dx_1dx_2ds_1 \\
&= \int_{s_1\in\Omega(s)}\int_{\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}}^{+\infty}\int_{\frac{U_j-\rho_j s_1}{\sqrt{1-\rho_j^2}}}^{+\infty}\phi(x_1)\phi(x_2)\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}dx_1dx_2ds_1\cdot\frac{1}{f_X(s)} \\
&= \int_{s_1\in\Omega(s)}\left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right]\left[1-\Phi\left(\frac{U_j-\rho_j s_1}{\sqrt{1-\rho_j^2}}\right)\right] \\
&\qquad\qquad\qquad\qquad \cdot\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1\cdot\frac{1}{f_X(s)}\ . \qquad \text{(III.1.62)}
\end{aligned}
$$

$$
\begin{aligned}
&E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_2^{(j)}>U_j\}}|X=s\right) \\
&= E\left[\mathbf{I}_{\{\rho_i S_1+\sqrt{1-\rho_i^2}\xi_i>U_i\}}\mathbf{I}_{\{\rho_j S_2+\sqrt{1-\rho_j^2}\xi_j'>U_j\}}|X=s\right] \\
&= E\left[\mathbf{I}_{\{\rho_i S_1+\sqrt{1-\rho_i^2}\xi_i>U_i\}}\mathbf{I}_{\{\rho_j g_s(S_1)+\sqrt{1-\rho_j^2}\xi_j'>U_j\}}|X=s\right] \\
&= \frac{1}{f_X(s)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{s_1\in\Omega(s)}\mathbf{I}_{\{\rho_i s_1+\sqrt{1-\rho_i^2}x>U_i\}}\mathbf{I}_{\{\rho_j g_s(s_1)+\sqrt{1-\rho_j^2}x'>U_j\}} \\
&\qquad\qquad \cdot\phi(x)\phi(x')\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}dxdx'ds_1 \\
&= \int_{s_1\in\Omega(s)}\int_{\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}}^{+\infty}\int_{\frac{U_j-\rho_j g_s(s_1)}{\sqrt{1-\rho_j^2}}}^{+\infty}\phi(x)\phi(x')\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}dxdx'ds_1\cdot\frac{1}{f_X(s)} \\
&= \int_{s_1\in\Omega(s)}\left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right]\left[1-\Phi\left(\frac{U_j-\rho_j g_s(s_1)}{\sqrt{1-\rho_j^2}}\right)\right] \\
&\qquad\qquad\qquad\qquad \cdot\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1\cdot\frac{1}{f_X(s)}\ . \qquad \text{(III.1.63)}
\end{aligned}
$$

Finally, $E(L_N^2|X=s)$ is calculated based on the formulas (III.1.57), (III.1.62) and (III.1.63).

So now $\sigma^2(Y|X=s)$ can be calculated through (III.1.51).

III.1.2.4.3 Calculate $\frac{\partial \sigma^2(Y|X=s)}{\partial s}$

In (III.1.51), we showed that

$$\sigma^2(Y|X=s) = E(L_N^2|X=s) - E^2(L_N|X=s) \ , \tag{III.1.64}$$

together with the results we proved in (III.1.55) and (III.1.58) , $\frac{\partial \sigma^2(Y|X=s)}{\partial s}$ can be calculated step by step by calculating each following terms:

$$\frac{\partial E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}|X=s\right)}{\partial s} \ , \tag{III.1.65}$$

$$\frac{\partial E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_1^{(j)}>U_j\}}|X=s\right)}{\partial s} \ , \tag{III.1.66}$$

$$\frac{\partial E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_2^{(j)}>U_j\}}|X=s\right)}{\partial s} \ . \tag{III.1.67}$$

Some preparations are required before the calculation is proceeded. First, the derivative of function $g_s(s_1)$ with respect to s is calculated as:

$$\begin{aligned}\frac{\partial g_s(s_1)}{\partial s} &= \frac{\partial l^{-1}(s - l(s_1))}{\partial s} \\ &= \frac{1}{l'(l^{-1}(s - l(s_1)))} \\ &= \frac{1}{l'(g_s(s_1))} \ .\end{aligned} \tag{III.1.68}$$

Second, define a new function $\beta(\cdot)$ as:

$$\beta(s, s_1) = \frac{\partial \phi(g_s(s_1))}{\partial s}$$

$$
\begin{aligned}
&= \phi'(g_s(s_1))\frac{\partial g_s(s_1)}{\partial s} \\
&= \phi'(g_s(s_1))\frac{1}{l'(g_s(s_1))} \ .
\end{aligned}
\tag{III.1.69}
$$

Third, find the formulas for $f_X'(s)$.

Equation (III.1.41) gives the density function of $f_X(s)$.

However, the values of $\lim_{x\to\pm\infty}\frac{l'(x)}{\phi(x)}$ are required before the analytic formula of $f_X'(s)$ is derived.

$$
\begin{aligned}
\lim_{x\to\pm\infty}\frac{l'(x)}{\phi(x)} &= \lim_{x\to\pm\infty}\sum_{i=1}^{N}\left\{u_i\frac{\rho_i}{\sqrt{1-\rho_i^2}}\exp\left[-\frac{1}{2}\left(\frac{U_i-\rho_i x}{\sqrt{1-\rho_i^2}}\right)^2+\frac{1}{2}x^2\right]\right\} \\
&= \lim_{x\to\pm\infty}\sum_{i=1}^{N}\left\{\frac{u_i\rho_i\exp\left[-\frac{1}{2(1-\rho_i^2)}\right]}{\sqrt{1-\rho_i^2}}\exp\left[(2\rho_i^2-1)x^2-2\rho_i U_i x+U_i^2\right]\right\} \\
&= 0 \text{ or } \infty \ .
\end{aligned}
\tag{III.1.70}
$$

From (III.1.70), $\lim_{x\to\pm\infty}\frac{l'(x)}{\phi(x)}$ is ∞ or 0 based on different sets of pre-determined $\{\rho_i\}$. If $\lim_{x\to\pm\infty}\frac{l'(x)}{\phi(x)}=0$, i.e. $\lim_{x\to\pm\infty}\frac{\phi(x)}{l'(x)}=\infty$, the continuity in a closed area condition of Proposition III.1.1 is not satisfied. So the formula of $f_X'(s)$ cannot be derived by applying Proposition III.1.1 when $s\neq\sum_{i=1}^N u_i$. In this case, the $f_X'(s)$ can only be calculated numerically. And then all the first derivatives of the conditional expectations have to be calculated numerically using simple numerically partial differential equation technique. In the rest of this chapter, it is assumed that, $\lim_{x\to\pm\infty}\frac{\phi(x)}{l'(x)}=0$.

With this assumption, Proposition III.1.1 can be applied as follows:

(i) If $2\sum_{i=1}^N u_i > s \geq \sum_{i=1}^N u_i$,

when $s\neq\sum_{i=1}^N u_i$,

$$
f_X'(s) = \frac{d\int_{\Omega(s)}\phi(s_1)\phi(g_s(s_1))\frac{1}{l'(g_s(s_1))}ds_1}{ds}
$$

$$
\begin{aligned}
&= -\phi(l^{-1}(s-\sum_{i=1}^{N} u_i))\phi(g_s(l^{-1}(s-\sum_{i=1}^{N} u_i))) \\
&\cdot \frac{1}{l'(g_s(l^{-1}(s-\sum_{i=1}^{N} u_i)))} \frac{1}{l'(l^{-1}(s-\sum_{i=1}^{N} u_i))} \\
&+ \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\frac{\partial \phi((g_s(s_1))}{\partial s} + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1 \\
&= -\phi(l^{-1}(s-\sum_{i=1}^{N} u_i))\left[\lim_{x\to+\infty} \frac{\phi(x)}{l'(x)}\right] \frac{1}{l'(l^{-1}(s-\sum_{i=1}^{N} u_i))} \\
&+ \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\frac{\partial \phi((g_s(s_1))}{\partial s} + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1 \\
&\text{(It is already assumed } \lim_{x\to+\infty} \tfrac{\phi(x)}{l'(x)} = 0) \\
&= \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\beta(s,s_1) + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1 \ ;
\end{aligned}
\tag{III.1.71}
$$

when $s = \sum_{i=1}^{N} u_i$, $\Omega(s) = (-\infty, +\infty)$

$$
f'_X(s) = \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\beta(s,s_1) + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1 \ . \tag{III.1.72}
$$

(ii) If $\sum_{i=1}^{N} u_i > s > 0$,

$$
\begin{aligned}
f'_X(s) &= \frac{d\int_{\Omega(s)} \phi(s_1)\phi(g_s(s_1))\frac{1}{l'(g_s(s_1))} ds_1}{ds} \\
&= \Bigg\{\phi(l^{-1}(s))\phi(g_s(l^{-1}(s)))\frac{1}{l'(g_s(l^{-1}(s)))} \\
&+ \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\frac{\partial \phi((g_s(s_1))}{\partial s} + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1\Bigg\} \\
&= \Bigg\{\phi(l^{-1}(s))\left[\lim_{x\to-\infty} \frac{\phi(x)}{l'(x)}\right] \\
&+ \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\frac{\partial \phi((g_s(s_1))}{\partial s} + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1\Bigg\}
\end{aligned}
$$

(It is already assumed $\lim_{x\to-\infty} \frac{\phi(x)}{l'(x)} = 0$)

$$= \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\beta(s,s_1) + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1 \ . \tag{III.1.73}$$

We find out that no matter in case (i) or (ii), the formula for $f'_X(s)$ is the same as follows

$$f'_X(s) = \int_{\Omega(s)} \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\beta(s,s_1) + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] ds_1 \ . \tag{III.1.74}$$

To simplify the remaining of the calculations, define a function $p(s, s_1)$ as

$$p(s,s_1) = \frac{1}{(l'(g_s(s_1)))^2}\phi(s_1)\left[l'(g_s(s_1))\beta(s,s_1) + \phi((g_s(s_1))\frac{\partial l'(g_s(s_1))}{\partial s}\right] \ . \tag{III.1.75}$$

So $f'_X(s)$ can be rewritten as

$$f'_X(s) = \int_{\Omega(s)} p(s,s_1)ds_1 \ . \tag{III.1.76}$$

Now we can proceed to the calculation of $\frac{\partial\sigma^2(Y|X=s)}{\partial s}$. From the results in (III.1.41), (III.1.57), (III.1.68), (III.1.69) and (III.1.76),

$$\frac{\partial E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}|X=s\right)}{\partial s}$$

$$= \frac{\partial\left\{\frac{1}{f_X(s)}\int_{s_1\in\Omega(s)}\left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right]\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1\right\}}{\partial s}$$

$$
= \frac{1}{f_X^2(s)} \left\{ \frac{\partial \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] \phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1}{\partial s} f_X(s) \right.
$$

$$
\left. + \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] \phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1 f_X'(s) \right\}
$$

$$
= \frac{1}{f_X^2(s)} \left\{ f_X(s) \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] p(s,s_1)ds_1 \right.
$$

$$
\left. + \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] \phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1 \int_{s_1\in\Omega(s)} p(s,s_1)ds_1 \right\} . \tag{III.1.77}
$$

From the results in (III.1.41),(III.1.62), (III.1.68), (III.1.69) and (III.1.76),

$$
\frac{\partial E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_1^{(j)}>U_j\}}|X=s\right)}{\partial s}
$$

$$
= \frac{1}{f_X^2(s)} \left\{ f_X(s) \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] \left[1-\Phi\left(\frac{U_j-\rho_j s_1}{\sqrt{1-\rho_j^2}}\right)\right] p(s,s_1)ds_1 \right.
$$

$$
+ \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] \left[1-\Phi\left(\frac{U_j-\rho_j s_1}{\sqrt{1-\rho_j^2}}\right)\right]
$$

$$
\left. \cdot\, \phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1 \cdot \int_{s_1\in\Omega(s)} p(s,s_1)ds_1 \right\} . \tag{III.1.78}
$$

From the results in (III.1.41),(III.1.63), (III.1.68), (III.1.69) and (III.1.76),

$$
\frac{\partial E\left(\mathbf{I}_{\{T_1^{(i)}>U_i\}}\mathbf{I}_{\{T_1^{(j)}>U_j\}}|X=s\right)}{\partial s}
$$

$$
\left. \cdot\, \phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1 f_X'(s) \right\}
$$

$$
= \frac{1}{f_X^2(s)} \left\{ f_X(s) \int_{s_1\in\Omega(s)} \left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right] \left\{ \left[1-\Phi\left(\frac{U_j-\rho_j g_s(s_1)}{\sqrt{1-\rho_j^2}}\right)\right] p(s,s_1) \right.\right.
$$

$$
+\frac{\rho_j}{\sqrt{1-\rho_j^2}}\phi\left(\frac{U_j-\rho_j g_s(s_1)}{\sqrt{1-\rho_j^2}}\right)\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-2}\Bigg\} ds_1
$$

$$
+\int_{s_1\in\Omega(s)}\left[1-\Phi\left(\frac{U_i-\rho_i s_1}{\sqrt{1-\rho_i^2}}\right)\right]\left[1-\Phi\left(\frac{U_j-\rho_j g_s(s_1)}{\sqrt{1-\rho_j^2}}\right)\right]
$$

$$
\cdot\,\phi(s_1)\phi(g_s(s_1))(l'(g_s(s_1)))^{-1}ds_1\cdot\int_{s_1\in\Omega(s)}p(s,s_1)ds_1\Bigg\}\ . \qquad \text{(III.1.79)}
$$

Finally $\frac{\partial\sigma^2(Y|X=s)}{\partial s}$ is derived using the results just calculated in (III.1.77), (III.1.78) and (III.1.79).

III.1.2.4.4 Calculate $\frac{dlnf_X(s)}{ds}$

With the formula of $f'_X(s)$ in (III.1.76), $\frac{dlnf_X(s)}{ds}$ is easily calculated as

$$
\frac{dlnf_X(s)}{ds}=\frac{1}{f_X(s)}f'_X(s)\ . \qquad \text{(III.1.80)}
$$

Finally, every component of the second order derivative in (III.1.50) has been derived, and $\alpha_q(L_N)$ could be calculated based on formula (III.1.12).

