# VARIATION OF CANONICAL HEIGHTS FOR SUBVARIETIES

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# Abstract

Let $X$ be a projective variety over a number field $K$, $\varphi : X \to X$ an endomorphism and $L \in \mathrm{Pic}(X)$ such that $\varphi^* L \cong L^{\otimes d}$.

If it is the case that $L$ is ample and $d \geq 2$ we say that $(\varphi, X, L)$ is a polarized (arithmetic) dynamical system. A single dynamical system is very well understood but much less is known for families. This is the focus of our work.

Call and Silverman in their seminal paper [CS93] built a canonical height $\hat{h}_\varphi : X \to \mathbb{R}_{\geq 0}$ satisfying

$$\hat{h}_\varphi(\varphi(x)) = d\hat{h}_\varphi(x) \text{ and } \hat{h}_\varphi(x) = h(x) + O(1),$$

where $h$ is a Weil height on $X$ with respect to $L$.

Let $(f, X, L)$ be a family of polarized dynamical systems over a curve $B$, and let $P : B \to X$ where we denote $P(t)$ by $P_t$. On all but finitely many fibres $(f_t, X_t, L_t)$ we may construct a canonical height $\widehat{h}_{f_t}$. Ingram [Ing21] proved

**Theorem 0.1.** Let $(f, X, L)$ be a family of polarized dynamical systems over a rational curve $B$, defined over $\mathbb{Q}$. Fix a degree one Weil height $h_B$ on $B$.

Then

$$\widehat{h}_{f_t}(P_t) = \widehat{h}_f(P) h_B(t) + O(h_B(t)^{\frac{1}{2}}).$$

Moriwaki [Mor00] introduced a height function $h^{\mathrm{Mor}}$ on all subvarieties $V \subset X$ that generalizes Weil heights for points. And as for points, we can define a canonical height for subvarieties. With all this at hand, we prove an analogous result to that of Ingram's. Concretely, we prove

**Theorem 0.2.** Let $(f, X, L)$ be a family of polarized dynamical systems over a rational curve $B$ and $V : B \to X$ a family of subvarieties of dimension $p$. Assume this dynamical system is defined over a number field $K$ and fix a degree one Weil height $h_B$ on $B$. Then

$$\hat{h}_{f_t}(V_t) = \hat{h}_f(V) h_B(t) + O(h_B(t)^{1 - \frac{1}{p+2}}),$$

where the implied constant depends only on the family, $\deg V$ and $h^{\mathrm{Mor}}(V)$.

# Acknowledgements

It is challenging to compile a complete list of everyone and everything that made this possible.

I am grateful for the support of my family and friends throughout the process, as well as the fruitful conversations, kind guidance, and enlightening lectures of the mathematicians who helped me reach the finish line. Lastly, I am grateful to belong to a generation that grew up in a world that now seems ancient and that can leverage technological advances for assistance rather than conceiving them as absolute truths.

***To all grandparents to ever live.***

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# Notation

| | |
|---|---|
| $\mathscr{L}^1(\mu)$ | space of functions that are integrable with respect to $\mu$. |
| $\mathscr{L}^1_{\mathrm{loc}}(\mu)$ | space of functions that are locally integrable with respect to $\mu$. |
| $\mathscr{L}^1(X)$ | space of functions that are integrable with respect to the Lebesgue measure. |
| $\mathscr{L}^1_{\mathrm{loc}}(X)$ | space of functions that are locally integrable with respect to the Lebesgue measure. |
| $\mathscr{C}(X)$ | space of continuous functions on $X$ |
| $\mathscr{C}_c(X)$ | space of continuous functions on $X$ with compact support |
| $\mathscr{C}^k(X)$ | space of functions of class $\mathscr{C}^k$ on $X$ |
| $\mathscr{C}^k_c(X)$ | space of functions of class $\mathscr{C}^k$ on $X$ with compact support |
| $\mathscr{C}^\infty(X)$ | space of functions of class $\mathscr{C}^\infty$ on $X$ |
| $\mathscr{C}^\infty_c(X)$ | space of functions of class $\mathscr{C}^\infty$ on $X$ with compact support |
| $\Omega^p_{[k]}(X)$ | space of $p$-forms of class $\mathscr{C}^k$ on $X$ |
| $\Omega^p(X)$ | space of $p$-forms of class $\mathscr{C}^\infty$ on $X$ |
| $\Omega^{p,q}_{[k]}(X)$ | space of $(p,q)$-forms of class $\mathscr{C}^k$ on $X$ |
| $\Omega^{p,q}(X)$ | space of $(p,q)$-forms of class $\mathscr{C}^\infty$ on $X$ |
| $\mathscr{D}^p_{[k]}(X)$ | space of $p$-forms of class $\mathscr{C}^k$ with compact support in $X$ |
| $\mathscr{D}^p(X)$ | space of $p$-forms of class $\mathscr{C}^\infty$ with compact support in $X$ |
| $\mathscr{D}^{p,q}_{[k]}(X)$ | space of $(p,q)$-forms of class $\mathscr{C}^k$ with compact support in $X$ |
| $\mathscr{D}^{p,q}(X)$ | space of $(p,q)$-forms of class $\mathscr{C}^\infty$ with compact support in $X$ |
| $\mathscr{A}(X)$ | space of distributions on $X$ |
| $\mathscr{A}'(X)$ | space of distributions on $X$ with compact support |
| $\mathscr{D}'(X)$ | space of currents on $X$ |
| $\mathscr{D}'_p(X)$ | space of currents of degree $p$ on $X$ |
| $\mathscr{E}'_p(X)$ | space of currents of degree $p$ with compact support in $X$ |
| $\mathscr{D}'_{p,q}(X)$ | space of currents of bidegree $(p,q)$ on $X$ |
| $\mathscr{E}'_{p,q}(X)$ | space of currents of bidegree $(p,q)$ with compact support in $X$ |
| $\log^+ z$ | $\max(\log z, 0)$ |
| $X^{\mathrm{an}}_v$ | analytic space associated to $X$ at the place $v$ |
| $X_{\mathrm{ad}}$ | coproduct of the family $(X^{\mathrm{an}}_v)_{v\in M_{\mathbb{Q}}}$ |

| | |
|---|---|
| $\lvert \cdot \rvert_v$ | $v-$adic absolute value |
| $\lVert \cdot \rVert$ | standard absolute value in $\mathbb{R}$ or $\mathbb{C}$. |
| $S^1$ | the unit circle |

# Chapter 1

# Introduction

Is there any human discipline whose giants are not required to master first the elementary building blocks of the discipline before they can become giants? It is often the case in our field that both developments of a given branch and the discovery of astonishing results are direct consequences of even just one of these building blocks.

We know that the set of natural numbers that are smaller than any other natural number is finite. In other words, the method of infinite descent (attributed to Fermat [Bus18] but implicit already in Euclid's Elements [Hea56]) works. It is hard to think of a more elementary building block in arithmetic. And yet it was used by Fermat himself to (sketch) a proof of Fermat's last theorem for $n = 4$ ([Hea10]).

Speaking now to the mathematician of our era let us rephrase the above by just saying that to approach problems in unlikely intersections we need to find an appropriate notion of a height function on a certain geometric object. That is the purpose of our work here.

We have made every effort to present a self-contained work. Although we did not accomplish this mission, the prerequisites a reader is expected to meet in order to successfully work through this text lies in one of the following three categories. Elementary notions used transversely throughout our work, which we will present starting in the next paragraph. Next, there are results that we use minimally, so we have decided to provide a reference when they are cited instead of including them. Lastly, unfortunately, a significant requirement is the group theory, ring theory, modules, and Galois theory learned in abstract algebra courses, as presented in parts I, II, III, and IV of Dummit and Foote's textbook [DF+04].

Let $\alpha$ be a root of a polynomial $p(t) \in \mathbb{Q}[t]$. The (naive) height $h(\alpha)$ of $\alpha$ may be thought intuitively as the number of bits required to store $\alpha$ on a computer. For

$\alpha = \frac{a}{b} \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ and $\gcd\{a, b\} = 1$,

$$h(\alpha) = \log \max\{|a|, |b|\}.$$

Unfortunately, as every number theorist should know after Lamé's claim to have proven Fermat's Last Theorem, we cannot do arithmetic with this definition; such $a, b$ are not unique in general:

$$6 = 2 \times 3 = \left(1 - \sqrt{-5}\right)\left(1 + \sqrt{-5}\right).$$

In general suppose $\alpha \in \overline{\mathbb{Q}}$ has degree $d$ over $\mathbb{Q}$ and let $p_\alpha(t) \in \mathbb{Z}[t]$ be the minimal primitive polynomial of $\alpha$. Writing $p_\alpha(t) = \sum_{j=0}^{d} a_j t^j$ we define $h(\alpha)$ as

$$h(\alpha) = \frac{1}{d}\big(\log |a_d| + \sum_{j=1}^{d} \log^+ |\alpha_j|\big),$$

where $\log^+ z = \max\{0, \log z\}$ and $\{\alpha_1, \ldots, \alpha_d\}$ is the set of conjugates of $\alpha$. This definition coincides with the one above when $\alpha = \frac{a}{b} \in \mathbb{Q}$ because $p_\alpha(t) = bt - a$ so $d = 1$ and $\log |a_d| + \sum_{j=1}^{d} \log^+ |\alpha_j| = \log \max\{|a|, |b|\}$. We also note that by definition the coefficients of this polynomial are relatively prime over $\mathbb{Z}$.

Our setting will be in the context of (projective) algebraic varieties. So we should define heights on it. Let $\mathbb{K}$ be an algebraically closed field. For any vector space $V$ over $\mathbb{K}$ we define an equivalence relation $\sim$ in $V$ by

$$x \sim y \iff \text{ there is } 0 \neq \lambda \in \mathbb{K} \text{ such that } x = \lambda y.$$

The projective space $\mathbb{P}^n$ (over $\mathbb{K}^n$) is defined as

$$\mathbb{P}^n = \left(\mathbb{K}^{n+1} / \sim\right) \setminus \{\vec{0}\}.$$

Elements $z \in \mathbb{P}^n$ are written in coordinates as $z = [z_0 : \ldots : z_n]$. Of course these coordinates are not unique. We also define the projectivization $\mathbb{P}(V)$ of $V$ is defined as

$$\mathbb{P}(V) = \left(V / \sim\right) \setminus \{\vec{0}\}.$$

Before introducing the already mentioned (naive) height function on a projective space let us note that there is a standard cover of a projective space by affine spaces

$$\mathbb{P}^n = \cup_{i=0}^n U_i, \text{ where } U_i = \{z = [z_0 : \ldots : z_n] \in \mathbb{A}^{n+1} : z_i \neq 0\}.$$

Now we can define heights in the projective space. In this introduction let us define them only over $\mathbb{P}^N(K)$ where $K$ is a number field. Let $M_K$ be a complete set of places on $K$. Set

$$n_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}],$$

and for all $v \in M_K$ normalize $|\ |_v$ to satisfy the product formula

$$\sum_{v \in M_K} n_v \log |x|_v = 0, \ \ x \neq 0.$$

Then for any $P \in \mathbb{P}^N(K)$, a Weil height is

$$h(P) = \sum_{v \in M_K} n_v \log \|P\|_v,$$

where $\|P\|_v := \max_{0 \leq j \leq N}\{|P_j|_v\}$.

We now recall some elementary notions in algebraic geometry. Again $\mathbb{K}$ is an algebraically closed field and we consider the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. The affine space $\mathbb{A}^n$ of dimension $n$ is just the set of points in $\mathbb{K}^n$. An algebraic set $Z \subset \mathbb{A}^n$ is defined as

$$Z = \{z \in \mathbb{A}^n : f_1(z) = \ldots = f_m(z) = 0 \text{ for } f_1, \ldots, f_m \in R\}.$$

In order to conceive $\mathbb{A}^n$ as a geometric object we endow it with the Zariski topology [1]. Here the closed subsets are the algebraic sets. We say that an algebraic set $Z \subset \mathbb{A}^n$ is an affine variety if it is irreducible in the Zariski topology inherited from $\mathbb{A}^n$ (that is if $Z = Z_1 \cup Z_2$ then either $Z_1 = \emptyset$ or $Z_2 = \emptyset$). An open set of an affine variety is called a quasi-affine variety. Let $I = \{f \in R : f(z) = 0 \forall z \in Z\}$. We call $R/I$ the coordinate ring of $Z$. It is an integral domain because $Z$ is a variety if and only if $I$ is a prime ideal.

For $I \subset R$ an ideal we denote by $\mathbb{V}(I) \subset \mathbb{A}^n$ the zero locus of all elements in $I$. Going in the other direction, for $Y \subset \mathbb{A}^n$ we denote by $\mathbb{I}(Y) = \langle f_1, \ldots f_m \rangle \subset R$ the ideal consisting of all polynomials that vanish at $Y$. We say the dimension of $X$ is the

[1] Oscar Zariski made a significant contribution to making the astonishingly beautiful field of algebraic geometry developed by the Italian school a rigorous one. This was at the extent of a level of abstraction that later on Alexander Grothendieck would show was far from being sufficient.

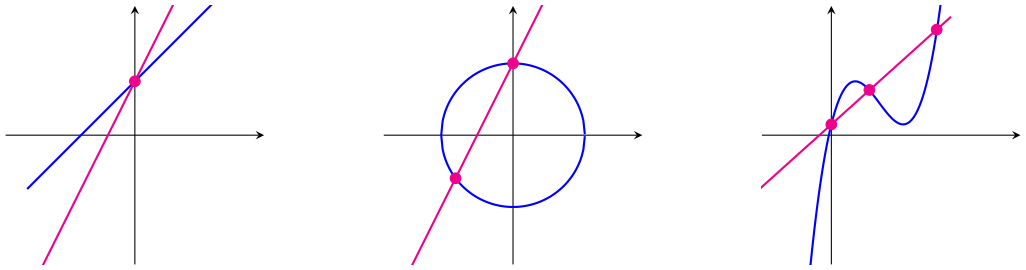


FIGURE 1.1: Expected intersections between a line and curves of degree $1, 2$ and $3$.

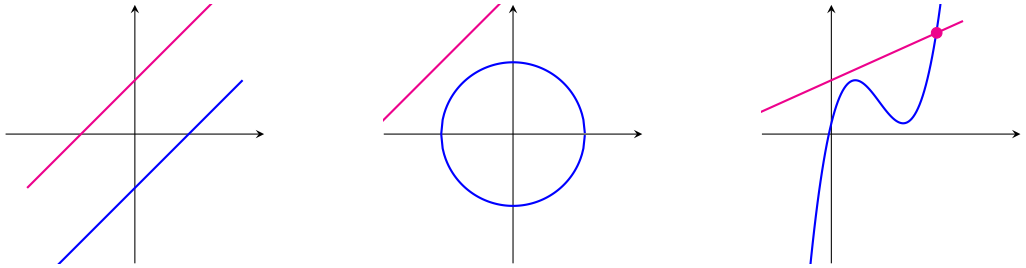


FIGURE 1.2: Unexpected intersections between a line and curves of degree $1, 2$ and $3$.

dimension of $\mathbb{I}(X)$ as vector space over $\mathbb{K}$ and we say $X$ is non singular if for every $p \in X$ we have $\dim T_pX = \dim X$ where $T_pX$ is the tangent space of $X$ at $p$.

One of the deepest results in this setting in terms of correspondences between ideals and varieties is Hilbert's Nullstellensatz:

**Theorem 1.1** (See Theorem 1.3A in [Har13])**.** Let $k$ be an algebraically closed field and $\mathfrak{a} \subset A = k[x_1, \ldots, x_n]$ an ideal. Then

$$\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}, \text{ where } \sqrt{\mathfrak{a}} \text{ is the radical of } \mathfrak{a}.$$

Let us note that the hypothesis of the field being algebraically closed is necessary. For example, take $\mathbb{K} = \mathbb{R}$ and $\mathfrak{a} = \langle x^2 + 1 \rangle$ then $\sqrt{\mathfrak{a}} = \mathfrak{a}$ because $x^2 + 1$ is irreducible (over $\mathbb{R}$). But $\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathbb{R}[x]$.

We have established a connection between algebra and geometry: (prime) ideals $I \subset R$ corresponds to affine varieties $Z \subset \mathbb{A}^n$. And even though this connection is far from being a satisfactory correspondence between rings and geometric objects (for instance there is no target object in the realm of geometry for the rings with zero divisors) there is an even more crucial inconvenience.

Note that the expected size of the intersections of a line with another line, a quadratic curve and a cubic curve (those curves given by polynomials of degree $1, 2$ and $3$ respectively) is $1, 2$ and $3$ respectively (see Figure 1.1). But if we consider a slightly different line then this expectation would fail to hold (see Figure 1.2).

That is why most mathematicians nowadays consider not affine varieties but projective varieties (the expert knows this is not the complete story. To get a correspondence between rings and geometric objects we learned to consider schemes and to understand moduli spaces we developed the notion of a stack). That is, varieties that are not conceived in an affine space but on a projective space.

Let $S = \mathbb{K}[x_0, \ldots, x_n]$ and consider $S_d \subset S$ the set of all linear combinations of monomials of total degree $d$ in the variables $x_0, \ldots, x_n$. We say that $f \in S_d$ is a homogeneous polynomial. As in the affine case, an algebraic set $Z \subset \mathbb{P}^n$ is defined as

$$Z = \{z \in \mathbb{P}^n : f_1(z) = \ldots = f_m(z) = 0 \text{ for } f_1, \ldots, f_m \in S \text{ homogeneous}\}.$$

We again endow $\mathbb{P}^n$ with is again the Zariski topology. A projective variety $Z \subset \mathbb{P}^n$ is an irreducible algebraic set of $\mathbb{P}^n$ and a quasi-projective variety is an open subset of a projective variety.

We conclude this crash course on elementary algebraic geometry with the how we relate varieties. Consider an affine variety $X$. A function $f : X \to \mathbb{K}$ is regular at $P \in X$ if there is an open neighborhood $U \subset X$ of $P$ and homogeneous polynomials $p, q \in S$ of the same degree such that $f \equiv \frac{p}{q}|_U$ and $q$ has no zeroes on $U$. We say that $f$ is regular if it is regular at every point. Identifying $\mathbb{K}$ with $\mathbb{A}^1$ yields that a regular function is continuous (see Lemma 3.1 in [Har13]). If $Y \subset \mathbb{P}^n$ is a quasi-projective variety the definition of regular function is exactly the same. Continuity is similar but we view $\mathbb{K}$ as $U_1$ or $U_2$ (recall $\mathbb{P}^1 = U_1 \cup U_2$) and therefore being open in $\mathbb{P}^1$.

Let $X, Y$ be both either affine, quasi-affine, projective or quasi-projective varieties. A function $\varphi : X \to Y$ is called a morphism if $f$ is continuous and for every $U \subset Y$ open and for every regular function $f : U \to \mathbb{K}$ the composition $f \circ \varphi : X \to \mathbb{K}$ is regular. If $\varphi$ admits an inverse morphism then we say $\varphi$ is an isomorphism. For instance the map from $\mathbb{A}^1$ to the curve $C : \{y^2 = x^3\} \subset \mathbb{C}^2$ that sends $t$ to $(t^2, t^3)$ is a morphism (moreover a homeomorphism!) but not an isomorphism. On the other hand, consider the map $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ that is the identity, except that the points 0 and 1 are swapped. This is a bijection on sets, and it's even a homeomorphism in the Zariski topology. But it is not given by a polynomial so it is not a morphism. Lastly, we say $f$ is rational if it is defined on an open subset of $X$ $f : U \subset X \to Y$. We write $f : X \dashrightarrow Y$. If there is a rational map $g : Y \dashrightarrow X$ inverse to $f$ we say $f$ is a birational map. As an example take $\varphi : \mathbb{A}^1 \dashrightarrow \mathbb{P}^1$ given by sending $t$ to $[t : 1]$. This is a birational map but it is not an isomorphism since $\mathbb{A}^1$ is not compact and $\mathbb{P}^1$ is.

The main achievement of our work is to develop a framework of using Moriwaki's perspective on heights to obtain an asymptotic for the variation of canonical heights in families (Theorem 5.39).

We now give an overview of our work aimed to the expert reader; for the definitions non-expert may refer to Chapter 5.

We define both the Philippon and Moriwaki heights for subvarieties $X \subset \mathbb{P}^N$, and we prove an explicit relation between these two notions (Theorem 5.34). Concretely, writing $\overline{\mathcal{O}(1)}_{\mathrm{tor}}$ for the canonical toric metric on $\mathcal{O}(1)$ (see the Section on Canonical Heights for the Subvarieties in 5 for the definitions), we prove

**Theorem** (Theorem 5.34) Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve and $X$ a projective variety of dimension $p$ over $K$.

$$\left| h^{\mathrm{Plpn},S}_{\overline{\mathcal{O}(1)}_{\mathrm{tor}}}(X) - h^{S}_{\overline{\mathcal{O}(1)}_{\mathrm{tor}}}(X) \right| \leq N \deg(X) \log 2 + \log p.$$

That is, we are finding an explicit constant for the difference of both heights in terms the dimension of the ambient space, and both the degree and dimension of our variety.

Continuing, Philippon's construction uses a multihomogeneous form associated to the variety (the generalized Chow form), and we use the previous relation to relate the height of this form to both the degree of our variety and the height of the parameter, as the parameter varies within a family.

We then adapt a classical telescoping-sum argument of Tate to relate the canonical height to the original height. Combining these ingredients, we arrive at our main result, which we now state.

**Main Theorem** (Theorem 5.39) Let $(f, X, L)$ be a family of polarized dynamical systems over a rational curve $B$, and let $V : B \to X$ be a family of subvarieties of dimension $p$. Assume the dynamical system is defined over a number field $K$, and fix a degree-one Weil height $h_B$ on $B$. Then

$$\hat{h}_{f_t}(V_t) = \hat{h}_f(V) h_B(t) + O(h_B(t)^{1-\frac{1}{p+2}}),$$

where the implied constant depends on the family, $\deg V$, and $h^{\mathrm{Mor}}(V)$.

This is an effective version and quantitative power saving of a theorem of Gauthier–Vigny [GV23], which the case for points was established by Call–Silverman [CS93]. The techniques used by Gauthier–Vigny differ from ours, but our proof is inspired by that of Call–Silverman.

We conclude this introduction by stating the structure of the document. Our work is divided in four chapters and the introduction. These chapters are

**Complex Geometry,**

**Moduli Spaces,**

**Intersection Theory, and**

**Heights**

in which we introduce the notions and properties that we will exploit to obtain our main result.

We also include two appendices in an attempt at self-containment:

**Category Theory**

**Scheme Theory,**

where we compile crucial notions used in our work, instead of including them in our chapters to facilitate the digestion of each chapter without overwhelming readers.

Without further ado, we hope you

Enjoy the read!

# Chapter 2

# Complex Geometry

As mentioned in the introduction in order to have Northcott's Theorem over a finitely generated field $k$ over $\mathbb{Q}$ we consider the Moriwaki height on it. This will require us to measure the size of an element $\phi \in \mathbb{P}_k^N$ on a projective space, with respect to a Kähler metric and by Kodaira's embedding theorem (see Theorem 2.17 below). The Kähler metric will tourn out to be the Fubini-Study metric. For the non-expert readers convenience and the comprehensiveness expected in a doctoral thesis, we build the theory needed to make sense of the above. For the section on complex analysis we follow [GK06] and [Ran95]. For the section on distributions and currents we follow [DS05] and [GS90]. For the rest we follow the presentation of the material found in [Sch02].

We remark that in order to introduce height functions from Moriwaki's perspective (see [Mor00]) we will require some elementary techniques from complex geometry. That is why in a thesis on arithmetic we begin the manuscript with some basics from that branch of mathematics.

## 2.1 Complex Analysis

Consider $f : U \subseteq \mathbb{C} \to \mathbb{C}$ a complex valued function and write $f(z) = u(x, y) + iv(x, y)$. We define

$$\begin{aligned} \frac{\partial f}{\partial z} &:= \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)f \\ \frac{\partial f}{\partial \bar{z}} &:= \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)f. \end{aligned} \tag{2.1}$$

We say that $f$ is holomorphic if for every $z_0 \in U$ the differential

$$df_{z_0} \in \hom(T_{U,z_0}, \mathbb{C})$$

is $\mathbb{C}$-linear, where the differential $df_{z_0}$ is defined as

$$df_{z_0} = \frac{\partial f}{\partial z}(z_0)dz + \frac{\partial f}{\partial \bar{z}}(z_0)d\bar{z},$$

where $z = x + iy, \bar{z} = x - iy$ are the complex coordinates on $U$.

Equivalently, and since Equation 2.1 yields

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

$f$ is holomorphic if $u, v$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.2}$$

These functions happen to be an interesting object to study for two reasons. First, either the real or imaginary parts of holomorphic functions contains essentially all the information of the function (see Theorem 2.2 below). Second, they are classically solutions to what are called elliptic differential equations.

Note that for such a function $f$ we have

$$\Delta u = \Delta v \equiv 0,$$

where $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

**Definition 2.1** (Harmonic)**.** A function $u : U \subset \mathbb{C} \to \mathbb{R}$ is called harmonic if $\Delta u \equiv 0$.

We show that all harmonic functions arise as real (or imaginary) parts of holomorphic functions.

**Theorem 2.2.** Let $D \subset \mathbb{C}$ be a domain (a non-empty connected and open subset). If $h$ is harmonic on $D$ then there exist a holomorphic function $f$ on $D$ such that $\mathrm{Re}(f) = h$. Moreover such $f$ is unique up to a constant.

*Proof.* We first show existence. Consider

$$g = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}.$$

It satisfies the Cauchy-Riemann equations so it is holomorphic on $D$. Pick any $z_0 \in D$.

We claim that

$$f := h(z_0) + \int_{z_0}^{z} g(w)dw$$

witnesses the assertion. Cauchy's Theorem yields that the integral defining $f$ is independent of the path because we can join any two paths to form a closed loop. And since $g$ is holomorphic and $\frac{df}{dz} = g$, also $f$ is holomorphic on $D$.

Put $\tilde{h} = \mathrm{Re}(f)$ so that $\frac{\partial}{\partial x}(\tilde{h} - h) = \frac{\partial}{\partial y}(\tilde{h} - h) = 0$. That is $h$ and $\tilde{h}$ differ only by a constant and setting $z = z_0$ the claim follows.

Let $f = h + ik$. Uniqueness follows easily since the Cauchy Riemann equations yield

$$\frac{df}{dz} = \frac{\partial h}{\partial x} + i\frac{\partial k}{\partial x} = \frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}.$$

□

Being harmonic carries profound consequences as expected from encoding all the information of holomorphic functions. We present two here. The Mean Value Property and the Maximum principle.

**Theorem 2.3** (Mean-Value Property)**.** Let $h$ be a harmonic function on an open neighbourhood of a closed disk $\overline{B}(z_0, r)$. Then

$$h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})d\theta.$$

*Proof.* Let $R > r$ so that $h$ is harmonic on $U = B(z_0, R)$. Theorem 2.2 yields $f$ holomorphic on $U$ such that $\mathrm{Re}(f) = h$. By Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

And by taking the real part to the equation above the result follows. □

We state the Maximum Principle for harmonic functions as another example of the relation between harmonic and holomorphic functions. The proof is a direct consequence

of the Maximum principle for holomorphic functions (Theorem 5.4.2 in [GK06]) and Theorem 2.2 above.

**Theorem 2.4** (The Maximum Principle for harmonic functions)**.** If $u : U \to \mathbb{R}$ is harmonic on a connected open set $U$ and if there is a point $P_0 \in U$ with the property that $u(P_0) = \sup_{Q \in U} u(Q)$, then $u$ is constant on $U$.

### 2.1.1 Potential Theory

The beautiful topic of arithmetic dynamics brings together several branches of mathematics. One of them is Analysis and in particular through Potential Theory.

Recall that the Laplacian operator is defined as

$$\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}. \tag{2.3}$$

Due to Definition 2.27 we can apply the Laplacian not only to functions but to more general objects like distributions (when applied to distributions [Ran95] calls it a generalized Laplacian).

Harmonic functions are solutions to Laplace's equation, $\Delta u = 0$. In classical potential theory, these functions model steady-state phenomena, such as temperature distribution or electrostatic potentials. We have already seen that in the complex plane harmonic functions are intimately connected to analytic functions: a real-valued function is harmonic if and only if it is locally the real part of an analytic function. This connection makes two-dimensional potential theory particularly rich.

### 2.1.2 Subharmonic functions

Subharmonic functions generalize harmonic functions. A function $u$ is subharmonic if its Laplacian is non-negative, $\Delta u \geq 0$ (in the sense of distributions for non-smooth functions).

**Definition 2.5.** We say that a function $u : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous if the set $\{x \in X : u(x) < \alpha\}$ is open in $X$ for every $\alpha \in \mathbb{R}$.

**Definition 2.6.** An upper semi-continuous function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic if for all sufficiently small $r > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta. \tag{2.4}$$

They satisfy a crucial property known as the maximum principle (Proposition 7.7.7 in [GK06]). Concretely, it states that a non-constant subharmonic function on a connected open domain cannot attain its maximum value in the interior of the domain. This principle is a cornerstone of potential theory and has far-reaching consequences in analysis.

**Remark 2.7** A function $u$ is harmonic if and only if both $u, -u$ are subharmonic.

We present some examples of subharmonic functions.

**Theorem 2.8.** Let $U$ be an open subspace of $\mathbb{C}$. Suppose $f$ is holomorphic on $U$ and $u, v$ are subharmonic. Then the following are subharmonic.

(i) $\log |f|$.

(ii) $\max\{u, v\}$.

(iii) $\alpha u + \beta v$ for all $\alpha, \beta \geq 0$.

*Proof.* Clearly items (ii) and (iii) satisfy the submean inequality 2.4.

For item (i) note that $h := \log |f|$ is upper semicontinuous. If $h(w) = -\infty$ the inequality 2.4 is trivially satisfied and if $h(w) \neq -\infty$ then $h$ is actually harmonic in a neighbourhood of $w$ so in particular that inequality holds. □

**Remark 2.9** Subharmonic functions also satisfy the Maximum Principle.

### 2.1.3 Potentials, Capacity and Equilibrium measure

Potential theory's core relies on the notions of potential (of course), energy and capacity.

Let $\mu$ be a finite Borel measure on $\mathbb{C}$ with compact support. We define the potential $V(\mu)$ of $\mu$ as

$$V(\mu, z) := \int \log |z - w| d\mu(w).$$

The energy $I(\mu)$ is defined as

$$I(\mu) := \int \int \log |z - w| d\mu(z) d\mu(w).$$

And for every compact $E \subset \mathbb{C}$, the capacity $c(E)$ of $E$ is defined as

$$c(E) := \sup_{\mu} \exp(I(\mu)),$$

where the supremum is taken over all probability measures on $\mathbb{C}$ supported on $E$. A measure for which this supremum is achieved is called an equilibrium measure.

**Theorem 2.10** (Frostman, Theorem 3.7.6 in [Ran95])**.** Let $K \subset C$ be a compact set such that $c(K) > 0$. Then $K$ has an equilibrium measure and this measure is unique.

## 2.2 Complex Manifolds

A complex manifold $X$ is a differentiable manifold which admits a complex structure on it. That is $X$ is covered by open sets $X_\alpha$ such that there are diffeomorphisms

$$\phi_\alpha : X_\alpha \to U_\alpha \subseteq \mathbb{C}^n,$$

and the $U_\alpha$ are open in $\mathbb{C}^n$ (in the euclidean topology). Moreover we ask that the transition diffeomorphisms $\phi_\alpha \circ \phi_\beta^{-1}$ are holomorphic. The set $\{(U_\alpha, \phi_\alpha)\}$ are called local charts.

### Examples

**Complex Tori:** Let $\Lambda$ be a lattice in $\mathbb{C}^m$, i.e., a discrete additive subgroup generated by a basis of $\mathbb{C}^m$ over $\mathbb{R}$. The group $\Lambda$ acts on $\mathbb{C}^m$ via translations, and this action is proper and has no fixed points. The quotient $T = \mathbb{C}^m/\Lambda$ forms a compact space. Moreover, there exists an $\mathbb{R}$-linear automorphism of $\mathbb{C}^m \cong \mathbb{R}^{2m}$ that sends $\Lambda$ to $\mathbb{Z}^{2m}$. This implies that the quotient $T$ is homeomorphic to $(\mathbb{R}/\mathbb{Z})^{2m} \cong (S^1)^{2m}$.

Clearly, $T$ has a differentiable structure such that the quotient map is a local diffeomorphism. We equip $T$ with a complex structure by considering holomorphic charts given by the local inverses of the quotient map.

**Riemann Surfaces:** Consider two-dimensional differentiable manifolds. When restricted to compact, oriented cases, these are classified by their genus $g$. Such a surface is diffeomorphic to a torus with $g$.

Moreover, every such surface $X$ can be equipped with a complex structure.

**Complex Projective Space:** The complex projective space $\mathbb{P}^n(\mathbb{C})$ consists of the set of complex lines in $\mathbb{C}^{n+1}$ or, equivalently, the quotient of $\mathbb{C}^{n+1}\setminus\{0\}$ by the equivalence relation identifying collinear vectors in $\mathbb{C}$. Its topology is the quotient topology, and its complex structure is constructed as follows:

For each $j$, consider the subset $\tilde{U}_j$ of $\mathbb{C}^{n+1}\setminus\{0\}$ consisting of points such that $z_j \neq 0$. Define $U_j$ as the image of $\tilde{U}_j$ in $\mathbb{P}^n(\mathbb{C})$. Any point in $U_j$ has a unique representative $z \in \tilde{U}_j$ satisfying $z_j = 1$. Thus, $U_j$ is homeomorphic to $\mathbb{C}^n$, providing holomorphic charts for $\mathbb{P}^n(\mathbb{C})$, which is covered by the $U_j$.

Transition maps between charts are holomorphic, as they correspond to ratios of coordinates. Specifically, if $z \in \mathbb{C}^{n+1}\setminus\{0\}$, and the representative in $U_i$ has coordinates $(z_1/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_{n+1}/z_i)$, then in the chart $U_k$, the representative has coordinates $(z_1/z_k, \ldots, z_{k-1}/z_k, z_{k+1}/z_k, \ldots, z_{n+1}/z_k)$. These transition maps are holomorphic hence $\mathbb{P}^n(\mathbb{C})$ is indeed a complex manifold.

**Vector Bundles:** A vector bundle over a (complex) manifold $M$ is a (complex) manifold $E$ (the total space) together with a map $\pi : E \longrightarrow M$ such that:

1. $\forall x \in M$, $\pi^{-1}(\{x\}) := F_x$ is a (complex) vector space.
2. $\forall x \in M$, $\exists U_x \subset M$ a neighbourhood of $x$ in $M$ and a diffeomorphism

$$\phi_x : \pi^{-1}(U_x) \longrightarrow U_x \times \mathbb{C}^n$$

such that for all $y \in U_x$

$$\phi_x^{-1} : \{y\} \times \mathbb{C}^n \longrightarrow F_y$$

is a linear isomorphism.

We call such $n$ the rank of the vector bundle. When $n = 1$ we say $E$ is a line bundle.

**Definition 2.11** (section of a vector bundle)**.** Let $X$ be a manifold and $E$ a vector bundle over $X$. We say a map $s : X \to E$ is a section of $E$ if $\pi(s(x)) = x$ for all $x \in X$.

## Calculus on Manifolds

Integration on manifolds is performed with respect to differential forms.

**Definition 2.12** (Differential $k$ form)**.** A $k-$differential form $\alpha$ is a section of the holomorphic bundle $\bigwedge^k(T_X)^*$, where $T_X$ is the holomorphic tangent bundle of $X$. We say $k$ is the degree of $\alpha$.

We will write $\Omega^k_{X,\mathbb{R}}$ for the real $k-$differential forms on $X$ and $\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for its complexification. When $k = 1$ we write simply $\Omega_{X,\mathbb{R}}$ and $\Omega_{X,\mathbb{C}} = \hom(T_X, \mathbb{C})$ the bundle whose fiber at each point $x \in X$ is the space of linear functionals $T_{X,x} \to \mathbb{C}$.

Let $M$ be a $k$-dimensional manifold with boundary, and $\phi : M \to U$ be a $C^1$ differentiable map. We assume that near the boundary of $M$, $\phi$ can be locally represented as a map from $[0,1] \times V$ into $U$, where $V$ is a suitable open set. Moreover, we require that $\phi$ can be extended to a $C^1$ map on a neighbourhood $(-\epsilon, 1+\epsilon) \times V$ for some $\epsilon > 0$ and denote this extension still by $\phi$.

Consider a continuous $k$-form $\alpha$ on $U$. The pullback of $\alpha$ under $\phi$, denoted as $\beta = \phi^*\alpha$, is a continuous $k$-form on $M$ defined as

$$\phi^*\alpha = \sum_I a_I \circ \phi \, d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k},$$

well defined by the extension hypothesis.

If $M$ is compact and oriented, we can define the integral of $\beta$ over $M$ as follows. Let $\{f_i\}$ be a partition of unity subordinate to a covering of $M$ by open sets $\{U_i\}$, where each $U_i$ is diffeomorphic to either $(0,1] \times (0,1)^{k-1}$ or $(0,1)^k$. Then $\beta = \sum_i f_i\beta$, and each $f_i\beta$ on $U_i$ can be extended to a continuous form on $[0,1]^k$.

Suppose that on each $U_i$, we have $f_i\beta = g_i(x_1, \ldots, x_k)\, dx_1 \wedge \cdots \wedge dx_k$ for some function $g_i$. Then we define the integral of $\beta$ over $M$ as:

$$\int_M \beta = \sum_i \int_{U_i} f_i\beta,$$

where

$$\int_{U_i} f_i\beta = \int_0^1 \cdots \int_0^1 g_i(x_1, \ldots, x_k)\, dx_1 \cdots dx_k.$$

As in vector calculus we have Stoke's formula.

**Theorem 2.13.** Let $M$ be a differentiable manifold and consider a differential form $\alpha$ on $M$. Then

$$\int_M \phi^* d\alpha = \int_{\partial M} \phi^*\alpha$$

where $\partial M$ denotes the boundary of $M$.

### The operators $\partial, \bar{\partial}$

An almost complex structure on $M$ is an endomorphism $I$ of the real tangent space $T_{M,\mathbb{R}}$ such that $I^2 = -\mathrm{Id}$.

**Definition 2.14** (Almost complex structure)**.** An almost complex structure $I$ on a manifold $X$ is integrable if there exist a complex structure on $X$ that induces $I$.

It is worth remarking that a complex manifold has naturally an almost complex structure. Specifically, if $X$ admits the complex structure $(X_\alpha, \phi_\alpha)$ then $d\phi_\alpha : TX_\alpha \to \mathbb{C}$ is $\mathbb{C}$-linear. On the other hand, as expected, not every almost complex manifold is complex (see [Ven66]).

Let $X$ be a complex manifold with local charts $\{(U_\alpha, \phi_\alpha)\}$. Note that $T_X$ on $U_\alpha$ is generated by elements

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big) \in T_{U_\alpha} \otimes \mathbb{C}$$

and so

$$T_X \subset T_{X,\mathbb{R}} \otimes \mathbb{C}.$$

Now consider the action of $I$ on $T_{X,\mathbb{R}} \otimes \mathbb{C}$. There is an eigenspace associated to the eigenvalue $i$ which we denote by $T_X^{(1,0)}$ and another eigenspace $T_X^{(0,1)}$ associated to $-i$.

We conclude that

$$T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C} = T_X^{(1,0)} \oplus T_X^{(0,1)}. \tag{2.5}$$

The above induces a decomposition

$$\Omega_{X,\mathbb{C}} = \Omega_X^{(1,0)} \oplus \Omega_X^{(0,1)}. \tag{2.6}$$

which then yields

$$\Omega_{X,\mathbb{C}}^k = \sum_{p+q=k} \Omega_X^{(p,q)}, \tag{2.7}$$

where $\Omega_X^{(p,q)} = \bigwedge^p \Omega_X^{(1,0)} \otimes \bigwedge^q \Omega_X^{(0,1)}$.

**Definition 2.15.** An element $\alpha \in \Omega_X^{(p,q)} \subset \Omega_{X,\mathbb{C}}^k$ is called a complex $k$-form of type $(p, q)$.

A $(p, q)$-form can be written locally as $\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$, and so

$$d\alpha = \sum_{I,J} d\alpha_{I,J} dz_I \wedge d\bar{z}_J,$$

is a sum of a form of type $(p, q+1)$ and a form of type $(p+1, q)$.

**Definition 2.16** (Differential $(p,q)$-form)**.** Let $\alpha$ be a differential $\mathscr{C}^1$ form of type $(p,q)$ on a complex manifold $X$. We define $\partial\alpha$ to be the component of type $(p+1,q)$ of $d\alpha$ and $\bar{\partial}\alpha$ the component of type $(p,q+1)$. In other words

$$d = \partial + \bar{\partial}.$$

### 2.2.1 Cohomology

The notions of both cohomology and homology date back to the work of Henri Poincaré in the late nineteenth century. The philosophy is to assign algebraic groups to real and/or complex manifolds as a way to study these manifolds. Remarkably just as it happens with algebraic varieties, this assignment turns out to be powerful. This similarity may be explained by Jean Pierre-Serre's *Géométrie Algébrique et Géométrie Analytique* theory now known as GAGA. The main result may be stated as follows.

The category of coherent algebraic sheaves on a complex projective variety $X$ and the category of coherent analytic sheaves on the corresponding analytic space $X^{\text{an}}$ are equivalent (see the Appendices for these definitions). Here $X^{\text{an}}$ denotes the complex points $X(\mathbb{C})$ of $X$ but with the usual complex topology and not the Zariski one.

It is not in the scope of our present work to develop this theory at the extent it deserves but fortunately many have done so already. We introduce these cohomologies, give an insight on their importance and provide a bibliography for the curious reader to refer to.

A differential complex is a direct sum $C = \bigoplus_{n\in\mathbb{Z}} C_n$ such that there is a sequence

$$\cdots \longrightarrow C_{m-1} \xrightarrow{d} C_m \xrightarrow{d} C_{m+1} \longrightarrow \cdots,$$

such that $d \circ d = 0$. The cohomology of the complex $C$ is

$$C = \bigoplus_{n\in\mathbb{Z}} H^n(C), \qquad H^n(C) = \big(\ker d \cap C_n\big)/\big(\text{Im}d \cap C_n\big).$$

When $C_n = \mathscr{D}^n(X)$ the real $n$-forms on a topological space $X$, we refer to the differential complex as the de Rham complex and the cohomology as the de Rham cohomology.

This cohomology allows one to study the structure of $X$ as a real manifold. Bott and Tu's book [BT13] is an excellent source for comprehensive treaty of this cohomology. The article [ET13] makes a quick introduction and build from scratch the tools required to proof the celebrated Grothendieck's algebraic de Rham theorem.

As we saw in previous sections of this chapter, complex manifolds are not only real manifolds as well but come equipped with a holomorphic complex structure. So to study complex manifolds is not enough to consider the de Rham complex but the Dolbeault complex i.e. when $C_n = \mathscr{D}^{p,q}_{p+q=n}$. The resulting cohomology is naturally called the Dolbeault cohomology. The reader could refer to Griffiths and Harris' book [GH14] for a comprehensive treaty.

#### Sheaf Cohomology

Sheaf cohomology generalizes classical cohomology by taking into account local data assigned to open sets in a space (or variety) and gluing this data together to form global sections. Recall from Definition B.5 that the sheaf cohomology groups $H^k(X, \mathcal{F})$ are computed from a sheaf $\mathcal{F}$ on a space $X$ as

$$H^k(X, \mathcal{F}) = H^k\left(X, \mathrm{Res}(\mathcal{I}^\bullet)\right),$$

where $\mathcal{I}^\bullet$ is an injective resolution of the sheaf $\mathcal{F}$ (see Proposition A.20). This means that we can compute the cohomology groups by resolving the sheaf into injective sheaves and then taking the cohomology of the resulting complex.

A short exact sequence of sheaves on $X$

$$0 \to A \to B \to C \to 0$$

yields a long exact sequence of the sheaf cohomology groups:

$$0 \to H^0(X, A) \to H^0(X, B) \to H^0(X, C) \to H^1(X, A) \to \cdots.$$

## 2.3 Kähler Geometry

Complex projective varieties are Kähler manifolds. This is Kodaira's Theorem.

**Theorem 2.17** (Theorem 0.1 in [Sch02])**.** A complex manifold is projective (meaning it admits a closed holomorphic embedding into a complex projective space) if and only if it admits a Kähler metric whose Kähler form is integral.

A Kähler structure on a complex manifold consists of three mutually compatible structures:

(i) A Riemannian metric $g$, i.e. a smoothly varying positive-definite quadratic form on each tangent space.

(ii) A symplectic form $w$, i.e. a smoothly varying non-degenerate skew-symmetric 2-form on each tangent space which is closed.

(iii) An integrable complex structure.

We pack these structures in a Hermitian structure $h$ i.e. $h$ is a sesquilinear form

$$h = g - iw,$$

where $g$ is a Riemannian metric and $w$ is a $(1,1)$-form called the Kähler form.

