HIGH ORDER ENERGY-CONSERVED SPLITTING FDTD METHODS FOR MAXWELL'S EQUATIONS

QIANG YUAN

A DISSERTATION SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTORATE OF PHILOSOPHY

GRADUATE PROGRAM IN
MATHEMATICS AND STATISTICS
YORK UNIVERSITY
TORONTO, ONTARIO

APRIL 2014

Abstract

Computation of Maxwell's equations has been playing an important role in many applications, such as the radio-frequencies, microwave antennas, aircraft radar, integrated optical circuits, wireless engineering and materials, etc. It is of particular importance to develop numerical methods to solve the equations effectively and accurately. During the propagation of electromagnetic waves in lossless media without sources, the energies keep constant for all time, which explains the physical feature of electromagnetic energy conservations in long term behaviors. Preserving the invariance of energies is an important issue for efficient numerical schemes for solving Maxwell's equations.

In my thesis, we first develop and analyze the spatial fourth order energy-conserved splitting FDTD scheme for Maxwell's equations in two dimensions. For each time stage, while the spatial fourth-order difference operators are used to approximate the spatial derivatives on strict interior nodes, the important feature is that on the near boundary nodes, we propose a new type of fourth-order boundary difference operators to approximate the derivatives for ensuring energy conservative. The proposed EC-S-FDTD-(2,4) scheme is proved to be energy-conserved, unconditionally stable and of fourth order convergence in space. Secondly, we develop and analyze a new time fourth order EC-S-FDTD scheme. At each stage, we construct a time fourth-order scheme for each-stage splitting equations by converting the third-order correctional temporal derivatives to the spatial third-order differential terms approximated further by the three central difference combination operators. The developed EC-S-FDTD-(4,4) scheme preserves energies in the discrete form and in the discrete variation forms and has both time and spatial fourth-order convergence and super-convergence. Thirdly, for the three dimensional Maxwell's equations, we develop high order energy-conserved splitting FDTD scheme by combining the symplectic splitting with the spatial high order near boundary difference operators and interior difference operators. Theoretical analyses including energy conservations, unconditional stability, error estimates and super-convergence are established for the three dimensional problems. Finally, an efficient Euler-based S-FDTD scheme is developed and analyzed to solve a very important application of Maxwell's equations in Cole-Cole dispersive medium. Numerical experiments are presented in all four parts to confirm our theoretical results.

Acknowledgements

First of all, the great thank should be given to my supervisor Professor Dong Liang. I would like to express my deepest and sincerest gratitude to him for the academic guidance, insight, inspiring discussions, and generous support, which lead to my successful Ph.D study at York University.

I am also grateful to the supervisory committee members, Professors Yuehua Wu and Hongmei Zhu for their valuable help and suggestions during my research.

I am grateful to my parents and my wife, Yanyan Zhang for their patience, support and *love*. Without them this work would never have come into existence.

Table of Contents

Abstract **ii**

Acknowledgements **iv**

Table of Contents **v**

List of Tables **viii**

List of Figures **xiii**

1 Introduction **1**

1.1 **Background** 1

1.2 **Work of Thesis** 4

2 The Spatial Fourth-order Energy-conserved S- FDTD Scheme for Maxwell's Equations **9**

2.1 **Introduction** 9

2.2 **Maxwell's equations and the spatial fourth-order EC-S-FDTD scheme** . . . 12

2.2.1 **Maxwell's equations** . . . 12

2.2.2 **The spatial fourth-order energy-conserved S-FDTD scheme** . . . 15

2.3 **Energy conservation in the discrete form** . . . 24

2.4 **Convergence analysis** . . . 31

2.5 **Numerical Experiments** . . . 45

3 The Time and Spatial High-Order Energy- conserved S-FDTD Scheme for Maxwell's Equations **56**

3.1 **Introduction** . . . 56

3.2 **Maxwell's Equations and Time Fourth-Order EC-S-FDTD Scheme** . . . 59

3.2.1 Maxwell's equations in two dimensions . . . 59

3.2.2 **The spatial high order difference operators** . . . 61

3.2.3 **The splitting method** . . . 64

3.2.4 **The EC-S-FDTD-(4,4) scheme** . . . 66

3.3 **Energy conservations** . . . 68

3.4 **Convergence and super-convergence** . . . 78

3.5 **Numerical experiments** 88

4 **The Spatial High-order Energy-conserved Splitting FDTD Method for Maxwell's Equations in Three Dimensions** **95**

4.1 **Introduction** 95

4.2 **Maxwell's equations and High-order EC-S-FDTD scheme** . 97

4.2.1 **Maxwell's equations in three dimensions** 97

4.2.2 **High order energy-conserved splitting FDTD scheme in 3D** 100

4.3 **Energy conservations** 109

4.4 **Convergence and super-Convergence** 118

4.5 **Numerical experiments** 130

5 **The S-FDTD method for Maxwell's equations in Cole-Cole dispersive medium** **140**

5.1 **Introduction** 140

5.2 **The Cole-Cole dispersive medium models** 142

5.3 **The Euler-based S-FDTD scheme** 144

5.4 **Stability and Convergence** 145

5.5 **Numerical Experiments** 155

6 **Conclusion** **160**

List of Tables

2.1 Relative Errors of EnergyI and EnergyII by EC-S-FDTD I&II, EC-S-FDTD-(2,4), and ADI-FDTD schemes. Parameters: $k_x = k_y = 1$, $\mu = \epsilon = 1$, $\Delta x = \Delta y = \Delta t = 1/N$, at $T = 1$. 48

2.2 Relative Errors of Energy I and Energy II by EC-S-FDTD I&II, EC-S-FDTD-(2,4), and ADI-FDTD schemes. Parameters: $k_y = 1$, $\mu = \epsilon = 1$ and $\Delta x = \Delta y = \Delta t$=0.01 at $T = 1$. 49

2.3 The convergence ratios of ErrorI in spatial step for different schemes. Parameters:$\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = k_y = 1$ and $\mu = \epsilon = 1$, at $T = 1$. 50

2.4 The convergence ratios of ErrorII in spatial step for different schemes. Parameters: $\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = k_y = 1$ and $\mu = \epsilon$ $= 1$, at $T = 1$. 51

2.5 The convergence ratios of ErrorI in time step by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$ and $\mu = \epsilon = 1$ at $T = 1$. 53

2.6 The convergence ratios of ErrorII in time step by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$ and $\mu = \epsilon = 1$, at $T = 1$. 53

2.7 The convergence ratios of divergence-free (DivI) in time step by EC-S-FDTD-I&II, EC-S-FDTD-(2,4), and ADI-FDTD. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$, $\mu = \epsilon = 1$, at $T = 1$. . . 54

2.8 The convergence ratios of divergence-free (DivII) in time step by EC-S-FDTD-I&II, EC-S-FDTD-(2,4), ADI-FDTD. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$, $\mu = \epsilon = 1$ at $T = 1$. . . 55

3.1 Relative errors of Energy I and Energy II by the different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$. 89

3.2 Relative errors of Energy I and Energy II by different schemes. Parameters: $k_y = 1$, $\Delta x = \Delta y = \Delta t$=0.01, $\mu = \epsilon = 1$, and $T = 1$. . . 90

3.3 Errors of energies in spatial variation forms of different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$. 91

3.4 ErrorI and ratios of solutions by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$. 92

3.5 ErrorII and ratios of solutions by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$. 93

3.6 Errors and ratios of divergence-free by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$. 94

4.1 Relative errors of EnergyI (REE-I) and EnergyII (REE-II) by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and with $\mu = \epsilon = 1$, $T = 1$. 133

4.2 Relative errors of EnergyI (I) and EnergyII (II) by EC-S-FDTDI&II (ECI&II), EC-S-FDTD-(2,4) (EC24) and ADI-FDTD (ADI) schemes. Parameters: $k_y = 2k_x$, $k_z = -3k_x$, $\Delta x = \Delta y = \Delta z = \Delta t$=0.02, $\mu = \epsilon = 1$, $T = 1$. 134

4.3 The ErrorI by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and with $\mu = \epsilon = 1$, $T = 1$ 135

4.4 The ErrorII by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters:$\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ 136

4.5 The ErrorI by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and EC-S-FDTD-(4,4) schemes. Parameters:$\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = 1$, $k_y = 2$, $k_z = -3$ 136

4.6 The ErrorII by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = 1$, $k_y = 2$, $k_z = -3$ 137

4.7 The error of divergence-free (DivI) by different schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ 137

4.8 The error of divergence-free (DivII) by different schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ 138

4.9 Relative energy errors in variation form by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and with $\mu = \epsilon = 1$, $T = 1$. 139

5.1 L^2-errors of the Euler-based S-FDTD scheme with $\Delta x = \Delta y = 1/N$, $\Delta t = 1/(N^2)$ and $\alpha = 0.5$ at T = 1. 158

5.2 L^2-errors of the Euler-based S-FDTD scheme with $\Delta x = \Delta y = 1/N$, $\Delta t = 1/(N^2)$ and $\alpha = 0.7$ at T = 1. 158

5.3 L^2-errors of the Euler-based S-FDTD scheme in time step with $\Delta x = \Delta y = 1/256$ and α=0.5 at T = 1. 159

5.4 L^2-errors of the Euler-based S-FDTD scheme in time step with $\Delta x = \Delta y = 1/256$ and α=0.7 at T = 1. 159

List of Figures

2.1 Numerical dispersion against the CFL number with N_λ =40 and $\theta = 65^o$*(a) and numerical dispersion against the number of points per wavelength* N_λ *with* $S = 2.4$ *and* $\theta = 65^o$ *(b).* 46

2.2 Numerical dispersion against the propagation angles ϕ with $N_\lambda = 40$ and $S = 3.5$. 47

2.3 ErrorI *(a) and ErrorII (b) in spatial step by different schemes when* T *= 1,* $N_x = N_y = N$, $N_t = N^2$, $k_x = k_y$ *= 1, and* $\mu = \epsilon$ *= 1.* 52

2.4 ErrorI *(a) and ErrorII (b) in time step by different schemes when* T *= 1,* $N_x = N_y = N_t = N$, $\mu = \epsilon$ *= 1,* $k_x = k_y = 1$ 54

5.1 Numerical solution P_x *(a) and exact solution* P_x *(b) with fixed* τ = 0.0005 *,* α=0.5 *and* h=1/64 *at* $T = 1$. 156

5.2 Numerical solution E_x *(a) and exact solution* E_x *(b) with fixed* τ = 0.0005 *,* α=0.5 *and* h=1/64 *at* $T = 1$. 156

5.3 Numerical solution H_z *(a) and exact solution* H_z *(b) with fixed* $\tau =$ 0.0005 , α=0.5 *and* h=1/64 *at* $T = 1$. 157

1 Introduction

1.1 Background

Maxwell's equations, which are a set of partial differential equations describing the relation of electric and magnetic fields, have been widely used in computational electromagnetics. Computation of Maxwell's equations has been playing an important role in many applications in modern society, such as the micro devices, the radio circuits, microwave circuits antennas, aircraft radar, integrated optical devices, telecommunication chips, wireless engineering and materials, the design of CPU in microelectronics, etc [6, 18, 19, 50, 37, 3, 13, 44, 22]. Recently, it is of particular importance to develop efficient numerical methods to solve the Maxwell's equations effectively and accurately.

For solving Maxwell's equations, the very popular method is the finite-difference time-domain (FDTD) method, which was first proposed by Yee [72] and further developed and analyzed by Taflove and Brodwin [61], Taflove and Hagness [62], Monk and Suli [47]. However, the FDTD method is only conditionally stable and leads to large computational cost and memory requirement in broadband applications with large domain and in high dimensions [61, 62]. Much progress has been made in the past two decades in improving the FDTD method including the ADI and splitting FDTD methods. Alternating direction implicit methods (ADI) have been very successful to solve PDEs. Peaceman, Douglas and Rachford [15, 54] proposed the ADI methods in 1950s, which are proved to be efficient in solving parabolic-type partial differential equations for saving memory and CPU cost (see [12, 16, 37, 11, 15, 54, 71, 34, 33, 26]). In order to overcome the limitation of stability and reduce the computational costs, the ADI or splitting FDTD schemes have been studied for solving Maxwell's equations in [5, 24, 25, 28, 36, 49, 50, 82, 68]. An ADI method combined with Yee's scheme was studied in [28] for two dimensional problems, but for three dimensional Maxwell's equations, it was hard to obtain the unconditional stability property. Papers [49, 50, 82] proposed ADI-FDTD schemes for Maxwell's equations, which were proved to be unconditionally stable and of second-order accuracy. The ADI-FDTD schemes were further analyzed in [27, 80, 81]. The iterated ADI-FDTD schemes were analyzed to solve Maxwell's curl equations in [67, 68]. Recently, combining the splitting technique with the staggered grids, two second-order splitting finite-difference time-domain methods (S-FDTDI and S-FDTDII) were proposed in [24] for Maxwell's equations in two dimensions. These two schemes bring much convenient in practical computation with a very simple structure and are unconditionally stable. The schemes have been applied to solve a scattering problem with PML boundary condition successfully.

During the propagation of electromagnetic waves in an electromagnetic field of lossless medium and without sources, the electromagnetic energies of the waves keep constant for all time. The physical invariance of energies is an important feature of the electromagnetic propagation, which describes its long term behavior. Keeping the invariance of electromagnetic energies in time is an important issue to construct efficient numerical schemes. However, the previous FDTD, ADI-FDTD and S-FDTD schemes break this invariance of energy conservations. Based on the Yee's grid and the splitting technique, [7, 8] developed two energy-conserved splitting FDTD schemes: EC-S-FDTDI and EC-S-FDTDII, where EC-S-FDTDI is first order in time and second order in space and EC-S-FDTDII is second order in both time and space. The EC-S-FDTD schemes have important properties: energy-conserved, unconditionally stable, non-dissipative and computationally efficient. Further, a symmetric energy-conserved splitting FDTD scheme (symmetric EC-S-FDTD) for Maxwell's equations was proposed in [9], which has the same properties of the EC-S-FDTDI and EC-S-FDTDII algorithms.

The second-order FDTD, ADI-FDTD, or splitting FDTD schemes have been used for electromagnetic problems with geometries of moderate electrical sizes. But, when computing modern problems of longer distance wave propagations and moderately high frequency wave propagations and/or large domains and large structures in nature, their numerical errors can no longer be considered as reasonable and the accuracy of second order is rendered questionable. It has led to a great attention to develop high order schemes which produce smaller dispersion or phase error for a given mesh resolution. At present, there is a pressing need for generalized FDTD methods which are simple and high order for large scale computations. In particular, no higher order S-FDTD schemes are available for preserving energy conservations. Thus, there are great interests to develop higher order energy-conserved splitting FDTD schemes for Maxwell's equations, which preserve physical laws of energy conservations.

1.2 Work of Thesis

In my thesis, we will focus on the development of high order energy-conserved splitting FDTD methods for solving Maxwell's equations.

In Chapter 2 of this thesis, we propose and analyze the spatial fourth-order energy-conserved splitting FDTD method with the accuracy of second order in time and fourth order in space (i.e. EC-S-FDTD-(2,4)). Although the second-order schemes have been applied with success for Maxwell's equation's with geometries of moderate electrical sizes. But, for computing large scale problems that requiring long-time integration and of wave propagations over longer distances, some high-order explicit FDTD schemes have been developed ([20, 32, 65, 69, 74, 83]). Associated with fourth-order interior finite difference schemes, one-side high-order approximations and extrapolations/interpolations for the near boundary points were constructed and used, which are suitably accurate relative to the interior difference operators. However, these explicit high-order FDTD schemes are conditionally stable and have prohibitive requirements of computational memories and computational costs, and they break energy conservations. In this part, we consider transverse electric/magnetic (TE/TM) models of Maxwell's equations in two dimensions. The scheme that we proposed is in three stages. For each stage, the approximations of derivatives in space at the strict interior points are defined by the spatial fourth order differences, which are obtained by a linear combination of the central differences, one with a spatial step and the other with three spatial steps. But for the near boundary nodes, the one-sided difference or extrapolations/interpolations schemes lose the energy conservation properties. To construct appropriate energy-conserved boundary difference schemes is a challenging and important task. We propose a new type of difference operators to approximate the derivatives at the near boundary nodes, by combining the PEC boundary conditions, original equations with Taylor's expansion, which ensure the each stage scheme to preserve energy conservation. We strictly prove the scheme to satisfy energy conversation and to be unconditionally stable. We prove that the scheme has the optimal error estimates of fourth order in space and of second order in time. We also obtain the error estimate of the approximation to divergence-free. Numerical dispersion analysis and numerical experiments are given to confirm our theoretical analyses results.

In Chapter 3, we develop and analyze the time high-order EC-S-FDTD scheme. The research in this chapter leads to a new time and spatial fourth-order energy-conserved S-FDTD scheme (i.e. EC-S-FDTD-(4,4)) for solving Maxwell's equations. Splitting of fourth-order in time provides a seven-stage time-stepping procedure. It is an important issue that at each stage, we construct a time and spatial fourth-order scheme for each splitting equation. We derive out the time high-order schemes to each-stage equations by converting the third-order correctional temporal derivatives to the spatial high-order derivatives, where the corresponding systems are with spatial third-order differential modified terms. Similarly to treat the first-order differential terms in Chapter 2, we approximate the spatial third-order differential operators on the strict interior points by the spatial fourth-order difference operators formed by a new linear combination of three central differences, one with a spatial step, the second with three spatial steps and the third with five spatial steps. Further, we construct new fourth-order near-boundary difference operators for the spatial third-order differential operators on the near boundary nodes for preserving energy conservations and the fourth-order accuracy in both time and space. We prove that the proposed EC-S-FDTD-(4,4) scheme satisfies energy conservations in the discrete forms and in the discrete variation forms, and the scheme is unconditionally stable in the discrete L_2-norm and in the discrete H_1-norm. We prove that the EC-S-FDTD-(4,4) scheme has the optimal error estimate of $O(\Delta t^4 + \Delta x^4 + \Delta y^4)$ in the discrete L_2-norm and the super-convergence of $O(\Delta t^4 + \Delta x^4 + \Delta y^4)$ in the discrete H_1-norm. The divergence-free of the scheme is also proved to be the fourth-order in both time and space.

In Chapter 4, we develop high-order energy-conserved splitting FDTD schemes for Maxwell's equations in three dimensions. The ADI-FDTD scheme for three dimensional Maxwell's equations in [50, 82] is unconditionally stable and has second order accuracy, but it does not satisfy the energy conservations. The EC-S-FDTD schemes for three dimensions in [8] are energy-conserved, but they are only of second order accuracy. In the TE or TM models of Maxwell's equations in two dimensions, there are three equations. Only the third equation needs to be split, the other two equations keep unchanged. However, for three dimensional Maxwell's equations, which are a complex system of six equations of the electromagnetic fields $\mathbf{E} = \{E_x, E_y, E_z\}$ and $\mathbf{H} = \{H_x, H_y, H_z\}$. All the six equations need to be split. Thus there is a further difficulty of developing high order accurate and energy-conserved FDTD schemes for three dimensional Maxwell's equations. Based on the staggered grids, and combining the splitting technique with the proposed spatial high order near boundary difference operators and high order interior difference operators, we develop the spatial high-order energy-conserved S-FDTD scheme for three dimensional Maxwell's equations. Theoretical analysis including energy conservations in the discrete form and in the discrete variation forms, unconditional stability and convergence are built for the three dimensional high-order EC-S-FDTD scheme. Optimal error estimates in the L_2-norm and superconvergences in the H_1-norm of the scheme are obtained. Numerical experiments are given to show the performance.

In Chapter 5, the S-FDTD schmeme is proposed to solve an important application problem of Maxwell's equations in Cole-Cole dispersive medium, which contains a fractional time derivative term [41, 35, 58]. The fractional time derivative model is very different from the standard models. The proposed Euler-based splitting FDTD scheme in this part is a two stage scheme. At each stage, for the splitting equations, the fractional time derivative is approximated by the Letnikov-typed difference operator and the spatial differential terms are approximated by the second order difference operators on the staggered meshes. The stability and convergence are strictly proved. We obtain that the proposed S-FDTD scheme for the Cole-Cole medium models is unconditionally stable and has the optimal error estimates of $O(\Delta t + \Delta x^2 + \Delta y^2)$. Numerical experiments show its performance.

2 The Spatial Fourth-order Energy-conserved S-FDTD Scheme for Maxwell's Equations

2.1 Introduction

For computing large scale problems, for problems requiring long-time integration, or for problems of wave propagations over longer distances, and moderately high frequency wave propagations, based on the staggered grids, the fourth-order explicit schemes were developed for solving Maxwell's equations in [20, 32, 65, 69, 74, 83], etc. The one-sided high-order difference or extrapolation/interpolation numerical boundary schemes were provided to be suitably accurate relative to the interior differences. However, the explicit higher-order schemes are conditionally stable and lead to prohibitive requirements of computational memory and computational cost.

In this chapter, we propose the spatial fourth-order energy-conserved splitting FDTD scheme (i.e. EC-S-FDTD-(2,4)) with fourth-order accuracy in space and second-order accuracy in time. We apply a second-order time-step splitting technique, leading to a three-stage time-splitting algorithm for Maxwell's equations. At each stage, on the Yee's staggered grid, we approximate the spatial differential operators by the spatial fourth-order difference operators obtained by a linear combination of two central differences, one with a spatial step and the other with three spatial steps. This obtains the spatial fourth-order scheme on the strict interior nodes of the domain. One important issue is to construct the numerical boundary difference schemes to be energy conservative and high-order relative to the interior difference schemes. It is because the high-order difference operators often have a large spatial stencil which cannot be used in the near boundary nodes. The one-sided differences and extrapolation/interpolation numerical boundary schemes normally break the property of energy conservations near the boundary. Appropriate energy-conserved numerical boundary difference schemes can be difficult to obtain, and this leads to a challenge of constructing energy-conserved higher-order S-FDTD schemes. We propose to construct the spatial fourth-order near boundary differences over the near boundary nodes by using the PEC boundary conditions, original equations and Taylor's expansion, which ensure the each-stage schemes to preserve the conservations of energy and to have fourth-order accuracy. The proposed EC-S-FDTD-(2,4) scheme has the significant properties that are energy-conserved, unconditionally stable, non-dissipative, high-order accurate, and computationally efficient. We strictly prove that the scheme satisfies energy conversations and is unconditionally stable. We analyze theoretically the convergence of the scheme by using the energy method and obtain the optimal-order error estimates of $O(\Delta t^2 + \Delta x^4 + \Delta y^4)$ in the discrete L_2-norm for the approximations of the electric and magnetic fields. Further, the divergence-free convergence is analyzed and we obtain the error estimate of the approximation of divergence-free. Numerical dispersion analysis verifies that the proposed scheme is non-dissipative. Numerical experiments show that the proposed scheme preserves energy conservations and has fourth-order accuracy in space and second-order in time. We also test numerical divergence-free and the divergence-free is of second order convergence in time step.

The sections are organized as follows. Section 2.2 gives a brief introduction of Maxwell's equations. Then the EC-S-FDTD-(2,4) is proposed. In Section 2.3, we prove the property of energy conservations. The truncation error and convergence analysis is given in Section 2.4. Numerical dispersion analysis and numerical experiments are presented in Section 2.5.

2.2 Maxwell's equations and the spatial fourth-order EC-S-FDTD scheme

We first introduce Maxwell's equations and give the two-dimensional transverse electric (TE) equations. We then present our spatial fourth-order energy-conserved splitting finite difference time domain scheme in this section.

2.2.1 Maxwell's equations

The Maxwell's equations in an isotropic, homogeneous and lossless medium are

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \tag{2.2.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{2.2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic fields; $\mathbf{D}$ and $\mathbf{B}$ are the electric displacement and magnetic flux density, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. (2.2.1) is Faraday's Law and (2.2.2) is Ampere's Law. In the absence of electric charge, the electric displacement and magnetic flux density satisfy divergence-free conditions (Gauss's Law)

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2.3}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{2.2.4}$$

where ϵ is the electric permittivity and μ is the magnetic permeability. The speed of the electromagnetic wave is $c = \frac{1}{\sqrt{\epsilon\mu}}$.

For the simplicity of notations, we shall focus on the two-dimensional transverse electric (**TE**) problems in a lossless medium and without sources and charges, where the electric field is a plane vector while the magnetic field is a scalar. Let the domain $\Omega = [0, a] \times [0, b]$ and the time period $T > 0$. The two-dimensional Maxwell's equations (TE) are

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\frac{\partial H_z}{\partial y}, \qquad (x, y) \in \Omega, t \in (0, T] \tag{2.2.5}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x}, \qquad (x, y) \in \Omega, t \in (0, T] \tag{2.2.6}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right), \qquad (x, y) \in \Omega, t \in (0, T] \tag{2.2.7}$$

where the electric and magnetic fields are $\mathbf{E} = (E_x(x, y, t), E_y(x, y, t))$ and H_z=$H_z(x, y, t)$. We consider the perfectly electric conducting (PEC) boundary condition:

$$(\mathbf{E}, 0) \times (\mathbf{n}, 0) = 0, \quad \text{on } (0, \mathbf{T}] \times \partial\Omega, \tag{2.2.8}$$

where $\mathbf{n}$ is the outward normal vector on the boundary. The initial conditions are given as

$$\mathbf{E}(x, y, 0) = \mathbf{E}_0(x, y) = (E_{x0}(x, y), E_{y0}(x, y)) \text{ and } H_z(x, y, 0) = H_{z0}(x, y). \tag{2.2.9}$$

It has been proved in [37] that for suitably smooth data, problem (2.2.5) - (2.2.9) has a unique smooth solution for all time, and if the initial fields satisfy divergence-free, the electric and magnetic fields always satisfy divergence-free for all time. For the problems in lossless medium, Poynting's theorem gives that electromagnetic waves satisfy energy conservations (see [6, 7, 18]).

Lemma 2.2.1. *(Energy conservation I) If* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *are the solutions of the Maxwell's equations (2.2.5)-(2.2.7) in lossless medium and satisfy the PEC boundary condition (2.2.8). Then it holds that for any* $t \geq 0$

$$\int_{\Omega} \left(\epsilon|\boldsymbol{E}(x,t)|^2 + \mu|\boldsymbol{H}(x,t)|^2\right) dx \equiv Constant. \tag{2.2.10}$$

This lemma describes that the energy of electromagnetic waves in lossless medium keeps constant for all time. There is another energy conservation in the time-variation form, which is given in the following lemma (see [7]).

Lemma 2.2.2. *(Energy conservation II) If* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *are the solutions of the Maxwell's equations (2.2.5)-(2.2.7) in lossless medium, and satisfy the PEC boundary condition (2.2.8). Then, it holds that*

$$\int_{\Omega} \left(\epsilon\left|\frac{\partial \boldsymbol{E}}{\partial t}\right|^2 + \mu\left|\frac{\partial \boldsymbol{H}}{\partial t}\right|^2\right) dx \equiv Constant. \tag{2.2.11}$$

The electromagnetic waves in lossless medium satisfy these two energy conservations (2.2.10) and (2.2.11), which are very important invariance for a long term propagation of electromagnetic waves. Thus, it is important to develop energy-conserved numerical schemes for solving Maxwell's equations in long term computations. Chen et al. [7] first proposed two second-order energy-conserved S-FDTD schemes (EC-S-FDTD I&II), which satisfy the energy conservations, unconditional stability, and have second-order accuracy in space step and time step. However, it is a challenging task to develop spatial higher-order energy-conserved S-FDTD schemes for the electromagnetic computations.

2.2.2 The spatial fourth-order energy-conserved S-FDTD scheme

We then propose the spatial fourth-order energy-conserved S-FDTD scheme for the two-dimensional Maxwell's equations (2.2.5) - (2.2.9).

For simplicity, take an uniformly staggered grid for the partition of the space domain Ω and the time interval $(0,T]$ by $\Delta x = \frac{a}{I}$, $\Delta y = \frac{b}{J}$, $\Delta t = \frac{T}{N}$; $x_i = i\Delta x$, $x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x$, $i = 0,1,\cdots,I-1$, $x_I = I\Delta x = a$; $y_j = j\Delta y$, $y_{j+\frac{1}{2}} = y_j + \frac{1}{2}\Delta y$, $j = 0,1,\cdots,J-1$, $y_J = J\Delta y = b$; $t^n = n\Delta t$, $t^{n+\frac{1}{2}} = t^n + \frac{1}{2}\Delta t$, $n = 0,1,\cdots,N-1$, $t^N = N\Delta t = T$; where $I>0$, $J>0$ and $N>0$ are integers.

The unknown grid function $\{E_{x_{i+\frac{1}{2},j}}\}$ is defined on nodes $(x_{i+\frac{1}{2}}, y_j), i = 0,1,\cdots,I-1, j = 0,1,\cdots,J$; $\{E_{y_{i,j+\frac{1}{2}}}\}$ is defined on nodes $(x_i, y_{j+\frac{1}{2}}), i = 0,1,\cdots,I, j = 0,1,\cdots,J-1$; and $\{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}$ is defined on nodes $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}), i = 0,1,\cdots,I-1; j = 0,1,\cdots,J-1$. Let $U^n_{\alpha,\beta} = U(n\Delta t, \alpha\Delta x, \beta\Delta y)$ be a grid function where $\alpha = i$ or $i+\frac{1}{2}$, and $\beta = j$ or $j+\frac{1}{2}$. We define the difference operators $\delta_t U$, $\delta_x U$, $\delta_y U$, and $\delta_u\delta_v U$ by

$$\delta_t U^n_{\alpha,\beta} = \frac{U^{n+\frac{1}{2}}_{\alpha,\beta} - U^{n-\frac{1}{2}}_{\alpha,\beta}}{\Delta t},\ \ \delta_x U^n_{\alpha,\beta} = \frac{U^n_{\alpha+\frac{1}{2},\beta} - U^n_{\alpha-\frac{1}{2},\beta}}{\Delta x},$$

$$\delta_y U^n_{\alpha,\beta} = \frac{U^n_{\alpha,\beta+\frac{1}{2}} - U^n_{\alpha,\beta-\frac{1}{2}}}{\Delta y}, \quad \delta_u\delta_v U^n_{\alpha,\beta} = \delta_u(\delta_v U^n_{\alpha,\beta}),$$

where u and v can be taken as the $x-$ and $y-$ directions respectively, and define the difference operators $\delta_{2,x}U$ and $\delta_{2,y}U$ with three spatial steps by

$$\delta_{2,x} U^n_{\alpha,\beta} = \frac{U^n_{\alpha+\frac{3}{2},\beta} - U^n_{\alpha-\frac{3}{2},\beta}}{3\Delta x}, \quad \delta_{2,y} U^n_{\alpha,\beta} = \frac{U^n_{\alpha,\beta+\frac{3}{2}} - U^n_{\alpha,\beta-\frac{3}{2}}}{3\Delta y}.$$

Now, we define the spatial fourth-order difference operator to $\frac{\partial}{\partial x}E_y$ for the strict interior nodes by a linear combination of two central differences, one with a spatial step and the other with three spatial steps above, as

$$\Lambda_x {E_y}^n_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{8}(9\delta_x - \delta_{2,x}){E_y}^n_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{2.2.12}$$

for $i = 1, 2, \cdots, I-2$ and $j = 0, 1, \cdots, J-1$.

The fourth-order difference operator (2.2.12) can be used to approximate the equations at the strict interior nodes with $i = 1, 2, \cdots, I-2$. However, when we treat the near boundary nodes with $i = 0$ and $i = I-1$, the function values in the definition of $\delta_{2,x}{E_y}^n_{i+\frac{1}{2},j+\frac{1}{2}}$ will go out the domain where ${E_y}_{-1,j+\frac{1}{2}}$ and ${E_y}_{I+1,j+\frac{1}{2}}$ are not defined. For constructing high-order difference operators on the near boundary nodes, one could use no symmetric one-sided difference/extrapolation operators by using more one-sided interior point values. But, these kind one-sided operators will make the scheme to break energy conservations. Thus, it is important to construct the high order difference operators on the near boundary nodes to have high-order accuracy in spatial step and to lead to one energy conserved scheme.

For this goal, we first give Lemma 2.2.3. Define some new notations as $x_{-1} = -\Delta x$, $x_{-\frac{1}{2}} = x_{-1} + \frac{1}{2}\Delta x$, $x_{I+1} = (I+1)\Delta x$, $x_{I+\frac{1}{2}} = x_I + \frac{1}{2}\Delta x$, $y_{-1} = -\Delta y$, $y_{-\frac{1}{2}} = y_{-1} + \frac{1}{2}\Delta y$, $y_{J+1} = (J+1)\Delta y$, and $y_{J+\frac{1}{2}} = y_J + \frac{1}{2}\Delta y$.

Lemma 2.2.3. *If the electric and magnetic fields $\{\boldsymbol{E}, H_z\}$ of the solution of system (2.2.5)-(2.2.9) are smooth enough and the initial field $\boldsymbol{E}_0$ is divergence-free, then the following relations hold*

$$\begin{aligned} E_x(x_{i+\frac{1}{2}}, y_{-1}, t) &= 2E_x(x_{i+\frac{1}{2}}, y_0, t) - E_x(x_{i+\frac{1}{2}}, y_1, t) + O(\Delta y^5), \quad (2.2.13)\\ E_x(x_{i+\frac{1}{2}}, y_{J+1}, t) &= 2E_x(x_{i+\frac{1}{2}}, y_J, t) - E_x(x_{i+\frac{1}{2}}, y_{J-1}, t) + O(\Delta y^5), \quad (2.2.14)\end{aligned}$$

$$\begin{aligned} E_y(x_{-1}, y_{j+\frac{1}{2}}, t) &= 2E_y(x_0, y_{j+\frac{1}{2}}, t) - E_y(x_1, y_{j+\frac{1}{2}}, t) + O(\Delta x^5), \quad (2.2.15)\\ E_y(x_{I+1}, y_{j+\frac{1}{2}}, t) &= 2E_y(x_I, y_{j+\frac{1}{2}}, t) - E_y(x_{I-1}, y_{j+\frac{1}{2}}, t) + O(\Delta x^5), \quad (2.2.16)\end{aligned}$$

$$\begin{aligned} H_z(x_{-\frac{1}{2}}, y_{j+\frac{1}{2}}, t) &= H_z(x_{\frac{1}{2}}, y_{j+\frac{1}{2}}, t) + O(\Delta x^5), \quad (2.2.17)\\ H_z(x_{I+\frac{1}{2}}, y_{j+\frac{1}{2}}, t) &= H_z(x_{I-\frac{1}{2}}, y_{j+\frac{1}{2}}, t) + O(\Delta x^5), \quad (2.2.18)\\ H_z(x_{i+\frac{1}{2}}, y_{-\frac{1}{2}}, t) &= H_z(x_{i+\frac{1}{2}}, y_{\frac{1}{2}}, t) + O(\Delta y^5), \quad (2.2.19)\\ H_z(x_{i+\frac{1}{2}}, y_{J+\frac{1}{2}}, t) &= H_z(x_{i+\frac{1}{2}}, y_{J-\frac{1}{2}}, t) + O(\Delta y^5). \quad (2.2.20)\end{aligned}$$

Proof. Because of the PEC condition, $E_x(x,0,t) = E_x(x,b,t) = 0, E_y(0,y,t) = E_y(a,y,t) = 0$, it holds that

$$\frac{\partial E_x(x,0,t)}{\partial t} = \frac{\partial E_x(x,b,t)}{\partial t} = 0, \ \frac{\partial E_y(0,y,t)}{\partial t} = \frac{\partial E_y(a,y,t)}{\partial t} = 0; \quad (2.2.21)$$

$$\frac{\partial^k E_x(x,0,t)}{\partial x^k} = \frac{\partial^k E_x(x,b,t)}{\partial x^k} = 0, \ \frac{\partial^k E_y(0,y,t)}{\partial y^k} = \frac{\partial^k E_y(a,y,t)}{\partial y^k} = 0, \tag{2.2.22}$$

for $k = 0, 1, \cdots, 4$.

From equations (2.2.5), (2.2.6), we have that $\frac{\partial H_z}{\partial y}(x,0,t) = \frac{\partial H_z}{\partial y}(x,b,t) = 0$ and $\frac{\partial H_z}{\partial x}(0,y,t) = \frac{\partial H_z}{\partial x}(a,y,t) = 0$. From (2.2.5) and (2.2.6), and the initial divergence-free $\nabla \cdot \mathbf{E}_0 = 0$, we then have that $\frac{\partial}{\partial x}E_x(x,y,t) + \frac{\partial}{\partial y}E_y(x,y,t) = 0$ for $t > 0$. Thus, $\frac{\partial E_x}{\partial x}(0,y,t) = 0$, $\frac{\partial E_x}{\partial x}(a,y,t) = 0$ and $\frac{\partial E_y}{\partial y}(x,0,t) = 0$, $\frac{\partial E_y}{\partial y}(x,b,t) = 0$. Further, taking derivative to (2.2.7) with respect to y-variable, we obtain that $\frac{\partial^2 E_x}{\partial y^2}(x,0,t) = \frac{\partial^2 E_x}{\partial y^2}(x,b,t) = 0$. Similarly, from (2.2.7), we get that $\frac{\partial^2 E_y}{\partial x^2}(0,y,t) = \frac{\partial^2 E_y}{\partial x^2}(a,y,t) = 0$. In the same way, we can further get that for $k = 0, 1$, and 2

$$\frac{\partial^{2k} E_x(x,0,t)}{\partial y^{2k}} = \frac{\partial^{2k} E_x(x,b,t)}{\partial y^{2k}} = 0, \ \frac{\partial^{2k} E_y(0,y,t)}{\partial x^{2k}} = \frac{\partial^{2k} E_y(a,y,t)}{\partial x^{2k}} = 0; \tag{2.2.23}$$

$$\frac{\partial^{2k+1} H_z(0,y,t)}{\partial x^{2k+1}} = \frac{\partial^{2k+1} H_z(a,y,t)}{\partial x^{2k+1}} = 0, \ \frac{\partial^{2k+1} H_z(x,0,t)}{\partial y^{2k+1}} = \frac{\partial^{2k+1} H_z(x,b,t)}{\partial y^{2k+1}} = 0. \tag{2.2.24}$$

Now, we prove (2.2.13). Using Taylor's expansion about $y = y_0$, it holds that

$$\begin{aligned} E_x(x,y_{-1},t) = {} & E_x(x,y_0,t) - \Delta y \frac{\partial E_x}{\partial y}(x,y_0,t) + \frac{1}{2!}(\Delta y)^2 \frac{\partial^2 E_x}{\partial y^2}(x,y_0,t) \\ & - \frac{1}{3!}(\Delta y)^3 \frac{\partial^3 E_x}{\partial y^3}(x,y_0,t) + \frac{1}{4!}(\Delta y)^4 \frac{\partial^4 E_x}{\partial y^4}(x,y_0,t) + O(\Delta y^5), \end{aligned} \tag{2.2.25}$$

$$\begin{aligned} E_x(x,y_1,t) = {} & E_x(x,y_0,t) + \Delta y \frac{\partial E_x}{\partial y}(x,y_0,t) + \frac{1}{2!}(\Delta y)^2 \frac{\partial^2 E_x}{\partial y^2}(x,y_0,t) \\ & + \frac{1}{3!}(\Delta y)^3 \frac{\partial^3 E_x}{\partial y^3}(x,y_0,t) + \frac{1}{4!}(\Delta y)^4 \frac{\partial^4 E_x}{\partial y^4}(x,y_0,t) + O(\Delta y^5). \end{aligned} \tag{2.2.26}$$

Adding (2.2.25) and (2.2.26), and using (2.2.23), we get (2.2.13). Similarly, we can get (2.2.14) - (2.2.20). This ends the proof. $\square$

Now, using the relationship of (2.2.15), we can derive that

$$\begin{aligned}\delta_{2,x}E_{y_{\frac{1}{2},j+\frac{1}{2}}}(t^n) &= \frac{E_{y_{2,j+\frac{1}{2}}}(t^n)-E_{y_{-1,j+\frac{1}{2}}}(t^n)}{3\Delta x}\\ &= \frac{E_{y_{1,j+\frac{1}{2}}}(t^n)+E_{y_{2,j+\frac{1}{2}}}(t^n)-2E_{y_{0,j+\frac{1}{2}}}(t^n)}{3\Delta x}+O(\Delta x^4)\end{aligned}$$

and thus, we can re-define the spatial fourth-order difference operator $\widetilde{\delta}_{2,x}E_y$ for the near boundary node with $i=0$ by

$$\widetilde{\delta}_{2,x}E^n_{y_{\frac{1}{2},j+\frac{1}{2}}} = \frac{E^n_{y_{1,j+\frac{1}{2}}}+E^n_{y_{2,j+\frac{1}{2}}}-2E^n_{y_{0,j+\frac{1}{2}}}}{3\Delta x}. \tag{2.2.27}$$

Similarly, for $i=I-1$, it can be defined by

$$\widetilde{\delta}_{2,x}E^n_{y_{I-\frac{1}{2},j+\frac{1}{2}}} = \frac{2E^n_{y_{I,j+\frac{1}{2}}}-E^n_{y_{I-1,j+\frac{1}{2}}}-E^n_{y_{I-2,j+\frac{1}{2}}}}{3\Delta x}. \tag{2.2.28}$$

Thus, we can define the difference operators to approximate $\frac{\partial}{\partial x}E_y$ for the near boundary nodes with $i=0$ and $i=I-1$ by

$$\begin{aligned}\widetilde{\Lambda}_x E_y{}^n_{\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{8}(9\delta_x-\widetilde{\delta}_{2,x})E_y{}^n_{\frac{1}{2},j+\frac{1}{2}}, &(2.2.29)\\ \widetilde{\Lambda}_x E_y{}^n_{I-\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{8}(9\delta_x-\widetilde{\delta}_{2,x})E_y{}^n_{I-\frac{1}{2},j+\frac{1}{2}}, &(2.2.30)\end{aligned}$$

for $j=0,1,\cdots,J-1$.

In the same way, we can define other difference operators for the near boundary nodes as

$$\widetilde{\delta}_{2,y} E^n_{x_{i+\frac{1}{2},\frac{1}{2}}} = \frac{E^n_{x_{i+\frac{1}{2},1}} + E^n_{x_{i+\frac{1}{2},2}} - 2E^n_{x_{i+\frac{1}{2},0}}}{3\Delta y},$$

$$\widetilde{\delta}_{2,y} E^n_{x_{i+\frac{1}{2},\frac{1}{2}}} = \frac{2E^n_{y_{i+\frac{1}{2},J}} - E^n_{y_{i+\frac{1}{2},J-1}} - E^n_{y_{i+\frac{1}{2},J-2}}}{3\Delta y},$$

$$\widetilde{\delta}_{2,x} H^n_{z_{1,j+\frac{1}{2}}} = \frac{H^n_{z_{2+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{\frac{1}{2},j+\frac{1}{2}}}}{3\Delta x}, \quad \widetilde{\delta}_{2,x} H^n_{z_{I-1,j+\frac{1}{2}}} = \frac{H^n_{z_{I-\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{I-2-\frac{1}{2},j+\frac{1}{2}}}}{3\Delta x},$$

$$\widetilde{\delta}_{2,y} H^n_{z_{i+\frac{1}{2},1}} = \frac{H^n_{z_{i+\frac{1}{2},2+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},\frac{1}{2}}}}{3\Delta y}, \quad \widetilde{\delta}_{2,y} H^n_{z_{i+\frac{1}{2},J-1}} = \frac{H^n_{z_{i+\frac{1}{2},J-\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},J-2-\frac{1}{2}}}}{3\Delta y}.$$

Further, we can similarly define the spatial fourth-order difference operators to approximate $\frac{\partial}{\partial y} E_x$, $\frac{\partial}{\partial x} H_z$ and $\frac{\partial}{\partial y} H_z$ for the strict interior nodes and the near boundary nodes by

$$\Lambda_y {E_x}^n_{i+\frac{1}{2},j+\frac{1}{2}} \tag{2.2.31}$$
$$= \frac{1}{8}(9\delta_y - \delta_{2,y}){E_x}^n_{i+\frac{1}{2},j+\frac{1}{2}},\ i = 0, 1, \cdots, I-1; j = 1, 2, \cdots, J-2,$$

$$\widetilde{\Lambda}_y {E_x}^n_{i+\frac{1}{2},j+\frac{1}{2}} \tag{2.2.32}$$
$$= \frac{1}{8}(9\delta_y - \widetilde{\delta}_{2,y}){E_y}^n_{i+\frac{1}{2},j+\frac{1}{2}},\ i = 0, 1, \cdots, I-1; j = 0\, and\, j = J-1,$$

$$\Lambda_x {H_z}^n_{i,j+\frac{1}{2}}$$
$$= \frac{1}{8}(9\delta_x - \delta_{2,x}){H_z}^n_{i,j}, \quad i = 2, 3, \cdots, I-2; j = 0, 1, \cdots, J-1, \tag{2.2.33}$$

$$\widetilde{\Lambda}_x {H_z}^n_{i,j+\frac{1}{2}}$$
$$= \frac{1}{8}(9\delta_x - \widetilde{\delta}_{2,x}){H_z}^n_{i,j}, \quad i = 1\, and\, I = I-1; j = 0, 1, \cdots, J-1, \tag{2.2.34}$$

$$\Lambda_y {H_z}^n_{i+\frac{1}{2},j}$$
$$= \frac{1}{8}(9\delta_y - \delta_{2,y}){H_z}^n_{i,j}, \quad i = 0, 1, \cdots, I-1; j = 2, 3, \cdots, J-2, \tag{2.2.35}$$

$$\widetilde{\Lambda}_y {H_z}^n_{i+\frac{1}{2},j}$$
$$= \frac{1}{8}(9\delta_y - \widetilde{\delta}_{2,y}){H_z}^n_{i,j}, \quad i = 0, 1, \cdots, I-1; j = 1\, and\, j = J-1. \tag{2.2.36}$$

Now, we consider the splitting procedure of the two dimensional Maxwell's equations (2.2.5) - (2.2.7) in each time interval $(t^n, t^{n+1}]$

$$\begin{cases} \frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\frac{\partial H_z}{\partial y} \\ \frac{1}{2}\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\frac{\partial E_x}{\partial y} \end{cases} \quad and \quad \begin{cases} \frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x} \\ \frac{1}{2}\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\frac{\partial E_y}{\partial x}. \end{cases} \tag{2.2.37}$$

By applying the spatial fourth-order difference operators (2.2.12) (2.2.29) (2.2.30) and (2.2.31)-(2.2.36), we can finally propose a spatial fourth-order energy-conserved splitting FDTD scheme for Maxwell's equations (2.2.7) - (2.2.9). The scheme can be called as the EC-S-FDTD-(2,4) scheme as it is fourth-order in space step and second-order in time step.

The spatial fourth-order energy-conserved splitting FDTD scheme (i.e. EC-S-FDTD-(2,4)) is defined as

Stage 1: Compute the variables E_x^* and H_z^* from E_x^n and H_z^n by that for the strict interior nodes, $i = 0, 1, \cdots, I-1$

$$\begin{aligned} \frac{E^*_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t} &= \frac{1}{4\epsilon}\Lambda_y\{H^*_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}}\}, \\ & j = 2, 3, \cdots, J-2, \end{aligned} \tag{2.2.38}$$

$$\begin{aligned} \frac{H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} &= \frac{1}{4\mu}\Lambda_y\{E^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\}, \\ & j = 1, 2, \cdots, J-2, \end{aligned} \tag{2.2.39}$$

and for the near boundary nodes, $i = 0, 1, \cdots, I-1$

$$\frac{E^*_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\widetilde{\Lambda}_y\{H^*_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}}\},\ j = 1,\ J-1, \tag{2.2.40}$$

$$\frac{H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\widetilde{\Lambda}_y\{E^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\},\ j = 0,\ J-1, \tag{2.2.41}$$

Stage 2: Compute the variables E_y^{n+1} and H_z^{**} from E_y^n and H_z^* by that for the strict interior nodes, $j = 0, 1, \cdots, J-1$

$$\begin{aligned}
\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} &= -\frac{1}{2\epsilon}\Lambda_x\{H^{**}_{z_{i,j+\frac{1}{2}}} + H^{*}_{z_{i,j+\frac{1}{2}}}\}, \\
& \quad i = 2, 3, \cdots, I-2, \qquad (2.2.42) \\
\frac{H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} &= -\frac{1}{2\mu}\Lambda_x\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}, \\
& \quad i = 1, 2, \cdots, I-2, \qquad (2.2.43)
\end{aligned}$$

and for the near boundary nodes, $j = 0, 1, \cdots, J-1$

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\widetilde{\Lambda}_x\{H^{**}_{z_{i,j+\frac{1}{2}}} + H^{*}_{z_{i,j+\frac{1}{2}}}\},\ i = 1,\ I-1, \qquad (2.2.44)$$

$$\frac{H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\mu}\widetilde{\Lambda}_x\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\},\ i = 0,\ I-1, \qquad (2.2.45)$$

Stage 3: Compute the variables E^{n+1}_x and H^{n+1}_z from H^{**}_z and E^{*}_x by that for the strict interior nodes, $i = 0, 1, \cdots, I-1$

$$\begin{aligned}
\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{*}_{x_{i+\frac{1}{2},j}}}{\Delta t} &= \frac{1}{4\epsilon}\Lambda_y\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{**}_{z_{i+\frac{1}{2},j}}\}, \\
& \quad j = 2, 3, \cdots, J-2 \qquad (2.2.46) \\
\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} &= \frac{1}{4\mu}\Lambda_y\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\}, \\
& \quad j = 1, 2, \cdots, J-2, \qquad (2.2.47)
\end{aligned}$$

and for the near boundary nodes, $i = 0, 1, \cdots, I-1$

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{*}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\widetilde{\Lambda}_y\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{**}_{z_{i+\frac{1}{2},j}}\},\ j = 1,\ J-1, \qquad (2.2.48)$$

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\widetilde{\Lambda}_y\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\},\ j = 0,\ J-1. \qquad (2.2.49)$$

The boundary conditions are given by

$$E^{*}_{x_{i+\frac{1}{2},0}} = E^{*}_{x_{i+\frac{1}{2},J}} = E^{n+1}_{x_{i+\frac{1}{2},0}} = E^{n+1}_{x_{i+\frac{1}{2},J}} = E^{n+1}_{y_{0,j+\frac{1}{2}}} = E^{n+1}_{y_{I,j+\frac{1}{2}}} = 0, \tag{2.2.50}$$

and the initial conditions are given by

$$E^{0}_{x_{\alpha,\beta}} = E_{x0}(\alpha\Delta x, \beta\Delta y); \quad E^{0}_{y_{\alpha,\beta}} = E_{y0}(\alpha\Delta x, \beta\Delta y);$$

$$H^{0}_{z_{\alpha,\beta}} = H_{z0}(\alpha\Delta x, \beta\Delta y). \tag{2.2.51}$$

Remark 1. This three-stage algorithm (2.2.38) - (2.2.51) can be easily solved. In the following sections, we will prove that it has fourth-order accuracy in space step and second-order accuracy in time step and it satisfies energy conservations I & II (2.2.10) and (2.2.11) in the discrete forms.

2.3 Energy conservation in the discrete form

In this section, we will prove the spatial fourth-order EC-S-FDTD scheme to satisfy two energy conservations in the discrete form. Let the discrete L^2-norms on the staggered grids be

$$\|U\|^2_{E_x} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|U_{i+\frac{1}{2},j}\right|^2 \Delta x\Delta y, \;\; \|V\|^2_{E_y} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|V_{i,j+\frac{1}{2}}\right|^2 \Delta x\Delta y,$$

$$\|W\|^2_{H} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|W_{i+\frac{1}{2},j+\frac{1}{2}}\right|^2 \Delta x\Delta y, \;\; \|\mathbf{F}\|^2_{E} = \|U\|^2_{E_x} + \|V\|^2_{E_y},$$

for grid functions $\mathbf{F} = (U, V)$ over electric field mesh and W over magnetic field mesh.