III.2 Numerical Results

The behavior of our two-period model is illustrated by comparing against the ASRF, ASRF plus standard (one period) GA, and one-period MC model, two-period MC model with and without portfolio rebalancing. The two-period MC model with portfolio rebalancing simulates the portfolio loss as outlined in III.1.9. If an asset defaults during the first period, the defaulted asset will be replaced with one having the same notional and rating in the second period. It can be viewed as simplified factor simulation model for IRC. This model serves as the benchmark to check how good our two-period conditional VaR behaves.

In order to understand the behavior of our model, we also benchmark against one-period MC model and two-period MC without portfolio rebalancing. In the latter, the defaulted asset is not replaced, which is the same assumption embedded in one-period MC and standard ASRF and GA.

In all numerical tests shown below, we assume a credit portfolio in which each asset is modeled as notional=LGD=1, a specific one-period default probability (which can be tied to rating from the modeling perspective) and a correlation within one-factor framework (i.e. each asset is correlated to one common factor). The VaR at any given percentile is expressed as a percentage of total notional of the portfolio.

III.2.1 ASRF with standard GA and Two-period Conditional VaR

The general behavior of two-period conditional VaR model are shown in Tables 26 and 27, where we show the 99.9 percentile VaR in different scenarios computed by different models. In Table 26, we assume a portfolio of 100 assets with uniform one-period PD being 1 percent and different levels of correlation. The correlation is fixed at 0.5 and the number of assets in the portfolio is changed in Table 27. The following observations can be summarized:

- Compared with one-period MC model, the ASRF and standard one-period GA behave as expected, which is, when the correlation approaches 1 and the portfolio become large, ASRF tends to converge with MC results and standard GA also serves as a reasonable approximation.
- ASRF does not converge to the two-period MC in the case of very large

portfolio and high correlations. This means that in the presence of liquidity horizon and portfolio rebalancing, the standard one-period ASRF is not practical enough as an analytical solution to credit portfolio VaR measure, .

- Compared with two-period MC model, the two-period conditional VaR model serves as a reasonable approximation. It has the same behavior as the standard GA in a sense that it converges to MC in the case of high correlation and large portfolio. As shown Appendix C, the order of each error terms is same as the standard one-period GA approximation.

Figure 25 shows the ratios of ASRF, ASRF plus standard GA, two-period conditional VaR to the two-period MC VaR w.r.t different number of assets. It can be seen that the two-period conditional VaR is a reasonable approximation to the full simulation model while at the same time, standard ASRF with and without standard GA is not enough.

One major observation from Tables 26 and 27 is the differences between two MC simulations. In order to understand the differences, we designed an additional MC model in which the portfolio is not rebalanced. The results of three MC models at the tail distributions (99 percentile and above) are shown in Figure 26. First of all, given all parameters are same, one-period MC has largest VaR at most percentile points. When we employ the two-period MC without portfolio rebalancing, VaR numbers at different percentile points become lower. When we switch on the portfolio rebalancing, VaR numbers become large but still smaller than that of one-period points. This behavior shows to two competing factors:

- The so called "'correlation leaking"' effect within multi-period factor modeling framework. Danniel Staumann [39] showed and discussed that the correlated default scenarios are different in one-period and multi-period simulation. In the example here a two-period MC simulation would cut off the possible joint default events in the first period and the second period. This will in general leads to less joint defaults given same level correlation.

- The portfolio rebalancing assumption at the end of first period means that, if an asset default in the first period and get replaced with a similar asset, it can default again. This will add more default scenarios compared with the one-period simulation. This is why even in the limitation of perfect correlation, one-period MC will be different from multi-period MC.

Due to the multiple defaults, the 100 percentile for two-period VaR is always higher than one-period since the entire portfolio can default multiple times. It is difficult to show in Figure 26 but if we increase the PD to seven percent, we can clearly see that the computational VaR by one-period and two-period models cross at the 99.9 percentile as shown in Figure 27. With higher PD, we have more chances that one asset default in the first period and the replaced asset also default in the second period.

Both Figure 26 and Figure 27 clearly show that the tail distributions in the presence of liquidity horizon and portfolio rebalancing are different from standard one-period models such as ASRF. It also shows that depending on the credit quality of the portfolio, standard ASRF can both be conservative and aggressive.

The two-period conditional VaR at different percentiles are shown in both Figure 26 and Figure 27. We can see that our analytical solution does capture the impact of liquidity horizon and portfolio rebalancing. It does provide a sensible comparable measure to the MC simulation.

III.2.2 Concentration Risk captured in the two-period conditional VaR model

The conditional VaR model is further assessed to see how good it is in capturing concentration risk in the presence of portfolio rebalancing. In our assessment, we designed two cases. In the first case, we create a portfolio of 50 assets with non-uniform PDs with the results shown in Table 28. Table 29 shows the case with non-uniform notional of the asset in the portfolio by change the weight of the one asset. We can see that in both cases the conditional VaR model behave reasonably better and closer to the MC results. More results with different notional weights are plotted in Figure 28. We can clearly see that it is necessary to do granularity adjustment to capture concentration risk and a two-period one does a better job than the standard one.

III.3 Discussion

This chapter provides a general two-period conditional VaR model for the credit portfolio that accounts for liquidity horizon and portfolio rebalance, as proposed in the IRC model for Basel 2.5. The portfolio is re-balanced at the end of the first period so that constant level of risk can be maintained. The Profit and Loss (P&L) and VaR contribution from the second period is conditional on the portfolio re-balance assumptions, in which for the credit portfolio is rating based. The methodology is an extension of GA model.

We have examined the numerical behavior of the model by benchmarking against one period MC model, two-period MC with and without portfolio rebalancing, standard ASRF, and standard (one-period) GA. Our major conclusions can be summarized as follows:

- As expected, when compared to two-period MC with portfolio rebalancing, our analytical model has a very similar behavior to the standard GA in capturing concentration risk of the credit portfolio.

- More importantly, the two-period conditional VaR model does captures the impact of liquidity horizon and portfolio rebalancing as confirmed by MC. The method can achieve a comparable measure for the standard MC based IRC/IDR model with much higher computational efficiency.

- It is shown that the standard one-period ASRF (with and without standard GA) is not enough to achieve a comparable risk measure when the liquidity horizon and portfolio rebalancing. The tail distribution with and without portfolio rebalancing are very different due to two competing factors. One is that the default correlation and its relationship between asset correlations are different for different time windows. The other factor is the portfolio rebalancing that allows multiple defaults. This addresses the fact that the defaulted asset will be replaced with another asset which can default again. We believe that this is an important feature for trading book that should be included in the capital calculation.

- Depending on the credit quality of portfolio and percentile, the ASRF will always be aggressive at the 100 percentile but can be both conservative and aggressive for other percentiles in the presence of portfolio rebalancing within the modeled time horizon. This means if we intend to come up with some analytical benchmark measure, we do need to factor in the portfolio rebalancing assumption as shown by our analysis.

In this dissertation we only considered one factor case, which can be expanded to a multi-factor as proposed by Pykhtin [34]. In the actual credit portfolio, different trades are assigned different liquidity horizons, which can be modeled in the current approach readily. For example, the longer liquidity horizon can be modeled by assuming no portfolio rebalancing at the end of the first period. Although rating is taken as the constant level of risk measure, we do not consider the rating migration P&L. Extending the current approach to include rating P&L is straightforward but the analytical solution for it is too complicated. In practice, rating migration to junk is the largest component in the rating migration P&L. We can treat migration to junk as default and merger the default probabilities. Also it may be worth to mention that in the latest Basel trading book fundamental review [2], IRC will be replaced with IDR, which is only default risk driven.

The model can also be extended to other types of risk factor by modeling the portfolio rebalancing assumption via some discretized values like rating ranks. The portfolio rebalancing assumption can be either exposure based like the case for credit portfolio discussed in this dissertation, or risk based, is defined as the sensitivity to the risk factors. In our opinion, this direction of research is important to address the liquidity modeling, and this was discussed in trading book fundamental review [2].

III.4 Infinite-period Analytic VaR Model

One remaining question regarding the two-period VaR model is what is the best liquidity horizon we should use. The answer is unknown to everyone. However, from our analysis, the analytic VaR can be achieved easily through central limit theorem. Although the VaR of infinite-period model is not practical and we can hardly find its financial application, we can use it as the boundary of these VaR based on different liquidity horizons. This result can at least tell us how the liquidity horizon can affect the VaR. So this test result can be very valuable for future research when the optimum liquidity horizon is concerned.

This infinite-period VaR calculation will be discussed in Appendix D and the formula for q-percentile VaR under infinite-period model is given in (D-29).

To compare this model with one-period and two-period default model, the numerical results of three models given the same parameters are showed in Table 30. The results are differentiated by various number of assets and default probability of the whole period. There is a valuable observation that the 99.9% VaR decreases to a much lower amount while the number of periods goes to infinity. The result suggests that in practice, it will be a very challenging topic to choose the proper liquidity horizon in the default model since the liquidity horizon can have huge impact on the VaR.

III.5 Chapter Conclusion

In this chapter, we provided a general framework of two-period conditional VaR model in the context of IRC modeling framework in which the liquidity horizon and constant level of risk are considered. In this dissertation, we successfully found an analytic approach to the two-period conditional VaR calculation by borrowing the GA technique. Considering all current IRC calculations rely on time consuming MC, and the banks still use more efficient but less practical ASRF (and its GA) as important measures for the effective capital management for banks. Our research is significant progress of expanding current IRC research on liquidity risk. Our research may have impact on the industry and improve their risk management level.

One concerning we have on the multi-period capital model is there is no optimum length liquidity horizon in any existing models. How does the liquidity horizon affect the VaR was unknown and the topic worth further research. So in this dissertation, we presented an exact analytic solution of infinite-period VaR which allows the liquidity horizon goes to zero. By comparing the infinite-period VaR result with two-period VaR result, we can clearly see the length of liquidity horizon matters. Although most IRC models are based on a finite period model, the exact infinite-period VaR calculation gives the boundary of VaR when liquidity horizon goes to 0. Our infinite-period analysis provides a great insight for the future research and regulations on how to choose the liquidity horizon properly.

IV Conclusion

In this dissertation, we have elaborated the multiple default models in two important credit risk issues Credit Valuation Adjustment (CVA) and Incremental Risk Charge (IRC), which received more and more attention after the 2008 financial crisis. The problem for each bank is the regulators kept changing their requirements constantly because there is not any widespread market standard acceptance. So the banks normally faced large pressures to fulfill these new regulations. From academic perspective, it is very valuable to investigate these significant concepts of credit risk for future reference.

First we focused on using PDEs on Credit Contingent Interest Rate Swap (CCIRS) pricing, a specific CVA pricing problem. Our research not only provided a successful multi-period PDEs solution in a multi-period model, it may also have an important impact on current CVA research framework. The previous research on CVA have not assessed the risk brought by the second and the subsequent defaults. By extending the pricing model to a two-default model and then a full model where we have no restriction on the number of defaults, we have successfully applied and numerically solved much more complicated two-dimensional PDEs. Because of the accuracy of PDE compared to the unstable Monte Carlo simulation, we have approached a thorough risk analysis using the results solved by PDE. This research suggests that the CVA of subsequent multiple defaults cannot be ignored.

Second, we focused on the second issue IRC, a new capital charge announced by BCBS. Our research is large progress of expanding current IRC research on liquidity risk and may also provide a guideline for future research on multi-period model. As introduced in this dissertation, IRC is essentially the credit VaR with liquidity horizon and constant level of risk, for which a new multi-period default model is required. The multi-period VaR can easily be calculated by Monte Carlo simulation but its accuracy and efficiency has always been criticized in the real business. We successfully found an analytic approach to the two-period conditional VaR calculation by borrowing the GA technique. The order of error term has been calculated as well and this provided an assessment of the accuracy of our analytic method. We also achieved a financially meaningful risk analysis by comparing the one, two-period ASRF term and VaR numbers. This analysis is very valuable on explaining the financial meanings and difference between one-period and two-period model. In the end, to test how the liquidity horizon affects VaR, we presented an accurate analytic solution for an infinite-period default model, which provided the boundary value of VaR by choosing different liquidity horizons. By comparing the VaR in one-period, two-period and infinite-period model, we complete our research and provide a valuable insight on how the liquidity horizon will affect VaR.

One thing we should mention is, in the multi-period model, systematic factors in each period are assumed to be independent. In the future research, we may expand our analytical approach by considering the dependence of the systematic factors in each period.

Compared to the current one-period CVA and numerical IRC calculation research, our research not only opened the door to a different dimensional research on these two important concepts but also provided guidelines for other potential multi-period credit risk research.

Table 1: 12 Month LIBOR Rate and CDS spread of different ratings (from May 1st, 2009 to April 30, 2014)

	LIBOR					
Date	12 Month					
20140430	0.5490%					
20140429	0.5490%					
20140428	0.5495%					
20140425	0.5495%					
20140424	0.5495%					
20140423	0.5483%					
...	...					
20090507	1.7813%					
20090506	1.8200%					
20090505	1.8589%					
20090504	1.8644%					
20090501	1.8644%					
variance	0.0007%					
mean	0.909%					
	CDS Spread					
Date	AAA	AA	A	BBB	BB	B
20140430	0.0938%	0.1670%	0.3881%	0.5722%	1.7952%	4.1154%
20140429	0.1105%	0.1679%	0.3985%	0.5913%	1.7995%	4.1502%
20140428	0.1113%	0.1685%	0.3992%	0.5894%	1.9206%	4.2556%
20140425	0.1113%	0.1673%	0.3826%	0.5697%	1.8139%	4.2349%
20140424	0.0968%	0.1676%	0.3807%	0.5806%	1.5016%	4.2173%

20140423	0.1111%	0.1686%	0.4037%	0.5774%	1.7221%	4.3761%
...	...	...	...	...	...	...
20090507	0.3075%	0.9621%	1.6332%	3.4747%	8.4460%	28.0827%
20090506	0.3599%	1.1833%	1.8355%	3.6819%	8.8693%	26.2862%
20090505	0.3695%	1.1359%	1.8929%	3.8222%	9.1412%	26.5660%
20090504	0.4343%	1.2053%	2.0017%	3.8649%	9.3918%	26.4036%
20090501	0.4366%	1.1579%	2.0539%	3.9778%	8.8810%	26.9790%
variance	0.0001%	0.0005%	0.0007%	0.0032%	0.0168%	0.2053%
mean	0.2476%	0.5264%	0.7041%	1.0387%	2.7020%	7.4352%

Table 2: Interest Rate Model Parameters estimations

r_0	b_1	σ_1	a_1
0.5490%	0.909%	0.038060013	1

Table 3: Hazard Rate Model Parameters estimations

	λ_0	b_2	σ_2	a_2
AAA	0.15633%	0.4127%	0.020113992	1
AA	0.27833%	0.8774%	0.032584086	1
A	0.64683%	1.1736%	0.035502957	1
BBB	0.95367%	1.7312%	0.060824805	1
BB	2.99200%	4.5034%	0.086378396	1
B	6.85900%	12.3920%	0.182026115	1

Table 4: Comparison of results obtained using PDE and Monte-Carlo methods for one default case (maturity=5, K=0.00909, 1 million simulation paths).