**Definition 2.18.** The fundamental form $\omega$ of an hermitian metric $h$ is defined as its imaginary part $Im(h)$.

Let $X$ be a Kähler manifold and $L$ a line bundle on $X$ with trivializations $(X_\alpha, \phi_\alpha)$ endowed with a Hermitian metric $h$. Then its fundamental form is given locally as

$$\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z_j}.$$

### 2.3.1 Fubini–Study metric

Let $X = \mathbb{P}^N(\mathbb{C})$ with the standard cover $U_i = \{z \in \mathbb{P}^N(\mathbb{C}) : z_i \neq 0\}$and $L = \mathcal{O}_{\mathbb{P}^N}(-1)$. By restricting the standard Hermitian metric on $\mathbb{C}^{N+1}$ to $\mathcal{O}_{\mathbb{P}^N}(-1) \subset \mathbb{P}^N(\mathbb{C}) \times \mathbb{C}^{N+1}$ we obtain a Hermitian metric $h$ on $\mathcal{O}_{\mathbb{P}^N}(-1)$.

Let $\sigma_i$ be the dual section to the one that on $\mathcal{O}_{\mathbb{P}^N}(1)$ assigns $z \in U_i$ to the unique vector on $\mathbb{C}^{N+1}$ where $z_i = 1$. Then

$$h(\sigma_i) = \frac{1}{1 + \sum_j |z_j|^2.}$$

**Definition 2.19** (Fubini-Study metric)**.** The metric above on $\mathcal{O}_{\mathbb{P}^N}(1)$ is called the Fubini-Study metric.

We describe the fundamental form $\omega_{FS}$ of the Fubini-Study metric.

$$\omega_{FS}|_{U_i} = \omega_i := \frac{i}{2\pi}\partial\overline{\partial}\log\Big(\sum_{j=1}^{N}|w_j|^2+1\Big).$$

The $\omega_i$ glue together because $h_i = g_{ij}h_j$ and $\partial\overline{\partial}\log|g_{ij}|^2 = 0$, where $g_{ij}$ are the transition matrices corresponding to the trivializations of $\mathcal{O}_{\mathbb{P}^N}(-1)$ corresponding to the cover $\{U_i\}$.

### 2.3.2 (First) Chern class

Let $X$ be a complex manifold. Recall that there is a short exact sequence of sheafs

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^{\times} \longrightarrow 0$$

And so the correponding long exact sequence of cohomology induces a map

$$c_1 : H^1(X, \mathcal{O}_X^{\times}) \to H^2(X, \mathbb{Z}).$$

Since $H^1(X, \mathcal{O}_X^{\times}) \cong \mathrm{Pic}(X)$, it yields that $c_1$ assigns to every line bundle $L$ over $X$ a well defined element in $H^2(X, \mathbb{Z})$. And this element is used to classify line bundles topologically.

**Definition 2.20** (First Chern class)**.** $c_1(L)$ is called the first Chern form of $L$.

**Remark 2.21** Let $[\omega_{FS}]$ denote the class of $\omega_{FS} \in H^2(X, \mathbb{Z})$.

$$[\omega_{FS}] = c_1(\mathcal{O}_{\mathbb{P}^N}(1)).$$

## 2.4 Distribution and currents

The main character in our work is an arithmetic variety. This is an astonishingly beautiful object that exhibits both arithmetic and geometric properties (see Example B.15)). The way we will connect both of these worlds is via the so called Green current on a complex manifold $X$. Currents are generalizations of distributions.

### 2.4.1 Distributions

Distributions were introduced by Schwartz [Sch57] to generalize the notion of a function.

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ and denote $|\alpha| = \alpha_1 + \ldots + \alpha_N$ and $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}$.

We endow the space $\mathscr{C}^k(X)$ of functions of class $\mathscr{C}^k$ functions on $X$ with a family of semi-norms as follows. If $K \subset X$ is compact, define for every function $\varphi$ in $\mathscr{C}^k(X)$

$$\|\varphi\|_{k,K} := \max_\alpha \left\{ \max_K |\partial^\alpha \varphi|, 0 \leq |\alpha| \leq k \right\}.$$

Consider the topology on $\mathscr{C}^k_C(X)$ associated to the previous family of semi-norms. That is, for all $f \in \mathscr{C}^k(X)$ and each semi-norm $p$, the topology we consider is the coarsest such that $p$ is continuous and also $g \to p(g-f)$ is continuous for all $g \in \mathscr{C}^k_c(X)$. We say that $(\varphi_n)$ converges to $\varphi$ in $\mathscr{C}^k_c(X)$ if there is a compact subset $K \subset X$ such that $\varphi_n$ and $\varphi$ are supported in $K$ and $\|\varphi_n - \varphi\|_{k,K}$ converge to 0.

It is more convenient to work with linear functionals defined on $\mathscr{C}^\infty_c(X)$ instead of the whole $\mathscr{C}^\infty(X)$. Theorem 2.29 below yields that we are not losing anything by doing so.

The elements of $\mathscr{C}^\infty_c(X)$ are called test functions. We say that a sequence $(\varphi_n)$ converges to $\varphi$ in $\mathscr{C}^\infty_c(X)$ if there is a compact subset $K \subset X$ such that $\varphi_n$ and $\varphi$ are supported in $K$ and $\|\varphi_n - \varphi\|_{k,K}$ tend to 0 for every $k \geq 0$.

**Definition 2.22** (Distributions)**.** A distribution on $X$ is a linear functional $T : \mathscr{C}^\infty_C(X) \to \mathbb{C}$, continuous with respect to the topology on $\mathscr{C}^\infty_c(X)$. The value of $T$ on $\varphi$ is denoted by $T(\varphi)$ or $\langle T, \varphi \rangle$. Denote by $\mathscr{A}(X)$ the set of distributions on $X$. We say that a sequence $(T_n) \subset \mathscr{A}(X)$ converges to $T \in \mathscr{A}(X)$ in the sense of distributions if $\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle$ for every function $\varphi \in \mathscr{C}^\infty_c(X)$.

**Example 2.23.** If $f \in \mathscr{L}^1_{\mathrm{loc}}(X)$ is a locally integrable function, it defines a distribution $T_f$ by:

$$\langle T_f, \varphi \rangle = \int_X f(x)\varphi(x)\, dx, \quad \forall \varphi \in \mathscr{C}^\infty_c(X)$$

**Proposition 2.24.** Let $T$ be a distribution on $X$ and let $K \subset X$ be a compact set. Then there exist a positive integer $k$ and a positive constant $c$ such that

$$|\langle T, \varphi \rangle| \leq c\|\varphi\|_{k,K} \quad \text{for every } \varphi \in \mathscr{C}^\infty_C(X) \text{ with } \operatorname{supp}(\varphi) \subset K.$$

*Proof.* $T$ is continuous if and only if there are semi-norms $p_1, p_2, \ldots, p_n$ defining the topology and $c > 0$ such that for all $\varphi$

$$\|T(\varphi)\| \leq c \max\{\|p_j(\varphi)\|\}. \tag{2.8}$$

On the other hand, $\|\varphi\|_{m,K} \leq \|\varphi\|_{m',K}$ for $m \leq m'$. Hence setting $p_j = ||\ \ ||_{m_j,K}$ and $k = \max\{m_j\}$ we have

$$\max\{\|\varphi\|_{m_j,K}\} \leq \|\varphi\|_{k,K} \tag{2.9}$$

Equations 2.8 and 2.9 together give the result. □

**Definition 2.25** (Order of a distribution)**.** If the integer $k$ in Proposition 2.24 can be chosen independently of $K$ we say that $T$ is of finite order and the smallest integer $k$ satisfying this property is called order of $T$.

**Example 2.26.** The Dirac delta distribution $\delta_a$ centred at a point $a \in X$ is defined by:

$$\langle \delta_a, \varphi \rangle = \varphi(a), \quad \forall \varphi \in \mathscr{C}_C^\infty(X)$$

Also we can define partial derivatives of distributions.

**Definition 2.27.** The partial derivative of a distribution $T \in \mathscr{A}(X)$ is defined by:

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle, \quad \forall \varphi \in \mathscr{C}_C^\infty(X)$$

More generally, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$:

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$$

And lastly we define convolutions as follows.

**Definition 2.28.** If $T \in \mathscr{A}(\mathbb{R}^n)$ and $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, the convolution $T * \psi$ is defined by:

$$(T * \psi)(x) = \langle T, \varphi \circ \tau_x \rangle$$

where $\tau_x(y) = x - y$.

We conclude this section with the following result.

**Theorem 2.29** (Theorem 2.1.11 in [DS05])**.** Let $T \in \mathscr{A}(X)$ have compact support. Then there is a unique continuous extension of $T$ defined on all $\mathscr{C}^\infty(X)$.

### 2.4.2 Currents

Currents were discovered by de Rham. They generalize the notion of distributions. In fact, distributions are currents of maximal degree.

**Definition 2.30.** A current of degree $p$ and of dimension $N-p$ on $X$ is a linear continuous functional $T : \Omega^{N-p}(X) \to \mathbb{C}$. If $\alpha$ is a form in $\Omega^{N-p}(X)$, the value of $T$ at $\alpha$ is denoted by $T(\alpha)$ or $\langle T, \alpha \rangle$. The form $\alpha$ is called test form. We say that a sequence $(T_n)$ of $p$-currents on $X$ converges weakly (or in the sense of currents) to a current $T$ if $\langle T_n, \alpha \rangle \to \langle T, \alpha \rangle$ for every $\alpha \in \Omega^{N-p}(X)$.

So distributions are currents of maximal degree. If $T$ is a current of degree 0, one can associate to $T$ a distribution $T'$ by

$$\langle T', \varphi \rangle := \langle T, \varphi dx_1 \wedge \ldots \wedge dx_N \rangle.$$

This correspondence between distributions and 0-currents is one-to-one. Hence we often identify distributions and currents of degree 0.

For the next two examples suppose $X$ is actually real.

**Example 2.31.** If $T$ is a $p$-form with coefficients in $\mathscr{L}^1_{loc}(X)$ then $T$ defines a current of degree $p$

$$\langle T, \alpha \rangle := \int_X T \wedge \alpha \quad \text{for } \alpha \in \Omega^{N-p}(X).$$

**Example 2.32.** Let $Y \subset \mathbb{R}^{N-p}$ be an open set and let $f : Y \to X$ be a proper injective map of class $\mathscr{C}^1$. Let $Z \subset Y$ be a Borel set and $Z' := f(Z)$. Define the current $[Z']$ of integration on $Z'$ by

$$\langle [Z'], \alpha \rangle = \int_{Z'} f^*(\alpha) \quad \text{for } \alpha \in \Omega^{N-p}(X).$$

In particular, when $X = Y = Z = Z'$ and $f = \mathrm{id}$ we get the 0-current $[X]$ of integration on $X$.

For non real manifolds we must consider both the holomorphic components and the antiholomorphic ones.

We have the following decomposition of spaces of $k$-forms as direct sums of spaces of $(p,q)$-forms

$$\Omega^r(X) = \bigoplus_{p+q=r} \Omega^{p,q}(X) \quad \text{and} \quad \mathscr{D}^r(X) = \bigoplus_{p+q=r} \mathscr{D}^{p,q}(X),$$

where $\mathscr{D}^r, \mathscr{D}^{p,q}$ denote the space of $C^\infty$ forms with compact support of degree $r$ and bidegree $(p,q)$ respectively. By duality, setting $\mathscr{D}'_r(X)$ to be the space of currents of degree $r$ and $\mathscr{E}'_r(X) \subset \mathscr{D}'_r(X)$ the subspace of currents of degree $r$ with compact support, we get

$$\mathscr{D}'_r(X) = \bigoplus_{p+q=r} \mathscr{D}'_{p,q}(X) \quad \text{and} \quad \mathscr{E}'_r(X) = \bigoplus_{p+q=r} \mathscr{E}'_{p,q}(X).$$

We can endow $\Omega(X)$ with an inner product that makes it a pre-Hilbert space as follows.

$$(\psi, \eta) = \int_X (\psi(z), \eta(z))\Phi(z), \text{ where } \Phi = \frac{\omega^{\dim X}}{\dim X!}$$

is the volume form associated to the $(1,1)$ fundamental form that comes from the metric of $X$. (see 2.18). This inner-product allows us to define the adjoint operator

$$\bar{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$$

by requiring that

$$(\bar{\partial}^* \psi, \eta) = (\psi, \bar{\partial}\eta).$$

### 2.4.3 Green Currents and Forms

In order to develop our theory we require the famous Hodge Theorem. We recall some definitions. We say that a differential form $\alpha$ on $X$ is harmonic if

$$\Delta\alpha \equiv 0, \text{ where } \Delta\alpha = (\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial})\alpha.$$

We denote the space of harmonic forms as $\mathscr{H}^{p,q}(X)$. Then the Hodge Theorem states

**Theorem 2.33** (Hodge Decomposition)**.** Let $M$ be a compact Kähler manifold. Then:

(i) The space of harmonic $(p,q)$-forms $\mathscr{H}^{p,q}(M)$ is finite-dimensional.

(ii) The orthogonal projection operator

$$\mathscr{H} : \Omega^{p,q}(M) \to \mathscr{H}^{p,q}(M) \tag{2.10}$$

is well-defined, and there exists a unique operator - the Green's operator

$$G : \Omega^{p,q}(M) \to \Omega^{p,q}(M) \tag{2.11}$$

satisfying:

$G(\mathscr{H}^{p,q}(M)) = 0$ (annihilates harmonic forms)

$\bar{\partial}G = G\bar{\partial}$ (commutes with $\bar{\partial}$)

$\bar{\partial}^*G = G\bar{\partial}^*$ (commutes with $\bar{\partial}^*$)

$\mathrm{Id} = \mathscr{H} + \Delta G$ (identity decomposition).

The identity $\mathrm{Id} = \mathcal{H} + \Delta G$ can be reformulated as the more explicit Hodge decomposition:

$$\psi = \mathcal{H}(\psi) + \bar{\partial}(\bar{\partial}^* G\psi) + \bar{\partial}^*(\bar{\partial} G\psi) \tag{2.12}$$

This directly yields the orthogonal direct-sum decomposition of the space of $(p, q)$-forms:

$$\Omega^{p,q}(M) = \mathcal{H}(M) \oplus \bar{\partial}\Omega^{p,q-1}(M) \oplus \bar{\partial}^*\Omega^{p,q+1}(M) \tag{2.13}$$

A key insight from this decomposition is that for any $\eta \in \Omega^{p,q}(M)$, the Laplace equation

$$\Delta\psi = \eta \tag{2.14}$$

has a solution if and only if $\mathscr{H}(\eta) = 0$ (i.e., $\eta$ is orthogonal to the harmonic forms). When this condition is satisfied, the unique solution orthogonal to $\mathscr{H}^{p,q}(M)$ is given by:

$$\psi = G(\eta) \tag{2.15}$$

Intuitively this last analysis can be thought of the Green operator being an inverse to the Laplacian. We will not prove Hodge's Theorem here but refer the reader to pages 84-100 in [GH14] for a rigorous construction and a concrete description of the Green operator.

We now move to Hodge Decomposition Theorem.

**Theorem 2.34** (Theorem 1.2.1 in [GS90])**.** Let $X$ be a compact Kähler manifold. Suppose $\eta \in \Omega^{p,q}(X)$, $p, q \geq 1$, is $d$-closed and is either $d$-, $\partial$-, or $\bar{\partial}$-exact. Then there exists

$$\gamma \in \Omega^{p-1,q-1}(X)$$

such that

$$dd^c\gamma = \frac{i}{2\pi}\partial\bar{\partial}\gamma = \eta.$$

*Proof.* Just take $\gamma = \pm\partial^*\overline{\partial}G\eta$ where $G$ is the Green operator from the Hodge Theorem. □

We formalize Equation 2.13 which is known as Hodge Decomposition Theorem.

**Theorem 2.35** (Hodge Decomposition )**.** Let $X$ be a complex manifold. Then:

(i) If $\gamma$ is a current on $X$ such that $\bar{\partial}\gamma$ is smooth (i.e., equals $[\varphi]$ for some $C^\infty$ form $\varphi$), then there exist currents $\alpha$ and $\beta$ such that $\gamma = \omega + \partial\alpha + \bar{\partial}\beta$ with $\omega$ smooth.

(ii) If $\omega$ is a $C^\infty$ form on $X$ such that $\omega = \partial u + \bar{\partial} v$ for currents $u$ and $v$, then there exist smooth currents $\alpha$ and $\beta$ such that $\omega = \partial\alpha + \bar{\partial}\beta$.

*Proof.* See Theorem 1.2.2 in [GS90]. □

Let $Y$ be a $p$-dimensional subvariety of a compact complex manifold $X$.

**Definition 2.36** (Green current)**.** A Green current for $Y$ is a current $g$ of type $(N-p-1, N-p-1)$ such that

$$dd^c g + \delta_Y = \omega$$

where $\delta_Y$ is the current of integration over $Y$ and $\omega$ is a smooth form of type $(p,p)$.

Such a Green current always exists if $X$ is compact and Kähler, since if $\omega$ is any real $(p,p)$ form that is cohomologous to $\delta_Y$, then $\delta_Y - \omega$ is exact and we can apply Theorem 2.34. We also define a Green form.

**Definition 2.37** (Green form)**.** If $|Y|$ is the support of $Y$ in $X$, we define a Green form for $Y$ to be a $C^\infty$ form $g$ on $X - |Y|$, locally $L^1$ on $X$, for which $[g]$ is a Green current for $Y$.

Lastly let us write:

$$\tilde{\Omega}^{p,q}(X) = \Omega^{p,q}(X)/(\partial\Omega^{p-1,q} + \bar{\partial}\Omega^{p,q-1}),$$

and

$$\tilde{\mathscr{D}}^{p,q}(X) = \mathscr{D}'_{p,q}(X)/(\partial\mathscr{D}'_{p-1,q} + \bar{\partial}\mathscr{D}'_{p,q-1}).$$

Observe that by (ii) of Theorem 2.35, the natural map $\tilde{\Omega}^{p,q}(X) \to \tilde{\mathscr{D}}^{p,q}(X)$ is an injection. The homomorphisms $\partial\bar{\partial}$ on both forms and currents factor through $\tilde{\Omega}^{p,q}(X)$ and $\tilde{\mathscr{D}}^{p,q}(X)$. By (i) of Theorem 1.2.2, the kernel of $\partial\bar{\partial} : \tilde{\mathscr{D}}^{p,q}(X) \to \mathscr{D}'_{p+1,q+1}(X)$ is contained in $\tilde{\Omega}^{p,q}(X)$. If $X$ is compact and Kähler, the (Dolbeault) cohomology group

$$H^{p,q}(X) = \big(\ker\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q+1}(X)\big)/\bar{\partial}(\Omega^{p,q-1}(X))$$

can be identified with the space of harmonic forms of type $(p,q)$ on $X$, and so we have two exact sequences

$$0 \to H^{p,q}(X) \to \tilde{\Omega}^{p,q}(X) \to B^{p+1,q+1}(X) \to 0$$

$$0 \to H^{p,q}(X) \to \tilde{\mathscr{D}}^{p,q}(X) \to \mathscr{B}^{p+1,q+1}(X) \to 0$$

Here $B^{\star,\star}(X)$ and $\mathscr{B}^{\star,\star}(X)$ are the spaces of exact forms and currents, respectively. It can be shown (see (iii) in Theorem 1.2.2 [GS90]) that there is a map from the top exact sequence to the bottom one making the whole diagram commute by sending $\tilde{\Omega}^{p,q}(X)$ to $\tilde{\mathscr{D}}^{p,q}(X)$ and $B^{p+1,q+1}(X)$ to $\mathscr{B}^{p+1,q+1}(X)$. Hence we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{p,q}(X) & \to & \tilde{\Omega}^{p,q}(X) & \longrightarrow & B^{p+1,q+1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{p,q}(X) & \to & \tilde{\mathscr{D}}^{p,q}(X) & \longrightarrow & \mathscr{B}^{p+1,q+1}(X) & \longrightarrow & 0 \end{array}$$

FIGURE 2.1: Relation between Forms and Currents

**Geometric construction of Green Currents**

We now give a geometric construction of Green currents. This construction will be used in Chapter 4 to define intersections of arithmetic cycles and is the purpose of us having introduced distributions, currents and Green currents or forms. We make use of the notion of an analytic cycle, a divisor and equivalence between cycles. These notions will be defined in the algebraic setting formally in Chapter 3 but for the reader's convenience we recall these notions here.

Let $\mathcal{Z}(X)$ be the free abelian group generated by submanifolds of $X$. We call elements of $\mathcal{Z}(X)$ analytic cycles. A divisor is a (finite) linear combination of codimension 1 submanifolds. For any meromorphic function $s$ on $X$ we define

$$\operatorname{div}(s) = \sum_{W \subset X \text{ codimension } 1} \operatorname{ord}_W(s)[W].$$

The elements $\operatorname{div}(s)$ generate a subgroup in $\mathcal{Z}(X)$ denoted by $\operatorname{Rat}(X)$. And for any submanifold $W \subset X$ we denote by $[W]$ the class of $W$ in the group $\mathcal{Z}(X)/\operatorname{Rat}(X)$.

Now our setting is as follows. Let $Y \subset X$ be a divisor and consider a line bundle $\mathscr{L}$ on $X$ such that for a meromorphic section $s$ of $\mathscr{L}$ we have $[Y] = \operatorname{div}(s)$. Choose a smooth Hermitian metric on $\mathscr{L}$, denoted $\|\cdot\|$. Then $\log\|s\|^2 \in \mathscr{L}^1(X)$, and by the

Poincaré-Lelong formula (see Chapter 3 in [GH14]),

$$dd^c([\log \|s\|^2]) = \delta_Y - \beta \tag{2.16}$$

for $\beta$ a closed smooth $(1,1)$-form on X. In other words $[-\log \|s\|^2]$ is a Green current for the divisor $Y$. For the next definition assume that $Y$ does not contain any irreducible component of $X$.

**Definition 2.38.** Let $Z$ be a blow-up of $X$ along $Y$ and denote by $\pi : Z \to X$ the blow-up map. A smooth form $\eta \in \Omega(X)$ with support on $X \setminus Y$ is said to be a form of logarithmic type (or log type) along $Y$ if $\pi$ is a projective morphism and there is a smooth form

$$\varphi \in \Omega(Z \setminus \pi^{-1}(Y))$$

such that

(i) $\pi^{-1}(Y) \subset Z$ is a divisor and $\pi$ is a $\mathscr{C}^\infty$ map over $X \setminus Y$,

(ii) $\eta = \pi\Big(\varphi|_{Z\setminus\pi^{-1}(Y)}\Big)$, and

(iii) For any point $x \in Z$, there is an open neighbourhood $U$ of $x$ such that

$$\pi^{-1}(Y) \cap U = \{z_1 = \cdots = z_k = 0\},$$

for $k \leq n$ and $z_j$ local coordinates in $U$. Moreover, there exist $\alpha_1, \ldots, \alpha_k, \beta \in \Omega^\infty(U)$ such that the $\alpha_j$'s are $\bar\partial$ and $\bar\partial$-closed and

$$\varphi|_U = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta. \tag{2.17}$$

**Theorem 2.39** (Log Type Green forms existence)**.** Let Y $\subset$ X be as above of codimension $n$. Then there exists a Green form of log type for Y.

This is Theorem 1.3.5 in [GS90]. For the reader's convenience we include here their proof.

*Proof.* Let $Z$ be a blow-up of $X$ along $Y$. Choose a line bundle $\mathscr{L}$ on $Z$, a holomorphic section $s$ in $\mathscr{L}$ with divisor $E$, and a smooth Hermitian metric on $\mathscr{L}$. Let

$$\beta = \delta_E - dd^c[\log \|s\|^2]$$

be the first Chern form of $\mathscr{L}$.

By Lemma 1.3.6 in [GS90] there is $\alpha \in \Omega^{n-1,n-1}(Z)$ such that

$$\begin{aligned} \pi_*(\delta_E \wedge \alpha) &= \delta_Y \\ &\Rightarrow dd^c([-\log\|s\|^2 \wedge \alpha]) + \delta_{\mathbf{E}} \wedge \alpha = \beta \wedge \alpha. \end{aligned} \tag{2.18}$$

And since $\delta_E \wedge \alpha$ represents $\pi^*[Y]$ in $H^{n,n}(Z, \mathbb{R})$ so does $\beta \wedge \alpha$.

Let $\omega$ be a closed form of type $(n, n)$ on X representing $[Y]$. The closed forms $\pi^*(\omega)$ and $\beta \wedge \alpha$ are cohomologous on $Z$, hence by Theorem 2.34 there exists a smooth $(n-1, n-1)$ form $\gamma$ on $Z$ such that

$$dd^c\gamma = \pi^*(\omega) - \beta \wedge \alpha.$$

Therefore

$$dd^c([-\log\|s\|^2 \wedge \alpha + \gamma]) + \delta_{\mathbf{E}} \wedge \alpha = \pi^*(\omega),$$

and

$$dd^c([g]) + \delta_Y = \omega,$$

where $g = \pi_*(-\log\|s\|^2 \wedge \alpha + \gamma)$ is a Green form of log type for $Y$. □

## 2.5 Berkovich Spaces

We will use this construction later on. This is an analytic space that solves the apparent issue that any algebraic closure of a non-Archimedian complete field is totally disconnected. And so analytical notions like a metric do not make sense. We take this from Silverman's book [Sil07] although we learned the topic through Benedetto's book [Ben19].

Let $K$ be field endowed with a non-Archimedian norm $\|\cdot\|_v$, and denote by $\mathbb{C}_v$ its algebraic closure.

For any $R > 0$, the Gauss norm $\|\cdot\|_R$ on $C_p[z]$ is defined by

$$\|f\|_R = \max_{n\geq 0} |c_n| R^n \quad \text{where} \quad f(z) = \sum_n c_n z^n \in C_p[z].$$

A $\|\cdot\|_R$-bounded seminorm on $C_p[z]$ is a nonconstant map

$$|\cdot| : C_p[z] \longrightarrow \mathbb{R}$$

with the following properties:

(i) $|f| \geq 0$ for all $f \in C_p[z]$.

(ii) $|fg| = |f| \cdot |g|$ for all $f, g \in C_p[z]$.

(iii) $|f + g| \leq \max\{|f|, |g|\}$ for all $f, g \in C_p[z]$.

(iv) $|f| \leq \|f\|_R$ for all $f \in C_p[z]$. (This is the boundedness condition.)

Berkovich [Ber90] proves that every bounded seminorm on $C_p[z]$, comes from one of the four types of Berkovich points.

| | Point | Seminorm |
|---|---|---|
| Type-I | $\xi_{a,0}$ | $\|f\|_{a,0} = \|f(a)\|$ |
| Types-II & III | $\xi_{a,r}$ | $\|f\|_{a,r} = \sup_{z \in D(a,r)} \|f(z)\|$ |
| Type-IV | $\xi_{a,r}$ | $\|f\|_{a,r} = \lim_{n \to \infty} \|f\|_{a,r_n}$ |

TABLE 2.1: Seminorms on $C_p[z]$ associated to Berkovich points.

Notice that the seminorm $|\cdot|_{a,1}$ corresponding to the Berkovich point $\xi_{0,1}$ is exactly the Gauss norm $|f|_{0,1} = \|f\|_1$, so following Baker and Rumely, we call $\xi_{0,1}$ the Gauss point.

Now that we know what are the points of the Berkovich space let us introduce the topology we endow it with.

**Definition 2.40.** The Gel'fond topology on the unit (closed) disk $\bar{D}^B$ is the weakest topology such that for every $f \in C_p[z]$ and every $B > 0$, the following sets are open:

$$U(f, B) = \{x \in D^B : |f|_x < B\} \quad \text{and} \quad V(f, B) = \{x \in D^B : |f|_x > B\}.$$

**Theorem 2.41** (Berkovich)**.** The Berkovich disk $D^B$ with the Gel'fond topology is a compact path-connected Hausdorff space.

Let us now provide a visualization of the Berkovich space.

We conclude with the construction of the Berkovich space associated to a scheme. Let $X$ be a $k$-scheme. The Berkovich space $X^{\text{an}}$ (as a set) associated with $X$ is defined as the

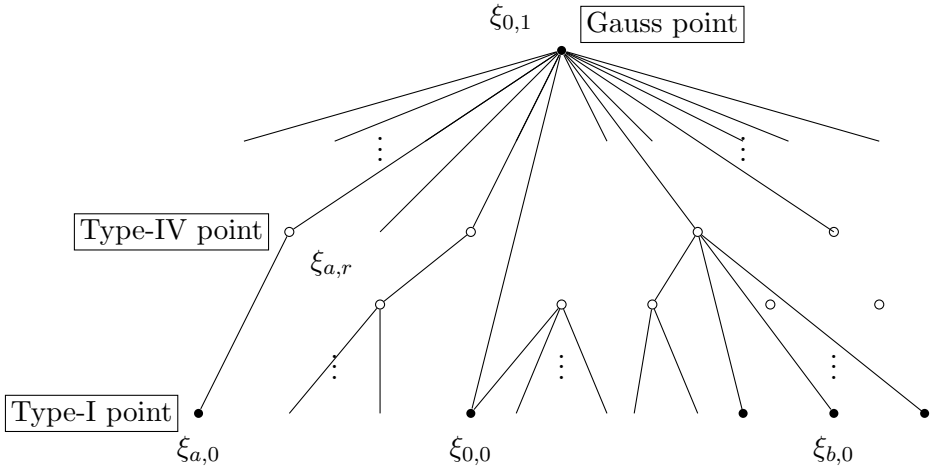



FIGURE 2.2: The Berkovich disk $D(0, B)$.

set of pairs $x = (p, |\cdot|_x)$, where $p$ is a scheme point of $X$ and $|\cdot|_x$ is an absolute value on the residue field $\kappa(p)$, which extends the absolute value $|\cdot|$ on $k$. For $x = (p, |\cdot|_x) \in X^{\mathrm{an}}$, we denote by $\kappa(x)$ the residue field $\kappa(p)$ and by $\widehat{\kappa}(x)$ the completion $\kappa(p)$ with respect to the absolute value $|\cdot|_x$.

Denote by $j : X^{\mathrm{an}} \to X$ the specialization map sending

$$(p, |\cdot|_x) \in X^{\mathrm{an}} \to p \in X.$$

The Berkovich topology on $X^{\mathrm{an}}$ is defined as the most coarse topology on $X^{\mathrm{an}}$ which makes the specification map $j : X^{\mathrm{an}} \to X$ and all functions of the form $|f|$ continuous, where $f$ runs over the set of all regular functions on Zariski open subsets of $X$.

We can put all places together and consider the topological space $X_{\mathrm{ad}} = \coprod_{v \in M_{\mathbb{Q}}} X_v^{\mathrm{an}}$, coproduct of the family $(X_v^{\mathrm{an}})_{v \in M_{\mathbb{Q}}}$. By construction, a function $\varphi$ on $X_{\mathrm{ad}}$ consists in a family $(\varphi_v)_{v \in M_{\mathbb{Q}}}$, where $\varphi_v$ is a function on $X_v^{\mathrm{an}}$, for every $v \in M_{\mathbb{Q}}$.

# Chapter 3

# Moduli Spaces

Our work is in the context of algebraic varieties not complex manifolds. From this chapter on, our main characters will be algebraic (projective) varieties. In what follows fix once and for all a projective algebraic variety $X$ of dimension $\dim X = n$.

The intention of this chapter is to set up a framework that will be appropriate for defining heights for subvarieties $Y \subseteq X$. This will come from exploring the following relation known to every STEM major after the first year of undergraduate studies. Let $H \subset \mathbb{R}^3$ be a $2-$dimensional vector subspace (a plane) and $\vec{v} \in \mathbb{R}^3$ its normal vector (see Figure 3.1). Then

$$H = \{\vec{u} \in \mathbb{R}^3 : \vec{v} \cdot \vec{u} = 0\}.$$

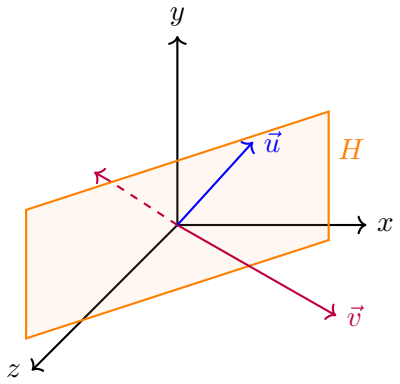



FIGURE 3.1: A plane and its normal vector

The intuition behind a moduli space comes natural when studying geometric structures. A moduli space is just a geometric space whose points represent isomorphism classes of certain mathematical objects. It's essentially a parameter space that classifies objects of a particular type up to some notion of equivalence. Here we study Grassmannians, Chow Varieties and Hilbert Schemes.

As an introduction, let us consider the first Chern class $c_1(L)$ of a line bundle $L$ on a variety $X$, which is an invariant of the line bundle (see the section on Kähler Geometry in the previous chapter). We will make use of $c_1$ to show that a particular Grassmannian is not isomorphic to a projective space. Specifically via the Adjunction formula. Given that it will only be until the next chapter when we will be endowed with enough tools to make a comprehensive study of the first chern class, we delay its formal introduction until then. As for now let us just say that $c_1(L)$ is the divisor class of any rational section $s$ of $L$.

To state the Adjunction formula we introduce the following. Let $D \subset X$ be a (Cartier) divisor. We define a line bundle $\mathcal{O}(D)$ as follows.

(a) Cover $X$ with open sets $\{U_\alpha\}$ such that on each $U_\alpha$, $D$ is defined by a rational function $f_\alpha$ (meaning $D|_{U_\alpha} = \mathrm{div}(f_\alpha)$).

(b) Define the line bundle $\mathcal{O}(D)$ by transition functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ on the intersections $U_\alpha \cap U_\beta$.

(c) These transition functions satisfy the cocycle condition $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$, giving a well-defined line bundle.

**Proposition 3.1** (Adjunct Formula)**.** Let $X$ be a variety of dimension $d$ and $Y \subset X$ a subvariety of dimension $(d-1)$. Then

$$\omega_Y = \omega_X|_Y \otimes \mathcal{O}(Y)|_Y.$$

## 3.1 Grassmannians

Let $V$ be a vector space of dimension $n$ over a field $\mathbb{K}$ with a chosen basis $e_1, \ldots, e_n$. The Grassmannian of dimension $k$ parametrizes the $k-$dimensional vector subspaces of $V$. Concretely

**Definition 3.2** (Grassmannian)**.** The Grassmannian $G(k,n)$ is defined as:

$$G(k,n) := \{\Lambda \subset V : \text{ such that } \dim \Lambda = k\}.$$

Or equivalently,

$$G(k,n) = \mathbb{G}(k-1, n-1) := \{\Lambda \subset \mathbb{P}(V) : \text{ such that } \dim \Lambda = k-1\}.$$

Since $\dim \Lambda = k$ there are $\{a_{i,j}\}_{1\leq i\leq n, 1\leq j\leq k}$ such that $\Lambda$ is the row space of the matrix

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \dots & a_{k,n} \end{bmatrix}$$

Since $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{1\leq i_1\leq \ldots \leq i_k\leq n}$ is a basis for $\bigwedge^k V$ then for $v_j = [a_{j,1}, \ldots, a_{j,n}]$ we can write

$$v_1 \wedge \ldots \wedge v_k = \sum_{1\leq i_1\leq \ldots \leq i_k\leq n} p_{i_1\ldots,i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

Note that the vector with entries $p_{i_1\ldots,i_k}$ is a well defined point in a projective space as coordinates is a well defined element of the projective space $\mathbb{P}(\Lambda^k V)$. This is because $A$ above is well defined up to a non-zero scalar. To see this note that if $\Omega$ is an invertible $k \times k$ matrix (with coefficients in $\mathbb{K}$ of course) then $A' = \Omega A$ and $A$ have the same row space. In other words,

The map

$$\Lambda \mapsto \{p_{i_1\ldots,i_k}\} \in \mathbb{P}(\Lambda^k V) \cong \mathbb{P}^{\binom{n}{k}-1} \text{ is injective.}$$

**Definition 3.3** (Plücker coordinates)**.** The $p_{i_1\ldots,i_k}$ above are called the Plücker coordinates of $\Lambda$ and the map $\Lambda \mapsto \{p_{i_1\ldots,i_k}\}$ the Plücker embedding.

Moreover the Plücker embedding gives a way to show that Grassmannians are (projective) varieties.

**Proposition 3.4.** The Grassmannian $G(k, n)$ is a projective variety.

*Proof.* We first show that $G(k, n)$ is an algebraic set i.e. it is the solution set of some polynomial equations. These equations are called the Plücker relations.

Note that for any $\zeta \in \bigwedge^k V$ we have that $e_i \wedge \zeta = 0$ if and only if $\zeta = e_i \wedge \zeta'$ for some $\zeta' \in \bigwedge^{k-1} V$. Inductively then $\zeta = w_1 \wedge \ldots \wedge w_k$ if and only if

$$\dim \ker \Big( \wedge \zeta : V \to \bigwedge^{k+1} V \Big) \geq k,$$

where $\wedge\zeta(w) = w \wedge \zeta$. In other words

$$G(k, n) = \{\zeta \in \bigwedge^k V : \mathrm{rk}(\wedge\zeta) \leq n - k.\}$$

And since the set in the right above is the zero set of the polynomial equations given by the $(n-k+1)-$minors of the matrix associated to $\wedge\zeta$ with respect to the basis $e_1, \ldots, e_n$, our first claim follows.

We must only see now that the ideal generated by these polynomials above is prime.

Consider the homomorphism of rings

$$\mathbb{K}[p_{i_1\ldots,i_k}]_{1\leq i_1\leq\ldots\leq i_k\leq n} \to \mathbb{K}[x_{i,j}]_{1\leq i\leq n, 1\leq j\leq k}$$

defined in the Plücker coordinates $p_{i_1\ldots,i_k}$ by sending $p_{i_1\ldots,i_k}$ to the corresponding Plücker coordinate of the generic matrix $(x_{i,j})$. The ideal is then the kernel of this map and since kernels of homomorphism of integral domains are prime ideals, we are done. □

We already know some Grassmannians. When $k=1$ then $G(k,n)=\mathbb{P}^{n-1}$. By the relation between $2-$dimensional and $1-$dimensional vector subspaces of $\mathbb{K}^3$ (see 3.1) we also may conceive $G(2,3)$ as $\mathbb{P}^2$. Let us now present an example of a Grassmannian that is not a projective space.

**Example 3.5.** Consider $G(2,4)$. The Plücker embedding sends $G(2,4)$ to $\mathbb{P}(\Lambda^2\mathbb{K}^4)\cong\mathbb{P}^5$ with coordinates given by the condition that $A$ has not a full rank where

$$A=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}$$

Since $A$ is not full ranked, we get $\det A'=0$ where

$$A'=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}$$

We are now ready to introduce the Plücker coordinates.

$$p_{1,2} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

$$p_{1,3} = \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} = a_{1,1}a_{2,3} - a_{1,3}a_{2,1}$$

$$p_{1,4} = \begin{vmatrix} a_{1,1} & a_{1,4} \\ a_{2,1} & a_{2,4} \end{vmatrix} = a_{1,1}a_{2,4} - a_{1,4}a_{2,1}$$

$$p_{2,3} = \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} = a_{1,2}a_{2,3} - a_{1,3}a_{2,2}$$

$$p_{2,4} = \begin{vmatrix} a_{1,2} & a_{1,4} \\ a_{2,2} & a_{2,4} \end{vmatrix} = a_{1,2}a_{2,4} - a_{1,4}a_{2,2}$$

$$p_{3,4} = \begin{vmatrix} a_{1,3} & a_{1,4} \\ a_{2,3} & a_{2,4} \end{vmatrix} = a_{1,3}a_{2,4} - a_{1,4}a_{2,3}$$

Since $\det A' = 0$, we get

$$p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0. \tag{3.1}$$

This polynomial is irreducible. And note that $G(2,4)$ is four dimensional: to determine a $2-$dimensional subspace of a four dimensional vector space we need 8 parameters but applying invertible linear transformations of $GL(2)$ two any pair of vectors will not alter the vector space they generate. This yields that the Plücker embedding realizes $G(2,4)$ as a hypersurface in $\mathbb{P}^5$ where $\mathbb{I}(G(2,4)) = \langle p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} \rangle$.

The fact that $G(2,4)$ is not isomorphic to $\mathbb{P}^4$ follows from the Adjunction formula together with the well known result (see Example 1.32 in [EH16])

$$\operatorname{div}(\omega_{\mathbb{P}^N}) = -(N+1)H, \quad \text{for } H = \mathbb{V}(X_0).$$

Concretely, and since $G(2,4)$ has codimension 1 inside $\mathbb{P}^5$, the Adjunction formula gives us

$$\omega_{G(2,4)} = \mathcal{O}_{\mathbb{P}^5}(-1)|_{G(2,4)} \neq \mathcal{O}_{\mathbb{P}^5}(-6) = \omega_{\mathbb{P}^5}.$$

As a projective variety, any Grassmannian $G(k,V)$ is covered by affine spaces. We claim that $G(k,V)$ is covered by affine spaces isomorphic to $\mathbb{A}^{k(n-k)}$. This is just due to the following observation (essentially the same observation than that of Figure 3.1 but for vector subspaces of dimension $k$ and not just 1):

Let $W \subset V$ be of dimension $k$ and fix an orthogonal complement $U$. Then every invertible linear transformation $T : V = W \bigoplus U \to W \bigoplus U$ such that $T(W) = W$ must satisfy $T(U) \cong U$.

Concretely, fix $A \subset V$ a vector subspace of dimension $n - k$.

The set

$$U_A = \{W \subset V : \text{ W is } 4 - \text{dimensional and } A \cap W = \vec{0}\}$$

is open in $G(k, V)$ (it is the complement of $\mathbb{V}(\mathbb{I}(A))$). Fix $\zeta \in U_A$ such that $V = A \bigoplus \zeta$. So $\zeta \subset V$ is $4-$dimensional and we have a diagram (see Figure 3.2) such that every $\Lambda \subset V$ for which $V = A \bigoplus \Lambda$, we have that $\Lambda$ is isomorphic to $\zeta$ via a map $\pi_2$.

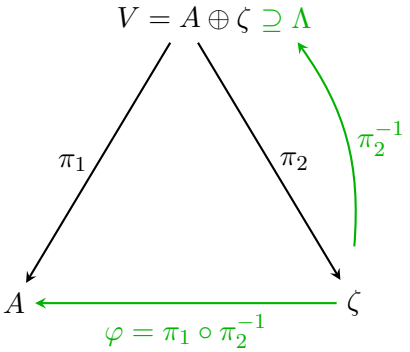



FIGURE 3.2: Decomposition of $k, (n-k)-$dimensional vector subspaces.

Then such a $\Lambda$ corresponds to the graph of the map $\varphi$ in the diagram. Conversely the graph on any map $\varphi : \zeta \to A$ yields a vector subspace $\Lambda \subset V$ such that $V = A \bigoplus \Lambda$ i.e. $\Lambda$ is $k$ dimensional. We have just proven that $U_A \cong \hom_{\mathbb{K}}(\zeta, A)$. Since $\hom_{\mathbb{K}}(\zeta, A) \cong \mathbb{A}^{k(n-k)}$ we are done.

As a projective variety, there is a graded ring $\mathcal{B} = \bigoplus_d \mathcal{B}_d$ which we and others refer to as the coordinate ring of the Grassmannian. Setting $\mathbb{I}\left(G(k,n)\right) \subset \mathbb{K}[p_{i_1 \ldots, i_k}]_{1 \leq i_1 \leq \ldots \leq i_k \leq n}$ to be the ideal generated by the Plücker relations, we have of course

$$B \cong \left(\mathbb{K}[p_{i_1 \ldots, i_k}]_{1 \leq i_1 \leq \ldots \leq i_k \leq n} / \mathbb{I}\left(G(k,n)\right)\right).$$

### 3.1.1 Associated Hypersurfaces

Suppose $X \subset \mathbb{P}^N$ has dimension $k$ and degree $D$. Consider the set $\mathcal{Z}(X)$ of all $(N-k-2)$-dimensional projective subspaces $L$ in $\mathbb{P}^{N-1}$ that intersect $X$. This is a subvariety in the Grassmannian $G(N-k-1, N)$ parametrizing all the $(N-k-2)$-dimensional projective subspaces in $\mathbb{P}^N$.

**Proposition 3.6** (Proposition 3.3.2 in [GKZ08])**.** The subvariety $\mathcal{Z}(X)$ is an irreducible hypersurface in $G(N-k-1,N)$.

*Proof.* We analyze the structure of $\mathcal{Z}(X)$ using the incidence variety framework. Define the correspondence

$$B(X) = \{(x, L) \mid x \in X, L \in G(N-k-1,N), x \in L\},$$

which has natural projections $q : \mathcal{Z}(X) \leftarrow B(X) \rightarrow X : p$. Here, $q$ forgets $x$, while $p$ forgets $L$.

The fiber of $p$ at $x \in X$ is the Grassmannian $G(N-k-2,N-1)$, making $p$ a Grassmannian fibration. Since $X$ is irreducible, so is $B(X)$. Now, a generic $(N-k-2)$-dimensional subspace $L$ intersects $X$ at a single point, the map $q$ is birational. Then $\mathcal{Z}(X)$ is also irreducible. Hence

$$\begin{aligned}\dim \mathcal{Z}(X) &= \dim B(X) = (N-k-2)k + (k-2) \\ &= k(N-k-1) - 2 = \dim G(N-k-1,N-1) - 1.\end{aligned}$$

i.e. $\mathcal{Z}(X)$ is indeed a hypersurface. □

**Definition 3.7** (Chow Form)**.** We call the element $R_X$ in the coordinate ring of $G(N-k,N)$ which vanishes at $\mathcal{Z}(X)$ the Chow form of $X$.