For analyzing energy conservations, we first list Lemmas 2.3.1 and 2.3.2 ([48]).

Lemma 2.3.1. *For $p \geq 1$, let $\{a_k\}_{k=1}^{p}$ and $\{b_k\}_{k=0}^{p}$ be two sequences. Then, it holds*

$$\sum_{k=1}^{p} a_k(b_k - b_{k-1}) = a_p b_p - a_1 b_0 - \sum_{k=1}^{p-1} b_k(a_{k+1} - a_k). \tag{2.3.1}$$

Lemma 2.3.2. *For $p \geq 1$, let $\{a_k\}_{k=1}^{p}$, $\{b_k\}_{k=0}^{p}$ be two sequences. Then, it holds*

$$\begin{aligned}\sum_{k=2}^{p-1} a_k(b_{k+1} - b_{k-2}) &= a_p b_{p-2} + a_{p-1} b_p + a_{p-2} b_{p-1} - a_1 b_2 - a_2 b_0 - a_3 b_1 \\ &\quad - \sum_{k=2}^{p-2} b_k(a_{k+2} - a_{k-1}).\end{aligned} \tag{2.3.2}$$

Proof. We have that

$$\begin{aligned}\sum_{k=2}^{p-1} a_k(b_{k+1} - b_{k-2}) &= \sum_{k=2}^{p-1} a_k(b_{k+1} - b_k + b_k - b_{k-1} + b_{k-1} - b_{k-2}) \\ &= \sum_{k=2}^{p-1} a_k(b_{k+1} - b_k) + \sum_{k=2}^{p-1} a_k(b_k - b_{k-1}) + \sum_{k=2}^{p-1} a_k(b_{k-1} - b_{k-2}).\end{aligned}$$

By using Lemma 2.3.1, the above equation can be changed to

$$\begin{aligned}&\sum_{k=2}^{p-1} a_k(b_{k+1} - b_{k-2}) = a_{p-1} b_p - a_2 b_2 - \sum_{k=2}^{p-2} b_{k+1}(a_{k+1} - a_k) \\ &\quad + a_{p-1} b_{p-1} - a_2 b_1 - \sum_{k=2}^{p-2} b_k(a_{k+1} - a_k) + a_{p-1} b_{p-2} - a_2 b_0 - \sum_{k=2}^{p-2} b_{k-1}(a_{k+1} - a_k) \\ &= a_{p-1} b_p - a_2 b_2 - [\sum_{k=2}^{p-2} b_k(a_k - a_{k-1}) + b_{p-1}(a_{p-1} - a_{p-2}) - b_2(a_2 - a_1)] \\ &\quad + a_{p-1} b_{p-1} - a_2 b_1 - \sum_{k=2}^{p-2} b_k(a_{k+1} - a_k) \\ &\quad + a_{p-1} b_{p-2} - a_2 b_0 - [\sum_{k=2}^{p-2} b_k(a_{k+2} - a_{k+1}) - b_{p-2}(a_p - a_{p-1}) + b_1(a_3 - a_2)].\end{aligned}$$

This leads to (2.3.2). □

With Lemmas 2.3.1 and 2.3.2, we have the following Lemma 2.3.3.

Lemma 2.3.3. *Let grid functions E_x, E_y and H_z be defined on staggered grids and E_x, E_y satisfy the PEC condition*

$$E_{x_{i+\frac{1}{2},0}} = E_{x_{i+\frac{1}{2},J}} = E_{y_{0,j+\frac{1}{2}}} = E_{y_{I,j+\frac{1}{2}}} = 0.$$

Then, we have that

$$\begin{aligned}
&\sum_{j=1}^{J-2} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \Lambda_y E_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + H_{z_{i+\frac{1}{2},\frac{1}{2}}} \widetilde{\Lambda}_y E_{x_{i+\frac{1}{2},\frac{1}{2}}} + H_{z_{i+\frac{1}{2},J-\frac{1}{2}}} \widetilde{\Lambda}_y E_{x_{i+\frac{1}{2},J-\frac{1}{2}}} \qquad (2.3.3)\\
&\qquad = -[\sum_{j=2}^{J-2} E_{x_{i+\frac{1}{2},j}} \Lambda_y H_{z_{i+\frac{1}{2},j}}, + E_{x_{i+\frac{1}{2},1}} \widetilde{\Lambda}_y H_{z_{i+\frac{1}{2},1}} + E_{x_{i+\frac{1}{2},J-1}} \widetilde{\Lambda}_y H_{z_{i+\frac{1}{2},J-1}}],\\
&\sum_{i=1}^{I-2} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \Lambda_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + H_{z_{\frac{1}{2},j+\frac{1}{2}}} \widetilde{\Lambda}_x E_{y_{\frac{1}{2},j+\frac{1}{2}}} + H_{z_{I-\frac{1}{2},j+\frac{1}{2}}} \widetilde{\Lambda}_x E_{y_{i+\frac{1}{2},I-\frac{1}{2}}} \qquad (2.3.4)\\
&\qquad = -[\sum_{i=2}^{I-2} E_{y_{i,j+\frac{1}{2}}} \Lambda_x H_{z_{i,j+\frac{1}{2}}} + E_{y_{1,j+\frac{1}{2}}} \widetilde{\Lambda}_x H_{z_{1,j+\frac{1}{2}}} + E_{y_{I-1,j+\frac{1}{2}}} \widetilde{\Lambda}_x H_{z_{I-1,j+\frac{1}{2}}}].
\end{aligned}$$

Proof. We give the proof of (2.3.3). In the same way, we can obtain (2.3.4).

$$\sum_{j=1}^{J-2} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \Lambda_y E_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + H_{z_{i+\frac{1}{2},\frac{1}{2}}} \widetilde{\Lambda}_y E_{x_{i+\frac{1}{2},\frac{1}{2}}} + H_{z_{i+\frac{1}{2},J-\frac{1}{2}}} \widetilde{\Lambda}_y E_{x_{i+\frac{1}{2},J-\frac{1}{2}}}$$
$$= \sum_{j=1}^{J-2} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \frac{1}{8}(9\delta_y E_{x_{i+\frac{1}{2},j+\frac{1}{2}}} - \delta_{2,y} E_{x_{i+\frac{1}{2},j+\frac{1}{2}}})$$
$$+ H_{z_{i+\frac{1}{2},\frac{1}{2}}} \frac{1}{8}(9\delta_y E_{x_{i+\frac{1}{2},\frac{1}{2}}} - \widetilde{\delta}_{2,y} E_{x_{i+\frac{1}{2},\frac{1}{2}}})$$
$$+ H_{z_{i+\frac{1}{2},J-\frac{1}{2}}} \frac{1}{8}(9\delta_y E_{x_{i+\frac{1}{2},J-\frac{1}{2}}} - \widetilde{\delta}_{2,y} E_{x_{i+\frac{1}{2},J-\frac{1}{2}}})$$
$$= \sum_{j=0}^{J-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \frac{9}{8}\delta_y E_{x_{i+\frac{1}{2},j+\frac{1}{2}}} - H_{z_{i+\frac{1}{2},\frac{1}{2}}} \frac{1}{8}\widetilde{\delta}_{2,y} E_{x_{i+\frac{1}{2},\frac{1}{2}}}$$
$$- H_{z_{i+\frac{1}{2},J-\frac{1}{2}}} \frac{1}{8}\widetilde{\delta}_{2,y} E_{x_{i+\frac{1}{2},J-\frac{1}{2}}} - \sum_{j=1}^{J-2} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \frac{1}{8}\delta_{2,y} E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}.$$

By using Lemmas 2.3.1 and 2.3.2, the right side term of the above equation becomes that

$$-\sum_{j=1}^{J-1} E_{x_{i+\frac{1}{2},j}} \frac{9}{8}\delta_y H_{z_{i+\frac{1}{2},j}} - H_{z_{i+\frac{1}{2},\frac{1}{2}}} \frac{1}{8}\widetilde{\delta}_{2,y} E_{x_{i+\frac{1}{2},\frac{1}{2}}} - H_{z_{i+\frac{1}{2},J-\frac{1}{2}}} \frac{1}{8}\widetilde{\delta}_{2,y} E_{x_{i+\frac{1}{2},J-\frac{1}{2}}}$$
$$-\frac{1}{8}(H_{z_{i+\frac{1}{2},J-\frac{1}{2}}} E_{x_{i+\frac{1}{2},J-2}} + H_{z_{i+\frac{1}{2},J-\frac{3}{2}}} E_{x_{i+\frac{1}{2},J}} + H_{z_{i+\frac{1}{2},J-\frac{5}{2}}} E_{x_{i+\frac{1}{2},J-1}}$$
$$- H_{z_{i+\frac{1}{2},-\frac{1}{2}}} E_{x_{i+\frac{1}{2},2}} - H_{z_{i+\frac{1}{2},\frac{3}{2}}} E_{x_{i+\frac{1}{2},0}} - H_{z_{i+\frac{1}{2},\frac{5}{2}}} E_{x_{i+\frac{1}{2},1}} - \sum_{j=2}^{J-2} E_{z_{i+\frac{1}{2},j}} \delta_{2,y} H_{x_{i+\frac{1}{2},j}})$$
$$= \quad -\sum_{j=1}^{J-1} E_{x_{i+\frac{1}{2},j}} \frac{9}{8}\delta_y H_{x_{i+\frac{1}{2},j}} + E_{x_{i+\frac{1}{2},1}} \frac{1}{8}\widetilde{\delta}_{2,y} H_{x_{i+\frac{1}{2},1}} + E_{x_{i+\frac{1}{2},J-1}} \frac{1}{8}\widetilde{\delta}_{2,y} H_{x_{i+\frac{1}{2},J-1}}$$
$$+ \sum_{j=2}^{J-2} E_{x_{i+\frac{1}{2},j}} \frac{1}{8}\delta_{2,y} H_{x_{i+\frac{1}{2},j}} = -[\sum_{j=1}^{J-2} E_{x_{i+\frac{1}{2},j}} \Lambda_y H_{z_{i+\frac{1}{2},j}} + E_{x_{i+\frac{1}{2},J-1}} \widetilde{\Lambda}_y H_{z_{i+\frac{1}{2},J-1}}].$$

This ends the proof. □

Now, we can derive the energy conservations in the discrete forms for our spatial fourth-order EC-S-FDTD scheme.

Theorem 2.3.1. *(Energy conservations) For* $n \geq 0$, $\boldsymbol{E}^n = \{(E^n_{x_{i+\frac{1}{2},j}}, E^n_{y_{i,j+\frac{1}{2}}})\}$ *and* $H^n_z = \{H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}$*, the solutions of the proposed scheme (2.2.38)-(2.2.51), satisfy the energy conservations in the discrete forms*

$$\left\|\epsilon^{\frac{1}{2}}\boldsymbol{E}^{n+1}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}H_z^{n+1}\right\|_H^2 = \left\|\epsilon^{\frac{1}{2}}\boldsymbol{E}^{n}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}H_z^{n}\right\|_H^2, \tag{2.3.5}$$

$$\left\|\epsilon^{\frac{1}{2}}\delta_t\boldsymbol{E}^{n+\frac{3}{2}}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{3}{2}}\right\|_H^2 = \left\|\epsilon^{\frac{1}{2}}\delta_t\boldsymbol{E}^{n+\frac{1}{2}}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\right\|_H^2. \tag{2.3.6}$$

Proof. Multiplying both sides of Eq. (2.2.38) with $\epsilon\Delta t\left(E^*_{x_{i+\frac{1}{2},j}} + E^n_{x_{i+\frac{1}{2},j}}\right)$ and multiplying both sides of Eq. (2.2.39) with $\mu\Delta t\left(H^*_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}}\right)$, we obtain that for $j = 2, 3, \cdots, J-2$ and $i = 0, 1, \cdots, I-1$

$$\epsilon\left[\left(E^*_{x_{i+\frac{1}{2},j}}\right)^2 - \left(E^n_{x_{i+\frac{1}{2},j}}\right)^2\right] = \frac{\Delta t}{4}\Lambda_y\left\{H^*_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}}\right\}\left(E^*_{x_{i+\frac{1}{2},j}} + E^n_{x_{i+\frac{1}{2},j}}\right), \tag{2.3.7}$$

and for $j = 1, 2, \cdots, J-2$ and $i = 0, 1, \cdots, I-1$

$$\begin{aligned}&\mu\left[\left(H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2 - \left(H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right] \\ &\quad= \frac{\Delta t}{4}\Lambda_y\left\{E^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}\left(H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right).\end{aligned} \tag{2.3.8}$$

Meanwhile, multiplying both sides of Eq. (2.2.40) with $\epsilon\Delta t\left(E^*_{x_{i+\frac{1}{2},j}} + E^n_{x_{i+\frac{1}{2},j}}\right)$, $j = 1$ and $j = J-1$, and multiplying both sides of Eq. (2.2.41) with $\mu\Delta t\Big(H^*_{z_{i+\frac{1}{2},j}}$

$+ \left. H^n_{z_{i+\frac{1}{2},j}}\right)$, $j = 0$ and $j = J-1$, we obtain that for $j = 1$ and $j = J-1$, and $i = 0, 1, \cdots, I-1$

$$\epsilon \left[\left(E^*_{x_{i+\frac{1}{2},j}} \right)^2 - \left(E^n_{x_{i+\frac{1}{2},j}} \right)^2 \right] = \frac{\Delta t}{4} \widetilde{\Lambda}_y \left\{ H^*_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}} \right\} \left(E^*_{x_{i+\frac{1}{2},j}} + E^n_{x_{i+\frac{1}{2},j}} \right), \tag{2.3.9}$$

and for $j = 0$ and $j = J-1$, and $i = 0, 1, \cdots, I-1$

$$\begin{aligned} &\mu \left[\left(H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right)^2 - \left(H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right)^2 \right] \\ &= \frac{\Delta t}{4} \widetilde{\Lambda}_y \left\{ E^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}} \right\} \left(H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right). \end{aligned} \tag{2.3.10}$$

For the above equations (2.3.7), (2.3.8), (2.3.9) and (2.3.10), we sum over all the terms over all i and j and then add these equations together. Noting that E^n_x and E^*_x satisfy the PEC boundary condition (2.2.50), and using Lemma 2.3.3, we have that

$$\sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left(\epsilon \left(E^*_{x_{i+\frac{1}{2},j}} \right)^2 + \mu \left(H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right)^2 \right) = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left(\epsilon \left(E^n_{x_{i+\frac{1}{2},j}} \right)^2 + \mu \left(H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right)^2 \right). \tag{2.3.11}$$

Similarly, from Eqns. (2.2.42)- (2.2.45), we can obtain that

$$\sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left(\epsilon \left(E^{n+1}_{y_{i,j+\frac{1}{2}}} \right)^2 + \mu \left(H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right)^2 \right) = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left(\epsilon \left(E^n_{y_{i,j+\frac{1}{2}}} \right)^2 + \mu \left(H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \right)^2 \right), \tag{2.3.12}$$

and from (2.2.46)- (2.2.49), it holds that

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n+1}_{x_{i+\frac{1}{2},j}}\right)^2+\mu\left(H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right)=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{*}_{x_{i+\frac{1}{2},j}}\right)^2+\mu\left(H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right). \tag{2.3.13}$$

Energy conservation (2.3.5) can be directly obtained by adding (2.3.11)-(2.3.13).

Let H_z^{*+1}, H_z^{**+1} and E_x^{*+1} be the intermediate values of H_z^*, H_z^{**} and E_x^* at time level $t = t^{n+1}$ respectively. We then have the intermediate value differences by $\delta_t H_z^{*+\frac{1}{2}} = \frac{H_z^{*+1}-H_z^*}{\Delta t}$, $\delta_t H_z^{**+\frac{1}{2}} = \frac{H_z^{**+1}-H_z^{**}}{\Delta t}$ and $\delta_t E_x^{*+\frac{1}{2}} = \frac{E_x^{*+1}-E_x^*}{\Delta t}$. Applying the difference operator δ_t to Eqns. (2.2.38)-(2.2.41), we have the following equations of $\delta_t E_x^*$ and $\delta_t H_z^*$. For $i = 0, 1, \cdots, I-1$

$$\begin{aligned}
\delta_t E^{*+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} - \delta_t E^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} &= \frac{\Delta t}{4\epsilon}\Lambda_y\{\delta_t H^{*+\frac{1}{2}}_{z_{i+\frac{1}{2},j}} + \delta_t H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j}}\},\ j = 2, 3, \cdots, J-2 \\
\delta_t H^{*+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \delta_t H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{\Delta t}{4\mu}\Lambda_y\{\delta_t E^{*+\frac{1}{2}}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \delta_t E^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\},\ j = 1, 3, \cdots, J-2;
\end{aligned}$$

and for $i = 0, 1, \cdots, I-1$

$$\begin{aligned}
\delta_t E^{*+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} - \delta_t E^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} &= \frac{\Delta t}{4\epsilon}\widetilde{\Lambda}_y\{\delta_t H^{*+\frac{1}{2}}_{z_{i+\frac{1}{2},j}} + \delta_t H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j}}\},\ j = 1\, and\, J-1, \\
\delta_t H^{*+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \delta_t H^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{\Delta t}{4\mu}\widetilde{\Lambda}_y\{\delta_t E^{*+\frac{1}{2}}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \delta_t E^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\},\ j = 0\, and\, J-1.
\end{aligned}$$

Similarly, we can have equations of $\{\delta_t E_y^{n+1},\ \delta_t H_z^{**}\}$ and $\{\delta_t E_x^{n+1},\ \delta_t H_z^{n+1}\}$ corresponding to (2.2.42) - (2.2.48). Moreover, $\delta_t E_x^*, \delta_t E_y^{n+1}, \delta_t E_x^{n+1}$ still satisfy the PEC boundary condition. Following the proof of (2.3.5), (2.3.6) can be proved from the above equations of $\delta_t \mathbf{E}$ and $\delta_t H_z$. □

From Theorem 2.3.1, we can obtain the following stability theorem.

Corollary 1. *(Unconditional Stability) The spatial fourth-order energy-conserved S-FDTD scheme (2.2.38)-(2.2.51) is unconditionally stable.*

2.4 Convergence analysis

In this section, we will analyze error estimates of the proposed scheme (2.2.38)-(2.2.51). In order to do it, we first introduce the (2,4)-order implicit Crank-Nicolson scheme and an equivalent splitting scheme and analyze their truncation errors.

The (2, 4)-order implicit Crank-Nicolson scheme for the Maxwell's equations can be written as that for the strict interior nodes

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{2\epsilon}\Lambda_y\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{n}_{z_{i+\frac{1}{2},j}}\},\ i = 0,1,\cdots,I-1; j = 2,3,\cdots,J-2, \tag{2.4.1}$$

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\Lambda_x\{H^{n+1}_{z_{i,j+\frac{1}{2}}} + H^{n}_{z_{i,j+\frac{1}{2}}}\},\ i = 2,3,\cdots,I-2; j = 0,1,\cdots,J-1, \tag{2.4.2}$$

$$\begin{aligned}\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} &= \frac{1}{2\mu}\left\{\Lambda_y\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} - \Lambda_x\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}\right\},\\ &\quad i = 1,2,\cdots,I-2; j = 1,2,\cdots,J-2,\end{aligned} \tag{2.4.3}$$

and for the near boundary nodes, at $j = 1$ and $j = J-1$

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{2\epsilon}\widetilde{\Lambda}_y\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{n}_{z_{i+\frac{1}{2},j}}\},\ i = 0,1,\cdots,I-1; \tag{2.4.4}$$

and at $i = 1$ and $i = I - 1$

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\widetilde{\Lambda}_x\{H^{n+1}_{z_{i,j+\frac{1}{2}}} + H^{n}_{z_{i,j+\frac{1}{2}}}\},\ j = 0, 1, \cdots, J-1, \tag{2.4.5}$$

and for $j = 0$ and $J - 1$, $i = 1, 2 \cdots, I - 2$,

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{2\mu}\left\{\widetilde{\Lambda}_y\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} - \Lambda_x\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}\right\}, \tag{2.4.6}$$

for $j = 0$ and $J - 1$, $i = 0$ and $i = I - 1$,

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{2\mu}\left\{\widetilde{\Lambda}_y\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} - \widetilde{\Lambda}_x\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}\right\}, \tag{2.4.7}$$

and the similar equations to (2.4.6), (2.4.7) can be obtained for $j = 1, 2, \cdots, J-2$, $i = 0$ and $I - 1$.

Let $\tau^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}$, $\tau^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}$, and $\tau^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$; $\widetilde{\tau}^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}$, $\widetilde{\tau}^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}$, and $\widetilde{\tau}^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$ be the truncation errors at the strict interior nodes and at the near boundary nodes respectively. Applying Taylor's expansion, the truncation errors at the strict interior nodes can be estimated by

$$\begin{aligned}
\tau_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} &= \Delta t^2\Big[\frac{1}{24}\frac{\partial^3 E_x(x_{i+\frac{1}{2}},y_j,t_{11})}{\partial t^3} - \frac{1}{4\epsilon}\frac{\partial^3 H_z(x_{i+\frac{1}{2}},y_j,t^{n+\frac{1}{2}})}{\partial y\partial t^2} \\
&\quad -\frac{\Delta t^2}{16\epsilon}\frac{\partial^5 H_z(x_{i+\frac{1}{2}},y_{12},t_{12})}{\partial y\partial t^4}\Big] + \frac{9}{16\epsilon}\Delta y^4\frac{\partial^5 H_z(x_{i+\frac{1}{2}},y_{11},t^{n+\frac{1}{2}})}{\partial y^5}, \\
\tau_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} &= \Delta t^2\Big[\frac{1}{24}\frac{\partial^3 E_y(x_i,y_{j+\frac{1}{2}},t_{21})}{\partial t^3} - \frac{1}{4\epsilon}\frac{\partial^3 H_z(x_i,y_{j+\frac{1}{2}},t^{n+\frac{1}{2}})}{\partial x\partial t^2} \\
&\quad -\frac{\Delta t^2}{16\epsilon}\frac{\partial^5 H_z(x_{22},y_{j+\frac{1}{2}},t_{22})}{\partial x\partial t^4}\Big] + \frac{9}{16\epsilon}\Delta x^4\frac{\partial^5 H_z(x_{21},y_{j+\frac{1}{2}},t^{n+\frac{1}{2}})}{\partial x^5}, \\
\tau_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} &= \Delta t^2\Big[\frac{1}{24}\frac{\partial^3 H_z(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t_{31})}{\partial t^3} - \frac{1}{4\mu}\frac{\partial^3 E_x(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t^{n+\frac{1}{2}})}{\partial y\partial t^2}\Big] \\
&\quad -\Delta t^2\Big[\frac{\Delta t^2}{16\mu}\frac{\partial^5 E_x(x_{i+\frac{1}{2}},y_{32},t_{32})}{\partial y\partial t^4} - \frac{1}{4\mu}\frac{\partial^3 E_y(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},t^{n+\frac{1}{2}})}{\partial x\partial t^2} \\
&\quad -\frac{\Delta t^2}{16\mu}\frac{\partial^5 E_y(x_{32},y_{j+\frac{1}{2}},t_{33})}{\partial y\partial t^4}\Big] + \frac{9}{16\epsilon}\Big[\Delta y^4\frac{\partial^5 E_x(x_{i+\frac{1}{2}},y_{31},t^{n+\frac{1}{2}})}{\partial y^5} \\
&\quad +\Delta x^4\frac{\partial^5 E_y(x_{31},y_{j+\frac{1}{2}},t^{n+\frac{1}{2}})}{\partial x^5}\Big].
\end{aligned}$$

Thus, we have the estimates that for the strict interior nodes

$$\Big\{|\tau_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}|, |\tau_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}|, |\tau_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}|\Big\} \leq C\{\Delta t^2 + \Delta x^4 + \Delta y^4\}, \tag{2.4.8}$$

where C is constant.

Noting the relation of (2.2.15), we get that

$$\begin{aligned}
\widetilde{\Lambda}_x E_{y_{\frac{1}{2},j+\frac{1}{2}}}^n &= \frac{1}{8}(9\delta_x - \widetilde{\delta}_{2,x})E_{y_{\frac{1}{2},j+\frac{1}{2}}}^n = \frac{-25E_{y_{0,j+\frac{1}{2}}}^n + 26E_{y_{1,j+\frac{1}{2}}}^n - E_{y_{2,j+\frac{1}{2}}}^n}{24\Delta x} \\
&= \frac{E_{y_{-1,j+\frac{1}{2}}}^n - 27E_{y_{0,j+\frac{1}{2}}}^n + 27E_{y_{1,j+\frac{1}{2}}}^n - E_{y_{2,j+\frac{1}{2}}}^n}{24\Delta x} + O(\Delta x^4).
\end{aligned}$$

Similarly, we have same results for other near-boundary difference operators. Thus, we can further get that the truncation errors $\widetilde{\tau}_x, \widetilde{\tau}_y$ and $\widetilde{\tau}_z$ at the near boundary nodes satisfy

$$\Big\{|\widetilde{\tau}_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}|, |\widetilde{\tau}_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}|, |\widetilde{\tau}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}|\Big\} \le C\{\Delta t^2 + \Delta x^4 + \Delta y^4\} \tag{2.4.9}$$

where C is constant.

In scheme (2.2.38)-(2.2.51), we eliminate the intermediate variables E_x^*, H_z^* and H_z^{**}, we can obtain an equivalent scheme. It holds that for the strict interior nodes

$$\frac{E_{x_{i+\frac{1}{2},j}}^{n+1} - E_{x_{i+\frac{1}{2},j}}^{n}}{\Delta t} = \frac{1}{2\epsilon}\Lambda_y\{H_{z_{i+\frac{1}{2},j}}^{n+1} + H_{z_{i+\frac{1}{2},j}}^{n}\} - \frac{\Delta t}{16\mu\epsilon}\Lambda_y\Lambda_y\{E_{x_{i+\frac{1}{2},j}}^{n+1} - E_{x_{i+\frac{1}{2},j}}^{n}\},$$
$$i = 0, 1, \cdots, I-1; j = 2, 3, \cdots, J-2, \tag{2.4.10}$$

$$\frac{E_{y_{i,j+\frac{1}{2}}}^{n+1} - E_{y_{i,j+\frac{1}{2}}}^{n}}{\Delta t} = -\frac{1}{2\epsilon}\Lambda_x\{H_{z_{i,j+\frac{1}{2}}}^{n+1} + H_{z_{i,j+\frac{1}{2}}}^{n}\} + \frac{\Delta t}{8\mu\epsilon}\Lambda_x\Lambda_y\{E_{x_{i,j+\frac{1}{2}}}^{n+1} - E_{x_{i,j+\frac{1}{2}}}^{n}\},$$
$$i = 2, 3, \cdots, I-2; j = 0, 1, \cdots, J-1, \tag{2.4.11}$$

$$\begin{aligned}
&\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}}{\Delta t} \\
&= \frac{1}{2\mu}\Big\{\Lambda_y\Big(E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\Big) - \Lambda_x\Big(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\Big)\Big\} \\
&\quad -\frac{\Delta t}{16\mu\epsilon}\Lambda_y\Lambda_y\{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\} + \frac{\Delta t^2}{32\mu^2\epsilon}\Lambda_y\Lambda_x\Lambda_y\{E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\} \\
&\quad i = 1, 2, \cdots, I-2; j = 1, 2, \cdots, J-2,
\end{aligned} \tag{2.4.12}$$

and for the near boundary nodes, at $j = 1$ and $j = J-1$

$$\frac{E_{x_{i+\frac{1}{2},j}}^{n+1} - E_{x_{i+\frac{1}{2},j}}^{n}}{\Delta t} = \frac{1}{2\epsilon}\widetilde{\Lambda}_y\{H_{z_{i+\frac{1}{2},1}}^{n+1} + H_{z_{i+\frac{1}{2},1}}^{n}\} - \frac{\Delta t}{16\mu\epsilon}\overline{\Lambda}_y\overline{\Lambda}_y\{E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} - E_{x_{i+\frac{1}{2},j-\frac{1}{2}}}^{n}\},$$
$$i = 0, 1, \cdots, I-1; \tag{2.4.13}$$

and at $i=1$ and $i=I-1$

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}}-E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} \;=\; -\frac{1}{2\epsilon}\widetilde{\Lambda}_x\{H^{n+1}_{z_{i,j+\frac{1}{2}}}+H^{n}_{z_{i,j+\frac{1}{2}}}\}+\frac{\Delta t}{8\mu\epsilon}\overline{\Lambda}_x\overline{\Lambda}_y\{E^{n+1}_{x_{i,j+\frac{1}{2}}}-E^{n}_{x_{i,j+\frac{1}{2}}}\},$$
$$j=0,1,\cdots,J-1; \tag{2.4.14}$$

and for $j=0$ and $J-1$, $i=1,2\cdots,I-2$,

$$\begin{aligned}&\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}\\ &\quad=\frac{1}{2\mu}\Big\{\widetilde{\Lambda}_y\Big(E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\Big)-\Lambda_x\Big(E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\Big)\Big\} \\ &\qquad-\frac{\Delta t}{16\mu\epsilon}\overline{\Lambda}_y\overline{\Lambda}_y\{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\frac{\Delta t^2}{32\mu^2\epsilon}\overline{\Lambda}_y\overline{\Lambda}_x\overline{\Lambda}_y\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}.\end{aligned} \tag{2.4.15}$$

and for $j=0$ and $J-1$, $i=0$ and $I-1$,

$$\begin{aligned}&\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}\\ &\quad=\frac{1}{2\mu}\Big\{\widetilde{\Lambda}_y\Big(E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\Big)-\widetilde{\Lambda}_x\Big(E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\Big)\Big\} \\ &\qquad-\frac{\Delta t}{16\mu\epsilon}\overline{\Lambda}_y\overline{\Lambda}_y\{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\frac{\Delta t^2}{32\mu^2\epsilon}\overline{\Lambda}_y\overline{\Lambda}_x\overline{\Lambda}_y\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}.\end{aligned} \tag{2.4.16}$$

The similar equations to (2.4.15)(2.4.16) can be obtained for $j=1,2,\cdots,J-2$, $i=0$ and $I-1$. Where the difference operator $\overline{\Lambda}_y$ is Λ_y when it is over the strict interior nodes, $\overline{\Lambda}_y$ is $\widetilde{\Lambda}_y$ when it is over the near boundary nodes. The same definition is for $\overline{\Lambda}_x$.

Noting the equivalent scheme (2.4.10) - (2.4.16), the scheme (2.2.38)-(2.2.51) can be regarded as the perturbation of the (2,4)-order implicit Crank-Nicolson scheme

(2.4.1) - (2.4.7). Let η_x, η_y and η_z be the truncation errors of our scheme (2.2.38)-(2.2.51). From the equivalent relations (2.4.10) - (2.4.16), we then have that for the strict interior nodes

$$\eta_{x\,i+\frac{1}{2},j}^{\,n+\frac{1}{2}} = \tau_{x\,i+\frac{1}{2},j}^{\,n+\frac{1}{2}} + \frac{\Delta t}{16\mu\epsilon}\Lambda_y\Lambda_y\{E_x(x_{i+\frac{1}{2}}, y_j, t^{n+1}) - E_x(x_{i+\frac{1}{2}}, y_j, t^n)\}, \tag{2.4.17}$$

$$\eta_{y\,i,j+\frac{1}{2}}^{\,n+\frac{1}{2}} = \tau_{y\,i+\frac{1}{2},j}^{\,n+\frac{1}{2}} - \frac{\Delta t}{8\mu\epsilon}\Lambda_x\Lambda_y\{E_x(x_i, y_{j+\frac{1}{2}}, t^{n+1}) - E_x(x_i, y_{j+\frac{1}{2}}, t^n)\}, \tag{2.4.18}$$

$$\begin{aligned}\eta_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{\,n+\frac{1}{2}} =&\tau_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{\,n+\frac{1}{2}} + \frac{\Delta t}{16\mu\epsilon}\Lambda_y\Lambda_y\{H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) - H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n)\}\\ &- \frac{\Delta t^2}{32\mu^2\epsilon}\Lambda_y\Lambda_x\Lambda_y\{E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) - E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n)\},\end{aligned} \tag{2.4.19}$$

and for the near boundary nodes

$$\eta_{x\,i+\frac{1}{2},j}^{\,n+\frac{1}{2}} = \widetilde{\tau}_{x\,i+\frac{1}{2},j}^{\,n+\frac{1}{2}} + \frac{\Delta t}{16\mu\epsilon}\overline{\Lambda}_y\overline{\Lambda}_y\{E_x(x_{i+\frac{1}{2}}, y_j, t^{n+1}) - E_x(x_{i+\frac{1}{2}}, y_j, t^n)\}, \tag{2.4.20}$$

$$\eta_{y\,i,j+\frac{1}{2}}^{\,n+\frac{1}{2}} = \widetilde{\tau}_{y\,i+\frac{1}{2},j}^{\,n+\frac{1}{2}} - \frac{\Delta t}{8\mu\epsilon}\overline{\Lambda}_x\overline{\Lambda}_y\{E_x(x_i, y_{j+\frac{1}{2}}, t^{n+1}) - E_x(x_i, y_{j+\frac{1}{2}}, t^n)\}, \tag{2.4.21}$$

$$\begin{aligned}\eta_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{\,n+\frac{1}{2}} =&\widetilde{\tau}_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{\,n+\frac{1}{2}} + \frac{\Delta t}{16\mu\epsilon}\overline{\Lambda}_y\overline{\Lambda}_y\{H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) - H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n)\}\\ &- \frac{\Delta t^2}{32\mu^2\epsilon}\overline{\Lambda}_y\overline{\Lambda}_x\overline{\Lambda}_y\{E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) - E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n)\}.\end{aligned} \tag{2.4.22}$$

From the truncation error expressions (2.4.17)-(2.4.22), we can have the following lemma.

Lemma 2.4.1. *(Truncation error) Assume that the solutions are smooth enough, for example,* $\boldsymbol{E}$ *and* $H \in \boldsymbol{C}^5([0,T]; [\boldsymbol{C}^5(\bar{\Omega})]^2)$*. Then the truncation errors of* $\eta_x^{n+\frac{1}{2}}$*,*

$\eta_y^{n+\frac{1}{2}}$, $\eta_z^{n+\frac{1}{2}}$ *are fourth order in space and second order in time*

$$max\left\{|\eta_x^{n+\frac{1}{2}}|, |\eta_y^{n+\frac{1}{2}}|, |\eta_z^{n+\frac{1}{2}}|\right\} \leq C_2(\epsilon,\mu)\{\Delta t^2 + \Delta x^4 + \Delta y^4\}, \tag{2.4.23}$$

where $C_2(\epsilon,\mu)$ is constant dependent of parameter ϵ and μ.

Now, we provide the convergence analysis for the scheme (2.2.38)-(2.2.51). Let $\mathcal{E}^n_{x_{\alpha,\beta}}$=$E_x(x_\alpha, y_\beta, t^n)$ - $E^n_{x_{\alpha,\beta}}$, $\mathcal{E}^n_{y_{\alpha,\beta}}$=$E_y(x_\alpha, y_\beta, t^n)$- $E^n_{y_{\alpha,\beta}}$ and $\mathcal{H}^n_{z_{\alpha,\beta}} = H_z(x_\alpha, y_\beta, t^n)$ - $H^n_{z_{\alpha,\beta}}$, where $E_x(x_\alpha, y_\beta, t^n)$, $E_y(x_\alpha, y_\beta, t^n)$ and $H_z(x_\alpha, y_\beta, t^n)$ are the exact solutions at point (x_α, y_β, t^n). Then we have the following theorem.

Theorem 2.4.1. *(Convergence) Let $E_x(x,y,t)$, $E_y(x,y,t)$ and $H_z(x,y,t)$ be the exact solutions of problem (2.2.5)-(2.2.8) and smooth enough. Let E^n_x, E^n_y and H^n_z be the solutions of the scheme (2.2.38)-(2.2.51) for $n \geq 0$. Then for any fixed time $T > 0$, there exists a positive constant $C_{\mu\epsilon}$, such that*

$$\begin{aligned}
&\max_{0\leq n\leq N}\left\{\|\epsilon^{\frac{1}{2}}[E(t^n) - E^n]\|_E^2 + \|\mu^{\frac{1}{2}}[H_z(t^n) - H_z^n]\|_H^2\right\}^{\frac{1}{2}} && (2.4.24)\\
&\quad\leq \left(\|\epsilon^{\frac{1}{2}}[E(t^0) - E^0]\|_E^2 + \|\mu^{\frac{1}{2}}[H_z(t^0) - H_z^0]\|_H^2\right)^{\frac{1}{2}} + C_{\mu\epsilon}T(\Delta t^2 + \Delta x^4 + \Delta y^4),\\
&\max_{0\leq n\leq N}\left\{\|\epsilon^{\frac{1}{2}}\delta t[E(t^{n+\frac{1}{2}}) - E^{n+\frac{1}{2}}]\|_E^2 + \|\mu^{\frac{1}{2}}\delta t[H_z(t^{n+\frac{1}{2}}) - H_z^{n+\frac{1}{2}}]\|_H^2\right\}^{\frac{1}{2}} && (2.4.25)\\
&\quad\leq \left(\|\epsilon^{\frac{1}{2}}\delta t[E(t^{\frac{1}{2}}) - E^{\frac{1}{2}}]\|_E^2 + \|\mu^{\frac{1}{2}}\delta t[H_z(t^{\frac{1}{2}}) - H_z^{\frac{1}{2}}]\|_H^2\right)^{\frac{1}{2}} + C_{\mu\epsilon}T(\Delta t^2 + \Delta x^4 + \Delta y^4).
\end{aligned}$$

Proof. Define $\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$, $\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$ and $\mathcal{E}^{*}_{z_{i+\frac{1}{2},j}}$, for the strict interior nodes, as that for $i = 1, 2, \cdots, I-2$; $j = 1, 2, \cdots, J-2$

$$\begin{aligned}
\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{2}(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}) - \frac{\Delta t}{8\mu}\Lambda_y(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}) \\
&\quad -\frac{\Delta t}{4\mu}\Lambda_x(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}) \\
\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{2}(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}) - \frac{\Delta t}{8\mu}\Lambda_y(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}) \\
&\quad +\frac{\Delta t}{4\mu}\Lambda_x(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}})
\end{aligned}$$

and for $i = 2, 3, \cdots, I-2$; $j = 2, 3, \cdots, J-2$

$$\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}} = \frac{1}{2}(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}} + \mathcal{E}^{n}_{x_{i+\frac{1}{2},j}}) + \frac{\Delta t^2}{16\mu}\Lambda_y\Lambda_x(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j}} + \mathcal{E}^{n}_{y_{i+\frac{1}{2},j}}) - \frac{\Delta t}{8\epsilon}\Lambda_y(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j}} - \mathcal{H}^{n}_{y_{i+\frac{1}{2},j}}).$$

For the near boundary nodes, define that at $i = 0$; $j = 1, 2, \cdots, J-2$ (and similarly at other nodes)

$$\begin{aligned}
\mathcal{H}^{**}_{z_{\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{2}(\mathcal{H}^{n+1}_{z_{\frac{1}{2},j+\frac{1}{2}}} + \mathcal{H}^{n}_{z_{\frac{1}{2},j+\frac{1}{2}}}) - \frac{\Delta t}{8\mu}\Lambda_y(\mathcal{E}^{n+1}_{x_{\frac{1}{2},j+\frac{1}{2}}} - \mathcal{E}^{n}_{x_{\frac{1}{2},j+\frac{1}{2}}}) - \frac{\Delta t}{4\mu}\widetilde{\Lambda}_x(\mathcal{E}^{n+1}_{y_{\frac{1}{2},j+\frac{1}{2}}} - \mathcal{E}^{n}_{y_{\frac{1}{2},j+\frac{1}{2}}}) \\
\mathcal{H}^{*}_{z_{\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{2}(\mathcal{H}^{n+1}_{z_{\frac{1}{2},j+\frac{1}{2}}} + \mathcal{H}^{n}_{z_{\frac{1}{2},j+\frac{1}{2}}}) - \frac{\Delta t}{8\mu}\Lambda_y(\mathcal{E}^{n+1}_{x_{\frac{1}{2},j+\frac{1}{2}}} - \mathcal{E}^{n}_{x_{\frac{1}{2},j+\frac{1}{2}}}) + \frac{\Delta t}{4\mu}\widetilde{\Lambda}_x(\mathcal{E}^{n+1}_{y_{\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{n}_{y_{\frac{1}{2},j+\frac{1}{2}}}),
\end{aligned}$$

and at $i = 1$; $j = 2, 3, \cdots, J-2$ (and similarly at other nodes)

$$\mathcal{E}^{*}_{x_{\frac{1}{2},j}} = \frac{1}{2}(\mathcal{E}^{n+1}_{x_{\frac{1}{2},j}} + \mathcal{E}^{n}_{x_{\frac{1}{2},j}}) + \frac{\Delta t^2}{16\mu}\Lambda_y\widetilde{\Lambda}_x(\mathcal{E}^{n+1}_{y_{\frac{1}{2},j}} + \mathcal{E}^{n}_{y_{\frac{1}{2},j}}) - \frac{\Delta t}{8\epsilon}\Lambda_y(\mathcal{H}^{n+1}_{z_{\frac{1}{2},j}} - \mathcal{H}^{n}_{y_{\frac{1}{2},j}}).$$

From (2.2.38)-(2.2.51) and (2.4.10)-(2.4.16), the error equations can be derived as

Stage 1 For the strict interior nodes, $i = 0, 1, \cdots, I-1$

$$\frac{\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}} - \mathcal{E}^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\Lambda_y\{\mathcal{H}^{*}_{z_{i+\frac{1}{2},j}} + \mathcal{H}^{n}_{z_{i+\frac{1}{2},j}}\} + \phi^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}}, \ j = 2, 3, \cdots, J-2, \quad (2.4.26)$$

$$\frac{\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=\frac{1}{4\mu}\Lambda_y\{\mathcal{E}^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}},\ j=1,2,\ldots,J-2; \tag{2.4.27}$$

and for the near boundary nodes, $i=0,2,\cdots,I-1$

$$\frac{\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}}-\mathcal{E}^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t}=\frac{1}{4\epsilon}\widetilde{\Lambda}_y\{\mathcal{H}^{*}_{z_{i+\frac{1}{2},j}}+\mathcal{H}^{n}_{z_{i+\frac{1}{2},j}}\}+\phi^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}},\ j=1\,and\,J-1, \tag{2.4.28}$$

$$\frac{\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=\frac{1}{4\mu}\widetilde{\Lambda}_y\{\mathcal{E}^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}},\ j=0\,and\,J-1. \tag{2.4.29}$$

Stage 2 For the strict interior nodes, $j=0,1,\cdots,J-1$

$$\frac{\mathcal{E}^{n+1}_{y_{i,j+\frac{1}{2}}}-\mathcal{E}^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t}=-\frac{1}{2\epsilon}\Lambda_x\{\mathcal{H}^{**}_{z_{i,j+\frac{1}{2}}}+\mathcal{H}^{*}_{z_{i,j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{3_{i,j+\frac{1}{2}}},\ i=2,3,\cdots,I-2, \tag{2.4.30}$$

$$\frac{\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=-\frac{1}{2\mu}\Lambda_x\{\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{4_{i+\frac{1}{2},j+\frac{1}{2}}},\ i=1,2,\cdots,I-2; \tag{2.4.31}$$

and for the near boundary nodes, $j=0,1,\cdots,J-1$

$$\frac{\mathcal{E}^{n+1}_{y_{i,j+\frac{1}{2}}}-\mathcal{E}^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t}=-\frac{1}{2\epsilon}\widetilde{\Lambda}_x\{\mathcal{H}^{**}_{z_{i,j+\frac{1}{2}}}+\mathcal{H}^{*}_{z_{i,j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{3_{i,j+\frac{1}{2}}},\ i=1\,and\,I-1, \tag{2.4.32}$$

$$\frac{\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=-\frac{1}{2\mu}\widetilde{\Lambda}_x\{\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{4_{i+\frac{1}{2},j+\frac{1}{2}}},\ i=0\,and\,I-1. \tag{2.4.33}$$

Stage 3 For the strict interior nodes, $i=0,1,\cdots,I-1$

$$\frac{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}}-\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}}}{\Delta t}=\frac{1}{4\epsilon}\Lambda_y\{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j}}+\mathcal{H}^{**}_{z_{i+\frac{1}{2},j}}\}+\phi^{n+\frac{1}{2}}_{5_{i+\frac{1}{2},j}},\ j=2,3,\cdots,J-2, \tag{2.4.34}$$

$$\frac{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=\frac{1}{4\mu}\Lambda_y\{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{6_{i+\frac{1}{2},j+\frac{1}{2}}},\ j=1,2,\ldots,J-2; \tag{2.4.35}$$

and for the near boundary nodes, $i=0,2,\cdots,I-1$

$$\frac{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}}-\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}}}{\Delta t}=\frac{1}{4\epsilon}\widetilde{\Lambda}_y\{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j}}+\mathcal{H}^{**}_{z_{i+\frac{1}{2},j}}\}+\phi^{n+\frac{1}{2}}_{5_{i+\frac{1}{2},j}},\ j=1\,and\,J-1, \tag{2.4.36}$$

$$\frac{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=\frac{1}{4\mu}\widetilde{\Lambda}_y\{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\}+\phi^{n+\frac{1}{2}}_{6_{i+\frac{1}{2},j+\frac{1}{2}}},\ j=0\,and\,J-1. \tag{2.4.37}$$

The following relations can be derived.

$$\phi^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}}=\frac{1}{2}\eta^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}},\phi^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}}=\frac{1}{2}\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}},\phi^{n+\frac{1}{2}}_{3_{i,j+\frac{1}{2}}}=\eta^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}},$$
$$\phi^{n+\frac{1}{2}}_{4_{i+\frac{1}{2},j+\frac{1}{2}}}=0,\phi^{n+\frac{1}{2}}_{5_{i+\frac{1}{2},j}}=\frac{1}{2}\eta^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}},\phi^{n+\frac{1}{2}}_{6_{i+\frac{1}{2},j+\frac{1}{2}}}=\frac{1}{2}\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}.$$

Thus, it holds that

$$max(|\phi_1^{n+\frac{1}{2}}|,|\phi_2^{n+\frac{1}{2}}|,|\phi_3^{n+\frac{1}{2}}|,|\phi_4^{n+\frac{1}{2}}|,|\phi_5^{n+\frac{1}{2}}|,|\phi_6^{n+\frac{1}{2}}|)\leq C(\Delta t^2+\Delta x^4+\Delta y^4). \tag{2.4.38}$$

From (2.4.26) - (2.4.29), we have that

$$\|\epsilon^{\frac{1}{2}}\mathcal{E}^*_x\|^2_{E_x}+\|\mu^{\frac{1}{2}}\mathcal{H}^*_z\|^2_H-\|\epsilon^{\frac{1}{2}}\mathcal{E}^n_x\|^2_{E_x}-\|\mu^{\frac{1}{2}}\mathcal{H}^n_z\|^2_H=\Delta t\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\Big\{\epsilon\phi^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}}(\mathcal{E}^*_{x_{i+\frac{1}{2},j}}+\mathcal{E}^n_{x_{i+\frac{1}{2},j}})$$
$$+\mu\phi^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}}(\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{H}^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}})\Big\}\Delta x\Delta y.$$

It can further obtained from the above equation that

$$\|\epsilon^{\frac{1}{2}}\mathcal{E}^*_x-\frac{\Delta t}{2\sqrt{\epsilon}}\phi_1\|^2_{E_x}+\|\mu^{\frac{1}{2}}\mathcal{H}^*_z-\frac{\Delta t}{2\sqrt{\mu}}\phi_2)\|^2_H$$
$$=\|\epsilon^{\frac{1}{2}}\mathcal{E}^n_x+\frac{\Delta t}{2\sqrt{\epsilon}}\phi_1\|^2_{E_x}+\|\mu^{\frac{1}{2}}\mathcal{H}^n_z+\frac{\Delta t}{2\sqrt{\mu}}\phi_2\|^2_H. \tag{2.4.39}$$

Similarly, from (2.4.30) - (2.4.33), we have that

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1} - \frac{\Delta t}{2\sqrt{\epsilon}}\phi_3\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{**} - \frac{\Delta t}{2\sqrt{\mu}}\phi_4\|_H^2 \\
&\quad = \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n} + \frac{\Delta t}{2\sqrt{\epsilon}}\phi_3\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{*} + \frac{\Delta t}{2\sqrt{\mu}}\phi_4\|_H^2,
\end{aligned} \tag{2.4.40}$$

and from (2.4.34) - (2.4.37), we have that

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1} - \frac{\Delta t}{2\sqrt{\epsilon}}\phi_5\|_{E_x}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{n+1} - \frac{\Delta t}{2\sqrt{\mu}}\phi_6\|_H^2 \\
&\quad = \|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{*} + \frac{\Delta t}{2\sqrt{\epsilon}}\phi_5\|_{E_x}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{**} + \frac{\Delta t}{2\sqrt{\mu}}\phi_6\|_H^2.
\end{aligned} \tag{2.4.41}$$

From (2.4.41), and using the triangle inequality of the discrete norm, we can obtain that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{n+1}\|_H^2\right)^{\frac{1}{2}} \\
&\quad \le \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1} - \frac{\Delta t}{2\sqrt{\epsilon}}\phi_5\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{n+1} - \frac{\Delta t}{2\sqrt{\mu}}\phi_6\|_H^2\right)^{\frac{1}{2}} \\
&\quad\quad + \left(\|\frac{\Delta t}{2\sqrt{\epsilon}}\phi_5\|_{E_x}^2\| + \|\frac{\Delta t}{2\sqrt{\mu}}\phi_6\|_H^2\|\right)^{\frac{1}{2}} \\
&\quad = \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{*} + \frac{\Delta t}{2\sqrt{\epsilon}}\phi_5\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{**} + \frac{\Delta t}{2\sqrt{\mu}}\phi_6\|_H^2\right)^{\frac{1}{2}} \\
&\quad\quad + \left(\|\frac{\Delta t}{2\sqrt{\epsilon}}\phi_5\|_{E_x}^2\| + \|\frac{\Delta t}{2\sqrt{\mu}}\phi_6\|_H^2\|\right)^{\frac{1}{2}} \\
&\quad \le \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{*}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{**}\|_H^2\right)^{\frac{1}{2}} \\
&\quad\quad + \left(\|\frac{\Delta t}{\sqrt{\epsilon}}\phi_5\|_{E_x}^2\| + \|\frac{\Delta t}{\sqrt{\mu}}\phi_6\|_H^2\|\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.4.42}$$

Similarly, from (2.4.40), we have that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{**}\|_H^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{*}\|_{E_x}^2\right)^{\frac{1}{2}} \\
&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{*}\|_H^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{*}\|_{E_x}^2\right)^{\frac{1}{2}}+\left(\|\frac{\Delta t}{\sqrt{\epsilon}}\phi_3\|_{E_x}^2\|+\|\frac{\Delta t}{\sqrt{\mu}}\phi_4\|_H^2\|\right)^{\frac{1}{2}},
\end{aligned}\tag{2.4.43}$$

and from (2.4.39), we have that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{*}\|_{E_x}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{*}\|_H^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2\right)^{\frac{1}{2}} \\
&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n}\|_{E_x}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n}\|_H^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2\right)^{\frac{1}{2}}+\left(\|\frac{\Delta t}{\sqrt{\epsilon}}\phi_1\|_{E_x}^2\|+\|\frac{\Delta t}{\sqrt{\mu}}\phi_2\|_H^2\|\right)^{\frac{1}{2}}.
\end{aligned}\tag{2.4.44}$$

Combining (2.4.42)(2.4.43) with (2.4.44), and applying the estimate (2.4.38) of truncation errors $\phi_1-\phi_6$, we obtain that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1}\|_{E_x}^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n+1}\|_H^2\right)^{\frac{1}{2}} \\
&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n}\|_{E_x}^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n}\|_H^2\right)^{\frac{1}{2}}+C_{\mu\epsilon}\Delta t(\Delta t^2+\Delta x^4+\Delta y^4).
\end{aligned}\tag{2.4.45}$$

Recursively using (2.4.45) from time level n to 1, we thus have that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1}\|_{E_x}^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n+1}\|_H^2\right)^{\frac{1}{2}} \\
&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{0}\|_{E_x}^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{0}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{0}\|_H^2\right)^{\frac{1}{2}}+C_{\mu\epsilon}T(\Delta t^2+\Delta x^4+\Delta y^4).
\end{aligned}$$

This proves (2.4.24). Further, applying time difference operators to the scheme (2.2.38)-(2.2.51) and the equivalent scheme (2.4.10)-(2.4.16), we can get the error equations of $\delta_t E_x^{n+\frac{1}{2}}$, $\delta_t E_y^{n+\frac{1}{2}}$ and $\delta_t H_z^{n+\frac{1}{2}}$. Similarly to the proof of (2.4.24), we can obtain (2.4.25). □

Finally, we analyze the approximation of the proposed scheme to the divergence-free property. We first prove that the scheme holds the modified divergence-free identity in the discrete form.