	PDE(ADI) time steps 600	PDE(ADI) time steps 2000	Monte Carlo
Price	\$ 2,236.22	\$ 2,235.02	\$ 2,204.58
Time (seconds)	1.19	3.62	648

Table 5: Comparison of results using PDEs and Monte-Carlo methods for the two default case. (maturity=5, K=0.00909, 1 million simulation paths)

	PDE timestep 600	PDE timestep 2000	MC timestep 2000
Price	\$ 2,264.26	\$ 2,263.50	\$ 2,223.51
Time (seconds)	4.07	14.68	660

Table 6: Price of CCIRS on different annual node (maturity=5, K=0.00909)

Year	$r(t)$	$\lambda(t)$	Price
0	0.5490%	0.6468%	2,264.26
1	0.5301%	0.5383%	1,293.70
2	0.6261%	0.6664%	705.00
3	0.7932%	1.0683%	401.82

4	0.9933%	0.8382%	146.92

Table 7: Comparison of the one-default, two-default and full prices (timestep=2000, maturity=5, K=0.00909).

	one-default	two-default	full
Price	\$ 2,235.02	\$ 2,263.50	\$ 2,263.51

Table 8: Comparison of the one-default, two-default and full prices w.r.t different correlation (timestep=600, maturity=5, K=0.01).

ρ	one-default	two-default	full
0	2,075	2,100	2,100
0.1	2,155	2,182	2,181
0.2	2,236	2,264	2,264
0.3	2,319	2,349	2,348
0.4	2,404	2,435	2,434
0.5	2,490	2,522	2,522
0.6	2,578	2,611	2,611
0.7	2,668	2,702	2,702
0.8	2,759	2,794	2,794

Table 9: Comparison of the one-default, two-default and full prices of different ratings.(maturity=10, K=0.01

Rating	one-default	two-default	infinity-default
AAA	303.76	310.24	310.29
AA	632.60	661.51	662.15
A	822.47	875.13	876.85
BBB	1,215.01	1,330.24	1,336.12
BB	2,548.34	3,185.29	3,279.92
B	4,224.85	6,964.76	8,283.93

Table 10: CCIRS price with different reverting speed of hazard rate (rating A)

a_2	one-default	two-default	full
0.50	840	891	892
0.70	835	887	888
0.90	827	879	881
1.10	819	871	873
1.30	812	865	866
1.50	806	859	860

Table 11: CCIRS price with different reverting speed of hazard rate (rating B)

a_2	one-default	two-default	full

0.50	4,479	7,134	8,382
0.70	4,372	7,092	8,381
0.90	4,270	7,008	8,320
1.10	4,184	6,923	8,248
1.30	4,114	6,846	8,178
1.50	4,056	6,779	8,114

Table 12: CCIRS price with different volatility of hazard rate(rating A)

σ_2	one-default	two-default	full
0.02	786	836	838
0.04	833	886	888
0.06	880	937	939
0.08	927	987	989
0.10	972	1,035	1,037

Table 13: CCIRS price with different volatility of hazard rate(rating B)

σ_2	one-default	two-default	full
0.10	3,973	6,573	7,803
0.20	4,278	7,045	8,384
0.30	4,559	7,445	8,894
0.40	4,801	7,754	9,304

Table 14: CCIRS price under different assumptions (a1=0.1, rating A)

b_1	one-default	two-default	full
0.60%	28,958	29,021	29,017
0.80%	57,022	57,006	56,998
1.00%	89,027	88,930	88,918
1.20%	122,530	122,365	122,348
1.40%	156,397	156,174	156,153
1.60%	190,142	189,868	189,842
1.80%	223,568	223,246	223,216
2.00%	256,595	256,228	256,194

Table 15: CCIRS price under different assumptions (a1=0.1, rating B)

b_1	one-default	two-default	full
0.60%	279,104	285,694	285,625
0.80%	516,726	519,715	518,818
1.00%	805,868	804,440	802,527
1.20%	1,107,340	1,102,014	1,099,080
1.40%	1,411,499	1,403,424	1,399,581
1.60%	1,716,088	1,705,702	1,701,004
1.80%	2,015,771	2,003,456	1,997,976
2.00%	2,313,791	2,299,663	2,293,428

Table 16: CCIRS price under different assumptions (a1=0.5, rating A)

b_1	one-default	two-default	full
0.60%	18,422	18,425	18,422
0.80%	50,912	50,820	50,812
1.00%	86,042	85,888	85,876
1.20%	121,013	120,808	120,792
1.40%	155,553	155,300	155,278
1.60%	189,630	189,329	189,304
1.80%	223,240	222,894	222,864
2.00%	256,388	255,998	255,963

Table 17: CCIRS price under different assumptions (a1=0.5, rating B)

b_1	one-default	two-default	full
0.60%	165,118	167,209	167,047
0.80%	458,605	453,699	452,074
1.00%	775,979	768,176	765,469
1.20%	1,091,566	1,081,910	1,078,361
1.40%	1,401,960	1,390,520	1,386,187
1.60%	1,708,185	1,694,906	1,689,799
1.80%	2,009,844	1,994,755	1,988,890
2.00%	2,307,735	2,290,852	2,284,243

Table 18: CCIRS price under different assumptions (a1=1, rating A)

b_1	one-default	two-default	full
0.60%	15,213	15,181	15,179
0.80%	50,162	50,053	50,045
1.00%	85,571	85,408	85,396
1.20%	120,579	120,365	120,348
1.40%	155,124	154,862	154,841
1.60%	189,201	188,892	188,866
1.80%	222,813	222,457	222,426
2.00%	255,963	255,562	255,527

Table 19: CCIRS price under different assumptions (a1=1, rating B)

b_1	one-default	two-default	full
0.60%	135,630	133,882	133,425
0.80%	452,671	445,975	444,031
1.00%	771,919	763,247	760,422
1.20%	1,086,414	1,075,819	1,072,185
1.40%	1,396,353	1,383,793	1,379,356
1.60%	1,702,193	1,687,699	1,682,474
1.80%	2,003,896	1,987,535	1,981,546
2.00%	2,301,607	2,283,420	2,276,685

Table 20: CCIRS price under different assumptions (a1=1.5, rating A)

b_1	one-default	two-default	full
0.60%	14,493	14,448	14,446
0.80%	49,923	49,809	49,801
1.00%	85,355	85,188	85,176
1.20%	120,360	120,143	120,126
1.40%	154,898	154,633	154,612
1.60%	188,969	188,656	188,630
1.80%	222,575	222,215	222,185
2.00%	255,720	255,316	255,281

Table 21: CCIRS price under different assumptions (a1=1.5, rating B)

b_1	one-default	two-default	full
0.60%	130,105	126,869	126,201
0.80%	450,915	443,942	441,951
1.00%	769,497	760,500	757,658
1.20%	1,083,581	1,072,560	1,068,895
1.40%	1,393,396	1,380,371	1,375,897
1.60%	1,699,130	1,684,159	1,678,898
1.80%	2,000,768	1,983,913	1,977,888
2.00%	2,298,370	2,279,668	2,272,898

Table 22: CCIRS price with different volatility of interest rate (rating A)

σ_1	one-default	two-default	full
0.02	25	27	27
0.04	997	1,060	1,061
0.06	3,797	3,984	3,988
0.08	7,974	8,308	8,316
0.10	12,921	13,405	13,415

Table 23: CCIRS price with different volatility of interest rate (rating B)

σ_1	one-default	two-default	full
0.02	112	218	290
0.04	5,181	8,438	9,973
0.06	21,458	31,854	36,061
0.08	47,282	66,491	73,668
0.10	78,937	107,272	117,388

Table 24: CCIRS price with different Maturity (rating A)

Maturity	one-default	two-default	full
5	560	572	572
7	765	792	793

9	824	869	870
11	821	883	885
13	805	884	888
15	751	841	847

Table 25: CCIRS price with different Maturity (rating B)

Maturity	one-default	two-default	full
5	4,511	5,506	5,637
7	5,219	7,141	7,621
9	4,667	7,235	8,259
11	3,797	6,677	8,355
13	2,999	5,965	8,378
15	2,268	5,016	7,953

Table 26: the Comparison of ASRF VaR, 1-period GA VaR, 2-period Conditional VaR and 2-period MC VaR with Different ρ (N=100, PD=1%, LGD=1)

ρ	ASRF VaR	1-period GA VaR	1-period MC VaR	2-period Cond. VaR	2-period MC VaR
0.1	3.96%	10.72%	8.00%	11.76%	8.00%
0.2	7.11%	10.72%	10.00%	10.14%	9.00%
0.3	11.84%	14.35%	14.00%	12.32%	12.00%
0.4	18.56%	20.49%	20.60%	17.09%	17.00%

0.5	27.77%	29.32%	29.00%	24.68%	24.10%
0.6	40.05%	41.31%	41.60%	35.76%	36.00%
0.7	55.97%	56.97%	57.00%	51.18%	50.70%
0.8	75.61%	76.35%	76.50%	71.64%	71.40%
0.9	95.20%	95.61%	95.60%	94.16%	94.30%

Table 27: the Comparison of ASRF VaR, 1-period GA VaR, 2-period Conditional VaR and 2-period MC VaR with Different Number of Assets N (ρ=0.5, PD=1%, LGD=1)

N	ASRF VaR	1-period GA VaR	1-period MC VaR	2-period Cond. VaR	2-period MC VaR
20	27.77%	35.53%	35.00%	31.49%	30.00%
40	27.77%	31.65%	32.50%	27.23%	27.53%
60	27.77%	30.36%	30.00%	25.82%	25.17%
80	27.77%	29.71%	31.25%	25.11%	25.25%
100	27.77%	29.32%	29.00%	24.68%	24.10%
200	27.77%	28.55%	28.50%	23.83%	23.95%
400	27.77%	28.16%	28.50%	23.41%	23.36%
500	27.77%	28.08%	27.80%	23.32%	23.50%
700	27.77%	27.99%	28.14%	23.23%	23.42%
1000	27.77%	27.92%	27.00%	23.15%	23.36%

Table 28: the Comparison of ASRF VaR, 1-period GA VaR, 2-period Conditional VaR and 2-period MC VaR w.r.t Mixed Default Probabilities ($\rho = 0.5$, N=50, LGD=1)

PD	ASRF VaR	1-period GA VaR	2-period Cond. VaR	MC VaR
mixed	31.17%	34.31%	31.70%	31.90%
5%	61.32%	64.70%	62.44%	62.47%
1%	27.77%	30.89%	26.38%	26.39%
0.10%	6.18%	8.60%	7.42%	7.80%

Table 29: the Comparison of ASRF VaR, 1-period GA VaR, 2-period Conditional VaR and 2-period MC VaR w.r.t Different Notional Weights of the first asset(the other assets are equally weighted, PD=1%, N=100, ρ=0.5, LGD=1)

weight	ASRF VaR	1-period GA VaR	2-period Cond. VaR	MC VaR
1%	27.77%	29.32%	24.68%	24.10%
11%	27.77%	30.89%	26.40%	26.28%
21%	27.77%	35.60%	31.55%	33.77%
31%	27.77%	43.44%	40.14%	42.85%
41%	27.77%	54.42%	52.17%	51.13%

Table 30: the Comparison of One, Two and Infinite Periods Analytic VaR w.r.t Different Default Probabilities

PD	1%	1%	1%	0.10%	0.10%	0.10%
Periods	1	2	∞	1	2	∞
N	VaR	VaR	VaR	VaR	VaR	VaR
20	35.53%	31.49%	11.81%	12.18%	11.22%	3.29%
40	31.65%	27.23%	8.94%	9.18%	8.05%	2.39%
60	30.36%	25.82%	7.67%	8.18%	7.00%	1.98%
80	29.71%	25.11%	6.91%	7.68%	6.47%	1.75%
100	29.32%	24.68%	6.39%	7.38%	6.15%	1.58%
200	28.55%	23.83%	5.11%	6.78%	5.52%	1.18%
400	28.16%	23.41%	4.20%	6.48%	5.20%	0.89%
500	28.08%	23.32%	3.97%	6.42%	5.14%	0.82%
700	27.99%	23.23%	3.67%	6.36%	5.07%	0.72%
1000	27.92%	23.15%	3.40%	6.30%	5.01%	0.64%

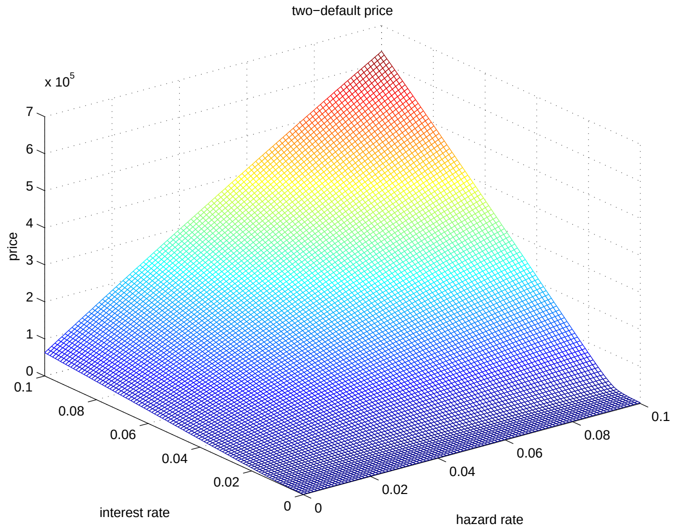

Figure 1: CCIRS price as a function of interest and hazard rates.