Concretely $R_X$ can be defined as follows.

Consider $L \subset \mathbb{K}^n$ a $k$-dimensional vector subspace and let $v_1, \ldots, v_k$ be a basis of $L$.

Let

$$M = \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,1} & v_{k,2} & \cdots & v_{k,n} \end{pmatrix}$$

be the matrix formed by the coordinates of the $v_i$'s. Since the $v_j$'s form a basis of $L$, $M$ is full rank. And for every $k \times n$ full rank matrix we assign the vector subspace of $\mathbb{K}^n$ generated by its rows. This allows us to unambiguously define the Stiefel coordinates of $L$ as $\{v_{i,j}\}$, the entries of $M$. We denote by $S(k,n)$ the (affine) variety of all full rank $k \times n$ complex matrices. Then

$$G(k,n) \cong S(k,n)/\mathrm{GL}(k). \tag{3.2}$$

Now fix a lifting of the associated hypersurface $\mathcal{Z}(X)$ to $S(k,n)$ and its closure $\tilde{\mathcal{Z}}(X)$ in the affine space $\mathrm{Mat}(k,n)$. By the above we may think of the elements in $\mathrm{Mat}(k,n)$ as subspaces of dimension at most $k$ and such a space $N$ is given by

$$N = \{f_1 = 0, \ldots, f_{n-k} = 0\}$$

where $f_j(x_1, \ldots, x_n) = \sum_i c_{ij} x_i$.

Thus $\tilde{\mathcal{Z}}(X)$ is the subvariety of $\mathbb{V}(\langle f_1, \ldots, f_k \rangle)$ such that

$$X \cap \{f_1 = \cdots = f_k = 0\} \neq \emptyset.$$

Since $\mathcal{Z}(X)$ is a hypersurface, it follows that $\tilde{\mathcal{Z}}(X)$ is also a hypersurface. The defining polynomial of $\tilde{\mathcal{Z}}(X)$ is a homogeneous polynomial $\tilde{R}_X(f_1, \ldots, f_k)$. We call $\tilde{R}_X$ the $X$-resultant since its vanishing expresses the compatibility of the system. It is just another way of writing the Chow form $R_X$.

## 3.2 Chow Varieties

Grassmannians have a rich geometric and combinatorial structure which has been exhaustively studied. One remarkable property is that they yield the existence of both Chow Varieties and Hilbert Schemes as we will see below. For our purposes though we find Chow Varieties (and specifically Generalized Chow Varieties) more appropriate because they capture the dynamical behaviour of a subvariety under the iteration of a morphism. The reason being that they take into account the degree of a subvariety.

### 3.2.1 Degree

The degree of $X \subset \mathbb{P}^N$ has been classically defined as the number of intersection points between $X$ and certain linear subspaces $L \subset \mathbb{P}^N$. This notion is not easy to use in practice. Therefore, we define the degree using the Hilbert polynomial. This also allows us to feed two birds with one scone, since we need this polynomial to define the Hilbert scheme.

**Definition 3.8** (Graded module)**.** A graded $S$-module is an $S$-module $M$ together with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $S_d M_e \subset M_{d+e}$ for all $d, e \in \mathbb{Z}$. The homogeneous components $M_d$ are called the graded components of $M$.

Let $M = \bigoplus_d M_d$ be a finitely generated graded $S-$module where $S = \mathbb{K}[x_0, \ldots, x_n]$ (i.e. $M$ is a homogeneous ideal or a quotient) and $S_d$ are the symmetric polynomials of degree $d$ in the variables $x_0, \ldots, x_n$.

The Hilbert function $\varphi_M : \mathbb{Z} \to \mathbb{N}$ is defined as

$$\varphi_M(t) := \dim_{\mathbb{K}}(M_l).$$

**Theorem 3.9** (Hilbert-Serre, Theorem 7.5 in [Har13])**.** Let $M$ be as above. Then there exists a unique polynomial $P_M(t) \in \mathbb{Q}[t]$ such that $P_M(l) = \varphi_M(l)$ for all sufficiently large $l$. Moreover, $\deg P_M = \dim \mathbb{V}(\mathrm{Ann}(M))$ where $\mathrm{Ann}(M) = \{s \in S : sM = 0\}$.

**Definition 3.10** (Hilbert Polynomial)**.** For a projective $\mathbb{K}$-variety $V \subseteq \mathbb{P}^n$, the Hilbert polynomial of $V$ is defined as the Hilbert polynomial of its coordinate ring $k[V]$.

**Example 3.11.** Consider $M = \mathbb{K}[\mathbb{P}^n] = \mathbb{K}[x_0, x_1, \ldots, x_n]$ and

$$M_d = \{\text{polynomials of degree } d\} \cup \{0\}.$$

A straightforward calculation shows that $\varphi_M(l) = \binom{l+n}{n}$. Therefore, the Hilbert polynomial is

$$P_M(t) = \binom{t+n}{n} = \frac{1}{n!}(t+n)(t+n-1)\ldots(t+1),$$

which has degree $n$ as expected.

**Definition 3.12** (Degree)**.** The degree, $\deg(V)$, of a projective $\mathbb{K}$-variety $V \subseteq \mathbb{P}^n$ is $r!h_r$, where $h_r$ is the coefficient of the $t^r$-term in the Hilbert polynomial of $V$ and $r$ is the dimension of $V$.

*Remark* 3.13. The degree depends on the ambient space. As an example consider the Veronese embedding

$$v_d \colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}, \quad [X_0 : \ldots : X_n] \mapsto \text{ all monomials of degree } d \text{ in } X_0, \ldots, X_n.$$

Consider a polynomial of degree $m$ on $\mathbb{P}^{\binom{n+d}{d}-1}$ and restrict it to the image of the Veronese map $v_d$. The degree of this polynomial becomes $dm$. Since, $P_{\mathbb{P}^n}(m) = \binom{n+m}{n}$ we conclude $P_{v_d(\mathbb{P}^n)}(m) = \binom{n+dm}{n}$. Expanding this as a polynomial, the lead term is $\frac{d^n}{n!}m^n$, and so the degree of $v_d(\mathbb{P}^n)$ is $d^n$. Nevertheless, it is well known (see page 53 in [SR94]) that $\mathbb{P}^n \cong v_d(\mathbb{P}^n)$.

**Proposition 3.14** (Proposition 7.6 in [Har13])**.** The degree function satisfies the following properties.

(a) If $Y \subseteq \mathbb{P}^r$, $Y \neq \emptyset$, then the degree of $Y$ is a positive integer.

(b) $\deg \mathbb{P}^m = 1$

(c) Let $Y = Y_1 \cup Y_2$, where $Y_1$ and $Y_2$ have the same dimension $r$, and where $\dim(Y_1 \cap Y_2) < r$. Then $\deg Y = \deg Y_1 + \deg Y_2$.

(d) If $H \subseteq \mathbb{P}^r$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree $d$, then $\deg H = d$.

*Proof.*

(a) Suppose $P_Y$ is a polynomial of degree $r = \dim Y$. We can write $P_Y(t)$ as

$$P_M(t) = c_0\binom{t}{r} + c_1\binom{t}{r-1} + \ldots + c_r, \tag{3.3}$$

where $\binom{t}{j} = \frac{1}{j!}t(t-1)\ldots(t-j+1)$ (see Proposition 7.3 in [Har13]).

Then $\deg Y = c_0 \in \mathbb{Z}$. It is positive because for $l \gg 0$, we have $P_Y(l) = \varphi_{S/\mathbb{I}(Y)}(l) \geq 0$.

(b) We already know that the Hilbert polynomial of $\mathbb{P}^n$ is $P(t) = \binom{t+n}{n}$. So its leading coefficient is $\frac{1}{n!}$ and the result follows.

(c) Let $I_1 = \mathbb{I}(Y_1)$ and $I_2 = \mathbb{I}(Y_2)$ so that $I = I_1 \cap I_2$ satisfies $I = \mathbb{I}(Y_1 \cup Y_2)$.

Consider the exact sequence:

$$0 \to S/I \to S/I_1 \oplus S/I_2 \to S/(I_1 + I_2) \to 0.$$

This gives us:

$$P_{S/I}(z) = P_{S/I_1}(z) + P_{S/I_2}(z) - P_{S/(I_1+I_2)}(z).$$

Since $\dim(Y_1 \cap Y_2) < r$, the leading term of $P_{S/(I_1+I_2)}(z)$ has degree less than $r$. Therefore, when comparing the coefficients of $z^r$, we get:

$$\frac{\deg Y}{r!} = \frac{\deg Y_1}{r!} + \frac{\deg Y_2}{r!}.$$

This yields $\deg Y = \deg Y_1 + \deg Y_2$, completing our proof.

(d) For a hypersurface $H$ defined by a homogeneous polynomial $f$ of degree $d$, we have an exact sequence of graded $S$-modules:

$$0 \to S(-d) \xrightarrow{f} S \to S/(f) \to 0.$$

This gives us the relation between Hilbert functions:

$$\varphi_{S/(f)}(l) = \varphi_S(l) - \varphi_S(l-d)$$

Substituting $\varphi_S(l) = \binom{l+r}{r}$, we get:

$$\varphi_{S/(f)}(l) = \binom{l+r}{r} - \binom{l-d+r}{r}.$$

The Hilbert polynomial $P_H(t)$ can be derived as:

$$P_H(t) = \binom{t+r}{r} - \binom{t-d+r}{r} = \frac{d}{(r-1)!}t^{r-1} + \text{lower degree terms.}$$

From this, we conclude that $\deg H = d$. □

### 3.2.2 Bezout's Theorem

For a $S-$module $M$, we denote by $M(l)$ the twisted module resulting by shifting $l$ places to the left i.e. the homogeneous component $M(l)_d = M_{l+d}(l)$. We refer the reader to Proposition 7.4 in [Har13] for the next paragraphs.

For every $S$-module $M$ there exists a filtration. That is a sequence of submodules $M_j$ $(0 \leq j \leq n)$ such that

$$0 = M_n \subset \ldots \subset M_0 = M, \text{ and}$$

$$M_{i-1}/M_i \cong (S/\mathfrak{p}_j)(l_j), \quad \text{for } \mathfrak{p}_j \text{ a homogeneous prime ideal and } l_j \in \mathbb{Z}.$$

The length of such a filtration is the number of links on it.

Let $\mathfrak{p}$ be a prime ideal of a graded $S$-module $M$ that is minimal with respect to the property that it contains $\mathrm{Ann}(M)$. We define the multiplicity of $M$ at $\mathfrak{p}$ denoted by $\mu_{\mathfrak{p}}(M)$ to be the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$ where $M_{\mathfrak{p}}, S_{\mathfrak{p}}$ denote the localizations of both $M, S$ at $\mathfrak{p}$ (see [AM18]) respectively.

Suppose $H$ is a hypersurface not containing our variety $X$. Then

$$X \cap H = Z_1 \cup \ldots \cup Z_s, \text{ where } \forall 1 \leq j \leq s(\dim Z_j = n-1).$$

Let $\mathfrak{p}_j = \mathbb{I}(Z_j)$. We define the intersection multiplicity of $X$ and $H$ along $Z_j$ to be

$$i(X, H; Z_j) := \mu_{\mathfrak{p}_j}\big(S/(\mathbb{I}(X) + \mathbb{I}(H)\big). \tag{3.4}$$

**Theorem 3.15.** Let $Y \subset \mathbb{P}^r$ be a variety of dimension $s \geq 1$ in $\mathbb{P}^r$, and let $H$ be a hypersurface not containing $Y$. Let $Z_1, ..., Z_t$ be the irreducible components of $Y \cap H$. Then:

$$\sum_{i=1}^{t} i(Y, H; Z_i) \cdot \deg Z_i = (\deg Y)(\deg H).$$

*Proof.* See Theorem 7.7 in [Har13] □

**Corollary 3.16** (Bezout's Theorem)**.** Let $Y, Z$ be two distinct curves in $\mathbb{P}^2$ with degrees $d, e$ respectively. Suppose $Y \cap Z = \{P_1, \ldots P_s\}$. Then

$$\sum_{j=1}^{s} i(Y, Z; P_j) = de.$$

*Proof.* A point has Hilbert polynomial 1 hence degree 1. □

### 3.2.3 Algebraic Cycles

The group of cycles on $X$, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties of $X$. The group $Z(X)$ is graded by dimension: we write $Z_k(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension $k$ (these are called $k$-cycles), so that $Z(X) = \bigoplus_k Z_k(X)$. A cycle $Z = \sum_j n_j V_j$, where the $V_j$ are subvarieties, is effective if the coefficients $n_i$ are all nonnegative. The $V_i$ for which $n_i \neq 0$ are the support of $Z$. The degree of $Z$ is $\sum_j n_j m_j$ where $d_j$ is the degree of $V_j$. A (Weil) divisor is just a $(n-1)$-cycle.

The rational equivalences among $k$-cycles, are generated as follows. For any $(k+1)$-dimensional subvariety $W$ of $X$, and any nonzero rational function $r \in K[W]^*$, define a

$k$-cycle on $X$ by

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V].$$

This generates a subgroup $\mathrm{Rat}_k X$, the subgroup of cycles rationally equivalent to 0.

Alternatively, $\mathrm{Rat}(X) \subset Z(X)$ is defined as the subgroup generated by

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle,$$

where $t_0, t_1 \in \mathbb{P}^1$ and $\Phi \subset \mathbb{P}^1 \times X$ is a subvariety not contained in any fiber of the projection $\mathbb{P}^1 \times X \to \mathbb{P}^1$. And we say that two cycles are rationally equivalent if their difference is in $\mathrm{Rat}(X)$.

We introduce now the Chow group via these two notions of rational equivalence. The fact that the two definitions are equivalent is Proposition 1.4 in [EH16]

**Definition 3.17** (Chow Group)**.** The Chow group $A(X)$ of $X$ is defined as

$$\begin{aligned} A(X) &= Z(X)/\mathrm{Rat}(X) \\ &= \bigoplus_k A_k(X), \quad \text{where } A_k(X) = Z_k(X)/\mathrm{Rat}_k X. \end{aligned} \tag{3.5}$$

Naturally if $D = \sum n_j[V_j]$ and $E = \sum m_j[V_j]$ then $D + E = \sum (n_j + m_j)V_j$.

Chow groups are hard to calculate in general. For instance (see 3.20 below) $A_1(X)$ is isomorphic to the group $\mathrm{Pic}(X)$. So if $X$ is an elliptic curve then $A_1(X)$ is an uncountable abelian group. Of course, there are cases in which we can calculate it. Let us see some examples.

**Proposition 3.18.** For the affine space $\mathbb{A}^n$, the Chow group is isomorphic to the free group generated by the class of the space itself:

$$A(\mathbb{A}^n) \cong \mathbb{Z} \cdot [\mathbb{A}^n]$$

*Proof.* Consider an arbitrary proper subvariety $Y \subset \mathbb{A}^n$. We prove that $Y$ is rationally equivalent to 0.

Start by choosing coordinates $z = (z_1, \ldots, z_n)$ such that the origin does not belong to $Y$ and define the set:

$$W^\circ = \{(t, tz) \in (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n \mid z \in Y\}$$

This set can be characterized as the vanishing locus of functions of the form $f(z/t)$ where $f$ vanishes on $Y$.

For each nonzero $t \in \mathbb{A}^1$, the fiber of $W^\circ$ over $t$ is precisely $tY$-the image of $Y$ under the scaling automorphism of $\mathbb{A}^n$. Let $W$ denote the closure of $W^\circ$ in $\mathbb{P}^1 \times \mathbb{A}^n$. The irreducibility of $W^\circ$ implies that $W$ is also irreducible.

The fiber of $W$ at $t = 1$ is precisely $Y$. Since the origin does not lie in $Y$, there exists a polynomial $g(z)$ that vanishes on $Y$ with nonzero constant term $c$. The function $G(t, z) = g(z/t)$ extends to a regular function on $(\mathbb{P}^1 \setminus 0) \times \mathbb{A}^n$ with constant value $c$ on the fiber above $t = \infty$. Consequently, the fiber of $W$ over $t = \infty$ is empty, establishing that $Y$ is rationally equivalent to zero. □

Next in line is the projective space of course. In the next chapter we will learn its full structure but for now let us exhibit the structure of $A_n(\mathbb{P}^n)$.

**Proposition 3.19.** Let $X$ be an irreducible variety of dimension $d$. Then $A_d(X) \cong \mathbb{Z}$.

*Proof.* This is a direct result of $X$ being irreducible since that yields both $Z_d(X) = \mathbb{Z}\cdot[X]$ and $\operatorname{Rat}_d X = 0$. □

Now let us show another relation between algebra and geometry. Recall that the Picard group $\operatorname{Pic}(X)$ is the group of isomorphism classes of line bundles on $X$.

**Theorem 3.20.** Let $X$ be a smooth projective variety. Then there is a natural isomorphism between the codimension-1 Chow group $A_{n-1}(X)$ and the Picard group $\operatorname{Pic}(X)$.

*Proof.* We will define explicit maps in both directions that are inverses to each other.

Define the map $\Phi : A_1(X) \to \operatorname{Pic}(X)$ as follows. For any divisor $D = \sum_i n_i D_i$ (where $D_i$ are prime divisors), recall the construction of the line bundle $\mathcal{O}(D)$:

(a) Cover $X$ with open sets $\{U_\alpha\}$ such that on each $U_\alpha$, $D$ is defined by a rational function $f_\alpha$ (meaning $D|_{U_\alpha} = \operatorname{div}(f_\alpha)$).

(b) Define the line bundle $\mathcal{O}(D)$ by transition functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ on the intersections $U_\alpha \cap U_\beta$.

(c) These transition functions satisfy the cocycle condition $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$, giving a well-defined line bundle.

If $D$ and $D'$ differ by a principal divisor, $D - D' = \operatorname{div}(h)$ for some rational function $h$, then $\mathcal{O}(D) \cong \mathcal{O}(D')$. This is because if $D|_{U_\alpha} = \operatorname{div}(f_\alpha)$ and $D'|_{U_\alpha} = \operatorname{div}(f'_\alpha)$, then $f'_\alpha = h \cdot f_\alpha$ for some rational function $h$. So $\Phi$ is well defined.

Now we define the map $\Psi : \operatorname{Pic}(X) \to A_1(X)$ as one would expect. Let $L$ be a line bundle on $X$ and let $s$ be any meromorphic section of $L$. Then $\Psi(L) = \operatorname{div}(s)$.

This map is well defined. If $s$ and $s'$ are two meromorphic sections of $L$, then $s' = f \cdot s$ for some rational function $f$, thus $\operatorname{div}(s') = \operatorname{div}(f) + \operatorname{div}(s)$.

It is clear that these maps are inverses to each other. □

**Definition 3.21.** Under the notation of the proof above we define the degree deg of a line bundle $L$ to be

$$\deg L = \deg \mathcal{O}(D).$$

We conclude this section with an overview of the behaviour of cycles under a morphism.

Let $\sum_j n_j V_j = Z \in Z_k(X)$ and $f : X \to Y$ a morphism. Suppose $f(V_j) = W_j$. If $\dim W_j < \dim V_j$ define $f_*(V_j) = 0$. Otherwise $f_*(V_j) = \deg(V_j/W_j)[W_j]$ where $\deg(V_j/W_j) = [\mathbb{K}(V) : \mathbb{K}(W)]$. By linearity we define the push forward of $Z$ as $f_*(Z) = \sum n_j \deg(V_j/W_j)[W_j]$.

Pullbacks of cycles, on the other hand are only defined when $f$ is flat. We recall that $f$ is flat if the induced map $f : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ where $y = f(x)$ makes $\mathcal{O}_{Y,y}$ a flat $\mathcal{O}_{X,x}$-module.

Recall that

$$\mathcal{O}_{X,x} = \{f : U \to \mathbb{K} \mid U \subset x \text{ is open}, X \in U, f \text{ is regular on } U\}/\sim$$

where $f : U \to \mathbb{K} \sim g : W \to \mathbb{K}$ if and only if $f = g$ on $U \cap W$.

If $f$ is flat then we define the pullback $f^*[W] := [f^{-1}(W)]$ for $W \in Z_k(Y)$.

We note that the equality $\deg V = \deg(V/W) \deg W$ does not hold in general. Take for an example the Veronese embedding of $V = \mathbb{P}^1$ in $\mathbb{P}^2$. We now $V$ has degree 1 but the degree depends on the ambient space. Indeed $\mathbb{P}^1$ seen as the subspace $(x^2)(y^2) - (xy)^2$ of $\mathbb{P}^2$ by the Veronese embedding has degree 2.

The coordinate rings of $V = \mathbb{P}^1$ and its image $W$ are

$$k[V] = k[x, y]$$
$$k[W] = k[x^2, xy, y^2]/(x^2y^2 - (xy)^2)$$

Thus $\deg(V/W) = 1$ since setting $w = x^2, u = xy, z = y^2$ the above yields

$$k(V) = k(x/y)$$
$$k(W) = k(w/z, u/z) = k(w/z) = k(x/y)$$

### 3.2.4 Chow Forms

We introduced last section the Chow form $R_X$ of $X$. This is a polynomial $R_X(f_0, \dots, f_n) \in \mathbb{K}[U_0, \dots, U_n]$ and is the zero locus of

$$\{f_0 = \cdots = f_n = 0\} \cap X \subset \mathbb{P}^N.$$

$R_X$ satisfies the following homogeneity property by just changing coordinates: For $d = \dim X$ and any $g = (g_{ij}) \in GL(n+1)$ we have

$$R_X\big(g_{00}f_1 + \cdots + g_{0n}f_n, \dots, g_{n0}f_0 + \cdots + g_{nn}f_n\big) = \det(g)^d R_X(f_0, \dots, f_n). \tag{3.6}$$

For any $k$-cycle $Z = \sum_j n_j V_j$ of degree $d$ we define

$$R_Z = \prod_j R_{V_j}^{n_j} \in \mathcal{B}_d. \tag{3.7}$$

The coordinates of $R_Z$ are called the Chow coordinates of $Z$ and are well defined up to a scalar in $\mathbb{K}$ (i.e. its coordinates are defined over the projective space). This homogeneous form lets us build a projective variety that parametrizes all such cycles.

Let $G(k, d, N)$ be the set of all $(k-1)$-dimensional algebraic cycles in $\mathbb{P}^N$ of degree $d$. Note that $G(k, 1, N)$ coincides with the Grassmannian $G(k, N)$.

**Theorem 3.22.** The map $X \mapsto R_X$ defines an embedding of $G(k, d, n)$ into the projective space $\mathbb{P}(\mathcal{B}_d)$ as a closed algebraic variety.

*Proof.* See Theorem 4.1.1 in [GKZ08]. □

The algebraic structure we endow $G(k,d,n)$ via this identification yields the projective variety that parametrizes all such cycles.

**Definition 3.23** (Chow Variety)**.** $G(k,d,n)$ is called the Chow variety and its embedding into $\mathbb{P}(\mathcal{B}_d)$ is called the Chow embedding.

We already mentioned that $G(k,n)$ is a Chow variety. Consider now the Chow variety $G(N-1,d,N)$, parametrizing cycles of degree $d$ and codimension 1 in $P^{N-1}$. Algebraic cycles of codimension 1 correspond to all nonzero homogeneous polynomials, (irreducible or not) of degree $d$. Thus the Chow variety $G(N-1,d,N)$ is the projective space of such polynomials:

$$G(N-1,d,N) = \mathbb{P}(\mathrm{Sym}^d \mathbb{K}^N).$$

Another example is $G(2,2,4)$ that parametrizes 1-dimensional algebraic cycles of degree 2 in $\mathbb{P}^3$. $G(2,2,4)$ parametrizes degree 2 curves in $\mathbb{P}^3$. These can be either smooth conics (degree 2 irreducible curves) or pairs of lines (possibly coincident).

The dimension of the Chow variety $G(2,2,4)$ is 8. To see this let us count the number of parameters needed to specify a degree 2 curve in $\mathbb{P}^3$:

(i) First, we need to choose a plane in $\mathbb{P}^3$, which requires 3 parameters (equivalently, a point in $\mathbb{P}^3$).

(ii) Within this plane (which is isomorphic to $\mathbb{P}^2$), we need to specify a degree 2 curve (a conic). A conic in $\mathbb{P}^2$ is given by a degree 2 homogeneous polynomial in 3 variables, which has 6 coefficients. Since we're working projectively, we have 5 independent parameters (dimension of $\mathbb{P}^5$).

Therefore, the dimension of $G(2,2,4)$ is indeed $3+5=8$. Note how this is different from the Grassmannian $G(2,4)$, which has dimension 4 and only parametrizes lines in $\mathbb{P}^3$.

### 3.2.5 Generalized Chow Forms

The Chow form may be seen as an algebraic criterion to detect intersections of hypersurfaces with a variety. But for our setting we need to consider not only codimension 1 subvarieties but arbitrary both dimension and degree. Again, due to the behaviour of a subvariety under the iteration of a morphism. For this purpose we introduce the notion of a generalized Chow form (defined first by Philippon [Phi86]) adapting the

Bost-Gillet-Soulé ([BGS94])) construction to the case of projective varieties over an algebraically closed field $k$. More precisely, we expand on their construction for the base scheme $T = \operatorname{Spec} k$. Let $X$ be a projective variety over $k$, and $E$ be a vector bundle of rank $N+1$ over $X$.

Let $r \in [0, N]$ be an integer, and $\mathbf{d} = (d_0, \ldots, d_r)$ be an $(r+1)$-tuple of positive integers. Let $\pi : \mathbb{P}(E) \to X$ be the projectivization of $E$, and for any $i$, let $S^{d_i}E$ be the $d_i$-th symmetric product of $E$.

We define $\mathbb{P}_i = \mathbb{P}(S^{d_i}E^\vee)$ and $\mathbb{P}_i^\vee = \mathbb{P}(S^{d_i}E)$. These projective bundles over $X$ parametrize degree $d_i$ hypersurfaces in the fibers of $\mathbb{P}(E)$. Note that these projective bundles $\mathbb{P}_i$ and $\mathbb{P}_i^\vee$ have relative dimensions

$$N_i = \operatorname{rk}(S^{d_i}E) - 1 = \binom{N+d_i}{d_i} - 1.$$

The product $\mathbb{P}_i \times_X \mathbb{P}_i^\vee$ contains the incidence subvariety $I_i$, which is the variety (smooth over $X$) whose points in $k$ are pairs $(x, u)$ where $x \in \mathbb{P}_i(k)$, $u \in \mathbb{P}_i^\vee(k)$, and $\tilde{u}(\tilde{x}) = 0$ for any representatives $\tilde{x} \in (S^{d_i}E)^\vee$ and $\tilde{u} \in S^{d_i}E$ of $x$ and $u$, respectively.

Let $\Gamma = I_0 \times \cdots \times I_r$ be the product of these incidence subvarieties. We have the following diagram:

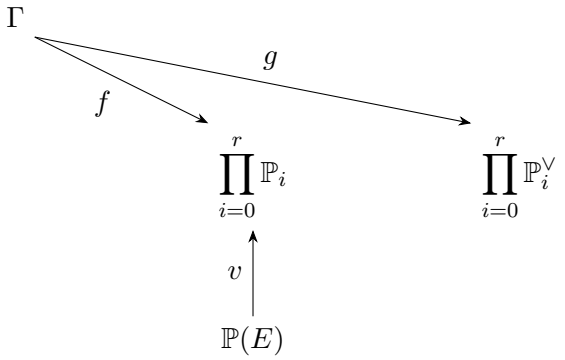



FIGURE 3.3: Construction of the Generalized Chow Form

Here, $f : \Gamma \to \prod_{i=0}^r \mathbb{P}_i$ and $g : \Gamma \to \prod_{i=0}^r \mathbb{P}_i^\vee$ are the projection maps, both of which are smooth and proper.

For $v_i : \mathbb{P}(E) \to \mathbb{P}_i$, we have the Veronese embedding attached to the very ample line bundle $\mathcal{O}(d_i)$ on $\mathbb{P}(E)$. Let $v : \mathbb{P}(E) \to \prod_{i=0}^r \mathbb{P}_i$ be the product of the $v_i$'s.

**Definition 3.24** (Generalized Chow Divisor/Form)**.** Given any subvariety $Z \subset X$ of codimension $r+1$, we define its generalized Chow divisor to be:

$$\mathrm{Ch}_{\mathbf{d}}(Z) = g_* f^* v_*(Z) = \Gamma_* v_*(Z) \tag{3.8}$$

This is a divisor in $\prod_{i=0}^{r} \mathbb{P}_i^\vee$.

We refer to the multihomogeneous form $\Phi_{\mathbf{d},Z} \in \bigotimes_{i=0}^{r} S^{\delta_i}\left(S^{d_i} E_K^\vee\right)^\vee$ for which $\mathrm{Ch}_{\mathbf{d}}(Z)$ is the divisor, the generalized Chow divisor of $Z$.

And lastly, we remark (Theorem 4.3.2 [BGS94])

$$\deg_i(\mathrm{Ch}_{\mathbf{d}}(Z)) = \left(\prod_{\substack{j=0 \\ j\neq i}}^{r} d_j\right) \deg(Z).$$

Let us see some examples.

**Example 3.25** (Classical Chow Form)**.** When $T = \mathrm{Spec}(k)$ for some field $k$ of characteristic zero, and $d_0 = \cdots = d_r = 1$, the Chow divisor $\mathrm{Ch}_{\mathbf{d}}(Z)$ is the divisor of the classical Chow form of $Z$. Its (geometric) points parametrize $(r+1)$-tuples of hyperplanes in $\mathbb{P}$ whose intersection meets the support of $Z$. It follows from this reference that $\mathrm{Ch}_{(1,\ldots,1)}(Z)$ is the classical Chow divisor of $Z$ when $k$ is algebraically closed of any characteristic.

When the characteristic of $k$ is zero, the generalized Chow form is what Philippon calls forme eliminante.

**Example 3.26** (Trivialization Case)**.** Suppose that $T = \mathrm{Spec}(A)$, where $A$ is a field or a principal ideal domain. Then $E$ may be trivialized, and we get identifications:

$$\mathbb{P} \simeq \mathbb{P}_A^N \text{ and } \prod_{i=0}^{r} \mathbb{P}_i^\vee \simeq \prod_{i=0}^{r} \mathbb{P}_A^{\binom{N+d_i}{d_i}-1} \tag{3.9}$$

Therefore, $\mathrm{Ch}_{\mathbf{d}}(Z)$ is the divisor of a multihomogeneous form $\Phi_{\mathbf{d},Z}$ with coefficients in $A$ in $\sum_{i=0}^{r} \binom{N+d_i}{d_i}$ variables (the coefficients of the "generic" homogeneous polynomials of degree $d_0, \ldots, d_r$ in $N+1$ variables), well defined up to multiplication by a unit in $A$.

**Example 3.27** (Resultants)**.** Consider the case where $K = \mathbb{Q}$, $E$ is trivial, and $r = N$. Then $\mathbb{P} = \mathbb{P}_{\mathbb{Z}}^N$, and $\mathrm{Ch}_{\mathbf{d}}(\mathbb{P}_{\mathbb{Z}}^N)$ is the divisor of a multihomogeneous polynomial with integer coefficients $R_{\mathbf{d}}$, whose variables are the coefficients of the "generic" homogeneous polynomials of degrees $d_0, \ldots, d_N$ in $N+1$ variables. This polynomial, well defined up

to a sign, coincides with the classical resultant of $N+1$ homogeneous polynomials of degrees $d_0, \ldots, d_N$ in $N+1$ variables.

## 3.3 Hilbert Scheme

We refer the reader to the appendices for a crash course on Categories and Schemes where we introduce all the fancy terms we will make use of in this section. Unsurprisingly the construction of the Hilbert Scheme is attributed to Grothendieck. And indeed he managed to build it using the language of representable functors in his seminal paper [Gro60]. The intuition behind the Hilbert Scheme is a moduli space for all the nice perturbations a subvariety $X \subset \mathbb{P}^N$ has. We proceed as follows. There exists $r$ such that $\dim H^0(\mathcal{I}_X(d)) = r$ for sufficiently large $d$ where $\mathcal{I}_X(d)$ is the twisted ideal sheaf. Then the Hilbert Scheme will be realized as closed subscheme of the Grassmannian $G(r, S_d)$ for $S_d$ all the homogeneous polynomials of degree $d$ in $N+1$ variables.

For the remainder of the chapter let $X \subset \mathbb{P}^N_k$ be a noetherian projective scheme over $\operatorname{Spec} k$.

For a coherent sheaf $\mathcal{F}$ on $X$ we define the twist of $\mathcal{F}$ by $d$ to be

$$\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d),$$

where $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$ and $\mathcal{O}_X(d) := \mathcal{O}_X(1)^{\otimes d}$. For a subscheme $Y \subset X$ and the ideal sheaf $\mathcal{F} = \mathcal{I}_X$, the twist $\mathcal{F}(d)$ corresponds to all degree $d$ polynomials that vanish on $Y$.

If $\mathcal{F}$ is finitely generated we define the Euler characteristic $\chi(X, \mathcal{F})$ to be

$$\chi(X, \mathcal{F}) := \sum_j (-1)^j \dim H^j(X, \mathcal{F}).$$

By induction on the dimension of the support of $\mathcal{F}$ it can be shown that there is a polynomial $P(t) \in \mathbb{Q}[t]$ such that for all $n \in \mathbb{Z}$

$$\chi(X, \mathcal{F}(n)) = P(n) \tag{3.10}$$

**Definition 3.28.** We call the polynomial $P$ above the Hilbert polynomial of the sheaf $\mathcal{F}$.

When $\mathcal{F} = \mathcal{O}_X$ the structure sheaf, $P = P_X$ is the Hilbert polynomial of $X$.

By Castelnuovo-Mumford regularity theorem (see Proposition 4.2 in [Str96]) there exist an integer $d_0$ such that for all $d \geq d_0$ we have that $\dim H^0(\mathcal{I}_Y(d_0))$ is constant for all projective schemes $Y \subset \mathbb{P}^N_k$ over $\operatorname{Spec} k$ with Hilbert polynomial $P_X$. Let us set

$$r := \dim H^0(\mathcal{I}_Y(d)). \tag{3.11}$$

### The Hilbert Functor

An algebraic family of closed subschemes of $X/S$, parameterized by $T$, is a closed subscheme $Z \subset X_T = X \times_S T$. The family is flat if the induced morphism $Z \to T$ is flat. A geometric fiber of the family is the pullback $(1 \times t)^* Z$ of $Z$ to $X \times_T \operatorname{Spec}(k(t))$, where $t$ :$\operatorname{Spec}(k(t)) \to T$ is a geometric point of $Z$ i.e. a morphism $t : \operatorname{Spec} k \to X$.

**Definition 3.29.** Let $\underline{\mathrm{Hilb}}_{X/S}(T)$ be the set of flat algebraic families of closed subschemes $Z$ of $X/S$ parameterized by $T$. If $T' \to T$ is any morphism, $Z \mapsto Z \times_T T'$ gives a map $\underline{\mathrm{Hilb}}_{X/S}(T) \to \underline{\mathrm{Hilb}}_{X/S}(T')$, which makes $\underline{\mathrm{Hilb}}_{X/S}$ a contravariant functor on the category of $S$-schemes.

If $\underline{\mathrm{Hilb}}_{X/S}$ is representable, the $S$-scheme $\mathrm{Hilb}_{X/S}$ representing it is called the Hilbert scheme of $X/S$.

When $P$ is the constant polynomial $P(t) = 1$, the Hilbert functor $\underline{\mathrm{Hilb}}^P_{\mathbb{P}^n_S}$ parametrizes families of points in $\mathbb{P}^n_S$. In this case, the Hilbert scheme is simply $\mathbb{P}^n_S$ itself.

When $P(t) = dt+1$ for some positive integer $d$, the Hilbert functor $\underline{\mathrm{Hilb}}^P_{\mathbb{P}^n_S}$ parametrizes families of curves of degree $d$ in $\mathbb{P}^n_S$.

Grothendieck's result may be stated as follows.

**Theorem 3.30** (Grothendieck)**.** Assume that $X$ is projective over $S$ and $P$ is a polynomial with integer values. Then $\mathrm{Hilb}^P_{X/S}$ exists and is projective over $S$.

We will not prove this result here. We refer the reader to [Str96] or Chapter 1 in [Kol13].

### A particular case...

Let us suppose that $X \subset \mathbb{P}^N$ is a variety with Hilbert polynomial $P_X$ and consider the Grassmannian $G = G(r, S_d)$ where $r$ is as in Equation 3.11 and $S_d$ is the homogeneous polynomials of degree $d$ in $N + 1$ variables

Let $\mathcal{Z} \subset G \times \mathbb{P}^N$ be defined as

$$\mathcal{Z} := \{([W], x) : W \subset S_d \text{ is an ideal and } F(x) = 0 \forall F \in W\}.$$

The Hilbert scheme $\text{Hilb}_X^{P_X}$ in this case is the subset $\mathcal{Y} \subset \mathcal{Z}$ consisting of pairs $([W], x)$ such that

(i) $\dim(S_d/W) = P_X(d)$ so that $\dim X = \dim W$.

(ii) $H^j(\mathcal{I}_W(d)) = 0$ for all $j > 0$ so that from Equation 3.10 we have $P_W = P_X$.

(iii) $W \otimes \mathcal{O}_{\mathbb{P}^N}(-d) \twoheadrightarrow \mathcal{I}_X = \mathbb{I}(X)$ since subvarieties $Y \subset \mathbb{P}^N$ correspond to radical homogeneuous ideals of $k[x_0, \ldots, x_N]$.

Note that the three conditions above and the ones defining $\mathcal{Z}$ are algebraic. So not only $\mathcal{Y} = \text{Hilb}_X^{P_X}$ is well defined but there is a finite set of regular functions in $k[G] \otimes [x_0, \ldots, x_N]$ for which $\mathcal{Y}$ is the zero locus.

# Chapter 4

# Intersection Theory

Let $k$ be a field. In the previous chapter we introduced the Chow Group $A(X)$ of a projective variety $X$ over $k$ (Definition 3.17) where for given two algebraic cycles $D = \sum n_j V_j$ and $E = D = \sum m_j V_j$

$$D + E = \sum (n_j + m_j) V_j.$$

There is a product operation we can define on $A(X)$ that makes it a ring. This operation tries to capture the intersection of the cycles $D$ and $E$ within $X$. We make some remarks of subtle issues that appear when trying to do so.

First note that we are forced to leave already the realm of varieties and work with schemes because the quotients of rings we used to define the intersection number $i(D, E; V_j)$ may have nilpotent elements. For instance take $D = \mathbb{V}(y - x^2) \subset \mathbb{A}^2$ and $E = \mathbb{V}(y) \subset \mathbb{A}^2$. Then

$$i(D, E; [(0,0)]) = \text{ length of } k[x,y]_0/(y - x^2, y) = \text{ length of } k[x]/(x^2).$$

Next note how these intersection numbers cannot be defined as we did in Definition 3.4. This only works properly when either $D$ or $E$ is a hyperplane. For instance consider $X = \mathbb{A}^4, D = \{x = y = 0\} \cup \{z = w = 0\}, E = \{x = z, y = w\}$ as in the following picture

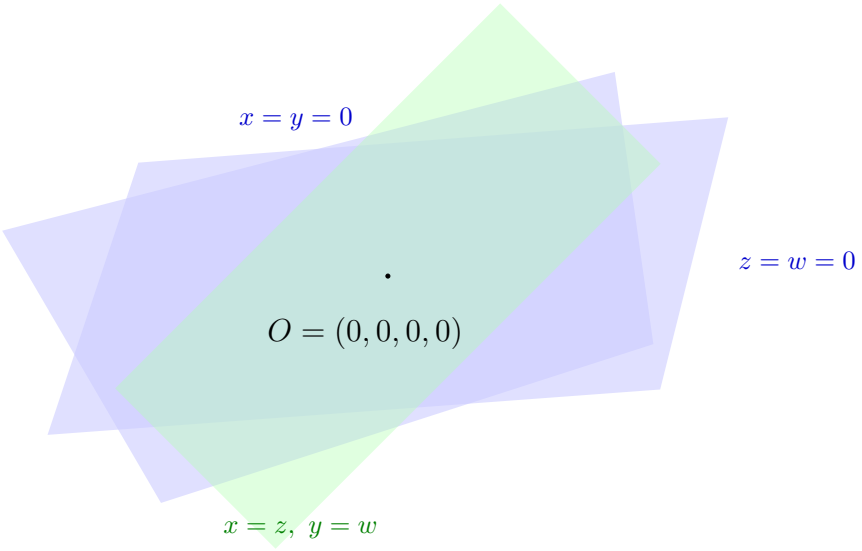



FIGURE 4.1: Two Algebraic Cycles failing to satisfy a intersection number definition

Then $i(D, E; [0, 0, 0, 0])$ should be 2 but since $k[x, y]_0/(x^2, xy, y^2)$ has basis $1, x, y$ we have

$$k[x, y, z, w]/\big(yz, yw, xz, xw, x - z, y - w\big) \cong k[x, y]_0/(x^2, xy, y^2) \text{ that has length 3.}$$

## 4.1 Chow Ring

Let $X$ be a non singular variety.

We say that two subvarieties $Y, W \subset X$ intersect transversely at a point $p$ if $Y, W$ are non singular at $p$ and

$$T_pY + T_pW = T_pX. \tag{4.1}$$

We say that $Y, W$ intersect generically transversely if $\operatorname{Supp} Y \cap \operatorname{Supp} W$ contains a Zariski dense open subset $U$ such that for every $p \in U$ Equation 4.1 above holds. For instance if $\dim Y + \dim W = \dim X$ then $Y$ and $W$ intersect generically transversely if and only if the intersection is finite and the intersection is transverse everywhere. If $\operatorname{Codim} Y + \operatorname{Codim} W < X$ then the intersection is generically transverse if and only if they are disjoint. We extend this definition to cycles $\alpha, \beta$ if every element in $\operatorname{Supp}(\alpha)$ is generically transverse to every element in $\operatorname{Supp}(\beta)$.

In order to introduce the intersection multiplicity formula let us set some notation. For $\mathcal{O}_X$ the structure sheaf of $X$ and any subscheme $Y \subset X$ we denote by $\mathcal{O}_{Y,Z}$ the localization at $Z \subset Y$ of the restriction of $\mathcal{O}_X$ to $\mathcal{O}_Y$.

Also, for two subschemes $S, T$ of a non singular scheme $X$ we say that $S, T$ are dimensionally transverse (sometimes called proper) if for every irreducible component $Y \subset S \cap T$ we have

$$\operatorname{Codim} Y = \operatorname{Codim} S + \operatorname{Codim} T. \tag{4.2}$$

If $X$ is a variety we extend this notion to cycles $\alpha, \beta$ if every element in $\operatorname{Supp}(\alpha)$ is dimensionally transverse to every element in $\operatorname{Supp}(\beta)$.

Note that being generically transverse implies being dimensionally transverse since a variety $X$ is non singular if and only if $\dim X = \dim T_p X$ at every $p \in X$.

**Definition 4.1** (Serre)**.** Let $S, T \subset X$ be two dimensionally transverse subschemes and $C \subset S \cap T$ an irreducible component. Then we define the intersection multiplicity $i(S, T; C)$ of $S$ and $T$ along $C$ to be

$$i(S,T;C) := \sum_{j=0}^{\dim X} (-1)^j \operatorname{length}_{\mathcal{O}_{S\cap T,C}} \left( \operatorname{Tor}_j^{\mathcal{O}_{X,C}} \left( \mathcal{O}_{S,C}, \mathcal{O}_{T,C} \right) \right).$$

A couple of remarks are due. But first let us point the reader to Definition A.23 for a quick overview on Tor. Now to our remarks. The way we defined Tor gives us directly that $i(S, T; C) = i(T, S; C)$. And note that for $j = 0$ we recover the definition we gave in the previous chapter for when either $S$ or $T$ was a hyperplane (in the case of varieties). So what this formula is really doing is adjusting the error term so that our theory is consistent with what we expect.

**Definition 4.2.** Let $X$ be a non singular variety. For any two dimensionally transverse subvarieties $Y, W \subset X$ we define their (intersection) product $[Y][W]$ as

$$[Y][W] = \sum_{Z \subset Y \cap W} i(Y, W; Z)[Z],$$

where the sum is over all irreducible components of the intersection.

We are ready to introduce the Chow ring.

**Theorem 4.3.** Let $X$ be a non singular variety, $A(X)$ its Chow group and let $\alpha = \sum_j n_j[V_j], \beta = \sum m_j[V_j] \in A(X)$. The following two operations make $A(X)$ into a commutative ring that we will denote by $CH(X)$.

(i) $[\alpha] + [\beta] = \Big[\sum_j (n_j + m_j)[V_j]\Big]$.

(ii) $[\alpha][\beta] = \Big[\sum_j n_j \sum_i m_i [V_j][V_i]\Big] = \sum_i \sum_j n_j m_i [V_j][V_i]$.

Moreover $CH(X) = \bigoplus_j CH_j(X)$ is a graded ring where the grading is given by codimension.