Lemma 2.4.2. *For the scheme (2.2.38)-(2.2.51), the following modified divergence-free identity holds that for the strict interior nodes,* $i = 2, 3, \cdots, I-2$; $j = 2, 3, \cdots, J-2$

$$\epsilon(\Lambda_x E^n_{x_{i,j}} + \Lambda_y E^n_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\Lambda_x\Lambda_y\Lambda_y E^n_{x_{i,j}} = \epsilon(\Lambda_x E^0_{x_{i,j}} + \Lambda_y E^0_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\Lambda_x\Lambda_y\Lambda_y E^0_{x_{i,j}}, \tag{2.4.46}$$

and for the near boundary nodes, at $i = 1$ *and* $I-1$; $j = 2, 3, \cdots, J-2$

$$\epsilon(\widetilde{\Lambda}_x E^n_{x_{i,j}} + \Lambda_y E^n_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\widetilde{\Lambda}_x\Lambda_y\Lambda_y E^n_{x_{i,j}} = \epsilon(\Lambda_x E^0_{x_{i,j}} + \Lambda_y E^0_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\widetilde{\Lambda}_x\Lambda_y\Lambda_y E^0_{x_{i,j}}, \tag{2.4.47}$$

and similar relations at other near boundary nodes.

Proof. From the scheme (2.2.38)-(2.2.51) or the equivalent scheme (2.4.10)-(2.4.16), we can get that for the strict interior nodes, for $i = 2, 3, \cdots, I-2$; $j = 2, 3, \cdots, J-2$

$$\begin{aligned}
\delta t(\Lambda_x E^{n+\frac{1}{2}}_{x_{i,j}} + \Lambda_y E^{n+\frac{1}{2}}_{y_{i,j}}) = \Lambda_x & \left(\frac{1}{2\epsilon}\Lambda_y(H^{n+1}_{i,j} + H^n_{i,j}) - \frac{\Delta t}{16\mu\epsilon}\Lambda_y\Lambda_y(E^{n+1}_{x_{i,j}} - E^n_{x_{i,j}})\right) \\
& - \Lambda_y\left(\frac{1}{2\epsilon}\Lambda_x(H^{n+1}_{i,j} + H^n_{i,j}) - \frac{\Delta t}{8\mu\epsilon}\Lambda_x\Lambda_y(E^{n+1}_{x_{i,j}} - E^n_{x_{i,j}})\right) \\
& = \frac{\Delta t}{16\mu\epsilon}\Lambda_x\Lambda_y\Lambda_y(E^{n+1}_{x_{i,j}} - E^n_{x_{i,j}}) = \frac{\Delta t^2}{16\mu\epsilon}\delta t\left(\Lambda_x\Lambda_y\Lambda_y E^{n+\frac{1}{2}}_{x_{ij,}}\right).
\end{aligned}$$

Similarly for the near boundary nodes, at $i = 1$; $j = 2, 3, \cdots, J-2$, we have that

$$\begin{aligned}\delta t(\widetilde{\Lambda}_x E_{x_{1,j}}^{n+\frac{1}{2}} + \Lambda_y E_{y_{1,j}}^{n+\frac{1}{2}}) &= \widetilde{\Lambda}_x \left(\frac{1}{2\epsilon}\Lambda_y (H_{1,j}^{n+1} + H_{,j}^{n}) - \frac{\Delta t}{16\mu\epsilon}\Lambda_y\Lambda_y (E_{x_{1,j}}^{n+1} - E_{x_{1,j}}^{n}) \right) \\ &\quad - \Lambda_y \left(\frac{1}{2\epsilon}\widetilde{\Lambda}_x (H_{1,j}^{n+1} + H_{1,j}^{n}) - \frac{\Delta t}{8\mu\epsilon}\widetilde{\Lambda}_x\Lambda_y (E_{x_{1,j}}^{n+1} - E_{x_{1,j}}^{n}) \right) \\ &= \frac{\Delta t^2}{16\mu\epsilon}\delta t \left(\widetilde{\Lambda}_x \Lambda_y \Lambda_y E_{x_{1,j}}^{n+\frac{1}{2}} \right),\end{aligned}$$

and similar relations can be obtained for other near boundary nodes. Summing over n for the above equations, we get the modified identities of divergence-free (2.4.46), (2.4.47) on strict interior nodes and near boundary nodes. This ends the proof. $\square$

For the approximate electric field $\mathbf{E}^n$, let

$$\begin{aligned}\|\nabla_h \cdot \mathbf{E}^n\|^2 &= \sum_{i=2}^{I-2}\sum_{j=2}^{J-2} (\Lambda_x E_{x_{i,j}}^{n} + \Lambda_y E_{y_{i,j}}^{n})^2 \Delta x \Delta y \\ &+ \sum_{j=2}^{J-2} [(\widetilde{\Lambda}_x E_{x_{1,j}}^{n} + \Lambda_y E_{y_{1,j}}^{n})^2 + (\widetilde{\Lambda}_x E_{x_{I-1,j}}^{n} + \Lambda_y E_{y_{I-1,j}}^{n})^2] \Delta x \Delta y \\ &+ \sum_{i=2}^{I-2} [(\Lambda_x E_{x_{i,1}}^{n} + \widetilde{\Lambda}_y E_{y_{i,1}}^{n})^2 + (\Lambda_x E_{x_{i,J-1}}^{n} + \widetilde{\Lambda}_y E_{y_{i,J-1}}^{n})^2] \Delta x \Delta y \\ &+ [(\widetilde{\Lambda}_x E_{x_{1,1}}^{n} + \widetilde{\Lambda}_y E_{y_{1,1}}^{n})^2 + (\widetilde{\Lambda}_x E_{x_{I-1,J-1}}^{n} + \widetilde{\Lambda}_y E_{y_{I-1,J-1}}^{n})^2] \Delta x \Delta y.\end{aligned} \tag{2.4.48}$$

Then, from the modified divergence-free identity in Lemma 2.4.2, we can easily prove the following theorem.

Theorem 2.4.2. *Let $\Delta t = \Delta x = \Delta y$. Suppose that the assumptions of Theorem 2.4.1 are satisfied. Then we have the following estimate of discrete divergence-free*

$$\|\nabla_h \cdot \mathbf{E}^n\|^2 \le C\Delta t^2 + 2\|\nabla_h \cdot \mathbf{E}^0\|^2. \tag{2.4.49}$$

2.5 Numerical Experiments

We first take the numerical dispersion analysis for our EC-S-FDTD-(2,4) scheme. Consider the two-dimensional Maxwell's equations' solution

$$E^n_{\alpha,\beta} = E_0\xi^n e^{-i(k_x\alpha\Delta x+k_y\beta\Delta y)},\ H^n_{\alpha,\beta} = H_{z0}\xi^n e^{-i(k_x\alpha\Delta x+k_y\beta\Delta y)}, \tag{2.5.1}$$

where ξ is the amplification factor and the k_x and k_y are wave numbers along the x-axis and y-axis. By computing, we have the equation of factor ξ

$$(\xi-1)(d_0\xi^2+2d_1\xi+d_0)=0 \tag{2.5.2}$$

where d_0 and d_1 are:

$$d_0 = \left(1-\frac{\Delta t^2}{4\mu\epsilon}v_y^2\right)^2+\frac{\Delta t^2}{\mu\epsilon}(v_y^2+u_x^2)+\frac{\Delta t^4}{16\mu^2\epsilon^2}v_y^4+\frac{\Delta t^4}{2\mu^2\epsilon^2}u_x^2v_y^2+\frac{\Delta t^6}{\mu^3\epsilon^3}u_x^2v_y^4,$$

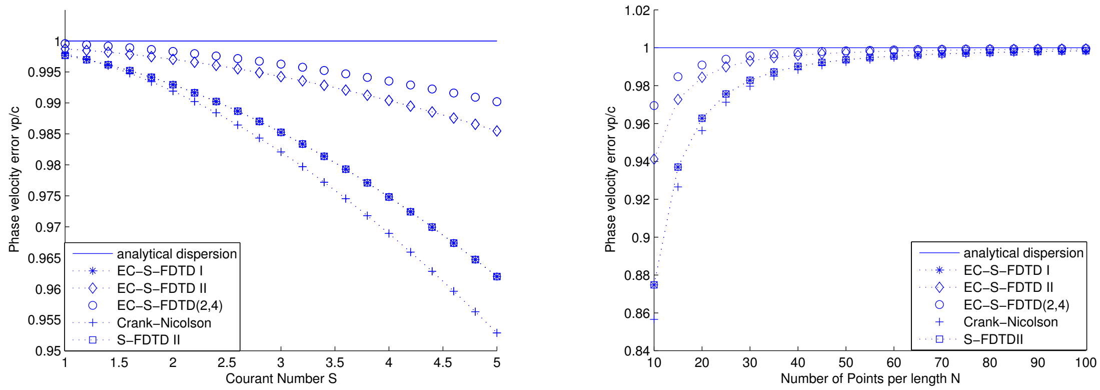
$$d_1 = -\left(1-\frac{\Delta t^2}{4\mu\epsilon}v_y^2\right)^2+\frac{\Delta t^2}{\mu\epsilon}(v_y^2+u_x^2)+\frac{\Delta t^4}{16\mu^2\epsilon^2}v_y^4+\frac{\Delta t^4}{2\mu^2\epsilon^2}u_x^2v_y^2+\frac{\Delta t^6}{\mu^3\epsilon^3}u_x^2v_y^4,$$

and u_x and v_y are defined by

$$u_x = \frac{sin(\frac{3}{2}k_x\Delta x)-27sin(\frac{1}{2}k_x\Delta x)}{24\Delta x},\quad v_y = \frac{sin(\frac{3}{2}k_y\Delta y)-27sin(\frac{1}{2}k_y\Delta y)}{24\Delta y}.$$

Clearly, the modulus of three roots of equation (2.5.2) are all equal to one. Thus, our EC-S-FDTD-(2,4) scheme is non-dissipative.

Let $c=\frac{1}{\sqrt{\mu\epsilon}}$ be the wave speed. Let $\Delta x=\Delta y=h$, $N_\lambda=\frac{\lambda}{h}$, and $S=\frac{c\Delta t}{h}$. Let v_p be the velocity of numerical wave. The phase error of the wave speed can be



(a) *(b)*

Figure 2.1: Numerical dispersion against the CFL number with N_λ =40 and $\theta = 65^o$*(a) and numerical dispersion against the number of points per wavelength* N_λ *with* $S = 2.4$ *and* $\theta = 65^o$ *(b).*

expressed as

$$\frac{v_p}{c} = \frac{N_\lambda}{2\pi S} arctan\left(\frac{|Im(\xi)|}{|Re(\xi)|}\right). \tag{2.5.3}$$

Figure 2.1 and Figure 2.2 show the comparisons of numerical phase errors against the wave courant number S, the number of points per wavelength N_λ and the propagation angles ϕ by our scheme and other schemes of Crank-Nicolson, S-FDTDII ([24]), EC-S-FDTDI and EC-S-FDTDII ([7]). It can be clearly seen that our EC-S-FDTD-(2,4) scheme is the best one whose numerical dispersion is the closest to the analytic solution 1.

Then, we will focus on the numerical study of energy conservation, conver-

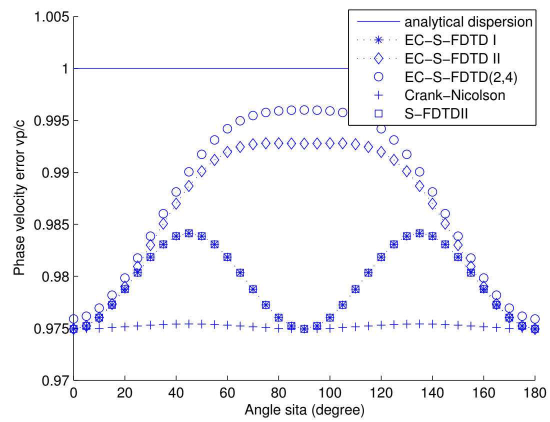


Figure 2.2: Numerical dispersion against the propagation angles ϕ with $N_\lambda = 40$ and $S = 3.5$.

gence, and convergence of divergence-free by comparing our EC-S-FDTD-(2,4) to other schemes of EC-S-FDTDI and EC-S-FDTDII ([7]), and ADI-FDTD ([49, 82]). Consider the Maxwell's equations (2.2.5)-(2.2.7) in a lossless medium with the domain $\Omega = [0,1] \times [0,1]$ surrounded by a perfect conductor. The exact solution of equations (2.2.5)-(2.2.7) is

$$E_x = \frac{k_y}{\epsilon\sqrt{\mu}\omega} cos(\omega\pi t) cos(k_x\pi x) sin(k_y\pi y), \tag{2.5.4}$$

$$E_y = -\frac{k_x}{\epsilon\sqrt{\mu}\omega} cos(\omega\pi t) sin(k_x\pi x) cos(k_y\pi y), \tag{2.5.5}$$

$$H_z = -\frac{1}{\sqrt{\mu}} sin(\omega\pi t) cos(k_x\pi x) cos(k_y\pi y), \tag{2.5.6}$$

where k_x and k_y satisfy the dispersion relation $\omega^2 = \frac{1}{\mu\epsilon}(k_x^2 + k_y^2)$. The exact energy of electromagnetic fields can be computed directly as $EnergyI = \big(\int_\Omega (\epsilon|\mathbf{E}(x,y,t)|^2 +$

$\mu|H_z(x,y,t)|^2)\ dxdy)^{\frac{1}{2}} = \frac{1}{2}$. Define the relative errors of energy by

$$\text{REE-I} = \max_{0\le n\le N} \frac{|(\|\epsilon^{\frac{1}{2}}\mathbf{E}^n\|_E^2 + \|\mu^{\frac{1}{2}}H_z^n\|_H^2)^{\frac{1}{2}} - EnergyI|}{EnergyI}, \tag{2.5.7}$$

and

$$\begin{aligned} &\text{REE-II} \\ &= \max_{0\le n\le N-1} \frac{|(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n+\frac{1}{2}}\|_E^2 + \|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\|_H^2)^{\frac{1}{2}} - (\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{\frac{1}{2}}\|_E^2 + \|\mu^{\frac{1}{2}}\delta_t H_z^{\frac{1}{2}}\|_H^2)^{\frac{1}{2}}|}{(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{\frac{1}{2}}\|_E^2 + \|\mu^{\frac{1}{2}}\delta_t H_z^{\frac{1}{2}}\|_H^2)^{\frac{1}{2}}}. \end{aligned} \tag{2.5.8}$$

Table 2.1: Relative Errors of EnergyI and EnergyII by EC-S-FDTD I&II, EC-S-FDTD-(2,4), and ADI-FDTD schemes. Parameters: $k_x = k_y = 1$, $\mu = \epsilon = 1$, $\Delta x = \Delta y = \Delta t = 1/N$, at $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD-(2,4)		ADI-FDTD	
N	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II
25	4.44e-16	2.00e-16	1.11e-16	4.01e-16	3.33e-15	3.01e-15	9.82e-04	9.75e-04
50	1.33e-15	2.00e-16	3.33e-16	1.40e-16	7.11e-15	6.20e-15	2.46e-04	2.46e-04
75	2.22e-15	6.00e-16	3.33e-16	2.20e-15	2.22e-15	2.20e-15	1.10e-04	1.10e-04
100	2.89e-15	8.00e-16	5.55e-16	3.60e-15	1.58e-14	1.38e-14	6.17e-05	6.17e-05
200	5.55e-15	8.00e-16	7.76e-16	7.60e-15	3.22e-14	2.84e-14	1.54e-05	1.54e-05

Table 2.1 clearly shows that EC-S-FDTD-(2,4), EC-S-FDTDI and EC-S-FDTDII schemes satisfy energy conservations I & II in the discrete forms. However, the ADI-FDTD does not satisfy energy conservations with the relative errors of 10^{-5}. In

Table 2.2: Relative Errors of Energy I and Energy II by EC-S-FDTD I&II, EC-S-FDTD-(2,4), and ADI-FDTD schemes. Parameters: $k_y = 1$, $\mu = \epsilon = 1$ and $\Delta x = \Delta y = \Delta t$=0.01 at $T = 1$.

Scheme	k_x=2k_y		k_x=5k_y		k_x=10k_y	
	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II
EC-S-FDTDI	1.11e-15	1.01e-15	3.33e-15	3.34e-15	4.44e-16	3.43e-16
EC-S-FDTDII	6.66e-16	5.06e-16	3.11e-15	3.12e-15	3.33e-16	4.57e-16
EC-S-FDTD-(2,4)	1.35e-14	1.19e-14	1.20e-14	1.19e-14	1.29e-14	1.29e-14
ADI-FDTD	9.87e-05	9.86e-05	1.19e-04	1.18e-04	1.22e-04	1.19e-04

Table 2.2, we set different $k_x = 2k_y, 5k_y$, and $10k_y$, numerical results show that EC-S-FDTD I & II, EC-S-FDTD-(2,4) schemes hold the properties of energy conservations while ADI-FDTD does not keep.

Let ErrorI and ErrorII be defined by

$$\text{ErrorI} = \max_{0\le n\le N} \frac{(\|\epsilon^{\frac{1}{2}}[\mathbf{E}(t^n) - E^n]\|_E^2 + \|\mu^{\frac{1}{2}}[H_z(t^n) - H_z^n]\|_H^2)^{\frac{1}{2}}}{EnergyI}, \tag{2.5.9}$$

$$\text{ErrorII} = \max_{0\le n\le N-1} \frac{(\|\epsilon^{\frac{1}{2}}[\delta_t\mathbf{E}(t^n) - \delta_t E^n]\|_E^2 + \|\mu^{\frac{1}{2}}[\delta_t H_z(t^n) - \delta_t H_z^n]\|_H^2)^{\frac{1}{2}}}{EnergyII}. \tag{2.5.10}$$

Table 2.3 and Table 2.4 show the convergence ratios of ErrorI and ErrorII by EC-S-FDTDI, EC-S-FDTDII, EC-S-FDTD-(2,4) and ADI-FDTD at time $t = 1$. It can be clearly seen that EC-S-FDTD-(2,4) is of fourth-order convergence in spatial

Table 2.3: The convergence ratios of ErrorI in spatial step for different schemes. Parameters:$\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = k_y = 1$ and $\mu = \epsilon = 1$, at $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD-(2,4)		ADI-FDTD	
N	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio
25	0.0034	-	0.0029	-	1.3411e-05	-	0.0029	-
50	8.4760e-04	2.0041	7.3130e-04	1.9875	8.3842e-07	3.9996	7.3154e-04	1.9870
75	3.7651e-04	2.0013	3.2485e-04	2.0013	1.6562e-07	3.9999	3.2495e-04	2.0014
100	2.1180e-04	1.9998	1.8274e-04	1.9998	5.2405e-08	3.9999	1.8275e-04	2.0007
200	5.2942e-05	2.0002	4.5677e-05	2.0003	3.2752e-09	4.0000	4.5679e-05	2.0003

step, however, other three methods are only of second-order convergence in spatial step. Further, Figure 2.3 shows that convergence lines of these four schemes.

Table 2.5 and Table 2.6 show the convergence ratios in the time step of $ErrorI$ and $ErrorII$ by the four schemes of EC-S-FDTDI, EC-S-FDTDII, EC-S-FDTD-(2,4) and ADI-FDTD at time $t = 1$ with $\Delta x = \Delta y = \Delta t$. From Table 2.5 and Table 2.6, it can be clearly seen that EC-S-FDTDII, EC-S-FDTD(2,4) and ADI-FDTD are second order in time but EC-S-FDTDI is first order in time, and meanwhile, EC-S-FDTD(2,4) has smaller $ErrorI$ and $ErrorII$ than EC-S-FDTDII and ADI-FDTD. Figure 2.4 also shows that EC-S-FDTD(2,4) is more accuracy since its error

Table 2.4: The convergence ratios of ErrorII in spatial step for different schemes. Parameters: $\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = k_y = 1$ and $\mu = \epsilon = 1$, at $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD-(2,4)		ADI-FDTD	
N	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio
25	0.0030	-	0.0030	-	1.3719e-05	-	0.0030	-
50	7.5897e-04	1.9828	7.4939e-04	2.0012	8.5816e-07	3.9988	7.4960e-04	2.0008
75	3.3715e-04	2.0012	3.3294e-04	2.0009	1.6954e-07	3.9996	3.3303e-04	2.0009
100	1.8963e-04	2.0003	1.8729e-04	1.9998	5.3646e-08	3.9998	1.8731e-04	2.0004
200	4.7401e-05	2.0002	4.6819e-05	2.0001	3.3543e-09	3.9994	4.6821e-05	2.0002

decreasing line is always below others.

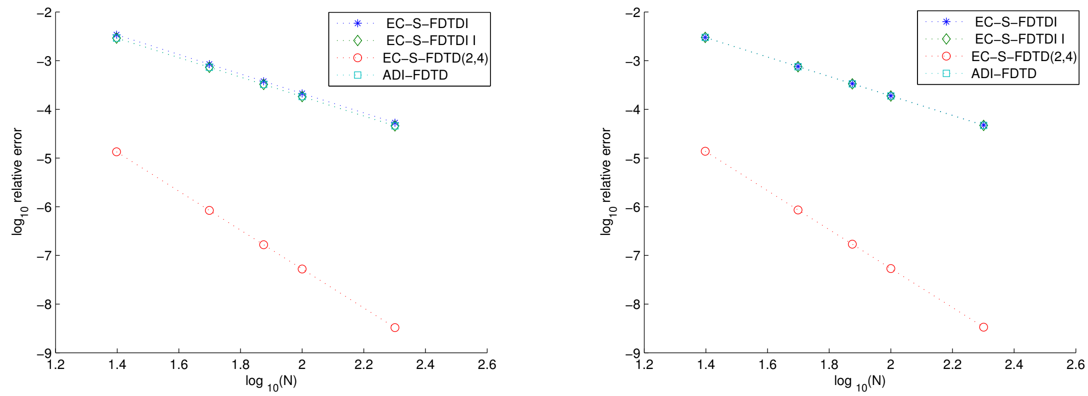
Let the discrete divergence-free errors be defined as that, for the spatial second order schemes

$$DivI = \max_{1\le i\le I-1, 1\le j\le J-1, 0\le n\le N} |\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}})|, \tag{2.5.11}$$

$$DivII = \max_{0\le n\le N} \left(\sum_{1\le i\le I-1} \sum_{1\le j\le J-1} \epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}})^2 \Delta x \Delta y \right)^{\frac{1}{2}}, \tag{2.5.12}$$

and for the spatial fourth-order scheme, the definitions of DivI and DivII in (2.5.11) (2.5.12) are changed by using Λ_x and Λ_y over strict interior nodes and $\widetilde{\Lambda}_x$ and $\widetilde{\Lambda}_y$ over the near boundary nodes to replace δ_x and δ_y.

Table 2.7 and Table 2.8 list the numerical results of DivI and DivII of different



(a) *(b)*

Figure 2.3: ErrorI *(a) and ErrorII (b) in spatial step by different schemes when* T *=* 1*,* $N_x = N_y = N$*,* $N_t = N^2$*,* $k_x = k_y = 1$*, and* $\mu = \epsilon = 1$*.*

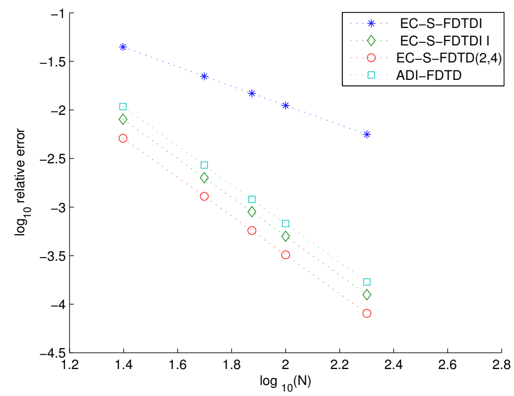
schemes with $\Delta x = \Delta y = \Delta t$ at time level $t = 1$. From these two tables, we see clearly that the errors of numerical divergence-free of EC-S-FDTD-(2,4), EC-S-FDTDII and ADI-FDTD is second order in time, but EC-S-FDTDI is first order in time.

Table 2.5: The convergence ratios of ErrorI in time step by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$ and $\mu = \epsilon = 1$ at $T = 1$.

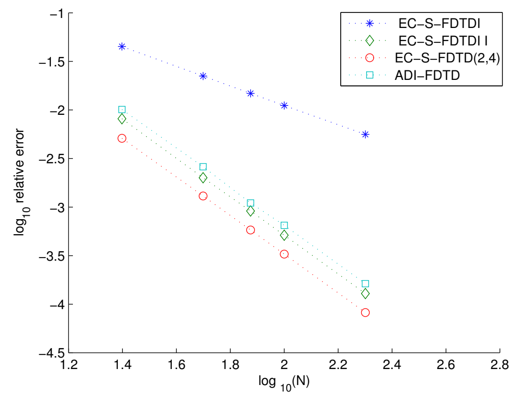
Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD-(2,4)		ADI-FDTD	
N	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio
25	0.0445	-	0.0080	-	0.0051	-	0.0108	-
50	0.0222	1.0032	0.0020	2.0000	1.2880e-03	1.9854	0.0027	2.0000
75	0.0148	1.0000	8.9566e-04	1.9813	5.7258e-04	1.9994	0.0012	2.0000
100	0.0111	1.0000	4.9897e-04	2.0035	3.2209e-04	1.9999	6.7599e-04	1.9949
200	0.0056	0.9871	1.2537e-04	1.9928	8.0527e-05	1.9999	1.6902e-04	1.9998

Table 2.6: The convergence ratios of ErrorII in time step by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$ and $\mu = \epsilon = 1$, at $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD-(2,4)		ADI-FDTD	
N	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio
25	0.0449	-	0.0081	-	0.0051	-	0.0101	-
50	0.0223	1.0097	0.0020	2.0179	0.0013	1.9720	0.0026	1.9577
75	0.0148	1.0111	9.1078e-04	1.9400	5.8111e-04	1.9858	0.0011	2.1215
100	0.0111	1.0065	5.1325e-04	1.9937	3.2745e-04	1.9939	6.4756e-04	1.8418
200	0.0056	1.0000	1.2865e-04	1.9962	8.2070e-05	1.9963	1.6246e-04	1.9949



(a)



(b)

Figure 2.4: ErrorI *(a) and ErrorII (b) in time step by different schemes when* $T = 1$, $N_x = N_y = N_t = N$, $\mu = \epsilon = 1$, $k_x = k_y = 1$.

Table 2.7: The convergence ratios of divergence-free (DivI) in time step by EC-S-FDTD-I&II, EC-S-FDTD-(2,4), and ADI-FDTD. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$, $\mu = \epsilon = 1$, at $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD-(2,4)		ADI-FDTD	
N	DivI	Ratio	DivI	Ratio	DivI	Ratio	DivI	Ratio
25	0.1961	-	0.0044	-	0.0044	-	0.0174	-
50	0.0986	0.9919	0.0011	2.0000	0.0011	2.0000	0.0043	2.0167
75	0.0657	1.0013	4.8687e-04	2.0102	4.8694e-04	2.0099	0.0019	2.0144
100	0.0493	0.9983	2.7401e-04	1.9982	2.7403e-04	1.9984	0.0011	1.8998
200	0.0247	0.9971	6.8512e-05	1.9998	6.8512e-05	1.9999	2.7404e-04	2.0050

Table 2.8: The convergence ratios of divergence-free (DivII) in time step by EC-S-FDTD-I&II, EC-S-FDTD-(2,4), ADI-FDTD. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, and with $k_x = k_y = 1$, $\mu = \epsilon = 1$ at $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	DivII	Ratio	DivII	Ratio	DivII	Ratio	DivII	Ratio
25	0.0984	-	0.0022	-	0.0022	-	0.0087	-
50	0.0493	1.0266	5.4763e-04	2.0062	5.4782e-04	2.0057	0.0022	1.9835
75	0.0329	0.9975	2.4354e-04	1.9985	2.4358e-04	1.9989	9.7400e-04	2.0095
100	0.0247	0.9965	1.3700e-04	1.9998	1.3701e-04	2.0001	5.4796e-04	1.9995
200	0.0123	1.0059	3.4255e-05	1.9998	3.4256e-05	1.9999	1.3702e-04	1.9997

3 The Time and Spatial High-Order Energy-conserved S-FDTD Scheme for Maxwell's Equations

3.1 Introduction

To achieve high order in time, the fourth-order Runge-Kutta schemes are used in [64, 73, 76], in which the time variables are not staggered. Another option in [20, 70] is the fourth order leap-frog time integrators derived. However, these explicit fourth-order FDTD schemes are conditionally stable and have prohibitive requirements of computational memories and computational costs.

In this chapter, we develop and analyze high-order energy-conserved splitting FDTD schemes by focusing on preserving energy conservations and high-order accuracy in both time and spatial steps. We propose a new and novel time and spatial fourth-order energy-conserved S-FDTD scheme (i.e. EC-S-FDTD-(4,4)) for solving Maxwell's equations. Firstly, constructing time fourth-order splitting leads to a seven-stage time splitting procedure for Maxwell's equations. At each stage, we will solve each-stage splitting equations on the Yee's staggered grid. But, if a time second-order scheme is still applied to each-stage equations, it can not obtain fourth-order accuracy in time for the seven-stage time splitting procedure globally. Thus, it is important to construct the time fourth-order scheme to each-stage equations. The first important feature is that for obtaining fourth-order accuracy in time to each-stage equations, we derive out the time fourth-order schemes to each-stage equations by converting the third-order correctional temporal derivatives to the spatial high-order derivatives, which lead to the systems with spatial third-order differential modified terms. Secondly, on the Yee's staggered grids, we approximate the spatial first-order differential operators in the strict interior points by the spatial fourth-order difference operators which are formed by a linear combination of two central differences, one with a spatial step and the other with three spatial steps, while the spatial third-order differential operators in the strict interior points are approximated by the spatial fourth-order difference operators obtained from a linear combination of three central differences, one with a spatial step, the second with three spatial steps and the third with five spatial steps. For the near boundary nodes, the one-sided high-order differences or extrapolations operators can not be used because they break the properties of energy conservations. The second important feature is that we propose the fourth-order near boundary difference operators for the spatial first-order differential operators and the spatial third-order differential operators, by using of the PEC boundary condition, original equations and Taylor's expansion, which have the same accuracy corresponding to the fourth-order interior difference operators and ensure the derived S-FDTD scheme energy conservative. The proposed EC-S-FDTD-(4,4) scheme has the significant properties: energy-conserved, unconditionally stable, fourth-order accurate in time and space, and computationally efficient. We strictly prove that the EC-S-FDTD-(4,4) scheme satisfies energy conservations in the discrete form and in the discrete variation form, and the scheme is unconditionally stable in the discrete L_2-norm and in the discrete H^1-norm. We then prove that the EC-S-FDTD-(4,4) scheme has the optimal fourth-order error estimates of $O\{\Delta t^4+\Delta x^4+\Delta y^4\}$ in the discrete L_2-norm and the super-convergence of $O\{\Delta t^4 + \Delta x^4 + \Delta y^4\}$ in the discrete H^1-norm. The approximation of divergence-free of the scheme is also proved to have fourth-order accuracy in both time and space. Numerical experiments confirm the theoretical results.

The rest of the chapter is organized as follows. In Section 3.2, Maxwell's equations are presented and the time and spatial fourth-order EC-S-FDTD scheme is proposed. In Section 3.3, we prove the properties of energy conservations. The convergence analysis is given in Section 3.4. Numerical experiments are presented in Section 3.5.

3.2 Maxwell's Equations and Time Fourth-Order EC-S-FDTD Scheme

We firstly present the Maxwell's equations in two dimensions, and then give our time fourth-order energy-conserved splitting FDTD scheme.

3.2.1 Maxwell's equations in two dimensions

Consider the two-dimensional transverse electric (**TE**) models with no sources in a lossless medium and that ϵ, μ are constant. The two dimensional Maxwell's equations are:

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\frac{\partial H_z}{\partial y}, \tag{3.2.1}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x}, \tag{3.2.2}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right), \tag{3.2.3}$$

where $\mathbf{E} = (E_x(x,y,t), E_y(x,y,t))$, H_z=$H_z(x,y,t)$ for $(x,y) \in \Omega = [0,a] \times [0,b]$ and $t \in (0,T]$. The perfectly electric conducting (**PEC**) boundary condition is provided:

$$(\mathbf{E}, 0) \times (\mathbf{n}, 0) = 0 \text{ on } (0, \mathbf{T}] \times \partial\Omega, \tag{3.2.4}$$

where $\mathbf{n}$ is the outward normal vector on the boundary. The initial conditions are:

$$\mathbf{E}(x,y,0) = \mathbf{E}_0(x,y) = (E_{x0}(x,y), E_{y0}(x,y)) \text{ and } H_z(x,y,0) = H_{z0}(x,y). \tag{3.2.5}$$

The solution satisfies the energy conservations (I&II) in Lemma 2.2.1 and Lemma 2.2.2, and the following energy conservations (III & IV) (see [23]).

Lemma 3.2.1. *(Energy Conservation III & IV) Let $\boldsymbol{E}$ and $\boldsymbol{H}$ be the solutions of the Maxwell's equations (3.2.1)-(3.2.5) in lossless medium and without charges, and satisfy the PEC boundary condition. If $\boldsymbol{E}$ and $\boldsymbol{H}$ are smooth enough, then*

$$\int_\Omega \left(\epsilon \left| \frac{\partial \boldsymbol{E}}{\partial u} \right|^2 + \mu \left| \frac{\partial \boldsymbol{H}}{\partial u} \right|^2 \right) dxdy \equiv Constant, \tag{3.2.6}$$

$$\int_\Omega \left(\epsilon \left| \frac{\partial^2 \boldsymbol{E}}{\partial t \partial u} \right|^2 + \mu \left| \frac{\partial^2 \boldsymbol{H}}{\partial t \partial u} \right|^2 \right) dxdy \equiv Constant, \tag{3.2.7}$$

where $u = x, y$.

Energy-conserved identities in Lemma 3.2.1 further explain that in a lossless medium and without sources, the electromagnetic waves also satisfy energy conservations in the variation form. For computing problems of longer distance wave propagations and moderately high frequency wave propagations in large domains and large structures, a great attention to develop the fourth-order FDTD schemes has recently been made. However, it is difficult to develop both time and spatial high-order energy-conserved S-FDTD schemes. In this chapter, we will develop a both time and spatial fourth-order S-FDTD scheme that preserves the important physical laws of energy conservations in Lemma 2.2.1, Lemma 2.2.2 and Lemma 3.2.1 in the discrete forms.

3.2.2 The spatial high order difference operators

Take the space domain Ω and time interval an uniformly staggered grid. Let $\Delta x = \frac{a}{I}$, $\Delta y = \frac{b}{J}$, $\Delta t = \frac{T}{N}$, where I, J and T are integers. Let $x_i = i\Delta x$, $x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x$, $i = 0, 1, \cdots, I-1$, $x_I = I\Delta x = a$, $y_j = j\Delta y$, $y_{j+\frac{1}{2}} = y_j + \frac{1}{2}\Delta y$, $j = 0, 1, \cdots, J-1$, $y_J = J\Delta y = b$, and $t^n = n\Delta t$, $t^{n+\frac{1}{2}} = t^n + \frac{1}{2}\Delta t$, $n = 0, 1, \cdots, N-1$, $t_N = N\Delta t = T$.

Let grid function $U^n_{\alpha,\beta} = U(n\Delta t, \alpha\Delta x, \beta\Delta y)$, where $\alpha = i$ or $i + \frac{1}{2}$, and $\beta = j$ or $j + \frac{1}{2}$. We define $\delta_x U$, $\delta_y U$, $\delta_{2,x} U$, $\delta_{2,y} U$, $\delta_{3,x} U$, $\delta_{3,y} U$ and $\delta_u\delta_v U$ as follows:

$$\delta_t U^n_{\alpha,\beta} = \frac{U^{n+\frac{1}{2}}_{\alpha,\beta} - U^{n-\frac{1}{2}}_{\alpha,\beta}}{\Delta t},\ \delta_x U^n_{\alpha,\beta} = \frac{U^n_{\alpha+\frac{1}{2},\beta} - U^n_{\alpha-\frac{1}{2},\beta}}{\Delta x},\ \delta_y U^n_{\alpha,\beta} = \frac{U^n_{\alpha,\beta+\frac{1}{2}} - U^n_{\alpha,\beta-\frac{1}{2}}}{\Delta y},$$
$$\delta_{2,x} U^n_{\alpha,\beta} = \frac{U^n_{\alpha+\frac{3}{2},\beta} - U^n_{\alpha-\frac{3}{2},\beta}}{\Delta x},\ \delta_{2,y} U^n_{\alpha,\beta} = \frac{U^n_{\alpha,\beta+\frac{3}{2}} - U^n_{\alpha,\beta-\frac{3}{2}}}{\Delta y},$$
$$\delta_{3,x} U^n_{\alpha,\beta} = \frac{U^n_{\alpha+\frac{5}{2},\beta} - U^n_{\alpha-\frac{5}{2},\beta}}{\Delta y},\ \delta_{3,y} U^n_{\alpha,\beta} = \frac{U^n_{\alpha,\beta+\frac{5}{2}} - U^n_{\alpha,\beta-\frac{5}{2}}}{\Delta y},\ \delta_u\delta_v U^n_{\alpha,\beta} = \delta_u(\delta_v U^n_{\alpha,\beta}),$$

where δ_u and δ_v can be taken as δ_x, $\delta_{2,x}$, $\delta_{3,x}$, δ_y, $\delta_{2,y}$ and $\delta_{3,y}$.

For the near boundary nodes, $\delta_{2,x} U$, $\delta_{3,x} U$, $\delta_{2,y} U$ and $\delta_{3,y} U$ may fall out of the domain. The one-sided difference or extrapolation operators can be used to construct high-order differences for the near boundary nodes by using more one-sided interior point values. But, these one-sided difference operators will break energy-conservations. We will construct the new near boundary difference operators which have the same high-order accuracy corresponding to the high-order interior difference operators and obtain an energy-conserved S-FDTD scheme. For doing this, we will first give Lemma 3.2.2, where some new notations are introduced as $x_{-i} = -i\Delta x$, $x_{-i+\frac{1}{2}} = x_{-i} + \frac{1}{2}\Delta x$, $x_{I+i} = (I+i)\Delta x$, $x_{I+i+\frac{1}{2}} = x_{I+i} + \frac{1}{2}\Delta x$, $y_{-j} = -j\Delta y$, $y_{-j+\frac{1}{2}} = y_{-j} + \frac{1}{2}\Delta y$, $y_{J+j} = (J+j)\Delta y$, and $y_{J+j+\frac{1}{2}} = y_{J+j} + \frac{1}{2}\Delta y$.

Lemma 3.2.2. *Let the electric and magnetic fields $\{\boldsymbol{E}(x,y,t), H_z(x,y,t)\}$ be the solution components of system (3.2.1) - (3.2.5) with PEC boundary condition, if $\boldsymbol{E}$ and H_z are smooth enough and the initial field $\boldsymbol{E}_0$ is divergence-free, then we have that for $j = 1, 2$*

$$\begin{aligned} E_x(x_{i+\frac{1}{2}}, y_{-j}, t) &= 2E_x(x_{i+\frac{1}{2}}, y_0, t) - E_x(x_{i+\frac{1}{2}}, y_j, t) + O(\Delta y^5), && (3.2.8) \\ E_x(x_{i+\frac{1}{2}}, y_{J+j}, t) &= 2E_x(x_{i+\frac{1}{2}}, y_J, t) - E_x(x_{i+\frac{1}{2}}, y_{J-j}, t) + O(\Delta y^5), && (3.2.9) \end{aligned}$$

and for $i = 0, 1, 2$

$$\begin{aligned} E_x(x_{-i-\frac{1}{2}}, y_j, t) &= E_x(x_{i+\frac{1}{2}}, y_j, t) + O(\Delta x^5), && (3.2.10) \\ E_x(x_{I+i+\frac{1}{2}}, y_j, t) &= E_x(x_{I-i-\frac{1}{2}}, y_j, t) + O(\Delta x^5), && (3.2.11) \end{aligned}$$

and similar relations for E_y at $x-$ and $y-$ directions and H_z at $x-$ and $y-$ directions on the near boundary nodes.

The near boundary difference operators should be suitably accurate relative to the interior high-order difference operators. With the help of Lemma 3.2.2 we can define the difference operators $\delta_{2,x}E_y$, $\delta_{3,x}E_y$ for the near boundary points (near left boundary) as

$$
\begin{aligned}
\delta_{2,x} E^n_{y_{\frac{1}{2},j+\frac{1}{2}}} &= \frac{E^n_{y_{1,j+\frac{1}{2}}} + E^n_{y_{2,j+\frac{1}{2}}} - 2E^n_{y_{0,j+\frac{1}{2}}}}{\Delta x}, \\
\delta_{3,x} E^n_{y_{\frac{1}{2},j+\frac{1}{2}}} &= \frac{E^n_{y_{2,j+\frac{1}{2}}} + E^n_{y_{3,j+\frac{1}{2}}} - 2E^n_{y_{0,j+\frac{1}{2}}}}{\Delta x}, \\
\delta_{3,x} E^n_{y_{1+\frac{1}{2},j+\frac{1}{2}}} &= \frac{E^n_{y_{1,j+\frac{1}{2}}} + E^n_{y_{4,j+\frac{1}{2}}} - 2E^n_{y_{0,j+\frac{1}{2}}}}{\Delta x},
\end{aligned}
$$

and $\delta_{2,x} H_z$, $\delta_{3,x} H_z$ on the near boundary nodes (near left boundary) as

$$
\delta_{2,x} H^n_{z_{1,j+\frac{1}{2}}} = \frac{H^n_{z_{\frac{5}{2},j+\frac{1}{2}}} - H^n_{z_{\frac{1}{2},j+\frac{1}{2}}}}{\Delta x},
$$

$$
\delta_{3,x} H^n_{z_{1,j+\frac{1}{2}}} = \frac{H^n_{z_{\frac{7}{2},j+\frac{1}{2}}} - H^n_{z_{\frac{3}{2},j+\frac{1}{2}}}}{\Delta x}, \ \delta_{3,x} H^n_{z_{2,j+\frac{1}{2}}} = \frac{H^n_{z_{\frac{9}{2},j+\frac{1}{2}}} - E^n_{y_{\frac{1}{2},j+\frac{1}{2}}}}{\Delta x}.
$$

Similarly, we define $\delta_{2,x} E^n_y$, $\delta_{3,x} E^n_y$, $\delta_{2,x} H^n_z$, and $\delta_{3,x} H^n_z$ on the near right boundary nodes. Further, we can similarly define $\delta_{2,y} E^n_x$, $\delta_{3,y} E^n_x$, $\delta_{2,y} H^n_z$, and $\delta_{3,y} H^n_z$ on the near boundary nodes (near top and bottom boundaries).

Finally, define Λ_x and Ξ_x to be the fourth-order difference operators to the first-order differential operator $\frac{\partial}{\partial x}$ and the third-order differential operator $\frac{\partial^3}{\partial x^3}$ on the strict interior nodes and the near boundary nodes:

$$
\Lambda_x E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}} \equiv \tfrac{27\delta_x - \delta_{2,x}}{24} E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}} = \tfrac{\partial E^n_y}{\partial x}\big|_{i+\frac{1}{2},j+\frac{1}{2}} + O(\Delta x^4), \tag{3.2.12}
$$

$$
\Lambda_x H^n_{z_{i,j+\frac{1}{2}}} \equiv \tfrac{27\delta_x - \delta_{2,x}}{24} H^n_{z_{i,j+\frac{1}{2}}} = \tfrac{\partial H^n_z}{\partial x}\big|_{i,j+\frac{1}{2}} + O(\Delta x^4), \tag{3.2.13}
$$

$$
\Xi_x E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}} \equiv \tfrac{-34\delta_x + 13\delta_{2,x} - \delta_{3,x}}{8\Delta x^2} E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}} = \tfrac{\partial^3 E^n_y}{\partial x^3}\big|_{i+\frac{1}{2},j+\frac{1}{2}} + O(\Delta x^4), \tag{3.2.14}
$$

$$
\Xi_x H^n_{z_{i,j+\frac{1}{2}}} \equiv \tfrac{-34\delta_x + 13\delta_{2,x} - \delta_{3,x}}{8\Delta x^2} H^n_{z_{i,j+\frac{1}{2}}} = \tfrac{\partial^3 H^n_z}{\partial x^3}\big|_{i,j+\frac{1}{2}} + O(\Delta x^4). \tag{3.2.15}
$$

Similarly, we can define $\Lambda_y E^n_{x_{i,j}}$, $\Lambda_y H^n_{z_{i+\frac{1}{2},j}}$, $\Xi_y E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}$ and $\Xi_y H^n_{z_{i+\frac{1}{2},j}}$ to approximate $\frac{\partial E^n_x}{\partial y}$, $\frac{\partial H^n_z}{\partial y}$, $\frac{\partial^3 E^n_x}{\partial y^3}$ and $\frac{\partial^3 H^n_z}{\partial y^3}$ on the strict interior nodes and the near boundary nodes.

3.2.3 The splitting method

Time splitting is based on symplectic integrator technique. For non-commutative operators A and B, let a set of real numbers $(c_1, c_2, \dots, c_k)$ and $(d_1, d_2, \dots, d_k)$ such that the following equality holds:

$$e^{\Delta t(A+B)} = \prod_{i=1}^{4} e^{(c_i 2\Delta t A)} e^{(d_i 2\Delta t B)} + O(\Delta t^5).$$

with

$$\begin{aligned} c_1 = c_4 &= \frac{1}{4(2-2^{\frac{1}{3}})},\ c_2 = c_3 = \frac{1-2^{\frac{1}{3}}}{4(2-2^{\frac{1}{3}})}, \\ d_1 = d_3 &= \frac{1}{2(2-2^{\frac{1}{3}})},\ d_2 = -\frac{2^{\frac{1}{3}}}{2(2-2^{\frac{1}{3}})},\ d_4 = 0. \end{aligned}$$

This leads to a seven-stage time splitting scheme for obtaining fourth-order time accuracy. For solving each-stage equations, we will further propose a fourth-order scheme in both time and space steps. The problem is that if a time second-order scheme is applied to each-stage equations, it can not obtain fourth-order accuracy in time for the seven-stage time splitting scheme globally. Thus, it is important and challenging to construct the time fourth-order scheme to each-stage equations so that we can obtain a time fourth-order energy-conserved S-FDTD scheme. The important feature is that we will derive a time fourth-order scheme to each-stage equations by applying the Taylor's expansions in which the third-order correctional temporal derivatives will be converted to the spatial derivatives. Thus, a new time fourth-order scheme to solve each-stage equations is proposed in the following.

Let $f(x)$ be a smooth enough function, we can obtain by using Taylor expansion that

$$\frac{f^{n+1} - f^n}{\Delta t} = \frac{1}{2}\frac{\partial(f^n + f^{n+1})}{\partial t} - \frac{\Delta t^2}{24}\frac{\partial^3\big(f^n + f^{n+1}\big)}{\partial t^3} + O(\Delta t^4). \tag{3.2.16}$$

Using this relation of (3.2.16), we approximate the following one-stage splitting equations in $(t_n,\, t_{n+1}]$

$$\begin{cases} \frac{\partial E_x}{\partial t} = \frac{2c}{\epsilon}\frac{\partial H_z}{\partial y} \\ \frac{\partial H_z}{\partial t} = \frac{2c}{\mu}\frac{\partial E_x}{\partial y} \end{cases} \tag{3.2.17}$$

where the c is one constant above. We can thus propose the time fourth-order scheme for the one-stage equations (3.2.17) as

$$\begin{cases} \frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \left(\frac{c}{\epsilon}\frac{\partial}{\partial y} - \frac{c^3\Delta t^2}{3\epsilon^2\mu}\frac{\partial^3}{\partial y^3}\right)(H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{n}_{z_{i+\frac{1}{2},j}}) \\ \frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \left(\frac{c}{\mu}\frac{\partial}{\partial y} - \frac{c^3\Delta t^2}{3\epsilon\mu^2}\frac{\partial^3}{\partial y^3}\right)(E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}). \end{cases} \tag{3.2.18}$$

Similarly, we have the time fourth-order schemes to other one-stage equations in the seven-stage splitting method. For further constructing the spatial schemes to each-stage equations (3.2.18), the proposed spatial fourth-order difference operators

Λ_x and Ξ_x for both the strict interior nodes and the near boundary nodes in Section 3.2.2 will ensure to obtain an energy-conserved high-order S-FDTD scheme in the next sub-section.