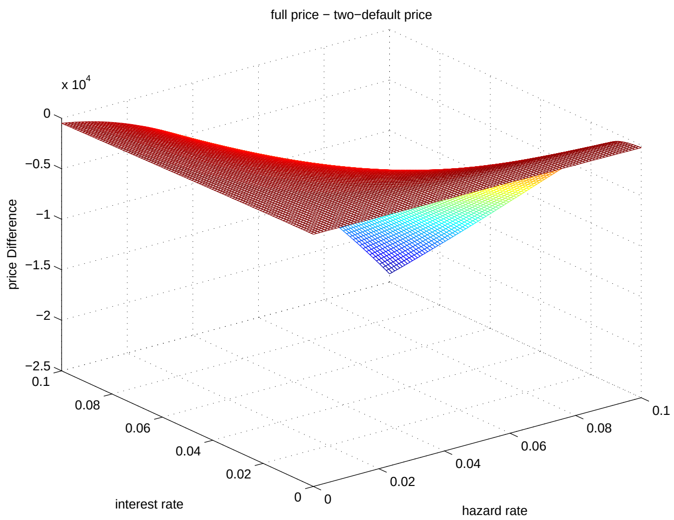

Figure 2: Difference of the full price and two-default price

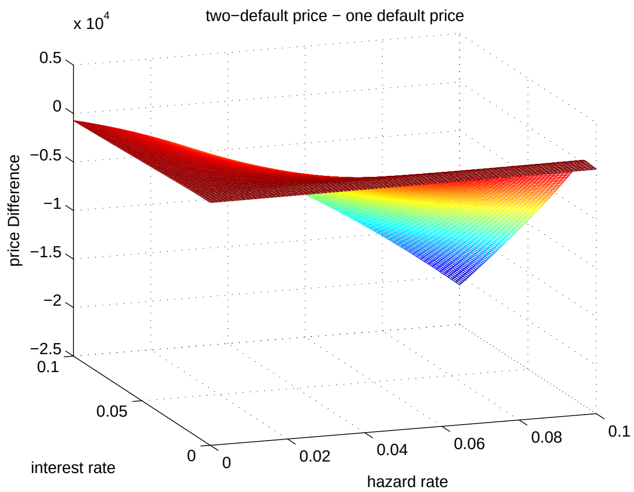

Figure 3: Difference of two-default and one-default price

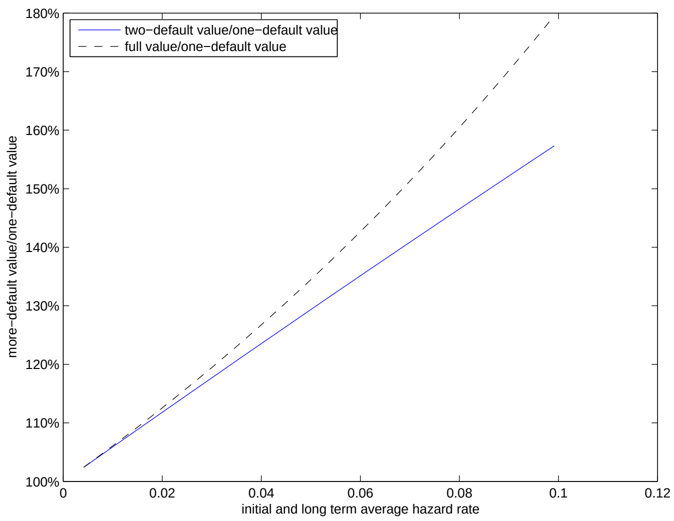

Figure 4: Price comparison with different initial and long term average hazard rates

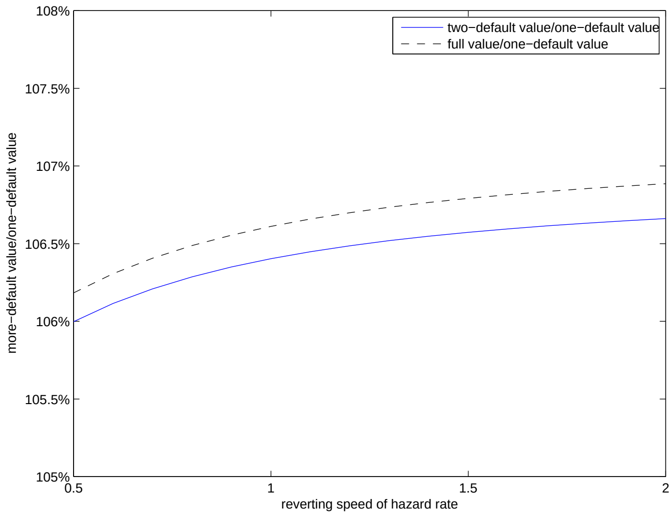

Figure 5: Price comparison with different reverting speed of hazard rate (For rating A)

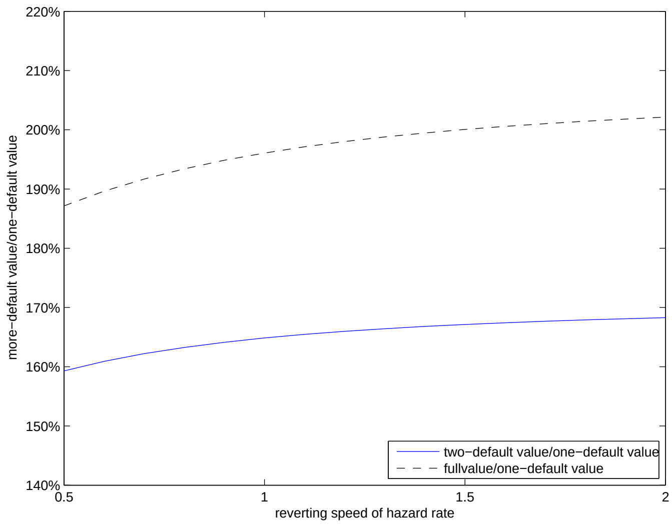

Figure 6: Price comparison with different reverting speed of hazard rate (For rating B)

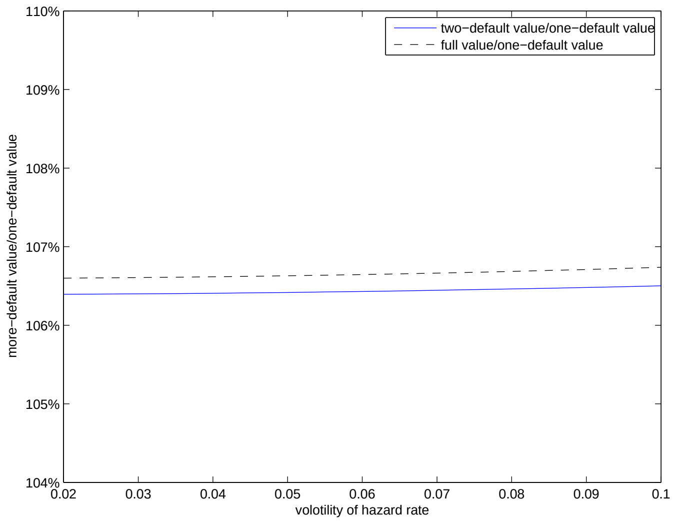

Figure 7: Price comparison with different volatility of hazard rate (For rating A)

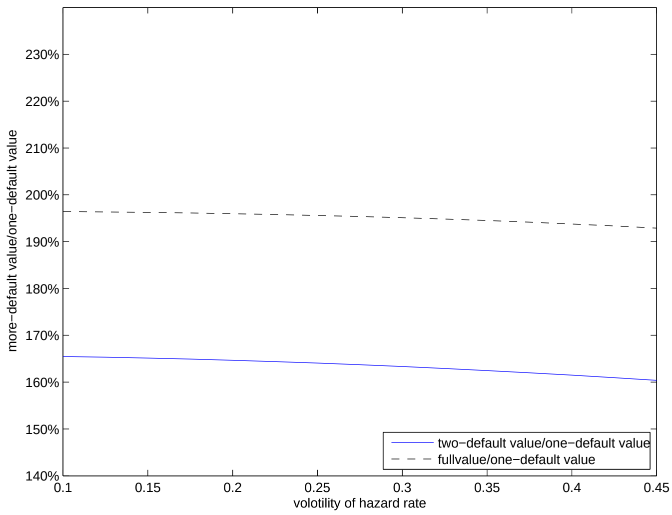

Figure 8: Price comparison with different volatility of hazard rate (For rating B)

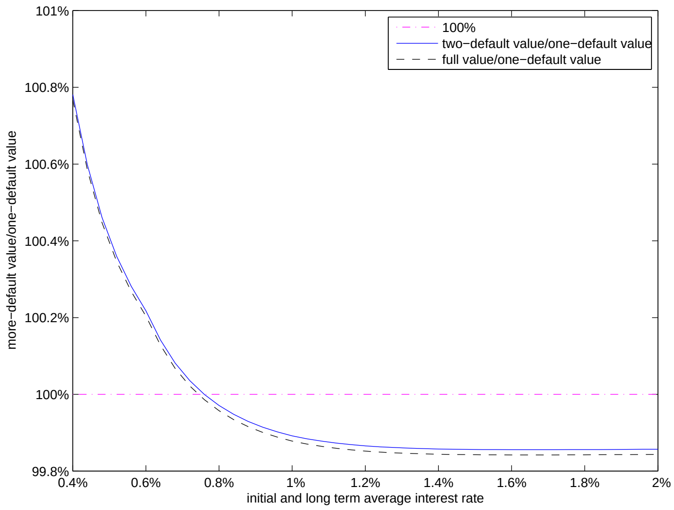

Figure 9: Price comparison with different long term average interest rates when a_1=0.1 (For rating A)

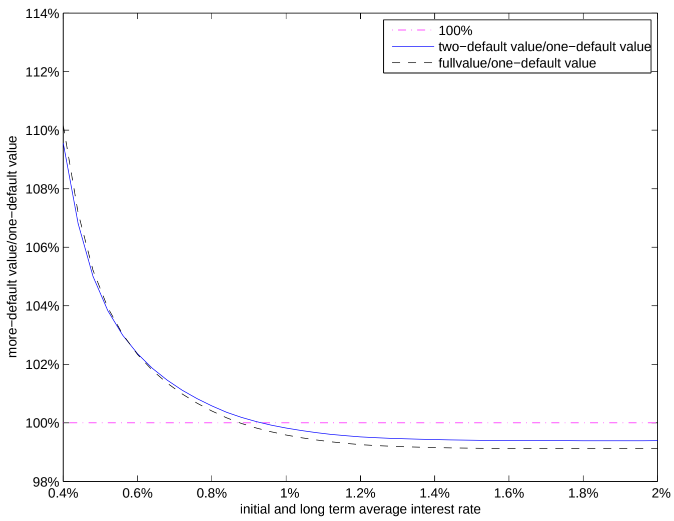

Figure 10: Price comparison with different long term average interest rates when a_1=0.1 (For rating B)

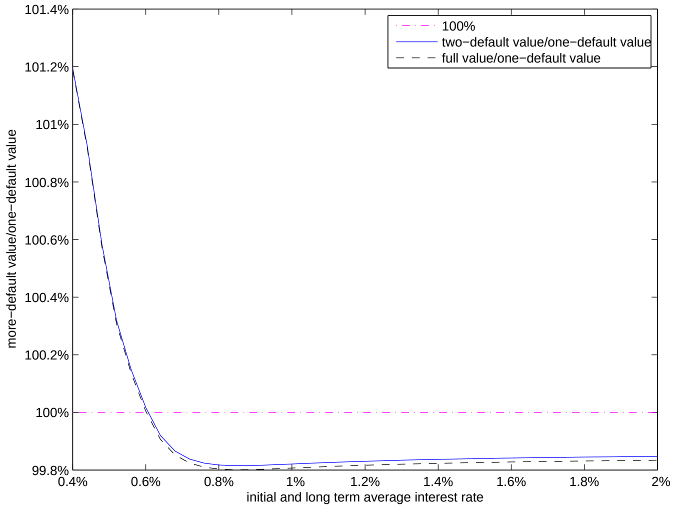

Figure 11: Price comparison with different long term average interest rates when a_1=0.5 (For rating A)

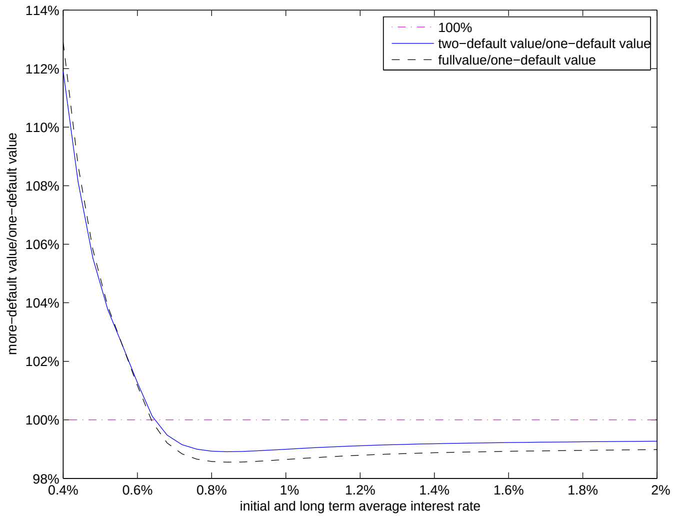

Figure 12: Price comparison with different long term average interest rates when a_1=0.5 (For rating B)

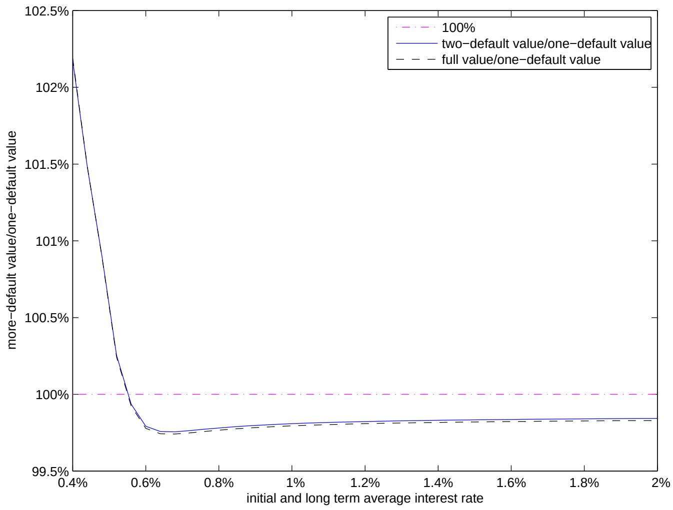

Figure 13: Price comparison with different long term average interest rates when a_1=1 (For rating A)