We refer the reader to Chapter 7 in [Ful13] for a proof of this. We do explain why the name of this ring. It is named after the Chinese mathematician Wei-Liang Chow, because it was by his Moving Lemma that all the theory came together. Clearly there are intersections between subvarieties that are not generically transverse. But Chow proved that for cycles in the Chow group (not varieties but cycles!), we can always find representatives that do intersect generically transversely. Concretely,

**Theorem 4.4** (Moving Lemma)**.** Let $X$ be a non singular, quasi-projective variety.

(i) Given cycles $\alpha, \beta$ on $X$ there is a cycle $\alpha'$ rationally equivalent to $\beta$ such that $\alpha'$ and $\beta$ intersect generically transversely.

(ii) The resulting class $[\alpha' \cap \beta] \in A(X)$ is independent of the choice of such an $\alpha'$.

The Chow ring satisfies the following property which we call excision.

**Proposition 4.5** (Excision)**.** Let $X$ be a scheme. If $Y \subset X$ is a closed subscheme and $U = X \setminus Y$ is its complement then the inclusion and restriction maps of cycles give a right exact sequence

$$\mathrm{CH}(Y) \to \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0.$$

Moreover, if $X$ is non singular, the map $\mathrm{CH}(X) \to \mathrm{CH}(U)$ is a ring homomorphism.

*Proof.* Proposition 1.14b in [EH16]. □

Now that we have developed the theory let us see some concrete examples. From the previous Chapter we know that $CH(\mathbb{A}^n) \cong \mathbb{Z}$. This together with the Excision property (Proposition 4.5) yields

**Corollary 4.6.** For $\emptyset \neq U \subset \mathbb{A}^n$, we have

$$CH(U) = \mathbb{Z}[U].$$

Our next goal is to study the Chow ring of the projective space. We do this via stratifications.

### Affine Stratifications

When a variety admits an affine stratification- a decomposition into affine spaces satisfying certain conditions- we can determine its Chow groups. This approach enables the computation of Chow rings for projective spaces and Grassmannians.

**Definition 4.7.** A scheme $X$ is stratified by a finite collection of irreducible, locally closed subschemes $\{U_i\}$ if:

(i) $X$ is the disjoint union of the $U_i$, and

(ii) for any $U_j$, if the closure $\overline{U_j}$ intersects $U_i$, then $\overline{U_j}$ contains $U_i$.

The subschemes $U_i$ are called the strata of the stratification, while their closures $Y_i := \overline{U_i}$ are termed the closed strata.

From the closed strata, one can recover the stratification via:

$$U_i = Y_i \setminus \bigcup_{Y_j \subsetneq Y_i} Y_j$$

A stratification of a scheme $X$ with strata $\{U_i\}$ is affine if each open stratum is isomorphic to some affine space $\mathbb{A}^k$ and quasi-affine if each $U_i$ is isomorphic to an open subset of some affine space $\mathbb{A}^k$.

For example, the complete flag $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$, induces an affine stratification of $\mathbb{P}^n$ where the closed strata are the projective spaces $\mathbb{P}^i$ and the open strata are the affine spaces $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i$.

The following proposition reveals the fundamental connection between stratifications and Chow groups.

**Proposition 4.8** (Proposition 1.17 in [EH16])**.** If a scheme $X$ admits a quasi-affine stratification, then its Chow ring $CH(X)$ is generated by the classes of the closed strata.

*Proof.* We proceed by induction on the number of strata. For a single stratum, this is Corollary 4.6.

Let $U_0$ be a minimal stratum. By the properties of the stratification, $U_0$ must be closed. Define $U := X \setminus U_0$, which is stratified by the remaining strata. By the induction hypothesis, $CH(U)$ is generated by the classes of the closures of these strata, while $CH(U_0)$ is generated by $[U_0]$.

By the excision property, we have the right-exact sequence:

$$\mathbb{Z} \cdot [U_0] = CH(U_0) \to CH(X) \to CH(X \setminus U_0) \to 0$$

Since the classes in $CH(U)$ of the closed strata in $U$ derive from the classes of the same closed strata in $X$, it follows that $CH(X)$ is generated by the classes of all closed strata. □

Now we are ready for the Chow ring of the projective space $\mathbb{P}^n$.

**Theorem 4.9** (Theorem 2.1 in [EH16])**.** The Chow ring of $\mathbb{P}^n$ has the structure:

$$CH(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$$

where $\zeta \in CH_1(\mathbb{P}^n)$ represents the rational equivalence class of a hyperplane.

More generally, the class of a variety of codimension $k$ and degree $d$ is $d\zeta^k$.

Consequently, for $0 \leq j \leq n$, we have $CH_j(\mathbb{P}^n) \cong \mathbb{Z}$, with each group generated by the class of an $(n-j)$-dimensional linear subspace.

*Proof.* Consider a complete flag $\{p\} \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ and use Proposition 4.8 to the affine stratification with strata $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1}$, to deduce that $A^k(\mathbb{P}^n)$ is generated by the class of $\mathbb{P}^{n-k}$, and consequently by the class of any $(n-k)$-dimensional linear subspace $L \subset \mathbb{P}^n$.

We know that $A^n(\mathbb{P}^n) = \mathbb{Z}$ (Proposition 3.19). Since a generic $(n-k)$-dimensional linear subspace intersects a generic $k$-dimensional linear subspace transversally at a single point, multiplication by the class of the latter induces a surjective map

$$CH_j(\mathbb{P}^n) \to CH_n(\mathbb{P}^n) = \mathbb{Z}.$$

Therefore, $CH_j(\mathbb{P}^n) = \mathbb{Z}$ for all $j$.

An $(n-k)$-dimensional linear subspace $L \subset \mathbb{P}^n$ arises as the transverse intersection of $k$ hyperplanes, which yields:

$$[L] = \zeta^k$$

where $\zeta \in CH_1(\mathbb{P}^n)$ represents the class of a hyperplane.

For the last part note that a subvariety $X \subset \mathbb{P}^n$ of dimension $n-k$ and degree $d$ intersects a generic $k$-dimensional linear subspace transversally at $d$ points. Hence, $\deg([X]\zeta^{n-k}) = d$. Given that $\deg(\zeta^n) = 1$, we conclude $[X] = d\zeta^k$. □

## 4.2 (First) Chern class(es)

Recall from the previous chapter that there is an isomorphism between line bundles and the Chow group of algebraic cycles of codimension 1 (Theorem 3.20). This isomorphism is of great importance in the study of intersection theory. In particular the map from $\Psi\colon \mathrm{Pic}(X) \to A(X)$ has a name. It is called the first Chern form $c_1$ of $\mathcal{L} \in \mathrm{Pic}(X)$.

The first Chern class is the bridge between algebraic geometry, topology, and analysis. It classifies line bundles, governs intersection numbers, and underlies Riemann-Roch Theorems, the classification of varieties and indexation of moduli spaces. Understanding $c_1$ is foundational for understanding the global geometry of algebraic varieties.

**Definition 4.10.** Let $X$ be a variety of dimension $d$ and $\mathcal{L} \in \mathrm{Pic}(X)$ a line bundle on $X$. For any rational section $s$ of $\mathcal{L}$ we define

$$c_1(\mathcal{L}) := \mathrm{div}(s).$$

By Theorem 3.20 this is well defined.

We note that $c_1$ is a linear operator. So even if the line bundle $\mathcal{L}$ does not have a non-zero section, we can find a twist $\mathcal{O}_X(n)$ such that $L \otimes \mathcal{O}_X(n)$ does have one and then define

$$c_1(\mathcal{L}) := c_1(\mathcal{L} \otimes \mathcal{O}_X(n)) - c_1(\mathcal{O}_X(n)).$$

As a first insight on the role the first Chern class plays in intersection theory we make the following remark.

$$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[c_1(\mathcal{O}(1))]/(c_1(\mathcal{O}(1))^{n+1}) \tag{4.3}$$

This follows from Theorem 4.9 and the fact that for every variety $X$, there is a group homomorphism

$$\zeta : A^1(X) \to H^2(X, \mathbb{Z}).$$

given by composing the natural map from $A^1(X)$ to the homology group $H_{n-2}(X, \mathbb{Z})$ with the one given by so Poincaré duality from $H_{n-2}(X, \mathbb{Z})$ to $H^2(X, \mathbb{Z})$. In particular $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ generates $H^2(\mathbb{P}^n, \mathbb{Z})$, and $c_1(\mathcal{O}(k)) = kH$ (we are making an abuse of notation here referring to $\zeta$ as $c_1$ interchangeably).

Next, and already latent in the proof of Theorem 4.9, we present the following.

**Proposition 4.11.** Suppose that $X$ is a smooth quasi-projective variety and $\mathcal{L}$ is a line bundle on $X$. If $Y_1, \ldots, Y_n$ are any subvarieties of $X$, then there is a cycle in the class of $c_1(\mathcal{L})$ that is generically transverse to each $Y_i$. If $X$ is smooth and $Y \subset X$ is any subvariety, then

$$c_1(\mathcal{L}) \cdot [Y] = c_1(\mathcal{L}|_Y).$$

*Proof.* We first exhibit a cycle in the class of $c_1(\mathcal{L})$ that is generically transverse to each $Y_i$.

Since $X$ is quasi-projective, there exists an ample line bundle $\mathcal{L}'$ on $X$ and also an integer $n$ for which both $\mathcal{L}'^{\otimes n}$ and $\mathcal{L}'^{\otimes n} \otimes \mathcal{L}$ become very ample line bundles. Denote $\mathcal{A} = \mathcal{L}'^{\otimes n}$ for simplicity.

Since $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{L}$ are very ample, their complete linear systems have no base points, i.e., for any finite set of points, there exist sections that do not vanish at those points.

By Bertini's theorem (Theorem 0.5 in [EH16]), for a very ample line bundle, the zero locus of a general section is smooth and intersects any given subvariety transversely at its smooth points. Specifically:

(i) There exists a section $\sigma \in H^0(X, \mathcal{A})$ whose zero locus $\mathrm{Div}(\sigma)$ is smooth and intersects each $Y_i$ transversely at smooth points of $Y_i$.

(ii) Similarly, there exists a section $\tau \in H^0(X, \mathcal{A} \otimes \mathcal{L})$ whose zero locus $\mathrm{Div}(\tau)$ is smooth and intersects each $Y_i$ transversely at smooth points of $Y_i$.

Since both $\mathrm{Div}(\tau)$ and $\mathrm{Div}(\sigma)$ are generically transverse to each $Y_i$. To conclude this first part note that for $\sigma, \tau$ we have:

$$\begin{aligned}
[\mathrm{div}(\sigma)] &= c_1(\mathcal{A}) = c_1(\mathcal{L}'^{\otimes n}) = n \cdot c_1(\mathcal{L}') \\
[\mathrm{div}(\tau)] &= c_1(\mathcal{A} \otimes \mathcal{L}) = c_1(\mathcal{A}) + c_1(\mathcal{L}) = n \cdot c_1(\mathcal{L}') + c_1(\mathcal{L})
\end{aligned}$$

Therefore:

$$\begin{aligned}
c_1(\mathcal{L}) &= [\mathrm{div}(\tau)] - [\mathrm{div}(\sigma)] \\
&= [\mathrm{div}(\tau) - \mathrm{div}(\sigma)]
\end{aligned}$$

i.e. the cycle $\mathrm{div}(\tau) - \mathrm{div}(\sigma)$ represents $c_1(\mathcal{L})$.

We now prove that $c_1(\mathcal{L}) \cdot [Y] = c_1(\mathcal{L}|_Y)$ for any subvariety $Y \subset X$.

Using the ring structure of $CH(X)$ we have

$$\begin{aligned}
c_1(\mathcal{L}) \cdot [Y] &= [\mathrm{Div}(\tau) - \mathrm{Div}(\sigma)] \cdot [Y] \\
&= [\mathrm{Div}(\tau) \cap Y] - [\mathrm{Div}(\sigma) \cap Y].
\end{aligned}$$

And since the restriction $s|_Y$ of a section $S$ to $Y$ is precisely the intersection of the divisor of $s$ with $Y$, we know

$$\begin{aligned}
\mathrm{Div}(\sigma) \cap Y &= \mathrm{Div}(\sigma|_Y), \text{and} \\
\mathrm{Div}(\tau) \cap Y &= \mathrm{Div}(\tau|_Y).
\end{aligned}$$

To conclude note that $\sigma|_Y$ is a section of $\mathcal{A}|_Y$ and $\tau|_Y$ is a section of $(\mathcal{A} \otimes \mathcal{L})|_Y = \mathcal{A}|_Y \otimes \mathcal{L}|_Y$ so that

$$\begin{aligned}
c_1(\mathcal{L}) \cdot [Y] &= [\mathrm{Div}(\tau|_Y)] - [\mathrm{Div}(\sigma|_Y)] \\
&= c_1((\mathcal{A} \otimes \mathcal{L})|_Y) - c_1(\mathcal{A}|_Y) \\
&= c_1(\mathcal{A}|_Y \otimes \mathcal{L}|_Y) - c_1(\mathcal{A}|_Y) \\
&= c_1(\mathcal{A}|_Y) + c_1(\mathcal{L}|_Y) - c_1(\mathcal{A}|_Y) \\
&= c_1(\mathcal{L}|_Y)
\end{aligned}$$

This completes the proof of the proposition. □

We continue our survey on the first Chern class with the following. Let $X \subset \mathbb{P}^n$ be a non-singular projective variety of dimension $k$ and $L$ a line bundle on $X$. For any homology class $\alpha \in H_{2n-2k}(\mathbb{P}^n, \mathbb{Z})$ the moving lemma gives us a cycle $A$ representing $\alpha$ and intersecting $X$ transversally at smooth points. Hence $X$ defines the linear functional

$$\langle -, [X] \rangle \colon H_{2n-2k}(\mathbb{P}^n, \mathbb{Z}) \to \mathbb{Z}, \text{ given by the intersection number } [A] \cdot [X].$$

We have

$$\deg X = \langle c_1(\mathcal{O}_X(1))^k, [X] \rangle =: \int_X c_1(\mathcal{O}_X(1))^k.$$

When $X$ is a curve and $L$ is a line bundle on $X$ we have

$$\deg(L) \ = \ \int_C c_1(L),$$

and the Riemann-Roch theorem reads

$$\dim H^0(C, L) - \dim H^1(C, L) \ = \ \deg(L) + 1 - g(C).$$

On a surface $S$, the intersection pairing

$$(c_1(L_1), c_1(L_2)) \ = \ \int_S c_1(L_1) \wedge c_1(L_2)$$

governs ampleness via the Nakai-Moishezon criterion (see Theorem V.1.10 in [Har13]) and the Hodge index theorem (see page 150 in [Sch02]).

We want to finish this introduction to (first) Chern classes by remarking how they show up in deep results as Grothendieck's Riemann Roch and the classification of algebraic varieties.

Let $X$ be a non singular $n$-dimensional variety. By the canonical bundle $\omega_X$ of $X$ we mean the top exterior power $\bigwedge^n \Omega_X$ of the cotangent bundle $\Omega_X$ of $X$; this is the line bundle whose sections are regular $n$-forms. By the canonical class we mean the first Chern class $c_1(\omega_X) \in A^1(X)$ of this line bundle and it is denoted by $K_X$. As a consequence to the Adjunction formula (Proposition 3.1) we have

$$\deg K_Y = \deg ((\, K_X + [Y])[Y]).$$

We have the following important results.

**Theorem 4.12** (Hirzebruch-Riemann-Roch)**.** Let $S$ be a surface and $L$ a line bundle over $S$. Then,

$$\chi(S, L) \;=\; \tfrac{1}{2}\, c_1(L)^2 \;-\; \tfrac{1}{2}\, c_1(L) \cdot c_1(K_S) \;+\; \chi(\mathcal{O}_S),$$

where $K_S$ is the canonical line bundle of $S$ and $\chi(\mathcal{O}_S)$ is the Euler characteristic of the structure sheaf $\mathcal{O}_S$ (see Definition B.7).

**Theorem 4.13** (Grothendieck-Riemann-Roch)**.** For $f : X \to Y$ and a line bundle $L$ on $X$,

$$\mathrm{ch}(L) = \exp\big(c_1(L)\big) = 1 + c_1(L) + \tfrac{1}{2}\, c_1(L)^2 + \cdots,$$

and

$$f_*\big(\mathrm{ch}(L) \cdot \mathrm{td}(X)\big) \;=\; \mathrm{ch}(f_* L) \cdot \mathrm{td}(Y).$$

where the Chern character of a vector bundle $E$ is the class in $H^{\mathrm{even}}(X; \mathbb{Q})$ given by

$$\mathrm{ch}(E) \;=\; \sum_{i=1}^{r} e^{\,x_i} \;=\; r \;+\; c_1(E) \;+\; \tfrac{1}{2}\big(c_1(E)^2 - 2c_2(E)\big) \;+\; \cdots.$$

and the Todd class of $E$ is the multiplicative characteristic class

$$\mathrm{td}(E) \;=\; \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} \;=\; 1 \;+\; \tfrac{1}{2}\, c_1(E) \;+\; \tfrac{1}{12}\big(c_1(E)^2 + c_2(E)\big) \;+\; \cdots.$$

(See Appendix A in [Har13] for both definitions).

We conclude with the classification of varieties Theorem.

A variety is Fano if $-c_1(K_X)$ is ample,

A variety is Calabi-Yau if $c_1(K_X) = 0$, and

A variety is of General type if $c_1(K_X)$ is big (Kodaira dimension $= \dim X$).

## 4.3 Arithmetic Intersection Theory

In the 1970s, Suren Arakelov [Ara74] initiated a profound extension of algebraic intersection theory into the realm of arithmetic geometry that ultimately endowed Faltings with the tools he needed to prove the Mordell conjecture. Working in analogy with the intersection theory on algebraic surfaces, Arakelov introduced an intersection theory on arithmetic surfaces. This established what is now called Arakelov geometry - the foundation of arithmetic intersection theory.

Arithmetic surfaces are arithmetic varieties of dimension 2. Let us recall the definition of an arithmetic variety.

**Definition 4.14.** An arithmetic variety $\mathcal{X}$ is a regular scheme, flat and projective over $\mathrm{Spec}(\mathbb{Z})$.

For such varieties, the fibers over closed points of $\mathrm{Spec}(\mathbb{Z})$ correspond to reductions modulo primes, while the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is a variety over $\mathbb{Q}$. Additionally, we can consider the complex manifold $\mathcal{X}(\mathbb{C})$ consisting of complex points of $X = \mathcal{X}_{\mathbb{Q}}$ [Sou+92]. We refer the reader to Example B.15 in the Appendices for a concrete example.

Classical intersection theory applies to varieties over fields (especially over $\mathbb{C}$ or finite fields), but number theorists needed an analogous theory for schemes over the ring of integers $\mathfrak{o}_K$ of a number field $K$. In the function field case (curves over finite fields), Weil had defined heights and degrees using all places of the function field, including infinite places corresponding to points at infinity on a curve. By analogy, for number fields one must include the Archimedean places (embeddings into $\mathbb{R}$ or $\mathbb{C}$) as analogues of points at infinity. Arakelov's insight was to treat those Archimedean places analytically by compactifying the arithmetic surface $\pi : \mathcal{X} \to \mathrm{Spec}\,\mathfrak{o}_K$ with certain data at infinity so that one can define an intersection product that includes Archimedean contributions.

Arakelov introduced the notion of a compactified divisor (or $c$-divisor) on an arithmetic surface. Since an arithmetic surface is proper over $\mathfrak{o}_K$, a divisor on $\mathcal{X}$ is either horizontal (whose projection to $\mathrm{Spec}\,\mathfrak{o}_K$ is surjective) or vertical (lying in special fibers above nonzero primes of $\mathfrak{o}_K$). Arakelov insight was to add to the group of divisors on $X$ an extra term for each Archimedean place $\sigma$ of $K$ (one from each complex conjugate pair).

**Definition 4.15.** A compactified divisor (or $c$-divisor) on an arithmetic surface $\mathcal{X}$ is a formal sum

$$D = D_{\mathrm{fin}} + \sum_{\sigma} r_\sigma X_\sigma \tag{4.4}$$

where $D_{\mathrm{fin}} = \sum k_i C_i$ ($k_i \in \mathbb{Z}$ and $C_i \subset \mathcal{X}$ are irreducible divisors) is a usual divisor on $\mathcal{X}$, and $X_\sigma$ represents the complex fiber at an archimedean place $\sigma$, and $r_\sigma \in \mathbb{R}$.

This definition yields a product formula for principal $c$-divisors $(f)_c$ for rational functions $f$ on $\mathcal{X}$, which include an Archimedean contribution defined using a fixed Hermitian metric on each complex fiber. Concretely, if $f \in K(\mathcal{X})^*$, Arakelov defined the Archimedean valuation

$$\nu_\sigma(f) = -\frac{1}{2\pi} \int_{X_\sigma} \log |f| d\mu_\sigma \tag{4.5}$$

(for a volume form $d\mu_\sigma$ normalized to volume 1 on $X_\sigma$), so that

$$\begin{aligned}(f)_c &= (f)_{\text{fin}} + \sum_{\sigma|\infty} \nu_\sigma(f) \cdot X_\sigma, \\ \text{where } (f)_{\text{fin}} &= \sum_{\mathfrak{p}\in \operatorname{Spec} \mathcal{O}_K} v_\mathfrak{p}(f) \log \#\left(\mathfrak{o}_K/\mathfrak{p}\right).\end{aligned} \tag{4.6}$$

Let $f^\sigma$ be the induced function on the complex fiber $X_\sigma$ by $f$ and also associate to every point $P \in \mathcal{X}$ a smooth real-valued function $\varphi_P$ which is to serve as the global equation of $P$ so that near $P$ we have

$$\varphi_P(z) = |t_P(z)| \cdot u(z),$$

where $t_P(z)$ is a local parameter near $P$, and $u(z)$ a smooth function with $u(P) \neq 0$.

Then,

$$|f^\sigma(z)| = \left( \prod_{Q\in\mathcal{X}} \varphi_Q(z)^{v_Q(f)} \right) \cdot u(z).$$

Since $u(z)$ never vanishes and satisfies $\Delta \log u = 0$, we have $u = c \in \mathbb{R}$. Taking logarithm and integrating:

$$c = \int_{X_\sigma} \log |f^\sigma| \, d\mu_\sigma = -v_\sigma(f)$$

Lastly note that a (horizontal) divisor $D \subset \mathcal{X}$ is of the form $\varepsilon(\operatorname{Spec} \mathfrak{o}_L)$ where $L$ is a finite extension of $K$. Hence

$$\sum_\mathfrak{p} v_\mathfrak{p}(f) + \sum_\sigma v_\sigma(f) = \sum_{\mathfrak{P}\in\operatorname{Spec}\mathcal{O}_L} -\log \|\varepsilon^* f\|_\mathfrak{P} + \sum_\sigma -\log \|\varepsilon^* f\|_\sigma = 0, \tag{4.7}$$

where the last equality is just the usual product formula.

This last equation shows that we indeed have a product formula for principal $c$-divisors.

Arakelov then defined an intersection pairing between $c$-divisors, incorporating both the usual intersection at finite places and a contribution from archimedean places using Green functions. We recall here for convenience the notion of a Green function.

**Definition 4.16.** A Green function on a Riemann surface $X$ is a function $g : X \times X \backslash \Delta \to \mathbb{R}$ (where $\Delta$ is the diagonal) satisfying:

1. $g$ is symmetric: $g(P,Q) = g(Q,P)$,

2. For fixed $P$, the function $g(P,\cdot)$ is harmonic on $X \setminus \{P\}$ meaning that it satisfies the Laplace equation:
$$\Delta g(P,\cdot) \equiv 0, \text{ and}$$

3. Near the diagonal, $g(P,Q) = -\log|z(P) - z(Q)| + O(1)$ where $z$ is a local coordinate.

For a divisor $D = \sum n_i P_i$ on $\mathcal{X}$, the associated Green function is (see [Ara74; Fal84])

$$g_D(Q) = \sum n_i g(P_i, Q).$$

Line bundles then take into account this archimedean compactification at infinity in the following way.

**Definition 4.17.** A Hermitian line bundle on an arithmetic variety $\mathcal{X}$ is a pair

$$\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$$

where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $\|\cdot\|$ is a hermitian metric on the complexification $\mathcal{L}_{\mathbb{C}}$.

Hermitian line bundles form a group $\widehat{\mathrm{Pic}}(\mathcal{X})$ under tensor product, which extends the usual Picard group of line bundles [Fal84; Sou+92].

Let us now introduce the intersection product in arithmetic varieties a lá Arakelov.

### 4.3.1 Arakelov's intersection product

For two divisors $D_1$ and $D_2$ on the arithmetic surface $\pi : \mathcal{X} \to \operatorname{Spec} \mathfrak{o}_K$, Arakelov defines their intersection product as:

$$(D_1 \cdot D_2) = \sum_{p \in \operatorname{Spec} \mathcal{O}_K} (D_1 \cdot D_2)_p + \sum_{\sigma|\infty} (D_1 \cdot D_2)_\sigma.$$

For finite places $p$, Arakelov defines $(D_1 \cdot D_2)_p = \sum_{P \in \pi^{-1}(p)} (D_1 \cdot D_2)_P$ with local intersection numbers

$$(D_1 \cdot D_2)_P = \text{length of } (\mathcal{O}_{P,V}/(f_1, f_2)) \cdot \log \#(\mathfrak{o}_K/\mathfrak{p})$$

where $f_1, f_2$ are local equations for $D_1, D_2$ at $P$.

For infinite places $(D_1 \cdot D_2)_\sigma$ uses the Green's function $G(P,Q)$ on the Riemann surface $X_\sigma$:

$$(P \cdot Q)_{X_\sigma} = -\log G(P,Q),$$

where $G$ satisfies $\frac{1}{2\pi}\Delta \log G(P,z) = \delta_P - d\mu$ with $d\mu$ the volume form.

To conclude this introduction let $D \subset \mathcal{X}$ be a divisor and set

$$\begin{aligned} ((f)_c \cdot D) &= ((f)_{\text{fin}} \cdot D) + \sum_{\sigma|\infty} v_\sigma(f) \cdot (X_\sigma \cdot D), \\ ((f)_{\text{fin}} \cdot D) &= \sum_{\mathfrak{p} \in \text{Spec}\mathfrak{o}_K} ((f)_{\text{fin}} \cdot D)_\mathfrak{p} + \sum_{\sigma|\infty} ((f)_{\text{fin}} \cdot D)_\sigma, \end{aligned} \tag{4.8}$$

By the product formula for principal $c$-divisors, we have $((f)_c, D) = 0$.

With all the ingredients on the table we state a fundamental result in Arakelov geometry which serves as a preamble to the connection between classical intersection theory and arithmetic intersection theory.

**Theorem 4.18** (Arakelov[Ara74], Gillet-Soulé [GS90])**.** Let $\mathcal{X}$ be an arithmetic variety. Then

(i) $c$-divisors modulo linear equivalence are isomorphic to $\widehat{\text{Pic}}(\mathcal{X})$.

(ii) The Arakelov intersection product corresponds to a pairing on $\widehat{\text{Pic}}(\mathcal{X})$

### 4.3.2 Higher-Dimensional Arithmetic Intersection Theory

In 1990, Gillet and Soulé introduced a comprehensive formalism for doing intersection theory on arithmetic varieties of arbitrary dimension [GS90].

Recall from Chapter 2 that on a complex manifold $M$, a current $g$ of type $(p-1,p-1)$ is called a Green current for a subvariety $Z \subset M$ (of codimension $p$) if it satisfies the partial differential equation $dd^c g + \delta_Z = \omega$, where $\delta_Z$ is the current of integration along $Z$ (a delta-current supported on $Z$) and $\omega$ is a smooth form representing the cohomology class Poincaré-dual to $Z$. Intuitively, $g$ is a potential for the current $\delta_Z$, analogous to a Green's function for $Z$.

In Arakelov geometry, one works with Green currents of log type, meaning that in local coordinates near the subvariety $Z$, the current is asymptotically like the logarithm of the norm of defining equations of $Z$. Gillet-Soulé proved an important existence theorem: for any cycle $Z$ on a compact Kähler manifold, one can find a Green current that is smooth off of $Z$ and has logarithmic singularities along $Z$.

**Definition 4.19.** For an arithmetic variety $\mathcal{X}$ of dimension $d$, the arithmetic Chow group $\widehat{\mathrm{CH}}^p(\mathcal{X})$ is defined as:

$$\widehat{\mathrm{CH}}^p(\mathcal{X}) = \frac{\{(Z, g_Z) \mid Z \text{ is a codimension } p \text{ cycle on } \mathcal{X}, g_Z \text{ is a Green current for } Z\}}{\text{Rational equivalence with Green currents}} \tag{4.9}$$

Here, a Green current $g_Z$ for a cycle $Z$ is a current on $\mathcal{X}(\mathbb{C})$ satisfying $dd^c g_Z + \delta_Z = \omega_Z$ where $\delta_Z$ is the current of integration over $Z(\mathbb{C})$ and $\omega_Z$ is a smooth form.

In this framework, a metrized (Hermitian) line bundle $\overline{L} = (L, |\cdot|)$ on $\mathcal{X}$ determines an element of $\widehat{\mathrm{CH}}^1(\mathcal{X})$ via its first Chern class: one chooses a local trivialization $s$ of $L$ and defines a Green current $g = -\log|s|^2$ (a smooth function on $\mathcal{X}(\mathbb{C})$ outside the zeros of $s$, with logarithmic singularities along $\mathrm{Div}(s)$). Then one sets

$$\widehat{c}_1(\overline{L}) := (\mathrm{Div}(s), -\log\|s\|^2) \tag{4.10}$$

which is an arithmetic divisor class independent of the choice of $s$.

More generally, any algebraic cycle plus a Green current gives an element of $\widehat{\mathrm{CH}}^p(X\mathcal{X})$. Two such pairs $(Z, g_Z)$ and $(Z', g_{Z'})$ are said to be equivalent if their difference is a principal arithmetic cycle, meaning it comes from the divisor of a rational function together with the $\log|\cdot|$ Green function.

**Theorem 4.20** (Gillet-Soulé)**.** There exists an intersection product

$$\widehat{\mathrm{CH}}^p(\mathcal{X}) \times \widehat{\mathrm{CH}}^q(\mathcal{X}) \to \widehat{\mathrm{CH}}^{p+q}(\mathcal{X})_{\mathbb{Q}} \tag{4.11}$$

which is commutative and associative.

For properly intersecting cycles $Z$ and $W$, the intersection $(Z, g_Z) \cdot (W, g_W)$ is represented by the pair $(Z \cdot W, g_Z * g_W)$ where $Z \cdot W$ is the usual intersection cycle and $g_Z * g_W$ is the star product of Green currents as in [GS90].

If $(Y, g_Y) \in \widehat{\mathrm{CH}}^p(X)$ and $(Z, g_Z) \in \widehat{\mathrm{CH}}^q(X)$ intersect properly, then one defines

$$(Y, g_Y) \cdot (Z, g_Z) := (Y \cdot Z, g_Y \wedge \delta_Z + \delta_Y \wedge g_Z + g_Y \wedge dd^c g_Z) \tag{4.12}$$

which intuitively is $Y \cap Z$ as an algebraic cycle together with a Green current obtained by a "*-product" of the currents [GS90].

**Definition 4.21.** The arithmetic degree is a homomorphism

$$\widehat{\deg} : \widehat{\mathrm{CH}}^d(\mathcal{X}) \to \mathbb{R} \tag{4.13}$$

defined by $\widehat{\deg}(Z, g_Z) = \deg(Z) + \int_{\mathcal{X}(\mathbb{C})} g_Z$, where $\deg(Z)$ is the usual degree of the 0-cycle $Z$.

Given a Hermitian vector bundle $\overline{\mathcal{E}}$ on $\mathcal{X}$, Gillet and Soulé defined arithmetic Chern classes [GS90]:

$$\widehat{c}_i(\overline{\mathcal{E}}) \in \widehat{\mathrm{CH}}^i(\mathcal{X}) \tag{4.14}$$

Let us give an example of a computation of the arithmetic Chow ring in a simple case. Set $A = \mathfrak{o}_F$ to be the ring of integers in a number field $F$, so that we have the exact sequence (see page 131 in [GS90])

$$\{1\} \to \mu(F) \to \mathfrak{o}_F^* \to \mathbf{R}^{r_1+r_2} \to \widehat{\mathrm{CH}}^1(X) \to \mathrm{Cl}(\mathfrak{o}_F) \to 0,$$

where $\mathrm{Cl}(\mathfrak{o}_F)$ is the ideal class group of $\mathfrak{o}_F$. This yields

$$\widehat{\mathrm{CH}}^1(X) \cong F^* \backslash J(F) / U_F,$$

where $J(F)$ is the idele group of $F$ (see [Neu13]) and $U_F$ is the maximal compact subgroup of $J(F)$. This isomorphism is obtained by sending the idele $(\alpha_v)$ to the arithmetic cycle $(Z, g)$, where $Z = \sum_v v(\alpha_v)[v]$ and $g = -\log|\alpha_v|^2$ when $v$ is real and $g = -2\log|\alpha_v|^2$ when $v$ is complex.

One of the fundamental outcomes of Gillet-Soulé's work is that any invariant of projective varieties that can be defined via intersection theory (Chern classes, degrees, etc.) has an analogue for arithmetic varieties. They developed, for instance, an arithmetic Riemann-Roch theorem and used it to compute arithmetic Chow groups in basic cases. Importantly, they demonstrated how to extend Chern class maps to this arithmetic setting, and defined arithmetic Chern characters and Todd classes, leading to an Arakelov version of the Grothendieck-Riemann-Roch theorem (fully proven later by Gillet, Rössler, and Soulé in [GRS08]). This allows one to compute arithmetic degrees of pushforwards (heights of projective varieties, as we will discuss later) in terms of more accessible data.

The Gillet-Soulé formalism confirmed that Arakelov's intersection theory was not a one-off phenomenon restricted to surfaces, but part of a general framework applicable in any dimension.

### 4.3.3 Chambert-Loir's inductive presentation

While the Gillet-Soulé theory is powerful, it is also technically demanding, relying on advanced machinery of currents and $K$-theory. To the extent that even Faltings in his acclaimed paper [Fal91] states that he will not need all the machinery from Gillet-Soulé to define his height functions on moduli spaces. One of the complications lies on the fact that the intersection product described above takes coefficients in $\mathbb{Q}$ and not $\mathbb{Z}$.

Antoine Chambert-Loir presents an alternative approach to arithmetic intersection theory inductively as follows.

**Theorem 4.22** (Theorem 2.5 in [Cha20])**.** Let $n = \dim(\mathcal{X})$ and let $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_n$ be hermitian line bundles on $\mathcal{X}$. There exists a unique family of linear maps:

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1) \dots \widehat{c}_1(\overline{\mathcal{L}}_d) \mid \cdot) : Z_d(\mathcal{X}) \to \mathbb{R}, \tag{4.15}$$

for $d \in \{0, \dots, n\}$ satisfying the following properties:

(i) Let $d \in \{1, \dots, n\}$, and $Z \subset \mathcal{X}$ and integral closed subscheme such that $\dim(Z) = d$. Then for every integer $m \neq 0$ and every meromorphic section $s$ of $\mathcal{L}_d^m|_Z$ we have

$$m\,\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1) \dots \widehat{c}_1(\overline{\mathcal{L}}_d) \mid Z) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1) \dots \widehat{c}_1(\overline{\mathcal{L}}_{d-1}) \mid \operatorname{div}(s)) + \int_{Z(\mathbb{C})} \log \|s\|^{-1} c_1(\overline{\mathcal{L}}_1) \dots c_1(\overline{\mathcal{L}}_{d-1}).$$

(ii) Let $z \in \mathcal{X}$ be a closed point (i.e an integral closed subscheme of dimension $d = 0$). Then

$$\widehat{\deg}(z) = \log(\#(\kappa(z))),$$

where $\kappa(z)$ is the residue field of $z$.

Moreover, these maps are multilinear and symmetric in the hermitian line bundles $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_n$ and only depend on their isomorphism classes in $\widehat{\operatorname{Pic}}(\mathcal{X})$.

**Example 4.23** (Example 2.7 in [Cha20])**.** Assume that Z is vertical and lies over a maximal ideal $(p)$ of $\operatorname{Spec}(\mathbb{Z})$. Then Z is a proper scheme over $\mathbb{F}_p$ and the analogous formula in classical intersection theory that together with inductive definition yield

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}_1) \dots \widehat{c}_1(\overline{\mathcal{L}}_d) \mid Z) = \deg\left(c_1(\mathcal{L}_1|_{\mathcal{X}_p}) \dots c_1(\mathcal{L}_d|_{\mathcal{X}_p}) \mid Z\right) \log(p).$$

**Example 4.24** (Example 2.8 in [Cha20])**.** Assume that $d = 1$ and that Z is horizontal, so that Z is the Zariski-closure in $\mathcal{X}$ of a closed point $z \in \mathcal{X}_{\mathbb{Q}}$. Let $\mathrm{F} = \kappa(z)$ be the residue field and let $\mathfrak{o}_F$ be its ring of integers; by properness of $\mathcal{X}$, the canonical morphism

Spec(F) $\to \mathcal{X}$ with image $z$ extends to a morphism $\varepsilon_z : \mathrm{Spec}(\mathfrak{o}_F) \to \mathcal{X}$, whose image is Z. The formula

$$\widehat{\deg}(\widehat{c}_1(\overline{\mathcal{L}}) \mid Z) = \widehat{\deg}(\varepsilon_z^* \overline{\mathcal{L}})$$

expresses the arithmetic intersection number as the arithmetic degree of the hermitian line bundle $\varepsilon_z^* \overline{\mathcal{L}}$ over $\mathrm{Spec}(\mathfrak{o}_F)$.

# Chapter 5

# Heights

Two self contained and comprehensive introduction to this topic may be found in the Diophantine Geometry book by Hindy and Silverman [HS13] and Heights in Diophantine Geometry by Bombieri and Gubler [BG06].

Let us fix a number field $K$ and a complete set of places $M_K$. We recall from the introduction that for an algebraic number $\alpha \in \overline{\mathbb{Q}}$ we define the naive height $h$ as

$$h(\alpha) = \frac{1}{d}\big(\log|a_d| + \sum_{j=1}^{d} \log^+|\alpha_j|\big),$$

where $p_\alpha(t) = \sum a_j t^j$ is the minimal polynomial of $\alpha$ in $\mathbb{Z}[t]$ and the $\alpha_j$'s are the conjugates of $\alpha$.

We can also define a (naive) height on the projective space $\mathbb{P}^N(K)$. For this, for all $v \in M_K$, set

$$n_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}],$$

and normalize $|\ |_v$ to satisfy the product formula

$$\sum_{v \in M_K} n_v \log|x|_v = 0, \ \ x \neq 0.$$

Then for any $P = [P_0 : \ldots : P_N] \in \mathbb{P}^N(K)$, the naive height $h(P)$ is defined as

$$h(P) = \sum_{v \in M_K} n_v \log\|P\|_v, \quad \text{where} \quad \|P\|_v := \max_{0 \leq j \leq N}\{|P_j|_v\}.$$

We can extend this definition to all algebraic numbers $\alpha \in \overline{\mathbb{Q}}$ by letting $K_\alpha$ be the field of definition of $\alpha$ and defining

$$h(\alpha) = \frac{1}{[K_\alpha : \mathbb{Q}]} \sum_{v \in M_K} n_v \log \|\alpha\|_v.$$

One of the most fundamental results in Diophantine Geometry is Northcott's Theorem.

**Theorem 5.1** (Northcott, Theorem 1.6.8 in [BG06])**.** Consider the naive height $h$ on $\overline{\mathbb{Q}}$. Then for any $B, D > 0$ we have

$$\#\{\alpha \in \overline{\mathbb{Q}} : h(\alpha) < B \text{ and } [K_\alpha : \mathbb{Q}] < D\} < \infty.$$

We will not prove this theorem here. A complete proof is contained in Bombieri and Gubler's book. We will just say that it follows from the following proposition that can also be found in Bombieri and Gubler.

**Proposition 5.2** (Proposition 1.6.6 in [BG06])**.** Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number and consider $p_\alpha$ the minimal polynomial of $\alpha$ over $\mathbb{Z}$. Then

$$\log M(p_\alpha) = \deg(\alpha) h(\alpha),$$

where $M(p_\alpha)$ is the Mahler measure of $p_\alpha$ and $\deg(\alpha) = [K_\alpha : \mathbb{Q}]$.

Northcott's Theorem serves as a tool to obtain finiteness results along the lines of what is called nowadays unlikely intersections in the setting of the Zilber-Pink Conjecture [Pil22]. This Conjecture is a general context for which Manin-Mumford (Raynaud Theorem [Ray83a]), Andre-Oort and Bogomolov are particular instances.

As an example let us exhibit a result due to Krönecker.

**Theorem 5.3** (Krönecker)**.** Let $h$ be the naive height on $\overline{\mathbb{Q}}$. Then for all $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$ the following are equivalent.

(i) $h(\alpha) = 0$.

(ii) $\alpha$ is a root of unity.

*Proof.* Recall that

$$h(\alpha) = \frac{1}{d}\big(\log |a_d| + \sum_{j=1}^{d} \log^+ |\alpha_j|\big).$$

So if $\alpha$ is a root of unity, say of order $q$ then $\alpha$ and all of its conjugates have norm 1. So by the equation above, and since the cyclotomic polynomial $\Phi_q(x)$ is given by

$$\Phi_q(x) = \prod_{\substack{1\leq k\leq q \\ \gcd(k,q)=1}} (x - \zeta_n^q)$$

we have indeed $h(\alpha) = 0$.

For the other direction we use Northcott's Theorem. Specifically note that $\alpha^m \in \mathbb{Q}(\alpha)$ and so

$$\deg \alpha^m \leq \deg \alpha.$$

And also, from Proposition 5.2 it follows that

$$h(\alpha^m) = mh(\alpha).$$

Then if $h(\alpha) = 0$ we can take $B = 0$ and $D = \deg \alpha$ in Northcott's Theorem to conclude that the set

$$\{\alpha, \alpha^2, \alpha^3, \ldots\} \text{ is finite .}$$

We conclude there are distinct natural numbers $r, s$ such that $\alpha^r = \alpha^s$ and the result follows. □

Heights are an intrinsically rich theory and form an essential framework for studying Diophantine Geometry. We now illustrate the latter.

## 5.1 Arithmetic Equidistribution

This section presents Arithmetic Equidistribution Theorems and applications to Unlikely Intersections in Algebraic and Diophantine Geometry. These theorems provide conditions under which the Galois orbits of generic points and points of small arithmetic height distribute uniformly with respect to a canonical measure on the ambient space. They yield powerful tools for fundamental questions like the Bogomolov conjecture.

There are several versions of Arithmetic Equidistribution Theorems. We present the following diagram (Figure 5.1) to outline their relationships.

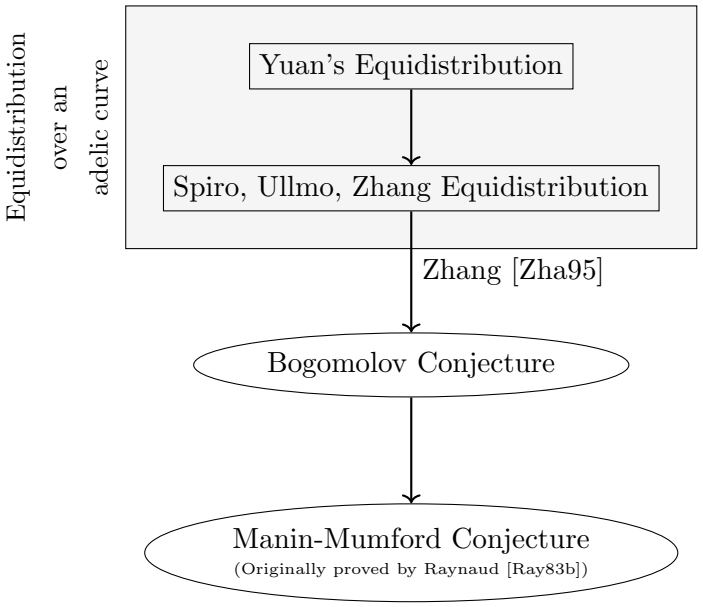



FIGURE 5.1: Diagram of the implications Arithmetic Equidistribution has in the realm of unlikely intersections.

Let us now state and prove Bilu's Equidistribution Theorem. Not only can we present it in such a way that it does not involve too much technology besides the fundamentals of complex potential theory (following Rumely's alternate proof [Rum99]), but it also allows a beautiful simple illustration (Figure 5.2).

**Theorem 5.4** (Bilu's Equidistribution Theorem)**.** Let $\{\alpha_n\} \subset \overline{\mathbb{Q}}^{\times}$ be a sequence of distinct algebraic numbers such that $h(\alpha_n) \to 0$. Put

$$\mu_n := \frac{1}{[\mathbb{Q}(\alpha_n) : \mathbb{Q}]} \sum_{\sigma : \mathbb{Q}(\alpha_n) \hookrightarrow \mathbb{C}} \delta_{\sigma(\alpha_n)}.$$

Then

$$\mu_n \xrightarrow{w} \mu,$$

where $\mu$ is the normalized Lebesgue measure on the unit circle $\mathbb{S}^1$. In other words, for every function $f \in \mathscr{C}_c(\mathbb{S}^1)$ we have

$$\lim_{n\to\infty} \int_{\mathbb{S}^1} f d\mu_n = \int_{\mathbb{S}^1} f d\mu.$$

To prove this Theorem we will use the a lemma that Rumely proves in his paper. We will follow the proof from the lecture notes from Math295 by Laura Demarco. To state this lemma we note that given that $h(\alpha_n) \to 0$, Theorem 5.3 above and Theorem A.4.2

in [Ran95] gives us a subsequence $\{\mu_{n_k}\} \subset \{\mu_n\}$ converging to a probability measure $\mu'$ on the unit circle $\mathbb{S}^1$. Passing to this subsequence we can assume $\mu_n \to \mu'$.