3.2.4 The EC-S-FDTD-(4,4) scheme

Based on the proposed time splitting technique in Section 3.2.3 and the proposed spatial fourth-order difference operators for the strict interior nodes and the near boundary nodes in Section 3.2.2, we propose the following time and spatial fourth-order splitting FDTD scheme. The scheme is defined as, for $n \geq 1$,

Stage 1: Compute the variables $E_x^{(1)}$ and $H_z^{(1)}$ from E_x^n and H_z^n:

$$\frac{E_{x_{i+\frac{1}{2},j}}^{(1)} - E_{x_{i+\frac{1}{2},j}}^{n}}{\Delta t} = \left(\frac{c_1}{\epsilon}\Lambda_y - \frac{c_1^3 \Delta t^2}{3\epsilon^2\mu}\Xi_y\right)(H_{z_{i+\frac{1}{2},j}}^{(1)} + H_{z_{i+\frac{1}{2},j}}^{n}), \quad (3.2.19)$$

$$\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}}{\Delta t} = \left(\frac{c_1}{\mu}\Lambda_y - \frac{c_1^3 \Delta t^2}{3\epsilon\mu^2}\Xi_y\right)(E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)} + E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}), \quad (3.2.20)$$

Stage 2: Compute the variables $E_y^{(1)}$ and $H_z^{(2)}$ from E_y^n and $H_z^{(1)}$:

$$\frac{E_{y_{i,j+\frac{1}{2}}}^{(1)} - E_{y_{i,j+\frac{1}{2}}}^{n}}{\Delta t} = -\left(\frac{d_1}{\epsilon}\Lambda_x - \frac{d_1^3 \Delta t^2}{3\epsilon^2\mu}\Xi_x\right)(H_{z_{i,j+\frac{1}{2}}}^{(2)} + H_{z_{i,j+\frac{1}{2}}}^{(1)}), \quad (3.2.21)$$

$$\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(2)} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}}{\Delta t} = -\left(\frac{d_1}{\mu}\Lambda_x - \frac{d_1^3 \Delta t^2}{3\epsilon\mu^2}\Xi_x\right)(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)} + E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}), \quad (3.2.22)$$

Stage 3: Compute the variables $E_x^{(2)}$ and $H_z^{(3)}$ from $E_x^{(1)}$ and $H_z^{(2)}$:

$$\frac{E_{x_{i+\frac{1}{2},j}}^{(2)} - E_{x_{i+\frac{1}{2},j}}^{(1)}}{\Delta t} = \left(\frac{c_2}{\epsilon}\Lambda_y - \frac{c_2^3 \Delta t^2}{3\epsilon^2\mu}\Xi_y\right)(H_{z_{i+\frac{1}{2},j}}^{(3)} + H_{z_{i+\frac{1}{2},j}}^{(2)}), \quad (3.2.23)$$

$$\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(3)} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(2)}}{\Delta t} = \left(\frac{c_2}{\mu}\Lambda_y - \frac{c_2^3 \Delta t^2}{3\epsilon\mu^2}\Xi_y\right)(E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{(2)} + E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}), \quad (3.2.24)$$

Stage 4: Compute the variables $E_y^{(2)}$ and $H_z^{(4)}$ from $E_y^{(1)}$ and $H_z^{(3)}$:

$$\frac{E^{(2)}_{y_{i,j+\frac{1}{2}}} - E^{(1)}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\left(\frac{d_2}{\epsilon}\Lambda_x - \frac{d_2^3\Delta t^2}{3\epsilon^2\mu}\Xi_x\right)(H^{(4)}_{z_{i,j+\frac{1}{2}}} + H^{(3)}_{z_{i,j+\frac{1}{2}}}), \quad (3.2.25)$$

$$\frac{H^{(4)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{(3)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\left(\frac{d_2}{\mu}\Lambda_x - \frac{d_2^3\Delta t^2}{3\epsilon\mu^2}\Xi_x\right)(E^{(2)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{(1)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}), \quad (3.2.26)$$

Stage 5: Compute the variables $E_x^{(3)}$ and $H_z^{(5)}$ from $E_x^{(2)}$ and $H_z^{(4)}$:

$$\frac{E^{(3)}_{x_{i+\frac{1}{2},j}} - E^{(2)}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \left(\frac{c_3}{\epsilon}\Lambda_y - \frac{c_3^3\Delta t^2}{3\epsilon^2\mu}\Xi_y\right)(H^{(5)}_{z_{i+\frac{1}{2},j}} + H^{(4)}_{z_{i+\frac{1}{2},j}}), \quad (3.2.27)$$

$$\frac{H^{(5)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{(4)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \left(\frac{c_3}{\mu}\Lambda_y - \frac{c_3^3\Delta t^2}{3\epsilon\mu^2}\Xi_y\right)(E^{(3)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{(2)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}), \quad (3.2.28)$$

Stage 6: Compute the variables E_y^{n+1} and $H_z^{(6)}$ from $E_y^{(2)}$ and $H_z^{(5)}$:

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{(2)}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\left(\frac{d_3}{\epsilon}\Lambda_x - \frac{d_3^3\Delta t^2}{3\epsilon^2\mu}\Xi_x\right)(H^{(6)}_{z_{i,j+\frac{1}{2}}} + H^{(5)}_{z_{i,j+\frac{1}{2}}}), \quad (3.2.29)$$

$$\frac{H^{(6)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{(5)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\left(\frac{d_3}{\mu}\Lambda_x - \frac{d_3^3\Delta t^2}{3\epsilon\mu^2}\Xi_x\right)(E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{(2)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}), \quad (3.2.30)$$

Stage 7: Compute the variables $E_x^{(n+1)}$ and H_z^{n+1} from $E_x^{(3)}$ and $H_z^{(6)}$:

$$\frac{E^{(n+1)}_{x_{i+\frac{1}{2},j}} - E^{(3)}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \left(\frac{c_4}{\epsilon}\Lambda_y - \frac{c_4^3\Delta t^2}{3\epsilon^2\mu}\Xi_y\right)(H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{(6)}_{z_{i+\frac{1}{2},j}}), \quad (3.2.31)$$

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{(6)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \left(\frac{c_4}{\mu}\Lambda_y - \frac{c_4^3\Delta t^2}{3\epsilon\mu^2}\Xi_y\right)(E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{(3)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}). \quad (3.2.32)$$

Where $E^{(k)}_{x_{i+\frac{1}{2},j}}$, $k = 1, 2, 3$, $E^{(k)}_{y_{i,j+\frac{1}{2}}}$, $k = 1, 2$, and $H^{(k)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$, $k = 1, 2, \cdots, 6$, are intermediate variables. The boundary conditions are given by

$$E^{(k)}_{x_{i+\frac{1}{2},0}} = E^{n}_{x_{i+\frac{1}{2},0}} = E^{(k)}_{x_{i+\frac{1}{2},J}} = E^{(k)}_{x_{i+\frac{1}{2},J}} = 0, \ k = 1, 2, 3, \quad (3.2.33)$$

$$E^{(k)}_{y_{0,j+\frac{1}{2}}} = E^{n}_{y_{0,j+\frac{1}{2}}} = E^{(k)}_{y_{I,j+\frac{1}{2}}} = E^{n}_{y_{I,j+\frac{1}{2}}} = 0, \ k = 1, 2, \quad (3.2.34)$$

and the initial conditions are given by

$$E^0_{x_{\alpha,\beta}} = E_{x_0}(\alpha\Delta x, \beta\Delta y);\ E^0_{y_{\alpha,\beta}} = E_{y_0}(\alpha\Delta x, \beta\Delta y);$$
$$H^0_{z_{\alpha,\beta}} = H_{z_0}(\alpha\Delta x, \beta\Delta y). \tag{3.2.35}$$

We will prove that the seven-stage scheme (3.2.19)-(3.2.35) satisfies the energy conservations in the following section. Thus, the scheme (3.2.19)-(3.2.35) can be called the EC-S-FDTD-(4,4) scheme.

3.3 Energy conservations

In this section, we will consider the discrete energy conservations of the EC-S-FDTD-(4,4) scheme. The discrete norms for grid functions **F**=(U,V) and W are defined on the staggered grids as:

$$\|U\|^2_{E_x} = \textstyle\sum_{i=0}^{I-1}\sum_{j=0}^{J}\left|U_{i+\frac{1}{2},j}\right|^2\Delta x\Delta y,\ \|V\|^2_{E_y} = \textstyle\sum_{i=0}^{I}\sum_{j=0}^{J-1}\left|V_{i,j+\frac{1}{2}}\right|^2\Delta x\Delta y,$$
$$\|W\|^2_{H_z} = \textstyle\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|W_{i+\frac{1}{2},j+\frac{1}{2}}\right|^2\Delta x\Delta y,\ \|\mathbf{F}\|^2_E = \|U\|^2_{E_x} + \|V\|^2_{E_y},$$

where the meshes are $\Omega_{E_x} = \{(x_{i+\frac{1}{2}}, y_j)|_{i=0}^{I-1},{}_{j=0}^{J}\}$, $\Omega_{E_y} = \{(x_i, y_{j+\frac{1}{2}})|_{i=0}^{I},{}_{j=0}^{J-1}\}$, and $\Omega_{H_z} = \{(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})|_{i=0}^{I-1},{}_{j=0}^{J-1}\}$. For the central difference operator δ, we define

$$\|\delta_x U\|^2_{\delta_x E_x} = \textstyle\sum_{i=1}^{I-1}\sum_{j=0}^{J}|\delta_x U_{i,j}|^2\Delta x\Delta y,$$
$$\|\delta_y U\|^2_{\delta_y E_x} = \textstyle\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|\delta_y U_{i+\frac{1}{2},j+\frac{1}{2}}\right|^2\Delta x\Delta y,$$

over meshes $\Omega_{\delta_x E_x} = \{(x_i, y_j)|_{i=1}^{I-1}{}_{,j=0}^{J}\}$ and $\Omega_{\delta_y E_x} = \{(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})|_{i=0}^{I-1}{}_{,j=0}^{J-1}\}$ respectively. Similarly, we can define $\|\delta_x V\|^2_{\delta_x E_y}$, $\|\delta_y V\|^2_{\delta_y E_y}$, $\|\delta_x W\|^2_{\delta_x H_z}$ and $\|\delta_y W\|^2_{\delta_y H_z}$ over meshes $\Omega_{\delta_x E_y} = \{(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})|_{i=0}^{I-1}{}_{,j=0}^{J-1}\}$, $\Omega_{\delta_y E_y} = \{(x_i, y_j)|_{i=0}^{I}{}_{,j=1}^{J-1}\}$, $\Omega_{\delta_x H_z} = \{(x_i, y_{j+\frac{1}{2}})|_{i=1}^{I-1}{}_{,j=0}^{J-1}\}$, and $\Omega_{\delta_y H_z} = \{(x_{i+\frac{1}{2}}, y_j)|_{i=0}^{I-1}{}_{,j=1}^{J-1}\}$ respectively. For fourth-order difference operator Λ defined differently for strict interior nodes and near boundary nodes, we can similarly define $\|\Lambda_x U\|^2_{\delta_x E_x}$, $\|\Lambda_y U\|^2_{\delta_y E_x}$, $\|\Lambda_x V\|^2_{\delta_x E_y}$, $\|\Lambda_y V\|^2_{\delta_y E_y}$, $\|\Lambda_x W\|^2_{\delta_x H_z}$ and $\|\Lambda_y W\|^2_{\delta_y H_z}$ respectively.

For analyzing energy conservations, we first give several lemmas.

Lemma 3.3.1. *Let $\{a_k\}_{k=1}^{p}$ and $\{b_k\}_{k=0}^{p}$ be two sequences. Then*

$$\begin{aligned}\sum_{k=3}^{p-2} a_k(b_{k+2} - b_{k-3}) = -a_1 b_3 - a_2 b_4 - a_3 b_0 - a_4 b_1 - a_5 b_2 + a_p b_{p-3} + a_{p-1} b_{p-4} \\ + a_{p-2} b_p + a_{p-3} b_{p-1} + a_{p-4} b_{p-2} - \sum_{k=3}^{p-3} b_k(a_{k+3} - a_{k-2}). \qquad (3.3.1)\end{aligned}$$

From Lemma 3.3.1, Lemma 2.3.1 and Lemma 2.3.2, we can have Lemma 3.3.2.

Lemma 3.3.2. *If grid functions E_x, E_y and H_z are defined on staggered grid and E_x, E_y satisfy the boundary conditions: $E_{x_{i+\frac{1}{2},0}} = E_{x_{i+\frac{1}{2},J}} = E_{y_{0,j+\frac{1}{2}}} = E_{y_{I,j+\frac{1}{2}}} = 0$, then it holds that*

$$\sum_{j=0}^{J_u-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \Lambda_u E_{v_{i+\frac{1}{2},j+\frac{1}{2}}} = -\sum_{j=1}^{J_u-1} E_{v_{i+\frac{1}{2},j}} \Lambda_u H_{z_{i+\frac{1}{2},j}}, \qquad (3.3.2)$$

$$\sum_{i=0}^{J_u-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \Xi_u E_{v_{i+\frac{1}{2},j+\frac{1}{2}}} = -\sum_{i=1}^{J_u-1} E_{v_{i,j+\frac{1}{2}}} \Xi_u H_{z_{i,j+\frac{1}{2}}}, \qquad (3.3.3)$$

where $u = y, x$; $J_u = J, I$; $v = x, y$.

Proof. We only give the proof of (3.3.3) for $u = y, J_u = J$ and $v = x$. Other relations in (3.3.2)-(3.3.3) can be obtained similarly. From the definition of Ξ_x, we have

$$\sum_{i=0}^{I-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \Xi_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} = \sum_{i=0}^{I-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \frac{-34\delta_x + 13\delta_{2,x} - \delta_{3,x}}{8\Delta x^2} E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}. \quad (3.3.4)$$

Considering the near boundary nodes and the strict interior nodes, using the relation (3.3.1) in Lemma 3.3.1 and the definition of $\delta_{3,x}$, we have that:

$$\begin{aligned}
\sum_{i=0}^{I-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \delta_{3,x} E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} &= \sum_{i=2}^{I-3} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \delta_{3,x} E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + H_{z_{\frac{1}{2},j+\frac{1}{2}}} \delta_{3,x} E_{y_{\frac{1}{2},j+\frac{1}{2}}} \\
&+ H_{z_{\frac{3}{2},j+\frac{1}{2}}} \delta_{3,x} E_{y_{\frac{3}{2},j+\frac{1}{2}}} + H_{z_{I-\frac{1}{2},j+\frac{1}{2}}} \delta_{3,x} E_{y_{I-\frac{1}{2},j+\frac{1}{2}}} + H_{z_{I-\frac{3}{2},j+\frac{1}{2}}} \delta_{3,x} E_{y_{I-\frac{3}{2},j+\frac{1}{2}}} \\
&= -\sum_{i=3}^{I-3} E_{y_{i,j+\frac{1}{2}}} \delta_{3,x} H_{z_{i,j+\frac{1}{2}}} - E_{y_{1,j+\frac{1}{2}}} \delta_{3,x} H_{z_{1,j+\frac{1}{2}}} - E_{y_{2,j+\frac{1}{2}}} \delta_{3,x} H_{z_{2,j+\frac{1}{2}}} \\
&- E_{y_{I-1,j+\frac{1}{2}}} \delta_{3,x} H_{z_{I-1,j+\frac{1}{2}}} - E_{y_{I-2,j+\frac{1}{2}}} \delta_{3,x} H_{z_{I-2,j+\frac{1}{2}}} \\
&= -\sum_{i=1}^{I-1} E_{y_{i,j+\frac{1}{2}}} \delta_{3,x} H_{z_{i,j+\frac{1}{2}}}. \qquad (3.3.5)
\end{aligned}$$

Similarly, from Lemma 2.3.1 and Lemma 2.3.2 we have that

$$\begin{aligned}
\sum_{i=0}^{I-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \delta_{2,x} E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} &= -\sum_{i=1}^{I-1} E_{y_{i,j+\frac{1}{2}}} \delta_{2,x} H_{z_{i,j+\frac{1}{2}}}, \qquad (3.3.6) \\
\sum_{i=0}^{I-1} H_{z_{i+\frac{1}{2},j+\frac{1}{2}}} \delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} &= -\sum_{i=1}^{I-1} E_{y_{i,j+\frac{1}{2}}} \delta_x H_{z_{i,j+\frac{1}{2}}}. \qquad (3.3.7)
\end{aligned}$$

(3.3.3) can be obtained by substituting the relations (3.3.5)-(3.3.7) into (3.3.4). This ends the proof. □

Theorem 3.3.1. *(Energy conservations I & II) For integer $n \geq 0$, let $\boldsymbol{E}^n = \{(E^n_{x_{i+\frac{1}{2},j}}, E^n_{y_{i,j+\frac{1}{2}}})\}$ and $H^n_z = \{H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}$ be the solutions of the EC-S-FDTD-(4,4) scheme*

(3.2.19) - (3.2.35). Then the following energy conservations hold:

$$\left\|\epsilon^{\frac{1}{2}}\boldsymbol{E}^{n+1}\right\|_E^2+\left\|\mu^{\frac{1}{2}}H_z^{n+1}\right\|_{H_z}^2=\left\|\epsilon^{\frac{1}{2}}\boldsymbol{E}^{n}\right\|_E^2+\left\|\mu^{\frac{1}{2}}H_z^{n}\right\|_{H_z}^2,\ n\geq 0, \tag{3.3.8}$$

$$\left\|\epsilon^{\frac{1}{2}}\delta_t\boldsymbol{E}^{n+\frac{3}{2}}\right\|_E^2+\left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{3}{2}}\right\|_{H_z}^2=\left\|\epsilon^{\frac{1}{2}}\delta_t\boldsymbol{E}^{n+\frac{1}{2}}\right\|_E^2+\left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\right\|_{H_z}^2,\ n\geq 0. \tag{3.3.9}$$

Proof. Multiplying both sides of (3.2.19) with $\epsilon\Delta t\left(E_{x_{i+\frac{1}{2},j}}^{(1)}+E_{x_{i+\frac{1}{2},j}}^{n}\right)$ and multiplying both sides of (3.2.20) with $\mu\Delta t\left(H_{z_{i+\frac{1}{2},j}}^{(1)}+H_{z_{i+\frac{1}{2},j}}^{n}\right)$, we obtain that

$$\begin{aligned}
&\epsilon\left[\left(E_{x_{i+\frac{1}{2},j}}^{(1)}\right)^2-\left(E_{x_{i+\frac{1}{2},j}}^{n}\right)^2\right]=c_1\left(E_{x_{i+\frac{1}{2},j}}^{(1)}+E_{x_{i+\frac{1}{2},j}}^{n}\right)\Lambda_y\left(H_{z_{i+\frac{1}{2},j}}^{(1)}+H_{z_{i+\frac{1}{2},j}}^{n}\right)\\
&\qquad-\frac{c_1^3\Delta t^2}{3\epsilon\mu}\left(E_{x_{i+\frac{1}{2},j}}^{(1)}+E_{x_{i+\frac{1}{2},j}}^{n}\right)\Xi_y\left(H_{z_{i+\frac{1}{2},j}}^{(1)}+H_{z_{i+\frac{1}{2},j}}^{n}\right), && (3.3.10)\\
&\mu\left[\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}\right)^2-\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)^2\right]\\
&\quad=c_1\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}+H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)\Lambda_y\Big(E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)} && (3.3.11)\\
&\qquad+E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\Big)-\frac{c_1^3\Delta t^2}{3\epsilon\mu}\left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}+H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)\Xi_y\left(E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}+E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right).
\end{aligned}$$

We sum over all the terms in the equations (3.3.10) and (3.3.11) and add these two equations together. Further, using the boundary condition (3.2.34), from Lemma 3.3.2, we can have that

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}(\epsilon(E_{x_{i+\frac{1}{2},j}}^{(1)})^2+\mu(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)})^2)=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}(\epsilon(E_{x_{i+\frac{1}{2},j}}^{n})^2+\mu(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n})^2). \tag{3.3.12}$$

Similarly, from equations (3.2.21) - (3.2.32), we can obtain other relations between $\{E_{y_{i,j+\frac{1}{2}}}^{(1)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(2)}\}$ and $\{E_{y_{i,j+\frac{1}{2}}}^{n},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)}\}$; $\{E_{x_{i+\frac{1}{2},j}}^{(2)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(3)}\}$ and $\{E_{x_{i+\frac{1}{2},j}}^{(1)},$

$H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(2)}\}$; $\{E_{y_{i,j+\frac{1}{2}}}^{(2)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(4)}\}$ and $\{E_{y_{i,j+\frac{1}{2}}}^{(1)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(3)}\}$; $\{E_{x_{i+\frac{1}{2},j}}^{(3)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(5)}\}$ and $\{E_{x_{i+\frac{1}{2},j}}^{(2)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(4)}\}$; $\{E_{y_{i,j+\frac{1}{2}}}^{(n+1)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(6)}\}$ and $\{E_{y_{i,j+\frac{1}{2}}}^{(2)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(5)}\}$; and $\{E_{x_{i+\frac{1}{2},j}}^{(n+1)},$ $H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(n+1)}\}$ and $\{E_{x_{i+\frac{1}{2},j}}^{(3)},\ H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(6)}\}$. Then, adding all the relations together with the boundary conditions (3.2.34) leads to conclusion (3.3.8).

Further, we denote $E_x^{(k)+1}$, $k = 1,2,3$, $E_y^{(k)+1}$, $k = 1,2$, and $H_z^{(k)+1}$, $k = 1,2,\cdots,6$, to be the intermediate values of $E_x^{(k)}$, $E_y^{(k)}$ and $H_z^{(k)}$ at time level $n+1$, respectively, and let $\delta_t E_x^{(k)+\frac{1}{2}} = \frac{E_x^{(k)+1}-H_z^{(k)}}{\Delta t}$, $\delta_t E_y^{(k)+\frac{1}{2}} = \frac{E_y^{(k)+1}-E_y^{(k)}}{\Delta t}$ and $\delta_t H_z^{(k)+\frac{1}{2}} = \frac{H_z^{(k)+1}-H_z^{(k)}}{\Delta t}$. Applying the operator δ_t to (3.2.19) - (3.2.20), we have the following equations:

Stage 1:

$$\begin{aligned}
\frac{\delta_t E_{x_{i+\frac{1}{2},j}}^{(1)+\frac{1}{2}} - \delta_t E_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}}{\Delta t} &= \left(\frac{c_1}{\epsilon}\Lambda_y - \frac{c_1^3\Delta t^2}{3\epsilon^2\mu}\Xi_y\right)(\delta_t H_{z_{i+\frac{1}{2},j}}^{(1)+\frac{1}{2}} + \delta_t H_{z_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}),\\
\frac{\delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)+\frac{1}{2}} - \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta t} &= \left(\frac{c_1}{\mu}\Lambda_y - \frac{c_1^3\Delta t^2}{3\epsilon\mu^2}\Xi_y\right)(\delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{(1)+\frac{1}{2}} + \delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}).
\end{aligned}$$

Stage 2 -**Stage 7** can be obtained similarly by applying the operator δt to (3.2.21) - (3.2.32) respectively. We notice that for the above equations, $\delta_t E_x$ and $\delta_t E_y$ still satisfy the PEC boundary conditions. Following the proof of (3.3.8), we can obtain (3.3.9). This complete the proof. □

In order to prove the energy conservations III and IV, we give another lemma.

Lemma 3.3.3. *If grid functions* E_x*,* E_y *and* H_z *are defined on staggered grid and* E_x*,* E_y *satisfy the boundary conditions:* $E_{x_{i+\frac{1}{2},0}} = E_{x_{i+\frac{1}{2},J}} = E_{y_{0,j+\frac{1}{2}}} = E_{y_{I,j+\frac{1}{2}}} = 0$*, then the following relations hold*

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x H_z \delta_x \mathbf{A}_y E_x)_{i,j+\frac{1}{2}} = -\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\delta_x E_x \delta_x \mathbf{A}_y H_z)_{i,j}, \tag{3.3.13}$$

$$\begin{aligned}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x H_z \delta_x \mathbf{A}_x E_y)_{i,j+\frac{1}{2}} = &-\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}(\delta_x E_y \delta_x \mathbf{A}_x H_z)_{i+\frac{1}{2},j+\frac{1}{2}} \\ &-\frac{1}{\Delta x^2}\sum_{j=0}^{J-1}\{(E_y \mathbf{A}_x H_z)_{1,j+\frac{1}{2}} + (E_y \mathbf{A}_x H_z)_{I-1,j+\frac{1}{2}}\},\end{aligned} \tag{3.3.14}$$

where $\mathbf{A} = \Lambda, \Xi$.

Proof. We give the proof of (3.3.14) for $\mathbf{A}_x = \Lambda_x$. Other (3.3.14) for $\mathbf{A}_x = \Xi_x$ and (3.3.13) for $\mathbf{A}_y = \Lambda_y, \Xi_y$ can be proved in a similar way. In (3.3.14), the left side is

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x H_z \delta_x \Lambda_x E_y)_{i,j+\frac{1}{2}} = \frac{1}{\Delta x}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\delta_x H_{z_{i,j+\frac{1}{2}}}\Lambda_x(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} - E_{y_{i-\frac{1}{2},j+\frac{1}{2}}}). \tag{3.3.15}$$

Noting of the identities $\delta_{2,x} E_{y_{i-\frac{1}{2},j+\frac{1}{2}}} = \delta_x(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E_{y_{i-\frac{1}{2},j+\frac{1}{2}}} + E_{y_{i-\frac{3}{2},j+\frac{1}{2}}})$, $\delta_{2,x} H_{z_{i+1,j+\frac{1}{2}}} = \delta_x(H_{z_{i+2,j+\frac{1}{2}}} + H_{z_{i+1,j+\frac{1}{2}}} + H_{z_{i,j+\frac{1}{2}}})$ for the strict interior nodes and the definition of Λ_x, one term of the right side of (3.3.15) can be re-organized as

$$\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\delta_x H_{z_{i,j+\frac{1}{2}}}\delta_{2,x}E_{y_{i-\frac{1}{2},j+\frac{1}{2}}}\\
&=\sum_{i=2}^{I-1}\sum_{j=0}^{J-1}\delta_x H_{z_{i,j+\frac{1}{2}}}\delta_x(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+E_{y_{i-\frac{1}{2},j+\frac{1}{2}}}+E_{y_{i-\frac{3}{2},j+\frac{1}{2}}})+\sum_{j=0}^{J-1}\delta_x H_{z_{1,j+\frac{1}{2}}}\delta_{2,x}E_{y_{\frac{1}{2},j+\frac{1}{2}}}\\
&=\sum_{i=1}^{I-3}\sum_{j=0}^{J-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\delta_x(H_{z_{i+2,j+\frac{1}{2}}}+H_{z_{i+1,j+\frac{1}{2}}}+H_{z_{i,j+\frac{1}{2}}})+\sum_{j=0}^{J-1}(\delta_x E_{y_{\frac{1}{2},j+\frac{1}{2}}}\delta_x H_{z_{2,j+\frac{1}{2}}}\\
&\quad-\delta_x E_{y_{\frac{3}{2},j+\frac{1}{2}}}\delta_x H_{z_{1,j+\frac{1}{2}}}+\delta_{2,x}E_{y_{\frac{1}{2},j+\frac{1}{2}}}\delta_x H_{z_{1,j+\frac{1}{2}}})+\sum_{j=0}^{J-1}(\delta_x E_{y_{I-\frac{3}{2},j+\frac{1}{2}}}\delta_x H_{z_{I-1,j+\frac{1}{2}}}\\
&\quad+\delta_x E_{y_{I-\frac{1}{2},j+\frac{1}{2}}}\delta_x H_{z_{I-1,j+\frac{1}{2}}}+\delta_x E_{y_{I-\frac{3}{2},j+\frac{1}{2}}}\delta_x H_{z_{I-2,j+\frac{1}{2}}})\\
&=\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\delta_{2,x}H_{z_{i+1,j+\frac{1}{2}}}+\sum_{j=0}^{J-1}\frac{1}{\Delta x^2}E_{y_{1,j+\frac{1}{2}}}(H_{z_{\frac{5}{2},j+\frac{1}{2}}}+H_{z_{\frac{3}{2},j+\frac{1}{2}}}-2H_{z_{\frac{1}{2},j+\frac{1}{2}}})\\
&\qquad\qquad+\sum_{j=0}^{J-1}\frac{1}{\Delta x^2}E_{y_{I-1,j+\frac{1}{2}}}(H_{z_{I-\frac{3}{2},j+\frac{1}{2}}}-H_{z_{I-\frac{1}{2},j+\frac{1}{2}}}).
\end{aligned}\tag{3.3.16}$$

Using (3.3.16) and the definition of Λ_x, we can get that

$$\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\delta_x H_{z_{i,j+\frac{1}{2}}}\Lambda_x E_{y_{i-\frac{1}{2},j+\frac{1}{2}}}=\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\Lambda_x H_{z_{i+1,j+\frac{1}{2}}}\\
&\quad-\sum_{j=0}^{J-1}\frac{1}{24\Delta x^2}E_{y_{1,j+\frac{1}{2}}}(H_{z_{\frac{5}{2},j+\frac{1}{2}}}+H_{z_{\frac{3}{2},j+\frac{1}{2}}}-2H_{z_{\frac{1}{2},j+\frac{1}{2}}})\\
&\quad-\sum_{j=0}^{J-1}\frac{1}{24\Delta x^2}E_{y_{I-1,j+\frac{1}{2}}}(H_{z_{I-\frac{3}{2},j+\frac{1}{2}}}-H_{z_{I-\frac{1}{2},j+\frac{1}{2}}})+\frac{27}{24}\sum_{j=0}^{J-1}\delta_x H_{z_{1,j+\frac{1}{2}}}\delta_x E_{y_{\frac{1}{2},j+\frac{1}{2}}}.
\end{aligned}$$

The similar relation can be obtained to the term of $\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\delta_x H_{z_{i,j+\frac{1}{2}}}\Lambda_x E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}$. Replacing these two relations into (3.3.15) and using the definition of operator $\delta_{2,x}H_z$ in the near boundary nodes, we finally obtain (3.3.14). This ends the proof. □

Theorem 3.3.2. *(Energy conservations III & IV in the δ-form) For $n \geq 0$, let $\boldsymbol{E}^n = \{(E^n_{x_{i+\frac{1}{2},j}}, E^n_{y_{i,j+\frac{1}{2}}})\}$ and $H^n_z = \{H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}$ be the solutions of the EC-S-FDTD-(4,4) scheme (3.2.19) - (3.2.35). Then the energy conservation properties in the δ-form hold:*

$$\begin{aligned} &\left\| \epsilon^{\frac{1}{2}} \delta_u E_x^{n+1} \right\|^2_{\delta_u E_x} + \left\| \epsilon^{\frac{1}{2}} \delta_u E_y^{n+1} \right\|^2_{\delta_u E_y} + \left\| \mu^{\frac{1}{2}} \delta_u H_z^{n+1} \right\|^2_{\delta_u H_z} \\ &= \left\| \epsilon^{\frac{1}{2}} \delta_u E_x^{n} \right\|^2_{\delta_u E_x} + \left\| \epsilon^{\frac{1}{2}} \delta_u E_y^{n} \right\|^2_{\delta_u E_y} + \left\| \mu^{\frac{1}{2}} \delta_u H_z^{n} \right\|^2_{\delta_u H_z}, \end{aligned} \tag{3.3.17}$$

$$\begin{aligned} &\left\| \epsilon^{\frac{1}{2}} \delta_t \delta_u E_x^{n+\frac{3}{2}} \right\|^2_{\delta_u E_x} + \left\| \epsilon^{\frac{1}{2}} \delta_t \delta_u E_y^{n+\frac{3}{2}} \right\|^2_{\delta_u E_y} + \left\| \mu^{\frac{1}{2}} \delta_t \delta_u H_z^{n+\frac{3}{2}} \right\|^2_{\delta_u H_z} \\ &= \left\| \epsilon^{\frac{1}{2}} \delta_t \delta_u E_x^{n+\frac{1}{2}} \right\|^2_{\delta_u E_x} + \left\| \epsilon^{\frac{1}{2}} \delta_t \delta_u E_y^{n+\frac{1}{2}} \right\|^2_{\delta_u E_y} + \left\| \mu^{\frac{1}{2}} \delta_t \delta_u H_z^{n+\frac{1}{2}} \right\|^2_{\delta_u H_z}, \end{aligned} \tag{3.3.18}$$

where $u = x, y$.

Proof. We give the proof of (3.3.18) for $u = x$. Others (3.3.18) for $u = y$ and (3.3.19) for $u = x, y$ can be further proved. Applying the operator δ_x to (3.2.19) and (3.2.20) in **Stages 1-7** of the EC-S-FDTD-(4,4) scheme, we can write the following equations

Stage 1: $1 \leq i \leq I-1$

$$\frac{\delta_x E^{(1)}_{x_{i,j}} - \delta_x E^n_{x_{i,j}}}{\Delta t} = \left(\frac{c_1}{\epsilon} \Lambda_y - \frac{c_1^3 \Delta t^2}{3\epsilon^2 \mu} \Xi_y \right) \left(\delta_x H^{(1)}_{z_{i,j}} + \delta_x H^n_{z_{i,j}} \right),$$

$$1 \leq j \leq J-1, \tag{3.3.19}$$

$$\frac{\delta_x H^{(1)}_{z_{i,j+\frac{1}{2}}} - \delta_x H^n_{z_{i,j+\frac{1}{2}}}}{\Delta t} = \left(\frac{c_1}{\mu} \Lambda_y - \frac{c_1^3 \Delta t^2}{3\epsilon \mu^2} \Xi_y \right) \left(\delta_x E^{(1)}_{x_{i,j+\frac{1}{2}}} + \delta_x E^n_{x_{i,j+\frac{1}{2}}} \right),$$

$$0 \leq j \leq J-1 \tag{3.3.20}$$

Stage 2: $0 \le j \le J-1$

$$\frac{\delta_x E^{(1)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} - \delta_x E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\left(\frac{d_1}{\epsilon}\Lambda_x - \frac{d_1^3\Delta t^2}{3\epsilon^2\mu}\Xi_x\right)\left(\delta_x H^{(2)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + \delta_x H^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right),$$
$$1 \le i \le I-2, \tag{3.3.21}$$

$$\frac{\delta_x H^{(2)}_{z_{i,j+\frac{1}{2}}} - \delta_x H^{(1)}_{z_{i,j+\frac{1}{2}}}}{\Delta t} = -\left(\frac{d_1}{\mu}\Lambda_x - \frac{d_1^3\Delta t^2}{3\epsilon\mu^2}\Xi_x\right)\left(\delta_x E^{(1)}_{y_{i,j+\frac{1}{2}}} + \delta_x E^{n}_{y_{i,j+\frac{1}{2}}}\right),$$
$$1 \le i \le I-1, \tag{3.3.22}$$

Stage 3 to **Stage 7** are similarly obtained.

We consider **Stage 2**. Multiplying both sides of (3.3.21) with $\epsilon\Delta t\Big(\delta_x E^{(1)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}$ $+\delta_x E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\Big)$ and multiplying both sides of (3.3.22) with $\mu\Delta t\left(\delta_x H^{(2)}_{z_{i,j+\frac{1}{2}}} + \delta_x H^{(1)}_{z_{i,j+\frac{1}{2}}}\right)$, we sum over all the terms in these two equations and add them together. It holds that

$$\begin{aligned}
&\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}(\epsilon(\delta_x E^{(1)}_{y_{i,j+\frac{1}{2}}})^2 - \mu(\delta_x H^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2) - \sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\mu(\delta_x H^{(2)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2 - \epsilon(\delta_x E^{n}_{y_{i,j+\frac{1}{2}}})^2) \\
&\quad = \frac{1}{\Delta x^2}\sum_{j=0}^{J-1}\left\{(E_y^{(1)} - E_y^{(n)})((d_1\Lambda_x - \frac{d_1^3\Delta t^2}{3\epsilon\mu}\Xi_x))(H_z^{(2)} + H_z^{(1)})\right\}_{1,j+\frac{1}{2}} \\
&\qquad + \frac{1}{\Delta x^2}\sum_{j=0}^{J-1}\left\{(E_y^{(1)} - E_y^{(n)})((d_1\Lambda_x - \frac{d_1^3\Delta t^2}{3\epsilon\mu}\Xi_x))(H_z^{(2)} + H_z^{(1)})\right\}_{I-1,j+\frac{1}{2}} \\
&\quad = -\frac{\epsilon}{\Delta x^2}\sum_{j=0}^{J-1}\left\{(E_y^{(1)})^2 - (E_y^{(n)})^2\right\}_{1,j+\frac{1}{2}} - \frac{\epsilon}{\Delta x^2}\sum_{j=0}^{J-1}\left\{(E_y^{(1)})^2 - (E_y^{(n)})^2\right\}_{I-1,j+\frac{1}{2}}.
\end{aligned}$$

From Lemma 3.3.3, and noticing that $E^{(1)}_{y_{0,j+\frac{1}{2}}} = E^{(n)}_{y_{0,j+\frac{1}{2}}} = E^{(1)}_{y_{I,j+\frac{1}{2}}} = E^{(n)}_{y_{I,j+\frac{1}{2}}} = 0$, we have that for **Stage 2**,

$$\|\epsilon^{\frac{1}{2}}\delta_x E_y^{(1)}\|^2_{\delta_x E_y} + \|\mu^{\frac{1}{2}}\delta_x H_z^{(2)}\|^2_{\delta_x H_z} = \|\epsilon^{\frac{1}{2}}\delta_x E_y^{n}\|^2_{\delta_x E_y} + \|\mu^{\frac{1}{2}}\delta_x H_z^{(1)}\|^2_{\delta_x H_z}. \tag{3.3.23}$$

Similarly, for **Stage 1**, we have that

$$\|\epsilon^{\frac{1}{2}}\delta_x E_x^{(1)}\|^2_{\delta_x E_x} + \|\mu^{\frac{1}{2}}\delta_x H_z^{(1)}\|^2_{\delta_x H_z} = \|\epsilon^{\frac{1}{2}}\delta_x E_x^{n}\|^2_{\delta_x E_x} + \|\mu^{\frac{1}{2}}\delta_x H_z^{n}\|^2_{\delta_x H_z}. \tag{3.3.24}$$

In the same way, we can have other relations for **Stage 3 - 7**. Finally, (3.3.18) for $u = x$ is obtained by adding these relations together. This ends the proof. □

We can similarly prove the following theorem in the Λ-form.

Theorem 3.3.3. *(Energy conservations III & IV in the Λ-form) For integer $n \geq 0$, let $\boldsymbol{E}^n = \{(E^n_{x_{i+\frac{1}{2},j}}, E^n_{y_{i,j+\frac{1}{2}}})\}$ and $H_z^n = \{H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}$ be the solutions of the EC-S-FDTD-(4,4) scheme (3.2.19) - (3.2.35). Then we have the energy conservation properties in the Λ-form*

$$\begin{aligned} &\left\|\epsilon^{\frac{1}{2}}\Lambda_u E_x^{n+1}\right\|^2_{\delta_u E_x} + \left\|\epsilon^{\frac{1}{2}}\Lambda_u E_y^{n+1}\right\|^2_{\delta_u E_y} + \left\|\mu^{\frac{1}{2}}\Lambda_u H_z^{n+1}\right\|^2_{\delta_u H_z} \\ &\qquad = \left\|\epsilon^{\frac{1}{2}}\Lambda_u E_x^{n}\right\|^2_{\delta_u E_x} + \left\|\epsilon^{\frac{1}{2}}\Lambda_u E_y^{n}\right\|^2_{\delta_u E_y} + \left\|\mu^{\frac{1}{2}}\Lambda_u H_z^{n}\right\|^2_{\delta_u H_z}, \end{aligned} \tag{3.3.25}$$

$$\begin{aligned} &\left\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u E_x^{n+\frac{3}{2}}\right\|^2_{\delta_u E_x} + \left\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u E_y^{n+\frac{3}{2}}\right\|^2_{\delta_u E_y} + \left\|\mu^{\frac{1}{2}}\delta_t\Lambda_u H_z^{n+\frac{3}{2}}\right\|^2_{\delta_u H_z} \\ &= \left\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u E_x^{n+\frac{1}{2}}\right\|^2_{\delta_u E_x} + \left\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u E_y^{n+\frac{1}{2}}\right\|^2_{\delta_u E_y} + \left\|\mu^{\frac{1}{2}}\delta_t\Lambda_u H_z^{n+\frac{1}{2}}\right\|^2_{\delta_u H_z} \end{aligned} \tag{3.3.26}$$

where $u = x, y$.

From Theorems 3.3.1 - 3.3.3, we have the following unconditional stability results.

Corollary 2. *(Unconditional stability) The EC-S-FDTD-(4,4) scheme defined by (3.2.19) - (3.2.35) with PEC boundary conditions are unconditionally stable in the discrete L_2-norm and in the discrete H^1-norm.*

3.4 Convergence and super-convergence

There is a difficulty of analyzing the truncation errors of the seven-stage EC-S-FDTD-(4,4) scheme. For the simplicity of notations, we define difference operators by

$$L_{c_k,y} = c_k\Lambda_y - \frac{c_k^3\Delta t^2}{3\epsilon\mu}\Xi_y,\ L_{d_k,x} = d_k\Lambda_x - \frac{d_k^3\Delta t^2}{3\epsilon\mu}\Xi_x,$$

where $k = 1,2,3,4$. We notice that $c_1 = c_4$, $c_2 = c_3$, $d_1 = d_3$, and $d_4 = 0$, thus, the operators are $L_{c_1,y} = L_{c_4,y}$, $L_{c_2,y} = L_{c_3,y}$ and $L_{d_1,x} = L_{d_3,x}$. Define the new intermediate variables $\tilde{E}_x^{(1)}$-$\tilde{E}_x^{(3)}$, $\tilde{E}_y^{(1)}$, $\tilde{E}_y^{(2)}$ and $\tilde{H}_z^{(1)}$-$\tilde{H}_z^{(6)}$ from exact solutions $E_x(t^n), E_y(t^n), H_z(t^n), E_x(t^{n+1}), E_y(t^{n+1})$ and $H_z(t^{n+1})$ by

$$\begin{aligned}
\tilde{E}_x^{(1)} &= \left(1+2\frac{\Delta t^2}{\epsilon\mu}L_{c1,y}^2+2\frac{\Delta t^4}{\epsilon^2\mu^2}L_{c1,y}^4\right)E_x(t^n)+\left(2\frac{\Delta t}{\epsilon}L_{c1,y}+2\frac{\Delta t^3}{\epsilon^2\mu}L_{c1,y}^3\right)H_z(t^n),\\
\tilde{H}_z^{(1)} &= \left(1+2\frac{\Delta t^2}{\epsilon\mu}L_{c1,y}^2+2\frac{\Delta t^4}{\epsilon^2\mu^2}L_{c1,y}^4\right)H_z(t^n)+\left(2\frac{\Delta t}{\mu}L_{c1,y}+2\frac{\Delta t^3}{\epsilon\mu^2}L_{c1,y}^3\right)E_x(t^n),
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_y^{(1)} &= \left(1+2\frac{\Delta t^2}{\epsilon\mu}L_{d1,x}^2+2\frac{\Delta t^4}{\epsilon^2\mu^2}L_{d1,x}^4\right)E_y(t^n)-\left(2\frac{\Delta t}{\epsilon}L_{d1,x}+2\frac{\Delta t^3}{\epsilon^2\mu}L_{d1,x}^3\right)\tilde{H}_z^{(1)},\\
\tilde{H}_z^{(2)} &= \left(1+2\frac{\Delta t^2}{\epsilon\mu}L_{d1,x}^2+2\Delta\frac{t^4}{\epsilon^2\mu^2}L_{d1,x}^4\right)\tilde{H}_z^{(1)}-\left(2\frac{\Delta t}{\mu}L_{d1,x}+2\frac{\Delta t^3}{\epsilon\mu^2}L_{d1,x}^3\right)E_y(t^n),
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_x^{(2)} &= \left(1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{c2,y}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{c2,y}^4\right) \tilde{E}_x^{(1)} + \left(2\frac{\Delta t}{\epsilon} L_{c2,y} + 2\frac{\Delta t^3}{\epsilon^2\mu} L_{c2,y}^3\right) \tilde{H}_z^{(2)}, \\
\tilde{H}_z^{(3)} &= \left(1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{c2,y}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{c2,y}^4\right) \tilde{H}_z^{(2)} + \left(2\frac{\Delta t}{\mu} L_{c2,y} + 2\frac{\Delta t^3}{\epsilon\mu^2} L_{c2,y}^3\right) \tilde{E}_x^{(1)},
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_x^{(3)} &= (1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{c4,y}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{c4,y}^4) E_x(t^{n+1}) - (2\frac{\Delta t}{\epsilon} L_{c4,y} + 2\frac{\Delta t^3}{\epsilon^2\mu} L_{c4,y}^3) H_z(t^{n+1}), \\
\tilde{H}_z^{(6)} &= (1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{c4,y}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{c4,y}^4) H_z(t^{n+1}) - (2\frac{\Delta t}{\mu} L_{c4,y} + 2\frac{\Delta t^3}{\epsilon\mu^2} L_{c4,y}^3) E_x(t^{n+1}),
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_y^{(2)} &= \left(1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{d3,x}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{d3,x}^4\right) E_y(t^{n+1}) + \left(2\frac{\Delta t}{\epsilon} L_{d3,x} + 2\frac{\Delta t^3}{\epsilon^2\mu} L_{d3,x}^3\right) \tilde{H}_z^{(6)}, \\
\tilde{H}_z^{(5)} &= \left(1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{d3,x}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{d3,x}^4\right) \tilde{H}_z^{(6)} + \left(2\frac{\Delta t}{\mu} L_{d3,x} + 2\frac{\Delta t^3}{\epsilon\mu^2} L_{d3,x}^3\right) E_y(t^{n+1}),
\end{aligned}$$

$$\tilde{H}_z^{(4)} = \left(1 + 2\frac{\Delta t^2}{\epsilon\mu} L_{c3,y}^2 + 2\frac{\Delta t^4}{\epsilon^2\mu^2} L_{c3,y}^4\right) \tilde{H}_z^{(5)}. + \left(2\frac{\Delta t}{\mu} L_{c2,y} + 2\frac{\Delta t^3}{\epsilon\mu^2} L_{c3,y}^3\right) \tilde{E}_x^{(3)}.$$

Replacing $\tilde{E}_x^{(1)}$-$\tilde{E}_x^{(3)}$, $\tilde{E}_y^{(1)}$, $\tilde{E}_y^{(2)}$ and $\tilde{H}_z^{(1)}$-$\tilde{H}_z^{(6)}$ into the seven-stages of the EC-S-FDTD scheme, it leads to the equivalent scheme with the local truncation errors of $\xi 1$-$\xi 14$. Among them, $\xi 1$-$\xi 6$ and $\xi 11$-$\xi 14$ can be computed directly by substituting the defined intermediate variables $\tilde{E}_x^{(1)}$-$\tilde{E}_x^{(3)}$, $\tilde{E}_y^{(1)}$, $\tilde{E}_y^{(2)}$ and $\tilde{H}_z^{(1)}$-$\tilde{H}_z^{(6)}$ into Stages 1 - 3 and Stages 6 - 7 as

$$\begin{aligned}
\xi 1_{i+\frac{1}{2},j} &= -2\frac{\Delta t^4}{\epsilon^3\mu^2} L_{c1,y}^5 H_z(t^n)_{i+\frac{1}{2},j}, \quad \xi 2_{i+\frac{1}{2},j+\frac{1}{2}} = -2\frac{\Delta t^4}{\epsilon^2\mu^3} L_{c1,y}^5 E_x(t^n)_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\xi 3_{i,j+\frac{1}{2}} &= 2\frac{\Delta t^4}{\epsilon^3\mu^2} L_{d1,x}^5 (\tilde{H}_z^{(1)})_{i,j+\frac{1}{2}}, \quad \xi 4_{i+\frac{1}{2},j+\frac{1}{2}} = 2\frac{\Delta t^4}{\epsilon^2\mu^3} L_{d1,x}^5 E_y(t^n)_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\xi 5_{i+\frac{1}{2},j} &= -2\frac{\Delta t^4}{\epsilon^3\mu^2} L_{c2,y}^5 (\tilde{H}_z^{(2)})_{i+\frac{1}{2},j}, \quad \xi 6_{i+\frac{1}{2},j+\frac{1}{2}} = -2\frac{\Delta t^4}{\epsilon^2\mu^3} L_{c2,y}^5 (\tilde{E}_x^{(1)})_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\xi 13_{i+\frac{1}{2},j} &= -2\frac{\Delta t^4}{\epsilon^3\mu^2} L_{c4,y}^5 H_z(t^{n+1})_{i+\frac{1}{2},j}, \xi 14_{i+\frac{1}{2},j+\frac{1}{2}} = -2\frac{\Delta t^4}{\epsilon^2\mu^3} L_{c4,y}^5 E_x(t^{n+1})_{i+\frac{1}{2},j+\frac{1}{2}}, \\
\xi 11_{i,j+\frac{1}{2}} &= -2\frac{\Delta t^4}{\epsilon^3\mu^2} L_{d3,x}^5 (\tilde{H}_z^{(6)})_{i,j+\frac{1}{2}}, \quad \xi 12_{i+\frac{1}{2},j+\frac{1}{2}} = -2\frac{\Delta t^4}{\epsilon^2\mu^3} L_{d3,x}^5 E_y(t^{n+1})_{i+\frac{1}{2},j+\frac{1}{2}}.
\end{aligned}$$

However, the $\xi 7$-$\xi 10$ terms for Stages 4 -5 have that

$$
\begin{aligned}
\xi 7_{i,j+\frac{1}{2}} = & \Big[\frac{E_y(t^{n+1})-E_y(t^n)}{\Delta t} + \frac{1}{2\epsilon}\Lambda_x(H_z(t^{n+1})+H_z(t^n)) \\
& -\frac{\Delta t^2}{24\epsilon^2\mu}\Xi_x(H_z(t^{n+1})+H_z(t^n)) - \frac{\Delta t^2}{24\epsilon^2\mu}\Lambda_x\Lambda_y^2(H_z(t^{n+1})+H_z(t^n))\Big] \\
& -\Big[4\frac{\Delta t^2}{\epsilon\mu}L_{d1,x}L_{c1,y}\left(\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t} - \frac{1}{2\epsilon}\Lambda_y(H_z(t^{n+1})+H_z(t^n))\right)\Big] \\
& +\Big[2\frac{\Delta t^2}{\epsilon\mu}L_{d1,x}^2\left(\frac{E_y(t^{n+1})-E_y(t^n)}{\Delta t} + \frac{1}{2\epsilon}\Lambda_x(H_z(t^{n+1})+H_z(t^n))\right)\Big] \\
& +\Big[2\frac{\Delta t^2}{\epsilon\mu}L_{d1,x}L_{d2,x}\left(\frac{E_y(t^{n+1})-E_y(t^n)}{\Delta t} + \frac{1}{2\epsilon}\Lambda_x(H_z(t^{n+1})+H_z(t^n))\right)\Big] \\
& -\Big[2\frac{\Delta t^2}{\epsilon\mu}L_{d2,x}(L_{c1,y}+L_{c2,y})\left(\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t} + \frac{1}{2\epsilon}\Lambda_y(H_z(t^{n+1})+H_z(t^n))\right)\Big] \\
& +\Big[2\frac{\Delta t^4}{\epsilon^2\mu^2}\Big(2L_{d1,x}L_{c1,y}^3 + 2L_{d1,x}^3L_{c1,y} + L_{d2,x}L_{c1,y}^3 + 2L_{d1,x}^2L_{d2,x}L_{c1,y} \\
& +2L_{d2,x}L_{c1,y}L_{c2,y}^2 + 2L_{d2,x}L_{c1,y}^2L_{c2,y} + L_{d2,x}L_{c2,y}^3\Big)\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t}\Big] \\
& -\Big[2\frac{\Delta t^4}{\epsilon^2\mu^2}\left(L_{d1,x}^4 + L_{d1,x}^3L_{d2,x} + 2L_{d1,x}L_{d2,x}L_{c2,y}^2\right)\frac{E_y(t^{n+1})-E_y(t^n)}{\Delta t}\Big] + O(\Delta t^4),
\end{aligned}
$$

$$
\begin{aligned}
\xi 8_{i+\frac{1}{2},j+\frac{1}{2}} &= \Big[\frac{H_z(t^{n+1})-H_z(t^n)}{\Delta t} - \frac{1}{2\mu}\Lambda_y(E_x(t^{n+1})+E_x(t^n)) \\
&+\frac{1}{2\mu}\Lambda_x(E_y(t^{n+1})+E_y(t^n)) - \frac{\Delta t^2}{24\epsilon\mu^2}(\Xi_y+\Lambda_x^2\Lambda_y)(E_x(t^{n+1})+E_x(t^n)) \\
&+\frac{\Delta t^2}{24\epsilon\mu^2}(\Xi_x+\Lambda_x\Lambda_y^2)(E_y(t^{n+1})+E_y(t^n))\Big] \\
&-\Big[2\frac{\Delta t^2}{\epsilon\mu}L_{d1,x}(L_{d1,x}+L_{d2,x})\Big(\frac{H_z(t^{n+1})-H_z(t^n)}{\Delta t} - \frac{1}{2\mu}\Lambda_y(E_x(t^{n+1})+E_x(t^n)) \\
&+\frac{1}{2\mu}\Lambda_x(E_y(t^{n+1})+E_y(t^n))\Big)\Big] - \Big[2\frac{\Delta t^2}{\epsilon\mu}(L_{c1,y}+L_{c2,y})^2\Big(\frac{H_z(t^{n+1})-H_z(t^n)}{\Delta t} \\
&-\frac{1}{2\mu}\Lambda_y(E_x(t^{n+1})+E_x(t^n)) + \frac{1}{2\mu}\Lambda_x(E_y(t^{n+1})+E_y(t^n))\Big)\Big] \\
&-\Big[2\frac{\Delta t^4}{\epsilon^2\mu^2}\Big(L_{c1,y}^4+2L_{d1,x}^2L_{c1,y}^2+L_{d1,x}^4+2L_{c1,y}^2L_{c2,y}^2+2L_{d1,x}^2L_{c2,y}^2+L_{c2,y}^4 \\
&+2L_{c1,y}^3L_{c2,y}+2L_{c1,y}L_{c2,y}^3\Big)\frac{H_z(t^{n+1})-H_z(t^n)}{\Delta t}\Big] + O(\Delta t^4),
\end{aligned}
$$

$$
\begin{aligned}
\xi 9_{i+\frac{1}{2},j} &= \Big[\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t} - \frac{1}{2\epsilon}\Lambda_y(H_z(t^{n+1})+H_z(t^n)) \\
&+\frac{\Delta t^2}{24\epsilon^2\mu}\Xi_y(H_z(t^{n+1})+H_z(t^n)) + \frac{\Delta t^2}{24\epsilon^2\mu}\Lambda_x^2\Lambda_y(H_z(t^{n+1})+H_z(t^n))\Big] \\
&+\Big[2\frac{\Delta t^2}{\epsilon\mu}(L_{c3,y}+L_{c4,y})^2\Big(\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t} - \frac{1}{2\epsilon}\Lambda_y(H_z(t^{n+1})+H_z(t^n))\Big)\Big] \\
&-\Big[\frac{\Delta t^2}{\epsilon\mu}L_{c3,y}L_{d3,x}\Big(\frac{E_y(t^{n+1})-E_y(t^n)}{\Delta t} + \frac{1}{2\epsilon}\Lambda_x(H_z(t^{n+1})+H_z(t^n))\Big)\Big] \\
&-\Big[\frac{\Delta t^4}{\epsilon^2\mu^2}\Big(L_{c1,y}^4+2L_{c1,y}^2L_{c2,y}^2+L_{c2,y}^4+2L_{c1,y}^3L_{c2,y}+2L_{d1,x}^2L_{c1,y}L_{c2,y} \\
&+2L_{c1,y}L_{c2,y}^3\Big)\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t}\Big] + O(\Delta t^4),
\end{aligned}
$$

and

$$
\begin{aligned}
\xi 10_{i+\frac{1}{2},j+\frac{1}{2}} &= \Big[\frac{\Delta t}{\mu}L_{c3,y}\Big(\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t} - \frac{1}{2\epsilon}\Lambda_y(H_z(t^{n+1})+H_z(t^n)) \\
&\quad + \frac{\Delta t^2}{24\epsilon^2\mu}\Xi_y(H_z(t^{n+1})+H_z(t^n)) + \frac{\Delta t^2}{24\epsilon^2\mu}\Lambda_x^2\Lambda_y(H_z(t^{n+1})+H_z(t^n))\Big)\Big] \\
&\quad - \Big[4\frac{\Delta t^2}{\epsilon\mu^2}L_{c3,y}^2L_{d3,x}\Big(\frac{E_y(t^{n+1})-E_y(t^n)}{\Delta t} + \frac{1}{2\epsilon}\Lambda_x(H_z(t^{n+1})+H_z(t^n))\Big)\Big] \\
&\quad + \Big[2\frac{\Delta t^2}{\epsilon\mu^2}L_{c3,y}(L_{c3,y}+L_{c4,y})^2\Big(\frac{E_x(t^{n+1})-E_x(t^n)}{\Delta t} \\
&\quad - \frac{1}{2\epsilon}\Lambda_y(H_z(t^{n+1})+H_z(t^n))\Big)\Big] + O(\Delta t^4).
\end{aligned}
$$

Applying Taylor's expansions, we have the following estimates of truncation errors.