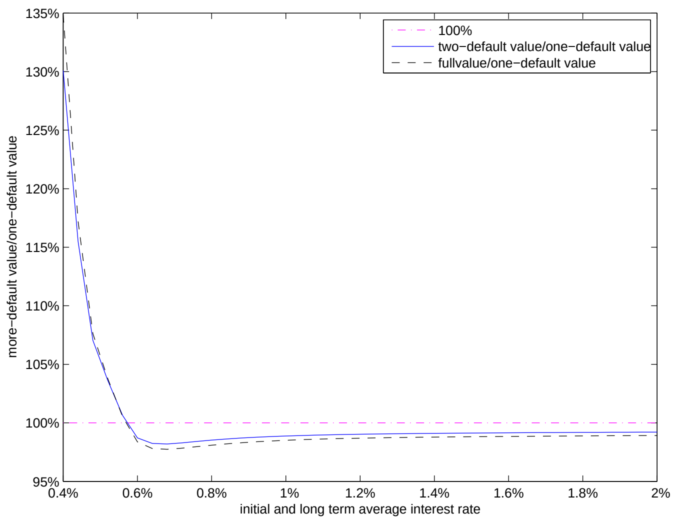

Figure 14: Price comparison with different long term average interest rates when a_1=1 (For rating B)

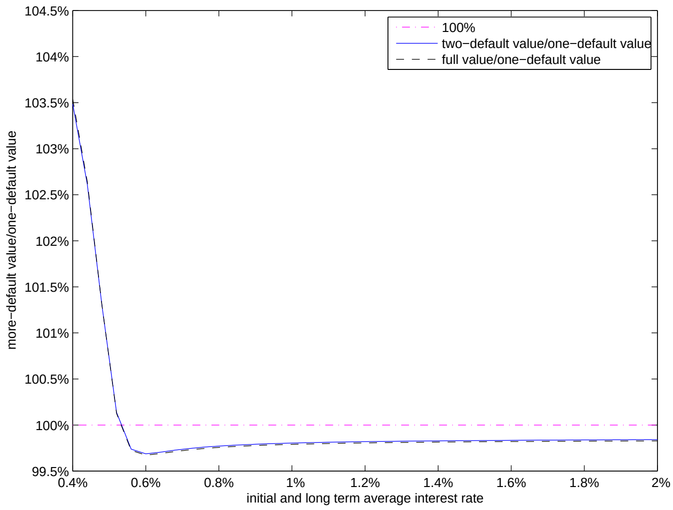

Figure 15: Price comparison with different long term average interest rates when a_1=1.5 (For rating A)

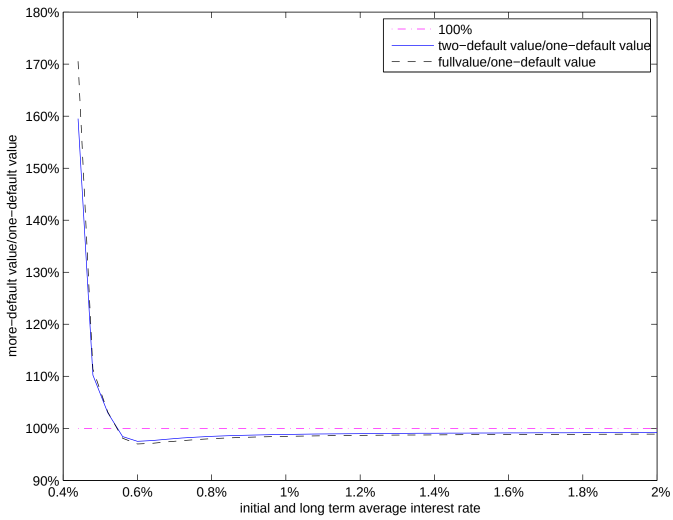

Figure 16: Price comparison with different long term average interest rates when a_1=1.5 (For rating B)

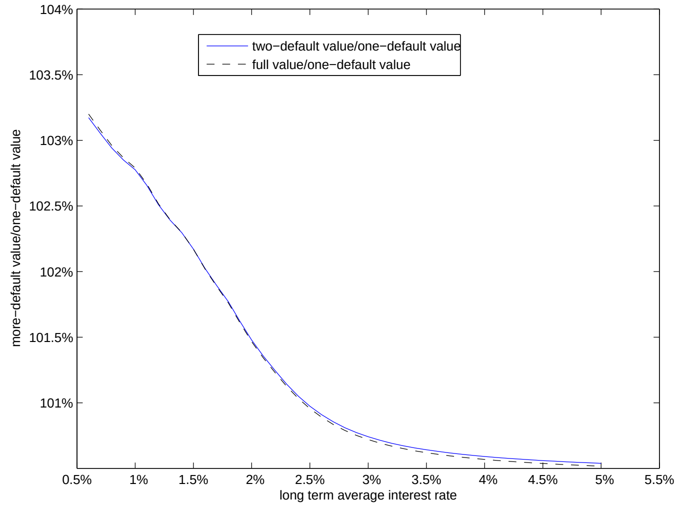

Figure 17: Price comparison when interest rate gradually increases (For rating A).

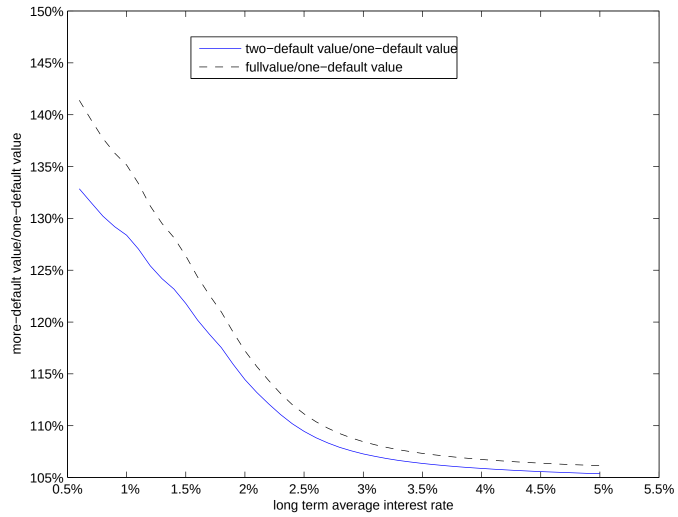

Figure 18: Price comparison when interest rate gradually increases (For rating B).

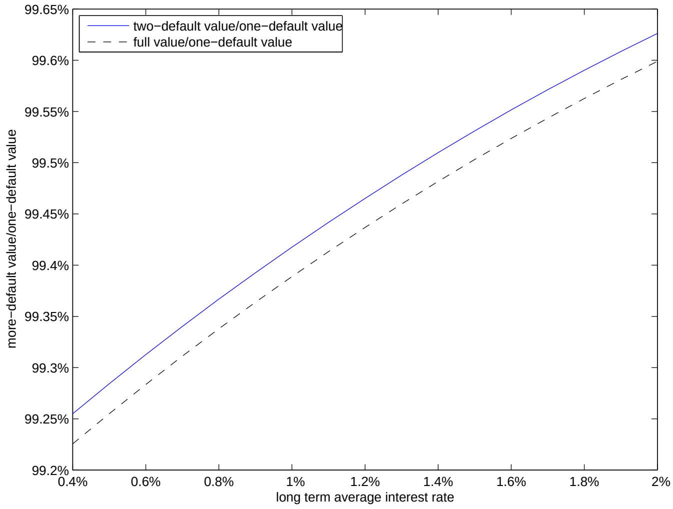

Figure 19: Price comparison when interest rate gradually decreases (For rating A).

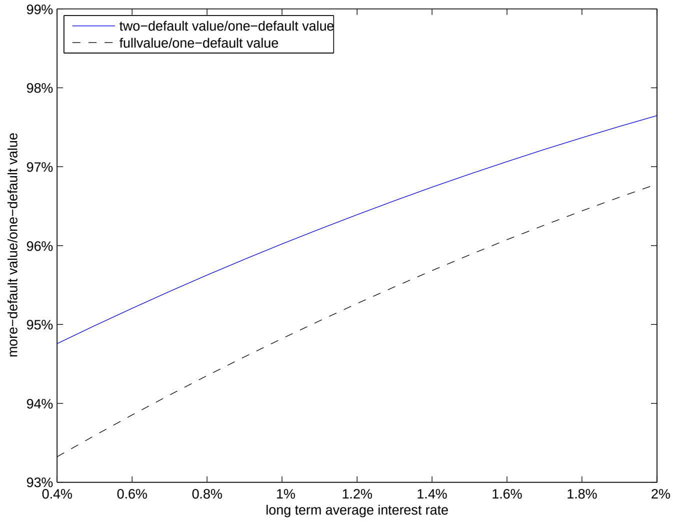

Figure 20: Price comparison when interest rate gradually decreases (For rating B).

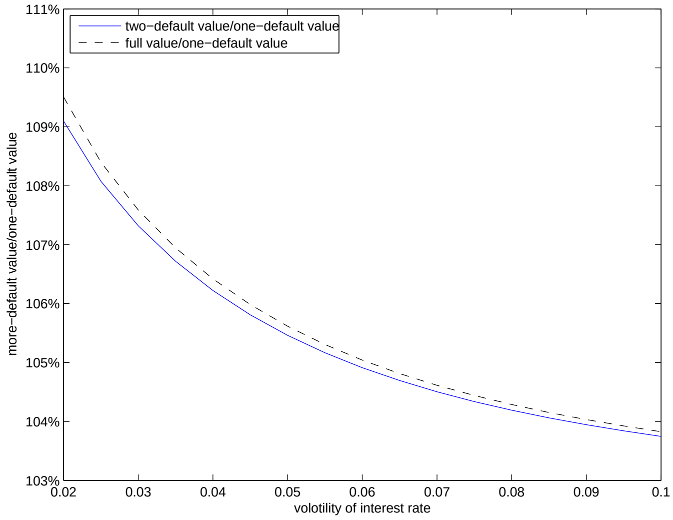

Figure 21: Price comparison with different volatility of interest rate (For rating A)

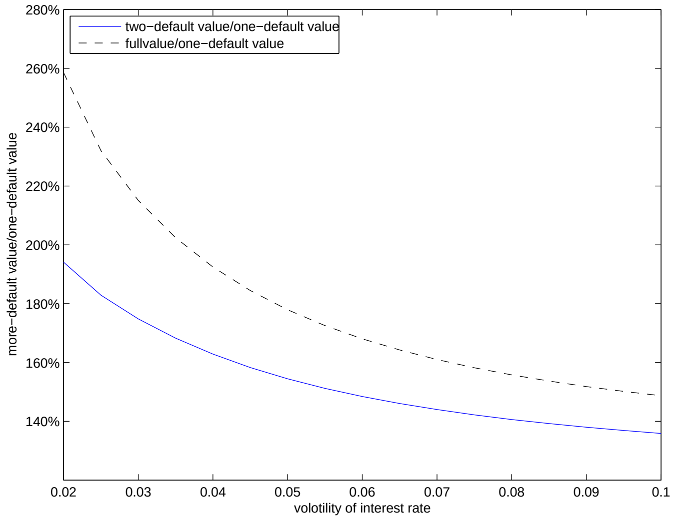

Figure 22: Price comparison with different volatility of interest rate (For rating B)

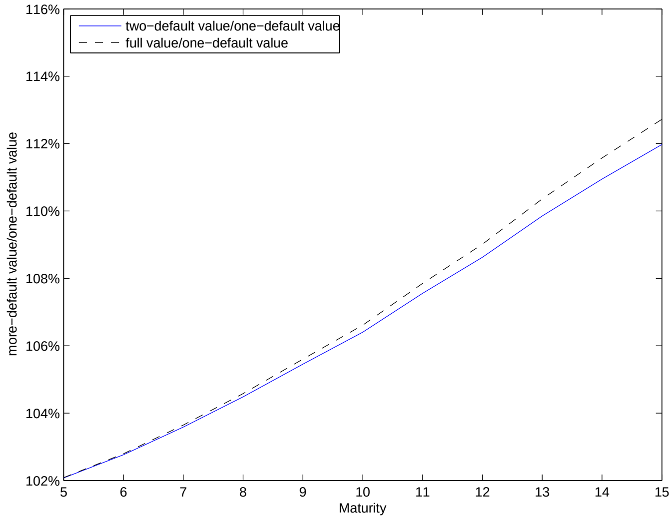

Figure 23: Price comparison with different Maturity (For rating A)

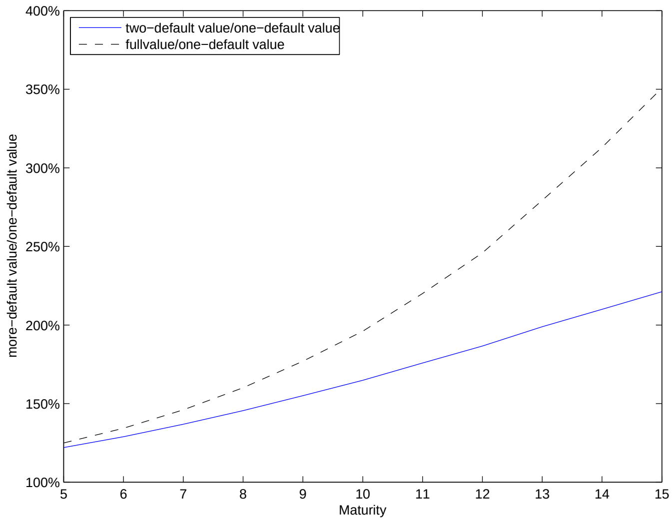

Figure 24: Price comparison with different Maturity (For rating B)