Let us set some notation for any measure $\nu$ on $\mathbb{C}$.

$$I_M(\nu) = \int\int_{\mathbb{C}\times\mathbb{C}} \max\{\log|z-w|, -M\} d\nu(z) d\nu(w),$$
$$I^*(\nu) = \int\int_{\mathbb{C}\times\mathbb{C}\setminus\Delta} \log|z-w| d\nu(z) d\nu(w),$$

where $\Delta$ is the diagonal in $\mathbb{C}\times\mathbb{C}$.

**Lemma 5.5.** Let $\mu_n, \mu'$ as above. Then

$$\limsup_{n\to\infty} I^*(\mu_n) \leq I(\mu').$$

*Proof.* Let $D_1, D_2 \subset \mathbb{C}$ be two disks such that $\mathbb{S}^1 \subsetneq D_1 \subsetneq D_2$ and choose $f \in \mathscr{C}_c(\mathbb{C})$ such that $f|_{D_1} \equiv 1$ and $\operatorname{Supp} f \subset D_2$.

Using the relation

$$1 = f(z)f(w) + f(z)\left(1-f(w)\right) + f(w)\left(1-f(z)\right) + \left(1-f(z)\right)\left(1-f(w)\right),$$

for every $M > 0$ we have

$$I_M(\mu_n) = \int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} d\mu_n(z) d\mu_n(w)$$

$$= \int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)f(w) d\mu_n(z) d\mu_n(w) \tag{5.1}$$

$$+2\int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)\left(1-f(w)\right) d\mu_n(z) d\mu_n(w) \tag{5.2}$$

$$+\int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} \left(1-f(z)\right)\left(1-f(w)\right) d\mu_n(z) d\mu_n(w) \tag{5.3}$$

where we have used the symmetry of the expressions to put together the terms corresponding to $f(z)(1-f(w))$ and $f(w)(1-f(z)$ )in Equation 5.2.

We estimate each term above separately. For the term in Equation 5.1, the weak convergence $\mu_n \xrightarrow{w} \mu'$ yields

$$\begin{aligned}&\int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)f(w)d\mu_n(z)d\mu_n(w)\\ &\longrightarrow \int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)f(w)d\mu'(z)d\mu'(w),\end{aligned}$$

and since $\operatorname{Supp}\mu' \subset \mathbb{S}^1$, we conclude

$$\begin{aligned}&\int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)f(w)d\mu_n(z)d\mu_n(w)\\ &\longrightarrow \int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} d\mu'(z)d\mu'(w).\end{aligned}$$

For the integrals in Equations 5.2 and 5.3 we use $h(\alpha_n) \to 0$ to get around the fact that the supports of the $\mu_n$'s are not uniformly bounded. Let $p_\alpha$ be the minimal polynomial of $\alpha_n$ over $\mathbb{Z}$ such that the leading coefficient is positive.

For Equation 5.2 note that for all $z \in D_2$ there is a uniform $L > 0$ such that

$$\log|z-w| \leq \log^+|w| + L \Longrightarrow \max\{\log|z-w|, -M\} \geq \log^+|w| + L.$$

Hence

$$\begin{aligned}&2\int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)\left(1-f(w)\right) d\mu_n(z)d\mu_n(w)\\ &\geq 2\int\int_{\mathbb{C}\times\mathbb{C}} \big(\log^+|w| + L\big) f(z)\left(1-f(w)\right) d\mu_n(z)d\mu_n(w)\\ &\geq 2\int_{\mathbb{C}\setminus D_1} \big(\log^+|w| + L\big) d\mu_n(w)\\ &\geq 2\left(\frac{1}{\deg\alpha_n}\sum \log^+|\alpha| + L\mu_n(\mathbb{C}\setminus D_1)\right)\\ &\geq 2h(\alpha_n) + 2L\mu_n(\mathbb{C}\setminus D_1).\end{aligned} \tag{5.4}$$

We finish this estimation by recalling that $\mu'$ is supported in the unit circle and $h(\alpha_n) \to 0$ so that

$$2\int\int_{\mathbb{C}\times\mathbb{C}} \max\left\{\log|z-w|, -M\right\} f(z)\left(1-f(w)\right) d\mu_n(z)d\mu_n(w) \longrightarrow 0. \tag{5.5}$$

We follow the previous strategy to estimate the integral in Equation 5.3. Pick any $L' > 0$ such that

$$\max\{\log|z-w|, -M\} \geq \log^+|z| + \log^+|w| + L'.$$

Then

$$\begin{aligned} &\int\int_{\mathbb{C}\times\mathbb{C}} \max\{\log|z-w|, -M\}(1-f(z))(1-f(w))\,d\mu_n(z)d\mu_n(w) \\ &\geq 2h(\alpha_n) + L'\mu_n(\mathbb{C}\setminus D_1)^2 \longrightarrow 0. \end{aligned} \tag{5.6}$$

Joining everything

$$\lim_{n\to\infty}\sup I_M(\mu_n) \leq \int\int_{\mathbb{C}\times\mathbb{C}} \max\{\log|z-w|, -M\}\,d\mu'(z)d\mu'(w). \tag{5.7}$$

Letting $M \to \infty$ the monotone convergence theorem yields

$$\lim_{n\to\infty}\sup I_M(\mu_n) \leq I(\mu'). \tag{5.8}$$

To conclude we note that

$$\begin{aligned} I^*(\mu_n) &\leq \int\int_{\mathbb{C}\times\mathbb{C}\setminus\Delta} \max\{\log|z-w|, -M\}d\mu_n(z)d\mu_n(w) \\ &= \int\int_{\mathbb{C}\times\mathbb{C}} \max\{\log|z-w|, -M\}d\mu_n(z)d\mu_n(w) - \frac{M}{\deg\alpha_n} \\ &= I_M(\mu_n) - \frac{M}{\deg\alpha_n}, \end{aligned} \tag{5.9}$$

hence

$$\lim_{n\to\infty}\sup I^*(\mu_n) \leq \lim_{n\to\infty}\sup I_M(\mu_n).$$

□

Let us now prove Bilu's equidistribution.

*Proof of Theorem 5.4.* Let $S = \{\sigma_1, \ldots, \sigma_N\}$ denote the set of all the embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ and put

$$D(\alpha_n) = \prod_{i \neq j} \big(\sigma^i(\alpha_n) - \sigma^j(\alpha_n)\big).$$

Note that for a rational prime $p$

$$\log|\sigma^i(\alpha_n) - \sigma^j(\alpha_n)| \leq \log^+|\sigma^i(\alpha_n)| + \log^+|\sigma^j(\alpha_n)|,$$

hence

$$\begin{aligned} \log|D(\alpha_n)|_p &= \sum_{i \neq j} \log|\sigma^i(\alpha_n) - \sigma^j(\alpha_n)|_p \\ &\leq 2\deg\alpha_n\big(\deg\alpha_n - 1\big)\big|\alpha_n\big|_p. \end{aligned} \tag{5.10}$$

For the Archimedean place

$$\frac{1}{(\deg\alpha_n)^2}\log|D(\alpha_n)| = I^*(\mu_n). \tag{5.11}$$

To get a contradiction suppose $\mu_n \overset{w}{\not\to} \mu$. Just as in Lemma we can assume without loss of generality that there is a probability measure $\mu'$ on $\mathbb{S}^1$ such that

$$\mu_n \overset{w}{\to} \mu' \quad \text{where} \quad \mu' \neq \mu. \tag{5.12}$$

Now note that by Northcott's Theorem, $h(\alpha_n) \to 0$ implies

$$\deg\alpha_n \to \infty.$$

And also by Frostman's Theorem (Theorem 2.10 above)

$$I(\mu') < 0. \tag{5.13}$$

The above together with the Lemma yields

$$\limsup_{n\to\infty} \frac{1}{\deg\alpha_n(\deg\alpha_n - 1)}\log(|D(\alpha_n)|) \leq I(\mu') < 0. \tag{5.14}$$

Next let $\varepsilon > 0$ be such that $I(\mu') < -3\varepsilon$. Then for a sufficiently large $n$, we have

$$\begin{aligned} \sum_{p \text{ prime}} |\alpha_n|_p < \epsilon \text{ and} \\ \frac{1}{\deg \alpha_n(\deg \alpha_n - 1)} \log(|D(\alpha_n)|) < I(\mu') + \varepsilon. \end{aligned} \tag{5.15}$$

It follows that for such $n$,

$$\begin{aligned} 0 &= \sum_{v \in M_{\mathbb{Q}}} \log(|D(\alpha_n)|_v) \\ &< 2 \deg \alpha_n(\deg \alpha_n - 1)(\sum_{p \text{ prime}} |\alpha_n|_p) + \deg \alpha_n(\deg \alpha_n - 1) \cdot (I(\mu') + \varepsilon) \\ &\leq \deg \alpha_n(\deg \alpha_n - 1) \cdot \big(I(\mu') + 3\varepsilon\big) < 0. \end{aligned} \tag{5.16}$$

The proof is complete. □

We want to provide a visualization of the convergence in this Theorem. To do so we have chosen integers $c_j$ lying in $[-10, 10]$ using the uniform distribution to get some polynomials of degrees $10, 19, 36$ and $77$ with integer coefficients.

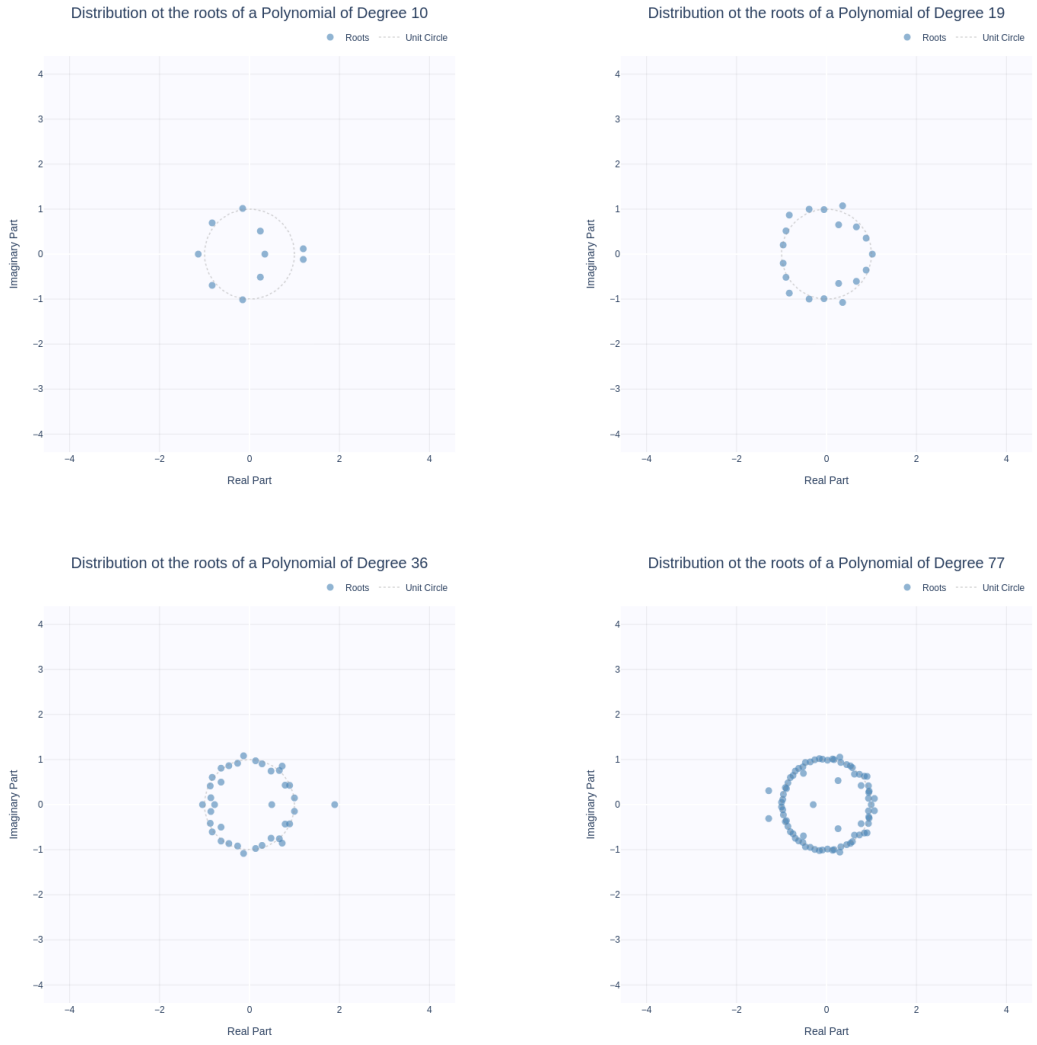



FIGURE 5.2: Visualization of the roots of random polynomials of degree 10, 19, 36, and 77.

### 5.1.1 Yuan-Zhang's (ergo Gauthier's, Yuan's, Szpiro-Ullmo-Zhang's) Theorem

Many authors have discovered more general Equidistributions Theorems. We mention the work of Szpiro-Ullmo-Zhang for abelian varieties [SUZ97] and more recently of Kühne [Küh21] and Gauthier [Gau21]. Yuan and Zhang recently proved another yet more general version. And Chen and Moriwaki [CM22] proved yet another version over an adelic curve. That implies all of the other Equidistribution Theorems of course.

**Theorem 5.6** (Theorem 5.4.3 in [YZ21])**.** Let $X$ be a (quasi-)projective variety over a number field $K$. Consider a nef adelic line bundle $\mathcal{L}$ on $X$ of positive degree. Suppose that $\{x_m\} \subset X(K)$ is a generic sequence (i.e. any subvariety $Y \subsetneq X$ contains only

finitely many points) with

$$h_{\mathcal{L}}(x_m) \to h_{\mathcal{L}}(X).$$

Then for each place $v \in K$ we have a weak convergence of measures

$$\mu_{m,v} \to \mu_{\mathcal{L},v}$$

in the Berkovich analytification $X_v^{an}$ where

$$\mu_{\mathcal{L},v} := \frac{1}{\deg_{\mathcal{L}}(X/F)} c_1(\mathcal{L})_v^n$$

is the equilibrium measure in $X_v^{\mathrm{an}}$.

Now, when applying these Theorems to unlikely intersections, Zhang [Zha95] used Szpiro-Ullmo-Zhang's Equidistribution Theorem to obtain

**Theorem 5.7** (Bogomolov Conjecture)**.** Let $X$ be a non-torsion subvariety of an abelian variety $A$. Then there is an $\epsilon > 0$ such that the set

$$\{x \in X(\mathbb{Q}) : \hat{h}_{\mathrm{NT}}(x) \leq \epsilon\}$$

is not Zariski dense where $\hat{h}_{\mathrm{NT}}$ is the Neron-Tate height in $A$.

As a corollary we have Raynaud's Theorem (previously known as the Manin-Mumford Conjecture). For any curve of genus $g \geq 2$ over a number field $K$ let $A$ be the Jacobian (and $L$ to any antisymmetric line bundle on $A$).

The last conjecture we mention in this setting is the Andre-Oort Conjecture recently solved by Pila, Shankar and Tsimerman [Pil+21].

## 5.2 Heights on Function Fields

One of our main playing fields in our work is not the number field $K$ itself, but its rational function field $K(t)$. The logarithmic Weil height over $K$ induces a naive coefficient height on $K(t)$ as follows.

Just like for number fields, we define heights on projective spaces. Let $f(t) \in k = K(t)$. Via the map $f(t) \mapsto (1 : f(t))$, we see $f(t)$ as a rational map on the projective space $\mathbb{P}^1$. And now we use that any such function is given in homogeneous coordinates by

$$(X, Y) \longmapsto (F(X, Y), G(X, Y))],$$

where $F, G \in K[X, Y]$ are homogeneous polynomials of the same degree and have no common zero in $\mathbb{P}^1(\overline{K})$. For each $\sigma \in M_K$, set

$$\|(F, G)\|_\sigma = \max\{|c|_\sigma : c \text{ is a coefficient of } F \text{ or } G\}.$$

**Definition 5.8** (Naive height)**.** With $f$ as above, we define its naive height by

$$h^{\text{naive}}(f) = \sum_{\sigma \in M_K} n_\sigma \log \|(F, G)\|_\sigma.$$

We note that this is well defined by the product formula.

As an example, suppose that $f(t) \in K[t]$ is a polynomial of degree $n$. Write

$$f(t) = \sum_{j=0}^{n} c_j t^j = c_n \prod_{j=1}^{n} (t - x_j).$$

In this case, we have that the lift map from above is given by

$$[(X, Y]) \mapsto (Y^n, \sum_{j=0}^{n} c_j X^j Y^{n-j}). \text{ Hence}$$

$$h^{\text{naive}}(f) = \sum_{\sigma \in M_K} n_\sigma \log \max\{1, \max_{0 \le j \le n} |c_j|_\sigma\}.$$

Equivalently, writing $\log^+(z) = \max\{0, \log(z)\}$, for the polynomial $f$ we are defining

$$h^{\text{naive}}(f) = \sum_{\sigma \in M_K} n_\sigma \log^+ \max_{0 \le j \le n} |c_j|_\sigma.$$

In other words, we are defining

$$h^{\text{naive}}(f) = h([1 : c_0 : \cdots : c_n]),$$

where $h$ is the height function on $\mathbb{P}_k^N$.

Now we present another height function.

**Definition 5.9** (Geometric height)**.** With $f$ as above, we define its geometric height $h^{\text{geom}}$ to be

$$h^{\text{geom}}(f) = \deg f = n.$$

**Definition 5.10** (Mahler Measure)**.** Let $f(t) \in K[t]$. We define the Mahler measure $m(f)$ of $f$ to be

$$m(f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

We state a well known fact (see Proposition 1.6.5 in [BG06])

**Proposition 5.11.** For $f(t)$ as above we have

$$m(f) = \log |c_n| + \sum_{j=1}^{n} \log^+ |x_j|,$$

And also a useful inequality (see Lemma 1.6.7 in [BG06])

**Proposition 5.12.** For $f(t)$ as above,

$$\log \max_j |c_j| - n \log 2 \leq m(f).$$

There is another measure arising from integrating $f(t)$ in which relies our subsequent work. We introduce it here. Let $\omega_{\mathrm{FS}}$ be the Fubini-Study form in the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.

Now, for $\omega_{\mathrm{FS}}$ the Fubini-Study form in the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ we have

**Lemma 5.13.** For $f(t)$ as above,

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log |f(t)| \mathrm{d}\omega_{\mathrm{FS}} \geq \frac{1}{2}|c_0| + \frac{1}{2}|c_n| + \frac{\log 2}{2} n.$$

*Proof.* Write $f(t) = \sum_j c_j t^j$, so that

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log |f(t)| \mathrm{d}\omega_{\mathrm{FS}} = \int_{\mathbb{P}^1_{\mathbb{C}}} \log |c_n| \mathrm{d}\omega_{\mathrm{FS}} + \sum_{j=1}^{n} \int_{\mathbb{P}^1_{\mathbb{C}}} \log |t - x_j| \omega_{\mathrm{FS}}$$

$$= \int_{\mathbb{P}^1_{\mathbb{C}}} \log |c_n| \mathrm{d}\omega_{\mathrm{FS}} + \sum_{j=1}^{n} \int_0^{\infty} \frac{r \log \max\{r, |x_j|\}}{(1+r^2)^2} \mathrm{d}r$$

where the last equality follows from Proposition 5.11.

Continuing

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|c_n| \mathrm{d}\omega_{\mathrm{FS}} + \sum_{j=1}^{n} \int_0^{\infty} \frac{r \log \max\{r, |x_j|\}}{(1+r^2)^2} \mathrm{d}r$$

$$= \log|c_n| + \sum_{j=1}^{n} \Big( \int_0^{|x_j|} \frac{r \log \max\{r, |x_j|\}}{(1+r^2)^2} \mathrm{d}r + \int_{|x_j|}^{\infty} \frac{r \log \max\{r, |x_j|\}}{(1+r^2)^2} \mathrm{d}r \Big)$$

$$= \log|c_n| + \sum_{j=1}^{n} \log|x_j|^2 + \frac{1}{4} \sum_{j=1}^{n} \log\left(1 + |x_j|^{-2}\right) = \log|c_n| + \frac{1}{4} \sum_{j=1}^{n} \log\left(1 + |x_j|^2\right). \quad (5.17)$$

On the other hand the Fubini-Study form is invariant under the transformation $z \mapsto \frac{1}{z}$ so that

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|f(t)| \mathrm{d}\omega_{\mathrm{FS}} = \int_{\mathbb{P}^1_{\mathbb{C}}} \log\left|f\left(\frac{1}{t}\right)\right| \mathrm{d}\omega_{\mathrm{FS}}.$$

Noting that

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|t^n| \mathrm{d}\omega_{\mathrm{FS}} = 0$$

we conclude

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|f(t)| \mathrm{d}\omega_{\mathrm{FS}} = \int_{\mathbb{P}^1_{\mathbb{C}}} \log\left|t^n f\left(\frac{1}{t}\right)\right| \mathrm{d}\omega_{\mathrm{FS}}.$$

Applying Equation 5.17 to $t^n f\left(\frac{1}{t}\right)$ yields

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|f(t)| \mathrm{d}\omega_{\mathrm{FS}} = \int_{\mathbb{P}^1_{\mathbb{C}}} \log\left|t^n f\left(\frac{1}{t}\right)\right| \mathrm{d}\omega_{\mathrm{FS}} = \log|c_0| + \frac{1}{4} \sum_{j=1}^{n} \log\left(1 + |x_j|^{-2}\right). \quad (5.18)$$

Joining Equation 5.17 and Equation 5.18 we get

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|f(t)| \mathrm{d}\omega_{\mathrm{FS}} = \frac{1}{2} \log|c_n| + \frac{1}{2} \log|c_0| + \frac{1}{4} \sum_{j=1}^{n} \log\left(\left(1 + |x_j|^2\right)\left(1 + |x_j|^{-2}\right)\right).$$

To conclude note that for any $0 \neq x \in \mathbb{R}$ we have $\left(1 + x^2\right)\left(1 + x^{-2}\right) \geq 4$. □

The proof of this naturally gives us another useful result.

**Proposition 5.14.** For $f(t)$ as above,

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|f(t)|\, d\omega_{FS} \;\geq\; m(f).$$

*Proof.* By Proposition 5.11, $m(f) = \log|c_n| + \sum_{j=1}^{n} \log^+|x_j|$. On the other hand, the computation in Lemma 5.13 gives

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log|f(t)|\, d\omega_{FS} = \log|c_n| + \frac{1}{2}\sum_{j=1}^{n} \log(1+|x_j|^2).$$

The result follows from $\frac{1}{2}\log(1+x^2) \geq \log^+|x|$ for all $x \in \mathbb{R}$. □

One of the most useful properties that a height function should have is finiteness of sets of bounded height. This is called the Northcott property (see section 4.5 in [BG06]).

Even when $K = \mathbb{Q}$ neither the naive nor the geometric height function endows $\mathbb{Q}(t)$ with the Northcott property (e.g. the naive height of polynomials of the form $t^n$ is identically zero). To do so we must bound both the degree and the naive height of a given rational function simultaneously. By considering a Moriwaki height on $h^{\mathrm{Mor}}$ on $K(t)$ we accomplish so. The Northcott property for $K(t)$ will then follow from setting $h^{Mor}(f) = h^{Mor}([1:f])$ for any $f \in K(t)$ as before. We proceed to define it in the next section.

## 5.3 Heights à la Moriwaki

Moriwaki in his seminal paper [Mor00] introduces a height function on fields that are finitely generated over $\mathbb{Q}$. As mentioned above, one of the many remarkable properties this height function has is that it satisfies the Northcott property.

And lastly, there is the Moriwaki height for the adelic curve $k = K(t)$. As before, for $x \in \mathbb{C}$ we set

$$\|f\|_x := \exp\left(\int_{\mathbb{P}^1_{\mathbb{C}}} \log|F_x(t)|^{-\mathrm{ord}_x(f)} \mathrm{d}\omega_{FS}\right),$$

where $F_x$ is the minimal polynomial of $x$ over the ring of algebraic integers $\mathfrak{o}_K$ and not just $\mathbb{Z}$. Then the Moriwaki height is given by

**Definition 5.15** (Moriwaki height)**.** Let $f(t) \in k$. We define the Moriwaki height $h^{Mor}$ of $f$ to be

$$h^{Mor}(f) := \sum_{\sigma \in M_K^\infty} \int_{\mathbb{P}^1_{\mathbb{C}}} \log^+ |f(t)|_\sigma \mathrm{d}\omega_{FS} + \sum_{x \in \mathbb{P}^1_K \setminus \{\infty\}} \log^+ \|f\|_x + \sum_{\mathfrak{p} \in M_K^0} \log^+ \|f\|_{\mathfrak{p}}.$$

### 5.3.1 Adelic Curves

In their book [CM20], Chen and Moriwaki introduce the notion of an adelic curve. This concept has as particular instances familiar objects to arithmeticians, such as number fields and function fields.

**Definition 5.16** (Adelic Curve)**.** Let $K$ be a field and $M_K$ a set of places in $K$. We say that the data

$$(K, (\Omega, \mathcal{A}, \nu), \Phi) \text{ is an adelic curve}$$

where

$$\Phi : \Omega \to M_K, \quad \omega \to |\ |_\omega$$

and the following are satisfied

(i) $\mathcal{A}$ is a $\sigma-$algebra on $\Omega$ and $\nu$ is a measure on $(\Omega, \mathcal{A})$.

(ii) For any $a \in K^\times$, the function $\omega \to \log |a|_\omega$ is $\mathcal{A}-$measurable and integrable with respect to $\nu$.

If the product formula

$$\int_\Omega \log |a|_\omega d\nu(\omega) = 0 \tag{5.19}$$

is satisfied for all $a \in K^\times$, then we say that the adelic curve is proper.

*Remark* 5.17. We do not require that the map $\Phi$ is either injective or surjective.

Let us now define a height function associated to an adelic curve.

Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve and $\bar{K}$ be an algebraic closure of $K$. Let $S \otimes_K \bar{K} = (\bar{K}, (\Omega_{\bar{K}}, \mathcal{A}_{\bar{K}}, \nu_{\bar{K}}), \phi_{\bar{K}})$ be the algebraic extension of $S$ by $\bar{K}$ (see Definition 3.4.1 in [CM20]).

**Definition 5.18.** For $a = [a_0 : \ldots : a_N] \in \mathbb{P}^N_{\bar{K}}$ we define the height $h_S(a)$ of $a$ with respect to the adelic curve $S$ to be

$$h_S(a_0, \ldots, a_n) := \int_{\Omega_{\bar{K}}} \log \left( \max\{|a_0|_v, \ldots, |a_n|_v\} \right) d\nu_{\bar{K}}(v). \tag{5.20}$$

By the product formula (Equation 5.19) this is well defined.

Let us check that number fields and function fields are proper adelic curves as claimed.

**Number Fields**

Let $K$ be a number field and $\Omega = M_K$. For any $\omega \in \Omega, |\ |_\omega$ the absolute value on $K$ associated to $\omega \in M_K$. Setting $\nu(\{\omega\}) = n_\omega = [K_\omega : \mathbb{Q}_\omega]$ the local degree of the completion of $\mathbb{Q}_\omega$, we can see a number field as an adelic curve. Furthermore, by the standard product formula, any number field is a proper adelic curve. Here $h_S$ is just the naive height.

**Function Fields**

Let $C$ be a curve defined over a field $F$ and consider the field $k = F(C)$ of rational functions on $C$. Let $\Omega$ be the set of closed points of $C$ and $\mathcal{A}$ the discrete $\sigma-$algebra on $\Omega$.

For a closed point $x$ consider the discrete valuation $\mathrm{ord}_x(\cdot) : k \to \mathbb{Z} \cup \{+\infty\}$ on $k$ that has as valuation ring the local ring $O_{C,x}$ and let $|\cdot|_x$ be the absolute value on $k$ defined as

$$\forall a \in k^\times, \quad |a|_x = \exp\big(-\mathrm{ord}_x(a)\big).$$

We set $\Phi : \Omega \to M_K$ to be the map that sends $x \mapsto |\cdot|_x$. Lastly, let $n_x := [F(x) : F]$ be the degree of the residue field of $x$ and equip the measure space $(\Omega, \mathcal{A})$ with the measure $\nu$ such that $\nu(\{x\}) = n_x$. The relation

$$\forall a \in K^\times, \quad \sum_{x \in \Omega} n_x \mathrm{ord}_x(a) = 0 \tag{5.21}$$

yields that the adelic curve $(k, (\Omega, \mathcal{A}, \nu), \phi)$ is proper. Here $h_S$ is just the degree of the function.

**Function field of an arithmetic variety**

For simplicity we let $k = \mathbb{Q}(T)$ be the field of rational functions on the arithmetic variety $\mathbb{P}^1_{\mathbb{Z}}$. We exhibit a set of absolute values on $k$ that makes it a proper adelic curve. Let $f(T) \in \mathbb{Q}(T)$.

If $f(T) = \frac{a(T)}{b(T)}$ for $a(T), b(T) \in \mathbb{Q}[T]$ then for any absolute value $|\cdot|$ we have

$$|f(T)| = \frac{|a(T)|}{|b(T)|}$$

so without loss of generality take $f(T) \in \mathbb{Q}[T]$ and write

$$f(T) = aF_{x_1}^{r_1} \dots F_{x_t}^{r_t} \in \mathbb{Q}(T), \tag{5.22}$$

where $F_{x_j}$ is the minimal polynomial of $x_j$ over $\mathbb{Q}$ and $r_j \in \mathbb{Z}$. Also, take all $F_{x_j} \in \mathbb{Z}[T]$ and monic.

We start by noting that as a set, $k$ is nothing but the field of rational functions on $\mathbb{P}^1_{\mathbb{Q}}$. It is the particular choice of absolute values we endow $k$ with here- in contrast to the ones from the previous example-that allows us to see $k$ as the rational functions on $\mathbb{P}^1_{\mathbb{Z}}$.

For any rational prime $p \in \mathbb{Q}$ let

$$|\ |_p : \mathbb{Q}(T) \to \mathbb{R}$$

be the unique absolute value on $k$ that extends the Gauss norm in $\mathbb{Q}[T]$, and for any $x \in \mathbb{C} \setminus \{\infty\}$, we introduce the following absolute value on $k$. For $x \in \mathbb{C}$ we set

$$\|f\|_x := \exp\left(\int_{\mathbb{P}^1_{\mathbb{C}}} \log |F_x(t)|^{-\mathrm{ord}_x(f)} \mathrm{d}\mu_{\mathrm{FS}}\right),$$

where $F_x$ is the minimal polynomial of $x$ over $\mathbb{Z}$.

Recall that $\mathbb{P}^1_{\mathbb{C}}$ is the result of gluing two affine copies $\mathbb{A}^1_{\mathbb{C}}$ via the map $z \mapsto \frac{1}{z}$. Let $\pi_1, \pi_2$ be the two projections from $\mathbb{P}^1_{\mathbb{C}}$ to each one of these copies of $\mathbb{A}^1_{\mathbb{C}}$. Denote by $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$ the set of $t \in \mathbb{P}^1_{\mathbb{C}}$ such that both $\pi_1(t)$ and $\pi_2(t)$ are transcendental. For any $t \in (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$, let $|\cdot|_t$ be the absolute value on $\mathbb{Q}(T)$ given by

$$\forall \varphi \in \mathbb{Q}(T), \quad |\varphi|_t := \|\varphi(t)\|.$$

Denote by $\Omega$ the disjoint union $\Omega_h \sqcup \mathcal{P} \sqcup (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$, where $\Omega_h$ is the set of all closed points of $\mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\}$, and $\mathcal{P}$ denotes the set of all prime numbers. Let $\phi : \Omega \to M_k$ be the map sending $\omega \in \Omega$ to $|\cdot|_\omega$. We equip $\Omega_h$ and $\mathcal{P}$ with the discrete $\sigma$-algebras, and $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$ with the restriction of the Borel $\sigma$-algebra on $\mathbb{P}^1_{\mathbb{C}}$. Let $\mathcal{A}$ be the $\sigma$-algebra on $\Omega$ generated by the above $\sigma$-algebras on $\Omega_h$, $\mathcal{P}$ and $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$ respectively. Let $\nu$ be the measure on $\Omega$ such that $\nu(\{x\}) = 1$ for $x \in \Omega_h$, that $\nu(\{p\}) = 1$ for any prime number $p$ and that the restriction of $\nu$ on $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$ coincides with the Fubini-Study measure $\mu_{\mathrm{FS}}$. Then for any $f \in \mathbb{Q}(T) \setminus \{0\}$ one has

$$\int_\Omega \log|f|_\omega \, d\nu(\omega) = \sum_{x\in\Omega_h} \log|f|_x + \sum_{p\in\mathcal{P}} \log|f|_p + \int_{(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}} \log\|f(t)\| \, d\mu_{\mathrm{FS}}. \quad (5.23)$$

Since $\mathbb{P}^1_{\mathbb{C}} \setminus (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}$ is negligible with respect to $\mu_{\mathrm{FS}}$, we get

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{tras}}} \log\|f(t)\| \, d\mu_{\mathrm{FS}} = \int_{\mathbb{P}^1_{\mathbb{C}}} \log\|f(t)\| \, d\mu_{\mathrm{FS}} \quad (5.24)$$

Now, all the $F'_{x_j}s$ are monic and have coefficients in $\mathbb{Z}$ and so

$$\text{for all } p \in \mathcal{P}, \;\; \log|F_{x_j}|_p = 0.$$

The above yields

$$\int_\Omega \log|f|_\omega d\nu(\omega) = \sum_{p\in\mathcal{P}} \log|a|_p + \int_{\mathbb{P}^1_{\mathbb{C}}} \log\|a\|\mu_{\mathrm{FS}}.$$

To conclude we recall that the Fubini-Study measure endows $\mathbb{P}^1$ with mass 1 (see page 119 in [Huy05]) and so

$$\int_\Omega \log|f|_\omega \nu(d(\omega)) = \sum_{p\in\mathcal{P}} \log|a|_p + \log\|a\| = 0$$

where the last equality is just the usual product formula on $\mathbb{Q}$.

Hence for $f \in \mathbb{Q}(t)$ we have

$$h_S(f) = \sum_{\sigma\in M^\infty_{\mathbb{Q}}} \int_{\mathbb{P}^1_{\mathbb{C}}} \log^+\|f(t)\| \mathrm{d}\mu_{\mathrm{FS}} + \sum_{x\in\mathbb{P}^1_{\mathbb{Q}}\setminus\{\infty\}} \log^+|f|_x + \sum_{\mathfrak{p}\in M^0_{\mathbb{Q}}} \log^+|f|_{\mathfrak{p}},$$

where $M^\infty_{\mathbb{Q}}$ and $M^0_{\mathbb{Q}}$ are the Archimedean and non-Archimedean absolute values on $\mathbb{Q}$ respectively.

The last example above, but for a general number field $K$, will play a central role for us. In some sense it is the cornerstone of our treatment of the variation of canonical heights for subvarieties. So it makes sense to give proper introduction and separate definition.

Recall that if we view $K(t)$ as a the function field of an algebraic curve, we do get a proper adelic curve, but as we mentioned already in the introduction, the height function will not satisfy Northcott's Theorem. So we view it as the function field of an arithmetic

variety as we did for $\mathbb{Q}(t)$. Archimedean absolute values on $K$ are defined in terms of the field embeddings $\sigma : K \to \mathbb{C}$. For any such embedding and any $\alpha \in K$ we define

$$|\alpha|_\sigma := \begin{cases} \|\sigma(\alpha)\| & \text{if } \sigma \text{ is real (i.e } \sigma(K) \subset \mathbb{R}), \\ \|\sigma(\alpha)\|^2 & \text{otherwise} \end{cases}$$

where $|\cdot|$ is the standard absolute value on $\mathbb{R}$ (or $\mathbb{C}$). If there are $r$ real embeddings and $s$ conjugate pairs of complex embeddings then there are $r + 2s$ field embeddings from $K$ to $\mathbb{C}$ in total. But as it turns out, these embeddings yield only $r + s$ Archimedean absolute values because conjugate complex embeddings define the same absolute value. Moreover this is the only way for two embeddings to define the same absolute value (see page 42 in [Rib99]). Denote by $M_K^\infty$ the Archimedean absolute values of $K$ and $M_K^0$ the non-Archimedean ones. Lastly, let $\Omega, \mathcal{A}, \nu, \phi$ be the result of replacing $\mathbb{Q}$ by $K$ and the rational primes $\mathcal{P}$ by all primes in $K$ in the construction of the adelic curve $\mathbb{Q}(t)$.

Then the adelic curve $(K, (\Omega, \mathcal{A}, \nu), \phi)$ gives a height function $h_S$ in $K(t)$ which we will refer to as the Moriwaki height $h^{Mor}$. Concretely,

**Definition 5.19** (Moriwaki Height for $K(t)$)**.** Let $f \in K(t)$. We define the Moriwaki height $h^{Mor}$ of $f$ to be

$$h^{Mor}(f) := \sum_{\sigma \in M_K^\infty} \int_{\mathbb{P}^1_{\mathbb{C}}} \log^+ \|f(t)\| \mathrm{d}\mu_{\mathrm{FS}} + \sum_{x \in \mathbb{P}^1_K \setminus \{\infty\}} \log^+ |f|_x + \sum_{\mathfrak{p} \in M_K^0} \log^+ |f|_{\mathfrak{p}}.$$

The Moriwaki height for $P = [P_0 : P_1 : \cdots : P_N] \in \mathbb{P}^N_{\overline{K(t)}}$ is defined as:

$$\begin{aligned} h^{\mathrm{Mor}}(P) := & \sum_{\sigma \in M_K^\infty} \int_{\mathbb{P}^1_{\mathbb{C}}} \log \max\{\|P_0(t)\|, \ldots, \|P_N(t)\|\} \, d\mu_{FS} \\ & + \sum_{x \in \mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\}} \log \max\{|P_0(t)|_x, \ldots, |P_N(t)|_x\} \\ & + \sum_{\mathfrak{p} \in M_K^0} \log \max\{|P_0(t)|_{\mathfrak{p}}, \ldots, |P_N(t)|_{\mathfrak{p}}\}. \end{aligned}$$

This height function has turned out to be impressively suited for dynamics. On the one hand it gives us the Northcott property for function fields as we stated above. On the other hand it satisfies a functoriality property that behaves well under iterations. Concretely,

**Theorem 5.20** (Adapted from Theorem 3.11 in [Sil07])**.** Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve and $\varphi : \mathbb{P}^N_S \to \mathbb{P}^N_S$ be a morphism of degree $d$. Then there exist

constants $C_1, C_2 > 0$, depending on $\varphi$, such that

$$dh^{\mathrm{Mor}}(P) - C_2 \leq h^{\mathrm{Mor}}(\varphi(P)) \leq dh^{\mathrm{Mor}}(P) + C_1$$

for all $P \in \mathbb{P}^N_S$.

*Proof.* To ease notation, for any absolute value $v \in \Omega$ and $m \in \mathbb{N}$, set

$$\delta_v(m) = \begin{cases} m & \text{if } v \in M_K^\infty \quad (v \text{ is Archimedean}), \\ 1 & \text{if } v \in M_K^0 \quad (v \text{ is non-Archimedean}). \end{cases}$$

The triangle inequality gives

$$\log|\varphi(P)|_v \leq \log\left|\delta_v\left(\binom{N+d}{d}\right)\right|_v + \log|\varphi|_v + d\log|P|_v. \tag{5.25}$$

Hence we can just set

$$C_1 = \log\left|\binom{N+d}{d}\right| + h^{\mathrm{Mor}}(\varphi).$$

For the lower bound, since $\varphi$ is a morphism, it has homogeneous polynomial components $[f_0, \ldots, f_M]$ with no common zeros in $\mathbb{P}^N(K(t))$. We invoke Ingram's effective Nullstellensatz (Lemma 4 in [Ing21]) to obtain an algebraic integer $a$ and polynomials $g_{ij} \in \mathfrak{o}_K[X_0, \ldots, X_N]$ such that setting $t = (N+1)(d-1)+1$ we have:

$$a\mathrm{Res}X_j^t = g_{j,0}f_0 + g_{j,1}f_1 + \cdots + g_{j,N}f_N \quad \text{for } j = 0, 1, ..., N.$$

For $P = [x_0 : x_1 : \ldots : x_N]$ and any absolute value $v$, these relations give us:

$$\begin{aligned} |P|_v^t &= \max_{0\leq i\leq N} |x_i|_v^t \\ &= \max_{0\leq i\leq N} |g_{i0}(P)f_0(P) + g_{i1}(P)f_1(P) + \cdots + g_{iM}(P)f_M(P)|_v \\ &\leq \delta_v(N+1) \max_{\substack{0\leq i\leq N \\ 0\leq j\leq N}} |g_{ij}(P)f_j(P)|_v \\ &\leq \delta_v(N+1) \max_{\substack{0\leq i\leq N \\ 0\leq j\leq N}} \delta_v\left(\binom{N+t-d}{t-d}\right)|g_{ij}|_v|P|_v^{t-d}|f_j(P)|_v \\ &\leq \delta_v\left((N+1)\binom{N+t-d}{t-d}\right)\left(\max_{\substack{0\leq i\leq N \\ 0\leq j\leq N}} |g_{ij}|_v\right)|P|_v^{t-d}|\varphi(P)|_v \end{aligned}$$

Dividing by $|P|_v^{e-d}$ and taking logarithms

$$d \log |P|_v \leq \log \left[ \delta_v \left( (N+1) \binom{N+t-d}{t-d} \right) \right] + \log \left( \max_{\substack{0 \leq i \leq N \\ 0 \leq j \leq N}} |g_{ij}|_v \right) + \log |\varphi(P)|_v. \quad (5.26)$$

Integrating with respect to all $v \in \Omega$ in the adelic curve and using the product formula yields,

$$dh^{\mathrm{Mor}}(P) - \log \left[ \left( (N+1) \binom{N+t-d}{t-d} \right) \right] - h^{\mathrm{Mor}}(g) \leq h^{\mathrm{Mor}}(\varphi(P)),$$

where $h^{\mathrm{Mor}}(g)$ denotes the Moriwaki projective height of the tuple of entries in all the coordinates of the $g_{i,j}$'s.

This completes the proof, showing an explicit error between the logarithmic height of $P$ times $d$ $\phi(P)$ and that of $d$. □

*Remark* 5.21. It is very important to note that even though we have not made $h^{\mathrm{Mor}}(g)$ explicit, the bounds are indeed effective (see Lemma 4 in [Ing24] for instance). The importance lies on the fact that our main result will be understanding an asymptotic relation so we will need to know explicit values of these bounds. Their order of magnitude will not be of crucial importance for us.

The reader may be referred to [KPS01] and [Bro87] for an overview on the history of the effective Nullstellensatz topic that we believe started with Hermann's work [Her19] a century ago.

### 5.3.2 Moriwaki heights vs. Naive heights

In this subsection, we will prove the following.

**Lemma 5.22.** Let $K$ be a number field, take $P = [P_0 : \ldots : P_N] \in \mathbb{P}^N_{K(t)}$ with no common factor and set $d = \max_{0 \leq j \leq N} \deg P_j$. Then

$$\frac{1}{6} \left( h^{\mathrm{naive}}(P) + h^{\mathrm{geom}}(P) \right) \leq h^{\mathrm{Mor}}(P) \leq 2 \left( h^{\mathrm{naive}}(P) + h^{\mathrm{geom}}(P) \right).$$

where $h^{\mathrm{geom}}(P) = \max_{0 \leq j \leq N} \deg P_j$.

This Lemma states that the Moriwaki height builds on two classical height notions for functions (naive or arithmetic and geometric), creating an appropriate height function to

study Diophantine Geometry in the setting of an Adelic curve, specifically with regards to Unlikely Intersections-type questions.