Lemma 3.4.1. *(Truncation errors) If the solution components $\{E, H_z\}$ of Maxwell's equations (3.2.1)-(3.2.5) are smooth enough, then the truncation errors are fourth order in time and space:*

$$
\max\{|\xi 1|, |\xi 2|, \cdots, |\xi 14|\} \le C\{\Delta t^4 + \Delta x^4 + \Delta y^4\} \tag{3.4.1}
$$

where C is a constant independent of Δt, Δx, and Δy.

We then analyze the convergence of the EC-S-FDTD-(4,4) scheme. Let error functions on the staggered grid be defined by

$$
\begin{aligned}
&\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}} = E_x(x_{i+\frac{1}{2}}, y_j, t^{n+1}) - E^{n+1}_{x_{i+\frac{1}{2},j}}, \quad \mathcal{E}^{n+1}_{y_{i,j+\frac{1}{2}}} = E_y(x_i, y_{j+\frac{1}{2}}, t^{n+1}) - E^{n+1}_{y_{i,j+\frac{1}{2}}}, \\
&\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} = H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) - H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}},
\end{aligned}
$$

and at intermediate levels

$$\mathcal{E}^{(k)}_{y_{i,j+\frac{1}{2}}} = \tilde{E}^{(k)}_{y_{i,j+\frac{1}{2}}} - E^{(k)}_{y_{i,j+\frac{1}{2}}},\ k=1,2;\ \mathcal{E}^{(l)}_{x_{i+\frac{1}{2},j}} = \tilde{E}^{(l)}_{x_{i+\frac{1}{2},j}} - E^{(l)}_{x_{i+\frac{1}{2},j}},\ l=1,2,3;$$

$$\mathcal{H}^{(m)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} = \tilde{H}^{(m)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{(m)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}},\ m=1,2,\cdots,6.$$

Then, from (3.2.19)-(3.2.35) and definitions above, the error equations of the EC-S-FDTD-(4,4) scheme can be derived as

Stage 1:

$$\epsilon\frac{\mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j}} - \mathcal{E}^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = L_{c1,y}\{\mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j}} + \mathcal{H}^{n}_{z_{i+\frac{1}{2},j}}\} + \xi 1_{i+\frac{1}{2},j}, \tag{3.4.2}$$

$$\mu\frac{\mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = L_{c1,y}\{\mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 2_{i+\frac{1}{2},j+\frac{1}{2}}; \tag{3.4.3}$$

Stage 2:

$$\epsilon\frac{\mathcal{E}^{(1)}_{y_{i,j+\frac{1}{2}}} - \mathcal{E}^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -L_{d1,x}\{\mathcal{H}^{(2)}_{z_{i,j+\frac{1}{2}}} + \mathcal{H}^{(1)}_{z_{i,j+\frac{1}{2}}}\} + \xi 3_{i,j+\frac{1}{2}}, \tag{3.4.4}$$

$$\mu\frac{\mathcal{H}^{(2)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -L_{d1,x}\{\mathcal{E}^{(1)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 4_{i+\frac{1}{2},j+\frac{1}{2}}; \tag{3.4.5}$$

Stage 3:

$$\epsilon\frac{\mathcal{E}^{(2)}_{x_{i+\frac{1}{2},j}} - \mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j}}}{\Delta t} = L_{c2,y}\{\mathcal{H}^{(3)}_{z_{i+\frac{1}{2},j}} + \mathcal{H}^{(2)}_{z_{i+\frac{1}{2},j}}\} + \xi 5_{i+\frac{1}{2},j}, \tag{3.4.6}$$

$$\mu\frac{\mathcal{H}^{(3)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{(2)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = L_{c2,y}\{\mathcal{E}^{(2)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 6_{i+\frac{1}{2},j+\frac{1}{2}}; \tag{3.4.7}$$

Stage 4:

$$\epsilon\frac{\mathcal{E}^{(2)}_{y_{i,j+\frac{1}{2}}} - \mathcal{E}^{(1)}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -L_{d2,x}\{\mathcal{H}^{(4)}_{z_{i,j+\frac{1}{2}}} + \mathcal{H}^{(3)}_{z_{i,j+\frac{1}{2}}}\} + \xi 7_{i,j+\frac{1}{2}}, \tag{3.4.8}$$

$$\mu\frac{\mathcal{H}^{(4)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{(3)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -L_{d2,x}\{\mathcal{E}^{(2)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{(1)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 8_{i+\frac{1}{2},j+\frac{1}{2}}; \tag{3.4.9}$$

Stage 5:

$$\epsilon\frac{\mathcal{E}^{(3)}_{x_{i+\frac{1}{2},j}} - \mathcal{E}^{(2)}_{x_{i+\frac{1}{2},j}}}{\Delta t} = L_{c3,y}\{\mathcal{H}^{(5)}_{z_{i+\frac{1}{2},j}} + \mathcal{H}^{(4)}_{z_{i+\frac{1}{2},j}}\} + \xi 9_{i+\frac{1}{2},j}, \quad (3.4.10)$$

$$\mu\frac{\mathcal{H}^{(5)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{(4)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = L_{c3,y}\{\mathcal{E}^{(3)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{(2)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 10_{i+\frac{1}{2},j+\frac{1}{2}}; \quad (3.4.11)$$

Stage 6:

$$\epsilon\frac{\mathcal{E}^{n+1}_{y_{i,j+\frac{1}{2}}} - \mathcal{E}^{(2)}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -L_{d3,x}\{\mathcal{H}^{(6)}_{z_{i,j+\frac{1}{2}}} + \mathcal{H}^{(5)}_{z_{i,j+\frac{1}{2}}}\} + \xi 11_{i,j+\frac{1}{2}}, \quad (3.4.12)$$

$$\mu\frac{\mathcal{H}^{(6)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{(5)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -L_{d3,x}\{\mathcal{E}^{(6)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{(5)}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 12_{i+\frac{1}{2},j+\frac{1}{2}}; \quad (3.4.13)$$

Stage 7:

$$\epsilon\frac{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}} - \mathcal{E}^{(3)}_{x_{i+\frac{1}{2},j}}}{\Delta t} = L_{c4,y}\{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j}} + \mathcal{H}^{(6)}_{z_{i+\frac{1}{2},j}}\} + \xi 13_{i+\frac{1}{2},j}, \quad (3.4.14)$$

$$\mu\frac{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{(6)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = L_{c4,y}\{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^{(3)}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\} + \xi 14_{i+\frac{1}{2},j+\frac{1}{2}}. \quad (3.4.15)$$

They satisfy the boundary conditions

$$\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},0}} = \mathcal{E}^{(l)}_{x_{i+\frac{1}{2},0}} = \mathcal{E}^{n+1}_{x_{i+\frac{1}{2},J}} = \mathcal{E}^{(l)}_{x_{i+\frac{1}{2},J}} = 0, \quad l = 1, 2, 3, \quad (3.4.16)$$

$$\mathcal{E}^{n+1}_{y_{0,j+\frac{1}{2}}} = \mathcal{E}^{(k)}_{y_{0,j+\frac{1}{2}}} = \mathcal{E}^{n+1}_{y_{I,j+\frac{1}{2}}} = \mathcal{E}^{(k)}_{y_{I,j+\frac{1}{2}}} = 0, \quad k = 1, 2. \quad (3.4.17)$$

Theorem 3.4.1. *(Convergence) Assume that* $\{E_x(t),\ E_y(t),\ H_z(t)\}$*, the exact solutions of (3.2.1)-(3.2.5), are smooth enough. Let* $\{E^n_x,\ E^n_y,\ H^n_z\}$ *be the numerical solutions of the EC-S-FDTD-(4,4) scheme (3.2.19)-(3.2.35). Then for any fixed* *time $T > 0$, there exist positive constants $C_{1\mu\epsilon}$ and $C_{2\mu\epsilon}$ such that*

$$\max_{0\le n\le N}(\|\epsilon^{\frac{1}{2}}(E(t^n)-E^n)\|_E^2+\|\mu^{\frac{1}{2}}(H_z(t^n)-H_z^n)\|_{H_z}^2)^{\frac{1}{2}}$$
$$\le (\|\epsilon^{\frac{1}{2}}(E(t^0)-E^0)\|_E^2+\|\mu^{\frac{1}{2}}(H_z(t^0)-H_z^0)\|_{H_z}^2)^{\frac{1}{2}}$$
$$+C_{1\mu\epsilon}T(\Delta t^4+\Delta x^4+\Delta y^4), \tag{3.4.18}$$
$$\max_{0\le n\le N}(\|\epsilon^{\frac{1}{2}}\delta t(E(t^{n+\frac{1}{2}})-E^{n+\frac{1}{2}})\|_E^2+\|\mu^{\frac{1}{2}}\delta t(H_z(t^{n+\frac{1}{2}})-H_z^{n+\frac{1}{2}})\|_{H_z}^2)^{\frac{1}{2}}$$
$$\le (\|\epsilon^{\frac{1}{2}}\delta t\left(E(t^{\frac{1}{2}})-E^{\frac{1}{2}}\right)\|_E^2+\|\mu^{\frac{1}{2}}\delta t\left(H_z(t^{\frac{1}{2}})-H_z^{\frac{1}{2}}\right)\|_{H_z}^2)^{\frac{1}{2}}$$
$$+C_{2\mu\epsilon}T(\Delta t^4+\Delta x^4+\Delta y^4). \tag{3.4.19}$$

Proof. From equations (3.4.2) and (3.4.3) at Stage 1, and by the energy method used in the proof of Theorem 3.3.1, we have that

$$\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(1)}\|_{E_x}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{(1)}\|_{H_z}^2\right)-\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n}\|_{E_x}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n}\|_{H_z}^2\right)$$
$$=\Delta t\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left[\xi 1_{i+\frac{1}{2},j}(\mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j}}+\mathcal{E}^{n}_{x_{i+\frac{1}{2},j}})+\xi 2_{i+\frac{1}{2},j+\frac{1}{2}}(\mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}})\right]\Delta x\Delta y,$$

which can be written as

$$\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(1)}-\frac{\Delta t}{2}\xi 1\|_{E_x}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{(1)}-\frac{\Delta t}{2}\xi 2\|_{H_z}^2=\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n}+\frac{\Delta t}{2}\xi 1\|_{E_x}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n}+\frac{\Delta t}{2}\xi 2\|_{H_z}^2.$$

Further, using the triangle inequality and the equation above leads to that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(1)}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{(1)}\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(1)} - \frac{\Delta t}{2}\xi 1\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{(1)} - \frac{\Delta t}{2}\xi 2\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\qquad + \left(\|\frac{\Delta t}{2}\xi 1\|_{E_x}^2 + \|\frac{\Delta t}{2}\xi 2\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\quad = \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n} + \frac{\Delta t}{2}\xi 1\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{n} + \frac{\Delta t}{2}\xi 2\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\qquad + \left(\|\frac{\Delta t}{2}\xi 1\|_{E_x}^2 + \|\frac{\Delta t}{2}\xi 2\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{n}\|_{H_z}^2\right)^{\frac{1}{2}} + \left(\|\Delta t\xi 1\|_{E_x}^2 + \|\Delta t\xi 2\|_{H_z}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, from (3.4.4) (3.4.5) at Stage 2, we have that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(1)}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{(1)}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{(2)}\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(1)}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{(1)}\|_{H_z}^2\right)^{\frac{1}{2}} + \left(\|\Delta t\xi 3\|_{E_x}^2 + \|\Delta t\xi 4\|_{H_z}^2\right)^{\frac{1}{2}},
\end{aligned}$$

and the similar relations for Stages 3 - 6. For Stage 7, from (3.4.14) (3.4.15) we have that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{n+1}\|_{H_z}^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{(3)}\|_{E_x}^2 + \|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2 + \|\mu^{\frac{1}{2}}\mathcal{H}_z^{(6)}\|_{H_z}^2\right)^{\frac{1}{2}} + \left(\|\Delta t\xi 13\|_{E_x}^2 + \|\Delta t\xi 14\|_{H_z}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

With these seven derived-relations for Stages 1- 7, and using Lemma 3.4.1, we obtain that,

$$\Big(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n+1}\|_{E_x}^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n+1}\|_{E_y}^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}_z^{n+1}\|_{H_z}^2\Big)^{\frac{1}{2}}$$

$$\leq\Big(\|\epsilon^{\frac{1}{2}}\mathcal{E}_x^{n}\|_{E_x}^2+\|\epsilon^{\frac{1}{2}}\mathcal{E}_y^{n}\|_{E_y}^2+\|\mu^{\frac{1}{2}}\mathcal{H}_z^{n}\|_{H_z}^2\Big)^{\frac{1}{2}}+C_1\Delta t(\Delta t^4+\Delta x^4+\Delta y^4).$$

Applying recursively (3.4.20) from time level n to 0, we finally get (3.4.18). Similarly, we can obtain (3.4.19). This ends the proof. $\square$

By combining with Lemma 3.3.2, we can obtain error estimates of $\delta_u E_x^n$, $\delta_u E_y^n$ and $\delta_u H_z^n$ where $u=x,y$.

Theorem 3.4.2. *(Super-convergence I) Assume that $\{E_x(t),E_y(t),H_z(t)\}$, the solutions of (3.2.1)-(3.2.5), are smooth enough. Let $\{E_x^n,\ E_y^n,\ H_z^n\}$ be the numerical solutions of the EC-S-FDTD-(4,4) scheme (3.2.19)-(3.2.35). Then, we have the following estimates*

$$(\|\epsilon^{\frac{1}{2}}\delta_u\mathcal{E}_x^n\|_{\delta_uE_x}^2+\|\epsilon^{\frac{1}{2}}\delta_u\mathcal{E}_y^n\|_{\delta_uE_y}^2+\|\mu^{\frac{1}{2}}\delta_u\mathcal{H}_z^n\|_{\delta_uH_z}^2)^{\frac{1}{2}} \tag{3.4.20}$$

$$\leq(\|\epsilon^{\frac{1}{2}}\delta_u\mathcal{E}_x^0\|_{\delta_uE_x}^2+\|\epsilon^{\frac{1}{2}}\delta_u\mathcal{E}_y^0\|_{\delta_uE_y}^2+\|\mu^{\frac{1}{2}}\delta_u\mathcal{H}_z^0\|_{\delta_uH_z}^2)^{\frac{1}{2}}+CT(\Delta t^4+\Delta x^4+\Delta y^4)$$

where $u=x,y$.

Similarly as Theorem 3.4.2, we have the following error estimates with fourth-order difference operators of Λ_x and Λ_y.

Theorem 3.4.3. *(Super-convergence II) Assume that $\{E_x(t),E_y(t),H_z(t)\}$, the solutions of Maxwell's equations (3.2.1)-(3.2.5), are smooth enough. Let $\{E_x^n,\ E_y^n,$*

$H_z^n\}$ *be numerical solutions of the EC-S-FDTD-(4,4) scheme (3.2.19)-(3.2.35). Then we have the following estimates:*

$$
\begin{aligned}
&(\|\epsilon^{\frac{1}{2}}\Lambda_u\mathcal{E}_x^n\|^2_{\delta_u E_x} + \|\epsilon^{\frac{1}{2}}\Lambda_u\mathcal{E}_y^n\|^2_{\delta_x E_y} + \|\mu^{\frac{1}{2}}\Lambda_u\mathcal{H}_z^n\|^2_{\delta_u H_z})^{\frac{1}{2}} \qquad (3.4.21)\\
&\leq (\|\epsilon^{\frac{1}{2}}\Lambda_u\mathcal{E}_x^0\|^2_{\delta_u E_x} + \|\epsilon^{\frac{1}{2}}\Lambda_u\mathcal{E}_y^0\|^2_{\delta_u E_y} + \|\mu^{\frac{1}{2}}\Lambda_u\mathcal{H}_z^0\|^2_{\delta_u H_z})^{\frac{1}{2}} + CT(\Delta t^4 + \Delta x^4 + \Delta y^4),
\end{aligned}
$$

where $u = x, y$.

Finally, from the super-convergence in Theorem 3.4.3, we have the following error estimate of divergence-free, if the initial approximations are of fourth order in spatial step.

Theorem 3.4.4. *(Convergence of divergence-free) Let* $\{E_x^n,\ E_y^n,\ H_z^n\}$ *be the numerical solutions of the EC-S-FDTD-(4,4) scheme (3.2.19)- (3.2.35). If the exact solutions of the Maxwell's equations are smooth enough, then the approximation of divergence-free of the electric field holds that*

$$\|\Lambda_x E_x^n + \Lambda_y E_y^n\|_{\delta_x E_x} \leq C(\Delta t^4 + \Delta x^4 + \Delta y^4). \qquad (3.4.22)$$

3.5 Numerical experiments

In this section, we present numerical experiments by focusing on the properties: (1) energy conservation, (2) accuracy, (3) divergence-free. Consider the problem in a lossless medium, $\Omega = [0, 1] \times [0, 1]$, surrounded by a perfect conductor. The exact solutions of equations (3.2.1)-(3.2.3) are

$$E_x = \frac{k_y}{\epsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\cos[k_x\pi(1-x)]\sin[k_y\pi(1-y)], \tag{3.5.1}$$

$$E_y = -\frac{k_x}{\epsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\sin[k_x\pi(1-x)]\cos[k_y\pi(1-y)], \tag{3.5.2}$$

$$H_z = -\frac{1}{\sqrt{\mu}}\sin(\omega\pi t)\cos[k_x\pi(1-x)]\cos[k_y\pi(1-y)]. \tag{3.5.3}$$

where k_x and k_y satisfy the dispersion relation $\omega^2 = \frac{1}{\mu\epsilon}(k_x^2 + k_y^2)$. The exact energy is directly computed as $\text{EnergyI} = \left(\int_\Omega(\epsilon|\mathbf{E}(x,t)|^2 + \mu\epsilon|H_z(x,t)|^2)dxdy\right)^{\frac{1}{2}} = \frac{1}{2}$.

Table 3.1: Relative errors of Energy I and Energy II by the different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		ADI-FDTD		EC-S-FDTD-(4,4)	
N	EnErI	EnErII	EnErI	EnErII	EnErI	EnErII	EnErI	EnErII
25	4.41e-16	2.10e-16	1.10e-16	4.11e-16	9.83e-4	9.76e-4	2.95e-14	2.58e-14
50	1.36e-15	2.05e-16	3.47e-16	1.45e-16	2.50e-4	2.59e-4	5.35e-14	4.76e-14
75	2.20e-15	5.96e-16	3.36e-16	2.31e-15	1.12e-4	1.12e-4	2.53e-14	2.26e-14
100	2.82e-15	8.10e-16	5.67e-16	3.63e-15	6.21e-5	6.20e-5	1.01e-13	8.98e-14
200	5.51e-15	8.03e-16	7.78e-16	7.55e-15	1.55e-5	1.56e-5	2.02e-13	1.80e-13

Define the relative errors of energy conservations:

$$\text{EnErI} = \max_{0\le n\le N}\frac{|(\|\epsilon^{\frac{1}{2}}\mathbf{E}^n\|^2 + \|\mu^{\frac{1}{2}}H_z^n\|^2)^{\frac{1}{2}} - \text{EnergyI}|}{\text{EnergyI}},$$

$$\text{EnErII} = \max_{0\le n\le N-1}\frac{|(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n+\frac{1}{2}}\|^2 + \|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\|^2)^{\frac{1}{2}} - (\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{\frac{1}{2}}\|^2 + \|\mu^{\frac{1}{2}}\delta_t H_z^{\frac{1}{2}}\|^2)^{\frac{1}{2}}|}{(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{\frac{1}{2}}\|^2 + \|\mu^{\frac{1}{2}}\delta_t H_z^{\frac{1}{2}}\|^2)^{\frac{1}{2}}}.$$

Table 3.2: Relative errors of Energy I and Energy II by different schemes. Parameters: $k_y = 1$, $\Delta x = \Delta y = \Delta t$=0.01, $\mu = \epsilon = 1$, and $T = 1$.

SCHEME	k_x=k_y		k_x=5k_y		k_x=10k_y	
	EnErI	EnErII	EnErI	EnErII	EnErI	EnErII
EC-S-FDTDI	2.82e-15	8.10e-16	3.33e-15	3.34e-15	4.44e-16	3.43e-16
EC-S-FDTDII	5.67e-16	3.63e-15	3.11e-15	3.12e-15	3.33e-16	4.57e-16
ADI-FDTD	6.19e-5	6.20e-5	1.20e-4	1.19e-4	1.24e-4	1.20e-4
EC-S-FDTD-(4,4)	1.01e-13	8.98e-14	1.57e-13	1.51e-13	1.45e-13	1.44e-13

In Table 3.1, EnErI and EnErII of our EC-S-FDTD-(4,4) are almost zero, i.e., in the relative error of 10^{-13}, which reach the machine precision. It shows that the EC-S-FDTD-(4,4) scheme stratifies the energy conservations. We can see clearly that the EC-S-FDTDI and EC-S-FDTDII ([7]) satisfy the energy conservations, while the ADI-FDTD ([49, 82]) breaks the energy conservations where the errors of energy only reach the error of 10^{-4}. Table 3.2 sets different wave numbers $k_x = 1k_y$, $5k_y$, and $10k_y$ with step sizes $\Delta x = \Delta y = \Delta t = 0.01$. The relative errors of EC-S-FDTD-(4,4) in the fourth row are 10^{-13} with different wave numbers, which show that the EC-S-FDTD-(4,4) scheme satisfies energy conservations while ADI-FDTD breaks the invariance of energy. We can see that the EC-S-FDTD-(4,4) holds the

the property of energy conservations for different frequency cases.

Table 3.3: Errors of energies in spatial variation forms of different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$.

Mesh	EC-S-FDTDII		ADI-FDTD		EC-S-FDTD-(4,4)			
N	$\mathrm{EnEr}_{\delta x}$	$\mathrm{EnEr}_{\delta y}$	$\mathrm{EnEr}_{\delta x}$	$\mathrm{EnEr}_{\delta y}$	$\mathrm{EnEr}_{\delta x}$	$\mathrm{EnEr}_{\delta y}$	$\mathrm{EnEr}_{\Lambda x}$	$\mathrm{EnEr}_{\Lambda y}$
25	1.24e-15	8.75e-16	0.0014	0.0014	4.20e-14	4.26e-14	4.26e-14	4.31e-14
50	1.99e-15	1.98e-15	3.55e-4	3.56e-4	8.06e-14	8.10e-14	8.08e-14	8.08e-14
75	3.77e-15	3.99e-15	1.56e-4	1.54e-4	3.82e-14	3.91e-14	3.86e-14	3.84e-14
100	5.55e-15	5.60e-15	8.97e-5	8.59e-5	1.53e-13	1.56e-13	1.56e-13	1.56e-13
200	1.31e-14	1.45e-14	1.01e-5	2.23e-5	3.10e-13	3.14e-13	3.15e-13	3.14e-13

Further, define the discrete energies and the errors of energies in the δ_x and δ_y forms:

$$\begin{aligned}
\mathrm{Energy}^n_{\delta_u} &= \left(\left\|\epsilon^{\frac{1}{2}}\delta_u E^n_x\right\|^2_{\delta_x E_x} + \left\|\epsilon^{\frac{1}{2}}\delta_u E^n_y\right\|^2_{\delta_x E_y} + \left\|\mu^{\frac{1}{2}}\delta_u H^n_z\right\|^2_{\delta_u H_z}\right)^{\frac{1}{2}}, \\
\mathrm{EnEr}_{\delta u} &= |\mathrm{Energy}^n_{\delta_u} - \mathrm{Energy}^0_{\delta_u}|,
\end{aligned}$$

where $u = x, y$, and similarly define $\mathrm{EnEr}_{\Lambda_x}$ and $\mathrm{EnEr}_{\Lambda_y}$. The errors of energies in the spatial variation forms are presented in Table 3.3, which shows that EC-S-FDTD-(4,4) and EC-S-FDTD I&II stratify the energy conservations in the discrete variation form but ADI-FDTD breaks the energy conservations in the discrete variation form.

Table 3.4: ErrorI and ratios of solutions by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		ADI-FDTD		EC-S-FDTD-(4,4)	
N	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio
25	0.0445	-	0.0080	-	0.0107	-	3.5846e-5	-
50	0.0222	1.003	0.0020	2.000	0.0026	2.041	2.2657e-6	3.983
75	0.0148	1.000	8.9566e-4	1.981	0.0012	1.906	4.4847e-7	3.994
100	0.0111	1.000	4.9897e-4	2.003	6.7600e-4	1.994	1.4200e-7	3.997
200	0.0056	0.987	1.2537e-4	1.992	1.6901e-4	1.999	8.8813e-9	3.999

Tables 3.4 - 3.5 give the errors and ratios of the numerical solutions of different schemes. The errors are defined by ErrorI $= \max_{0\leq n\leq N}(\|\epsilon^{\frac{1}{2}}[\mathbf{E}(t^n) - E^n]\|_E^2 + \|\mu^{\frac{1}{2}}[H_z(t^n) - H_z^n]\|_H^2)^{\frac{1}{2}}/\text{EnergyI})$, ErrorII $= \max_{0\leq n\leq N-1}(\|\epsilon^{\frac{1}{2}}\delta_t(\mathbf{E}(t^n) - E^n)\|_E^2 + \|\mu^{\frac{1}{2}}\delta_t(H_z(t^n) - H_z^n)\|_H^2)^{\frac{1}{2}}/\text{EnergyII})$. Results in Tables 3.4 and 3.5 indicate that EC-S-FDTD-(4,4) is the most accurate, EC-S-FDTDII and ADI-FDTD are much less accurate, but EC-S-FDTDI is the worst. It is shown clearly that EC-S-FDTD-(4,4) is fourth-order in both time and spatial steps. However, EC-S-FDTDII and ADI-FDTD are second-order in both time and space and the EC-S-FDTDI is only first-order in time.

Table 3.5: ErrorII and ratios of solutions by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$.

Mesh	EC-S-FDTDI		EC-S-FDTDII		ADI-FDTD		EC-S-FDTD-(4,4)	
N	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio
25	0.0449	-	0.0081	-	0.0102	-	3.5224e-5	-
50	0.0223	1.009	0.0020	2.017	0.0027	1.917	2.2523e-6	3.967
75	0.0148	1.011	9.1078e-4	1.940	0.0011	2.214	4.4740e-7	3.986
100	0.0111	1.006	5.1325e-4	1.993	6.4757e-4	1.841	1.4191e-7	3.991
200	0.0056	1.000	1.2865e-4	1.996	1.6245e-4	1.995	8.8980e-9	3.995

Finally, for the error of divergence-free, we let $\mathrm{DivEr}_\delta = \max_{0 \le n \le N} ||\epsilon^{\frac{1}{2}}(\delta_x E_x^n + \delta_y E_y^n)||$, and similarly let DivEr_Λ by just replacing δ_x and δ_y by Λ_x and Λ_y. Numerical errors of the divergence-free are presented in Table 3.6. We can see clearly that the convergence of divergence-free of our ES-S-FDTD-(4,4) is also fourth-order in both time and space in the discrete L_2-norm. However, EC-S-FDTD-II and ADI-FDTD is second-order in time and the EC-FDTD-I is first-order in time. It has shown the important feature that the EC-FDTD-(4,4) has the super-convergence to the divergence-free.

Table 3.6: Errors and ratios of divergence-free by different schemes. Parameters: $\Delta x = \Delta y = \Delta t = 1/N$, $k_x = k_y = 1$, $\mu = \epsilon = 1$, and $T = 1$.

Mesh	EC-S-FDTDII		ADI-FDTD		EC-S-FDTD-(4,4)			
N	DivEr$_\delta$	Ratio	DivEr$_\delta$	Ratio	DivEr$_\delta$	Ratio	DivEr$_\Lambda$	Ratio
25	0.0022	-	0.0086	-	3.8079e-5	-	3.8008e-5	-
50	5.4763e-4	2.006	0.0022	1.966	2.3804e-6	3.999	2.3800e-6	3.997
75	2.4354e-4	1.998	9.7402e-4	2.009	4.7027e-7	3.999	4.7026e-7	3.999
100	1.3700e-4	1.999	5.4797e-4	1.999	1.4879e-7	4.000	1.4879e-7	4.000
200	3.4255e-5	1.999	1.3703e-4	1.999	9.2991e-9	4.000	9.2992e-9	4.000

4 The Spatial High-order Energy-conserved Splitting FDTD Method for Maxwell's Equations in Three Dimensions

4.1 Introduction

In this chapter, we focus on the development and analysis of high order energy-conserved splitting FDTD schemes for three dimensional Maxwell's equations. The two dimensional Maxwell's equations are simple transverse electric (**TE**) and transverse magnetic (**TM**) models. In the (**TE**) or (**TM**) model, there are only three equations, and among them, only one equation needs to be split in the construction of EC-S-FDTD schemes, and the other two equations are not changed. However, the three-dimensional Maxwell's equations have six equations of the electric field $E = \{E_x, E_y, E_z\}$ and the magnetic field $H = \{H_x, H_y, H_z\}$, and every equation needs to be split for constructing the splitting FDTD scheme. Thus it is difficult to develop and theoretically analyze high-order Energy-conserved splitting S-FDTD schemes for the three dimensional problems.

In this chapter, we develop and analyze the three-dimensional spatial high order energy-conserved splitting schemes. Based on the staggered grids, the proposed scheme is a three-stage scheme. At each stage, the spatial differential operators are approximated by the spatial fourth-order difference operators on the strict interior nodes which are a linear combination of two central differences, one with a spatial step and the other with three spatial steps. On the other hand, the one-sided high-order differences and extrapolations/interpolations are normally applied to the near boundary nodes [64, 74, 70]. However, the corresponding high order near boundary operators break the property of energy conservations near the boundaries. It is difficult to construct boundary difference operators and this leads to a challenge of constructing energy-conserved higher-order S-FDTD schemes. We propose to construct the spatial fourth-order near boundary differences over the near boundary nodes by using the PEC boundary conditions, original equations and Taylor's expansion, which ensure the each-stage schemes to preserve the conservations of energy and to have fourth-order accuracy. We strictly prove that the scheme satisfies energy conversation and is unconditionally stable. We obtain the optimal-order error estimate of $O(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)$ in the discrete L_2-norm. We also prove that the scheme preserves the energies in the discrete variation forms and obtain the super-convergence of $O(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)$ in the discrete H_1-norm. Further, we obtain the error estimate of the approximation of divergence-free. Numerical experiments confirm the theoretical results.

The paper is organized as follows. In Section 4.2, the Maxwell's equations in three dimensions are introduced and the spatial fourth-order EC-S-FDTD scheme is proposed. In Section 4.3, we prove energy conservations. The error estimates are analyzed in Section 4.4. Numerical experiments are presented in Section 4.5.

4.2 Maxwell's equations and High-order EC-S-FDTD scheme

We first give the Maxwell's equations in three dimensions, and then propose our high order energy-conserved splitting FDTD scheme for the three dimensional problems.

4.2.1 Maxwell's equations in three dimensions

Consider the three-dimensional Maxwell's equations with no source and in a lossless medium, which are described as:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{4.2.1}$$

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \tag{4.2.2}$$

where $\mathbf{E} = (E_x(x,y,z,t),\, E_y(x,y,z,t),\, E_z(x,y,z,t))$, $\mathbf{H} = (H_x(x,y,z,t)$ and $H_y(x,y,z,t)$, $H_z(x,y,z,t))$, $(x,y,z) \in \Omega = [0,a] \times [0,b] \times [0,c]$, $t \in (0,T]$, denote the electric and magnetic fields. $\mathbf{D}$ and $\mathbf{B}$ are the electric displacement and the magnetic flux density

$$\mathbf{D} = \epsilon\ \mathbf{E},\ \ \mathbf{B} = \mu\ \mathbf{H},\ \ \nabla \cdot \mathbf{B} = 0,\ \ \nabla \cdot \mathbf{D} = 0, \tag{4.2.3}$$

where ϵ is the electric permittivity and μ is the magnetic permeability.

Assume that the perfectly electric conducting (**PEC**) boundary condition is provided

$$\mathbf{E} \times \mathbf{n} = 0,\ \text{ or } \mathbf{H} \times \mathbf{n} = 0,\ \ (x, y, z) \in \partial\Omega, \tag{4.2.4}$$

where $\mathbf{n}$ is the outward normal vector on the boundary. The initial conditions are

$$\mathbf{E}(x, y, z, 0) = \mathbf{E}_0(x, y, z), \mathbf{H}(x, y, z, 0) = \mathbf{H}_0(x, y, z). \tag{4.2.5}$$

It has been proved in [37] that for suitable smooth data, problem (4.2.1)-(4.2.5) has a unique solution, and if the initial fields satisfy divergence-free the electric and magnetic fields always satisfy divergence-free for all time.

In order to construct our scheme, we rewrite (4.2.1-4.2.2) in the matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E} \\ \\ \mathbf{H} \end{pmatrix} = \mathbf{A}\begin{pmatrix} \mathbf{E} \\ \\ \mathbf{H} \end{pmatrix},$$

where $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{curl} \\ \\ -\mathbf{curl} & 0 \end{pmatrix},$$

and the **curl** is

$$\mathbf{curl} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}.$$

The **curl** operator has two kinds of splittings

$$\mathbf{curl} = \mathbf{curl}_{+} + \mathbf{curl}_{-} \text{ and } \mathbf{curl} = \mathbf{curl}_{x} + \mathbf{curl}_{y} + \mathbf{curl}_{z},$$

where

$$\mathbf{curl}_{+} = \begin{pmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \mathbf{curl}_{-} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{curl}_{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \mathbf{curl}_{y} = \begin{pmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \end{pmatrix}, \mathbf{curl}_{z} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus two decompositions of $\mathbf{A}$ are obtained as

Splitting I:

$$\mathbf{A} = \mathbf{A}_{+} + \mathbf{A}_{-},$$

where

$$\mathbf{A}_{+} = \begin{pmatrix} 0 & \mathbf{curl}_{+} \\ \mathbf{curl}_{+}^{T} & 0 \end{pmatrix}, \mathbf{A}_{-} = \begin{pmatrix} 0 & \mathbf{curl}_{-} \\ \mathbf{curl}_{-}^{T} & 0 \end{pmatrix};$$

Splitting II:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z,$$

where

$$\mathbf{A}_x = \begin{pmatrix} 0 & \mathbf{curl}_x \\ \mathbf{curl}_x^T & 0 \end{pmatrix}, \mathbf{A}_y = \begin{pmatrix} 0 & \mathbf{curl}_y \\ \mathbf{curl}_y^T & 0 \end{pmatrix}, \mathbf{A}_z = \begin{pmatrix} 0 & \mathbf{curl}_z \\ \mathbf{curl}_z^T & 0 \end{pmatrix}.$$

4.2.2 High order energy-conserved splitting FDTD scheme in 3D

Take an uniformly staggered grid of the space domain $\Omega \subset R^3$ and the time interval $(0, T]$. Let $\Delta x = \frac{a}{I},\ \Delta y = \frac{b}{J},\ \Delta z = \frac{c}{K},\ \Delta t = \frac{T}{N}$; $x_i = i\Delta x,\ x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x,\ i = 0, 1, \cdots, I-1,\ x_I = I\Delta x = a$; $y_j = j\Delta y,\ y_{j+\frac{1}{2}} = y_j + \frac{1}{2}\Delta y,\ j = 0, 1, \cdots, J-1,\ y_J = J\Delta y = b$; $z_k = k\Delta z,\ z_{k+\frac{1}{2}} = z_k + \frac{1}{2}\Delta z,\ k = 0, 1, \cdots, K-1,\ z_K = K\Delta z = c$; $t^n = n\Delta t,\ t^{n+\frac{1}{2}} = t^n + \frac{1}{2}\Delta t,\ n = 0, 1, \cdots, N-1,\ t^N = N\Delta t = T$; where $I > 0$, $J > 0$, $K > 0$ and $N > 0$ are positive integers.

The grid function $\{E_{x_{i+\frac{1}{2},j,k}}\}$ is defined on nodes $(x_{i+\frac{1}{2}}, y_j, z_k), i = 0, 1, \cdots, I-1, j = 0, 1, \cdots, J, k = 0, 1, \cdots, K$. Similarly, the grid functions $\{E_{y_{i,j+\frac{1}{2},k}}\}$, $\{E_{z_{i,j,k+\frac{1}{2}}}\}$, $\{H_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}\}$, $\{H_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}\}$ and $\{H_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}\}$ are defined on the staggered mesh. Let $U^n_{\alpha,\beta,\gamma} = U(n\Delta t, \alpha\Delta x, \beta\Delta y, \gamma\Delta y)$, $\alpha = i$ or $i + \frac{1}{2}$, $\beta = j$ or $j + \frac{1}{2}$ and $\gamma = k$ or $k + \frac{1}{2}$. We define the difference operators $\delta_x U$, $\delta_y U$, $\delta_z U$ and $\delta_u \delta_v U$ by

$$\delta_t U^n_{\alpha,\beta,\gamma} = \frac{U^{n+\frac{1}{2}}_{\alpha,\beta,\gamma} - U^{n-\frac{1}{2}}_{\alpha,\beta,\gamma}}{\Delta t},\ \delta_x U^n_{\alpha,\beta,\gamma} = \frac{U^n_{\alpha+\frac{1}{2},\beta,\gamma} - U^n_{\alpha-\frac{1}{2},\beta,\gamma}}{\Delta x},$$

$$\delta_y U^n_{\alpha,\beta,\gamma} = \frac{U^n_{\alpha,\beta+\frac{1}{2},\gamma} - U^n_{\alpha,\beta-\frac{1}{2},\gamma}}{\Delta y},\ \delta_z U^n_{\alpha,\beta,\gamma} = \frac{U^n_{\alpha,\beta,\gamma+\frac{1}{2}} - U^n_{\alpha,\beta,\gamma-\frac{1}{2}}}{\Delta y},$$

$$\delta_u\delta_v U^n_{\alpha,\beta,\gamma} = \delta_u(\delta_v U^n_{\alpha,\beta,\gamma})$$

where u and v can be taken as x, y and z directions, and define the difference operators $\delta_{2,x}U$, $\delta_{2,y}U$ and $\delta_{2,z}U$ with three spatial steps by.

$$\delta_{2,x}U^n_{\alpha,\beta,\gamma} = \frac{U^n_{\alpha+\frac{3}{2},\beta,\gamma} - U^n_{\alpha-\frac{3}{2},\beta,\gamma}}{3\Delta x},\ \delta_{2,y}U^n_{\alpha,\beta,\gamma} = \frac{U^n_{\alpha,\beta+\frac{3}{2},\gamma} - U^n_{\alpha,\beta-\frac{3}{2},\gamma}}{3\Delta y},$$

$$\delta_{2,z}U^n_{\alpha,\beta,\gamma} = \frac{U^n_{\alpha,\beta,\gamma+\frac{3}{2}} - U^n_{\alpha,\beta,\gamma-\frac{3}{2}}}{3\Delta z}$$

Now, we define the spatial fourth-order difference operator $\frac{\partial}{\partial x}E_y$ for the strict interior nodes by a linear combination of two central differences, one with a spatial step and the other with three spatial steps above, as

$$\Lambda_x E^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k}} = \frac{1}{8}(9\delta_x - \delta_{2,x})E^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}, \tag{4.2.6}$$

for $i = 1, 2, \cdots, I-2,\ j = 0, 1, \cdots, J-1$ and $k = 0, 1, \cdots, K$. The fourth-order difference operator (4.2.6) can be used to approximate the equations at the strict interior nodes with $i = 1, 2, \cdots, I-2$. However, when we treat the near boundary nodes with $i = 0$ and $i = I-1$, the function values in the definition of $\delta_{2,x}E^n_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}$ will go out of the domain where $E^n_{y_{-1,j+\frac{1}{2},k}}$ and $E^n_{y_{I+1,j+\frac{1}{2},k}}$ are not defined. For constructing high-order difference operators on the near boundary nodes, one could use one-sided difference/extrapolation operators by using more one-sided interior point values. But, these kind one-sided operators will make the scheme to break energy conservations. Thus, it is important to construct the high order difference operators on the near boundary nodes to have high-order accuracy in spatial step and to lead to one energy conserved scheme. Let

$$x_{-1} = -\Delta x,\ x_{-\frac{1}{2}} = x_{-1} + \frac{1}{2}\Delta x,\ x_{I+1} = (I+1)\Delta x,\ x_{I+\frac{1}{2}} = x_I + \frac{1}{2}\Delta x,$$

$$y_{-1} = -\Delta y,\ y_{-\frac{1}{2}} = y_{-1} + \frac{1}{2}\Delta y,\ y_{J+1} = (J+1)\Delta y,\ y_{J+\frac{1}{2}} = y_J + \frac{1}{2}\Delta y,$$

$$z_{-1} = -\Delta z,\ z_{-\frac{1}{2}} = z_{-1} + \frac{1}{2}\Delta z,\ z_{J+1} = (K+1)\Delta z,\ z_{K+\frac{1}{2}} = z_K + \frac{1}{2}\Delta z.$$

Before we propose the spatial fourth-order energy-conserved S-FDTD scheme for the three-dimensional Maxwell's equations (4.2.1) - (4.2.5), we first give the following lemma.

Lemma 4.2.1. *If the solution components $\{\boldsymbol{E}, \boldsymbol{H}\}$ of the system (4.2.1)-(4.2.5) are smooth enough, and the initial fields $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are divergence-free, then it holds that*

$$\begin{aligned} E_x(x_{i+\frac{1}{2}}, y_{-1}, z_k, t) &= 2E_x(x_{i+\frac{1}{2}}, y_0, z_k, t) - E_x(x_{i+\frac{1}{2}}, y_1, z_k, t) \\ &\quad + O(\Delta y^5), \end{aligned} \tag{4.2.7}$$

$$\begin{aligned} E_x(x_{i+\frac{1}{2}}, y_{J+1}, z_k, t) &= 2E_x(x_{i+\frac{1}{2}}, y_J, z_k, t) - E_x(x_{i+\frac{1}{2}}, y_{J-1}, z_k, t) \\ &\quad + O(\Delta y^5), \end{aligned} \tag{4.2.8}$$

$$\begin{aligned} E_x(x_{i+\frac{1}{2}}, y_j, z_{-1}, t) &= 2E_x(x_{i+\frac{1}{2}}, y_j, z_0, t) - E_x(x_{i+\frac{1}{2}}, y_j, z_1, t) \\ &\quad + O(\Delta z^5), \end{aligned} \tag{4.2.9}$$

$$
\begin{aligned}
E_x(x_{i+\frac{1}{2}}, y_j, z_{K+1}, t) &= 2E_x(x_{i+\frac{1}{2}}, y_j, z_K, t) - E_x(x_{i+\frac{1}{2}}, y_j, z_{K-1}, t) \\
&\quad + O(\Delta z^5), \qquad (4.2.10)
\end{aligned}
$$

$$
\begin{aligned}
E_x(x_{-\frac{1}{2}}, y_j, z_k, t) &= E_x(x_{\frac{1}{2}}, y_j, z_k, t) + O(\Delta x^5), && (4.2.11)\\
E_x(x_{I+\frac{1}{2}}, y_j, z_k, t) &= E_x(x_{I-\frac{1}{2}}, y_j, z_k, t) + O(\Delta x^5), && (4.2.12)\\
H_z(x_{-\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k, t) &= H_z(x_{\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k, t) + O(\Delta x^5), && (4.2.13)\\
H_z(x_{I+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k, t) &= H_z(x_{I-\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k, t) + O(\Delta x^5), && (4.2.14)\\
H_z(x_{i+\frac{1}{2}}, y_{-\frac{1}{2}}, z_k, t) &= H_z(x_{i+\frac{1}{2}}, y_{\frac{1}{2}}, z_k, t) + O(\Delta y^5), && (4.2.15)\\
H_z(x_{i+\frac{1}{2}}, y_{J+\frac{1}{2}}, z_k, t) &= H_z(x_{i+\frac{1}{2}}, y_{J-\frac{1}{2}}, z_k, t) + O(\Delta y^5), && (4.2.16)\\
H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{-1}, t) &= 2H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_0, t) - H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_1, t) \\
&\quad + O(\Delta z^5), && (4.2.17)\\
H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{K+1}, t) &= 2H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_K, t) - H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{K-1}, t) \\
&\quad + O(\Delta z^5), && (4.2.18)
\end{aligned}
$$

and similar relations to E_y, E_z, H_x, H_y hold.

Proof. From the PEC condition, $E_x(x, 0, z, t) = E_x(x, b, z, t) = E_x(x, y, 0, t) = E_x(x, y, c, t) = 0$, $E_y(0, y, z, t) = E_y(a, y, z, t) = E_y(x, y, 0, t) = E_y(x, y, c, t) = 0$, $E_z(0, y, z, t) = E_z(a, y, z, t) = E_z(0, y, z, t) = E_z(a, y, z, t) = 0$, $H_x(0, y, z, t) = H_x(a, y, z, t) = 0$, $H_y(x, 0, z, t) = H_y(x, b, z, t) = 0$ and $H_z(x, y, 0, t) = H_z(x, y, c, t) =$

0, it holds that

$$\frac{\partial E_x(x,0,z,t)}{\partial t} = \frac{\partial E_x(x,b,z,t)}{\partial t} = 0, \quad \frac{\partial E_y(0,y,z,t)}{\partial t} = \frac{\partial E_y(a,y,z,t)}{\partial t} = 0; \tag{4.2.19}$$

$$\frac{\partial^l E_x(x,0,z,t)}{\partial x^l} = \frac{\partial^l E_x(x,b,z,t)}{\partial x^l} = 0, \quad \frac{\partial^l E_y(0,y,z,t)}{\partial y^l} = \frac{\partial^l E_y(a,y,z,t)}{\partial y^l} = 0; \tag{4.2.20}$$

$$\frac{\partial^l H_x(0,y,z,t)}{\partial y^l} = \frac{\partial^l H_x(a,y,z,t)}{\partial y^l} = 0, \quad \frac{\partial^l H_x(0,y,z,t)}{\partial z^l} = \frac{\partial^l H_x(a,y,z,t)}{\partial z^l} = 0; \tag{4.2.21}$$

$$\frac{\partial^l H_y(x,0,z,t)}{\partial x^l} = \frac{\partial^l H_y(x,b,z,t)}{\partial x^l} = 0, \quad \frac{\partial^l H_y(x,0,z,t)}{\partial z^l} = \frac{\partial^l H_y(x,b,z,t)}{\partial z^l} = 0; \tag{4.2.22}$$

$$\frac{\partial^l H_z(x,y,0,t)}{\partial x^l} = \frac{\partial^l H_z(x,y,c,t)}{\partial x^l} = 0, \quad \frac{\partial^l H_z(x,y,0,t)}{\partial y^l} = \frac{\partial^l H_z(x,y,c,t)}{\partial y^l} = 0; \tag{4.2.23}$$

where $l = 0, 1, \cdots, 4$. From the first equation in (4.2.1), we have that $\frac{\partial H_z}{\partial y}(x,0,z,t) = \frac{\partial H_z}{\partial y}(x,b,z,t) = 0$. Using (4.2.1) and the initial divergence-free $\nabla \cdot \mathbf{E}_0(x,y,z) = 0$, we have that $\nabla \cdot \mathbf{E}(x,y,z,t) = 0$ for $t > 0$. Thus, $\frac{\partial E_y}{\partial y}(x,0,z,t) = 0$, $\frac{\partial E_y}{\partial y}(x,b,z,t) = 0$. Further,taking derivative to the third equation in (4.2.2) with respect to y-variable, we obtain that $\frac{\partial^2 E_x}{\partial y^2}(x,0,t) = \frac{\partial^2 E_x}{\partial y^2}(x,b,t) = 0$. Similarly, we have that $\frac{\partial^2 E_x}{\partial z^2}(x,y,0,t) = \frac{\partial^2 E_x}{\partial z^2}(x,y,c,t) = 0$. In the same way, we can further get that for $l = 0, 1$, and 2

$$\frac{\partial^{2l} E_x(x,0,z,t)}{\partial y^{2l}} = \frac{\partial^{2l} E_x(x,b,z,t)}{\partial y^{2l}} = 0, \quad \frac{\partial^{2l} E_x(x,y,0,t)}{\partial z^{2l}} = \frac{\partial^{2l} E_x(x,y,c,t)}{\partial z^{2l}} = 0; \tag{4.2.24}$$

$$\frac{\partial^{2l+1}H_z(x,0,z,t)}{\partial y^{2l+1}} = \frac{\partial^{2l+1}H_z(x,b,z,t)}{\partial y^{2l+1}} = 0, \quad \frac{\partial^{2l+1}H_z(0,y,z,t)}{\partial x^{2l+1}}\frac{\partial^{2l+1}H_z(a,y,z,t)}{\partial x^{2l+1}} = 0. \tag{4.2.25}$$

Further using these relations above, we get the results (4.2.7)-(4.2.18) for E_x, H_z.