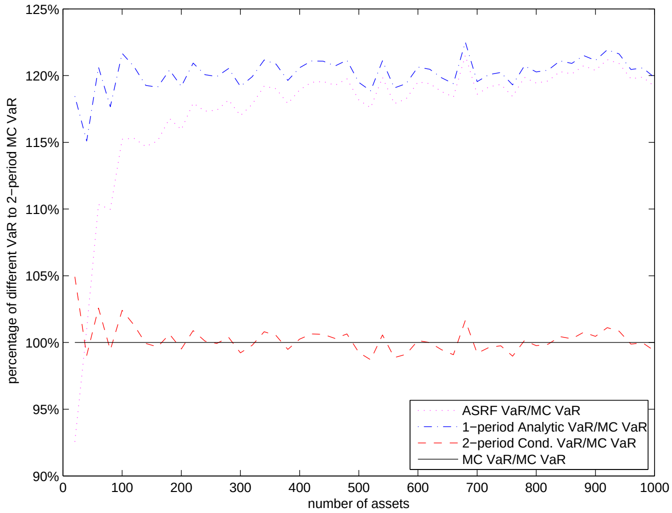

Figure 25: The comparison of ratio of ASRF, One-period, two-period analytic VaR to 2-period MC VaR w.r.t different N (20 to 1000) when ρ=0.5, PD=1%

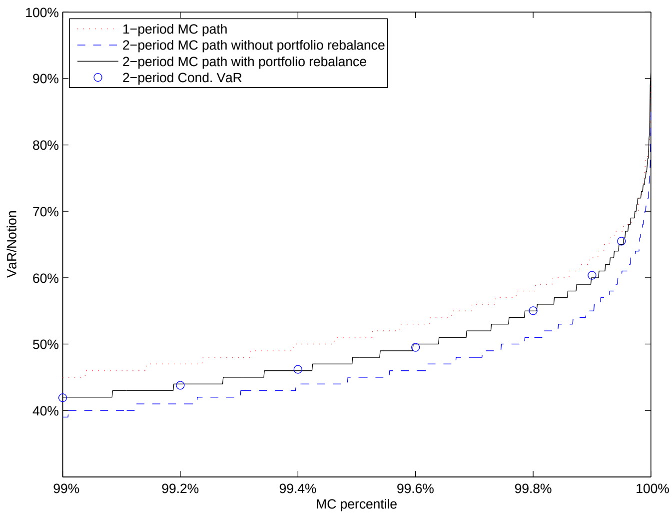

Figure 26: One-period, two-period MC paths and two-period MC paths without rebalancing when N=100, ρ=0.5, PD=5%

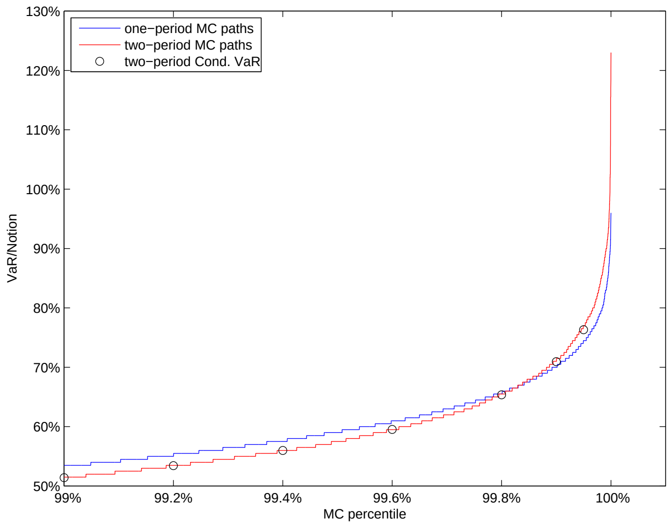

Figure 27: One-period and two-period MC paths value and 2-period Conditional VaR, when PD=7%, $N = 200$, $\rho = 0.5$

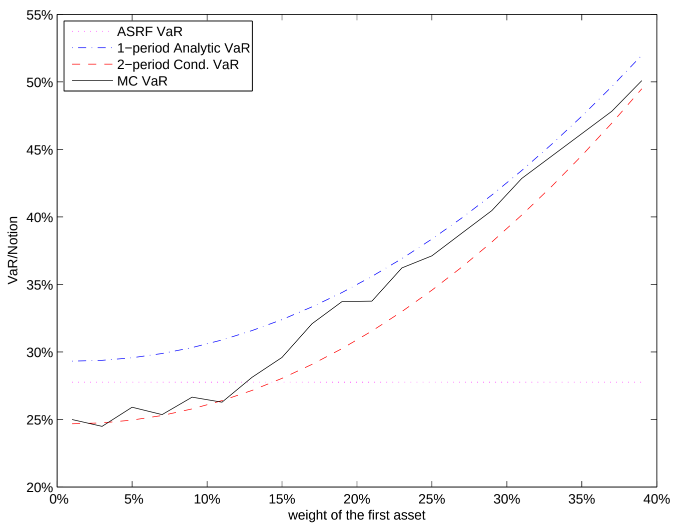

Figure 28: The comparison of ratio of ASRF, one-period, two-period analytic VaR to two-period MC VaR w.r.t different weight of the first asset (1% to 40%) when N=100, ρ=0.5, PD=0.1%

A Convergence Analysis

The solutions obtained by numerical methods are usually not the exact solutions of the problem. There are two types of errors: round-off errors and truncation errors. Roundoff errors came from the limitation of a finite-state machine which cannot display the infinite real numbers. Truncation errors are resulting from the difference of the approximate solution and the exact solution. Once an error is generated, it will generally propagate through the calculation. So the convergence test is a very important criterion to guarantee the numerical solution moves towards the real solution. The solution of the discretized problem converges to the solution of the continuous problem as the grid size goes to zero, and the speed of convergence is one of the factors of the efficiency of the method.

In this appendix, we will carry out a convergence investigation of one-default, two-default and full model PDEs solutions. Many different experiments are performed with different size of interest rate, hazard rate and time steps. Since the analytic solution is not available in these experiments, we choose the results gained from the finest grid as our reference solution. Then we compute the absolute error between the reference solution and the solution obtained on the coarser grid. The convergence rate is the divisions of two corresponding absolute errors.

- Convergence test for the price under one and two defaults models.

To calculate the convergence rate, we perform eight experiments with varying grids. In the grids, r and λ steps are 45, 68, 101, 152, 228, 342. So the r and λ decrease by the ratio of 1.5. Then Δr and $\Delta\lambda \approx$ 0.00222, 0.00147, 0.00099, 0.00066, 0.00044, 0.00029 and 0.0001949. The maturity is 5 years and the time steps is set to be 600 (so Δt is about 0.00833). The parameters are chosen from Table 2 and Rating A row in Table 3. $K = b1$, $M = 5$ and $\rho = 0.2$.

Table 30 shows the convergence analysis for r and λ. From this table we see that our numerical algorithm converges faster when the step sizes get smaller and the convergence rate is around $1.5^2 = 2.25$, which gives the order of our algorithm is about two.

Table 30: Convergence Test

Steps			Value			error		
r	λ	time	one	two	full	one	two	full
45	45	600	2,319	2,347	2,347	105.70	104.40	104.00
68	68	600	2,268	2,296	2,296	54.40	53.40	53.00
101	101	600	2,238	2,266	2,266	24.90	23.90	23.50
152	152	600	2,224	2,252	2,252	10.60	9.60	9.30
228	228	600	2,220	2,248	2,248	6.30	5.30	5.00
342	342	600	2,217	2,245	2,245	4.00	3.00	2.60
513	513	600	2,216	2,244	2,244	2.90	2.00	1.60
			error percentage			Convergence rate		
r	λ	time	one	two	full	one	two	full
45	45	600	4.7757%	4.6557%	4.6375%			
68	68	600	2.4579%	2.3814%	2.3633%	1.94	1.96	1.96
101	101	600	1.1250%	1.0658%	1.0479%	2.18	2.23	2.26
152	152	600	0.4789%	0.4281%	0.4147%	2.35	2.49	2.53
228	228	600	0.2846%	0.2364%	0.2230%	1.68	1.81	1.86
342	342	600	0.1807%	0.1338%	0.1159%	1.58	1.77	1.92

513	513	600	0.1310%	0.0892%	0.0713%	1.38	1.50	1.62
Reference Solution								
r	λ	time	one	two	full			
546	546	20000	2,213	2,242	2,243			

• Remarks

From Table 30 we see that our algorithms converge to the exact solution when the step sizes go to zero. This gives us confidence that our numerical methods for the PDE of PG and PS are convergent and trustable. Hence the numerical results we have obtained in this dissertation are good.

B Proposition Required for 2-Period Analytic VaR solution

Proposition B.1 . *Let X be defined as (III.1.14), S_1, S_2 are the same independent systematic factors as previously defined. Function $l(s)$ is defined as (III.1.15). Function $g_s(\cdot)$ is defined as (III.1.39). Interval $\Omega(s)$ is defined as (III.1.40).*

Assume $X_1, X_2, ...X_n, S_1, S_1$ are i.i.d and their common density function is $f(\cdot)$ and common distribution function is $F(\cdot)$. Then we could have:

$$
\begin{aligned}
& E[h(X_1, X_2, ...X_n, S_1, S_2)|X = s] \\
&= \Bigg[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{\Omega(s)} h(x_1, x_2, ..., x_n, y, g_s(y)) \\
&\qquad \cdot f(x_1)f(x_2)...f(x_n)f(y)f(g_s(y)) \frac{\partial g_t(y)}{\partial t}\bigg|_{t=s} dx_1 dx_2 ... dx_n dy\Bigg] \\
&\qquad \cdot \left[\int_{\Omega(s)} f(y)f(g_s(y)) \frac{\partial g_t(y)}{\partial t}\bigg|_{t=s} dy\right]^{-1} \\
&= \Bigg[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{\Omega(s)} h(x_1, x_2, ..., x_n, y, g_s(y)) \\
&\qquad \cdot f(x_1)f(x_2)...f(x_n)f(y)f(g_s(y))(l'(g_s(y)))^{-1} dx_1 dx_2 ... dx_n dy\Bigg]
\end{aligned}
$$

$$\cdot \left[\int_{\Omega(s)} f(y) f(g_s(y)) (l'(g_s(y)))^{-1} dy \right]^{-1} .$$

(B-1)

Proof First we know if $X = s$, then $S_2 = g_s(S_1)$. And

$$E[h(X_1, X_2, ...X_n, S_1, S_2)|X = s]$$

$$= E[h(X_1, X_2, ...X_n, S_1, g_s(S_1))|X = s]$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x_1, x_2, ..., x_n, y, g_s(y))$$

$$\cdot f_{X_1, X_2, ..., X_n, S_1|X}(x_1, x_2, ..., x_n, y|s) dx_1 dx_2 ... dx_n dy ,$$

(B-2)

where $f_{X_1,X_2,...,X_n,S_1|X}(x_1, x_2, ..., x_n, y|x)$ is the conditional density function and it satisfies:

$$f_{X_1,X_2,...,X_n,S_1|X}(x_1, x_2, ..., x_n, y|x) = \frac{f(x_1, x_2, ..., x_n, y, x)}{f_X(x)} . \quad \text{(B-3)}$$

Here $f(x_1, x_2, ..., x_n, y, x)$ is the joint density function of $X_1, X_2, ..., X_n, S_1$ and X. $f_X(x)$ is the marginal density function of X and it is already derived in equation (III.1.41).

Now let's get

$$f(x_1, x_2, ..., x_n, y, x) = \frac{\partial F(x_1, x_2, ..., x_n, y, x)}{\partial x_1 \partial x_2 ... \partial x_n \partial y \partial x} . \quad \text{(B-4)}$$

$$F(x_1, x_2, ..., x_n, y, x) P(X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n, S_1 \le y, X \le x)$$

$(X_1, X_2, ...X_n, S_1, S_2$ are independent)

$P(X_1 \leq x_1)P(X_2 \leq x_2)...P(X_n \leq x_n)P(S_1 \leq y, X \leq x)$

$F(x_1)F(x_2)...F(x_n)P(S_1 \leq y, X \leq x)$.

(B-5)

Now as always, we need to discuss in two cases:

case 1: $x > \sum_{i=1}^{N}(u_i)$;

case 2: $0 < x \leq \sum_{i=1}^{N}(u_i)$.

In case 1, let $\tilde{s}(x) = l^{-1}(x - \sum_{i=1}^{N}(u_i))$.

So, if $y \leq \tilde{s}(x)$, X will be always less than x,

$$P(S_1 \leq y, X \leq x) = P(S_1 \leq y) = F(y) \; ; \tag{B-6}$$

if $y > \tilde{s}(x)$,

$$\begin{aligned}
&P(S_1 \leq y, X \leq x) \\
&= P(X \leq x, S_1 \leq y, S_2 \leq \tilde{s}(x)) + P(X \leq x, S_1 \leq \tilde{s}(x), S_2 > \tilde{s}(x)) \\
&+ P(X \leq x, \tilde{s}(x) < S_1 \leq y, S_2 > \tilde{s}(x)) \\
&= F(\tilde{s}(x))F(y) + F(\tilde{s}(x))(1 - F(\tilde{s}(x))) + P(\tilde{s}(x) < S_1 \leq y, \tilde{s}(x) < S_2 \leq g_x(S_1)) \\
&= (1 - F(\tilde{s}(x)) + F(y))F(\tilde{s}(x)) + \int_{\tilde{s}(x)}^{y} f(s_1)[F(g_x(s_1)) - F(\tilde{s}(x))]ds_1 \; . \qquad \text{(B-7)}
\end{aligned}$$

From (B-6) and (B-7) we can get the joint density function as:

$$
\begin{aligned}
& f(x_1, x_2, ..., x_n, y, x) \\
&= f(x_1)f(x_2)...f(x_n) \\
&\cdot \left[f(y)f(\tilde{s}(x))\tilde{s}'(x) + f(y)f(g_x(y)) \frac{\partial g_t(y)}{\partial t}\bigg|_{t=x} - f(y)f(\tilde{s}(x))\tilde{s}'(x) \right] I(y > \tilde{s}(x)) \\
&= f(x_1)f(x_2)...f(x_n)f(y)f(g_x(y)) \frac{\partial g_t(y)}{\partial t}\bigg|_{t=x} I(y > \tilde{s}(x)) \\
&= f(x_1)f(x_2)...f(x_n)f(y)f(g_x(y))(l'(g_x(y)))^{-1} I(y > \tilde{s}(x)) \ .
\end{aligned} \tag{B-8}
$$

So from (III.1.41) and (B-8), we have

$$
\begin{aligned}
& E[h(X_1, X_2, ...X_n, S_1, S_2)|X = s] \\
&= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x_1, x_2, ..., x_n, y, g_s(y)) \frac{f(x_1, x_2, ..., x_n, y, s)}{f_X(s)} dx_1 dx_2 ... dx_n dy \\
&= \left[\int_{\Omega(s)} f(y)f(g_s(y))(l'(g_s(y)))^{-1} dy \right]^{-1} \left[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x_1, x_2, ..., x_n, y, g_s(y)) \right. \\
&\left. \cdot f(x_1)f(x_2)...f(x_n)f(y)f(g_s(y))(l'(g_s(y)))^{-1} I(y > \tilde{s}(s)) dx_1 dx_2 ... dx_n dy \right] \\
&= \left[\int_{\Omega(s)} f(y)f(g_s(y))(l'(g_s(y)))^{-1} dy \right]^{-1} \left[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{\Omega(s)} h(x_1, x_2, ..., x_n, y, g_s(y)) \right. \\
&\left. \cdot f(x_1)f(x_2)...f(x_n)f(y)f(g_s(y))(l'(g_s(y)))^{-1} dx_1 dx_2 ... dx_n dy \right] \ .
\end{aligned} \tag{B-9}
$$

This result is the same as the result in (B-1).