Without loss of generality, assume that the coordinates $P_j(t)$ of $P \in \mathbb{P}^N_k$ are polynomials. Recall that $h^{\text{naive}}(P(t))$ denotes the naive Weil height of the projective point determined by the list of all the coefficients appearing in the $P_j$'s. And that the geometric height $h^{\text{geom}}(P)$ of $P$ is the maximum degree among the $P_j$'s. Lastly, the Moriwaki height for $P = [P_0(t) : P_1(t) : \cdots : P_N(t)] \in \mathbb{P}^N_{\bar{k}}$ is defined as:

$$\begin{aligned} h^{\text{Mor}}(P) := &\sum_{\sigma \in M_K^\infty} \int_{\mathbb{P}^1_{\mathbb{C}}} \log \max\{|P_0(t)|_\sigma, \ldots, |P_N(t)|_\sigma\}\, d\omega_{FS} \\ &+ \sum_{x \in \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}} \log \max\{|P_0(t)|_x, \ldots, |P_N(t)|_x\} \\ &+ \sum_{\mathfrak{p} \in M_K^0} \log \max\{\|P_0\|_\mathfrak{p}, \ldots, \|P_N\|_\mathfrak{p}\}. \end{aligned}$$

*(Proof of Lemma 5.22).* We assume, moreover that the $P_j$'s have coefficients in $\mathfrak{o}_K$. In particular, the absolute values of the form $|\cdot|_x$ coming from orders of the poles of the functions in the coordinates yield no contribution. That is,

$$h^{\text{Mor}}(P) = \sum_{\sigma \in M_K^\infty} \int_{\mathbb{P}^1_{\mathbb{C}}} \log \max\{|P_0(t)|_\sigma, \ldots, |P_N(t)|_\sigma\}\, d\omega_{FS} + \sum_{\mathfrak{p} \in M_K^0} \log \max\{\|P_0\|_\mathfrak{p}, \ldots, \|P_N\|_\mathfrak{p}\}.$$

For $|t| \geq 1$ and every $j$, the triangle inequality yields $|P_j(t)| \leq (n+1)\|P_j\||t|^n$, while for $|t| < 1$ we have $|P_j(t)| \leq (n+1)\|P_j\|$. Hence

$$\log \max\{|P_0(t)|_\sigma, \ldots, |P_N(t)|_\sigma\} \leq \log \max_j \|P_j\|_\sigma + n \log^+ |t| + \log(n+1).$$

From previous calculations or just integrating and summing over all places:

$$h^{\text{Mor}}(P) \leq h^{\text{naive}}(P) + \frac{\log 2}{2} n + \log(n+1) \leq 2\left(h^{\text{naive}}(P) + h^{\text{geom}}(P)\right).$$

Now we move to the lower bound. For this set $S = \sum_{\mathfrak{p} \in M_K^0} \max_j \|P_j\|_\mathfrak{p}$ and note that for every $j$ and every $\sigma \in M_K^\infty$,

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log \max_i |P_i(t)|_\sigma\, d\omega_{FS} \;\geq\; \int_{\mathbb{P}^1_{\mathbb{C}}} \log |P_j(t)|_\sigma\, d\omega_{FS} \;\geq\; m(P_j) \;\geq\; \log \|P_j\|_\sigma - n \log 2,$$

where the second inequality is Proposition 5.14 and the third Proposition 5.12.

Summing then over all $\sigma \in M_K^\infty$, we get

$$h^{\mathrm{Mor}}(P) - S \;\geq\; \big(h^{\mathrm{naive}}(P) - S\big) - [K:\mathbb{Q}]\, n \log 2,$$

and hence

$$h^{\mathrm{Mor}}(P) \;\geq\; h^{\mathrm{naive}}(P) - [K:\mathbb{Q}]\, n \log 2. \tag{5.27}$$

On the other hand, Lemma 5.13, at each $\sigma \in M_K^\infty$, gives

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \log \max_j |P_j(t)|_\sigma \, d\omega_{FS} \;\geq\; \frac{1}{2} \log |c_{n,j_0}|_\sigma + \frac{1}{2} \log |c_{0,j_0}|_\sigma + \frac{\log 2}{2}\, n.$$

Summing over all $\sigma \in M_K^\infty$ we get:

$$h^{\mathrm{Mor}}(P) - S \;\geq\; \frac{1}{2} \sum_{\sigma \in M_K^\infty} \log |c_{n,j_0}|_\sigma \;+\; \frac{1}{2} \sum_{\sigma \in M_K^\infty} \log |c_{0,j_0}|_\sigma \;+\; \frac{\log 2}{2}\, n;. \tag{5.28}$$

And by the product formula we also have

$$\sum_{\sigma \in M_K^\infty} \log |c_\alpha|_\sigma \;=\; \sum_{\sigma \in M_K^0} -\log |c_\alpha|_\sigma \;\geq\; -\sum_{\mathfrak{p} \in M_K^0} \log \max_j \|P_j\|_{\mathfrak{p}} \;=\; -S.$$

This and Equation 5.28 yield

$$h^{\mathrm{Mor}}(P) \;\geq\; \tfrac{1}{2}(-S) + \tfrac{1}{2}(-S) + [K:\mathbb{Q}]\, \tfrac{\log 2}{2}\, n + S \;=\; [K:\mathbb{Q}]\, \frac{\log 2}{2}\, n.$$

That is,

$$h^{\mathrm{Mor}}(P) \;\geq\; [K:\mathbb{Q}]\, \frac{\log 2}{2}\, n. \tag{5.29}$$

Taking $\frac{1}{6}(5.27) + \frac{5}{6}(5.29)$:

$$h^{\mathrm{Mor}}(P) \;\geq\; \frac{1}{6}\, h^{\mathrm{naive}}(P) + \left( -\frac{[K:\mathbb{Q}]\, \log 2}{6} + \frac{5\,[K:\mathbb{Q}]\, \log 2}{12} \right) n \;=\; \frac{1}{6}\, h^{\mathrm{naive}}(P) + \frac{3[K:\mathbb{Q}]\, \log 2}{4}\, n.$$

Since $[K:\mathbb{Q}] \geq 1$ and $\frac{3 \log 2}{4} > \frac{1}{6}$, the result follows. □

As claimed, this Theorem gives us the Northcott property.

**Corollary 5.23** (Northcott Theorem for the adelic curve $K(t)$)**.** For all $M \in \mathbb{R}_{\geq 0}$ it holds

$$\#\{P \in \mathbb{P}^N_{K(t)} : h^{Mor}(P) \leq M\} < \infty.$$

*Proof.* If $h^{Mor}(P) \leq M$ then there are constants $C_1, C_2 > 0$ such that both $C_1 d$ and $C_2 h(f) \leq M$. Then this reduces to the usual Northcott Theorem (e.g. see Theorem 3.7 in [Sil07]). □

## 5.4 Heights for Subvarieties

At the time of writing this, we have only found two essentially different approaches to define heights for subvarieties. On the one hand we can use a geometric perspective and attempt to translate the problem to that of points, by assingning to every subvariety its generalized Chow form. On the other hand we can use Arakelov Geometry and do intersection theory. The first is the one due to Philippon and the second is the one due to Gillet-Soulé classically. It provided the framework for Faltings to prove the Mordell Conjecture. So we understand why that approach has seen a greater development by the community. The approach we follow, due to Chen-Moriwaki (see [CM21]) but already present in Moriwaki's seminal paper [Mor00] or Gubler's M-fields theory, belongs in the second realm but the adelic curve groundfield is more modern. Concretely, what we do is arithmetic intersection theory on adelic curves.

### 5.4.1 *Philippon*

Let $U \subset X$ be a subvariety of a projective variety $X$ of dimension $p$. Just as a plane in $\mathbb{R}^3$ can be described in terms of the vectors it cointains, it is also determined by a 1-dimensional subspace (the normal vector; see 3.1), a subvariety can be characterized intrinsically by what it contains and also by what is needed to "fill out" the ambient space.

Then, in general, a linear subspace of dimension $\dim X - p$ (or equivalently the intersection of $\dim X - p$ hyperplanes) will not intersect $U$. This is what the Chow Form from Chapter 3 captures. But for us, since we are interested in iterating, we consider not the Chow Form but the generalized Chow Form (we refer the reader to Chapter 3 for this as well).

For $s = (s_0, \dots s_p)$ let $\Phi_{s,U}$ be the generalized Chow Form associated to $U$. Then Philippon defines the height $h^{\mathrm{Plpn}}(U)$ of $U$ as

$$h^{\mathrm{Plpn}}(U) := \sum_{v \in M_K \text{finite}} \log \left\| \Phi_{s,U} \right\|_v + \sum_{\sigma : K \to \mathbb{C}} \int_{(S^N)^{r+1}} \log \left| \sigma(\Phi_{s,U})(\vec{t_0}, \dots, \vec{t_r}) \right| (d\mu_N)^{r+1}, \tag{5.30}$$

where $\mu_N$ is the probability measure (and so it has mass 1) in the unit ball $S^N$ and the vectors $\vec{t_j}$ denote the set of the generic coefficients of homogeneous polynomials of degree $d_j$.

### 5.4.2 *Arakelov (and Gillet-Soulé)*

The motivation for this approach is based on Andre Weil's idea that Arithmetic resembles doing geometry in a family. That is what the notion of a model captures in this setting.

Let $k$ be a complete field with respect to a metric $\|\cdot\|$ (e.g the completion of $\mathbb{Q}_p$) and $Z$ be an algebraic variety of dimension $s$ over $k$.

A model $\mathscr{Z}$ of $Z$ over $\operatorname{Spec}\mathcal{O}_k$ is a flat projective scheme over $\operatorname{Spec}\mathcal{O}_k$ together with an isomorphism from

$$\mathscr{Z} \otimes_{\mathcal{O}_k} k \to Z.$$

Similarly, for every line bundle $L$ on $X$ a model $\mathscr{L} = (\mathscr{L}, e, \phi)$ of $L$ is a line bundle over $\mathscr{Z}$ (i.e. a line bundle over the finite places and a metric at the archimedian places) such that $\phi$ is an isomorphism from

$$\mathscr{L} \otimes_{\mathcal{O}_k} k \to L^{\otimes e} \text{ (or } eL \text{ depending on the author).}$$

We note that there is a relationship between models (of line bundles) and metrics. And so it is very natural to consider a metric at the archimedian place. Let $s \in H^0(L, Z)$ be a section of $L$ and $z \in Z \setminus \operatorname{div} s$. Then the model $\mathscr{L}$ induces a metric as follows

$$\|s(z)\|_{\mathscr{L}} = \inf_{a \in k^*} \{\|a\|^{\frac{-1}{e}} : as^e \text{ extends to a section of } \mathscr{L}\}. \tag{5.31}$$

Recall that our original setting is in a projective algebraic variety $X$ over a number field $K$. And so in order to connect it with the above notion we must consider not one metric but a family of metrics. We do this via the notion of an adelic metric.

### 5.4.3 Adelic Metrics

Let $X$ be a projective variety over $\mathbb{Q}$ and $L$ a line bundle over $X$. We define

$$X_v^{\text{an}} = \begin{cases} X(\mathbb{C}) \text{ if } v = \infty \\ X^{\text{an}} \text{ over } \mathbb{C}_p \text{ if } v = p \end{cases} \tag{5.32}$$

A continuous adelic metric on $L$ is a family $\{\|\cdot\|_v\}_{M_\mathbb{Q}}$ where $\|\cdot\|_v$ is a continuous metric on $L_v = L \otimes_\mathbb{Z} \mathbb{Q}_v$ such that there is a model $\mathscr{X}$ of $X$ over $U \subset \operatorname{Spec}\mathbb{Z}$ and a model $\mathscr{L}$ of $L$ such that for all $u \in U$ the metric $\|\cdot\|_u$ is induced by the model as in Equation 5.31. We refer to $\overline{L} = (L, \|\cdot\|_v)$ as an adelic metrized line bundle.

**Definition 5.24.** Let $\overline{L} = (L, \|\cdot\|_v)$ be an adelic metrized line bundle on $X$ and $s$ a section of $L$. For any $x \in X \setminus \operatorname{div} s$ we define the height $h_{\overline{L}}$ associated to $\overline{L}$ as

$$h_{\overline{L}}(x) = -\sum_{v \in M_\mathbb{Q}} \log \|s(x)\|_v.$$

A couple of remarks are due. First, note that since any rational section is just a function on the coordinates of the point, this is well defined by the product formula. And second, note that the sum is actually a finite sum because

$$\overline{\{x\}} \not\subset \operatorname{div} s$$

and so the intersection is finite because the model $\mathscr{X}$ is flat and proper over $\operatorname{Spec}\mathbb{Z}$ (see Figure 5.3).

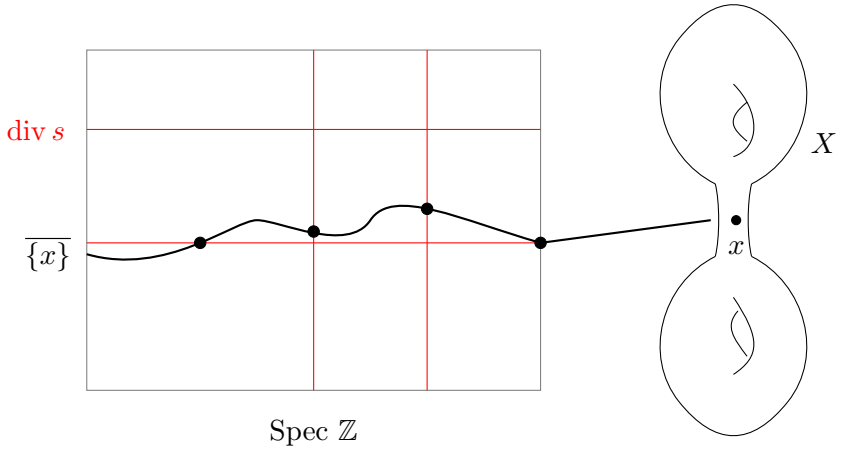



FIGURE 5.3: Intersection between $\operatorname{div} s$ and $\overline{\{x\}}$

It is only at this finite set of intersection points that the induced metric is different from 1 because otherwise there are not zeros or poles and so $s$ already extends to a section of $\mathscr{L}$.

We illustrate these notions in an example that we learned from a series of lectures that Jose Ignacio Burgos Gil gave at the International Centre for Theoretical Sciences in February of 2023.

**Example 5.25.** Let $X = \mathbb{P}^1_{\mathbb{Q}}$ and $L = \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}(1)$. For every $\alpha = [x_0 : y_0] \in X$ we have $h(\alpha) = \log \max\{|x_0|, |y_0|\}$. We show how to obtain this same result via the framework we have established.

First let us pick the players that will come into play. Since

$$H^0(L, X) = \{aX + bY : a, b \in \mathbb{Q}\},$$

the function $s = X$ is a regular section of $L$ with a zero at $z = [0 : 1]$ and no poles.

And we set the model of $X$ to be $\mathscr{X} = \mathbb{P}^1_{\mathbb{Z}}$, the model of $L$ to be $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ and the open subset $U = \operatorname{Spec} \mathbb{Z} \setminus \eta$.

For every $p \in U$ there are

$$r \geq 0 \quad \text{and} \quad a, b \in \mathbb{Z}$$

with $\gcd(a, b) = \gcd(a, p) = \gcd(b, p) = 1$ such that

$$\alpha = [p^r a : b] \quad \text{or} \quad \alpha = [a : p^r b].$$

For the first case and $r > 0$ note that $p^{-r}s$ still extends to a section of $\mathscr{L}$ and so

$$\|s(\alpha)\|_p = |p^{-r}|_p^{-1} = |x_0|_p.$$

For the latter case, or when $r = 0$, we have

$$\|s(\alpha)\|_p = 1.$$

Joining everything

$$\|s(\alpha)\|_p = \begin{cases} |x_0|_p, & \text{if } |x_0|_p < |y_0|_p \\ 1, & \text{otherwise.} \end{cases}$$

Or equivalently

$$\|s(\alpha)\|_p = \frac{|x_0|_p}{\max\{|x_0|_p, |y_0|_p\}}.$$

Now, for the Archimedean place we have freedom to choose any metric. Set

$$\|s(\alpha)\|_\infty = \frac{|x_0|}{\max\{|x_0|, |y_0|\}}.$$

Then

$$h_{\overline{L}}(\alpha) = \sum_{v \in M_{\mathbb{Q}}} -\log \|s(x)\|_v = \log \max\{|x_0|, |y_0|\} \text{ as claimed.}$$

We conclude by noticing that our choice of metric at the Archimedean place gives of course different height functions. For instance we can set

$$\|s(\alpha)\|_\infty = \begin{cases} \frac{|x_0|}{\sqrt{|x_0|^2+|y_0|^2}} \\ \frac{|x_0|}{|x_0|+|y_0|} \\ \frac{|x_0|}{\max\{|x_0|,|y_0|\}}, \end{cases} \qquad \text{yielding} \quad h_{\overline{L}}(\alpha) = \begin{cases} \log \sqrt{|x_0|^2 + |y_0|^2} \\ \frac{|x_0|}{|x_0|+|y_0|} \\ \log \max\{|x_0|, |y_0|\}, \end{cases}$$

which are three distinct height functions on $\mathbb{P}^1_{\mathbb{Q}}$.

Not all height functions in this language will arise from models. We enlarge our framework by considering smooth metrics and admissible metrics.

A continuous adelic metric is smooth if $U = \operatorname{Spec} \mathbb{Z}$ and at the Archimedean place the metric is smooth. These are the metrics considered by Gillet-Soulé in their arithmetic intersection theory. We refer to the line bundle endowed with theis structure as a continuous metrized line bundle.

Finally, after a few preliminary chapters, we are ready to define heights of subvarieties the way is most used in the literature today.

**Definition 5.26.** Let $X$ be a projective variety over $\mathbb{Q}$ and $L$ a line bundle on $X$. Consider $\mathscr{X}, \mathscr{L}$ models over $\operatorname{Spec} \mathbb{Z}$ of $X, L$ respectively.

Suppose that $\mathscr{L} = (\mathscr{L}, \|\cdot\|_v)$ is a continuous metrized line bundle. For every subvariety $Y \subset X$ we set $\mathscr{Y}$ to be the closure of $Y$ inside $\mathscr{X}$ and define

$$h^{\text{Arak}}(Y) = \frac{\widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}})^{p+1} \mid \mathscr{Y}\right)}{((p+1)\deg_{\mathscr{L}}(Y))}. \tag{5.33}$$

Soulè (Theorem 3 in [Sou89]) calculated an explicit relation these two height functions.

**Theorem 5.27.** Let $X$ be a projective variety defined over a number field $K$ and $U \subset X$ a subvariety of dimension $r$. Then

$$h^{\text{Arak}}(U) = h^{\text{Plpn}}(U) + \frac{1}{2}(r+1)\left(\sum_{j=1}^{N}\frac{1}{j}\right)[K:\mathbf{Q}]\deg(U). \tag{5.34}$$

We say that a smooth metric (on $\mathscr{L}$) is semipositive if

(i) $\mathscr{L}$ is relatively nef (meaning each fiber is nef).

(ii) $c_1(L, \|\cdot\|_\infty)$ is a positive $(1,1)$ form.

And we say that a continuous adelic metric is admissible if for each $v \in M_{\mathbb{Q}}$ there is a sequence $\{\|\cdot\|_{v,n}\}_{n\in\mathbb{N}} \to \|\cdot\|_v$ such that

(i) there is an open $U \subset \mathbb{Z}$ such that for all $U \in U$ the sequence is constant i.e.

$$\|\cdot\|_{u,n} = \|\cdot\|_u, \text{ and}$$

(ii) for all $v \in M_{\mathbb{Q}}$

$$\log\frac{\|\cdot\|_{v,n}}{\|\cdot\|_v} \to 0 \quad \text{uniformly.}$$

We refer to the line bundle endowed with this structure as an admissible metrized line bundle. Zhang proved that we can define heights with respect to admissible adelic metrics. Concretely,

**Theorem 5.28** (Theorem 1.4 in [Zha95])**.** Let $X$ be a projective variety over $\mathbb{Q}, L$ a line bundle on $X$ and $Y \subset X$ a subvariety. Consider $\mathscr{X}, \mathscr{L}$ models over $\operatorname{Spec}\mathbb{Z}$ of $X$ and $L$ respectively and $\mathscr{Y}$ the closure of $Y$ inside $\mathscr{X}$.

Suppose that $\overline{\mathscr{L}} = (\mathscr{L}, \|\cdot\|_v)$ is an admissible line bundle and let $\overline{\mathscr{L}_m} = (\mathscr{L}, \|\cdot\|_{v,m})$ be any sequence of semipositive metrized line bundles converging to $\overline{\mathscr{L}}$. Then

$$\lim_{m\to\infty} h_{\overline{\mathscr{L}_m}}(Y) \text{ exists and is independent from the sequence.}$$

### 5.4.4 *Arithmetic Intersection Theory on Adelic Curves*

Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve, $X$ a projective variety of dimension $r$ over $K$, and $\overline{L}_0, \ldots, \overline{L}_r$ are integrable (i.e. can be written as $\overline{L} = \overline{L}' \otimes (\overline{L}'')^\vee$ where $\overline{L}'$ and $\overline{L}''$ are both ample and semi-positive) adelic line bundles on $X$.

Matching Chamber-Loir's presentation of the work developed by Arakelov and then generalized by Gillet-Soulè (Theorem 4.22 above), Chen-Moriwaki built (Definition 4.4.3 in [CM21])a multilinear pairing

$$(\overline{L}_0, \dots, \overline{L}_r) \to (\overline{L}_1 \cdots \overline{L}_r)_S,$$

which, again inspiring ourselves from Chamber-Loir's presentation, summarizes the arithmetic intersection theory over adelic curves required to define heights. The ultimate tool in use to give the inductive definition of this pairing is a certain natural (in the sense of Bedford-Taylor [BT76] for the Archimedean case and then Chambert-Loir [Cha03] for the non-Archimedean case) measure on adelic line bundles. The non-Archimedean case was already introduced above in the section on measures on Berkovich spaces in Chapter 2. Let us though give a brief summary of the measure in this case for the reader's convenience.

Let $k$ be a complete non-Archimedean field, and write $k^0$ for its valuation ring and $\mathfrak{m}$ for its maximal ideal. Let $X/k$ be a projective integral variety of dimension $d$, and $L$ a line bundle on $X$. Consider also $\mathscr{X}$, a normal projective $k^\circ$-model of $X$. We need to consider the special fiber of $\mathscr{X}$. This is given by

$$\mathscr{X}_s = \sum_j m_j Y_j.$$

Let $\mathscr{L}_i$ be line bundles on $\mathscr{X}$ inducing model metrics $\varphi_i$ on line bundles $L_i$ on $X$. The measure then is given by

$$\mu_{\overline{L}_1,\dots,\overline{L}_d} = \sum_j m_j \, \deg\big(c_1(\mathscr{L}_1|_{Y_j}) \cdots c_1(\mathscr{L}_d|_{Y_j}) \mid [Y_j]\big)\, \delta_{\xi_j},$$

where $\xi_j$ is the Berkovich point specializing to the generic point of $Y_j$.

Explicitly, this pairing is given recursively as follows (see Proposition 4.4.4 in [CM21]). Write $s_j$ for a meromorphic section of $\overline{\mathscr{L}_j}$ and suppose $\operatorname{div}(s_d) = \sum_j a_j Z_j$. Then

$$\begin{aligned}(\overline{L}_0 \cdots \overline{L}_d)_S = &\int_\Omega \left( \int_{X_\omega^{\mathrm{an}}} -\log \|s_d\|_{\varphi_{d,\omega}}([\mathbf{x}])\, \mathrm{d}\left(\mu_{\overline{L_0},\dots,\overline{L_{d-1}}}(x)\right)\right)(\mathrm{d}\nu(\omega)) \\ &+ \sum_{i=1}^n a_i \big(\, \overline{L}_0\big|_{Z_i} \cdots \overline{L}_{d-1}\big|_{Z_i}\,\big)_S. \end{aligned} \tag{5.35}$$

Moreover, this pairing satisfies the projection formula.

**Theorem 5.29** (Projection formula (Theorem 4.4.9 in [CM21]))**.** Let $f : Y \to X$ be a morphism of projective schemes over $K$ and $\overline{L}_0, \ldots, \overline{L}_l$ be integrable adelic line bundles on $X$. For an $l$-cycle $Z$ on $Y$,

$$(f^*(\overline{L}_0) \cdots f^*(\overline{L}_l) \mid Z)_S = (\overline{L}_0 \cdots \overline{L}_l \mid f_*(Z))_S.$$

With all this at hand, when $\overline{L}_i = \overline{L}$ for all $i$, we define the height $h_{\overline{L}}(U)$ of $U$ as

**Definition 5.30** (Moriwaki Height of a Subvariety)**.**

$$h^S_{\overline{L}}(U) = \frac{(\overline{L}^{r+1}|U)_S}{(L^r|U)},$$

where $L$ is the line bundle underlying $\overline{L}$ and $(L^r|U)$ and the classical intersection degree.

A reasonable first question someone could ask themselves after seeing this definition is how is this related to the previous definitions for heights that we have seen so far. Chen-Moriwaki answer this in what for us is a deep Theorem. They label it as Proposition 4.5.1 in their book [CM21].

Let $\mathfrak{F}$ be a finitely generated field over $\mathbb{Q}$ of transcendence degree $n$. A normal projective arithmetic variety $\mathcal{B}$ is called an arithmetic model of $\mathfrak{F}$ if the function field of $\mathcal{B}$ is isomorphic to $\mathfrak{F}$.

Consider $\overline{H}$ a $C^\infty$-hermitian line bundle on $B$. $\overline{H}$ is said to be vertically nef if $H$ is relatively nef with respect to $B \to \mathrm{Spec}(\mathbb{Z})$, and $c_1(\overline{H})$ is semipositive on $B(\mathbb{C})$. We say $\overline{H}$ is horizontally nef if for all integral and closed one-dimensional subschemes $\Gamma$ flat over $\mathbb{Z}$, $\widehat{\deg}\left(\overline{H}\big|_\Gamma\right) \geq 0$. If $\overline{H}$ is both vertically and horizontally nef, we say $\overline{H}$ is nef.

**Definition 5.31.** Let $\mathfrak{F}$ be a finitely generated field over $\mathbb{Q}$ of transcendence degree $n$ and $\mathcal{B}$ an arithmetic model of $\mathfrak{F}$. Consider $\overline{H}_1, \ldots, \overline{H}_n$ nef $C^\infty$-hermitian line bundles on $\mathcal{B}$. We call the data $(\mathcal{B}; \overline{H_1}, \ldots, \overline{H_n})$ a polarization of $\mathfrak{F}$.

Coming back to our variety $X$, we choose a projective arithmetic variety $\mathscr{X}$ and a morphism $\pi : \mathscr{X} \to \mathscr{B}$ with generic fiber equal to $X$. Let $L_0, \ldots, L_r$ be integrable line bundles on $X$ and let $\overline{\mathscr{L}}_0, \ldots, \overline{\mathscr{L}}_r$ be metrized line bundles on $\mathscr{X}$ such that $\mathscr{L}_0, \ldots, \mathscr{L}_r$ coincide with $L_0, \ldots, L_r$. Note that, for each $w \in \Omega$, $\overline{\mathscr{L}}_i$ induces a local metric $\varphi_{i,w}$ of $L_{i,w}$, that is, if $w \in \Omega_\infty$, then

$$\varphi_{i,w} = h_i|_{\pi^{-1}(w)}, \quad \text{where } h_i \text{ is the metric on } \mathscr{L}_i.$$

And for $w \in \Omega \setminus \Omega_\infty$, we set $\varphi_{i,w}$ to be the model metric induced by the model $(\mathscr{X}, \mathscr{L}_i)$. We denote the adelic metrized line bundle $\{(L_{i,w}, \varphi_{i,w})\}_{w\in\Omega}$ by $\overline{L}_i$.

**Theorem 5.32** (Proposition 4.5.1 in [CM21])**.** With the notation from above, we have

$$(\overline{L}_0 \cdots \overline{L}_r)_S = (\overline{\mathscr{L}_0} \cdots \overline{\mathscr{L}_r} \cdot \pi^* \overline{H_1} \cdots \pi^* \overline{H_n}) \tag{5.36}$$

We note that the dimension of the base $\mathcal{B}$ is $r+1$, so

$$\dim \mathscr{X} = (r+1) + n.$$

Hence, the classical arithmetic intersection number on $\mathscr{X}$ needs $r+1+n$ metrized line bundles: $r+1$ line bundles coming from the generic fiber $X$, and the $n$ pullbacks of the polarization. What the result above is saying in English is that in the adelic curve setting, the latter $n$ line bundles get absorbed by the measure space on the induced adelic curve.

As a corollary to this theorem, we get that the Moriwaki height coincides with the one coming from arithmetic intersection theory.

Moriwaki [Mor00] confirms, via a more explicit definition of this height function, that this height function also coincides with the naive Weil height in the number field case.

Expanding on the remark on the measure space on the induced adelic curve, another natural question we could ask is what happens to the Philippon height $h^{\mathrm{Plpn}}$ when instead of the set of places $M_K$ we consider the set of places $\Omega$ of the induced adelic curve.

**Definition 5.33** (Philippon Height on an Adelic Curve)**.** Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve and $X$ a projective variety over $K$.

For every subvariety $U$ of $X$ of dimension $p$, we define the Philippon height $h^{\mathrm{Plpn}}(U)$ of $U$ as

$$h^{\mathrm{Plpn},S}(U) = \int_\Omega \log \|\Phi_{s,U}\|_\omega \, d\nu(\omega),$$

where $\Phi_{\boldsymbol{s},U}$ is the generalized Chow form associated to $U, \boldsymbol{s} = (s_0, \ldots, s_p)$ and the local height $\log \|\Phi_{\boldsymbol{s},U}\|_\omega$ is given by:

$$\begin{aligned} m_\omega^S(\Phi_{s,U}) &:= \log \|\Phi_{s,U}\|_\omega \\ &= \begin{cases} \text{Gauss norm of } \Phi_{\boldsymbol{s},U}, & \text{if } \omega \text{ is non-Archimedean} \\ m(\Phi_{\boldsymbol{s},U}) = \displaystyle\int_{(S^N)^{r+1}} \log \left|\Phi_{\boldsymbol{s},U}(\vec{t}_0, \ldots, \vec{t}_r)\right| \, (d\mu_N)^{r+1}, & \text{if } \omega \text{ is Archimedean.} \end{cases} \end{aligned}$$

We note that this is well defined by Theorem 4.2.7 in [CM21].

Naturally, we ask ourselves how does $h_{\overline{L}}^{S}(U)$ relate to $h_{\overline{L}}^{\mathrm{Plpn},S}(U)$. But before answering this question, let us introduce toric canonical metric on $\mathbb{P}^N$ (see Examples 2.28 and 2.32 in[BPS14]). We will denote by $\overline{\mathcal{O}(1)}_{\mathrm{tor}}$ the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^N$, equipped with the canonical toric metric, defined at each place by

$$\|s\|_{\overline{\mathcal{O}(1)}_{\mathrm{tor}},v}(\mathbf{x}) = \frac{|s(\mathbf{x})|_v}{\|\mathbf{x}\|_v},$$

for any meromorphic section $s$ of $\mathcal{O}(1)$. These will be the default adelic metrics on $\mathbb{P}^N$ or $\check{\mathbb{P}}^N \cong \mathbb{P}^N$ that we will use unless otherwise specified.

We note that this is an approachable (the type of metrizations considered by Zhang [Zha95]) metric. Indeed, consider the sequence of semipositive smooth metrics $\|\cdot\|_m$ on $\mathcal{O}(1)^{\otimes m}$ given by a model for non-Archimedean places, and the Fubini-Study metric given by

$$\|s\|_{m,\infty}[x_0 : \ldots : x_N] = \frac{|s([x_0 : \ldots : x_n])|}{(\sum_i |x_i|^{2m})^{1/2m}},$$

for Archimedean ones. This family of metrics converges uniformly to the canonical toric metric.

We are now ready to present our first result.

**Theorem 5.34** (Ingram, Jaramillo-Martinez)**.** Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve that is either a number field, a function field or the function field of an arithmetic variety. And let $X$ a projective variety of dimension $p$ over $K$. Then

$$\left| h_{\overline{\mathcal{O}(1)}_{\mathrm{tor}}}^{\mathrm{Plpn},S}(X) - h_{\overline{\mathcal{O}(1)}_{\mathrm{tor}}}^{S}(X) \right| \leq N \deg(X) \log 2 + \log p.$$

That is, we are finding an explicit constant for the difference of both heights in terms the dimension of the ambient space, and both the degree and dimension of our variety.

*Proof.* To reduce notation, let us skip the subscript $\overline{\mathcal{O}(1)}_{\mathrm{tor}}$ for now. We first use the triangle inequality to get

$$\left| h^{\mathrm{Plpn},S}(X) - h^{S}(X) \right| \leq \left| h^{\mathrm{Plpn},S}(X) - h^{S}(\Phi_{\boldsymbol{s},X}) \right| \tag{5.37}$$

$$+ \left| h^{S}(\Phi_{\boldsymbol{s},X}) - h^{S}(X) \right|, \tag{5.38}$$

where

$$\Phi_{\boldsymbol{s},X} = \Phi_{\boldsymbol{s},X}(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_r), \qquad \boldsymbol{u}_i = (u_{i,0}, \ldots, u_{i,N}),$$

is the generalized Chow form of $X$.

To bound (5.37), we invoke Lemma 1.6.10 in [BG06] to get at Archimedean places $\omega$:

$$-\left(\frac{N}{2}\log\big(\deg(X)+1\big)+\log r+\log^+\lceil\frac{1}{N}|\boldsymbol{s}|\rceil+1\right)\leq\log\|\Phi_{\boldsymbol{s},X}\|_\omega-m_\omega^S(\Phi_{\boldsymbol{s},X})$$
$$\leq N\deg(X)\log 2+\log^+\lceil\frac{1}{N}|\boldsymbol{s}|\rceil+1,$$

where $|\boldsymbol{s}|=\sum_{j=0}^p s_j$.

At non-Archimedean places $\omega$, we have $m_\omega^S(\Phi_{\boldsymbol{s},X})$ agreeing with the Gauss norm of $\Phi_{\boldsymbol{s},X}$ and so the above gives

$$\Big|h^{\mathrm{Plpn},S}(X)-h^S(\Phi_{\boldsymbol{s},X})\Big|\leq N\deg(X)\log 2+\log r+\log^+\lceil\frac{1}{N}|\boldsymbol{s}|\rceil+1. \tag{5.39}$$

We prove now that the difference in Equation (5.38) is actually zero. The idea here is no stranger in science. We replace our original setting by a more complex one, but then what ends up happening is that the original subject of study becomes much simpler in this new setting. Concretely, consider the following diagram:

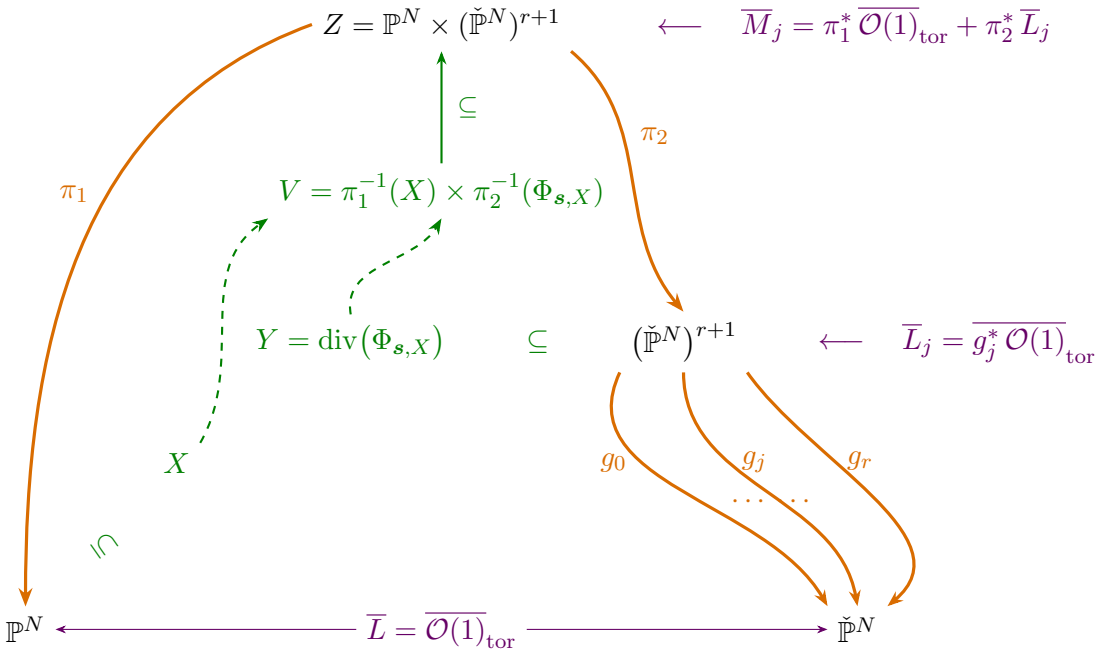



FIGURE 5.4: Map of the proof of Theorem 5.34

In words, we are leveraging on the fact that $\operatorname{div}(\Phi_{\boldsymbol{s},X}) \subseteq (\check{\mathbb{P}}^N)^{p+1}$, so that it becomes natural to consider the moduli space $Z = \mathbb{P}^N \times (\check{\mathbb{P}}^N)^{p+1}$ parametrizing pairs $(X, \Phi_{\boldsymbol{s},X})$. We then consider certain line bundles and make extensive use of the projection formula (Theorem 5.29) and the Bezout-type formula for $(\overline{L}_0 \cdots \overline{L}_r \mid \quad)_S$ to get our result.

Let us introduce the line bundles we will make use of. Consider $(\check{\mathbb{P}}^N)^{p+1}$ with natural projections $g_j : (\check{\mathbb{P}}^N)^{p+1} \to \check{\mathbb{P}}^N$, and

$$\forall 0 \leq j \leq p, \quad \overline{L_j} = g_j^* \, \overline{\mathcal{O}(1)}_{\mathrm{tor}},$$

the pullback of $\mathcal{O}(1)$ endowed with the canonical toric on $\check{\mathbb{P}}^N$ to $(\check{\mathbb{P}}^N)^{p+1}$, metrized again with the canonical toric metric.

Now, for $Z = \mathbb{P}^N \times (\check{\mathbb{P}}^N)^{p+1}$, with projections $\pi_1 : Z \to \mathbb{P}^N$ and $\pi_2 : Z \to (\check{\mathbb{P}}^N)^{p+1}$, we write

$$\forall 0 \leq j \leq p, \quad \big(\overline{M_j}\big)_{\mathrm{tor}} \, \pi_1^* \, \overline{\mathcal{O}(1)}_{\mathrm{tor}} + \pi_2^* \, \overline{L}_j,$$
$$\text{endowed with the canonical toric metric.}$$

Lastly, we consider a polarization $(\mathscr{B}; \overline{\mathscr{H}})$ of $K$ and normal arithmetic models $\mathscr{X}, \mathscr{Z}$ of $X, Z$, respectively, and line bundles $\overline{\mathscr{L}_j}, \overline{\mathscr{M}_j}$ on $\mathscr{X}, \mathscr{Z}$, with generic fibers $\overline{L}_j, \overline{M}_j$.

We continue with the proof. Note that the generalized Chow form $\Phi_{\boldsymbol{s},X}$ is in fact a section of $\big(\deg X\big) \sum_{j=0}^p \overline{L}_j^{\otimes s_j}$.

Hence, by Proposition 4.4.4 (3) in [CM21], writing $Y = \operatorname{div}\big(\Phi_{\boldsymbol{s},X}\big)$, we have

$$\begin{aligned}\left(\overline{L}_0^{\otimes N} \cdots \overline{L}_p^{\otimes N} \ \ \big(\deg X\big) \sum_{j=0}^p \overline{L_j}^{\otimes s_j}\right)_S &= \left(\overline{L}_0^{\otimes N} \cdots \overline{L}_p^{\otimes N} \mid Y\right)_S \\ &\quad - \int_{\omega \in \Omega} \int_{X^{\mathrm{an},\omega}} \log \big\|\Phi_{\boldsymbol{s},X}\big\|_\omega \ d\mu_{\overline{L}_0,\ldots,\overline{L}_p}(x) \, d\nu(\omega).\end{aligned} \tag{5.40}$$

But, on the other hand, by multilinearity (and symmetry) of that pairing, we have

$$\left(\overline{L}_0^{\otimes N}\cdots\overline{L}_p^{\otimes N}\ \ (\deg X)\sum_{j=0}^{p}\overline{L_j}^{\otimes s_j}\right)_S = |\boldsymbol{s}|\deg(X)\Big(p+1\Big)\left(\overline{L}_0^{\otimes N+1}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N}\mid(\check{\mathbb{P}}^N)^{p+1}\right)_S - \int_{\omega\in\Omega}\int_{X^{\mathrm{an},\omega}}\log\left\|\Phi_{\boldsymbol{s},X}\right\|_\omega\, d\mu_{\overline{L}_0,\ldots,\overline{L}_p}(x)\, d\nu(\omega) \tag{5.41}$$

Now we claim that

$$\left(\overline{L}_0^{\otimes N+1}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N}\mid(\check{\mathbb{P}}^N)^{p+1}\right)_S = 0.$$

Indeed, consider the hyperplane $H:\{u_{0,0}=0\}$ (recall the coordinates of $(\check{\mathbb{P}}^N)^{p+1}$ are $\boldsymbol{u}_0=u_{0,0},\ldots,u_{0,N},\ldots,\boldsymbol{u}_p=u_{p,0},\ldots,u_{p,N}$). Then

$$\int_{\omega\in\Omega}\int_{X^{\mathrm{an},\omega}}\log\left\|u_{0,0}\right\|_\omega\, d\mu_{\overline{L}_0,\ldots,\overline{L}_p}(x)\, d\nu(\omega) + \left(\overline{L}_0^{\otimes N+1}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N}\mid(\check{\mathbb{P}}^N)^{p+1}\right)_S = \left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N}\mid H\right)_S. \tag{5.42}$$

Recall that the measure $\mu_{\overline{L}_0,\ldots,\overline{L}_p}$ above is a linear combination of Gauss points (that yield the sup norm) in the non-Archimedean case, and by Theorem 3.8.7 in [CM21], the Mahler measure on each set of coordinates $\boldsymbol{u}_j$ in the Archimedean case. And the metrization we are using on our bundles, is the canonical toric metric on $\mathbb{P}^N$ is given locally by

$$\|s\|_{\overline{\mathcal{O}(1)}_{\mathrm{tor}},v}(\mathbf{x}) = \frac{|s(\mathbf{x})|_v}{\|\mathbf{x}\|_v},\quad s \text{ a section of } O(1),$$

so that, for $s=u_{0,0}$ we have,

$$\|u_{0,0}\|_{\overline{\mathcal{O}(1)}_{\mathrm{tor}},\omega}(\mathbf{x}) = \frac{|u_{0,0}(\mathbf{x})|_\omega}{\|\mathbf{x}\|_\omega} = \frac{|u_{0,0}|_\omega}{\|\mathbf{x}\|_\omega}.$$

Hence the integral in Equation (5.42) above is zero. Inductively, and since $H$ is invariant under the squaring map $[x_0:\cdots x_N]\to[x_0^2:\cdots x_N^2]$, we also get:

$$\left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N} \mid H\right)_S = \lim_{k\to\infty}\frac{1}{2^k}h_{\overline{L}}^S(f^k(H)) = 0. \tag{5.43}$$

The expert reader will notice that this is the canonical height of $H$ with respect to the squaring map. Assuming, of course, the enormous act of faith the reader must make to trust that in fact the limit above makes sense, we have proven the claim. We reassure the reader that we will prove that this make sense in the next section. We continue with our proof.

Equations (5.40) and (5.41) then yield

$$\int_{\omega\in\Omega}\int_{X^{\mathrm{an},\omega}} \log\left\|\Phi_{\boldsymbol{s},X}\right\|_\omega \, d\mu_{\overline{L}_0,\ldots,\overline{L}_p}(x)\, d\nu(\omega) = \left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N} \mid Y\right)_S.$$

And since, as noted above, the measure defined by $\left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N} \mid (\check{\mathbb{P}}^N)^{p+1}\right)_S$ is the Mahler measure in the Archimedean case, we have,

$$h_{\overline{L}}^{\mathrm{Plpn},S}(X) = \left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N} \mid Y\right)_S. \tag{5.44}$$

Now that we have cooked up the appearance of the Philippon height in the adelic curve setting, as an intersection pairing on $(\check{\mathbb{P}}^N)^{p+1}$, we move to the top of the latter in the diagram from Figure 5.4.