$\square$

Now, using the relationship of (4.2.7) , we can derive

$$\begin{aligned}
\delta_{2,y}E_{x_{i+\frac{1}{2},\frac{1}{2},k}}(t^n) &= \frac{E_{x_{i+\frac{1}{2},2,k}}(t^n) - E_{x_{i+\frac{1}{2},-1,k}}(t^n)}{3\Delta x} \\
&= \frac{E_{x_{i+\frac{1}{2},1,k}}(t^n) + E_{x_{i+\frac{1}{2},2,k}}(t^n) - 2E_{x_{i+\frac{1}{2},0,k}}(t^n)}{3\Delta y} + O(\Delta y^4).
\end{aligned}$$

and the relationship of (4.2.8), we obtain

$$\delta_{2,y}E_{x_{i+\frac{1}{2},J-\frac{1}{2},k}}(t^n) = \frac{2E_{x_{i+\frac{1}{2},J,k}}(t^n) - E_{x_{i+\frac{1}{2},J-1,k}}(t^n) - E_{x_{i+\frac{1}{2},J-2,k}}(t^n)}{3\Delta y} + O(\Delta y^4).$$

Similarly, with the relationship of (4.2.9)-(4.2.12), we derive the following relations:

$$\delta_{2,z}E_{x_{i+\frac{1}{2},j,\frac{1}{2}}}(t^n) = \frac{E_{x_{i+\frac{1}{2},j,1}}(t^n) + E_{x_{i+\frac{1}{2},j,2}}(t^n) - 2E_{x_{i+\frac{1}{2},j,0}}(t^n)}{3\Delta z} + O(\Delta z^4).$$

$$\delta_{2,z}E_{x_{i+\frac{1}{2},j,K-\frac{1}{2}}}(t^n) = \frac{2E_{x_{i+\frac{1}{2},j,K}}(t^n) - E_{x_{i+\frac{1}{2},j,K-1}}(t^n) - E_{x_{i+\frac{1}{2},j,K-2}}(t^n)}{3\Delta z} + O(\Delta z^4).$$

$$\delta_{2,x}E_{x_{1,j,k}}(t^n) = \frac{E_{x_{\frac{5}{2},j,k}}(t^n) - E_{x_{\frac{1}{2},j,k}}(t^n)}{3\Delta x} + O(\Delta x^4).$$

and

$$\delta_{2,x}E_{x_{I-1,j,k}}(t^n) = \frac{E_{x_{I-\frac{1}{2},j,K}}(t^n) - E_{x_{I-\frac{5}{2},j,k}}(t^n)}{3\Delta x} + O(\Delta x^4).$$

Thus, we can re-define the spatial fourth-order difference operator $\delta_{2,y}E_x$ for the near boundary node with $j = 0$ by

$$\delta_{2,y}E^n_{x_{i+\frac{1}{2},\frac{1}{2},k}} = \frac{E^n_{x_{i+\frac{1}{2},1,k}} + E^n_{x_{i+\frac{1}{2},2,k}} - 2E^n_{x_{i+\frac{1}{2},0,k}}}{3\Delta y}, \tag{4.2.26}$$

similarly, we define

$$\delta_{2,y}E^n_{x_{i+\frac{1}{2},J-\frac{1}{2},k}} = \frac{2E^n_{x_{i+\frac{1}{2},J,k}} - E^n_{x_{i+\frac{1}{2},J-1,k}} - E^n_{y_{i+\frac{1}{2},J-2,k}}}{3\Delta y}, \tag{4.2.27}$$

$$\delta_{2,z}E^n_{x_{i+\frac{1}{2},j,\frac{1}{2}}} = \frac{E^n_{x_{i+\frac{1}{2},j,1}} + E^n_{x_{i+\frac{1}{2},j,2}} - 2E^n_{x_{i+\frac{1}{2},j,0}}}{3\Delta z}, \tag{4.2.28}$$

$$\delta_{2,z}E^n_{x_{i+\frac{1}{2},j,K-\frac{1}{2}}} = \frac{2E^n_{x_{i+\frac{1}{2},j,K}} - E^n_{x_{i+\frac{1}{2},j,K-1}} - E^n_{x_{i+\frac{1}{2},j,K-2}}}{3\Delta z}. \tag{4.2.29}$$

$$\delta_{2,x}E^n_{x_{1,j,k}} = \frac{E^n_{x_{\frac{5}{2},j,k}} - E^n_{x_{\frac{1}{2},j,k}}}{3\Delta x}, \tag{4.2.30}$$

and

$$\delta_{2,x}E^n_{x_{I-1,j,k}} = \frac{E^n_{x_{I-\frac{1}{2},j,K}} - E^n_{x_{I-\frac{5}{2},j,k}}}{3\Delta x}. \tag{4.2.31}$$

Thus, we can define the difference operators to approximate $\frac{\partial}{\partial y}E_x$ for the near boundary nodes with $j = 0$ and $j = J - 1$ by

$$\Lambda_y E^n_{x_{i+\frac{1}{2},\frac{1}{2},k}} = \frac{1}{8}(9\delta_y - \delta_{2,y})E^n_{x_{i+\frac{1}{2},\frac{1}{2},k}}, \tag{4.2.32}$$

$$\Lambda_y E^n_{y_{i+\frac{1}{2},J-\frac{1}{2},k}} = \frac{1}{8}(9\delta_y - \delta_{2,y})E^n_{y_{i+\frac{1}{2},J-\frac{1}{2},k}}, \tag{4.2.33}$$

for $i = 0, 1, \cdots, I - 1$ and $k = 0, 1, \cdots, K$.

Similarly, for approximating $\frac{\partial}{\partial z}E_x$ for the near boundary nodes with $z = 0$ and $z = K - 1$ by

$$\Lambda_z E^n_{x_{i+\frac{1}{2},j,\frac{1}{2}}} = \frac{1}{8}(9\delta_z - \delta_{2,z})E^n_{x_{i+\frac{1}{2},j,\frac{1}{2}}}, \tag{4.2.34}$$

$$\Lambda_z E^n_{x_{i+\frac{1}{2},j,K-\frac{1}{2}}} = \frac{1}{8}(9\delta_z - \delta_{2,z})E^n_{x_{i+\frac{1}{2},j,K-\frac{1}{2}}}, \tag{4.2.35}$$

for $i = 0, 1, \cdots, I-1$ and $j = 1, 2, \cdots, J-1$, and for approximating $\frac{\partial}{\partial x}E_x$ for the near boundary nodes with $i = 1$ and $i = I - 1$,

$$\Lambda_x E^n_{x_{1,j,k}} = \frac{1}{8}(9\delta_x - \delta_{2,x})E^n_{x_{1,j,k}}, \tag{4.2.36}$$

$$\Lambda_x E^n_{x_{I-1,j,k}} = \frac{1}{8}(9\delta_x - \delta_{2,x})E^n_{x_{I-1,j,k}}, \tag{4.2.37}$$

for $j = 1, 2, \cdots, J-1$ and $k = 1, 2 \cdots, K-1$.

In the same way, we can define other difference operators as $\Lambda_x E_y$, $\Lambda_y E_y$, $\Lambda_z E_y$, $\Lambda_x E_z$, $\Lambda_y E_z$, $\Lambda_z E_z$, $\Lambda_x H_x$, $\Lambda_y H_x$, $\Lambda_z H_x$, $\Lambda_x H_y$, $\Lambda_y H_y$, $\Lambda_z H_y$, $\Lambda_x H_z$, $\Lambda_y H_z$ and $\Lambda_z H_z$ on the near boundary nodes.

In this chapter, we only use the **splitting I** for constructing our scheme. We can similarly propose the scheme based on the **splitting II**. Based on the operators Λ_x, Λ_y and Λ_z defined above, we have the approximate **curl**$_+$ and **curl**$_-$ operators. Let

$$\mathbf{curl}_{+,h} = \begin{pmatrix} 0 & 0 & \Lambda_y \\ \Lambda_z & 0 & 0 \\ 0 & \Lambda_x & 0 \end{pmatrix}, \mathbf{curl}_{-,h} = \begin{pmatrix} 0 & -\Lambda_z & 0 \\ 0 & 0 & -\Lambda_x \\ -\Lambda_y & 0 & 0 \end{pmatrix}$$

Then

$$\mathbf{A}_h = \mathbf{A}_{+,h} + \mathbf{A}_{-,h},$$

where

$$\mathbf{A}_{+,h} = \begin{pmatrix} 0 & \mathbf{curl}_{+,h} \\ \mathbf{curl}_{+,h}^T & 0 \end{pmatrix}, \mathbf{A}_{-,h} = \begin{pmatrix} 0 & \mathbf{curl}_{-,h} \\ \mathbf{curl}_{-,h}^T & 0 \end{pmatrix}.$$

We omit the subscripts i, j and k, and define $\mathbf{E}^{(1)} = (E_x^{(1)}, E_y^{(1)}, E_z^{(1)})$, $\mathbf{E}^{(2)} = (E_x^{(2)}, E_y^{(2)}, E_z^{(2)})$, $\mathbf{H}^{(1)} = (H_x^{(1)}, H_y^{(1)}, H_z^{(1)})$, and $\mathbf{H}^{(2)} = (H_x^{(2)}, H_y^{(2)}, H_z^{(2)})$. We propose the spatial high order energy-conserved splitting FDTD scheme, i.e. EC-S-FDTD-(2,4), for the three dimensional Maxwell's equations.

Stage 1:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\mathbf{E}^{(1)} \\ \mu\mathbf{H}^{(1)} \end{pmatrix} - \begin{pmatrix} \epsilon\mathbf{E}^{n} \\ \mu\mathbf{H}^{n} \end{pmatrix}\right] = \frac{1}{4}\mathbf{A}_{+,h}\left[\begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{H}^{(1)} \end{pmatrix} + \begin{pmatrix} \mathbf{E}^{n} \\ \mathbf{H}^{n} \end{pmatrix}\right] \tag{4.2.38}$$

Stage 2:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\mathbf{E}^{(2)} \\ \mu\mathbf{H}^{(2)} \end{pmatrix} - \begin{pmatrix} \epsilon\mathbf{E}^{(1)} \\ \mu\mathbf{H}^{(1)} \end{pmatrix}\right] = \frac{1}{2}\mathbf{A}_{-,h}\left[\begin{pmatrix} \mathbf{E}^{(2)} \\ \mathbf{H}^{(2)} \end{pmatrix} + \begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{H}^{(1)} \end{pmatrix}\right] \tag{4.2.39}$$

Stage 3:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\mathbf{E}^{n+1} \\ \mu\mathbf{H}^{n+1} \end{pmatrix} - \begin{pmatrix} \epsilon\mathbf{E}^{(2)} \\ \mu\mathbf{H}^{(2)} \end{pmatrix}\right] = \frac{1}{4}\mathbf{A}_{+,h}\left[\begin{pmatrix} \mathbf{E}^{n+1} \\ \mathbf{H}^{n+1} \end{pmatrix} + \begin{pmatrix} \mathbf{E}^{(2)} \\ \mathbf{H}^{(2)} \end{pmatrix}\right] \tag{4.2.40}$$

The PEC boundary conditions are

$$
\begin{aligned}
&E^m_{x_{i+\frac{1}{2},0,k}} = E^m_{x_{i+\frac{1}{2},J,k}} = E^m_{x_{i+\frac{1}{2},j,0}} = E^m_{x_{i+\frac{1}{2},j,K}} = 0,\\
&E^m_{y_{0,j+\frac{1}{2},k}} = E^m_{y_{I,j+\frac{1}{2},k}} = E^m_{y_{i,j+\frac{1}{2},0}} = E^m_{y_{i,j+\frac{1}{2},K}} = 0;\\
&E^m_{z_{0,j+\frac{1}{2},k}} = E^m_{z_{I,j+\frac{1}{2},k}} = E^m_{z_{i+\frac{1}{2},j,0}} = E^m_{z_{i+\frac{1}{2},j,K}} = 0;\\
&H^m_{x_{0,j+\frac{1}{2},k+\frac{1}{2}}} = H^m_{x_{I,j+\frac{1}{2},k+\frac{1}{2}}} = H^m_{y_{i+\frac{1}{2},0,k+\frac{1}{2}}} = H^m_{y_{i+\frac{1}{2},J,k+\frac{1}{2}}} = 0;\\
&H^m_{z_{i+\frac{1}{2},j+\frac{1}{2},0}} = H^m_{z_{i+\frac{1}{2},j+\frac{1}{2},K}} = 0
\end{aligned}
\tag{4.2.41}
$$

where $m = (1), (2)$ and $n+1$. The initial conditions are given by

$$
\begin{aligned}
&E^0_{x_{\alpha,\beta,\gamma}} = E^0_x(\alpha\Delta x, \beta\Delta y, \gamma\Delta z);\ E^0_{y_{\alpha,\beta,\gamma}} = E^0_y(\alpha\Delta x, \beta\Delta y, \gamma\Delta z);\\
&E^0_{z_{\alpha,\beta,\gamma}} = E^0_z(\alpha\Delta x, \beta\Delta y, \gamma\Delta z);\ H^0_{x_{\alpha,\beta,\gamma}} = H^0_x(\alpha\Delta x, \beta\Delta y, \gamma\Delta z);\\
&H^0_{y_{\alpha,\beta,\gamma}} = H^0_y(\alpha\Delta x, \beta\Delta y, \gamma\Delta z);\ H^0_{z_{\alpha,\beta,\gamma}} = H^0_z(\alpha\Delta x, \beta\Delta y, \gamma\Delta z).
\end{aligned}
\tag{4.2.42}
$$

4.3 Energy conservations

We now prove the discrete energy conservations of the EC-S-FDTD-(2,4) scheme in three dimensions.

For grid functions $U^n_{\alpha,\beta,\gamma}$, $W^n_{\alpha,\beta,\gamma}$, the discrete norms are defined on the staggered grids as:

$$
\|U\|^2_{E_x} = \sum_{i=0}^{I-1}\sum_{j=0}^{J}\sum_{k=0}^{K}\left|U_{i+\frac{1}{2},j,k}\right|^2\Delta v, \quad \|W\|^2_{H_x} = \sum_{i=0}^{I}\sum_{j=0}^{J-1}\sum_{k=0}^{K-1}\left|W_{i,j+\frac{1}{2},k+\frac{1}{2}}\right|^2\Delta v,
$$

where $\Delta v = \Delta x \Delta y \Delta z$, the meshes are $\Omega_{E_x} = \{(x_{i+\frac{1}{2}}, y_j, z_k)|_{i=0}^{I-1}\,,_{j=0}^{J}\,,_{k=0}^{K}\}$ and $\Omega_{H_x} = \{(x_i, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}})\ |_{i=0}^{I},_{j=0}^{J-1}\,,_{k=0}^{K-1}\}$ respectively. Other norms $\|U\|_{E_y}^2$, $\|U\|_{E_z}^2$, $\|W\|_{H_y}^2$, $\|W\|_{H_z}^2$ can be similarly defined. We then define norms $\|\mathbf{E}^n\|_E^2$ and $\|\mathbf{H}^n\|_H^2$ as

$$\|\mathbf{E}^n\|_E^2 = \|E_x^n\|_{E_x}^2 + \|E_y^n\|_{E_y}^2 + \|E_z^n\|_{E_z}^2,\ \|\mathbf{H}^n\|_H^2 = \|H_x^n\|_{H_x}^2 + \|H_y^n\|_{H_y}^2 + \|H_z^n\|_{H_z}^2.$$

For the difference operators δ, Λ, we define

$$\|\delta_x U\|_{\delta_x E_x}^2 = \textstyle\sum_{i=1}^{I-1} \sum_{j=0}^{J} \sum_{k=0}^{K} \left|\delta_x U_{i,j,k}\right|^2 \Delta v,$$

$$\|\delta_x W\|_{\delta_x H_x}^2 = \textstyle\sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \left|\delta_x W_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}\right|^2 \Delta v,$$

and

$$\|\Lambda_x U\|_{\Lambda_x E_x}^2 = \textstyle\sum_{i=1}^{I-1} \sum_{j=0}^{J} \sum_{k=0}^{K} \left|\Lambda_x U_{i,j,k}\right|^2 \Delta v,$$

$$\|\Lambda_x W\|_{\Lambda_x H_x}^2 = \textstyle\sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \left|\Lambda_x W_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}\right|^2 \Delta v,$$

where the meshes are $\Omega_{\delta_x E_x} = \{(x_i, y_j, z_k)|_{i=1}^{I-1}\,,_{j=0}^{J}\,,_{k=0}^{K}\}$, $\Omega_{\delta_x H_x} = \{(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}})\ |_{i=0}^{I-1}\,,_{j=0}^{J-1}\,,_{k=0}^{K-1}\}$, $\Omega_{\Lambda_x E_x} = \{(x_i, y_j, z_k)|_{i=1}^{I-1}\,,_{j=0}^{J}\,,_{k=0}^{K}\}$, and $\Omega_{\Lambda_x H_x} = \{(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}})\ |_{i=0}^{I-1}\,,_{j=0}^{J-1}\,,_{k=0}^{K-1}\}$. We can similarly define other norms $\|\delta_u U\|_{\delta_u E_x}^2$, $\|\delta_u U\|_{\delta_u E_y}^2$, $\|\delta_u U\|_{\delta_u E_z}^2$, $\|\delta_u U\|_{\delta_u H_x}^2$, $\|\delta_u U\|_{\delta_u H_y}^2$, $\|\delta_u U\|_{\delta_u H_z}^2$, $\|\Lambda_u U\|_{\Lambda_u E_x}^2$, $\|\Lambda_u U\|_{\Lambda_u E_y}^2$, $\|\Lambda_u U\|_{\Lambda_u E_z}^2$, $\|\Lambda_u U\|_{\Lambda_u H_x}^2$, $\|\Lambda_u U\|_{\Lambda_u H_y}^2$ and $\|\Lambda_u U\|_{\Lambda_u H_z}^2$ respectively. Thus, we have that

$$\|\delta_u \mathbf{E}^n\|_{\delta_u E}^2 = \|\delta_u E_x^n\|_{\delta_u E_x}^2 + \|\delta_u E_y^n\|_{\delta_u E_y}^2 + \|\delta_u E_z^n\|_{\delta_u E_z}^2,$$

$$\|\delta_u \mathbf{H}^n\|_{\delta_u H}^2 = \|\delta_u H_x^n\|_{\delta_u H_x}^2 + \|\delta_u H_y^n\|_{\delta_u H_y}^2 + \|\delta_u H_z^n\|_{\delta_u H_z}^2,$$

and

$$\|\Lambda_u \mathbf{E}^n\|^2_{\Lambda_u E} = \|\Lambda_u E_x^n\|^2_{\Lambda_u E_x} + \|\Lambda_u E_y^n\|^2_{\Lambda_u E_y} + \|\Lambda_u E_z^n\|^2_{\Lambda_u E_z},$$

$$\|\Lambda_u \mathbf{H}^n\|^2_{\Lambda_u H} = \|\Lambda_u H_x^n\|^2_{\Lambda_u H_x} + \|\Lambda_u H_y^n\|^2_{\Lambda_u H_y} + \|\Lambda_u H_z^n\|^2_{\Lambda_u H_z},$$

where u = x,y or z. We first give the following lemma.

With Lemma 2.3.1 and 2.3.2 , we have Lemma 4.3.1.

Lemma 4.3.1. *Let $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{H} = (H_x, H_y, H_z)$ be the solution components of EC-S-FDTD-(2,4)scheme 4.2.38-4.2.42 in three dimensions. Then we have that*

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(H_z\Lambda_x E_y)_{i+\frac{1}{2},j+\frac{1}{2},k} = -\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(E_y\Lambda_x H_z)_{i,j+\frac{1}{2},k},$$

and similar relations for operator Λ_y and Λ_z.

The energy conservations of EC-S-FDTD-(2,4) scheme 4.2.38-4.2.42 can be proved.

Theorem 4.3.1. *(Energy conservations I&II) Let $\boldsymbol{E}^n$=($E^n_{x_{i+\frac{1}{2},j,k}}$, $E^n_{y_{i,j+\frac{1}{2},k}}$, $E^n_{z_{i,j,k+\frac{1}{2}}}$) and $\boldsymbol{H}^n = (H^n_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}, H^n_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}, H^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k}})$ be the solutions of EC-S-FDTD-(2,4) scheme (4.2.38)-(4.2.42) for three dimensional Maxwell's equations. Then the energy conservations hold that for $n \geq 0$*

$$\left\|\epsilon^{\frac{1}{2}}\boldsymbol{E}^{n+1}\right\|^2_E + \left\|\mu^{\frac{1}{2}}\boldsymbol{H}^{n+1}\right\|^2_H = \left\|\epsilon^{\frac{1}{2}}\boldsymbol{E}^{n}\right\|^2_E + \left\|\mu^{\frac{1}{2}}\boldsymbol{H}^{n}\right\|^2_H, \tag{4.3.1}$$

$$\left\|\epsilon^{\frac{1}{2}}\delta_t\boldsymbol{E}^{n+\frac{3}{2}}\right\|^2_E + \left\|\mu^{\frac{1}{2}}\delta_t\boldsymbol{H}^{n+\frac{3}{2}}\right\|^2_H = \left\|\epsilon^{\frac{1}{2}}\delta_t\boldsymbol{E}^{n+\frac{1}{2}}\right\|^2_E + \left\|\mu^{\frac{1}{2}}\delta_t\boldsymbol{H}^{n+\frac{1}{2}}\right\|^2_H. \tag{4.3.2}$$

Proof. The **Stage 1** of the EC-S-FDTD-(2,4) can be written as

$$\frac{E^{(1)}_{x_{i+\frac{1}{2},j,k}} - E^{n}_{x_{i+\frac{1}{2},j,k}}}{\Delta t} = \frac{1}{4\epsilon}\Lambda_y\{H^{(1)}_{z_{i+\frac{1}{2},j,k}} + H^{n}_{z_{i+\frac{1}{2},j,k}}\}, \tag{4.3.3}$$

$$\frac{E^{(1)}_{y_{i,j+\frac{1}{2},k}} - E^{n}_{y_{i,j+\frac{1}{2},k}}}{\Delta t} = \frac{1}{4\epsilon}\Lambda_z\{H^{(1)}_{x_{i,j+\frac{1}{2},k}} + H^{n}_{x_{i,j+\frac{1}{2},k}}\}, \tag{4.3.4}$$

$$\frac{E^{(1)}_{z_{i,j,k+\frac{1}{2}}} - E^{n}_{z_{i,j,k+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\epsilon}\Lambda_x\{H^{(1)}_{y_{i,j,k+\frac{1}{2}}} + H^{n}_{y_{i,j,k+\frac{1}{2}}}\}, \tag{4.3.5}$$

$$\frac{H^{(1)}_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}} - H^{n}_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\Lambda_z\{E^{(1)}_{y_{i,j+\frac{1}{2},k+\frac{1}{2}}} + E^{n}_{y_{i,j+\frac{1}{2},k+\frac{1}{2}}}\}, \tag{4.3.6}$$

$$\frac{H^{(1)}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} - H^{n}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\Lambda_x\{E^{(1)}_{z_{i+\frac{1}{2},j,k+\frac{1}{2}}} + E^{n}_{z_{i+\frac{1}{2},j,k+\frac{1}{2}}}\}, \tag{4.3.7}$$

$$\frac{H^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}}{\Delta t} = \frac{1}{4\mu}\Lambda_y\{E^{(1)}_{x_{i+\frac{1}{2},j+\frac{1}{2},k}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2},k}}\}. \tag{4.3.8}$$

By multiplying both sides of equation (4.3.5) with $\epsilon\Delta t\left(E^{(1)}_{z_{i,j,k+\frac{1}{2}}} + E^{n}_{z_{i,j,k+\frac{1}{2}}}\right)$ and then multiplying both sides of (4.3.7) with $\mu\Delta t\left(H^{(1)}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} + H^{n}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}\right)$, we get two equations, then sum over all terms in the two equations together. Noting that E^n_z and $E^{(1)}_z$ satisfy the PEC boundary condition, we use Lemma 4.3.1 and obtain that

$$\epsilon\|E^{(1)}_z\|^2_{E_z} + \mu\|H^{(1)}_y\|^2_{H_y} = \epsilon\|E^n_z\|^2_{E_z} + \mu\|H^n_y\|^2_{H_y} \tag{4.3.9}$$

Similarly, from (4.3.6) and (4.3.4), (4.3.3) and (4.3.8), we obtain that

$$\epsilon\|E^{(1)}_y\|^2_{E_y} + \mu\|H^{(1)}_x\|^2_{H_x} = \epsilon\|E^n_y\|^2_{E_y} + \mu\|H^n_x\|^2_{H_x} \tag{4.3.10}$$

and

$$\epsilon\|E^{(1)}_x\|^2_{E_x} + \mu\|H^{(1)}_z\|^2_{H_z} = \epsilon\|E^n_x\|^2_{E_x} + \mu\|H^n_z\|^2_{H_z}. \tag{4.3.11}$$

Therefore, by (4.3.9)-(4.3.11), we have that

$$\epsilon\|\mathbf{E}^{(1)}\|_E^2 + \mu\|\mathbf{H}^{(1)}\|_H^2 = \epsilon\|\mathbf{E}^n\|_E^2 + \mu\|\mathbf{H}^n\|_H^2. \tag{4.3.12}$$

Similarly, for **Stage 2** and **Stage 3**, we obtain that

$$\epsilon\|\mathbf{E}^{(2)}\|_E^2 + \mu\|\mathbf{H}^{(2)}\|_H^2 = \epsilon\|\mathbf{E}^{(1)}\|_E^2 + \mu\|\mathbf{H}^{(1)}\|_H^2. \tag{4.3.13}$$

$$\epsilon\|\mathbf{E}^{n+1}\|_E^2 + \mu\|\mathbf{H}^{n+1}\|_H^2 = \epsilon\|\mathbf{E}^{(2)}\|_E^2 + \mu\|\mathbf{H}^{(2)}\|_H^2. \tag{4.3.14}$$

The equation (4.3.1) can be obtained directly by eliminating the intermediate variables $\mathbf{E}^{(1)}$, $\mathbf{H}^{(1)}$, $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ in equations (4.3.12), (4.3.13) and (4.3.14). We denote $\mathbf{H}^{(1)+1}$, $\mathbf{E}^{(1)+1}$ and $\mathbf{E}^{(2)+1}$ as the intermediate values $\mathbf{H}^{(1)}$, $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ at time level $n+1$ respectively, then we have $\delta_t\mathbf{H}^{(1)+\frac{1}{2}} = \frac{\mathbf{H}^{(1)+1}-\mathbf{H}^{(1)}}{\Delta t}$, $\delta_t\mathbf{E}^{(1)+\frac{1}{2}} = \frac{\mathbf{E}^{(1)+1}-\mathbf{E}^{(1)}}{\Delta t}$ and $\delta_t\mathbf{E}^{(2)+\frac{1}{2}} = \frac{\mathbf{E}^{(2)+1}-\mathbf{E}^{(2)}}{\Delta t}$. For the EC-S-FDTD-(2,4) scheme (4.2.38)-(4.2.42), we have the following equations.

Stage 1:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\delta_t\mathbf{E}^{(1)-\frac{1}{2}} \\ \mu\delta_t\mathbf{H}^{(1)-\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} \epsilon\delta_t\mathbf{E}^{n-\frac{1}{2}} \\ \mu\delta_t\mathbf{H}^{n-\frac{1}{2}} \end{pmatrix}\right] = \frac{1}{4}\mathbf{A}_{+,h}\left[\begin{pmatrix} \delta_t\mathbf{E}^{(1)-\frac{1}{2}} \\ \delta_t\mathbf{H}^{(1)-\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} \delta_t\mathbf{E}^{n-\frac{1}{2}} \\ \delta_t\mathbf{H}^{n-\frac{1}{2}} \end{pmatrix}\right]$$

Stage 2:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\delta_t\mathbf{E}^{(2)-\frac{1}{2}} \\ \mu\delta_t\mathbf{H}^{(2)-\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} \epsilon\delta_t\mathbf{E}^{(1)-\frac{1}{2}} \\ \mu\delta_t\mathbf{H}^{(1)-\frac{1}{2}} \end{pmatrix}\right] = \frac{1}{2}\mathbf{A}_{-,h}\left[\begin{pmatrix} \delta_t\mathbf{E}^{(2)-\frac{1}{2}} \\ \delta_t\mathbf{H}^{(2)-\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} \delta_t\mathbf{E}^{(1)-\frac{1}{2}} \\ \delta_t\mathbf{H}^{(1)-\frac{1}{2}} \end{pmatrix}\right]$$

Stage 3:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\delta_t\mathbf{E}^{n+\frac{1}{2}} \\ \\ \mu\delta_t\mathbf{H}^{n+\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} \epsilon\delta_t\mathbf{E}^{(2)-\frac{1}{2}} \\ \\ \mu\delta_t\mathbf{H}^{(2)-\frac{1}{2}} \end{pmatrix}\right] = \frac{1}{4}\mathbf{A}_{+,h}\left[\begin{pmatrix} \delta_t\mathbf{E}^{n+\frac{1}{2}} \\ \\ \delta_t\mathbf{H}^{n+\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} \delta_t\mathbf{E}^{(2)-\frac{1}{2}} \\ \\ \delta_t\mathbf{H}^{(2)-\frac{1}{2}} \end{pmatrix},\right]$$

and $\delta_t\mathbf{E}$ still satisfy the PEC boundary conditions. Similarly, following the proof of (4.3.1), we obtain (4.3.2). □

Corollary 3. *(Unconditionally stable) The EC-S-FDTD(2,4) scheme in three dimensions is unconditionally stable.*

In order to show energy conservations in discrete variation forms, we give Lemma 4.3.2.

Lemma 4.3.2. *Let grid functions $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{H} = (H_x, H_y, H_z)$ be the solution components of EC-S-FDTD-(2,4) scheme (4.2.38)-(4.2.42) for three dimensional Maxwell's equations. Then we have that*

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\Lambda_y H_z \Lambda_y \Lambda_x E_y)_{i+\frac{1}{2},j,k} = -\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\sum_{k=1}^{K-1}(\Lambda_y E_y \Lambda_y \Lambda_x H_z)_{i,j,k}, \quad (4.3.15)$$

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\Lambda_x H_z \Lambda_x \Lambda_x E_y)_{i,j+\frac{1}{2},k} = -\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\Lambda_x E_y \Lambda_x \Lambda_x H_z)_{i+\frac{1}{2},j+\frac{1}{2},k}. \quad (4.3.16)$$

and similar relations for operator Λ_x, Λ_y and Λ_z.

Proof. We only give the proof of (4.3.16). Then (4.3.15) can be proved in a similar way. With the definition of operator Λ_x, equation (4.3.16) can be written equivalently as

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\frac{1}{8}(9\delta_x-\delta_{2,x})H_z\frac{1}{8}(9\delta_x-\delta_{2,x})\Lambda_x E_y)_{i,j+\frac{1}{2},k}$$
$$=-\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\frac{1}{8}(9\delta_x-\delta_{2,x})E_y\frac{1}{8}(9\delta_x-\delta_{2,x})\Lambda_x H_z)_{i+\frac{1}{2},j+\frac{1}{2},k}\quad(4.3.17)$$

In order to prove (4.3.17), we first derive that

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x H_z\delta_x\Lambda_x E_y)_{i,j+\frac{1}{2},k}=-\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x E_y\delta_x\Lambda_x H_z)_{i+\frac{1}{2},j+\frac{1}{2},k}.\quad(4.3.18)$$

By the definition of δ_x, the left side of (4.3.18) is

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x H_z\delta_x\Lambda_x E_y)_{i,j+\frac{1}{2},k}$$
$$=\frac{1}{\Delta x}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\Lambda_x(E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}-E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}).\quad(4.3.19)$$

Using the definition of Λ_x, we get that for the second term on the right side of (4.3.19)

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\Lambda_x E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}$$
$$=\frac{1}{8}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}(9\delta_x-\delta_{2,x})E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}.\quad(4.3.20)$$

Noting that the identities $\delta_{2,x}E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}=\frac{1}{3}\delta_x(E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}+E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}+E_{y_{i-\frac{3}{2},j+\frac{1}{2},k}})$ and $\delta_{2,x}H_{z_{i+1,j+\frac{1}{2},k}}=\frac{1}{3}\delta_x(H_{z_{i+2,j+\frac{1}{2},k}}+H_{z_{i+1,j+\frac{1}{2},k}}+H_{z_{i,j+\frac{1}{2},k}})$ on the strict interior nodes, and for $i=1$ that is near boundary node for $\delta_{2,x}$, then the second term on the right side of (4.3.20) can be organized as

$$
\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\delta_{2,x}E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}\\
&\quad=\sum_{i=2}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\delta_{2,x}E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}+\sum_{j=0}^{J-1}\delta_x H_{z_{1,j+\frac{1}{2},k}}\delta_{2,x}E_{y_{\frac{1}{2},j+\frac{1}{2},k}}\\
&\quad=\frac{1}{3}\sum_{i=2}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\delta_x(E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}+E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}}+E_{y_{i-\frac{3}{2},j+\frac{1}{2},k}})\\
&\qquad+\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{1,j+\frac{1}{2},k}}\delta_{2,x}E_{y_{\frac{1}{2},j+\frac{1}{2},k}}.
\end{aligned}
\tag{4.3.21}
$$

In (4.3.21), we further use Lemma 2.3.1 for the first term on the right side and use the definition of $\delta_{2,x}$ for the second term, with the boundary condition, (4.3.21) are equal to

$$
\begin{aligned}
&=\frac{1}{3}\sum_{i=1}^{I-3}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}\delta_x(H_{z_{i+2,j+\frac{1}{2},k}}+H_{z_{i+1,j+\frac{1}{2},k}}+H_{z_{i,j+\frac{1}{2},k}})\\
&\quad+\frac{1}{3}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x E_{y_{I-\frac{3}{2},j+\frac{1}{2},k}}\delta_x H_{z_{I-1,j+\frac{1}{2},k}}+\delta_x E_{y_{I-\frac{1}{2},j+\frac{1}{2},k}}\delta_x H_{z_{I-1,j+\frac{1}{2},k}}\\
&\quad+\delta_x E_{y_{I-\frac{3}{2},j+\frac{1}{2},k}}\delta_x H_{z_{I-2,j+\frac{1}{2},k}})+\frac{1}{3}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x E_{y_{\frac{1}{2},j+\frac{1}{2},k}}\delta_x H_{z_{2,j+\frac{1}{2},k}}\\
&\quad-\delta_x E_{y_{\frac{3}{2},j+\frac{1}{2},k}}\delta_x H_{z_{1,j+\frac{1}{2},k}}+\delta_{2,x}E_{y_{\frac{1}{2},j+\frac{1}{2},k}}\delta_x H_{z_{1,j+\frac{1}{2},k}})\\
&=\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}\delta_{2,x}H_{z_{i+1,j+\frac{1}{2},k}}\\
&\quad+\frac{1}{3}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\frac{1}{\Delta x^2}E_{y_{I-1,j+\frac{1}{2},k}}(H_{z_{I-\frac{3}{2},j+\frac{1}{2},k}}-H_{z_{I-\frac{1}{2},j+\frac{1}{2},k}})\\
&\quad+\frac{1}{3}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\frac{1}{\Delta x^2}E_{y_{1,j+\frac{1}{2},k}}(H_{z_{\frac{5}{2},j+\frac{1}{2},k}}+H_{z_{\frac{3}{2},j+\frac{1}{2},k}}-2H_{z_{\frac{1}{2},j+\frac{1}{2},k}}).
\end{aligned}
\tag{4.3.22}
$$

With (4.3.22), and similarly to treat the first term on the right side of (4.3.20), equation (4.3.20) can thus be written as

$$\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\Lambda_x E_{y_{i-\frac{1}{2},j+\frac{1}{2},k}} = \sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}\Lambda_x H_{z_{i+1,j+\frac{1}{2},k}}\\
&\quad -\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\frac{1}{24\Delta x^2}E_{y_{1,j+\frac{1}{2},k}}(H_{z_{\frac{5}{2},j+\frac{1}{2},k}}+H_{z_{\frac{3}{2},j+\frac{1}{2},k}}-2H_{z_{\frac{1}{2},j+\frac{1}{2},k}})\\
&\quad -\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\frac{1}{24\Delta x^2}E_{y_{I-1,j+\frac{1}{2},k}}(H_{z_{I-\frac{3}{2},j+\frac{1}{2},k}}-H_{z_{I-\frac{1}{2},j+\frac{1}{2},k}})\\
&\quad +\frac{27}{24}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{1,j+\frac{1}{2},k}}\delta_x E_{y_{\frac{1}{2},j+\frac{1}{2},k}}.
\end{aligned}\tag{4.3.23}$$

Similarly, we have that for the first term on the right side of (4.3.19)

$$\begin{aligned}
&\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{i,j+\frac{1}{2},k}}\Lambda_x E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}} = \sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x E_{y_{i+\frac{1}{2},j+\frac{1}{2},k}}\Lambda_x H_{z_{i,j+\frac{1}{2},k}}\\
&\quad -\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\frac{1}{24\Delta x^2}E_{y_{I-1,j+\frac{1}{2},k}}(H_{z_{I-\frac{5}{2},j+\frac{1}{2},k}}+H_{z_{I-\frac{3}{2},j+\frac{1}{2},k}}-2H_{z_{I-\frac{1}{2},j+\frac{1}{2},k}})\\
&\quad -\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\frac{1}{24\Delta x^2}E_{y_{1,j+\frac{1}{2},k}}(H_{z_{\frac{3}{2},j+\frac{1}{2},k}}-H_{z_{\frac{1}{2},j+\frac{1}{2},k}})\\
&\quad +\frac{27}{24}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\delta_x H_{z_{I-1,j+\frac{1}{2},k}}\delta_x E_{y_{I-\frac{1}{2},j+\frac{1}{2},k}}.
\end{aligned}\tag{4.3.24}$$

Substituting (4.3.23) and (4.3.24) into (4.3.19), and using the definition of operator $\Lambda_x H_{z_{i,j+\frac{1}{2}}}$ on the near boundary nodes for $i = 0, 1, I-1, I$, we finally obtain (4.3.18). Similarly, we can also obtain that

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x H_z \delta_{2,x}\Lambda_x E_y)_{i,j+\frac{1}{2},k} = -\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_x E_y \delta_{2,x}\Lambda_x H_z)_{i+\frac{1}{2},j+\frac{1}{2},k}. \tag{4.3.25}$$

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_{2,x}H_z\delta_x\Lambda_x E_y)_{i,j+\frac{1}{2},k} = -\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_{2,x}E_y\delta_x\Lambda_x H_z)_{i+\frac{1}{2},j+\frac{1}{2},k}, \tag{4.3.26}$$

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_{2,x}H_z\delta_{2,x}\Lambda_x E_y)_{i,j+\frac{1}{2},k} = -\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}(\delta_{2,x}E_y\delta_{2,x}\Lambda_x H_z)_{i+\frac{1}{2},j+\frac{1}{2},k}. \tag{4.3.27}$$

From (4.3.18),(4.3.25),(4.3.26) and (4.3.27), we have (4.3.16). This ends the proof.

□

Theorem 4.3.2. *(Energy conservation III&IV) Let* $\boldsymbol{E}^n = \{(E^n_{x_{i+\frac{1}{2},j,k}}, E^n_{y_{i,j+\frac{1}{2},k}}, E^n_{z_{i,j,k+\frac{1}{2}}})\}$ *and* $\boldsymbol{H}^n = \{(H^n_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}, H^n_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}, H^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k}})\}$ *be the solution components of the EC-S-FDTD-(2,4) scheme (4.2.38)-(4.2.42) for three dimensional Maxwell's equations. Then the energy conservation properties in the discrete* $\Lambda-$*form hold that for* $n \geq 0$

$$\left\|\epsilon^{\frac{1}{2}}\Lambda_u\boldsymbol{E}^{n+1}\right\|^2_{\Lambda_u E} + \left\|\mu^{\frac{1}{2}}\Lambda_u\boldsymbol{H}^{n+1}_z\right\|^2_{\Lambda_u H} = \left\|\epsilon^{\frac{1}{2}}\Lambda_u\boldsymbol{E}^{n}\right\|^2_{\Lambda_u E} + \left\|\mu^{\frac{1}{2}}\Lambda_u\boldsymbol{H}^{n}_z\right\|^2_{\Lambda_u H}, \tag{4.3.28}$$

$$\left\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u E^{n+\frac{3}{2}}\right\|^2_{\Lambda_u E} + \left\|\mu^{\frac{1}{2}}\delta_t\Lambda_u H_z^{n+\frac{3}{2}}\right\|^2_{\Lambda_u H} = \left\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u E^{n+\frac{1}{2}}\right\|^2_{\Lambda_u E} + \left\|\mu^{\frac{1}{2}}\delta_t\Lambda_u H_z^{n+\frac{1}{2}}\right\|^2_{\Lambda_u H}, \tag{4.3.29}$$

where $u = x$, y *or* z.

4.4 Convergence and super-Convergence

Then, we give the truncation errors of EC-S-FDTD-(2,4), and analyze convergence and super-Convergence.

In order to estimate the truncation errors for the EC-S-FDTD-(2,4) scheme in three dimensions, we construct the intermediate variables $\tilde{\mathbf{E}}^{(i)}$ and $\tilde{\mathbf{H}}^{(i)}$ for $i = 1, 2$ as

$$\begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(1)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(1)} \end{pmatrix} = (I + \frac{\Delta t}{2} cA_{+,h} + \frac{\Delta t^2}{8}(cA_{+,h})^2) \begin{pmatrix} \sqrt{\epsilon}\mathbf{E}(t^n) \\ \sqrt{\mu}\mathbf{H}(t^n) \end{pmatrix} \tag{4.4.1}$$

$$\begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(2)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(2)} \end{pmatrix} = (I - \frac{\Delta t}{2} cA_{+,h} + \frac{\Delta t^2}{8}(cA_{+,h})^2) \begin{pmatrix} \sqrt{\epsilon}\mathbf{E}(t^{n+1}) \\ \sqrt{\mu}\mathbf{H}(t^{n+1}) \end{pmatrix}. \tag{4.4.2}$$

Then we have that

$$\begin{aligned} &\frac{1}{\Delta t}\left[\begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(1)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(1)} \end{pmatrix} - \begin{pmatrix} \sqrt{\epsilon}\mathbf{E}(t^n) \\ \sqrt{\mu}\mathbf{H}(t^n) \end{pmatrix}\right] \\ &= \frac{c}{4}A_{+,h}\left[\begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(1)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(1)} \end{pmatrix} + \begin{pmatrix} \sqrt{\epsilon}\mathbf{E}(t^n) \\ \sqrt{\mu}\mathbf{H}(t^n) \end{pmatrix}\right] + \vec{\xi}, \end{aligned} \tag{4.4.3}$$

$$\begin{aligned} &\frac{1}{\Delta t}\left[\begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(2)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(2)} \end{pmatrix} - \begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(1)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(1)} \end{pmatrix}\right] \\ &= \frac{c}{2}A_{-,h}\left[\begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(2)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(2)} \end{pmatrix} + \begin{pmatrix} \sqrt{\epsilon}\tilde{\mathbf{E}}^{(1)} \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(1)} \end{pmatrix}\right] + \vec{\eta}, \end{aligned} \tag{4.4.4}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+1})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+1})\end{pmatrix}-\begin{pmatrix}\sqrt{\epsilon}\tilde{\mathbf{E}}^{(2)}\\ \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(2)}\end{pmatrix}\right]$$

$$=\frac{c}{4}A_{+,h}\left[\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+1})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+1})\end{pmatrix}+\begin{pmatrix}\sqrt{\epsilon}\tilde{\mathbf{E}}^{(2)}\\ \\ \sqrt{\mu}\tilde{\mathbf{H}}^{(2)}\end{pmatrix}\right]+\vec{\zeta}, \qquad (4.4.5)$$

where $\vec{\xi}$, $\vec{\eta}$, $\vec{\zeta}$ are truncation errors.

Lemma 4.4.1. *Suppose that the exact solutions* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *of Maxwell's equations are smooth enough:* $\boldsymbol{E} \in (C^3[0,T];[C^5(\Omega)]^3)$ *and* $\boldsymbol{H} \in (C^3[0,T];[C^5(\Omega)]^3)$*. Then we have the truncation estimation*

$$\max_n\{\|\vec{\xi}\|_{\ell^\infty}, \|\vec{\eta}\|_{\ell^\infty}, \|\vec{\zeta}\|_{\ell^\infty}\} \le C(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4). \qquad (4.4.6)$$

Proof. By direct computation, we have that

$$\vec{\xi}=-\frac{\Delta t^2}{32}(cA_{+,h})^3\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^n)\\ \\ \sqrt{\mu}\mathbf{H}(t^n)\end{pmatrix},$$

and

$$\vec{\zeta}=-\frac{\Delta t^2}{32}(cA_{+,h})^3\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+1})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+1})\end{pmatrix}.$$

Now we compute $\vec{\eta}$. Substituting equations (4.4.1) and (4.4.2) into equation (4.4.4), we have that

$$\vec{\eta} = \frac{1}{\Delta t}\left(I - \frac{\Delta t}{2}cA_{+,h} + \frac{\Delta t^2}{8}(cA_{+,h})^2\right)\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+1})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+1})\end{pmatrix}$$
$$- \frac{1}{\Delta t}\left(I + \frac{\Delta t}{2}cA_{+,h} + \frac{\Delta t^2}{8}(cA_{+,h})^2\right)\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n})\end{pmatrix}$$
$$- \frac{c}{2}A_{-,h}\left(I - \frac{\Delta t}{2}cA_{+,h} + \frac{\Delta t^2}{8}(cA_{+,h})^2\right)\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+1})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+1})\end{pmatrix}$$
$$- \frac{c}{2}A_{-,h}\left(I + \frac{\Delta t}{2}cA_{+,h} + \frac{\Delta t^2}{8}(cA_{+,h})^2\right)\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n})\end{pmatrix}.$$

This equation can be written in the following form,

$$\vec{\eta} = \begin{pmatrix}\sqrt{\epsilon}\delta_t\mathbf{E}(t^{n+\frac{1}{2}})\\ \\ \sqrt{\mu}\delta_t\mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix} - cA_h\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+\frac{1}{2}})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix} + \frac{\Delta t^2}{8}(cA_{+,h})^2\begin{pmatrix}\sqrt{\epsilon}\delta_t\mathbf{E}(t^{n+\frac{1}{2}})\\ \\ \sqrt{\mu}\delta_t\mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix}$$
$$+\frac{\Delta c^2t^2}{4}A_{-,h}A_{+,h}\begin{pmatrix}\sqrt{\epsilon}\delta_t\mathbf{E}(t^{n+\frac{1}{2}})\\ \\ \sqrt{\mu}\delta_t\mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix} - \frac{\Delta c^3t^2}{8}A_{-,h}(A_{+,h})^2\begin{pmatrix}\sqrt{\epsilon}\mathbf{E}(t^{n+\frac{1}{2}})\\ \\ \sqrt{\mu}\mathbf{H}(t^{n+\frac{1}{2}})\end{pmatrix}.$$

We can see that the first part on the right side of the above equation is the truncation error of quasi Crank-Nicolson scheme. Others terms are second order accuracy in time step. The matrix operators are four order accuracy in space step, thus, we have the conclusion (4.4.6). □

Now, we provide the convergence analysis for the EC-S-FDTD-(2,4) scheme in three dimensions.

Theorem 4.4.1. *(Convergence) If $\boldsymbol{E}$, $\boldsymbol{H}$, the exact solution components of Maxwell's equations (4.2.1)-(4.2.5) in three dimensions, are smooth enough. Let $\boldsymbol{E}^n$ and $\boldsymbol{H}^n$ the numerical solutions of the EC-S-FDTD-(2,4) scheme. Then for any fixed time $T > 0$, we have the following error estimates*

$$\begin{aligned}
&\max_{0\le n\le N}\{\|\epsilon^{\frac{1}{2}}[\boldsymbol{E}(t^n)-\boldsymbol{E}^n]\|_E^2+\|\mu^{\frac{1}{2}}[\boldsymbol{H}(t^n)-\boldsymbol{H}^n]\|_H^2\}\\
&\le \|\epsilon^{\frac{1}{2}}[\boldsymbol{E}(t^0)-\boldsymbol{E}^0]\|_E^2+\|\mu^{\frac{1}{2}}[\boldsymbol{H}(t^0)-\boldsymbol{H}^0]\|_H^2 \qquad (4.4.7)\\
&+C_{\mu\epsilon}T(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)^2
\end{aligned}$$

and

$$\begin{aligned}
&\max_{0\le n\le N}\{\|\epsilon^{\frac{1}{2}}[\delta t\boldsymbol{E}(t^{n+\frac{1}{2}})-\delta t\boldsymbol{E}^{n+\frac{1}{2}}]\|_E^2+\|\mu^{\frac{1}{2}}[\delta t\boldsymbol{H}(t^{n+\frac{1}{2}})-\delta t\boldsymbol{H}^{n+\frac{1}{2}}]\|_H^2\}\\
&\le \|\epsilon^{\frac{1}{2}}[\delta t\boldsymbol{E}(t^{\frac{1}{2}})-\delta t\boldsymbol{E}^0]\|_E^2+\|\mu^{\frac{1}{2}}[\delta t\boldsymbol{H}(t^{\frac{1}{2}})-\delta t\boldsymbol{H}^{\frac{1}{2}}]\|_H^2 \qquad (4.4.8)\\
&+C_{\mu\epsilon}T(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)^2,
\end{aligned}$$

where $C_{\mu\epsilon}$ is independent of Δt, Δx, Δy and Δz.

Proof. Let error functions be

$$\mathcal{E}^n_{w_{\alpha,\beta,\gamma}}=\mathbf{E}_w(x_\alpha,y_\alpha,z_\alpha,t^n)-\mathbf{E}^n_{w_{\alpha,\beta,\gamma}},\ \mathcal{H}^n_{w_{\alpha,\beta,\gamma}}=\mathbf{H}_w(x_\alpha,y_\alpha,z_\alpha,t^n)-\mathbf{H}^n_{w_{\alpha,\beta,\gamma}},$$

and

$$\mathcal{E}^{(i)}=\tilde{\mathbf{E}}^{(i)}-\mathbf{E}^{(i)},\ \mathcal{H}^{(i)}=\tilde{\mathbf{H}}^{(i)}-\mathbf{H}^{(i)},$$

where $w=x,y,z$, the $\tilde{\mathbf{E}}^{(i)}$ and $\tilde{\mathbf{H}}^{(i)}$ for $i=1,2$ are defined in equations (4.4.1) and (4.4.2) from exact solutions.