Now let us prove (B-1) still holds in case 2.

If $y \geq l^{-1}(x)$, we have:

$$
\begin{aligned}
& P(S_1 \leq y, X \leq x) \\
&= P(X \leq x, S_1 < l^{-1}(x)) + P(X \leq x, y \geq S_1 \geq l^{-1}(x)) \\
&= P(S_1 < l^{-1}(x), S_2 < g_x(S_1)) + 0 \\
&= \int_{-\infty}^{l^{-1}(x)} f(s_1) F(g_x(S_1)) ds_1 \ . \qquad \text{(B-10)}
\end{aligned}
$$

If $y < l^{-1}(x)$, we have:

$$
\begin{aligned}
& P(S_1 \leq y, X \leq x) \\
&= P(S_1 \leq y, S_2 < g_x(S_1)) \\
&= \int_{-\infty}^{y} f(s_1) F(g_x(S_1)) ds_1 \ . \qquad \text{(B-11)}
\end{aligned}
$$

From (B-10) and (B-11) we can get the joint density function as:

$$
\begin{aligned}
& f(x_1, x_2, ..., x_n, y, x) \\
&= f(x_1) f(x_2) ... f(x_n) \left[f(y) f(g_x(y)) \frac{\partial g_t(y)}{\partial t} \bigg|_{t=x} \right] I(y < l^{-1}(x)) \\
&= f(x_1) f(x_2) ... f(x_n) f(y) f(g_x(y)) (l'(g_x(y)))^{-1} I(y < l^{-1}(x)) \ . \qquad \text{(B-12)}
\end{aligned}
$$

So from (III.1.41) and (B-12), we have

$$
\begin{aligned}
& E[h(X_1, X_2, ... X_n, S_1, S_2) | X = s] \\
= & \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x_1, x_2, ..., x_n, y, g_s(y)) \frac{f(x_1, x_2, ..., x_n, y, s)}{f_X(s)} dx_1 dx_2 ... dx_n dy
\end{aligned}
$$

$$= \left[\int_{\Omega(s)} f(y)f(g_s(y))(l'(g_s(y)))^{-1}dy\right]^{-1} \left[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x_1, x_2, ..., x_n, y, g_s(y))\right.$$

$$\left.\cdot f(x_1)f(x_2)...f(x_n)f(y)f(g_s(y))(l'(g_s(y)))^{-1}I(y < l^{-1}(x))dx_1dx_2...dx_ndy\right]$$

$$= \left[\int_{\Omega(s)} f(y)f(g_s(y))(l'(g_s(y)))^{-1}dy\right]^{-1} \left[\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{\Omega(s)} h(x_1, x_2, ..., x_n, y, g_s(y))\right.$$

$$\left.\cdot f(x_1)f(x_2)...f(x_n)f(y)f(g_s(y))(l'(g_s(y)))^{-1}dx_1dx_2...dx_ndy\right]. \tag{B-13}$$

This result is the same as the result in (B-1). ∎

C Order of 2-Period Analytic VaR Adjustment

The two-period conditional VaR is based on the value of the second derivative in Taylor expansion (III.1.12). This appendix provides the order of error term in this approximation and the higher order derivative, i.e. the order of each derivative of VaR of a fine-grained portfolio. The full Taylor expansion is

$$\alpha_q(L_N) = \alpha_q(X) + \sum_{m=1}^{+\infty} \frac{\partial^m}{\partial \varepsilon^m} \alpha_q(X + \varepsilon Y) \Big|_{\varepsilon=0} .$$

Without loss of generality, simply assume $\sum u_i = 1$. Since the portfolio is fine-grained, it is reasonable to assume that

$$\exists C > 0, \text{for all } N, \text{ s.t. } \max(u_i) \leq \frac{C}{N} . \tag{C-1}$$

A special case of fine-grained portfolio is the homogenous portfolio, in which the trades have the same default probability, the same exposure and the same correlations, so each $u_i = \frac{1}{N}$.

To proceed, a few new notations and a proposition will be presented.

For any integer m, if p is a partition of m, denote by $p \prec m$, then p can be indicated by

$$p = 1^{e_{p1}}, 2^{e_{p2}}, ..., m^{e_{pm}} \ , \tag{C-2}$$

where e_i is the frequency of the number i in the partition and then

$$m = e_{p1} + 2e_{p2} + ... + me_{pm} \ . \tag{C-3}$$

The number of summands of p is expressed by $|p|$, which is the sum

$$|p| = e_{p1} + e_{p2} + ... + e_{pm} \ . \tag{C-4}$$

The notation $\hat{p}$ indicates the partition when each summand of a partition p is increased by 1, i.e.

$$\hat{p} = 1^{e_{p1}+1}, 2^{e_{p2}+1}, ..., m^{e_{pm}+1} \ . \tag{C-5}$$

Proposition C.1 *Denote*

$$\alpha_p = \frac{m!}{\prod_{i=1}^{m}[(i!)^{e_{pi}} e_{pi}!]} \ . \tag{C-6}$$

Then the mth order derivative of VaR is

$$\begin{aligned} \left.\frac{\partial^m \alpha_q(X+\varepsilon Y)}{\partial \varepsilon^m}\right|_{\varepsilon=0} =&(-1)^m \Bigg\{ \sum_{p \prec m, u \prec s \leq |p|-1} \Bigg[\frac{\alpha_p \alpha_{\hat{u}} (|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!} \cdot (-f_X(x))^{-|p|-|u|} \\ &\cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f_X(x)}{dx^i} \right]^{e_{ui}} \right) \\ &\cdot \frac{d^{|p|-1-s}}{dx^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}(E(Y^i|X=x) f_X(x))}{dx^{i-1}} \right]^{e_{pi}} \right) \Bigg] \Bigg\} \Bigg|_{x=\alpha_q(X)} . \end{aligned} \tag{C-7}$$

Proof See section 4.5.6.2.4 in Hibbeln, 2010 [27]. ∎

Now Proposition C.1 can be used to get the order of each derivative with respect to N when $m \geq 2$ (note the first derivative is 0).

In equation (C-7), the number of trades N is independent of this part

$$\frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!} \cdot (-f_X(x))^{-|p|-|u|} \cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f_X(x)}{dx^i} \right]^{e_{ui}} \right) \Bigg|_{x=\alpha_q(X)} . \quad \text{(C-8)}$$

So the mth order derivative of VaR can be written as a simplified form

$$\frac{\partial^m \alpha_q(X+\varepsilon Y)}{\partial \varepsilon^m} \Bigg|_{\varepsilon=0} = g \left(\sum_{p \prec m} \prod_{i=1}^{m} [\mu_i(Y|X=x)]^{e_{pi}} \right) \Bigg|_{x=\alpha_q(X)} , \quad \text{(C-9)}$$

where g is a function that is independent of N and μ_i is the ith moment about the origin. Denote η_i is the ith moment about the mean. Remember $X = E(L_N|S_1, S_2)$ and $Y = L_N - E(L_N|S_1, S_2)$, so it is possible to rewrite

$$\begin{aligned}
&\sum_{p \prec m} \prod_{i=1}^{m} [\mu_i(Y|X=x)]^{e_{pi}} \Bigg|_{x=\alpha_q(X)} \\
&= \sum_{p \prec m} \prod_{i=1}^{m} [E[(L_N - E(L_N|S_1,S_2))^i|X=x]]^{e_{pi}} \Bigg|_{x=\alpha_q(X)} \\
&\quad (\text{Since } \sigma(X) = \sigma(E(L_N|S_1,S_2)) \subseteq \sigma(S_1,S_2), \text{by tower property}) \\
&= \sum_{p \prec m} \prod_{i=1}^{m} [E[E[(L_N - E(L_N|S_1,S_2))^i|S_1,S_2]|X=x]]^{e_{pi}} \Bigg|_{x=\alpha_q(X)} \\
&= \sum_{p \prec m} \prod_{i=1}^{m} [E[\eta_i(L_N|S_1,S_2)|X=x]]^{e_{pi}} \Bigg|_{x=\alpha_q(X)} .
\end{aligned} \quad \text{(C-10)}$$

Now lets find the order of $\eta_i(L_N|S_1, S_2)$ with respect to the number of trades

N. Since the η_i is additive to independent random variables only when $i \leq 3$ and $\eta_1 \equiv 0$, two situations will be discussed here.

In the first case, $i = 2$ or 3. Then

$$\eta_i(L_N|S_1, S_2) = \eta_i \left(\sum_{j=1}^{N} \left[u_j \mathbf{I}_{\{T_1^{(j)} > U_j\}} + u_j \mathbf{I}_{\{T_2^{(j)} > U_j\}} \right] \Bigg| S_1, S_2 \right) . \tag{C-11}$$

Conditioning on $\sigma(S_1, S_2)$, all $\mathbf{I}_{\{T_1^{(j)} > U_j\}}$ and $\mathbf{I}_{\{T_2^{(j)} > U_j\}}$ are independent, so equation (C-11) can be simplified as

$$\begin{aligned} \eta_i(L_N|S_1, S_2) &= \eta_i \left(\sum_{j=1}^{N} \left[u_j \mathbf{I}_{\{T_1^{(j)} > U_j\}} + u_j \mathbf{I}_{\{T_2^{(j)} > U_j\}} \right] \Bigg| S_1, S_2 \right) \\ &= \sum_{j=1}^{N} (u_j)^i \left[\eta_i \left(\mathbf{I}_{\{T_1^{(j)} > U_j\}} \Bigg| S_1, S_2 \right) + \eta_i \left(\mathbf{I}_{\{T_2^{(j)} > U_j\}} \Bigg| S_1, S_2 \right) \right] . \end{aligned} \tag{C-12}$$

For any S_1 and S_2, it is obvious $\left| \eta_i \left(\mathbf{I}_{\{T_1^{(j)} > U_j\}} \Big| S_1, S_2 \right) \right| < 1$, so

$$\begin{aligned} |\eta_i(L_N|S_1, S_2)| &\leq 2 \sum_{j=1}^{N} (u_j)^i \\ &\text{(by assumption } (C-1)) \\ &\leq 2N \cdot \left(\frac{C}{N} \right)^i \\ &= O\left(\frac{1}{N^{i-1}} \right) . \end{aligned} \tag{C-13}$$

In the second case, $i > 3$. Then

$$\eta_i(L_N|S_1, S_2) = \sum_{p \prec i, e_{p_1} = 0} a_p \prod_{j=2}^{i} (\kappa_j(L_N|S_1, S_2))^{e_{pj}} ,$$

where κ_j is the cumulants and a_p is the coefficient which can be found through the Faà di Bruno's formula. One proposition is required here to get the order of $\eta_i(L_N|S_1, S_2)$ with respect to N.

Proposition C.2

$$|\kappa_i(L_N|S_1, S_2)| \leq O\left(\frac{1}{N^{i-1}}\right) \ . \tag{C-14}$$

Proof κ_i is additive to the independent random variables. So

$$\begin{aligned}\kappa_i(L_N|S_1, S_2) &= \kappa_i\left(\sum_{j=1}^{N}\left[u_j \mathbf{I}_{\{T_1^{(j)}>U_j\}} + u_j \mathbf{I}_{\{T_2^{(j)}>U_j\}}\right]\middle|S_1, S_2\right) \\ &= \sum_{j=1}^{N}(u_j)^i\left[\kappa_i\left(\mathbf{I}_{\{T_1^{(j)}>U_j\}}\middle|S_1, S_2\right) + \kappa_i\left(\mathbf{I}_{\{T_2^{(j)}>U_j\}}\middle|S_1, S_2\right)\right] \ .\end{aligned} \tag{C-15}$$

Because

$$\begin{aligned}\left|\kappa_i\left(\mathbf{I}_{\{T_1^{(j)}>U_j\}}\middle|S_1, S_2\right)\right| &= \left|\sum_{p\prec i} b_p^{(1)} \prod_{j=1}^{i}\left[\mu_j\left(\mathbf{I}_{\{T_1^{(j)}>U_j\}}\middle|S_1, S_2\right)\right]^{e_{pj}}\right| \\ &\leq \sum_{p\prec i}|b_p^{(1)}|\end{aligned} \tag{C-16}$$

and similarly,

$$\left|\kappa_i\left(\mathbf{I}_{\{T_2^{(j)}>U_j\}}\middle|S_1, S_2\right)\right| \leq \sum_{p\prec i}|b_p^{(2)}| \ , \tag{C-17}$$

then

$$
\begin{aligned}
|\kappa_i(L_N|S_1, S_2)| &\leq \sum_{j=1}^{N} (u_j)^i \left(\sum_{p \prec i} |b_p^{(1)}| + \sum_{p \prec i} |b_p^{(2)}| \right) \\
&\quad \text{(by assumption } (C-1)) \\
&\leq \left(\sum_{p \prec i} |b_p^{(1)}| + \sum_{p \prec i} |b_p^{(2)}| \right) \cdot N \cdot \left(\frac{C}{N} \right)^i \\
&= O\left(\frac{1}{N^{i-1}} \right) \quad . \qquad \text{(C-18)}
\end{aligned}
$$