Consider the functions $f_j$ given by

$$f_j = \sum_{i=0}^{N} x_i^{s_i} u_{j,i}, \qquad \text{noting that they are actually sections of } M_j = \pi_1^*(\mathcal{O}(1)) + \pi_2^* L_j.$$

We have

$$\pi_2^{-1}(\mathrm{div}(\Phi_{\boldsymbol{s},X})) = \bigcap_{j=0}^{p}\mathrm{div}(f_j) =: W'.$$

And so we consider

$$W := W' \cap \pi_1^{-1}(X) = \bigcap_{j=0}^{p}\mathrm{div}(f_j)\cap\pi_1^{-1}(X).$$

On the one hand, writing $\mathscr{W}$ for an arithmetic normal model of $W$ and by abuse of notation use both $\pi_1, \pi_2$ to denote the projections from $\mathscr{W}$ to $\mathscr{X}, \mathscr{Z}$, respectively, Theorem 5.32 yields

$$\begin{aligned}\deg(X)h^S(X) &= \left(\overline{L_0}^N \cdots \overline{L_r}^N \middle| (\check{\mathbb{P}}^N)^p\right)\left(\overline{\mathcal{O}(1)_{\mathrm{tor}}}^{\otimes(p+1)} \middle| X\right)_S. \\ &= \left(\pi_2^*\overline{\mathscr{L}_0}^{\otimes N} \cdots \pi_2^*\overline{\mathscr{L}_r}^{\otimes N}\pi_1^*\overline{\mathscr{O}(1)_{\mathrm{tor}}}^{\otimes(p+1)}\pi^*(\overline{\mathscr{H}}) \middle| \mathscr{W}\right). \end{aligned} \tag{5.45}$$

Similarly,

$$\begin{aligned}\left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N} \mid Y\right)_S &= \left(\overline{\mathscr{L}}_0^{\otimes N}\overline{\mathscr{L}}_1^{\otimes N}\cdots\overline{\mathscr{L}}_p^{\otimes N}\pi^*\overline{\mathscr{H}} \mid \mathscr{Y}\right) \\ &= \left(\pi_2^*\overline{\mathscr{L}}_0^{\otimes N}\pi_2^*\overline{\mathscr{L}}_1^{\otimes N}\cdots\pi_2^*\overline{\mathscr{L}}_p^{\otimes N}\pi^*\overline{\mathscr{H}} \mid \mathscr{W}\right). \end{aligned} \tag{5.46}$$

Before relating now Equations (5.45) and (5.46), we note that by Claim 4.5.2 in [CM21], we have moreover that for any $\epsilon_j \in \{0,1\}$, with $i = \sum_{j=0}^p \epsilon_j$,

$$\left(\pi_2^*\overline{\mathscr{L}_0}^{\otimes N+\epsilon_0}\cdots\pi_2^*\overline{\mathscr{L}_r}^{\otimes N+\epsilon_r}\pi_1^*\overline{\mathscr{O}(1)_{\mathrm{tor}}}^{\otimes(p+1-i)}\pi^*(\overline{\mathscr{H}}) \middle| \mathscr{W}\right) = 0, \tag{5.47}$$

unless $i = 0$.

This translates Equation (5.45) to

$$\begin{aligned}&\deg(X)h^S(X) = \\ &\qquad\left(\pi_2^*\overline{\mathscr{L}_0}^{\otimes N}\cdots\pi_2^*\overline{\mathscr{L}_r}^{N}\left(\pi_1^*\overline{\mathcal{O}(1)}+\pi_2^*\overline{\mathscr{L}_0}\right)\cdots\left(\pi_1^*\overline{\mathcal{O}(1)}+\pi_2^*\overline{\mathscr{L}_r}\right)\pi^*\left(\overline{\mathscr{H}}\right) \middle| \mathscr{W}\right).\end{aligned} \tag{5.48}$$

It is Equation (5.48) that we relate to (5.46).

The $M_j$'s have the $f_j$'s as sections. Consider

$$W_0 = \pi_1^{-1}(X), \quad \text{and} \quad W_{j+1} = W_j \cap \mathrm{div}(f_j),$$

so that $W_{p+1} = W$.

The same argument as above for the measure $\mu_{\overline{L}_0,\dots,\overline{L}_p}$ yields

$$\begin{aligned}\deg(X)h^S(X) &= \left(\pi_2^*\overline{\mathscr{L}_0}^{\otimes N}\cdots\pi_2^*\overline{\mathscr{L}_r}^{N}\cdots\big(\pi_1^*\overline{\mathcal{O}(1)}+\pi_2^*\overline{\mathscr{L}_r}\big)\pi^*\big(\overline{\mathscr{H}}\big)\middle|\mathscr{W}_0\right)\\ &\vdots\\ &= \left(\pi_2^*\overline{\mathscr{L}_0}^{\otimes N}\cdots\pi_2^*\overline{\mathscr{L}_r}^{N}\big(\pi_1^*\overline{\mathcal{O}(1)}+\pi_2^*\overline{\mathscr{L}_0}\big)\pi^*\big(\overline{\mathscr{H}}\big)\middle|\mathscr{W}\right)\\ &= \left(\overline{L}_0^{\otimes N}\overline{L}_1^{\otimes N}\cdots\overline{L}_p^{\otimes N}\mid Y\right)_S.\end{aligned}$$

Choosing $\boldsymbol{s} = (1,\dots,1)$ we get

$$h^S(\Phi_{\boldsymbol{s},X}) = h^S(X).$$

That is, with the canonical toric metrizations, Philippon and Moriwaki heights agree. We compare this to the analogous result in [BGS94] which shows that Philippon and Arakelov heights agree when the metrizations are induced from a model in the non-Archimedean case and Fubini-Study in the Archimedean case. □

As a corollary, we have the following result.

**Corollary 5.35.** Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve and $X$ a projective variety of dimension $p$ over $K$. There is a positive constant $C = C(N, \dim X, \deg X)$ depending linearly on $N, \dim X, \deg X$, such that

$$\left|h^{\mathrm{Plpn},S}_{\overline{\mathcal{O}(1)}_{\mathrm{tor}}}(X) - h^{S}_{\overline{\mathcal{O}(1)}_{\mathrm{tor}}}(X)\right| \leq C.$$

Moreover, if $X$ is defined over the function field $K(t)$,

$$h^{\mathrm{geom}}(X) = \frac{h^{\mathrm{geom}}(\Phi_X)}{\deg X}.$$

*Proof.* The first claim follows directly. The second claim follows from noticing that the places at which the two heights differ are only the archimedean ones. □

## 5.5 Canonical Heights

Let $V$ be a projective variety over a number field $K$, $\varphi : V \to V$ an endomorphism and $L \in \mathrm{Pic}(V)$ such that $\varphi^* L \cong L^{\otimes d}$. Call and Silverman in their seminal paper ([CS93]) built a canonical height $\hat{h}_\varphi : V \to \mathbb{R}_{\geq 0}$ satisfying

$$\hat{h}_\varphi(\varphi(x)) = d\hat{h}_\varphi(x) \text{ and } \hat{h}_\varphi(x) = h(x) + O(1), \tag{5.49}$$

where $h$ is a Weil height on $V$ with respect to $L$. Specifically,

**Definition 5.36** (Canonical Height)**.** The canonical height $\hat{h}_\varphi$ with respect to $\varphi$ is defined as

$$\hat{h}_\varphi(x) := \lim_{k\to\infty} \frac{1}{\deg \varphi} h(\varphi^{\circ k}(x))$$

Call and Silverman showed that this height is well defined and satisfies Equation (5.49). This height function is remarkable because it characterizes the points $x \in V$ whose forward orbit under $\varphi$ is finite, generalizing to all projective varieties the classical work of Neron and Tate on elliptic curves.

If it is the case that $L$ is ample and $d \geq 2$ we say that $(\varphi, V, L)$ is a polarized (arithmetic) dynamical system. A single dynamical system is very well understood but much less is known for families.

Let $(f, X, L)$ be a family of polarized dynamical systems over a curve $B$, and let $P : B \to X$ where we denote $P(t)$ by $P_t$. On all but finitely many fibres $(f_t, X_t, L_t)$ we may construct a canonical height $\widehat{h}_{f_t}$. Ingram (Theorem 1 in [Ing21]) proves

**Theorem 5.37.** Let $(f, X, L)$ be a family of polarized dynamical systems over a rational curve $B$, defined over $\mathbb{Q}$. Fix a degree one Weil height $h_B$ on $B$.

Then

$$\widehat{h}_{f_t}(P_t) = \widehat{h}_f(P) h_B(t) + O(h_B(t)^{\frac{1}{2}}).$$

A key ingredient is the following.

**Lemma 5.38.** Let $P_t = P(t) \in \mathbb{P}^N_{K(t)}$ and $h_B$ any Weil height of degree 1 on the base $B$. Then we have

$$|h(P_t) - h^{\mathrm{geom}}(P) h_B(t)| \leq 308(h^{\mathrm{Mor}}(P)^2 + \log(N+1) + 1).$$

*Proof.* We use repeatedly Lemma 5.22, Lemma 7 in [Ing21] and the fact that for every $x \in \mathbb{R}$, we have

$$x \leq \frac{1}{2}(x^2 + 1).$$

First, if $h^{\text{geom}}(P) = 0$, then $h(P_t) = h^{\text{naive}}(P) \leq 6h^{\text{Mor}}(P)$ yielding

$$|h(P_t) - h^{\text{geom}}(P)h_B(t)| \leq 6(h^{\text{Mor}}(P) + \log(N+1) + 1).$$

Now suppose $h^{\text{geom}}(P) > 0$. Then we have

$$\begin{aligned} h^{\text{geom}}(P)h_B(t) - h(P_t) &\leq 4h^{\text{geom}}(P)h^{\text{naive}}(P) \\ &\quad + 8h^{\text{geom}}(P)\log 2 + \log h^{\text{geom}}(P) + \log(N+1). \\ &\leq 308\big(h^{\text{Mor}}(P)^2 + \log(N+1) + 1\big). \end{aligned}$$

To conclude,

$$\begin{aligned} h(P_t) - h^{\text{geom}}(P)h_B(t) &\leq h^{\text{naive}}(P) + \log h^{\text{geom}}(P) \\ &\leq 7\big(h^{\text{Mor}}(P)^2 + \log(N+1) + 1\big). \end{aligned}$$

□

We now move to the case of subvarieties of positive dimension.

### 5.5.1 Canonical Heights for Subvarieties

Our goal now is to prove an analogous result to Theorem 5.37 for subvarieties of positive dimension. Concretely, we will prove the following.

**Theorem 5.39** (Ingram, Jaramillo-Martinez)**.** Let $(f, X, L)$ be a family of polarized dynamical systems over a rational curve $B$ and $V : B \to X$ a family of subvarieties of dimension $p$. Assume this dynamical system is defined over a number field $K$ and fix a degree one Weil height $h_B$ on $B$. Then

$$\hat{h}_{f_t}(V_t) = \hat{h}_f(V)h_B(t) + O(h_B(t)^{1-\frac{1}{p+2}}),$$

where the implied constant depends on the family, $\deg V$ and $h^{\text{Mor}}(V)$.

We note that when $p = 0$ we recover Theorem 5.37. Before we proceed we need to set the framework.

In the setting for points we considered a family of polarized dynamical systems $(f, X, L)$ over a curve $B$ and everything is defined over a number field $K$. Now we additionally consider a family of subvarieties of dimension $p$.

$$V : B \to \mathbb{P}^M_{K(t)}, \qquad V(t) =: V_t \subset \mathbb{P}^M.$$

Chen-Moriwaki build the canonical height as the canonical metrization of the line bundle $L$ with respect to $f$ in such a way that $f^*\bar{L} = d\bar{L} + \widehat{\mathrm{div}}(s)$, where $\widehat{\mathrm{div}}(s)$ is in the sense of their adelic curve framework and not the more classical one from Gillet-Soulé's arithmetic intersection theory. From this it follows that assuming that $\hat{h}_f(V)$ exists, we have $\hat{h}(f(V)) = d\hat{h}_f(V)$. Moreover, using Theorems 5.32 and 5.28 above settle the existence once and for all.

**Lemma 5.40.** Let $X \subset \mathbb{P}^N_{K(t)}$ be a projective variety of dimension $p$ and let $\phi_X$ be its generalized Chow form. There is a non-empty open set $U \subset B$ such that for every $t \in U(\overline{K})$ one has

$$\left|h^s(X_t) - h^{\mathrm{geom}}(X)\, h_B(t)\right| = O_{N,p}\left((h^{\mathrm{Mor}}(X)^2 + 1)\deg X\right).$$

*Proof.* If $p = 0$, then $X$ is just a point of $\mathbb{P}^N_{K(t)}$, and the statement is exactly Lemma 5.38 above. We may therefore assume $p > 0$.

Let $\mathscr{X} \subset \mathbb{P}^N \times B$ be the Zariski closure of $X$, and let $U \subset B$ be a non-empty open set over which $\mathscr{X} \to B$ is flat. In particular, for every $t \in U(\overline{K})$, the fiber $X_t$ has dimension $p$ and degree $\deg X$.

Let $(\phi_X)_t$ denote the result of specializing $\phi_X$ at $t$. After shrinking $U$ further, we may assume that the coefficients of a representative of $\phi_X$ are regular on $U$. Thus $(\phi_X)_t$ is defined and non-zero for every $t \in U(\overline{K})$.

If a fiber is not reduced, write

$$[X_t] = \sum_j m_{j,t}[W_{j,t}],$$

where $m_{j,t}$ is the local multiplicity of $X_t$ at the generic point of $W_{j,t}$. In this case we write

$$\phi_{X_t} := \prod_j \phi_{W_{j,t}}^{m_{j,t}}$$

up to multiplication by a non-zero scalar. This is the convention for these generalized Chow forms in [CM21, Proposition 1.6.2 and Definition 1.6.3].

We claim that

$$(\phi_X)_t = \phi_{X_t}.$$

Indeed, let $\mathrm{Ch}(\cdot)$ denote the generalized Chow divisor. Bost–Gillet–Soulé define $\mathrm{Ch}(\cdot)$ over any Noetherian regular base and show that it commutes with flat base change. They also prove that, under certain conditions, the same formula holds for the restriction of a closed embedding, see [BGS94, §4.3.2, formula (4.3.5) and the paragraph after it].

For $t \in U(\overline{K})$, choose a finite extension $K'/K$ over which $t$ is rational, and let

$$i_t : \operatorname{Spec} K' \hookrightarrow U_{K'}$$

be the corresponding closed point. We keep the same notation after base change to $K'$. Since $U_{K'}$ is a smooth curve, $i_t$ is a closed embedding. Moreover, since $\mathscr{X} \to U$ is a flat morphism, the condition in [BGS94, §4.3.2, formula (4.3.5)] is satisfied, and hence

$$i_t^*[\mathscr{X}] = [X_t],$$

where by abuse of notation, here $(\cdot)^*$ denotes the Gysin pullback. (see [Ful13, §6.2 and §20.3])

Formula (4.3.5) in [BGS94], by recalling that $\mathrm{Ch}(\cdot) = g_* f^* v_*(\cdot)$, then yields

$$i_t^* \mathrm{Ch}([\mathscr{X}]) = \mathrm{Ch}([X_t]).$$

Passing to their defining multihomogeneous forms gives the claim (up to a scalar of course). In particular, on $U$ we have

$$h^s\big((\phi_X)_t\big) = h^s(\phi_{X_t}).$$

Now, view $\phi_X$ as a point of some projective space $\mathbb{P}^M_{K(t)}$. Since $\phi_X$ is multihomogeneous of degree $\deg X$ in $p+1$ blocks of $N+1$ variables, we may take

$$M + 1 \le \binom{N + \deg X}{\deg X}^{p+1} \le 2^{(p+1)(N+\deg X)}.$$

Therefore

$$\log(M+1) \le (p+1)(N+\deg X)\log 2 \le (p+1)(N+1)\deg X \log 2,$$

because $\deg X \ge 1$.

Applying Lemma 5.38 to the point $\phi_X \in \mathbb{P}^M_{K(t)}$ gives

$$\big|h^s(\phi_{X_t}) - h^{\mathrm{geom}}(\phi_X)\, h_B(t)\big| \le 308\big(h^{\mathrm{Mor}}(\phi_X)^2 + \log(M+1) + 1\big). \tag{5.50}$$

Next we compare the height of a variety with the height of its Chow form. Corollary 5.35 above yields

$$h^{\mathrm{geom}}(X) = \frac{h^{\mathrm{geom}}(\phi_X)}{\deg X},$$

and, Equation (5.39) yields for every $t \in U(\overline{K})$,

$$\left|h^s(X_t) - \frac{h^s(\phi_{X_t})}{\deg X}\right| \le (N \deg X + p)\log 2.$$

Therefore,

$$\begin{aligned}\big|h^s(X_t) - h^{\mathrm{geom}}(X)\, h_B(t)\big| &\le \left|h^s(X_t) - \frac{h^s(\phi_{X_t})}{\deg X}\right| + \frac{1}{\deg X}\big|h^s(\phi_{X_t}) - h^{\mathrm{geom}}(\phi_X)\, h_B(t)\big| \\ &\le (N \deg X + p)\log 2 + \frac{308}{\deg X}\big(h^{\mathrm{Mor}}(\phi_X)^2 + \log(M+1) + 1\big).\end{aligned}$$

Using the bound for $\log(M+1)$ above, we obtain the explicit estimate

$$\begin{aligned}&\big|h^s(X_t) - h^{\mathrm{geom}}(X)\, h_B(t)\big| \le \\ &\qquad (N \deg X + p)\log 2 + \frac{308}{\deg X} h^{\mathrm{Mor}}(\phi_X)^2 + 308\big((p+1)(N+1)\log 2 + 1\big).\end{aligned} \tag{5.51}$$

Estimates in the proof of Theorem 5.34 yield

$$\left|\frac{h^{\mathrm{Mor}}(\phi_X)}{\deg X} - h^{\mathrm{Mor}}(X)\right| \le N\log 2 \quad \text{and hence } h^{\mathrm{Mor}}(\phi_X) \le \big(h^{\mathrm{Mor}}(X) + N\log 2\big)\deg X.$$

Then

$$\frac{308}{\deg X} h^{\mathrm{Mor}}(\phi_X)^2 \le 308 \deg X \big(h^{\mathrm{Mor}}(X) + N\log 2\big)^2.$$

Using the elementary inequality

$$(u+a)^2 \le (1+a^2)(u^2+1) \qquad (u, a \in \mathbb{R}),$$

with $u = h^{\mathrm{Mor}}(X)$ and $a = N\log 2$, we get

$$\frac{308}{\deg X} h^{\mathrm{Mor}}(\phi_X)^2 \le 308\big(1 + N^2\log^2 2\big)\deg X\big(h^{\mathrm{Mor}}(X)^2 + 1\big).$$

Also, since $\deg X \geq 1$,

$$\begin{aligned}&(N \deg X + p)\log 2 + 308\big((p+1)(N+1)\log 2 + 1\big)\\ \leq &\Big((N+p)\log 2 + 308\big((p+1)(N+1)\log 2 + 1\big)\Big)\deg X\big(h^{\mathrm{Mor}}(X)^2 + 1\big)\end{aligned}$$

Combining this with (5.51), we get

$$\big|h^s(X_t) - h^{\mathrm{geom}}(X)\,h_B(t)\big| \leq C_{N,p}\,\deg X\,\big(h^{\mathrm{Mor}}(X)^2 + 1\big),$$

where one can take

$$C_{N,p} = 308\big(1 + N^2\log^2 2\big) + (N+p)\log 2 + 308\big((p+1)(N+1)\log 2 + 1\big). \qquad \square$$

This previous Lemma and Theorem 5.34 give the following corollary.

**Corollary 5.41.** Let $X \subset \mathbb{P}^N_{K(t)}$ be a projective variety of dimension $p$ and let $\phi_X$ be its generalized Chow form.

There is a non-empty open set $U \subset B$ such that for every $t \in U(\overline{K})$ one has

$$\big|h^S(X_t) - h^S(X)\,h_B(t)\big| = O_{N,p}\left((h^{\mathrm{Mor}}(X)^2 + 1)\deg X\right).$$

Before proving theorem 5.39 we need to prove the following lemma.

**Lemma 5.42.** Let $f\colon \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ over an adelic curve $S$. For any subvariety $V \subseteq \mathbb{P}^N$ of dimension $p$,

$$\big|\hat{h}_f(V) - h(V)\big| = O(h^{\mathrm{naive}}(f)).$$

Moreover, when $S$ is either a number field, a function field or the function field of an arithmetic variety, we can state this precisely as follows.

$$\big|\hat{h}_f(V) - h(V)\big| \leq C_1\,h^{\mathrm{naive}}(f) + C_2,$$

where

$$C_1 = \frac{(p+1)(N+1)d^{N+1}}{d-1} \qquad \text{and} \qquad C_2 = \frac{(p+1)c_{N,d}}{d-1},$$

with

$$c_{N,d} = (N+1)d^{N+1}\big(\log(N+1) + d\log d\big) + \log\left((N+1)\binom{N+m-d}{m-d}\binom{N+d}{d}\right),$$

and

$$m = (N+1)(d-1)+1 \leq (N+1)d^N.$$

*Proof.* We prove the latter statement for simplicity.

Let

$$L_k = (f^k)^* \mathcal{O}_{\mathbb{P}^N}(1),$$

endowed with adelic metrics $\varphi_{k,\omega}$, where $\varphi_{0,\omega}$ is the canonical toric metric and

$$\varphi_{k+1,\omega} = f^* \varphi_{k,\omega}.$$

Let $L_\infty$ denote $\mathcal{O}_{\mathbb{P}^N}(1)$ endowed with the canonical adelic metric

$$\varphi_{\infty,\omega} = \lim_{k\to\infty} d^{-k} \varphi_{k,\omega}.$$

By construction, $\hat{h}_f(V)$ is the height of $V$ with respect to $L_\infty$, while $h(V)$ is the height with respect to $L_0$.

The outline of the proof now is to bound the adelic metric on the trivial line bundle $L_{k+1} - dL_k$ using Ingram's effective Nullstellensatz (Lemma 4 in [Ing24]), and then use the classical Tate's telescoping argument to get our claim.

We start with the bound on the adelic metric. Choose a homogeneous lift

$$F = [F_0 : \cdots : F_N]$$

of $f$, where each $F_i$ has degree $d$. For each place $\omega \in \Omega$, define

$$\lambda_\omega(\mathbf{X}) = d \log \|\mathbf{X}\|_\omega - \log \|F(\mathbf{X})\|_\omega, \qquad \mathbf{X} \in \mathbb{A}^{N+1} \setminus \{0\},$$

with

$$\|\mathbf{X}\|_\omega = \max\{|X_0|_\omega, \ldots, |X_N|_\omega\}, \qquad \|F(\mathbf{X})\|_\omega = \max_{0\leq i\leq N} |F_i(\mathbf{X})|_\omega,$$

noting that on the chart $U_i = \{x_i \neq 0\}$, if $s_i$ denotes the standard section of $\mathcal{O}_{\mathbb{P}^N}(1)$, then

$$\lambda_\omega(\mathbf{X}) = \log |f^* s_i|_{\varphi_{1,\omega}} - \log |F_i|_{\varphi_{0,\omega}^{\otimes d}} = d \log \|\mathbf{X}\|_\omega - \log \|F(\mathbf{X})\|_\omega.$$

Let us estimate $\lambda_\omega$. Fix $\omega$, and write

$$\epsilon_\omega = \begin{cases} 1, & \text{if } \omega \text{ is archimedean,} \\ 0, & \text{if } \omega \text{ is non-archimedean.} \end{cases}$$

Since $F_0, \dots, F_N$ have no common zero on $\mathbb{A}^{N+1}$, we have

$$\mathrm{Res}(F) \neq 0.$$

Now we invoke the effective Nullstellensatz as stated by Ingram to get $a \in \mathcal{O}_K \setminus \{0\}$ and polynomials

$$A_{ij} \in \mathcal{O}_K[\mathbf{F}, \mathbf{X}]$$

such that

$$a\ \mathrm{Res}(F)\, X_j^m = \sum_{i=0}^{N} F_i(\mathbf{X})\, A_{ij}(\mathbf{F}, \mathbf{X}), \qquad m = (N+1)(d-1)+1, \tag{5.52}$$

and, at each archimedean place, the coefficients of the $A_{ij}$ satisfy the uniform bound

$$\log\|A_{ij}\|_\omega \leq (N+1)d^{N+1}\big(\log(N+1) + d\log d\big).$$

Let $B_{ij}$ be the specialization of $A_{ij}$ at our chosen lift $F$. Then $B_{ij}$ is a polynomial in $\mathbf{X}$ alone, of degree $m-d$, and we set

$$\|B\|_\omega = \max_{0\leq i,j\leq N} \|B_{ij}\|_\omega.$$

Take $\mathbf{X} \neq 0$, and choose $j$ such that

$$|X_j|_\omega = \|\mathbf{X}\|_\omega.$$

Equation (5.52) yields

$$|a|_\omega\, |\,\mathrm{Res}(F)|_\omega\, \|\mathbf{X}\|_\omega^m \leq (N+1)^{\epsilon_\omega} \binom{N+m-d}{m-d}^{\epsilon_\omega} \|B\|_\omega\, \|\mathbf{X}\|_\omega^{m-d}\, \|F(\mathbf{X})\|_\omega.$$

After rearranging, this gives

$$\lambda_\omega(\mathbf{X}) \leq -\log|a|_\omega - \log|\,\mathrm{Res}(F)|_\omega + \log\|B\|_\omega + \epsilon_\omega\left(\log(N+1) + \log\binom{N+m-d}{m-d}\right). \tag{5.53}$$

On the other hand, the triangle inequality gives

$$\|F(\mathbf{X})\|_\omega \leq \binom{N+d}{d}^{\epsilon_\omega} \|F\|_\omega\, \|\mathbf{X}\|_\omega^d. \text{ Hence}$$

$$-\lambda_\omega(\mathbf{X}) \leq \log\|F\|_\omega + \epsilon_\omega \log\binom{N+d}{d}. \tag{5.54}$$

Now write

$$\rho_\omega^+ = \sup_{\mathbf{X}\neq 0} \lambda_\omega(\mathbf{X}), \qquad \rho_\omega^- = \sup_{\mathbf{X}\neq 0} \big(-\lambda_\omega(\mathbf{X})\big).$$

Then

$$\sup_{\mathbf{X}\neq 0} |\lambda_\omega(\mathbf{X})| \leq \rho_\omega^+ + \rho_\omega^-.$$

We remark that this is the idea behind Chen-Moriwaki's measure in the adelic curve setting being dominated; we need to bound from both ends.

From (5.53) we get

$$\rho_\omega^+ \leq -\log|a|_\omega - \log|\operatorname{Res}(F)|_\omega + \log\|B\|_\omega + \epsilon_\omega\left(\log(N+1) + \log\binom{N+m-d}{m-d}\right).$$

Integrating over $\Omega$ and using the product formula on the proper adelic curve $S$ yields

$$\int_\Omega \rho_\omega^+ \,\nu(d\omega) \leq \int_\Omega \log\|B\|_\omega \,\nu(d\omega) + \log(N+1) + \log\binom{N+m-d}{m-d}. \tag{5.55}$$

Now note that specializing at $F$ yields

$$\int_\Omega \log\|B\|_\omega \,\nu(d\omega) \leq \big((N+1)d^{N+1}-1\big)h^{\text{naive}}(f) + (N+1)d^{N+1}\big(\log(N+1) + d\log d\big).$$

And hence, substituting into (5.55), we obtain

$$\begin{aligned}\int_\Omega \rho_\omega^+ \,\nu(d\omega) \leq{}& \big((N+1)d^{N+1}-1\big)h^{\text{naive}}(f)\\ &+ (N+1)d^{N+1}\big(\log(N+1) + d\log d\big) + \log(N+1) + \log\binom{N+m-d}{m-d}.\end{aligned} \tag{5.56}$$

Now we estimate $\rho_\omega^-$. For this, we begin noticing that Equation (5.54) yields

$$\rho_\omega^- \leq \log\|F\|_\omega + \epsilon_\omega \log\binom{N+d}{d}.$$

Integrating over $\Omega$ gives

$$\int_\Omega \rho_\omega^- \,\nu(d\omega) \leq h^{\text{naive}}(f) + \log\binom{N+d}{d}. \tag{5.57}$$

Combining (5.56) and (5.57), we arrive at

$$\int_\Omega \sup_{\mathbf{X}\neq 0} \bigl|\lambda_\omega(\mathbf{X})\bigr|\,\nu(d\omega) \le (N+1)d^{N+1}\,h^{\mathrm{naive}}(f) + c_{N,d}. \tag{5.58}$$

From here the rest is the usual telescoping argument. For every $\omega$ and every $k \ge 0$,

$$d^{-(k+1)}\bigl(\varphi_{k+1,\omega} - d\varphi_{k,\omega}\bigr) = d^{-(k+1)}\,\lambda_\omega \circ F^k,$$

$$\Longrightarrow \Bigl\| d^{-(k+1)}\bigl(\varphi_{k+1,\omega} - d\varphi_{k,\omega}\bigr)\Bigr\|_{\sup} \le d^{-(k+1)} \sup_{\mathbf{X}\neq 0} \bigl|\lambda_\omega(\mathbf{X})\bigr|.$$

Summing the geometric series gives

$$\|\varphi_{\infty,\omega} - \varphi_{0,\omega}\|_{\sup} \le \sum_{k=0}^{\infty} d^{-(k+1)} \sup_{\mathbf{X}\neq 0} \bigl|\lambda_\omega(\mathbf{X})\bigr| = \frac{1}{d-1} \sup_{\mathbf{X}\neq 0} \bigl|\lambda_\omega(\mathbf{X})\bigr|.$$

Integrating over $\Omega$ and using (5.58), we get

$$\int_\Omega \|\varphi_{\infty,\omega} - \varphi_{0,\omega}\|_{\sup}\,\nu(d\omega) \le \frac{(N+1)d^{N+1}\,h^{\mathrm{naive}}(f) + c_{N,d}}{d-1}. \tag{5.59}$$

Finally, $L_\infty - L_0$ is the trivial bundle endowed with the metric $\varphi_{\infty,\omega} - \varphi_{0,\omega}$. By multilinearity of the arithmetic intersection pairing,

$$\hat{h}_f(V) - h(V) = \sum_{j=0}^{p} \frac{1}{\deg(V)} \widehat{\deg}\Bigl(\hat{c}_1(L_\infty)^{\,p-j}\,\hat{c}_1(L_\infty - L_0)\,\hat{c}_1(L_0)^{\,j} \mid V\Bigr).$$

Each summand is bounded in absolute value by the right-hand side of (5.59), so

$$\bigl|\hat{h}_f(V) - h(V)\bigr| \le (p+1)\int_\Omega \|\varphi_{\infty,\omega} - \varphi_{0,\omega}\|_{\sup}\,\nu(d\omega) \le \frac{(p+1)\bigl((N+1)d^{N+1}\,h^{\mathrm{naive}}(f) + c_{N,d}\bigr)}{d-1}.$$

That is,

$$\bigl|\hat{h}_f(V) - h(V)\bigr| \le \frac{(p+1)(N+1)d^{N+1}}{d-1}\,h^{\mathrm{naive}}(f) + \frac{(p+1)c_{N,d}}{d-1} = C_1\,h^{\mathrm{naive}}(f) + C_2.$$

□

And now we finally prove theorem 5.39.

*Proof of Theorem 5.39.* We first note that the Theorem is true in general if and only if it is true when $X$ is projective space. Indeed, if $f\colon X \to X$ satisfy $f^*L \cong L^{\otimes d}$ over

$K(t)$, By an argument of Fakhruddin, there are

$$\text{an embedding } i\colon X \hookrightarrow \mathbb{P}^N \quad \text{and a morphism } F\colon \mathbb{P}^N \to \mathbb{P}^N$$

such that $L^{\otimes e} \cong i^*\mathcal{O}(1)$ for some integer $e \geq 1$ and $i \circ f = F \circ i$. Moreover, If $\overline{\mathcal{O}(1)}$ is metrized with the canonical metric, then one sees that the metrized line bundle

$$\overline{L} := i^*\overline{\mathcal{O}(1)},$$

induces the canonical height with respect to $f$ for $f^*L = L^{\otimes d}$ and this characterizes the canonical height. Hence

$$e\hat{h}_{X,L,f}(Y) = \hat{h}_{X,L^{\otimes e},f}(Y) = \hat{h}_{\mathbb{P}^N,\mathcal{O}(1),F}(i(Y)),$$

for any $Y \subseteq X$.

As before, let $\mathscr{X} \subset \mathbb{P}^N \times B$ be the Zariski closure of $X$, and let $U \subset B$ be a non-empty open set over which $\mathscr{X} \to B$ is flat. In particular, for every $t \in U(\overline{K})$, the fiber $X_t$ has dimension $p$ and degree $\deg X$. Moreover, $i\colon X \to \mathbb{P}^N$ specializes to an embedding, and our claim remains true on the fibres, so the result for $(X, f)$ follows from the result for $(\mathbb{P}^N, \mathcal{O}(1))$.

So we may assume without loss of generality that $X = \mathbb{P}^N$.

Write

$$r = \dim Y, \qquad \delta_Y = \deg(Y), \qquad H_Y = h^{\mathrm{Mor}}(Y).$$

By Lemma 5.42 there is a constant $A_f \geq 0$ such that

$$\left|h^{\mathrm{Mor}}(Z) - \hat{h}_f^{\mathrm{Mor}}(Z)\right| \leq A_f$$

for every cycle $Z$. Hence, if we set

$$M_Y := 1 + \delta_Y\big(H_Y + 1\big)^2,$$

then after enlarging the constant if necessary we have

$$h^{\mathrm{Mor}}(f^k(Y))^2 \deg(f^k(Y)) + 1 \leq C_0\, M_Y\, d^{(r+2)k}$$

for a constant $C_0 = C_0(f)$, since $\deg(f^k(Y)) = d^{rk} \deg Y$ (and $M_Y$ depends on $h^{\mathrm{Mor}}(Y)^2$ of course).

Now fix $t \in U(\overline{K})$. For any $k$

$$\begin{aligned}\big|\hat{h}_{f_t}(Y_t) - \hat{h}_f^{\mathrm{geom}}(Y)\, h_B(t)\big| &= d^{-k}\big|\hat{h}_{f_t}(f_t^k(Y_t)) - \hat{h}_f^{\mathrm{geom}}(f^k(Y))\, h_B(t)\big| \\ &\le d^{-k}\big|\hat{h}_{f_t}(f_t^k(Y_t)) - h(f_t^k(Y_t))\big| \qquad \text{(A)} \\ &\quad + d^{-k}\big|h(f_t^k(Y_t)) - h^{\mathrm{geom}}(f^k(Y))\, h_B(t)\big| \qquad \text{(B)} \\ &\quad + d^{-k}\big|h^{\mathrm{geom}}(f^k(Y)) - \hat{h}_f^{\mathrm{geom}}(f^k(Y))\big|\, h_B(t). \qquad \text{(C)}\end{aligned}$$

By Lemma 5.42 there is a constant $C_1 = C_1(f)$ such that the term in $(C)$ is at most

$$d^{-k} C_1\, h_B(t).$$

The same Lemma together with $h^{\mathrm{naive}}(f_t) = O(h_B(t))$, tells us that there is a constant $C_2 = C_2(f)$ such that the term in $(A)$ is at most

$$d^{-k} C_2\, h_B(t).$$

For $(B)$, Lemma 5.40 gives

$$\big|h(f_t^k(Y_t)) - h^{\mathrm{geom}}(f^k(Y))\, h_B(t)\big| \le C_3\Big(h^{\mathrm{Mor}}(f^k(Y))^2 \deg(f^k(Y)) + 1\Big) \le C_3 C_0\, M_Y\, d^{(r+2)k}.$$

Combining these estimates yields

$$\big|\hat{h}_{f_t}(Y_t) - \hat{h}_f^{\mathrm{geom}}(Y)\, h_B(t)\big| \le d^{-k}\Big(C_4\, h_B(t) + C_5\, M_Y\, d^{(r+2)k}\Big),$$

for constants$C_4, C_5$ depending only on $f$.

This already handles the case $h_B(t) \le 1$: taking $k = 0$, we obtain

$$\big|\hat{h}_{f_t}(Y_t) - \hat{h}_f^{\mathrm{geom}}(Y)\, h_B(t)\big| \le C_6\, M_Y$$

for a constant $C_6 = C_6(f)$.

Assume now that $h_B(t) > 1$. For each $t$, choose (the unique) $k \ge 0$ such that

$$d^{(r+2)k} \le h_B(t) < d^{(r+2)(k+1)}. \qquad \text{(D)}$$

Then

$$d^{-k} \le d\, h_B(t)^{-1/(r+2)} \qquad \text{and} \qquad d^{(r+1)k} \le h_B(t)^{(r+1)/(r+2)}.$$

Applying these inequalities to the previous bound gives

$$\begin{aligned}\left|\hat{h}_{f_t}(Y_t) - \hat{h}_f^{\mathrm{geom}}(Y)\, h_B(t)\right| &\leq d^{-k} C_4\, h_B(t) + C_5\, M_Y\, d^{(r+1)k} \\ &\leq C_7\, M_Y\, h_B(t)^{(r+1)/(r+2)}\end{aligned}$$

for another constant $C_7 = C_7(f)$. □

We conclude this chapter with a couple remarks on our result, its history and future directions.

First, we note that our effective result is a power saving improvement and can be used to control the height $h_B$ of the parameters $t$ for which $V_t$ is preperiodic.

Next, we recall that when $V_t$ is a point $P_t$, Call-Silverman[CS93] proved the non-effective analogue of our result

$$\hat{h}_{f_t}(P_t) = \hat{h}_f(P) h_B(t) + o(h_B(t)).$$

More recently, Gauthier-Vigny[GV23] proved the same asymptotic in the general case.

Lastly, we conclude with what we feel is the next natural step in our work. And that is, instead of using the Philippon height and an adelic line bundle with the toric metric, we believe we can rely solely on the adelic curve framework that Chen-Moriwaki have developed.

# Appendix A

# Category Theory

Category theory emerged as a formal mathematical discipline in the 1940s, thanks to the pioneering work of Eilenberg and Mac Lane in their seminal paper "General Theory of Natural Equivalences" [EM45]. Their framework encapsulated ideas that were already implicit in various branches of mathematics, such as the concept of a morphism between algebraic varieties. In the 1950s and 1960s, Alexander Grothendieck revolutionized algebraic geometry by applying categorical methods - most notably through his theory of schemes and the development of étale cohomology-. By the 1970s, category theory had expanded dramatically, influencing diverse fields ranging from logic and computer science to physics. The concept of topoi, introduced by Grothendieck and later refined by others, provided new insights into mathematical logic and set theory. Today, category theory serves as a unifying language across mathematics, with applications extending into theoretical physics, computer science, and even linguistics [CSC10].

For the sake of self containment of the present work we replicate here the presentation found in [Har20] of the material.

## A.1 Preliminaries

**Definition A.1** (Category)**.** A category $\mathcal{C}$ consists of:

(i) A class of objects, denoted $\mathrm{Ob}(\mathcal{C})$.

(ii) For each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of morphisms (or arrows) $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

(iii) For each object $A \in \mathcal{C}$, an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

(iv) A composition law: for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, there exists a morphism $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$.

These must satisfy:

Associativity: If $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Identity: For each morphism $f : A \to B$, we have $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

**Example A.2.** The following are examples of categories.

The sets form a category (that we denote by $\mathcal{S}et$), taking as morphisms the usual functions.

The groups form a category (denoted $\mathcal{G}rp$), the morphisms being the group homomorphisms. The same is true for abelian groups, or for rings (the morphisms are then the ring homomorphisms). We denote by $\mathcal{A}b$ and $\mathcal{A}nn$ these categories.

If $R$ is a ring, the left $R$-modules form a category that we denote by $\mathcal{M}od_R$. The corresponding category of right $R$-modules is denoted by $\mathrm{Mod}_{R^{\mathrm{op}}}$. We have an analogous result for right $R$-modules.

Topological spaces form a category $\mathcal{T}op$, where the morphisms are continuous functions.

Note that in general $\mathrm{Ob}(\mathcal{C})$ is not a set; otherwise, we very quickly run into paradoxes of the type the set of all sets. A category whose objects form a set is called a small category.

**Definition A.3.** An arrow (morphism) $f\colon A \to B$ in a category $\mathcal{C}$ is called an isomorphism if there exists a morphism

$$g\colon B \to A \quad \text{(necessarily unique, denoted by } g = f^{-1})$$

satisfying

$$f \circ g = \mathrm{Id}_B \quad \text{and} \quad g \circ f = \mathrm{Id}_A.$$

A monomorphism in $\mathcal{C}$ is a morphism $f\colon B \to C$ such that for every object $E$ and every pair of morphisms $a, b\colon E \to B$, the equality

$$f \circ a = f \circ b$$

implies

$$a = b.$$

An epimorphism is defined by reversing the directions of the arrows in this condition.

For example, in the category $\mathcal{A}b$ (resp. in $\mathcal{M}od_R$, in $Top$), the monomorphisms are the injective morphisms. The epimorphisms in $Ab$ and $\mathcal{M}od_R$ are the surjective morphisms, but this is no longer true for example in the category of rings (for example take the inclusion $\mathbb{Z} \to \mathbb{Q}$) or Hausdorff topological spaces (take for example the inclusion $\mathbb{Q} \to \mathbb{R}$).

**Definition A.4.** We say that an object $A$ of $\mathcal{C}$ is initial (resp. final) if for any object $B$ of $\mathcal{C}$, there exists one and only one morphism $A \to B$ (resp. one and only one morphism $B \to A$). An object which is both initial and final is called a zero-object (or simply zero, or null object), that we generally denote by 0.

Note that two initial (resp. final) objects are isomorphic in $\mathcal{C}$ and the isomorphism between them is unique. We can then talk of the initial (resp. final) object of $\mathcal{C}$ when it exists. In $\mathcal{S}et$, the only initial object is $\emptyset$, the final objects are the singletons (hence there is no zero). In $\mathcal{A}b$ or $\mathcal{M}od_R$, the trivial group (resp. the trivial $R$-module) 0 is a zero-object.

**Definition A.5.** Let $\mathcal{C}$ be a category which has a zero 0. If $A$ and $B$ are two objects of $\mathcal{C}$, we denote again by 0 the morphism $A \to B$ which is the composite of $A \to 0$ and $0 \to B$.

(i) A kernel of a morphism $f : B \to C$ is a morphism $i : A \to B$ satisfying $fi = 0$, and which is universal for this property (meaning: for any morphism $i' : A' \to B$ with $fi' = 0$, there exists a unique morphism $g : A' \to A$ such that $i' = ig$).

(ii) A cokernel of a morphism $f : B \to C$ is a morphism $p : C \to D$ such that $pf = 0$, and universal for that property (meaning: for any morphism $p' : C \to D'$ with $p'f = 0$, there exists a unique morphism $q : D \to D'$ such that $p' = q \circ p$).

The notions of kernel and cokernel in the category $Ab$ or $\mathcal{M}od_R$ coincide with the usual notions.

**Definition A.6.** Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is the category whose objects are the same as those of $\mathcal{C}$, but the morphisms and their composites are reversed.

For any morphism $f : A \to B$ in $\mathcal{C}$, we thus have a morphism $f^{\mathrm{op}} : B \to A$ in $\mathcal{C}^{\mathrm{op}}$. The morphism $f$ is a monomorphism if and only if $f^{\mathrm{op}}$ is an epimorphism (and vice versa).

Relations between categories are not described using functions but rather with functors. The revolutionary concept of a natural functor was introduced by Eilenberg and Mac Lane in their seminal paper mentioned in the introduction.

## A.2 Functors

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is given by:

(i) A function on objects: $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.

(ii) A function on morphisms: for each $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

These must satisfy:

Functoriality: $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms $f, g$ in $\mathcal{C}$.

Identity preservation: $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathcal{C}$.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between categories. A natural transformation $\eta : F \to G$ consists of a collection of morphisms

$$\eta_X : F(X) \to G(X)$$

in $\mathcal{D}$, one for each object $X$ in $\mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

That is, $\eta_Y \circ F(f) = G(f) \circ \eta_X$ for all $f : X \to Y$ in $\mathcal{C}$.

A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

We define in an obvious way the composite $G \circ F$ of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, and the identity functor $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$.

For instance, there is a functor from $\mathcal{G}rp$ to $\mathcal{A}b$ by associating to every group $G$ its abelianization $G^{\mathrm{ab}}$. Also for a field $k$ there is a contravariant functor from the category of $k$-vector spaces to itself by associating to every $k$-vector space $E$ its dual $E^*$.

**Definition A.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let $F$ and $G$ be two functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation (or a morphism of functors) $\eta$ from $F$ to $G$ is the assignment for any object $A$ of $\mathcal{C}$ of a morphism $\eta_A : F(A) \to G(A)$ in $\mathcal{D}$, so that for any morphism $f : A \to A'$ in $\mathcal{C}$, the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

If furthermore all of the $\eta_A$ are isomorphisms, we say that $\eta$ is a natural isomorphism. An equivalence of categories is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mathrm{Id}_{\mathcal{C}} \simeq G \circ F$ and $\mathrm{Id}_{\mathcal{D}} \simeq F \circ G$.

**Example A.8.** Let us see some examples of natural transformations.

Let $T$ be the functor from $Ab$ to itself that associates to every abelian group $A$ its torsion subgroup $T(A)$. Then the inclusion $T(A) \to A$ induces a natural transformation from $T$ to $\mathrm{Id}_{Ab}$.

Let $k$ be a field. Let $\mathcal{C}$ be the category of finite-dimensional $k$-vector spaces. The contravariant functor $E \mapsto E^*$ is an equivalence of categories from $\mathcal{C}$ to its opposite category.

The forgetful functor which to a $\mathbb{Z}$-module associates the underlying abelian group is an equivalence of categories between $\mathcal{M}od_{\mathbb{Z}}$ and $\mathcal{A}b$.

**Definition A.9** (Faithful Functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be faithful if for all $A, B \in \mathcal{C}$, the map $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective. It is fully faithful if this map is bijective for all $A, B \in \mathcal{C}$.

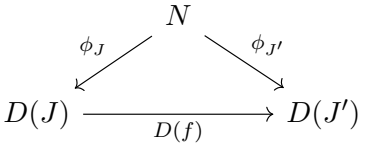
There is a remarkable result that explains Grothendieck's ideas of studying structures by their relations with other structures.