The error equations of EC-S-FDTD-(2,4) can be written as

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\mathcal{E}^{(1)} \\ \mu\mathcal{H}^{(1)} \end{pmatrix} - \begin{pmatrix} \epsilon\mathcal{E}^{n} \\ \mu\mathcal{H}^{n} \end{pmatrix}\right] = \frac{1}{4}A_{+,h}\left[\begin{pmatrix} \mathcal{E}^{(1)} \\ \mathcal{H}^{(1)} \end{pmatrix} + \begin{pmatrix} \mathcal{E}^{n} \\ \mathcal{H}^{n} \end{pmatrix}\right] + \vec{\xi}, \tag{4.4.9}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\mathcal{E}^{(2)} \\ \mu\mathcal{H}^{(2)} \end{pmatrix} - \begin{pmatrix} \epsilon\mathcal{E}^{(1)} \\ \mu\mathcal{H}^{(1)} \end{pmatrix}\right] = \frac{1}{2}A_{-,h}\left[\begin{pmatrix} \mathcal{E}^{(2)} \\ \mathcal{H}^{(2)} \end{pmatrix} + \begin{pmatrix} \mathcal{E}^{(1)} \\ \mathcal{H}^{(1)} \end{pmatrix}\right] + \vec{\eta}, \tag{4.4.10}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\mathcal{E}^{n+1} \\ \mu\mathcal{H}^{n+1} \end{pmatrix} - \begin{pmatrix} \epsilon\mathcal{E}^{(2)} \\ \mu\mathcal{H}^{(2)} \end{pmatrix}\right] = \frac{1}{4}A_{+,h}\left[\begin{pmatrix} \mathcal{E}^{n+1} \\ \mathcal{H}^{n+1} \end{pmatrix} + \begin{pmatrix} \mathcal{E}^{(2)} \\ \mathcal{H}^{(2)} \end{pmatrix}\right] + \vec{\zeta}. \tag{4.4.11}$$

We first analyze the first error equation (4.4.9). Write the error equation (4.4.9) in a component form, where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)^T$:

$$\epsilon\frac{\mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j,k}} - \mathcal{E}^{n}_{x_{i+\frac{1}{2},j,k}}}{\Delta t} = \frac{1}{4}\Lambda_y\{\mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j,k}} + \mathcal{H}^{n}_{z_{i+\frac{1}{2},j,k}}\} + \xi_{1_{i+\frac{1}{2},j,k}}, \tag{4.4.12}$$

$$\mu\frac{\mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}} - \mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}}{\Delta t} = \frac{1}{4}\Lambda_y\{\mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j+\frac{1}{2},k}} + \mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2},k}}\} + \xi_{2_{i+\frac{1}{2},j+\frac{1}{2},k}}, \tag{4.4.13}$$

$$\epsilon\frac{\mathcal{E}^{(1)}_{y_{i,j+\frac{1}{2},k}} - \mathcal{E}^{n}_{y_{i,j+\frac{1}{2},k}}}{\Delta t} = -\frac{1}{4}\Lambda_x\{\mathcal{H}^{(1)}_{x_{i,j+\frac{1}{2},k}} + \mathcal{H}^{n}_{x_{i,j+\frac{1}{2},k}}\} + \xi_{3_{i,j+\frac{1}{2},k}}, \tag{4.4.14}$$

$$\mu\frac{\mathcal{H}^{(1)}_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}} - \mathcal{H}^{n}_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}}}{\Delta t} = -\frac{1}{4}\Lambda_x\{\mathcal{E}^{(1)}_{y_{i,j+\frac{1}{2},k+\frac{1}{2}}} + \mathcal{E}^{n}_{y_{i,j+\frac{1}{2},k+\frac{1}{2}}}\} + \xi_{4_{i,j+\frac{1}{2},k+\frac{1}{2}}}, \tag{4.4.15}$$

$$\epsilon\frac{\mathcal{E}^{(1)}_{z_{i,j,k+\frac{1}{2}}} - \mathcal{E}^{n}_{z_{i,j,k+\frac{1}{2}}}}{\Delta t} = \frac{1}{4}\Lambda_y\{\mathcal{H}^{(1)}_{y_{i,j,k+\frac{1}{2}}} + \mathcal{H}^{(n)}_{y_{i,j,k+\frac{1}{2}}}\} + \xi_{5_{i,j,k+\frac{1}{2}}}, \tag{4.4.16}$$

$$\mu\frac{\mathcal{H}^{(1)}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} - \mathcal{H}^{n}_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}}}{\Delta t} = \frac{1}{4}\Lambda_y\{\mathcal{E}^{(1)}_{z_{i+\frac{1}{2},j,k+\frac{1}{2}}} + \mathcal{E}^{(n)}_{z_{i+\frac{1}{2},j,k+\frac{1}{2}}}\} + \xi_{6_{i+\frac{1}{2},j,k+\frac{1}{2}}}. \quad (4.4.17)$$

From (4.4.12) and (4.4.13), similarly to the proof of energy conservation theorem, we have that

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)}_x\|^2_{E_x} + \|\mu^{\frac{1}{2}}\mathcal{H}^{(1)}_z\|^2_{H_z} - \|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}_x\|^2_{E_x} - \|\mu^{\frac{1}{2}}\mathcal{H}^{n}_z\|^2_{H_z} \\
&= \Delta t\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{K-1}\{\epsilon\xi_{1_{i+\frac{1}{2},j,k}}(\mathcal{E}^{(1)}_{x_{i+\frac{1}{2},j,k}} + \mathcal{E}^{n}_{x_{i+\frac{1}{2},j,k}}) \\
&\quad + \mu\xi_{2_{i+\frac{1}{2},j+\frac{1}{2},k}}(\mathcal{H}^{(1)}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}} + \mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2},k}})\}\Delta x\Delta y\Delta z.
\end{aligned}$$

The above equation can be written as

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)}_x - \frac{\Delta t}{2}\sqrt{\epsilon}\xi_1\|^2_{E_x} + \|\mu^{\frac{1}{2}}\mathcal{H}^{(1)}_z - \frac{\Delta t}{2}\sqrt{\mu}\xi_2\|^2_{H_z} \\
&= \|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}_x + \frac{\Delta t}{2}\sqrt{\epsilon}\xi_1\|^2_{E_x} + \|\mu^{\frac{1}{2}}\mathcal{H}^{n}_z + \frac{\Delta t}{2}\sqrt{\mu}\xi_2\|^2_{H_z}. \qquad (4.4.18)
\end{aligned}$$

Similarly, from (4.4.14) and (4.4.15),

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)}_y - \frac{\Delta t}{2}\sqrt{\epsilon}\xi_3\|^2_{E_y} + \|\mu^{\frac{1}{2}}\mathcal{H}^{(1)}_x - \frac{\Delta t}{2}\sqrt{\mu}\xi_4\|^2_{H_x} \\
&= \|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}_y + \frac{\Delta t}{2}\sqrt{\epsilon}\xi_3\|^2_{E_y} + \|\mu^{\frac{1}{2}}\mathcal{H}^{n}_x + \frac{\Delta t}{2}\sqrt{\mu}\xi_4\|^2_{H_x}, \qquad (4.4.19)
\end{aligned}$$

and from (4.4.16) and (4.4.17)

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)}_z - \frac{\Delta t}{2}\sqrt{\epsilon}\xi_5\|^2_{E_z} + \|\mu^{\frac{1}{2}}\mathcal{H}^{(1)}_y - \frac{\Delta t}{2}\sqrt{\mu}\xi_6\|^2_{H_y} \\
&= \|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}_z + \frac{\Delta t}{2}\sqrt{\epsilon}\xi_5\|^2_{E_z} + \|\mu^{\frac{1}{2}}\mathcal{H}^{n}_y + \frac{\Delta t}{2}\sqrt{\mu}\xi_6\|^2_{H_y}. \qquad (4.4.20)
\end{aligned}$$

Thus, from (4.4.18), (4.4.19) and (4.4.20), we have for ((4.4.9)) that

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)} - \frac{\Delta t}{2}\sqrt{\epsilon}\xi_E\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{(1)} - \frac{\Delta t}{2}\sqrt{\mu}\xi_H\|_H^2 \\
&\quad = \|\epsilon^{\frac{1}{2}}\mathcal{E}^{n} + \frac{\Delta t}{2}\sqrt{\epsilon}\xi_E\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{n} + \frac{\Delta t}{2}\sqrt{\mu}\xi_H\|_H^2, \qquad (4.4.21)
\end{aligned}$$

where the vector $\xi_E = (\xi_1, \xi_3, \xi_5)^T$ and $\xi_H = (\xi_2, \xi_4, \xi_6)^T$.

Similarly, we have that from (4.4.10)

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(2)} - \frac{\Delta t}{2}\sqrt{\epsilon}\eta_E\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{(2)} - \frac{\Delta t}{2}\sqrt{\mu}\eta_H\|_H^2 \\
&\quad = \|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)} + \frac{\Delta t}{2}\sqrt{\epsilon}\eta_E\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{(1)} + \frac{\Delta t}{2}\sqrt{\mu}\eta_H\|_H^2, \qquad (4.4.22)
\end{aligned}$$

and from (4.4.11)

$$\begin{aligned}
&\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n+1} - \frac{\Delta t}{2}\sqrt{\epsilon}\zeta_E\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{n+1} - \frac{\Delta t}{2}\sqrt{\mu}\zeta_H\|_H^2 \\
&\quad = \|\epsilon^{\frac{1}{2}}\mathcal{E}^{(2)} + \frac{\Delta t}{2}\sqrt{\epsilon}\zeta_E\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{(2)} + \frac{\Delta t}{2}\sqrt{\mu}\zeta_H\|_H^2. \qquad (4.4.23)
\end{aligned}$$

From the relation (4.4.23) and the triangle inequality of the norm, we obtain that

$$\begin{aligned}
&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n+1}\|_E^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{n+1}\|_H^2\right)^{\frac{1}{2}} \\
&\le \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n+1} - \frac{\Delta t}{2}\mu^{\frac{1}{2}}\zeta_E\|_E^2 + \|\epsilon^{\frac{1}{2}}\mathcal{H}^{n+1} - \frac{\Delta t}{2}\mu^{\frac{1}{2}}\zeta_H\|_H^2\right)^{\frac{1}{2}} \\
&\quad + \left(\|\frac{\Delta t}{2}\epsilon^{\frac{1}{2}}\zeta_E\|_E^2 + \|\frac{\Delta t}{2}\mu^{\frac{1}{2}}\zeta_H\|_H^2\right)^{\frac{1}{2}} \\
&= \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(2)} + \frac{\Delta t}{2}\epsilon^{\frac{1}{2}}\zeta_E\|_E^2 + \|\epsilon^{\frac{1}{2}}\mathcal{H}^{(2)} + \frac{\Delta t}{2}\mu^{\frac{1}{2}}\zeta_H\|_H^2\right)^{\frac{1}{2}} \\
&\quad + \left(\|\frac{\Delta t}{2}\epsilon^{\frac{1}{2}}\zeta_E\|_E^2 + \|\frac{\Delta t}{2}\mu^{\frac{1}{2}}\zeta_H\|_H^2\right)^{\frac{1}{2}} \\
&\le \left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(2)}\|_E^2 + \|\epsilon^{\frac{1}{2}}\mathcal{H}^{(2)}\|_H^2\right)^{\frac{1}{2}} + \left(\|\Delta t\epsilon^{\frac{1}{2}}\zeta_E\|_E^2 + \|\Delta t\mu^{\frac{1}{2}}\zeta_H\|_H^2\right)^{\frac{1}{2}}. \qquad (4.4.24)
\end{aligned}$$

Similarly, from the relations (4.4.22) and (4.4.23) and the triangle inequality of the norm, we obtain that

$$\begin{aligned}&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(2)}\|_E^2+\|\mu^{\frac{1}{2}}\mathcal{H}^{(2)}\|_H^2\right)^{\frac{1}{2}}\\&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{(1)}\|_H^2\right)^{\frac{1}{2}}+\left(\|\Delta t\epsilon^{\frac{1}{2}}\eta_E\|_E^2+\|\Delta t\mu^{\frac{1}{2}}\eta_H\|_H^2\right)^{\frac{1}{2}},\end{aligned}$$

$$\begin{aligned}&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{(1)}\|_E^2+\|\mu^{\frac{1}{2}}\mathcal{H}^{(1)}\|_H^2\right)^{\frac{1}{2}}\\&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{n}\|_H^2\right)^{\frac{1}{2}}+\left(\|\Delta t\epsilon^{\frac{1}{2}}\xi_E\|_E^2+\|\Delta t\mu^{\frac{1}{2}}\xi_H\|_H^2\right)^{\frac{1}{2}}.\end{aligned}\tag{4.4.25}$$

Combining (4.4.24)-(4.4.25), we have that

$$\begin{aligned}&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n+1}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{n+1}\|_H^2\right)^{\frac{1}{2}}\\&\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{n}\|_H^2\right)^{\frac{1}{2}}+\left(\|\Delta t\sqrt{\epsilon}\xi_E\|_E^2+\|\Delta t\sqrt{\mu}\xi_H\|_H^2\right)^{\frac{1}{2}}\\&\quad+\left(\|\Delta t\sqrt{\epsilon}\eta_E\|+\|\Delta t\sqrt{\mu}\eta_H\|_H^2\right)^{\frac{1}{2}}+\left(\|\Delta t\sqrt{\epsilon}\zeta_E\|_E^2+\|\Delta t\sqrt{\mu}\zeta_H\|_H^2\right)^{\frac{1}{2}}.\end{aligned}\tag{4.4.26}$$

We know the truncation errors of ξ,η,ζ, we get that

$$\begin{aligned}&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n+1}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{n+1}\|_H^2\right)^{\frac{1}{2}}\\&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{n}\|_H^2\right)^{\frac{1}{2}}+C\Delta t(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4).\end{aligned}\tag{4.4.27}$$

Recursively applying (4.4.27) from time level n to 0, we finally have that

$$\begin{aligned}&\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{n+1}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{n+1}\|_H^2\right)^{\frac{1}{2}}\\&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{0}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{0}\|_H^2\right)^{\frac{1}{2}}+Cn\Delta t(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)\\&\quad\le\left(\|\epsilon^{\frac{1}{2}}\mathcal{E}^{0}\|_E^2+\|\epsilon^{\frac{1}{2}}\mathcal{H}^{0}\|_H^2\right)^{\frac{1}{2}}+CT(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4)\end{aligned}$$

We complete the proof of (4.4.7). The further analysis can obtain (4.4.7). $\square$

Then, we consider super-convergence of the $\mathbf{E}^n$ and $\mathbf{H}^n$ in discrete H^1 norm. Applying operator Λ_x to the equations (4.2.38)-(4.2.40), the Λ_x-EC-S-FDTD-(2,4) scheme is then,

Stage 1:

$$\frac{1}{\Delta t}\Lambda_x\left[\begin{pmatrix}\epsilon\mathbf{E}^{(1)}\\ \mu\mathbf{H}^{(1)}\end{pmatrix}-\begin{pmatrix}\epsilon\mathbf{E}^{n}\\ \mu\mathbf{H}^{n}\end{pmatrix}\right]=\frac{1}{4}A_{+,h}\Lambda_x\left[\begin{pmatrix}\mathbf{E}^{(1)}\\ \mathbf{H}^{(1)}\end{pmatrix}+\begin{pmatrix}\mathbf{E}^{n}\\ \mathbf{H}^{n}\end{pmatrix}\right],\tag{4.4.28}$$

Stage 2:

$$\frac{1}{\Delta t}\Lambda_x\left[\begin{pmatrix}\epsilon\mathbf{E}^{(2)}\\ \mu\mathbf{H}^{(2)}\end{pmatrix}-\begin{pmatrix}\epsilon\mathbf{E}^{(1)}\\ \mu\mathbf{H}^{(1)}\end{pmatrix}\right]=\frac{1}{2}A_{-,h}\Lambda_x\left[\begin{pmatrix}\mathbf{E}^{(2)}\\ \mathbf{H}^{(2)}\end{pmatrix}+\begin{pmatrix}\mathbf{E}^{(1)}\\ \mathbf{H}^{(1)}\end{pmatrix}\right],\tag{4.4.29}$$

Stage 3:

$$\frac{1}{\Delta t}\Lambda_x\left[\begin{pmatrix}\epsilon\mathbf{E}^{n+1}\\ \mu\mathbf{H}^{n+1}\end{pmatrix}-\begin{pmatrix}\epsilon\mathbf{E}^{(2)}\\ \mu\mathbf{H}^{(2)}\end{pmatrix}\right]=\frac{1}{4}A_{+,h}\Lambda_x\left[\begin{pmatrix}\mathbf{E}^{n+1}\\ \mathbf{H}^{n+1}\end{pmatrix}+\begin{pmatrix}\mathbf{E}^{(2)}\\ \mathbf{H}^{(2)}\end{pmatrix}\right].\tag{4.4.30}$$

The boundary values for this scheme are from the PEC boundary condition, same as that in (4.2.41), and initial value are given as in (4.2.42).

The corresponding truncation error equations are

Stage 1:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(1)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(1)} \end{pmatrix} - \begin{pmatrix} \sqrt{\epsilon}\Lambda_x E(t^n) \\ \\ \sqrt{\mu}\Lambda_x H(t^n) \end{pmatrix}\right]$$
$$= \frac{1}{4}A_{+,h}\left[\begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(1)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(1)} \end{pmatrix} + \begin{pmatrix} \sqrt{\epsilon}\Lambda_x E(t^n) \\ \\ \sqrt{\mu}\Lambda_x H(t^n) \end{pmatrix}\right] + \vec{\varphi}, \qquad (4.4.31)$$

Stage 2:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(2)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(2)} \end{pmatrix} - \begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(1)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(1)} \end{pmatrix}\right]$$
$$= \frac{1}{2}A_{-,h}\left[\begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(2)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(2)} \end{pmatrix} + \begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(1)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(1)} \end{pmatrix}\right] + \vec{\psi}, \qquad (4.4.32)$$

Stage 3:

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \sqrt{\epsilon}\Lambda_x E(t^{n+1}) \\ \\ \sqrt{\mu}\Lambda_x H(t^{n+1}) \end{pmatrix} - \begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(2)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(2)} \end{pmatrix}\right]$$
$$= \frac{1}{4}A_{+,h}\left[\begin{pmatrix} \sqrt{\epsilon}\Lambda_x E(t^{n+1}) \\ \\ \sqrt{\mu}\Lambda_x H(t^{n+1}) \end{pmatrix} + \begin{pmatrix} \sqrt{\epsilon}\Lambda_x\tilde{E}^{(2)} \\ \\ \sqrt{\mu}\Lambda_x\tilde{H}^{(2)} \end{pmatrix}\right] + \vec{\chi}. \qquad (4.4.33)$$

With the definition of intermediate variables and the truncation analysis, it holds that

$$\|\vec{\varphi}\| + \|\vec{\psi}\| + \|\vec{\chi}\| \le \{\Delta t^2 + \Delta x^4 + \Delta y^4 + \Delta z^4\}.$$

Thus, the following error equations are

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\Lambda_x\mathcal{E}^{(1)} \\ \mu\Lambda_x\mathcal{H}^{(1)} \end{pmatrix} - \begin{pmatrix} \epsilon\Lambda_x\mathcal{E}^{n} \\ \mu\Lambda_x\mathcal{H}^{n} \end{pmatrix}\right] = \frac{1}{4}A_{+,h}\left[\begin{pmatrix} \Lambda_x\mathcal{E}^{(1)} \\ \Lambda_x\mathcal{H}^{(1)} \end{pmatrix} + \begin{pmatrix} \Lambda_x\mathcal{E}^{n} \\ \Lambda_x\mathcal{H}^{n} \end{pmatrix}\right] + \vec{\varphi} \tag{4.4.34}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\Lambda_x\mathcal{E}^{(2)} \\ \mu\Lambda_x\mathcal{H}^{(2)} \end{pmatrix} - \begin{pmatrix} \epsilon\Lambda_x\mathcal{E}^{(1)} \\ \mu\Lambda_x\mathcal{H}^{(1)} \end{pmatrix}\right] = \frac{1}{2}A_{-,h}\left[\begin{pmatrix} \Lambda_x\mathcal{E}^{(2)} \\ \Lambda_x\mathcal{H}^{(2)} \end{pmatrix} + \begin{pmatrix} \Lambda_x\mathcal{E}^{(1)} \\ \Lambda_x\mathcal{H}^{(1)} \end{pmatrix}\right] + \vec{\psi} \tag{4.4.35}$$

$$\frac{1}{\Delta t}\left[\begin{pmatrix} \epsilon\Lambda_x\mathcal{E}^{n+1} \\ \mu\Lambda_x\mathcal{H}^{n+1} \end{pmatrix} - \begin{pmatrix} \epsilon\Lambda_x\mathcal{E}^{(2)} \\ \mu\Lambda_x\mathcal{H}^{(2)} \end{pmatrix}\right] = \frac{1}{4}A_{+,h}\left[\begin{pmatrix} \Lambda_x\mathcal{E}^{n+1} \\ \Lambda_x\mathcal{H}^{n+1} \end{pmatrix} + \begin{pmatrix} \Lambda_x\mathcal{E}^{(2)} \\ \Lambda_x\mathcal{H}^{(2)} \end{pmatrix}\right] + \vec{\chi}. \tag{4.4.36}$$

From (4.4.34)-(4.4.36), similarly to the proof of Theorem 4.4.1, we have the following super-convergence theorem.

Theorem 4.4.2. *(Super-convergence) Assume that* $(\boldsymbol{E}, \boldsymbol{H})$ *the solution components of Maxwell's equations (4.2.1)-(4.2.5) in three dimensions are smooth enough. Let* $\boldsymbol{E}^n$*,* $\boldsymbol{H}^n$ *be the solutions of EC-S-FDTD-(2,4) and let* $\mathcal{E}^n = \boldsymbol{E}(t^n) - \boldsymbol{E}^n$*,* $\mathcal{H}^n = \boldsymbol{H}(t^n) - \boldsymbol{H}^n$*. be the errors. Then we have the following estimations:*

$$\begin{aligned} &(\|\epsilon^{\frac{1}{2}}\Lambda_u\mathcal{E}^n\|^2_{\Lambda_u E} + \|\mu^{\frac{1}{2}}\Lambda_u\mathcal{H}^n\|^2_{\Lambda_u H})^{\frac{1}{2}} \\ &\quad\leq (\|\epsilon^{\frac{1}{2}}\Lambda_u\mathcal{E}^0\|^2_{\Lambda_u E} + \|\mu^{\frac{1}{2}}\Lambda_u\mathcal{H}^0\|^2_{\Lambda_u H})^{\frac{1}{2}} + CT(\Delta t^2 + \Delta x^4 + \Delta y^4 + \Delta z^4), \end{aligned} \tag{4.4.37}$$

and

$$(\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u\mathcal{E}^{n+\frac{1}{2}}\|^2_{\Lambda_u E}+\|\mu^{\frac{1}{2}}\delta_t\Lambda_u\mathcal{H}^{n+\frac{1}{2}}\|^2_{\Lambda_u H})^{\frac{1}{2}} \tag{4.4.38}$$
$$\leq (\|\epsilon^{\frac{1}{2}}\delta_t\Lambda_u\mathcal{E}^{\frac{1}{2}}\|^2_{\Lambda_u E}+\|\mu^{\frac{1}{2}}\delta_t\Lambda_u\mathcal{H}^{\frac{1}{2}}\|^2_{\Lambda_u H})^{\frac{1}{2}}+CT(\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4),$$

where C is a constant independent of Δt, Δx, Δy, and Δz, $u = x, y$, or z.

Using Theorem 4.4.2, we have estimation of divergence-free.

Theorem 4.4.3. *Let $\boldsymbol{E}^n$, and $\boldsymbol{H}^n$ be the solutions of the EC-S-FDTD-(2,4) scheme. If the exact solution of the Maxwell's equations in three dimensions is smooth enough, the error of divergence-free is estimated by*

$$\|\Lambda_x E_x^n+\Lambda_y E_y^n+\Lambda_z E_z^n\| \leq C\{\Delta t^2+\Delta x^4+\Delta y^4+\Delta z^4\}. \tag{4.4.39}$$

Proof. The left side of (4.4.39) can be written as

$$\|\Lambda_x E_x^n+\Lambda_y E_y^n+\Lambda_z E_z^n\| \leq \|\Lambda_x E_x(t^n)+\Lambda_y E_y(t^n)+\Lambda_z E_z(t^n)\|$$
$$+\|\Lambda_x(E_x^n-E_x(t^n))+\Lambda_y(E_y^n-E_y(t^n))+\Lambda_z(E_z^n-E_z(t^n))\|.$$

Then (4.4.39) can be obtained with Theorem 4.4.2 and the fourth order accuracy of operators of Λ_x, Λ_y and Λ_z. □

4.5 Numerical experiments

Finally, we present numerical experiments by EC-S-FDTD-(2,4) in three dimensions, comparing to EC-S-FDTDI and EC-S-FDTDII ([8]), 3D-ADI-FDTD ([49, 50, 82]).

We compute by our EC-S-FDTD-(2,4) the three-dimensional Maxwell equations with PEC boundary condition, where domain $\Omega = [0,1] \times [0,1] \times [0,1]$ surrounded by a perfect conductor. The exact solution is

$$E_x = \frac{k_y - k_z}{\epsilon\sqrt{\mu}\omega} cos(\omega\pi t) cos(k_x\pi x) sin(k_y\pi y) sin(k_z\pi z), \tag{4.5.1}$$
$$E_y = \frac{k_z - k_x}{\epsilon\sqrt{\mu}\omega} cos(\omega\pi t) sin(k_x\pi x) cos(k_y\pi y) sin(k_z\pi z), \tag{4.5.2}$$
$$E_z = \frac{k_x - k_y}{\epsilon\sqrt{\mu}\omega} cos(\omega\pi t) sin(k_x\pi x) sin(k_y\pi y) cos(k_z\pi z), \tag{4.5.3}$$
$$H_x = \frac{1}{\sqrt{\mu}} sin(\omega\pi t) sin(k_x\pi x) cos(k_y\pi y) cos(k_z\pi z), \tag{4.5.4}$$
$$H_y = \frac{1}{\sqrt{\mu}} sin(\omega\pi t) cos(k_x\pi x) sin(k_y\pi y) cos(k_z\pi z), \tag{4.5.5}$$
$$H_z = \frac{1}{\sqrt{\mu}} sin(\omega\pi t) cos(k_x\pi x) cos(k_y\pi y) sin(k_z\pi z), \tag{4.5.6}$$

where $k_x = 1$, $k_y = 2$ $k_z = -3$, $\mu = 1$, $\epsilon = 1$ and $\omega^2 = \frac{1}{\mu\epsilon}(k_x^2 + k_y^2 + k_z^2)$.

First, we check the performance of energy-conserved properties. In the constant electric permittivity case, the exact energy of solution can be computed directly as:

$$\text{EnergyI} = \left(\int_\Omega \epsilon|\mathbf{E}|^2 + \mu_\Omega \epsilon |\mathbf{H}|^2 \right)^{\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{8}}.$$

We define the relative errors of energy conversations:

$$\text{REE-I} = \max_{0 \le n \le N} \frac{|(\|\epsilon^{\frac{1}{2}}\mathbf{E}^n\|^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^n\|^2)^{\frac{1}{2}} - \text{EnergyI}|}{\text{EnergyI}}, \tag{4.5.7}$$

REE-II (4.5.8)

$$= \max_{0\le n\le N-1} \frac{|(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n+\frac{1}{2}}\|^2+\|\mu^{\frac{1}{2}}\delta_t\mathbf{H}^{n+\frac{1}{2}}\|^2)^{\frac{1}{2}}-(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{\frac{1}{2}}\|^2+\|\mu^{\frac{1}{2}}\delta_t\mathbf{H}^{\frac{1}{2}}\|^2)^{\frac{1}{2}}|}{(\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{\frac{1}{2}}\|^2+\|\mu^{\frac{1}{2}}\delta_t\mathbf{H}^{\frac{1}{2}}\|^2)^{\frac{1}{2}}}.$$

In Table 4.1, the REE-I and REE-II of EC-S-FDTDI&II and EC-S-FDTD-(2,4) are almost zero, i.e., most less than the relative error of 10^{-14}, which are of the machine precision. However, the results of ADI-FDTD ([49, 50]) are only 10^{-4}. Table 4.2 sets different wave numbers $k_x = 2, 3, 5, 10$, $k_y = 2k_x$, $k_z = -3k_x$, $\Delta x = \Delta y = \Delta z = \Delta t$=0.02. The relative errors of EC-S-FDTDI&II and EC-S-FDTD-(2,4) are 10^{-14} with different wave numbers. By the Tables 4.1 and 4.2, we can say that EC-S-FDTD-(2,4) hold the the property of energy conservation.

Secondly, Table 4.3 - Table 4.6 give the numerical results to show the accuracy of scheme EC-S-FDTD-(2,4). The relative errors are defined as:

$$\text{ErrorI} = \max_{0\le n\le N} \frac{(\|\epsilon^{\frac{1}{2}}[\mathbf{E}(t^n)-E^n]\|_E^2+\|\mu^{\frac{1}{2}}[\mathbf{H}(t^n)-\mathbf{H}^n]\|_H^2)^{\frac{1}{2}}}{\text{EnergyI}}, \tag{4.5.9}$$

$$\text{ErrorII} = \max_{0\le n\le N-1} \frac{(\|\epsilon^{\frac{1}{2}}[\delta_t\mathbf{E}(t^n)-\delta_tE^n]\|_E^2+\|\mu^{\frac{1}{2}}[\delta_t\mathbf{H}(t^n)-\delta_t\mathbf{H}^n]\|_H^2)^{\frac{1}{2}}}{\text{EnergyII}}, \tag{4.5.10}$$

Accuracy and convergence ratios are shown in Table 4.3 and Table 4.4 at time $T = 1$ with $\Delta x = \Delta y$=$\Delta z = \Delta t$. Both tables indicate that EC-S-FDTDII, ADI-FDTD and EC-S-FDTD-(2,4) are of second order accuracy in time step, while EC-S-FDTDI is of first order accuracy in time step.

The accuracies in space are listed in Tables 4.5 and 4.6. From both tables, it clearly shows that EC-S-FDTD-(2,4) has the convergence of fourth order, but con-

Table 4.1: Relative errors of EnergyI (REE-I) and EnergyII (REE-II) by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and with $\mu = \epsilon = 1$, $T = 1$.

Partition	EC-S-FDTDI-1		EC-S-FDTDII-1		EC-S-FDTD(2,4)		ADI-FDTD	
N	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II
25	1.81e-16	2.53e-16	3.63e-16	6.30e-16	6.53e-15	6.53e-15	1.93e-03	2.21e-03
50	1.63e-15	1.74e-15	7.25e-16	1.12e-15	1.20e-14	1.26e-14	5.51e-04	5.70e-04
75	3.08e-15	2.97e-15	1.99e-15	2.23e-15	2.90e-15	2.84e-15	2.51e-04	2.54e-04
100	3.26e-15	3.21e-15	4.35e-15	4.94e-15	2.7376e-14	2.96e-14	1.42e-04	1.44e-04
200	3.08e-15	3.21e-15	8.70e-15	9.51e-15	9.26e-15	8.52e-15	3.59e-05	3.60e-05

vergence of EC-S-FDTDI , EC-S-FDTDII and ADI-FDTD are only second order. The results in Tables 4.3 - 4.6 confirm our theoretical analysis for convergence.

For checking the error of divergence-free, we define two kinds of formulas:

$$\text{DivI} = \max_{1\le i,j,k\le N-1, 0\le n\le N} |\epsilon(\Lambda_x E^n_{x_{i,j,k}} + \Lambda_y E^n_{y_{i,j,k}} + +\Lambda_z E^n_{z_{i,j,k}})|, \tag{4.5.11}$$

$$\begin{aligned} &\text{DivII} \\ &= \max_{0\le n\le N}\left(\sum_{1\le i\le N-1}\sum_{1\le j\le N-1}\sum_{1\le k\le N-1}\epsilon(\Lambda_x E^n_{x_{i,j,k}} + \Lambda_y E^n_{y_{i,j,k}} + \Lambda_z E^n_{z_{i,j,k}})^2 \Delta x\Delta y\Delta z\right)^{\frac{1}{2}}. \end{aligned} \tag{4.5.12}$$

For the above definitions, DivI and DivII with δ_x, δ_y and δ_z are used for EC-S-FDTDI&II and ADI-FDTD. We use fourth-order operators Λ_x, Λ_y and Λ_z to replace

Table 4.2: Relative errors of EnergyI (I) and EnergyII (II) by EC-S-FDTDI&II (ECI&II), EC-S-FDTD-(2,4) (EC24) and ADI-FDTD (ADI) schemes. Parameters: $k_y = 2k_x$, $k_z = -3k_x$, $\Delta x = \Delta y = \Delta z = \Delta t$=0.02, $\mu = \epsilon = 1$, $T = 1$.

Partition	k_x=2		k_x=3		k_x=5		k_x=10	
N	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II	REE-I	REE-II
ECI	1.45e-15	1.14e-15	9.06e-16	6.97e-16	7.25e-16	5.72e-16	2.54e-15	2.02e-15
ECII	3.63e-16	2.52e-16	1.63e-15	1.90e-15	1.63e-15	1.79e-15	7.25e-16	4.43e-16
EC24	1.20e-14	1.12e-14	1.34e-14	1.32e-14	1.50e-14	1.47e-14	1.78e-14	1.70e-14

δ_x, δ_y and δ_z,respectively in the definition of DivI and DivII in the EC-S-FDTD-(2,4) item. From Tables 4.7 and Table 4.8, we see that the errors of numerical divergence-free of EC-S-FDTDII, ADI-FDTD and EC-S-FDTD-(2,4) are second order in time, but EC-S-FDTDI is first order in time.

Finally, we check the energy conservation in the variation forms. we define the discrete energies and the errors of energies in the δ_x, δ_y and δ_z forms:

$$
\begin{aligned}
\text{Energy}^n_{\delta_u} &= \left(\left\|\epsilon^{\frac{1}{2}}\delta_u \mathbf{E}^n\right\|^2_{\delta_u \mathbf{E}} + \left\|\mu^{\frac{1}{2}}\delta_u \mathbf{H}^n\right\|^2_{\delta_u \mathbf{H}}\right)^{\frac{1}{2}}, \\
\text{EnEr}_{\delta u} &= |\text{Energy}^n_{\delta_u} - \text{Energy}^0_{\delta_u}|,
\end{aligned}
$$

where $u = x, y, z$, and similarly define EnEr_{Λ_x}, EnEr_{Λ_y} and EnEr_{Λ_z}. The errors of energies in the spatial variation forms are presented in Table 4.5, which shows

Table 4.3: The ErrorI by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and with $\mu = \epsilon = 1$, $T = 1$

Partition	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio
25	0.2077	-	0.1493	-	0.0995	-	0.1864	-
50	0.0739	1.4909	0.0379	1.9779	0.0250	1.9928	0.0475	1.9724
75	0.0445	1.2510	0.0169	1.9919	0.0111	2.0025	0.0212	1.9896
100	0.0320	1.1462	0.0095	2.0023	0.0063	1.9688	0.0119	2.0073
200	0.0154	1.0551	2.3915e-3	1.9900	1.5860e-3	1.9900	2.9826e-3	1.9963

that EC-S-FDTD-(2,4) and EC-S-FDTD I&II stratify the energy conservations in the discrete variation form but ADI-FDTD breaks the energy conservations in the discrete variation form.

Table 4.4: The ErrorII by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters:$\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$

Partition	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio
25	0.1890	-	0.1430	-	0.0942	-	0.1782	-
50	0.0649	1.5421	0.0371	1.9465	0.0241	1.9667	0.0464	1.9413
75	0.0390	1.2561	0.0166	1.9834	0.0108	1.9796	0.0208	1.9788
100	0.0280	1.1518	0.0094	1.9768	0.0061	1.9857	0.0117	2.0000
200	0.0134	1.0632	2.3663e-3	1.9900	1.5456e-3	1.9806	2.9485e-03	1.9885

Table 4.5: The ErrorI by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and EC-S-FDTD-(4,4) schemes. Parameters:$\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = 1$, $k_y = 2$, $k_z = -3$

Partition	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio	ErrorI	Ratio
25	0.0543	-	5.4194e-2	-	9.2642e-4	-	5.426e-2	-
50	0.0136	1.9973	1.3542e-2	2.0007	5.8197e-5	3.9926	1.3546e-2	2.0020
75	6.0270e-3	2.0071	6.0180e-3	2.0003	1.1507e-5	3.9976	6.0188e-3	2.0007
100	3.3899e-3	2.0003	3.3850e-3	2.0001	3.6419e-6	3.9990	3.3852e-3	2.0004
200	8.4764e-4	1.9997	8.4582e-4	2.0007	2.2753e-7	4.0006	8.4647e-4	1.9997

Table 4.6: The ErrorII by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = 1/N$, $\Delta t = 1/N^2$, $k_x = 1$, $k_y = 2$, $k_z = -3$

Partition	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio	ErrorII	Ratio
25	0.0543	-	5.4218e-2	-	9.2824e-4	-	5.4280e-2	-
50	0.0136	1.9973	1.3580e-2	1.9973	5.8350e-5	3.9917	1.3584e-2	1.9985
75	6.0452e-3	1.9997	6.0374e-3	1.9992	1.1538e-5	3.9974	6.0382e-3	1.9996
100	3.4006e-3	1.9998	3.3964e-3	1.9996	3.6521e-06	3.9987	3.3967e-3	1.9998
200	8.4998e-4	2.0003	8.4902e-4	2.0001	2.2834e-7	3.9995	8.4901e-4	2.0003

Table 4.7: The error of divergence-free (DivI) by different schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$

Partition	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	DivI	Ratio	DivI	Ratio	DivI	Ratio	DivI	Ratio
25	1.1703	-	0.0157	-	0.0812	-	0.3135	-
50	0.5269	1.1513	3.2958e-3	2.2521	0.0203	1.9982	0.0804	1.9632
75	0.3423	1.0638	1.4012	2.1095	9.0087e-3	2.0037	0.0359	1.9885
100	0.2548	1.0262	8.0134e-4	1.9424	5.0671e-3	2.0002	0.0203	1.9817
200	0.1240	1.0390	2.0340e-4	1.9780	1.2672e-3	1.9995	5.0728e-3	2.0006

Table 4.8: The error of divergence-free (DivII) by different schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$

Partition	EC-S-FDTDI		EC-S-FDTDII		EC-S-FDTD(2,4)		ADI-FDTD	
N	DivII	Ratio	DivII	Ratio	DivII	Ratio	DivII	Ratio
25	0.4162	-	0.0278	-	0.0284	-	0.1115	-
50	0.1867	1.1565	8.3925e-3	1.7279	7.1734e-3	1.9852	0.0285	1.9680
75	0.1213	1.0636	4.0134e-3	1.8194	3.1907e-3	1.9980	0.0127	1.9935
100	0.0901	1.0336	2.3028e-3	1.9309	1.7945e-3	2.0005	0.0072	1.9727
200	0.0425	1.0841	5.9373e-4	1.9555	4.4828e-4	1.9994	1.7935e-3	2.0052

Table 4.9: Relative energy errors in variation form by EC-S-FDTDI&II, EC-S-FDTD-(2,4) and ADI-FDTD schemes. Parameters: $\Delta x = \Delta y = \Delta z = \Delta t = 1/N$, $k_x = 1$, $k_y = 2$, $k_z = -3$ and with $\mu = \epsilon = 1$, $T = 1$.

Mesh N		25	50	75	100	200
EC-S-FDTDI	$\text{EnEr}_{\delta x}$	4.6198e-15	6.5301e-14	1.9193e-13	2.9548e-13	4.2534e-13
	$\text{EnEr}_{\delta y}$	0	7.1866e-14	2.0796e-13	3.1398e-13	4.5568e-13
	$\text{EnEr}_{\delta z}$	0	6.9746e-14	2.1337e-13	3.3252e-13	4.8914e-13
EC-S-FDTDII	$\text{EnEr}_{\delta x}$	9.2395e-15	4.5710e-14	1.5993e-13	4.6169e-13	8.5685e-13
	$\text{EnEr}_{\delta y}$	9.2578e-15	3.9200e-14	1.5997e-13	4.2480e-13	9.0290e-13
	$\text{EnEr}_{\delta z}$	1.2384e-14	5.2309e-14	1.8137e-13	4.3105e-13	8.9045e-13
EC-S-FDTD-(2,4) 2nd-diff	$\text{EnEr}_{\delta x}$	1.4321e-13	5.6159e-13	1.9192e-13	2.6224e-12	9.2713e-12
	$\text{EnEr}_{\delta y}$	1.4349e-13	5.6186e-13	1.7597e-13	2.6412e-12	9.3612e-12
	$\text{EnEr}_{\delta z}$	1.4242e-13	5.6668e-13	1.9204e-13	2.6602e-12	8.6490e-12
EC-S-FDTD-(2,4) 4nd-diff	$\text{EnEr}_{\Lambda x}$	5.7709e-15	2.3577e-14	7.9964e-15	1.1028e-13	5.1267e-13
	$\text{EnEr}_{\Lambda y}$	6.0275e-15	2.3577e-14	7.5521e-15	1.0977e-13	5.4712e-13
	$\text{EnEr}_{\Lambda z}$	5.8142e-15	2.3698e-14	8.2925e-15	1.1148e-13	4.8712e-13
ADI-FDTD	$\text{EnEr}_{\delta x}$	0.0301	0.0150	0.0101	7.5620e-3	3.7821e-3
	$\text{EnEr}_{\delta y}$	0.0301	0.0151	0.0101	7.5620e-3	3.7822e-3
	$\text{EnEr}_{\delta z}$	0.0301	0.0151	0.0101	7.5620e-3	3.7820e-3

5 The S-FDTD method for Maxwell's equations in Cole-Cole dispersive medium

5.1 Introduction

In electromagnetics, the dispersive medium, such as biological tissue, soil, plasma, radar absorbing material, and optical fiber, etc, is described as a medium whose permittivity or permeability depends on the wave frequency. The study of the propagation of electromagnetic waves in dispersive media is very important.

The computation of Maxwell's equations with dispersive media by using the FDTD scheme started in 1990 ([62]). Some further studies and applications of modeling of dispersive media by FDTD were done in [52, 45, 20]. The TDFE (Time-domain finite element method) is another method to compute Maxwell's equations with dispersive media, which was first studied in [31] in 2001. Since 2003, various methods [4, 29, 38, 39, 40, 43, 66] have been developed and analyzed for three popular dispersive media models: the cold plasma model, the Debye model, and the Lorentz model. However, it is difficult to extent to solve the so-called Cole-Cole dispersive medium models [1, 55, 63, 57, 56], due to the fractional time derivative term in the Cole-Cole models. The fractional time derivative model is quite different from the standard dispersive media models. Most recently, paper [41] proposed two type fully-discrete Finite Element methods for solving Maxwell's equations in a cole-cole dispersive medium: the Crank-Nicolson type and the leap-frog type.

In this chapter, we propose to combine the splitting technique and FDTD to treat the fractional derivative equations of the Maxwell's equations in Cole-Cole dispersive medium in two dimensions. Our proposed splitting FDTD scheme is a two-stage scheme, in which each stage is a Euler-based scheme. The fractional time derivative is approached by the Letnikov-typed difference approximate operator, while the spatial second-order difference operators are used to approximate the spatial differential term for each stage splitting equations. We prove that the proposed scheme is unconditionally stable. We analyze theoretically the convergence of the scheme and obtain error estimates of $O\{\Delta t + \Delta x^2 + \Delta y^2\}$. The Numerical experiments confirm the theoretical results.

This chapter starts with a brief background of the cole-cole dispersive medium models. Our Euler-based S-FDTD scheme is proposed in Section 5.3. We give the theoretical analysis of stability and convergence in Section 5.4. Finally, numerical experiments are given in Section 5.5.

5.2 The Cole-Cole dispersive medium models

The Cole-Cole dispersive medium is named from Kenneth S. Cole and Robert H. Cole [10]. Its relative permittivity is represented by:

$$\epsilon_r(\omega) = \epsilon_\infty + (\epsilon_s - \epsilon_\infty)/(1 + (j\omega\tau_0)^\alpha),\ 0 < \alpha < 1, \tag{5.2.1}$$

where ϵ_∞, ϵ_s, τ_0 are the infinite-frequency permittivity, the static permittivity, and the relaxation time, respectively. ω denotes a general frequency.

Let $\mathbf{E}$ be the electric field and $\mathbf{P}$ be the induced polarization field. The relation between $\mathbf{P}$ and $\mathbf{E}$ is

$$\mathbf{P} = \epsilon_0(\epsilon_r - \epsilon_\infty)\mathbf{E} = \epsilon_0(\epsilon_s - \epsilon_\infty)/(1 + (j\omega\tau_0)^\alpha)\mathbf{E}, \tag{5.2.2}$$

where ϵ_0 is the permittivity in the free space.

The frequency domain of the relation in equation (5.2.2) can be changed into time domain by the inverse Fourier transform,

$$\tau_0^\alpha \frac{\partial^\alpha \mathbf{P}(t)}{\partial t^\alpha} + \mathbf{P}(t) = \epsilon_0(\epsilon_s - \epsilon_\infty)\mathbf{E}(t), \tag{5.2.3}$$

where $\frac{\partial^\alpha \mathbf{P}(t)}{\partial t^\alpha}$ represents the Letnikov time fractional derivative given by

$$\begin{aligned} \frac{\partial^\alpha \mathbf{P}(t)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \mathbf{P}(s) ds \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (s)^{-\alpha} \mathbf{P}(t-s) ds. \end{aligned} \tag{5.2.4}$$

The corresponding Maxwell's equations are

$$\epsilon_0\epsilon_\infty \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{P}}{\partial t}, \tag{5.2.5}$$

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}. \tag{5.2.6}$$

We assume a perfect boundary condition provided

$$n \times \mathbf{E} = 0 \text{ on } \partial\Omega \times (0, T), \tag{5.2.7}$$

and the initial conditions are

$$\mathbf{E}(x, y, 0) = \mathbf{E}_0(x, y), \mathbf{H}(x, y, 0) = \mathbf{H}_0(x, y), \mathbf{P}(x, y, 0) = \mathbf{P}_0(x, y) = 0, (x, y) \in \Omega, \tag{5.2.8}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\mathbf{P}$ is the induced polarization field. μ_0 is the permeability of free space.

With the PEC boundary condition, for the solution of model (5.2.3)-(5.2.6), we can derive that

$$\begin{aligned} div(E \times H) &= -\frac{1}{2}\frac{d}{dt}(\epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty\|E\|^2 + \mu_0\|H\|^2) + \|P\|^2) \\ &\quad + \tau_0^\alpha(\frac{\partial^\alpha}{\partial t^\alpha}P(t), \frac{\partial}{\partial t}P(t)) = 0, \end{aligned} \tag{5.2.9}$$

and further integrating equation (5.2.9) from 0 to t, we obtain that

$$\begin{aligned} &\epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty\|E(t)\|^2 + \mu_0\|H(t)\|^2) + \|P(t)\|^2 + 2\tau_0^\alpha \int_0^t (\frac{\partial^\alpha P(s)}{\partial s^\alpha} \cdot \frac{\partial P(s)}{\partial s})ds \\ &= \epsilon_0(\epsilon_s - \epsilon_\infty)(\epsilon_0\epsilon_\infty\|E(0)\|^2 + \mu_0\|H(0)\|^2) + \|P(0)\|^2. \end{aligned} \tag{5.2.10}$$

5.3 The Euler-based S-FDTD scheme

The Maxwell's equations in the Cole-Cole dispersive medium (5.2.3)-(5.2.6) can be split into the following form in time interval $(t_n, t_{n+1}]$:

$$\begin{cases} \tau_0^\alpha \frac{\partial^\alpha \mathbf{P}_x(t)}{\partial t^\alpha} + \mathbf{P}_x(t) = \epsilon_0(\epsilon_s - \epsilon_\infty)\mathbf{E}_x(t) \\ \epsilon_0\epsilon_\infty \frac{\partial \mathbf{E}_x}{\partial t} = \frac{\partial \mathbf{H}_z}{\partial y} - \frac{\partial \mathbf{P}_x}{\partial t} \\ \mu_0 \frac{\partial \mathbf{H}_z}{\partial t} = \frac{\partial \mathbf{E}_x}{\partial y} \end{cases} \quad \begin{cases} \tau_0^\alpha \frac{\partial^\alpha \mathbf{P}_y(t)}{\partial t^\alpha} + \mathbf{P}_y(t) = \epsilon_0(\epsilon_s - \epsilon_\infty)\mathbf{E}_y(t) \\ \epsilon_0\epsilon_\infty \frac{\partial \mathbf{E}_y}{\partial t} = -\frac{\partial \mathbf{H}_z}{\partial x} - \frac{\partial \mathbf{P}_x}{\partial t} \\ \mu_0 \frac{\partial \mathbf{H}_z}{\partial t} = -\frac{\partial \mathbf{E}_x}{\partial x} \end{cases} \tag{5.3.1}$$

The fractional derivative $\frac{\partial^\alpha P(t)}{\partial t^\alpha}$ is approximated by:

$$\frac{\partial^\alpha P(t)}{\partial t^\alpha}|_{t=t_{n+1}} \approx \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{l=0}^{n} (P(t_{n+1-l}) - P(t_{n-l}))b_l \equiv \tilde{\partial}_t^a P^{n+1}, \tag{5.3.2}$$

where $b_l = (l+1)^{1-\alpha} - l^{1-\alpha}$ and $l \geq 0$.

The Euler-based S-FDTD scheme is proposed as

Stage 1:

$$\tau_0^\alpha \tilde{\partial}_t^a (P^{n+1}_{x_{i+\frac{1}{2},j}}) + P^{n+1}_{x_{i+\frac{1}{2},j}} = \epsilon_0(\epsilon_s - \epsilon_\infty) E^{n+1}_{x_{i+\frac{1}{2},j}}, \tag{5.3.3}$$

$$\epsilon_0\epsilon_\infty \frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \delta_y H^*_{z_{i+\frac{1}{2},j}} - \frac{P^{n+1}_{x_{i+\frac{1}{2},j}} - P^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t}, \tag{5.3.4}$$

$$\mu_0 \frac{H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \delta_y E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}. \tag{5.3.5}$$

Stage 2:

$$\tau_0^\alpha \tilde{\partial}_t^a (P^{n+1}_{y_{i,j+\frac{1}{2}}}) + P^{n+1}_{y_{i,j+\frac{1}{2}}} = \epsilon_0(\epsilon_s - \epsilon_\infty) E^{n+1}_{y_{i,j+\frac{1}{2}}}, \tag{5.3.6}$$

$$\epsilon_0\epsilon_\infty \frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\delta_x H^{n+1}_{z_{i,j+\frac{1}{2}}} - \frac{P^{n+1}_{y_{i,j+\frac{1}{2}}} - P^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t}, \tag{5.3.7}$$

$$\mu_0 \frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\delta_x E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}. \tag{5.3.8}$$

The boundary conditions are given by

$$E^{n+1}_{x_{i+\frac{1}{2},0}} = E^{n+1}_{x_{i+\frac{1}{2},J}} = E^{n+1}_{y_{0,j+\frac{1}{2}}} = E^{n+1}_{y_{I,j+\frac{1}{2}}} = 0, \tag{5.3.9}$$

and the initial conditions are given by

$$E^0_{x_{\alpha,\beta}} = E_{x^0}(\alpha\Delta x, \beta\Delta y);\ E^0_{y_{\alpha,\beta}} = E_{y^0}(\alpha\Delta x, \beta\Delta y);\ H^0_{z_{\alpha,\beta}} = H_{z^0}(\alpha\Delta x, \beta\Delta y);$$

$$P^0_{x_{\alpha,\beta}} = P_{x^0}(\alpha\Delta x, \beta\Delta y) = 0;\ P^0_{y_{\alpha,\beta}} = P_{y^0}(\alpha\Delta x, \beta\Delta y) = 0. \tag{5.3.10}$$

5.4 Stability and Convergence

In this section, we analyze the stability and convergence of the Euler-based S-FDTD scheme (5.3.3)-(5.3.10). We first give two lemmas.