This ends the proof. ∎

With Proposition C.2,

$$
\begin{aligned}
|\eta_i(L_N|S_1, S_2)| &= \left| \sum_{p \prec i, e_{p_1}=0} a_p \prod_{j=2}^{i} (\kappa_j(L_N|S_1, S_2))^{e_{pj}} \right| \\
&\leq \sum_{p \prec i,\ e_{p_1}=0} |a_p| O\left(\frac{1}{N^{i-|p|}} \right) \\
&= O\left(\frac{1}{N^{i-|p|}} \right) \Bigg|_{p \prec i,\ e_{p_1}=0} \\
&\quad \text{(Since } i > 3, \text{ then } i - |p| \geq 2) \\
&= O\left(\frac{1}{N^2} \right) \qquad \text{(C-19)}
\end{aligned}
$$

Now combine the conclusions in (C-13) and (C-19), and apply them on (C-10). If $m = 2$,

$$
\left| \sum_{p \prec m} \prod_{i=1}^{m} [\mu_i(Y|X = x)]^{e_{pi}} \Big|_{x=\alpha_q(X)} \right|
$$

$$
\begin{aligned}
&= E[\eta_2(L_N|S_1,S_2)|X=x]|_{x=\alpha_q(X)} \\
&\leq O\left(\frac{1}{N}\right) \quad . \qquad \text{(C-20)}
\end{aligned}
$$

If $m \geq 3$,

$$
\begin{aligned}
&\left|\sum_{p\prec m}\prod_{i=1}^{m}[\mu_i(Y|X=x)]^{e_{pi}}\bigg|_{x=\alpha_q(X)}\right| \\
=&\left|\sum_{p\prec m}\prod_{i=1}^{m}[E[\eta_i(L_N|S_1,S_2)|X=x]]^{e_{pi}}\bigg|_{x=\alpha_q(X)}\right| \\
\leq&\sum_{p\prec m} O\left(\frac{1}{N^{2|p|}}\right) \\
=&O\left(\frac{1}{N^{2|p|}}\right)\bigg|_{p\prec m} \\
&\text{(Since } |p| \geq 1\text{)} \\
\leq&O\left(\frac{1}{N^2}\right) \qquad \text{(C-21)}
\end{aligned}
$$

So it is proved that the order of the second derivative is at least $O\left(\frac{1}{N}\right)$, and the order of higher derivative is at least $O\left(\frac{1}{N^2}\right)$.

D Infinite-Period Analytic VaR

D.1 General M-period Model

Similar to the two-period default model (III.1.9), the M-period model is defined

$$L_N^{(M)} = \sum_{j=1}^{M} \left[\sum_{i=1}^{N} u_i \mathbf{I}_{\{T_i^{(j)} > U_i\}} \right] , \tag{D-1}$$

where

$$T_i^{(j)} = \rho_i S_j + \sqrt{1 - \rho_i^2} \xi_i^{(j)} , \tag{D-2}$$

u_i is the loss given default of asset i, all the $\xi_i^{(j)}$ are the idiosyncratic factors which are independent across each other and across each systematic factor S_j, and ρ_i is the positive correlation between asset factor $T_i^{(j)}$ and systematic factor S_i. Note ρ_i is the same in each jth period since the trade has the same behavior over the systematic factor in any time before the maturity. U_i is the threshold to determine if the default of the ith trade will happen. The default level is assumed to be constant, so U_i is the same in each jth period. Again, all S_j and $\xi_i^{(j)}$ are i.i.d. standard normal random variables.

D.2 Analytic VaR by Central Limit Theorem

Denote

$$X_j = \sum_{i=1}^{N} u_i \mathbf{I}_{\{T_i^{(j)} > U_i\}} \ . \tag{D-3}$$

Then

$$L_N^{(M)} = \sum_{j=1}^{M} X_j \ . \tag{D-4}$$

It is easy to find out all X_j are i.i.d.. By central limit theorem,

$$\frac{\sqrt{M}(L_N^{(M)}/M - \mu)}{\sigma} \xrightarrow{d} N(0,1), \text{ when } M \to \infty \ , \tag{D-5}$$

where $\mu = E(X_j)$ and $\sigma^2 = \sigma^2(X_j)$. This means

$$L_N^{(M)} \xrightarrow{d} N(\mu M, \sigma^2 M), \text{ when } M \to \infty \ . \tag{D-6}$$

So when M is large enough, the VaR of this portfolio, $\alpha_q(L_N^{(M)})$ can be approximated as

$$\alpha_q(L_N^{(M)}) \approx \mu M + \sqrt{M\sigma^2}\Phi^{-1}(q) \ . \tag{D-7}$$

D.3 Expectation and Variance of X_j

Now it is necessary to find the expectation and variance of X_j.

$$\begin{aligned}\mu &= E(X_j) \\ &= \sum_{i=1}^{N} u_i E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}}\right) \\ &= \sum_{i=1}^{N} u_i P(T_i^{(j)} > U_i) \\ &= \sum_{i=1}^{N} u_i PD_i^{(j)} \ , \end{aligned} \tag{D-8}$$

where $PD_i^{(j)}$ is the default probability of the ith trade in the jth period. In the multi-period model, default probability is assumed to be on a constant level. And PD_i, the default probability of the ith trade within the whole period is given. By the property of survival probability

$$\begin{aligned}&\prod_{k=1}^{M} P(\text{the } i\text{th trade survives in the } k\text{th period}) \\ &= P(\text{the } i\text{th trade survives in all periods})\end{aligned} \tag{D-9}$$

i.e.

$$\prod_{k=1}^{M}(1 - PD_i^{(k)}) = (1 - PD_i^{(j)})^M = 1 - PD_i \ . \tag{D-10}$$

Then

$$PD_i^{(j)} = 1 - (1 - PD_i)^{\frac{1}{M}} \ . \tag{D-11}$$

So we have

$$\mu = \sum_{i=1}^{N} u_i [1 - (1 - PD_i)^{\frac{1}{M}}] \ . \tag{D-12}$$

The variance is a little more complicated.

$$\sigma^2 = E(X_j^2) - (E(X_j))^2 \ . \tag{D-13}$$

$E(X_j)$ is given by (D-8), and $E(X_j^2)$ can be calculated as

$$\begin{aligned} E(X_j^2) &= \sum_{i=1}^{N} u_i^2 E\left(\mathbf{I}^2_{\{T_i^{(j)} > U_i\}}\right) + \sum_{i,k=1(i \neq k)}^{N} u_i u_k E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}} \mathbf{I}_{\{T_k^{(j)} > U_k\}}\right) \\ &= \sum_{i=1}^{N} u_i^2 E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}}\right) + \sum_{i,k=1(i \neq k)}^{N} u_i u_k E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}} \mathbf{I}_{\{T_k^{(j)} > U_k\}}\right) \ , \end{aligned} \tag{D-14}$$

where $E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}} \mathbf{I}_{\{T_k^{(j)} > U_k\}}\right)$ is

$$\begin{aligned} & E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}} \mathbf{I}_{\{T_k^{(j)} > U_k\}}\right) \\ &= P((T_i^{(j)} > U_i) \cap (T_k^{(j)} > U_k)) \\ &= P(T_i^{(j)} > U_i | T_k^{(j)} > U_k) P(T_k^{(j)} > U_k) \\ &= \int_{-\infty}^{\infty} \phi(s) \left[1 - \Phi\left(\frac{U_i - \rho_i s}{\sqrt{1 - \rho_i^2}}\right)\right] \left[1 - \Phi\left(\frac{U_k - \rho_k s}{\sqrt{1 - \rho_k^2}}\right)\right] ds \cdot PD_k^{(j)} \\ &= \int_{-\infty}^{\infty} \phi(s) PD_i^{(j)}(s) PD_k^{(j)}(s) ds PD_k^{(j)} \ . \end{aligned} \tag{D-15}$$

$PD_i^{(j)}(s)$ is the default probability of the ith trade in the jth period given the systematic factor is equal to s. Again, by the property of survival probability, we have

$$(1 - PD_i^{(j)}(s))^M = 1 - P(\text{the } i\text{th trade is default}|S_1 = s, S_2 = s, ..., S_M = s) \ . \tag{D-16}$$

Assume

$$\widetilde{PD}_i(s) = P(\text{the } i\text{th trade is default}|S_1 = s, S_2 = s, ..., S_M = s) \ , \tag{D-17}$$

then

$$PD_i^{(j)}(s) = 1 - (1 - \widetilde{PD}_i(s))^{\frac{1}{M}} \ . \tag{D-18}$$

So

$$\begin{aligned} & E\left(\mathbf{I}_{\{T_i^{(j)} > U_i\}} \mathbf{I}_{\{T_k^{(j)} > U_k\}}\right) = \\ & \int_{-\infty}^{\infty} \phi(s)[1 - (1 - \widetilde{PD}_i(s))^{\frac{1}{M}}][1 - (1 - \widetilde{PD}_k(s))^{\frac{1}{M}}] ds PD_k^{(j)} \ . \end{aligned} \tag{D-19}$$

Finally, the formula of variance is derived

$$\begin{aligned} \sigma^2 = & \sum_{i=1}^{N} u_i^2 [1 - (1 - PD_i)^{\frac{1}{M}}] \\ & + \sum_{i,k=1(i \neq k)}^{N} u_i u_k \int_{-\infty}^{\infty} \phi(s)[1 - (1 - \widetilde{PD}_i(s))^{\frac{1}{M}}][1 - (1 - \widetilde{PD}_k(s))^{\frac{1}{M}}] ds \\ & \cdot [1 - (1 - PD_k)^{\frac{1}{M}}] - \mu^2 \ . \end{aligned} \tag{D-20}$$

D.4 Infinite periods

Now the only question remains is when M goes to infinity, does the limit of $\alpha_q(L_N^{(M)})$ exist? By (D-7),

$$\lim_{M\to\infty} \alpha_q(L_N^{(M)}) = \lim_{M\to\infty} (M\mu + \sqrt{M\sigma^2}\Phi^{-1}(q)) \ . \tag{D-20}$$

First, let us see if $\lim_{M\to\infty} \mu M$ exist.

$$\begin{aligned}
\lim_{M\to\infty} M\mu &= \sum_{i=1}^{N} u_i \lim_{M\to\infty} M[1-(1-PD_i)^{\frac{1}{M}}] \\
&\quad (\text{let } Y = \frac{1}{M}) \\
&= \sum_{i=1}^{N} u_i \lim_{Y\to 0} \frac{1-(1-PD_i)^Y}{Y} \\
&\quad (\text{by L}'\text{Hopital}'\text{s rule}) \\
&= \sum_{i=1}^{N} u_i \lim_{Y\to 0} -(1-PD_i)^Y ln(1-PD_i) \\
&= \sum_{i=1}^{N} u_i(-ln(1-PD_i)) \ .
\end{aligned} \tag{D-21}$$

Second, let us see if $\lim_{M\to\infty} M\sigma^2$ exist.

$$\begin{aligned}
\lim_{M\to\infty} M\sigma^2 = & \sum_{i=1}^{N} u_i^2 \lim_{M\to\infty} M[1-(1-PD_i)^{\frac{1}{M}}] \\
& + \sum_{i,k=1(i\neq k)}^{N} u_i u_k \lim_{M\to\infty} \left\{ M \int_{-\infty}^{\infty} \phi(s)[1-(1-\widetilde{PD}_i(s))^{\frac{1}{M}}] \right. \\
& \left. \cdot [1-(1-\widetilde{PD}_k(s))^{\frac{1}{M}}]ds \cdot [1-(1-PD_k)^{\frac{1}{M}}] \right\} - \lim_{M\to\infty} M\mu^2 \ .
\end{aligned} \tag{D-22}$$

The limit of the first and the third part is easier, similarly to the derivation of $\lim_{M\to\infty} M\mu$,

$$\sum_{i=1}^{N} u_i^2 \lim_{M\to\infty} M[1-(1-PD_i)^{\frac{1}{M}}] = \sum_{i=1}^{N} u_i^2(-ln(1-PD_i)) \ ; \tag{D-23}$$

$$\lim_{M\to\infty} M\mu^2 = 0 \ . \tag{D-24}$$

For the limit of the second part, **Lebesgue's dominated convergence theorem** is applied. Since

$$\left|\phi(s)[1-(1-\widetilde{PD}_i(s))^{\frac{1}{M}}][1-(1-\widetilde{PD}_k(s))^{\frac{1}{M}}]\right| \leq \phi(s) \ , \tag{D-25}$$

then

$$\begin{aligned} &\lim_{M\to\infty} \int_{-\infty}^{\infty} \phi(s)[1-(1-\widetilde{PD}_i(s))^{\frac{1}{M}}][1-(1-\widetilde{PD}_k(s))^{\frac{1}{M}}]ds \\ =&\int_{-\infty}^{\infty} \lim_{M\to\infty} \phi(s)[1-(1-\widetilde{PD}_i(s))^{\frac{1}{M}}][1-(1-\widetilde{PD}_k(s))^{\frac{1}{M}}]ds \\ =&0 \ . \end{aligned} \tag{D-26}$$

The limit of the rest of the second part is already known,

$$\lim_{M\to\infty} M[1-(1-PD_k)^{\frac{1}{M}}] = -ln(1-PD_k) \ . \tag{D-27}$$

So the limit of the second part is $-ln(1-PD_k)\cdot 0 = 0$.

Finally, we have

$$\lim_{M\to\infty} M\sigma^2 = \sum_{i=1}^{N} u_i^2(-ln(1-PD_i)) \ . \tag{D-28}$$

Substitute the results in (D-21) and (D-28),

$$\lim_{M\to\infty} \alpha_q(L_N^{(M)}) = \sum_{i=1}^{N} u_i(-ln(1-PD_i)) + \sqrt{\sum_{i=1}^{N} u_i^2(-ln(1-PD_i))}\Phi^{-1}(q)\ . \quad \text{(D-29)}$$

In other words, the result in (D-29) can also be used as the approximation of the VaR when M is chosen large enough.

Now the analytic VaR of a portfolio based on large enough time-step have been solved successfully. [1]

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