**Theorem A.10** (Yoneda's Lemma)**.** Given a category $\mathscr{C}$, we can produce a new category, called the functor category of $\mathscr{C}$, where the objects are contravariant functors $\mathscr{C} \to \mathrm{Sets}$, and the morphisms are natural transformations of such functors. We have a functor (which we call $\mathfrak{h}$) from $\mathscr{C}$ to its functor category, which sends $A$ to $\mathrm{h}_A$. Yoneda's Lemma states that this is a fully faithful functor, called the Yoneda embedding.

## A.3 Limits and Colimits

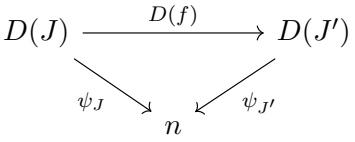
In order to translate set theoretic notions like cartesian products and disjoint unions into this new language we use limits and colimits. A diagram $J$ in a category $\mathcal{C}$ is a functor $D : J \to \mathcal{C}$.

Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. A cone over $D$ with summit or apex $N$ is a collection of morphisms $\{\phi_J : N \to D(J)\}_{J \in \mathcal{J}}$ such that for every morphism $f : J \to J'$ in $\mathcal{J}$, the following diagram commutes:



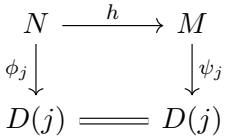

That is, for all $f : J \to J'$, we have $D(f) \circ \phi_J = \phi_{J'}$.

A cocone or cone under $D$ with nadir $n$ is a collection of morphisms $\{\psi_J : D(J) \to n\}_{J \in \mathcal{J}}$ such that for every morphism $f : J \to J'$ in $\mathcal{J}$, the following diagram commutes:




That is, for all $f : J \to J'$, we have $\psi_{J'} \circ D(f) = \psi_J$.

For any diagram $D$ there is functor $\mathrm{Cone}(-, D) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ that sends $N \in \mathcal{C}$ to the set of cones over $D$ with summit $N$. Let $(N, \{\phi_j\}_{j \in J})$ and $(M, \{\psi_j\}_{j \in J})$ be two cones over $D$. A morphism of cones $h : (N, \{\phi_j\}) \to (M, \{\psi_j\})$ is a morphism $h : N \to M$ in $\mathcal{C}$ such that for every object $j$ in $J$, the following diagram commutes:




That is, for all $j \in J$, we have $\psi_j \circ h = \phi_j$.

A limit diagram is any final object in the category of cones over $D$. A colimit is an initial object. If $J$ is discrete we call the limit a product $\prod$ and the colimit a coproduct $\amalg$. For topological spaces the product is the cartesian product (endowed with the product topology). For Sets the coproduct is the disjoint union.

When the category $J$ is $\omega$ we say the limit is direct and we denote it by $\to$ and when it is $\omega^{\mathrm{op}}$ we call the limit an inverse limit and denote it by $\leftarrow$. For instance the $p$-adic integers $\mathbb{Z}_p$ are defined as

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

Elements of $\mathbb{Z}_p$ can also be viewed as formal power series:

$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \mid a_i \in \{0, 1, \ldots, p-1\} \right\}$$

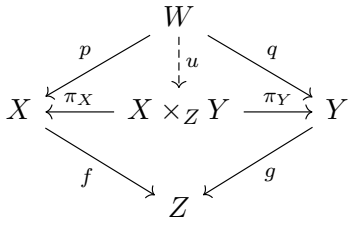
Where the coefficients $a_i$ are digits in base $p$, and the series converges in the $p$-adic topology.

Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a category $\mathcal{C}$ with common codomain $Z$. Consider the diagram $D : \mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is the category $\bullet \leftarrow \bullet \rightarrow \bullet$, and $D$ maps this to $X \xrightarrow{f} Z \xleftarrow{g} Y$ in $\mathcal{C}$.

The fibered product or pullback $X \times_Z Y$ is defined as the limit of this diagram $D$. Equivalently, it is the terminal object in the category of cones over $D$.

Explicitly, a fibered product consists of:

1. An object $X \times_Z Y$ in $\mathcal{C}$
2. Morphisms $\pi_X : X \times_Z Y \to X$ and $\pi_Y : X \times_Z Y \to Y$ such that $f \circ \pi_X = g \circ \pi_Y$
3. Universal property: For any object $W$ with morphisms $p : W \to X$ and $q : W \to Y$ satisfying $f \circ p = g \circ q$, there exists a unique morphism $u : W \to X \times_Z Y$ such that $\pi_X \circ u = p$ and $\pi_Y \circ u = q$, as shown in the diagram:




## A.4 Abelian Categories

**Definition A.11.** An additive category $\mathcal{C}$ is a category whose sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ (for $A, B$ objects of $\mathcal{C}$) are endowed with a structure of abelian group, satisfying the following properties:

(i) The composition of morphisms is distributive with respect to the addition: if $A, B, C, D$ are objects of $\mathcal{C}$ and $f : A \to B$, $g : B \to C$, $g' : B \to C$, $h : C \to D$ are morphisms, we have

$$h(g + g')f = hgf + hg'f \in \mathrm{Hom}(A, D).$$

(ii) $\mathcal{C}$ has a zero-object 0.

(iii) If $A$ and $B$ are objects of $\mathcal{C}$, the product $A \times B$ exists.

In particular, in an additive category, for any object $A$, $\mathrm{Hom}(A, A)$ is endowed with the structure of a ring (the unit being the identity morphism).

**Definition A.12.** An abelian category (denoted by $\mathcal{C}$) is an additive category additionally satisfying:

(i) Every morphism in $\mathcal{C}$ has a kernel and a cokernel.

(ii) Every monomorphism is the kernel of its cokernel.

(iii) Every epimorphism is the cokernel of its kernel.

We easily see that in an abelian category, every morphism $f : A \to B$ factors uniquely as $f = i \circ p$, where $p : A \to B'$ is an epimorphism and $i : B' \to B$ is a monomorphism, with $B'$ defined as the kernel of the cokernel of $f$. The subobject $B' := \mathrm{Im}\ f$ of $B$ is called the *image* of $f$. In particular, the notion of exact sequence

$$\cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots$$

is given by the usual property that the image of a map is the kernel of the next one. Injective morphisms (= with zero kernel) coincide with monomorphisms and surjective morphisms (= with zero cokernel) with epimorphisms. The category of abelian groups $\mathcal{A}b$ is abelian. If $R$ is a ring, the category $\mathcal{M}od_R$ is abelian. The opposite category of an abelian category is abelian.

An additive functor $F$ between two abelian categories $\mathcal{C}$ and $\mathcal{D}$ is a functor such that if $A$ and $B$ are objects in $\mathcal{C}$, the map induced between abelian groups $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a morphism of abelian groups.

**Definition A.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor defined between two abelian categories. We say that $F$ is left exact (resp. right exact) if for every exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0,$$

the sequence

$$0 \longrightarrow F(A_1) \longrightarrow F(A_2) \longrightarrow F(A_3)$$

(resp. the sequence $F(A_1) \to F(A_2) \to F(A_3) \to 0$) is exact. We will say that $F$ is exact if it is both left and right exact. We similarly define the notion of contravariant left exact (resp. right exact, resp. exact) functor $F' : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ has the same property.

If $\mathcal{C}$ is an abelian category and $A$ is an object of $\mathcal{C}$, the covariant functor $\mathrm{Hom}_{\mathcal{C}}(A, .)$ from $\mathcal{C}$ to $Ab$ is left exact. Let $R$ be a commutative ring. Let $A$ be an $R$-module. The functor $. \otimes_R A$ from $\mathcal{M}od_R$ to itself is right exact. We say that $A$ is flat if this functor is exact. For example it is the case if $R$ is a principal ideal domain and $A$ is torsion-free. Also if $G$ is a finite group, the functor $A \mapsto A^G$ (from the category of $G$-modules to the category of abelian groups) is left exact.

**Definition A.14.** Let $\mathcal{C}$ be an abelian category. A functor $F : \mathcal{C} \to Ab$ is called representable by an object $A$ of $\mathcal{C}$ if there exists a natural isomorphism from $F$ to the functor $\mathrm{Hom}_{\mathcal{C}}(A, .)$. A contravariant functor from $\mathcal{C}$ to $Ab$ is called representable by an object $I$ from $\mathcal{C}$ if it is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(., I)$.

## A.5 Derived Functors

Let $\mathcal{C}$ be an abelian category.

**Definition A.15.** An object $I$ in $\mathcal{C}$ is called injective if the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(., I)$ (from $\mathcal{C}$ to $\mathcal{A}b$) is exact. An object $P$ in $\mathcal{C}$ is called projective if the covariant functor $\mathrm{Hom}_{\mathcal{C}}(P, .)$ is exact. We say that $\mathcal{C}$ has enough injectives (resp. enough projectives) if any object is isomorphic to a subobject of an injective object (resp. for any object $A$, there exists a surjection $P \to A$ with $P$ projective).

The category $\mathcal{A}b$ has enough injectives and projectives. The injectives in $\mathcal{A}b$ are the divisible groups. More generally, the projectives in the category $\mathcal{M}od_R$ are projective modules in the usual sense, i.e., direct factors of a free module. Thus the category $\mathcal{M}od_R$ has enough projectives. It also has enough injectives. (see Corollary 2.3.11 in [Sza+09]).

The category $\mathcal{S}h_X$ of sheaves of abelian groups on a topological space $X$ has enough injectives.

**Definition A.16.** A complex $A^\bullet$ in $\mathcal{C}$ is a family of objects $A^i, i \in \mathbb{Z}$ and morphisms (called coboundaries, one also sometimes calls them (co)differentials) $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for every $i$ (when the objects are only defined on a certain interval, ex. $i \geq 0$, we set $A^i = 0$ for other superscripts $i$). A morphism of complexes $f : A^\bullet \to B^\bullet$ is a family of morphisms $f^i : A^i \to B^i$ that commute with the coboundaries $d^i$.

The complexes in $\mathcal{C}$ form an abelian category. We call a morphism of degree $d$ between two complexes $A^\bullet$ and $B^\bullet$ a morphism from $A^\bullet$ to the shifted complex $B^\bullet[d]$ (defined by $(B^\bullet[d])^i = B^{i+d}$).

**Definition A.17** (Cohomology object)**.** The $i$th cohomology object of a complex $A^\bullet$ is

$$h^i(A^\bullet) := \text{Ker } d^i/\text{Im } d^{i-1}.$$

If $f : A^\bullet \to B^\bullet$ is a morphism of complexes, it induces a natural map $h^i(f) : h^i(A^\bullet) \to h^i(B^\bullet)$.

An application of the snake lemma is (Theorem 1.3.1 in [Sza+09]):

**Theorem A.18.** Let

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes. Then we have natural maps $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ that give rise to a long exact sequence

$$\cdots \longrightarrow h^i(A^\bullet) \xrightarrow{h^i(f)} h^i(B^\bullet) \xrightarrow{h^i(g)} h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet) \longrightarrow \cdots$$

We can also define homology by working with boundaries instead of coboundaries

**Definition A.19** (Homology object)**.** Let $A^\bullet$ be a complex and $d_j \colon A_j \to A_{j-1}$ maps such that $d_j \circ d_{j+1} = 0$. The $j$-th homology object $h_j(A^\bullet)$ is defined as

$$h_j(A^\bullet) := \ker d_j / \operatorname{Im} d_{j+1}.$$

The following result (Proposition A.43 in [Har20]) will be crucial for us when defining cohomology for sheaves.

**Proposition A.20.** Let $\mathcal{C}$ be an abelian category with enough injectives. Then every object $A$ of $\mathcal{C}$ admits an injective resolution; i.e., there exists a complex

$$I^\bullet = 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

(defined in degrees $i \geq 0$) and a morphism $A \to I^0$ such that all the objects of $I^\bullet$ are injective and we have an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

We have an analogous result if $\mathcal{C}$ has enough projectives, replacing injective resolutions by projective resolutions, i.e., exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

where each $P_i$ is projective.

We deduce the following definition.

**Definition A.21** (Derived Functor)**.** Let $\mathcal{C}$ be an abelian category with enough injectives. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant left exact functor. The (right) derived functors $R^iF, i \geq 0$ are defined as follows: for any object $A$ of $\mathcal{C}$, fix an injective resolution $I^\bullet$ of $A$ and set

$$R^iF(A) = h^i(F(I^\bullet)).$$

Analogously we define the left derived functors of a right exact covariant functor in a category with enough projectives by considering the homology and not the cohomology.

**Example A.22.** Let us see some examples of this important construction.

(i) Let $G$ be a finite group. For any $G$-module $A$, the cohomology groups $H^i(G, A)$ are derived functors of the functor $A \mapsto A^G$ (which is right exact) from the category of $G$-modules to that of abelian groups. Same holds true if $G$ is profinite.

(ii) If $X$ is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on $X$, then the groups $H^i(X, \mathcal{F})$ are derived functors of the functor "global sections" $\Gamma(X, .) : \mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ from the category of sheaves of abelian groups on $X$ to $\mathcal{A}b$.

We conclude this appendix with a construction that Serre used to define intersection multiplicities and allowed us to study intersection theory in a consistent way.

**Definition A.23.** Let $R$ be a commutative ring and $A, B$ two $R-$modules. We define $\mathrm{Tor}_j^R(A, B)$ to be the left derived functors of $M \to M \otimes B$ as well as those of $N \to A \otimes N$.

In other words, take a resolution

$$\cdots \to R^{\oplus n_1} \to R^{\oplus n_0} \to B \to 0.$$

We can do this by taking generators of $B$ giving us $R^{\oplus n_0} \to B \to 0$. Then choose generators of the Kernel and so on. Drop the last element of this complex and then tensor with $A$ (exactness may be lost!). $\mathrm{Tor}_j^R(A, B)$ is the $j$-th homology object of this complex.

The name Tor denotes torsion. And is justified by the fact that if $A, B$ are abelian groups then $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion abelian group and $\mathrm{Tor}_j^{\mathbb{Z}}(A, B) = \{0\}$ for $j \geq 2$. Also $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion subgroup of $B$ and if either one of $A$ or $B$ is torsion free then $\mathrm{Tor}_{\mathbb{Z}}^0(A, B) = 0$.

# Appendix B

# Schemes

After the work Oscar Zariski there were several attempts to set up a more general framework for algebraic geometry in the 1950's. And we could say that Grothendieck's notion of scheme (coined by Chevalley for a different structure) won by natural selection since it turned out to be the most useful one.

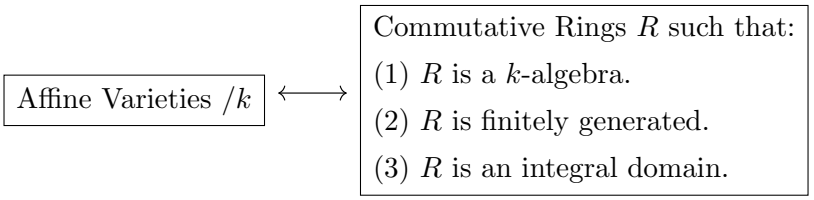
Sometimes in mathematics generalizations take place just for the sake of it. Because finding universal truths is the ultimate goal. So generalizing an idea or result can be an art expression conceived just for the art itself. But in the case of algebraic geometry, it was required to build a more general framework so there would be an actual bridge between commutative rings and geometric structures. We elaborate on this.

Let $R$ be a commutative ring. We define $\operatorname{Spec} R$ as the set of prime ideals of $R$. Initially people defined the points of $\operatorname{Spec} R$ as the set of maximal ideals. The issue is that under a ring homomorphism, the preimage of a maximal ideal need not be a maximal ideal. But the preimage of a prime ideal is always prime.

We consider the Zariski topology on $\operatorname{Spec} R$ where a base of open sets is given by

$$D_f = \{m \in \operatorname{Spec} R : f \notin m\}.$$

These basic open sets may be thought intuitively as functions on $R$ that do not vanish at $f$. Fix a field $k$. There is a one-to-one correspondence between




The correspondence is then given by identifying an affine variety with the maximal ideals of a commutative ring. But we claim that these three properties are too restrictive even to just study varieties. First, it has been known for over a century the similarities between $k[x]$ and $\mathbb{Z}$. But the ring $\mathbb{Z}$ is not an algebra over $k$. Secondly, it is natural to consider $k[x]$ as a subset of the field of rational functions $k(x)$ or the localization $k[x]_{\langle p(x)\rangle}$ at the irreducible polynomial $p(x)$. But none of these is finitely generated over $k$. Lastly consider $V = \{y - x^2 = 0\} \cap \{y = 0\} \subset \mathbb{A}^2$. Then $k[V] = k[x, y]/\langle y - x^2, y\rangle \cong k[x]/\langle x^2\rangle$ which is not an integral domain.

## B.1 Sheaves

Sheaves emerged as a groundbreaking mathematical construct in the 1950s through the pioneering work of Jean Leray. This formalism was subsequently refined and expanded by Henri Cartan, Jean-Pierre Serre, and others, who demonstrated the remarkable versatility of sheaf theory in algebraic geometry (see [Ser55]). Their innovative application of sheaves provided mathematicians with a powerful framework that elegantly unified local and global perspectives, transforming our approach to geometric structures. Serre's foundational work, in particular, illuminated how sheaf cohomology could reveal profound connections between algebraic varieties and their topological properties, opening new avenues for research that continue to resonate throughout modern mathematics.

The motivation is as follows. Given a topological space $X$, we can assign to every open set $U \subset X$ the set of continuous functions from $U$ to $k$.

**Definition B.1** (Presheaf)**.** Let $X$ be a topological space. A presheaf of abelian groups on $X$ is a map $\mathcal{F} : X \to$ Abelian Groups such that

(i) For each open set $U \subseteq X$, $\mathcal{F}(U)$ is an abelian group.

(ii) Given $V \subset U \subset X$ open sets, there is a map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ called the restriction map. $\rho_{UV}$ is the identity map.

(iii) Given $W \subset V \subset U \subset X$ open sets we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We can similarly define presheaves of sets, rings or modules. To see this let $\mathrm{Open}(X)$ be the category whose objects are the open subsets of $X$ and whose morphisms are the inclusion maps. And also let $\mathcal{C}$ be a category (such as the category of sets, abelian groups, rings, or modules).

A presheaf $\mathcal{F}$ of $\mathcal{C}$-objects on $X$ is a contravariant functor

$$\mathcal{F} : \mathrm{Open}(X) \to \mathcal{C}.$$

Elements of $\mathcal{F}(U)$ are called sections. We also denote this set by $\Gamma(U, \mathcal{F})$.

And for any $P \in X$, we define the stalk $\mathcal{F}_P$ of $\mathcal{F}$ at $P$ to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets $U$ containing $P$, via the restriction maps $\rho$. That is $\mathcal{F}_P = \varinjlim_{P \in U} \mathcal{F}(U)$.

For instance if we consider a complex manifold we naturally have the presheaf of holomorphic functions on it. For a smooth manifold we have the presheaf of smooth functions. For a topological manifold we have the presheaf of continous functions. And for an algebraic variety we have the presheaf of regular functions. The stalks in the first three examples are the germs of the given functions. For the last example the stalk at $P$ is the localization of the coordinate ring of the variety at the ideal generated by the irreducible polynomial corresponding to $P$.

Another example is as follows. Let $X$ be a topological space and $A$ an abelian group. Consider $\mathcal{F}$ defined as follows

$$\mathcal{F}(x) = \begin{cases} 0, & \text{if } U = \emptyset \\ A, & \text{if } U \neq \emptyset \end{cases} \tag{B.1}$$

Then $\mathcal{F}$ is a presheaf of abelian groups on $X$ called the constant presheaf.

**Remark B.2** When studying a Riemann surface, the choice of presheaf reflects our mathematical focus. Viewing it as a complex manifold, we employ the presheaf of holomorphic functions, capturing its conformal and analytic properties. As a real smooth manifold, we utilize the presheaf of smooth functions, highlighting differential structure and geometric characteristics. When treated purely as a topological manifold, the presheaf of continuous functions preserves only topological features. Each presheaf encodes precisely the lens we are seeing the Riemann surface through.

**Definition B.3** (Morphism between presheaves)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of abelian groups on $X$. A morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ consists of a homomorphism of abelian groups

$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U$, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes.

A sheaf is a presheaf that captures the local nature of continuous functions. Concretely,

**Definition B.4** (Sheaf)**.** A sheaf $\mathcal{F}$ on a topological space $X$ is a presheaf that satisfies the following additional conditions:

(i) (Separability) For any open set $U \subseteq X$ and any open cover $\{U_i\}_{i \in I}$ of $U$, if $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$.

(ii) (Gluing) For any open set $U \subseteq X$, any open cover $\{U_i\}_{i \in I}$ of $U$, and any collection of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

In other words, a sheaf is a presheaf where we can detect whether a section is zero by checking locally and we can glue compatible local sections into a unique global section.

The constant presheaf is not a sheaf if we choose $A$ such that $A \neq A^2$ but all the other examples are indeed sheaves because (holomorphic/smooth/continous/regular) functions are defined locally.

Although it will not play a role for us work, there is an important property that a sheaf can have which is crucial for Serre's GAGA principles. That is the notion of being flasque (french for flabby). A sheaf $\mathcal{F}$ on a topological space $X$ is flasque if the restrictions maps $\rho XU$ are surjective. Not every sheaf satisfies this property. To see this take $X = \mathbb{R}$ with the euclidean topology and $\mathcal{F}$ to be the sheaf of continuous functions. And note that $\sec : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is continuous but cannot be extended to a continuous function on $\mathbb{R}$.

We conclude this section with the framework of cohomology for sheaves. An injective resolution of a sheaf $\mathcal{F}$ on a space $X$ is a complex of injective sheaves:

$$\cdots \to \mathcal{I}_2 \to \mathcal{I}_1 \to \mathcal{I}_0 \to 0,$$

where each $\mathcal{I}_n$ is an injective sheaf, and the sequence is exact. This means that $\mathcal{I}_n$ is injective in the category of sheaves, and the sequence of morphisms is exact.

Using an injective resolution of a sheaf $\mathcal{F}$, (which exists by Proposition A.20) the sheaf cohomology groups $H^k(X, \mathcal{F})$ are given by the cohomology of the complex:

**Definition B.5** (Sheaf Cohomology)**.**

$$H^k(X, \mathcal{F}) = H^k\left(X, \mathrm{Res}(\mathcal{I}^\bullet)\right),$$

where $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$.

$H^0(X, \mathcal{F})$ is the space of global sections of the sheaf $\mathcal{F}$, representing data defined globally on the space $X$.

## B.2 Ringed Spaces

A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf of rings $\mathcal{O}_X$ on $X$. We refer to $\mathcal{O}_X$ as the structure sheaf. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on $Y$. The ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring.

A morphism of locally ringed spaces is a morphism $(f, f^\#)$ of ringed spaces, such that for each point $P \in X$, the induced map (see below) of local rings $f^\#_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings. We explain this last condition. First of all, given a point $P \in X$, the morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$, for every open set $V$ in $Y$. As $V$ ranges over all open neighborhoods of $f(P)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of $P$. Taking direct limits, we obtain a map

$$\mathcal{O}_{Y,f(P)} = \varinjlim_{V \in f(P)} \mathcal{O}_Y(V) \to \varinjlim_{V \in f(P)} \mathcal{O}_X(f^{-1}V),$$

and the latter limit maps to the stalk $\mathcal{O}_{X,P}$. Thus we have an induced homomorphism $f^\#_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$. We require that this be a local homomorphism: If $A$ and $B$ are local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively, a homomorphism $\varphi : A \to B$ is called a local homomorphism if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

An isomorphism of locally ringed spaces is a morphism with a two-sided inverse. Thus a morphism $(f, f^{\#})$ is an isomorphism if and only if $f$ is a homeomorphism of the underlying topological spaces, and $f^{\#}$ is an isomorphism of sheaves.

## B.3 Coherent Sheaves

A sheaf of modules over a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ where $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each open set $U$, such that the restriction maps respect the module structure.

Let $R$ and $M$ an $R$ -module. We define a sheaf $\widetilde{M}$ -the sheaf associated to $M$ on $\mathrm{Spec}(R)$ -as follows. Let $U \subset \mathrm{Spec}(R)$. Then

$$\widetilde{M}(U) = \{s : U \to \amalg_{\mathfrak{p} \in U} M_{\mathfrak{p}}\}$$

such that $s(\mathfrak{p}) \in M_p$ and for all $\mathfrak{p} \in U$ there is a neighborhood $V$ of $\mathfrak{p}$ contained in $U$ and elements $m \in M, f \in R$ such that for each $\mathfrak{q} \in V$

$$f \notin \mathfrak{q}, s(\mathfrak{q}) = \frac{m}{f} \in M_{\mathfrak{q}}.$$

**Definition B.6** (Coherent Sheaves)**.** Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is quasi coherent if $X$ can be covered by affine spaces $U_j = \mathrm{Spec}(A_j)$ such that for each $j$ there is an $A_j$-module $M_j$ with $\mathcal{F}|_{U_j} \cong \widetilde{M_j}$. It is coherent if each $M_j$ can be chosen finitely generated.

The structure sheaf is coherent. Moreover if $X = \mathrm{Spec}(R)$ is affine and $Y \subset X$ is defined by an ideal $\mathfrak{a} \subset R$ and $i : Y \to X$ is the inclusion then $i_*\mathcal{O}_Y$ is also a coherent $\mathcal{O}_X$-module. The map $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$ allows us to define the ideal sheaf of $Y$ denoted by $\mathcal{I}_Y$ as the kernel of $i^{\#}$.

**Definition B.7** (Euler characteristic)**.** Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a coherent sheaf on $X$. The Euler characteristic of $\mathcal{F}$ is defined as

$$\chi(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F}),$$

where $H^i(X, \mathcal{F})$ are the sheaf cohomology groups of $\mathcal{F}$.

## B.4 Affine Schemes

In the introduction we have provided the underlying set and the topology of Spec $R$ for a given ring $R$. Let us now endow it with a sheaf of rings.

**Definition B.8** (Spec $R$)**.** Spec $R = \{\mathfrak{p} \subseteq R : \mathfrak{p} \text{ is a prime ideal in } R\}$ and we consider the topology given by declaring $D_f = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$ to be basic open sets.

We consider the structure sheaf to be

$$\mathcal{O}(D_f) = R[f^{-1}].$$

If $Z \subset \operatorname{Spec} R$ is an irreducible component then $z \in Z$ is called a generic point if $Z = \overline{\{z\}}$.

**Proposition B.9.** Spec $R$ is a locally ringed space.

*Proof.* If Spec $R$ was a ringed space then it is a locally ringed space because the stalk at any point $P$ is

$$\mathcal{O}_P = \varinjlim_{P \in D_{f_j}} \mathcal{O}(D_{f_j}) = \varinjlim_{P \in D_{f_j}} R[f_j^{-1}] = R_P, \text{ where } R_P \text{ is the localization of } R \text{ at } P.$$

We prove now that Spec $R$ is a ringed space. First note that $\mathcal{O}$ is a presheaf by mapping $R[f^{-1}]$ to $R[g^{-1}]$ via $f \to g$. Also given that the $D_f$'s form a base for the topology, it is enough to show the separability and gluing property in the case

$$D_f = \cup_{\alpha \in I} D_{f_\alpha}.$$

And by considering $R[f^{-1}]$ instead of $R$ we can assume without loss of generality that $D_f = \operatorname{Spec} R$.

This last reduction yields $\langle f_\alpha \rangle_{\alpha \in I} = R$ so that in particular there exist $a_1, \ldots, a_t \in R$ such that

$$\sum_{j=1}^{t} a_j f_j = 1. \tag{B.2}$$

To prove separability let $r \in R$ be such that $r|_{R[f_\alpha^{-1}]} \equiv 0$ for all $\alpha \in I$. Where to ease notation we denote $r|_{R[f_\alpha^{-1}]} = \rho_{D_{f_\alpha} R}(r) \in R[f_\alpha^{-1}]$. This means that there is $n_\alpha \in \mathbb{Z}$ such that $f^{n_\alpha} r = 0$. But

$$D_{f_\alpha} = D_{f_\alpha^{n_\alpha}} \tag{B.3}$$

so that in fact $f_j r = 0$. Equation B.2 then yields

$$r = r\sum_{j=1}^{t} a_j f_j = \sum_{j=1}^{t} a_j \big(f_j r\big) = 0.$$

To finish take

$$\frac{r_\alpha}{f_\alpha^{n_\alpha}} \in R[f_\alpha^{-1}], \text{ such that } \frac{r_\alpha}{f_\alpha^{n_\alpha}} = \frac{r_\beta}{f_\beta^{n_\beta}} \in R[f_\alpha^{-1}, f_\beta^{-1}].$$

We want to find $r \in R$ such that for all $\alpha \in I$ we have $r = \frac{r_\alpha}{f_\alpha^{n_\alpha}}$ or equivalently

$$\text{There is } m_\alpha \in \mathbb{Z} \text{ such that } f_\alpha^{m_\alpha}\big(f_\alpha^{n_\alpha} r - r_\alpha\big) = 0.$$

By setting $s_\alpha = r_\alpha f_\alpha$ and appealing again to Equation B.3 again we want $r \in R$ such that

$$f_j r = s_j.$$

The equality $\frac{r_\alpha}{f_\alpha^{n_\alpha}} = \frac{r_\beta}{f_\beta^{n_\beta}}$ yields the existence of $m_{\alpha\beta} \in \mathbb{Z}$ such that

$$\big(f_\alpha f_\beta\big)^{m_{\alpha\beta}}\Big(f_\alpha^{n_\alpha} r_\beta - r_\alpha f_\beta^{n_\beta}\Big) = 0.$$

We use Equation B.3 twice to conclude

$$s_\beta f_\alpha - s_\alpha f_\beta = 0. \tag{B.4}$$

Define $r$ as

$$r = \sum_{j=1}^{t} a_j s_j.$$

Equations B.2 and B.3 yield $f_\alpha r = s_\alpha$. This concludes the proof.

□

**Remark B.10** Equation B.2 above shows in particular that Spec $R$ is compact.

**Definition B.11** (Affine Scheme)**.** An affine Scheme $S = (X, \mathcal{O})$ is a locally ringed space that is isomorphic (as a locally ringed space) to Spec $R$ for some commutative ring $R$.

**Example B.12.** Let $k$ be a field. Then Spec$(k)$ is a point as a set and the structure sheaf is $k$ itself. More generally , the spectrum of a direct sum of rings is just the disjoint union of spectra of its components and each of these is open in the whole spectrum.

**Example B.13.** Assume $k$ is algebraically closed. We define

$$\mathbb{A}^1_k := \text{Spec}(k[x]). \tag{B.5}$$

Then the closed points of $\mathbb{A}^1_k$ can be identified with $\langle x - a\rangle$ for each $a \in k$ and there is the generic point corresponding to $\{0\}$. Then $\mathbb{A}^1_k$ is just the ordinary $x$-axis together with a generic point. The stalk of $\mathcal{O}_{\mathbb{A}^1_k}$ at $\langle x - a\rangle$ is the localization $k[x]_{\langle x-a\rangle}$. The stalk at $\{0\}$ is just the function field $k(x)$.

More generally we define the affine plane $\mathbb{A}^n_k$ to be

$$\mathbb{A}^n_k := \text{Spec}(k[x_1, \ldots, x_n]). \tag{B.6}$$

The set of closed points again are in one-to-one correspondence to ordered $n-$tuples in $k^n$ and there is also a generic point $\eta \in \mathbb{A}^n_k$ such that $\overline{\eta} = \mathbb{A}^n_k$.

But this time there are more generic points. One for each irreducible subvariety $W \subset \mathbb{A}^n_k$.

The above example illustrates that even though there are cases for which the closed points of a scheme correspond to a variety, there are more points in a scheme: one for each irreducible subvariety of the underlying topological space of closed points.

**Example B.14.** Consider $R = \text{Spec}(\mathbb{Z})$.

There is one closed point for each prime number, plus a generic point $[(0)]$. The stalk at $\langle p\rangle$ is $\mathbb{Z}_{(p)}$ and at $[(0)]$ is $\mathbb{Q}$. The non-empty open sets of Spec$(\mathbb{Z})$ are the complement of finitely many primes $p_1, \ldots, p_n$. If $m = \prod p_i$, then this is the distinguished open set $\text{Spec}(\mathbb{Z})_{\langle m\rangle}$.

Note that $\mathbb{Z}$ is a principal ideal domain just like $k[x]$ and so it can be visualized as

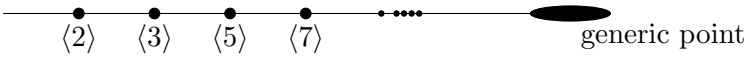



FIGURE B.1: Spectrum of the integers as a scheme

**Example B.15.** We consider now the arithmetic surface $\mathrm{Spec}(\mathbb{Z}[x])$. The prime ideals in $\mathbb{Z}[x]$ are:

(i) $(0)$.

(ii) principal prime ideals $\langle f \rangle$, where $f$ is either a prime $p_i$ or a primitive irreducible polynomial over $\mathbb{Q}$.

(iii) maximal ideals $(p, f)$, where $p$ is a prime and $f$ is a monic integral polynomial irreducible modulo $p$.

This last example is called an arithmetic surface because although defined as a geometric structure, it encodes arithmetic in the following sense. Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial and $p \in \mathbb{Z}$ a prime number. Then both the factorization of $f(x)$ when we reduce its coefficients modulo $p$, and whether or not $f(x)$ has roots in $\mathbb{F}_p$ can be deduced from the intersection of the points corresponding to $\langle f(x) \rangle$ and $\langle p \rangle$ in $\mathrm{Spec}(\mathbb{Z}[x])$. For instance let $f(x) = x^2 + 1$. Then $\pmod 2$, $f$ has a double root and it factors as $(x+1)^2$. There are no roots $\pmod 3$ and so $x^2+1$ is its factorization. Lastly, $\pmod 5$ it factors as $(x+1)(x+2)$ and so 2 and 3 are simple roots.

$\mathrm{Spec}(\mathbb{Z}[x])$ should be pictured as follows (see page 75 in [Mum04]):

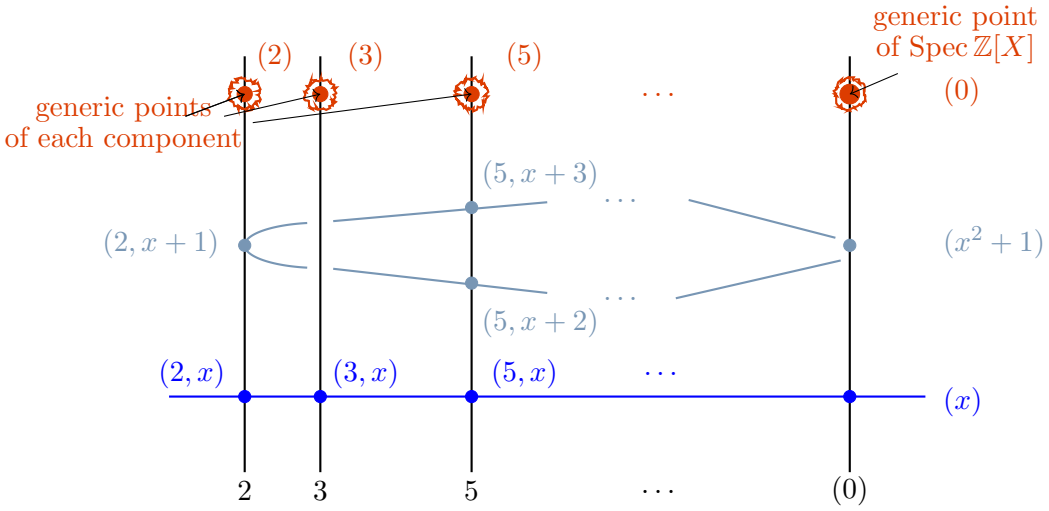



FIGURE B.2: The Arithmetic surface $\mathrm{Spec}(\mathbb{Z}[x])$

## B.5 Schemes

We are finally equipped to introduce schemes.

**Definition B.16.** A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that for every $x \in X$ there is an open neighborhood $U_x$ of $x$ such that $(U_x, \mathcal{O}_X|_{U_x})$ is an affine scheme.

All the examples above of affine schemes are of course examples of schemes as well. To exhibit the existence of non affine scheme let us recall how to build a new scheme from other schemes by gluing them together.

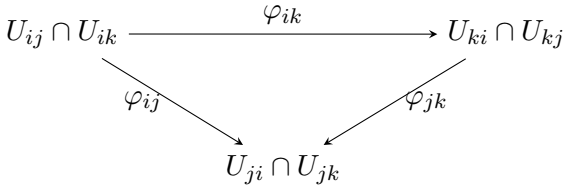
Let $(X_i, \mathcal{O}_i)$ be family of a locally ringed spaces. For each pair $i, j \in I$ consider $U_{ij} \subset X_i$ an open subset and for each pair $i, j \in I$, let

$$\varphi_{ij} : U_{ij} \to U_{ji}.$$

Assume that $U_{ii} = X_i$ and for each triple $i, j, k \in I$

(i) $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$.

(ii) The following diagram commutes.




**Proposition B.17.** Given any family as above, there exists a locally ringed space $X$ and open subspaces $U_i \subset X$ together with isomorphisms $\varphi_i : X_i \to U_i$ of locally ringed spaces such that

(1) $X = \bigcup_{i \in I} U_i$,

(2) $\varphi_i(U_{ij}) = U_i \cap U_j$, and

(3) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

*Proof.* The proof is an explicit construction of $X$ based on the presentation from [De +].

Let the points of $X$ be given by

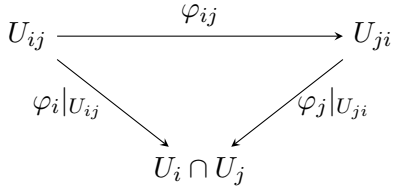
$$X = (\cup X_i)/\sim$$

where for $x_i \in X_i$ and $x_j \in X_j$, we say $x_i \sim x_j$ if $x_i \in U_{ij}, x_j \in U_{ji}$ and $\varphi_{ij}(x_i) = x_j$.

This is an equivalence relation. Note that the two properties that define the family, together with the assumption that $U_{ii} = X_i$ gives us that $\varphi_{ii}$ is the identity in $X_{ii}$ hence the relation is symmetric and reflexive. On the other hand transitivity follows from the commutativity of the diagram in property (ii).

Define $\varphi_i : X_i \to X$ to be the inclusion maps $U_i = \varphi_i(X_i) \subset X$ tits image. Note that $\varphi_i : X_i \to U_i$ is a bijection. In fact, if we endow $X$ with the topology where $U \subset X$ is open if and only if $\varphi_i^{-1}(U)$ is open for all $i$, $\varphi_i$ is a homeomorphism. Indeed, $U_i$ is open since $\varphi_j^{-1}(U_i) = U_{ji}$ which is open in $X_j$ for all $j$. Moreover, for any open set $W \subset X_i$ the image $\varphi_i(W) \subset U_i$ is open because $\varphi_j^{-1}(\varphi_i(W)) = \varphi_{ji}^{-1}(W \cap U_{ij})$.

To obtain a locally ringed space note that we have built a commutative diagram




where $\varphi_{ij}$ is an isomorphism of ringed spaces, and so we get an isomorphisms of sheaves of rings

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j}.$$

In other words, $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}$ and the gluing property of sheaves yields a sheaf of rings $\mathcal{O}_X$ on $X$ such that $(X, \mathcal{O}_X)$ is a locally ringed space since the stalks of $\mathcal{O}_X$ are equal to the stalks of $\mathcal{O}_i$ at corresponding points. □

**Definition B.18.** We say that the space $X$ above is the scheme obtained by gluing together the schemes $(X_i, \mathcal{O}_{X_i})$

**Example B.19** (Line with two origins)**.** Let $X_1 = X_2 = \mathbb{A}^1_k$ and $U_1 = U_2 = \mathbb{A}^1_k \setminus \{0\}$ where 0 is the point in $\mathbb{A}^1_k$ that corresponds to $\langle x \rangle \subset k[x]$. Let $\varphi_1 = \varphi_2$ to be the identity maps. And consider $X$ to be the scheme obtained by gluing together $X_1$ and $X_2$.

If $X$ was an affine scheme then $X \cong \mathrm{Spec}(k[x]) = \mathbb{A}^1_k$ but $X \ncong \mathbb{A}^1_k$.

We now provide the analogous construction of projective varieties in the context of schemes.

**Example B.20.** Let $S = \bigoplus_{n\geq 0} S_n$ be a graded ring. We define a scheme

$$\text{Proj}(S) = \{\mathfrak{p} \subset S : \bigoplus_{n>0} S_n \not\subseteq \mathfrak{p}\}.$$

The base of the topology is given by

$$D_f = \{\mathfrak{p} \subset S : f \text{ is homogeneous of positive degree and } f \notin \mathfrak{p}\}.$$

The structure sheaf is given by $\mathcal{O}(D_f) = S[f^{-1}]^0$ where $S[f^{-1}]^0$ denotes the homogeneous part of degree 0.

Note that if $S = k[x_0, \ldots, x_n]$ then the closed points of $\text{Proj}(S)$ are in one-to-one correspondence between the ideals that are maximal with respect to the property that are not contained in the irrelevant ideal $\langle x_0, \ldots, x_n\rangle$. For instance $[1 : 0 : \ldots : 0]$ corresponds to $\langle x_1, \ldots, x_n\rangle \not\subseteq \langle x_0, \ldots, x_n\rangle$.

**Definition B.21.** For $S = k[x_0, \ldots, x_n]$ we denote $\text{Proj}(S)$ by

$$\mathbb{P}^n_k = \text{Proj}(S).$$

So $\mathbb{P}^1_k$ is one way to glue two affine lines $\mathbb{A}^1_k$ and the line with two origins is another.

Let us now introduce properties that schemes and morphisms between them may or may not have so we can define an abstract variety. We begin with properties of schemes themselves.

**Definition B.22.** We say that a scheme $S = (X, \mathcal{O}_X)$ is connected if the underlying topological set is connected. We say it is reduced if for every open set $U \subset X$ the ring $\mathcal{O}_X(U)$ has no nilpotents. And we say $S$ is integral if $\mathcal{O}_X(U)$ is an integral domain.

In terms of properties of morphisms between schemes we introduce the following.

**Definition B.23.** A morphism $f : X \to Y$ of schemes is locally of finite type if there exists a covering of $Y$ by open affine subsets $V_i = \text{Spec } B_i$, such that for each $i$, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \text{Spec } A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra.

We say it is of finite type if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the $U_{ij}$.

Lastly, $f$ is finite if there exists a covering of $Y$ by open affine subsets $V_i = \text{Spec } B_i$, such that for each $i$, $f^{-1}(V_i)$ is affine, equal to Spec $A_i$, where $A_i$ is a $B_i$-algebra which is a finitely generated $B_i$-module.

The last notion we require follows Grothendieck's general philosophy of developing algebraic geometry in a relative setting (see the next section for an expansion on his ideas).

**Definition B.24.** Let $S$ be a fixed scheme. A scheme over $S$ is a scheme $X$, together with a morphism $X \to S$. If $X$ and $Y$ are schemes over $S$, a morphism of $X$ to $Y$ as schemes over $S$, (also called an $S$-morphism) is a morphism $f : X \to Y$ which is compatible with the given morphisms to $S$. We denote by $\mathfrak{Sch}(S)$ the category of schemes over $S$. If $R$ is a ring we denote by $\mathfrak{Sch}(R)$ the category of schemes over Spec$(R)$.

Now we are ready to define an abstract variety.

**Definition B.25** (Abstract variety)**.** We say a scheme $S$ is an abstract variety if it is integral, separated and of finite type over an algebraically closed field.

## B.6 Grothendieck

A well-known anecdote recounts a visit by G.H. Hardy to Srinivasa Ramanujan, who was hospitalized with tuberculosis. Hardy remarked that the taxi he had taken bore the unremarkable number 1729. Ramanujan immediately corrected him, pointing out that it is the smallest number expressible as the sum of two distinct pairs of nonzero cubes:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

To Ramanujan, numbers were not just abstract entities but objects of deep mathematical significance.

A contrasting story involves Alexander Grothendieck, who is said to have mistakenly identified 57 as a prime number.[1] His approach to algebraic geometry was fundamentally different from that of the Italian school. He developed a new mathematical framework centered on categories, functors, and abstract structures, significantly influencing modern mathematics.

[1] Hermann Weyl made a similar error (see [Wey51]).

Grothendieck's perspective emphasized not just individual schemes but their relationships with one another. Over time, his insights have proven invaluable, with results such as the Yoneda lemma reinforcing the idea that an object's properties are best understood through its interactions with others.

We have already seen how there are schemes such that their closed points form a variety. Moreover we have defined an abstract variety in the language of schemes. We conclude this appendix stating and not proving a concrete identification between varieties and certain schemes.

Let $X$ be a scheme. For any scheme $T$, put

$$h_X(T) = \{\text{morphisms } t : T \to X\}.$$

Then $h_X$ is a contravariant functor on the category of Schemes to the category of Sets. It is called the functor of points of $X$. A morphism $t : T \to X$ is a point of $X$ with values in $T$.

Then we have the following relation.

**Proposition B.26** (Proposition 2.6 in [Har13])**.** Let $k$ be an algebraically closed field. There is a natural fully faithful functor

$$\tau : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$$

from the category of varieties over $k$ to schemes over $k$. For any variety $V$, its topological space is homeomorphic to the set of closed points of $\tau(V)$, and its sheaf of regular functions is obtained by restricting the structure sheaf of $\tau(V)$ via this homeomorphism.

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