Lemma 5.4.1. *Let $b_l = (l+1)^{1-\alpha} - (l)^{1-\alpha}$, for $l \geq 0$ and $0 < \alpha < 1$, then sequence $\{b_i\}_{i=0}^{\infty}$ is positive and convex.*

Proof. We can write $b_l = (1-\alpha)\int_l^{l+1} x^{-\alpha}dx$, $b_{l+1} = (1-\alpha)\int_l^{l+1}(x+1)^{-\alpha}dx$ and $b_{l+2} = (1-\alpha)\int_l^{l+1}(x+2)^{-\alpha}dx$, then $b_l - 2b_{l+1} + b_{l+2} = (1-\alpha)\int_l^{l+1}(x^{-\alpha} - 2(x+1)^{-\alpha} + (x+2)^{-\alpha})dx \geq 0$. □

By Lemma 5.4.1, we have Lemma 5.4.2.

Lemma 5.4.2. *The sequence used in (5.3.2) is positive and convex, and let $w_0 = b_0/2$, $w_i = b_i$, for $i > 0$. Then*

$$\Sigma_{n=1}^{N}\Sigma_{j=1}^{n} w_{n-j}\phi_j\phi_n \geq 0, \forall(\phi_1, \phi_2, \cdots, \phi_N) \in R^N, N \geq 1.$$

Proof. See LEMMA 4.3 in [46]. □

With Lemma 5.4.2, we get the stability.

Theorem 5.4.1. *For the numerical solution of the scheme (5.3.3)-(5.3.10), we have the unconditional stability:*

$$\begin{aligned}
&\epsilon_0(\epsilon_s-\epsilon_\infty)[\epsilon_0\epsilon_\infty \parallel E^{n+1} \parallel^2 +\mu_0 \parallel H_z^{n+1} \parallel^2] + \|P^{n+1}\|^2 \\
&\quad + \frac{1}{\Gamma(2-\alpha)}(\frac{\tau_0}{\tau})^\alpha \sum_{i=0}^{n} \|P^{i+1}-P^i\|^2 \\
&\quad \leq \epsilon_0(\epsilon_s-\epsilon_\infty)[\epsilon_0\epsilon_\infty \parallel E^0 \parallel^2 +\mu_0 \parallel Hz^0 \parallel^2] + \|P^0\|^2. \qquad (5.4.1)
\end{aligned}$$

Proof. Multiplying (5.3.4), (5.3.5) by E_x^{n+1}, H_z^*, respectively, adding them together, and using PEC boundary conditions, we have that

$$\epsilon_0\epsilon_\infty(E_x^{n+1}-E_x^n, E_x^{n+1}) + \epsilon_0\epsilon_\infty(P_x^{n+1}-P_x^n, E_x^{n+1}) + \mu_0(H_z^*-H_z^n, H_z^*) = 0. \quad (5.4.2)$$

Combining equation (5.4.2) with equations (5.3.3) , we get that

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(E_x^{n+1}-E_x^n, E_x^{n+1}) + \mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(H_z^*-H_z^n, H_z^*) \\
&\quad + (P_x^{n+1}-P_x^n, P_x^{n+1}) + (P_x^{n+1}-P_x^n, \tau_0^\alpha \tilde{\partial}_t^a P_x^{n+1}) = 0. \qquad (5.4.3)
\end{aligned}$$

Noting the inequality

$$(a-b, a) \geq a^2 - (a^2+b^2)/2 \geq (a^2-b^2)/2, \qquad (5.4.4)$$

from equation (5.4.3), we can have that

$$\frac{1}{2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(\|E_x^{n+1}\|^2-\|E_x^n\|^2)+\frac{1}{2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\|H_z^*\|^2-\|H_z^n\|^2)$$
$$+\frac{1}{2}(\|P_x^{n+1}\|^2-\|P_x^n\|^2)+(P_x^{n+1}-P_x^n,\tau_0^\alpha\tilde{\partial}_t^a P_x^{n+1})\le 0. \tag{5.4.5}$$

Similarly, from Stage 2, we get

$$\frac{1}{2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(\|E_y^{n+1}\|^2-\|E_y^n\|^2)+\frac{1}{2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\|H_z^{n+1}\|^2-\|H_z^*\|^2)$$
$$+\frac{1}{2}(\|P_y^{n+1}\|^2-\|P_y^n\|^2)+(P_y^{n+1}-P_y^n,\tau_0^\alpha\tilde{\partial}_t^a P_y^{n+1})\le 0. \tag{5.4.6}$$

Combining equations (5.4.5) and (5.4.6), we obtain that

$$\frac{1}{2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(\|E_x^{n+1}\|^2-\|E_x^n\|^2+\|E_y^{n+1})^2-(E_y^n\|^2)$$
$$+\frac{1}{2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\|H_z^{n+1}\|^2-\|H_z^n\|^2) \tag{5.4.7}$$
$$+\frac{1}{2}(\|P_x^{n+1}\|^2-\|P_x^n\|^2+\|P_y^{n+1}\|^2-\|P_y^n\|^2)$$
$$+(P_x^{n+1}-P_x^n,\tau_0^\alpha\tilde{\partial}_t^a P_x^{n+1})+(P_y^{n+1}-P_y^n,\tau_0^\alpha\tilde{\partial}_t^a P_y^{n+1})\le 0. \tag{5.4.8}$$

Taking the summation of equation (5.4.8) for n+1 level, we have

$$\frac{1}{2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(\|E_x^{n+1}\|^2-\|E_x^0\|^2+\|E_y^{n+1}\|^2-\|E_y^0\|^2)$$
$$+\frac{1}{2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\|H_z^{n+1}\|^2-\|H_z^0\|^2)+\frac{1}{2}(\|P_x^{n+1}\|^2-\|P_x^0\|^2+\|P_y^{n+1}\|^2-\|P_y^0\|^2)$$
$$+\sum_{k=1}^{n+1}\Big[(P_x^k-P_x^{k-1},\tau_0^\alpha\tilde{\partial}_t^a P_x^k)+(P_y^k-P_y^{k-1},\tau_0^\alpha\tilde{\partial}_t^a P_y^k)\Big]\le 0. \tag{5.4.9}$$

We estimate the last two term in equation (5.4.9). By (5.3.2), they can be directly written as

$$\begin{aligned}
&\sum_{k=1}^{n+1}(P^k - P^{k-1}, \tau_0^\alpha \tilde{\partial}_t^a P^k) \\
&\quad = \frac{1}{\Gamma(2-\alpha)}(\frac{\tau_0}{\tau})^\alpha \sum_{k=1}^{n+1}(\sum_{j=1}^{k} b_{k-j}(P^j - P^{j-1}), P^k - P^{k-1}) \\
&\quad = \frac{1}{\Gamma(2-\alpha)}(\frac{\tau_0}{\tau})^\alpha \sum_{k=1}^{n+1}(\sum_{j=1}^{k} w_{k-j}(P^j - P^{j-1}), P^k - P^{k-1}) \\
&\qquad + \frac{1}{\Gamma(2-\alpha)}(\frac{\tau_0}{\tau})^\alpha \sum_{k=1}^{n+1} \frac{b_0}{2}(P^k - P^{k-1}), P^k - P^{k-1}), \qquad (5.4.10)
\end{aligned}$$

where $w_0 = b_0/2$, $w_i = b_i$, for $i > 0$.

By Lemma 5.4.2, from (5.4.10) we have that

$$\sum_{k=1}^{n+1}(P^k - P^{k-1}, \tau_0^\alpha \tilde{\partial}_t^a P^k) \geq \frac{1}{2\Gamma(2-\alpha)}(\frac{\tau_0}{\tau})^\alpha \sum_{k=1}^{n+1}(P^k - P^{k-1}, P^k - P^{k-1}) \quad (5.4.11)$$

Then from equation (5.4.9), we have the result (5.4.1). □

For analysis of the convergence, we define: $\mathcal{E}^n_{w_{\alpha,\beta}} = E_w(x_\alpha, y_\beta, t^n)$ - $E^n_{w_{\alpha\beta}}$, $\mathcal{H}^n_{z_{\alpha,\beta}} = H_z(x_\alpha, y_\beta, t^n)$ - $H^n_{z_{\alpha\beta}}$ and $\mathcal{P}^n_{w_{\alpha,\beta}} = P_w(x_\alpha, y_\beta, t^n)$ - $P^n_{w_{\alpha\beta}}$, where $E_w(x_\alpha, y_\beta, t^n)$, $H_z(x_\alpha, y_\beta, t^n)$ and $P_w(x_\alpha, y_\beta, t^n)$ are exact solutions at point (x_α, y_β, t^n), $E^n_{w_{\alpha\beta}}$, $H^n_{z_{\alpha\beta}}$, $P^n_{w_{\alpha\beta}}$ are numerical solutions and w can be x or y.

Theorem 5.4.2. *(Convergence) Let $E_x(x,y,t)$, $E_y(x,y,t)$, $H_z(x,y,t)$ be the exact solutions of system (5.2.5)-(5.2.8) and smooth enough. The numerical solutions of scheme(5.3.3)-(5.3.8) are E^n_x, E^n_y and H^n_z for $n \geq 0$. Then for any fixed time*

$T > 0$, *there is a positive constant* C, *such that*

$$\max_{0\le n\le N}\{\|\bar{\epsilon}[\boldsymbol{E}(t^n)-\boldsymbol{E}^n]\|^2+\|\bar{\mu}[H_z(t^n)-H_z^n]\|^2+\|[\boldsymbol{P}(t^n)-\boldsymbol{P}^n]\|^2$$
$$\le (\|\bar{\epsilon}[\boldsymbol{E}(t^0)-\boldsymbol{E}^0]\|^2+\|\bar{\mu}[H_z(t^0)-H_z^0]\|^2+\|[\boldsymbol{P}(t^0)-\boldsymbol{P}^0]\|^2)$$
$$+C(\Delta t+\Delta x^2+\Delta y^2)^2, \tag{5.4.12}$$

and

$$\max_{0\le n\le N}\{\|\bar{\epsilon}[\delta_t\boldsymbol{E}(t^{n+\frac{1}{2}})-\delta_t\boldsymbol{E}^{n+\frac{1}{2}}]\|^2+\|\bar{\mu}[\delta_t H_z(t^{n+\frac{1}{2}})-\delta_t H_z^{n+\frac{1}{2}}]\|^2$$
$$+\|[\delta_t\boldsymbol{P}(t^{n+\frac{1}{2}})-\delta_t\boldsymbol{P}^{n+\frac{1}{2}}]\|^2$$
$$\le (\|\bar{\epsilon}[\delta_t\boldsymbol{E}(t^{\frac{1}{2}})-\delta_t\boldsymbol{E}^{\frac{1}{2}}]\|^2+\|\bar{\mu}[\delta_t H_z(t^{\frac{1}{2}})-\delta_t H_z^{\frac{1}{2}}]\|^2+\|[\delta_t\boldsymbol{P}(t^{\frac{1}{2}})-\delta_t\boldsymbol{P}^{\frac{1}{2}}]\|^2)$$
$$+C(\Delta t+\Delta x^2+\Delta y^2)^2, \tag{5.4.13}$$

where $\bar{\epsilon}=\epsilon_0\sqrt{\epsilon_\infty}\sqrt{\epsilon_0-\epsilon_\infty}$ *and* $\bar{\mu}=\sqrt{\epsilon_0\mu_0}\sqrt{\epsilon_0-\epsilon_\infty}$.

Proof. Define $\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$ as:

$$\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{2}(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}})+\frac{\Delta t}{\mu}(\delta_y\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+\delta_x\mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}).$$

The error equations of the scheme (5.3.3)-(5.3.8) can be written as

Stage 1:

$$\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}^{n+1}_{x_{i+\frac{1}{2},j}}+\mathcal{P}^{n+1}_{x_{i+\frac{1}{2},j}}=\epsilon_0(\epsilon_s-\epsilon_\infty)\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}}+\xi^{n+1}_{1_{i+\frac{1}{2},j}}, \tag{5.4.14}$$
$$\epsilon_0\epsilon_\infty\frac{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}}-\mathcal{E}^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t}=\delta_y\mathcal{H}^*_{z_{i+\frac{1}{2},j}}-\frac{\mathcal{P}^{n+1}_{x_{i+\frac{1}{2},j}}-\mathcal{P}^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t}+\xi^{n+1}_{2_{i+\frac{1}{2},j}}, \tag{5.4.15}$$
$$\mu_0\frac{\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{H}^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t}=\delta_y\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}+\xi^{n+1}_{3_{i+\frac{1}{2},j+\frac{1}{2}}}. \tag{5.4.16}$$

Stage 2:

$$\tau_0^\alpha \tilde{\partial}_t^a (P_{y_{i,j+\frac{1}{2}}}^{n+1}) + P_{y_{i,j+\frac{1}{2}}}^{n+1} = \epsilon_0(\epsilon_s - \epsilon_\infty) E_{y_{i,j+\frac{1}{2}}}^{n+1} + \xi_{4_{i,j+\frac{1}{2}}}^{n+1}, \tag{5.4.17}$$

$$\epsilon_0\epsilon_\infty \frac{\mathcal{E}_{y_{i,j+\frac{1}{2}}}^{n+1} - \mathcal{E}_{y_{i,j+\frac{1}{2}}}^{n}}{\Delta t} = -\delta_x \mathcal{H}_{z_{i,j+\frac{1}{2}}}^{n+1} - \frac{\mathcal{P}_{y_{i,j+\frac{1}{2}}}^{n+1} - \mathcal{P}_{y_{i,j+\frac{1}{2}}}^{n}}{\Delta t} + \xi_{5_{i,j+\frac{1}{2}}}^{n+1}, \tag{5.4.18}$$

$$\mu_0 \frac{\mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} - \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*}}{\Delta t} = -\delta_x \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + \xi_{6_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1}. \tag{5.4.19}$$

Substituting the intermediate variables and exact solutions into the scheme (5.3.3)-(5.3.8), after computation, we have the truncation errors

$$\max(\xi_1, \xi_2, \cdots, \xi_6) \le C(\Delta t + \Delta x^2 + \Delta y^2). \tag{5.4.20}$$

Multiplying (5.4.15), (5.4.16) with $\mathcal{E}_x^{n+1}$, $\mathcal{H}_z^*$, respectively, and adding them together, it holds that

$$\begin{aligned}
&\epsilon_0\epsilon_\infty(\mathcal{E}_x^{n+1} - \mathcal{E}_x^n, \mathcal{E}_x^{n+1}) + \epsilon_0\epsilon_\infty(\mathcal{P}_x^{n+1} - \mathcal{P}_x^n, \mathcal{E}_x^{n+1}) + \mu_0(\mathcal{H}_z^* - \mathcal{H}_z^n, \mathcal{H}_z^*)\\
&= \Delta t(\mathcal{E}^{n+1}, \xi_2^{n+1}) + \Delta t(\mathcal{H}_z^*, \xi_3^{n+1}),
\end{aligned} \tag{5.4.21}$$

Combining equation (5.4.21) with equation (5.4.14), we get

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s - \epsilon_\infty)(\mathcal{E}_x^{n+1} - \mathcal{E}_x^n, \mathcal{E}_x^{n+1}) + \mu_0\epsilon_0(\epsilon_s - \epsilon_\infty)(\mathcal{H}_z^* - \mathcal{H}_z^n, \mathcal{H}_z^*)\\
&\quad +(\mathcal{P}_x^{n+1} - \mathcal{P}_x^n, \mathcal{P}_x^{n+1}) + (\mathcal{P}_x^{n+1} - \mathcal{P}_x^n, \tau_0^\alpha \tilde{\partial}_t^a \mathcal{P}_x^{n+1})\\
&= \Delta t\epsilon_0(\epsilon_s - \epsilon_\infty)(\mathcal{E}_x^{n+1}, \xi_2^{n+1}) + \Delta t\epsilon_0(\epsilon_s - \epsilon_\infty)(\mathcal{H}_z^*, \xi_3^{n+1}) + \Delta t(\mathcal{P}_x^{n+1}, \xi_1^{n+1}).
\end{aligned} \tag{5.4.22}$$

Using the inequality shown in (5.4.4), we have the following inequality from the equation (5.4.22):

$$\begin{aligned}
&\frac{1}{2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(\|\mathcal{E}_x^{n+1}\|^2-\|\mathcal{E}_x^n\|^2)+\frac{1}{2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\|\mathcal{H}_z^*\|^2-\|\mathcal{H}_z^n\|^2)\\
&\quad+\frac{1}{2}(\|\mathcal{P}_x^{n+1}\|^2-\|\mathcal{P}_x^n\|^2)+(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1}) \qquad (5.4.23)\\
&\quad\le\Delta t\epsilon_0(\epsilon_s-\epsilon_\infty)(\mathcal{E}_x^{n+1},\xi_2^{n+1})+\Delta t\epsilon_0(\epsilon_s-\epsilon_\infty)(\mathcal{H}_z^*,\xi_3^{n+1})+\Delta t(\mathcal{P}_x^{n+1},\xi_1^{n+1}).
\end{aligned}$$

It can be written as

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)(\|\mathcal{E}_x^{n+1}\|^2-\|\mathcal{E}_x^n\|^2)+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)(\|\mathcal{H}_z^*\|^2-\|\mathcal{H}_z^n\|^2)\\
&\quad+(\|\mathcal{P}_x^{n+1}\|^2-\|\mathcal{P}_x^n\|^2)+2(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1}) \qquad (5.4.24)\\
&\le 2\Delta t\epsilon_0(\epsilon_s-\epsilon_\infty)(\mathcal{E}_x^{n+1},\xi_2^{n+1})+2\Delta t\epsilon_0(\epsilon_s-\epsilon_\infty)(\mathcal{H}_z^*,\xi_3^{n+1})+2\Delta t(\mathcal{P}_x^{n+1},\xi_1^{n+1}).
\end{aligned}$$

By the Schwartz inequality,

$$\begin{aligned}
&(1-\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+(1-\Delta t)\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^*\|^2\\
&\quad+(1-\Delta t)\|\mathcal{P}_x^{n+1}\|^2+2(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1})\\
&\le\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2\\
&\quad+\|\mathcal{P}_x^n\|^2+C_1\Delta t(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.25)
\end{aligned}$$

Dividing both sides of (5.4.25) by $(1-\Delta t)$, we have

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^*\|^2\\
&\quad+\|\mathcal{P}_x^{n+1}\|^2+\frac{2}{1-\Delta t}(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1})\\
&\le\frac{1}{1-\Delta t}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+\frac{1}{1-\Delta t}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2\\
&\quad+\frac{1}{1-\Delta t}\|\mathcal{P}_x^n\|^2+C_1\frac{\Delta t}{1-\Delta t}(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.26)
\end{aligned}$$

Similarly, by the error equations in Stage 2, we obtain that

$$\begin{aligned}
&(1-\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^{n+1}\|^2+(1-\Delta t)\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^{n+1}\|^2\\
&\quad+(1-\Delta t)\|\mathcal{P}_y^{n+1}\|^2+2(\mathcal{P}_y^{n+1}-\mathcal{P}_y^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_y^{n+1})\\
&\le \epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^n\|^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^*\|^2\\
&\quad+\|\mathcal{P}_y^n\|^2+C_1\Delta t(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.27)
\end{aligned}$$

Eliminating the intermediate variables $\mathcal{H}_z^*$ in inequalities (5.4.26) and (5.4.27), we obtain

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+(1-\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^{n+1}\|^2\\
&\quad+(1-\Delta t)\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^{n+1}\|^2+\|\mathcal{P}_x^{n+1}\|^2+(1-\Delta t)\|\mathcal{P}_y^{n+1}\|^2\\
&\quad+\frac{2}{1-\Delta t}(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1})+2(\mathcal{P}_y^{n+1}-\mathcal{P}_y^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_y^{n+1})\\
&\le \frac{1}{1-\Delta t}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^n\|^2\\
&\quad+\frac{1}{1-\Delta t}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2+\frac{1}{1-\Delta t}\|\mathcal{P}_x^n\|^2+\|\mathcal{P}_y^n\|^2\\
&\quad+C_1(\frac{\Delta t}{1-\Delta t}+\Delta t)(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.28)
\end{aligned}$$

By (5.4.28), we can have that

$$
\begin{aligned}
&(1-\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+(1-\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^{n+1}\|^2\\
&\quad+(1-\Delta t)\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^{n+1}\|^2+(1-\Delta t)\|\mathcal{P}_x^{n+1}\|^2+(1-\Delta t)\|\mathcal{P}_y^{n+1}\|^2\\
&\quad+\frac{2}{1-\Delta t}(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1})+2(\mathcal{P}_y^{n+1}-\mathcal{P}_y^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_y^{n+1})\\
&\le\frac{1}{1-\Delta t}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+\frac{1}{1-\Delta t}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^n\|^2\\
&\quad+\frac{1}{1-\Delta t}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2+\frac{1}{1-\Delta t}\|\mathcal{P}_x^n\|^2+\frac{1}{1-\Delta t}\|\mathcal{P}_y^n\|^2\\
&\quad+C_1(\frac{\Delta t}{1-\Delta t}+\Delta t)(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.29)
\end{aligned}
$$

Dividing (5.4.29) by $1-\Delta t$ on both sides, we get

$$
\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^{n+1}\|^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^{n+1}\|^2\\
&\quad+\|\mathcal{P}_x^{n+1}\|^2+\|\mathcal{P}_y^{n+1}\|^2+\frac{2}{(1-\Delta t)^2}(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1})\\
&\quad+\frac{2}{1-\Delta t}(\mathcal{P}_y^{n+1}-\mathcal{P}_y^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_y^{n+1})\\
&\le\frac{1}{(1-\Delta t)^2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+\frac{1}{(1-\Delta t)^2}\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^n\|^2\\
&\quad+\frac{1}{(1-\Delta t)^2}\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2+\frac{1}{(1-\Delta t)^2}\|\mathcal{P}_x^n\|^2+\frac{1}{(1-\Delta t)^2}\|\mathcal{P}_y^n\|^2\\
&\quad+C_2\Delta t(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.30)
\end{aligned}
$$

Let $\Delta t\le\Delta_0<1$, it holds that $\frac{1}{(1-\Delta t)^2}\le 1+C_3\Delta t$, from inequality (5.4.29), we have that

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^{n+1}\|^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^{n+1}\|^2\\
&\quad+\|\mathcal{P}_x^{n+1}\|^2+\|\mathcal{P}_y^{n+1}\|^2+\frac{2}{(1-\Delta t)^2}(\mathcal{P}_x^{n+1}-\mathcal{P}_x^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{n+1})\\
&\quad+\frac{2}{1-\Delta t}(\mathcal{P}_y^{n+1}-\mathcal{P}_y^n,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_y^{n+1})\\
&\le(1+C_3\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+(1+C_3\Delta t)\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^n\|^2\\
&\quad+(1+C_3\Delta t)\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2+(1+C_3\Delta t)\|\mathcal{P}_x^n\|^2+(1+C_3\Delta t)\|\mathcal{P}_y^n\|^2\\
&\quad+C\Delta t(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.31)
\end{aligned}$$

Summing (5.4.31) from time level n to 0 we can obtain

$$\begin{aligned}
&\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^{n+1}\|^2+\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^{n+1}\|^2+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^{n+1}\|^2\\
&\quad+\|\mathcal{P}_x^{n+1}\|^2+\|\mathcal{P}_y^{n+1}\|^2+\frac{2}{(1-\Delta t)^2}\sum_{i=1}^{n}(\mathcal{P}_x^{i+1}-\mathcal{P}_x^i,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_x^{i+1})\\
&\quad+\frac{2}{1-\Delta t}\sum_{i=0}^{n}(\mathcal{P}_y^{i+1}-\mathcal{P}_y^i,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}_y^{i+1})\\
&\le C_3\Delta t\sum_{i=0}^{n}\left[\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_x^n\|^2+\epsilon_0^2\epsilon_\infty(\epsilon_s-\epsilon_\infty)\|\mathcal{E}_y^n\|^2\right.\\
&\quad\left.+\mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\|\mathcal{H}_z^n\|^2+\|\mathcal{P}_x^n\|^2+\|\mathcal{P}_y^n\|^2\right]\\
&\quad+Cn\Delta t(\Delta t+\Delta x^2+\Delta y^2)^2. \qquad (5.4.32)
\end{aligned}$$

Noticing $\Delta t=T/N$, using discrete Gronwall's inequality and using the estimation of $\sum_{i=0}^{n}(\mathcal{P}^{i+1}-\mathcal{P}^i,\tau_0^\alpha\tilde{\partial}_t^a\mathcal{P}^{i+1})$ as shown in the proof of Theorem 5.4.2, we can get the result (5.4.13). We can similarly get the proof of (5.4.14). □

5.5 Numerical Experiments

Consider Maxwell's equations in Cole-Cole medium. Take $\epsilon_0\epsilon_\infty = \mu_0 = \tau_0 = \epsilon_0(\epsilon_s - \epsilon_\infty) = 1$ and set T=1. The exact solutions are

$$H_z(x,y,t) = -[\frac{2}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)}t^{3-\alpha} + \frac{1}{3}t^3] \cdot 2\pi cos\pi x cos\pi y,$$

$$\mathbf{P}(x,y,t) = t^2\mathbf{w}(x,y),$$

$$\mathbf{E}(x,y,t) = [\frac{2}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)}t^{2-\alpha} + t^2]\mathbf{w}(x,y),$$

where

$$\mathbf{w}(x,y) = (-cos\pi x sin\pi y \quad sin\pi x cos\pi y)^T.$$

The Maxwell's equations in the Cole-Cole media satisfy

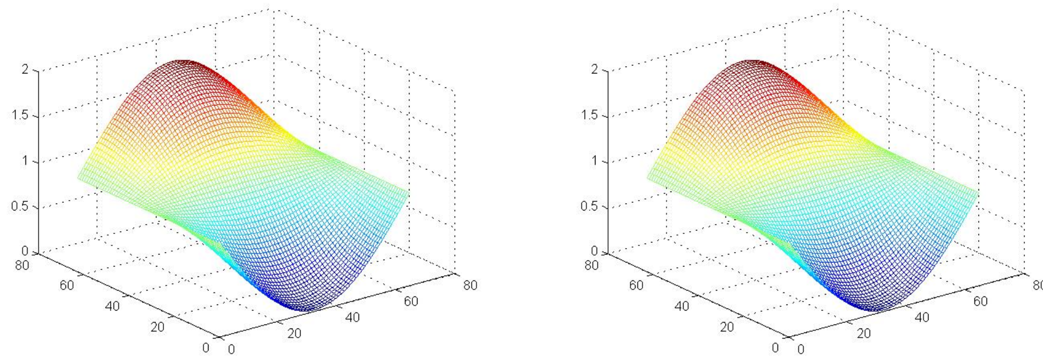
$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} - \nabla \times H = \mathbf{f}(x,y,t),$$

where

$$\begin{aligned}\mathbf{f}(x,y,t) = \{&\frac{2}{\Gamma(1-\alpha)(1-\alpha)}t^{1-\alpha} + 4t \\ &+ 2\pi^2[\frac{2}{\Gamma(1-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)} + \frac{1}{3}t^3]\}\mathbf{w}(x,y),\end{aligned}$$

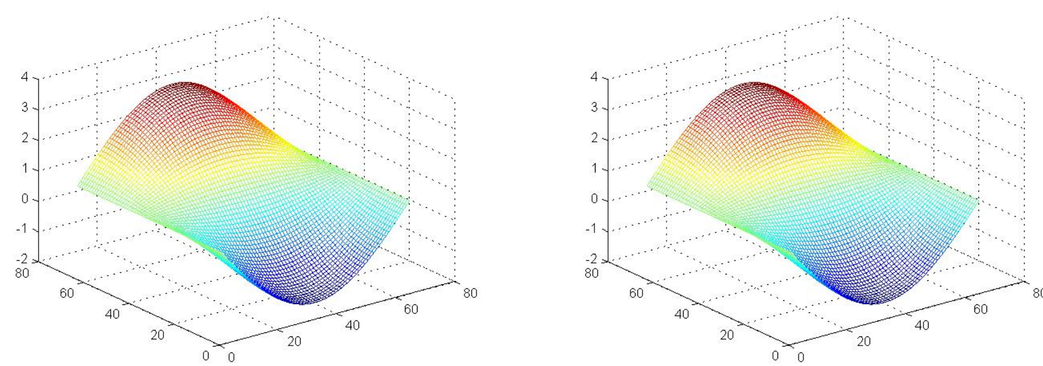
and (5.2.3), (5.2.6).

We first fix τ=0.0005, α=0.5 and h=1/64, and compute the P_x, E_x, H_z at $T=1$. The Figures 5.1-5.3 show the pictures of numerical results and exact solutions. From the figures, we can see that numerical results and real data fit very well.



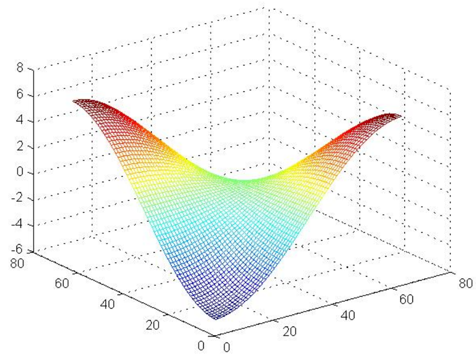
(a) *(b)*

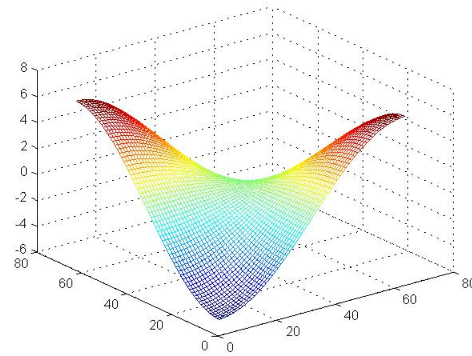
Figure 5.1: Numerical solution P_x *(a) and exact solution* P_x *(b) with fixed* $\tau = 0.0005$, α=0.5 *and* h=1/64 *at* $T = 1$.



(a) *(b)*

Figure 5.2: Numerical solution E_x *(a) and exact solution* E_x *(b) with fixed* $\tau = 0.0005$, α=0.5 *and* h=1/64 *at* $T = 1$.





(a) *(b)*

Figure 5.3: Numerical solution H_z *(a) and exact solution* H_z *(b) with fixed* τ = 0.0005 , α=0.5 *and* h=1/64 *at* T = 1.

In Table 5.1 and Table 5.2, we test the accuracy of the scheme in spatial step with different N where Δt=$1/(N^2)$ and Δx=Δy= $1/N$ in spatial step. These two tables show that the accuracies in L^2 norm all the test items are of order two, which confirms that our scheme is second order in space. The tests for time step are given in Table 5.3 and Table 5.4 where $\Delta x = \Delta y = \frac{1}{256}$ at T = 1, the convergenc rate of the scheme is first order in time step as the two tables showed.

Table 5.1: L^2-errors of the Euler-based S-FDTD scheme with $\Delta x = \Delta y = 1/N$, Δt=$1/(N^2)$ and α=0.5 at T = 1.

N	H error	Rate	E error	Rate	P error	Rate	$Total$ error	Rate
4	0.193180	-	0.144599	-	0.053701	-	0.247207	-
8	0.048593	1.9911	0.037211	1.9583	0.013470	1.9952	0.062669	1.9799
16	0.012146	2.0003	0.009362	1.9908	0.003371	1.9985	0.015702	1.9968
32	0.003034	2.0011	0.002344	1.9978	0.000843	1.9996	0.003926	1.9998
64	0.000758	2.0010	0.000586	2.000	0.000211	1.9983	0.000981	2.0001

Table 5.2: L^2-errors of the Euler-based S-FDTD scheme with $\Delta x = \Delta y = 1/N$, Δt=$1/(N^2)$ and α=0.7 at T = 1.

N	H error	Rate	E error	Rate	P error	Rate	$Total$ error	Rate
4	0.202471	-	0.170287	-	0.057857	-	0.270813	-
8	0.050304	2.0090	0.044225	1.9450	0.014203	2.0263	0.068469	1.9838
16	0.012455	2.0139	0.011156	1.9870	0.003528	2.0093	0.017089	2.0024
32	0.003094	2.0092	0.002795	1.9969	0.000880	2.0033	0.004261	2.0038
64	0.000770	2.0065	0.000699	1.9995	0.000220	2.0000	0.001063	2.0031

Table 5.3: L^2-errors of the Euler-based S-FDTD scheme in time step with $\Delta x = \Delta y = 1/256$ and α=0.5 at T = 1.

Δt	H error	Rate	E error	Rate	P error	Rate	$Total$ error	Rate
1/4	0.577035	-	0.570517	-	0.232273	-	0.844044	-
1/8	0.322167	0.8409	0.316648	0.8494	0.117495	0.9832	0.466758	0.8546
1/16	0.170532	0.9178	0.165017	0.9403	0.059714	0.9765	0.244698	0.9317
1/32	0.087768	0.9583	0.083979	0.9745	0.030123	0.9872	0.125152	0.9673
1/64	0.044524	0.9791	0.042342	0.9879	0.015150	0.9915	0.063283	0.9838

Table 5.4: L^2-errors of the Euler-based S-FDTD scheme in time step with $\Delta x = \Delta y = 1/256$ and α=0.7 at T = 1.

Δt	H error	Rate	E error	Rate	P error	Rate	$Total$ error	Rate
1/4	0.582597	-	0.628539	-	0.270187	-	0.898600	-
1/8	0.326754	0.8343	0.356367	0.8186	0.122382	1.1426	0.498741	0.8494
1/16	0.171682	0.9285	0.187778	0.9243	0.060249	1.0224	0.261468	0.9317
1/32	0.087340	0.9750	0.096134	0.9659	0.030116	1.0004	0.133330	0.9716
1/64	0.043804	0.9956	0.048635	0.9831	0.015139	0.9923	0.067181	0.9889

6 Conclusion

In this thesis, firstly, we proposed the spatial fourth-order energy-conserved splitting finite-difference time-domain scheme (i.e. EC-S-FDTD-(2,4)) for solving Maxwell's equations. One important issue is to construct the numerical boundary difference schemes to be energy conservative and high-order relative to the interior difference schemes. The one-sided differences and extrapolation/interpolation numerical boundary schemes normally fail to satisfy energy conservations. At each stage, we proposed the energy-conserved and fourth-order accurate FDTD operators for both the near boundary nodes and the strict interior nodes. The developed scheme was proved to be energy-conserved, unconditionally stable, and to have fourth-order convergence in space step and second-order convergence in time step. The convergence of numerical divergence-free was also analyzed. Numerical experiments showed that the developed scheme conserves energy and is of fourth-order accuracy in space and second-order in time.

Secondly, the aim of Chapter 3 has been to develop and analyze energy-conserved and time and spatial high-order S-FDTD techniques for solving Maxwell's equations. This goal has been fulfilled by proposing and analyzing a new and novel time and spatial fourth-order S-FDTD scheme. As proved, the most important feature is that the proposed scheme satisfies the energy conservations in the discrete form and in the discrete variation forms and it has the optimal fourth-order accuracy in both time and space in the discrete L_2-norm and the super-convergence of forth-order in the discrete H^1-norm. We also proved another important prominent quality of the scheme that its convergence of the divergence-free is of fourth-order as well. Numerical experiments were presented to confirm the theoretical results.

Thirdly, in Chapter 4, we focused on the development and analysis of efficient high-order energy-conserved splitting FDTD schemes for three-dimensional Maxwell's equations. The original Maxwell's equations in three dimensions were decomposed into twelve one-dimensional splitting equations in each time interval. Based on the splitting of the operator equations and Yee's staggered space-time grids, the spatial fourth-order energy-conserved splitting FDTD scheme was developed. We proposed to construct the spatial fourth-order near boundary differences over the near boundary nodes by using the PEC boundary conditions, original equations and Taylor's expansion, which ensured the each-stage schemes to preserve the conservations of energies and to have fourth-order accuracy. We obtained the optimal-order error estimates of spatial fourth-order and time second order in the discrete L_2-norm for the approximations of the electric and magnetic fields and the same ordered super-convergence in the discrete H^1-norm.

Finally, the Euler-based S-FDTD scheme was developed to solve Maxwell's equations in Cole-Cole dispersive medium. The computation of the propagation of electromagnetic waves in dispersive media is a very important application. Due to the fractional time derivative term in the models, it is difficult and important to develop the efficient splitting FDTD scheme. Our developed Euler-based S-FDTD scheme is a two-stage scheme. We proved the unconditional stability, and analyzed theoretically the convergence of the scheme and obtained the optimal-order error estimates in the discrete L_2-norm. Numerical experiments showed the performance of the scheme.

The high-order S-FDTD schemes developed in this study can be extended to solve other type wave problems and these schemes can be used in applications in electromagnetic industry.

Bibliography

[1] Abdullah, H.H., Elsadek, H.A., ElDeeb, H.E., Bagherzadeh, N., *Fractional Derivatives Based Scheme for FDTD Modeling of nth-Order Cole-Cole Dispersive Media*, IEEE Antenn. Wireless Propag. Lett., 11 (2012), 281-284.

[2] Almeida, A.M., *Hamiltonian systems: Chaos and quantization*, Cambridge monographs on mathematical physics., (1992), Cambridge Univ. Press.

[3] Ammari, H., Kang, H. and Kim, E., *Approximate boundary conditions for patch antennas mounted on thin dielectric layers*, Commun. Comput., 1(2006), 1076-1095.

[4] Banks, H.T., Bokil, V.A., and Gibson, N.L., *Analysis of stability and dispersion in a finite element method for Debye and Lorentz media*, Numer. Methods Partial Differential Equations, 25 (2009), 885-917.

[5] Botchev, M.A., Faragó, I., and Horáth, R., *Application of the Operator Splitting to the Maxwell Equations with the Source Term*, Appl. Numer. Math., 59 (2009), 522-541.

[6] Brekhovskikh, L.M., *Aves in Layered Media, 2nd edn*, Academic Press, New York, 1980.

[7] Chen, W.B., Li, X.J. and Liang, D., *Energy-conserved splitting FDTD methods for Maxwell's equations*, Numer. Math., 108 (445-485), 2008.

[8] Chen, W.B., Li, X.J., and Liang, D., *Energy-conserved splitting finite-difference time-domian methods for Maxwell's equations in three dimensions*, Siam J. Numer. Anal., 48 (2010), 1530-1554.

[9] Chen, W.B., Li, X.J., and Liang, D., *Symmetric energy-conserved splitting FDTD scheme for the Maxwell's equations*, Commun. Comput. Phys., 6 (2009), 804-825.

[10] Cole, K.S.; Cole, R.H., *Dispersion and Absorption in Dielectrics - I Alternating Current Characteristics*, J. Chem. Phys., 9 (1941), 341-352.

[11] Crandall, C.G., and Majda, A., *The method of fractional steps for conservation laws*, Numer. Math., 34 (1980), 285-314.

[12] Dai,W.,Raja, N., *A new ADI scheme for solving three-dimensional parabolic equations with first-order derivatives and variable coefficients*, J. Comput. Anal. Appl., 122 (1995), 223-250.

[13] Davidson, D., *Computational Electromagnetics for RF and Microwave Engineering*, Cambridge University Press, 2005.

[14] Diamanti, N. and Giannopoulos, A., *Implementation of ADI-FDTD subgrids in ground penetrating radar FDTD models*, Journal of Applied Geophysics, 67 (2009), 309-317.

[15] Douglas, J.Jr. and Rachford, J.H.H., *On the numerical solutions of heat conduction problems in two and three space variables*, Trans. Amer. Math. Soc., 82(1956), 421-439.

[16] Douglas, J.Jr., Kim, S., *Improved accuracy for locally one-dimensional methods for parabolic equations*, Math. Models Methods Appl. Sci., 11(2001), 1563-1579.

[17] Du, C. and Liang, D., *An Efficient S-DDM Iterative Approach for Compressible Contamination Fluid Flows in Porous Media*, J. Comput. Phys., 229 (2010), 4501-4521.

[18] Eyges,L., *The Classical Electromagnetic Fields*, Addison-Wesley, Reading, 1972.

[19] Eyges,L., *The Classical Electromagnetic Fields*, 456, Interscience, New York, 1957.

[20] Fang, J., *Time domain finite difference computation for Maxwell's equations*, Ph.D. dissertation, Univ. California Berkeley, CA, 1989.

[21] Forest, E., Ruth, Ronald D., *Fourth-order symplectic integration*, Physica D, 43 (1990), 105-117.

[22] Fornberg, B., *Some numerical techniques for Maxwell's equations in different types of geometries* Topics in Computational Wave Propagation, Lecture notes in Computational Science and Engineering 31, Springer Verlag (2003), 265-299.

[23] Gao, L., and Liang, D., *New Energy-Conserved Identities and Superconvergence of the Symmetric EC-S-FDTD Scheme for Maxwell's Equations in 2D*, Commun. Comput. Phys., 11 (2012), 1673-1696.

[24] Gao, L., Zhang, B. and Liang, D., *The splitting-difference time-domain methods for Maxwell's equations in two dimensions*, J. Comput. Appl. Math., 205(2007), 207-230.

[25] Garcia, S.G., Lee, T.W. and Hagness, S.C., *On the accuracy of the ADI-FDTD method*, IEEE Antennas Wireless Compon. Lett., 13 (2002), 31-34.

[26] Ge, Y.B., Tian, Z.F. and Zhang, J., *An exponential high-order compact ADI method for 3D unsteady convection-diffusion problems*, Numerical. Methods for PDEs, 29 (2013), 186-205.

[27] Gedney, S., Liu, G., Roden, J.A., and Zhu, A., *Perfectly matched layer media with CFS for an unconditional stable ADI-FDTD method*, IEEE Trans. Antennas Propagation 49 (2001), 1554-1559.

[28] Holland, R., *Implicit three-dimensional finite differencing of Maxwell's equations*, IEEE rans. Nucl. Sci., 31 (1984), 1322-1326.

[29] Huang, Y., and Li, J. *Interior penalty discontinuous Galerkin method for Maxwells equations in cold plasma*, J. Sci. Comput., 41 (2009), 321-340.

[30] Huttunen, T. and Monk, P., *The use of plane waves to approximate wave propagation in anisotropic media*, J. Comp. Math., 25(2007), 350-367.

[31] Jiao, D. and Jin, J.M., *Time-domain finite-element modling of dispersive media*, IEEE Microw. Wirel. Compon. Lett., 11 (2001), 220-222.

[32] Jurgens, H.M. and Zingg, W., *Numerical solution of the time-domain Maxwell equations using high-accuracy finite difference methods*, SIAM J. Sci. Comput., 22 (2000), 1675-1696.

[33] Karaa, S. and Zhang, J., *High order ADI method for solving unsteady convection-diffusion problems*, J. Comput. Phys., 198(1) (2004), 1-9

[34] Karlsen, K.H. and Lie, K.A., *An unconditionally stable splitting scheme for a class of nonlinear parabolic equations*, IMA J. Numer. Anal., 19 (1999), 1-28.

[35] Kashiwa, T., Ohtomo, Y. and Fukai, I., *A finite-difference time-domain formulation for transient propagation in dispersive media associated with Cole-Cole's circular ARC law*, Microw. Opt. Technol. Lett., 3 (1990), 416-419.

[36] Lee, J. and Fornberg, B., *A split step approach for the 3-D Maxwell's equations*, J. Comput. Appl. Math., 158 (2003), 485-505.

[37] Leis, R., *Initial Boundary Value Problems in Mathematical Physics*, Wiley, New York, 1986.

[38] Li, J., *Unified analysis of leap-frog methods for solving time-domain Maxwells equations in dispersive media*, J. Sci. Comput., 47 (2011), 1-26.

[39] Li, J., and Chen, Y., *Analysis of a time-domain finite element method for 3-D Maxwells equations in dispersive media*, Comput. Methods Appl. Mech. Engrg., 195 (2006), 4220-4229.

[40] Li, J., and Zhang, Z., *Unified analysis of time domain mixed finite element methods for Maxwells equations in dispersive media*, J. Comput. Math., 28 (2010), 693-710.

[41] Li, J., Huang, Y.Q. and Lin, Y.P., *Develping finite element method for Maxwell's equations in a Cole-Cole dispersive medium*, SIAM J. Sci. Comput. 33 (2011), 3153-3174.

[42] Liang, D. and Yuan, Q., *The spatial fourth-order energy-conserved S-FDTD scheme for Maxwells equations*, J. Comput. Phys., 243(2013), 344-364.

[43] Lu, T., Zhang, P. and Cai, W., *Discontinuous Galerkin methods for dispersive and lossy Maxwell's equations and PML boundary conditions*, J. Comput. Phys., 200(2004), 549-580.

[44] Lu, Y., *Some techniques for computing wave propagation in optical waveguides*, Commun. Comput.Phys., 1(2006), 1056-1075.

[45] Luebbers, R.J. and Hunsberger, F., *FDTD for Nth-order dispersive media*, IEEE Trans. Antennas. Propag., 40 (1992), 1297-1301.

[46] McLean, W. and Thome, V., *Numerical solution of an evolution equation with a positive type memory term*, J. Austral. Math. Soc. Ser. B, 35 (1993), 23-70.

[47] Monk, P. and Suli, E., *A convergence analysis of Yee's scheme on nonuniform grids*, SIAM J. Numer. Anal., 31 (1994), 1056-1075.

[48] Morton, K.W. and Mayers, D.F., *Numerical Solution of Partial Differential Equations*, second ed., Cambridge University Press, NewYork, 2005.

[49] Namiki, T., *A new FDTD algorithm based on alternating direction implicit method*, IEEE Trans. Microw. Theory Tech., 47 (2003-2007), 1999.

[50] Namiki, T., *3-D ADI-FDTD method-unconditionally stable time-domain Maxwells equations*, IEEE Trans. Microw. Theory Tech., 48 (1743-1748), 2000 .

[51] Nachman, A., *A brief perspective on computational electromagnetics*, J. Comput. Phys., 129 (1996), 237-239.

[52] Nickisch, L.J. and Franke, P.M., *Finite-difference time-domain solution of Maxwell's equations for the dispersive ionosphere*, IEEE Antennas Propagat. Mag., 34 (1992), 33-39.

[53] Pani, A.K., Fairweather, G., and Fernandes, R.I., *Alternating direction implicit orthogonal spline collocation methods for an evolution equation with a positive-type memory term*, SIAM J. Numer. Anal., 46 (2008), 344-364.

[54] Peaceman, D. and Rachford, J.H.H., *The numerical solution of parabolic and elliptic difference*, J. SIAM 42(1955), 28-41.

[55] Petropoulos, P.G., *On the time-domain response of Cole-Cole dielectrics*, IEEE Trans. Antennas and Propagation, 53 (2005), 3741-3746.

[56] Rekanos, I.T. and Papadopoulos, T.G., *FDTD Modeling of Wave Propagation in ColeCCole Media With Multiple Relaxation Times*, IEEE ANTENNAS AND WIRELESS PROPAGATION LETTERS, 9 (2010), 67-69.

[57] Rekanos, I.T. and Papadopoulos, T.G., *An Auxiliary Differential Equation Method for FDTD Modeling of Wave Propagation in Cole-Cole Dispersive Media*, IEEE Trans. Antennas and Propagation, 58 (2010), 3666-3674.

[58] Schuster, J.W. and Luebbers, R.J., *An FDTD algorithm for transient propagation in biological tissue with a Cole-Cole dispersion relation*, IEEE Antennas and Propagation Society International Symposium, 4 (1998), 1988 - 1991.

[59] Singh, G., E. L. Tan and Z. N. Chen, *A Split-Step FDTD Method for 3-D Maxwell's Equations in General Anisotropic Media*, IEEE Trans. Antennas Propag., 58 (2010), 3647-3657.

[60] Strong, G., *On the construction and comparison of difference scheme*, SIAM J. Numer. Anal., 5 (1968), 506-517.

[61] Taflove, A. and Brodwin, M.E., *Numerical solution of steady-state electromagnetic scattering problems using the time-dependent Maxwell's equations*, IEEE Trans. Microwave Theory Tech., 23(1975), 623-630.

[62] Taflove, A. and Hagness, S., *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, second ed., Artech House, Boston, MA, 2000.

[63] Torres, F., Vaudon, P. and Jecko, B., *Application of fractional derivatives to the FDTD modeling of pulse propagation in a Cole-Cole dispersive medium*, Microwave and Optical Technol. Lett., 13 (1996), 300-304.

[64] Turkel, E. and Yefet, A., *Fourth order method for Maxwell equations on a staggered mesh*, Antennas and Propagation Society International Symposium, IEEE., 4 (1997), 2156-2159.

[65] Turkel, E., and Yefet, A., *On the construction of a high order difference scheme for complex domains in a Cartesian grid*, Appl. Numer. Math., 33 (2000), 113-124.

[66] Wang, B., and Zhang, Z., *Error analysis of a discontinuous Galerkin method for Maxwell equations in dispersive media*, J. Comput. Phys., 229 (2010), 8552-8563.

[67] Wang, S., Teixeira, F., Chen, J., *An iterative ADI-FDTD with reduced splitting error, IEEE Microwave Wireless* Comp. Lett., 15 (2005), 1531-1535.

[68] Welfert, B.D., *Analysis of iterated ADI-FDTD schemes for Maxwell curl equations*, J. Comput. Phys., 222 (2007), 9-27.

[69] Xie, Z., Chan, C.H. and Zhang, B., *An explicit fourth-order staggered finite-difference time-domain method for Maxwell's equations*, J. Comput. Appl. Math., 147 (2002), 75-98.

[70] Xie, Z., Chan, C.H. and Zhang, B., *An explicit fourth-order orthogonal curvilinear staggered-grid FDTD method for Maxwell's equations*, J. Comput. Phys., 175 (2002), 739-763.

[71] Yanenko, N.N, *The method of fractional steps: The solution of Problems of Mathematical Physics in Several Variables*, M. Holt, ed., Springer-Verlag, New York, 1971.

[72] Yee, K., *Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media*, IEEE Trans. Antennas and Propagation, 14(1966), 302-307.

[73] Yefet, A. and Turkel, E., *fourth order compact implicit method for the Maxwell equations with discontinuous coefficients*, App. Numer. Math., 33 (2000), 125-134.

[74] Yefet, A. and Petropoulos, P.G., *A non-dissipative staggered fourth-order accurate explicit finite difference scheme for the time-domain Maxwell's equation*, J. Comput. Phys., 168 (2001), 286-315.

[75] Yoshida, H., *Construction of higher order symplectic integrators*, Physics Letters A, 150 (1990), 262-268.

[76] Young, J.L., Gaitonde, D. and Shang, J.J.S., *Toward the construction of a fourth-order difference scheme for transient EM Wave simulation: staggered grid approach*, IEEE Trans. Antennas Propag. AP-45 (1997), 1573-1580.

[77] Yuan, Q., and Liang, D., *A New Multiple Sub-domain RS-HDMR Method and its Application to Tropospheric Alkane Photochemistry Model*, Int. J. Numer. Anal. Modeling, Ser. B, 2(2011), 73-90.

[78] Yuan, Q., and Liang, D., *The time and spatial high-order Energy-conserved S-FDTD scheme for Maxwell's equations*, Submitted (Jul, 2013) to Numer. Math.

[79] Yuan, Q., and Liang, D., *The Spatial High-order Energy-conserved Splitting FDTD Method for Maxwell's Equations in Three Dimensions*, Submitted (2013) to APPL. NUMER. MATH.

[80] Zhang, F. and Chen, Z., *Numerical dispersion analysis of the unconditionally stable ADI-FDTD method*, IEEE Trans. Microwave Theory Tech., 49 (2000), 1006-1009.

[81] Zhao, S. and Wei, G.W., *High-order FDTD methods via derivative matching for Maxwells equations with material interfaces*, J. Compu. Phys. Sci., 200 (2004), 60-103.

[82] Zheng, F., Chen, Z., Zhang, J., *Toward the development of a three-dimensional unconditionally stable finite-difference time-domian method*, IEEE Trans, Microw theory Tech., 48 (2000), 1550-1558.

[83] Zygiridis, T.T. and Tsiboukis, T.D., *Optimized three-dimensional FDTD discretizations of Maxwell's equations on Cartesian grids*, J. Comput. Phys., 226 (2007), 2372-2388.