

ENERGY-SAVING TRAJECTORY AND CONTROL DESIGN FOR QUADROTORS WITH SLUNG PAYLOADS

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Abstract

Quadrotors have promising applications such as payload transportation, which can change the future of the package delivery industry. However, many challenges block the way of implementing payload transportation in reality. Slung payload vibrations and quadrotor's energy consumption are among the major challenges, which are related to each other because payload vibrations affect energy consumption.

In this dissertation, the kinematics, dynamics, and energy models are first developed for both a single quadrotor and a transportation system consisting of a quadrotor with a slung payload. The proposed energy model is novel and introduces the concepts of power and energy quotients that, unlike the existing models, do not depend on quadrotor-related parameters such as motor and propeller parameters. This is the first energy model for such a transportation system.

Second, this dissertation focuses on polynomial trajectories, where a generic framework to design feasible polynomial trajectories of arbitrary degree with a large number of waypoints is presented. This allows for extending the capabilities of polynomial trajectories to overcome some kinematic limitations associated with continuous-path trajectories, e.g., arbitrary kinematic constraints.

Third, extensive vibration analyses of the transportation system and polynomial trajectories are conducted. As a result, a novel controller-independent payload vibration

reduction method is proposed. The proposed method is more generic than the existing methods, e.g., anti-swing controllers.

Fourth, the effects of polynomial trajectories, payload mass, and cable length on quadrotor's energy consumption are studied. The comparison with an energy-minimized trajectory shows that polynomial trajectories are not only energy-efficient, but their design is simpler than energy-minimized trajectories and does not require quadrotor-related parameters.

Lastly, a robust energy-saving sliding mode controller with input saturation is designed for the transportation system. The experimental results show that the proposed controller is robust and energy-efficient when, qualitatively, compared with an existing energy-saving controller. The proposed controller is the first energy-saving controllers for such a transportation system.

This dissertation opens the door for package delivery with quadrotors by providing the first energy analysis, and energy-saving trajectories and controllers for quadrotors with slung payloads.

To my parents, wife, and kids

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Nomenclature

Subscripts

$(\cdot)_0$	Initial value of its argument
$(\cdot)_f$	Final or total value of its argument
$(\cdot)_p$	Payload-related parameter, value or quantity
$(\cdot)_q$	Quadrotor-related parameter, value or quantity
$(\cdot)_{av}$	Average value of its argument
$(\cdot)_a$	Minimum acceleration trajectory-related parameter, value or quantity
$(\cdot)_c$	Minimum crackle trajectory-related parameter, value or quantity
$(\cdot)_d$	Desired value
$(\cdot)_j$	Minimum jerk trajectory-related parameter, value or quantity
$(\cdot)_{max}$	Maximum or peak value of its argument
$(\cdot)_{min}$	Minimum value of its argument

$(\cdot)_m$	Mean value of its argument
$(\cdot)_{per}$	Perturbed value of its argument
$(\cdot)_s$	Minimum snap trajectory-related parameter, value or quantity

Topology

$\mathbb{R}^{n \times m}$	Set of real numbers of a dimension $n \times m$
$\mathbb{S}^2$	Topological space of a sphere
$SO3$	Special orthogonal group

Frames of Reference

$\mathcal{B}$	Body-fixed frame
$\mathcal{I}$	Inertial frame

Functions

$(\cdot), (\ddot{\cdot}), (\dddot{\cdot})$	First, second, and third time derivatives of its argument, respectively
$(\cdot)^\top$	Transpose of a matrix or vector
$(\cdot)^{(n)}$	n th time derivative of its argument (used to provide notation consistency in case of higher time derivatives)
$(\cdot)^\times$	Function returns a skew-symmetric matrix of its argument
$[\cdot]^{\circ 2}$	Function returns the Hadamard square power of its argument, i.e., element-wise square.
$ \cdot $	Function returns the absolute value of its argument

$\text{diag}(\cdot)$	Diagonalization function returns a diagonal matrix of a vector
$\text{di}(\cdot)$	Function returns a vector consisting of diagonal elements of a matrix
$\text{Sat}(\cdot)$	Saturation function of a vector
$\text{sat}(\cdot)$	Saturation function of a scalar
$\text{sign}(\cdot)$	Signum function
$\text{vec}(\cdot)$	Vectorization function converts a matrix into a column vector
$\ \cdot\ $	L -2 norm of a vector
$\text{sup}(\cdot)$	Supremum function

Scalars

α	positive constant depends on air density, and motor and propeller parameters
β	Force ratio (constant for non-aggressive flights)
δ	Period of a periodic function
γ	Constant for non-aggressive flights and equal to α/β
$\gamma_1, \gamma_2, \dots, \gamma_5$	Positive constants
γ_6	Positive constants < 1
$\mathcal{L}$	The Lagrangian
$\mathcal{T}$	Kinetic energy
$\mathcal{U}$	Potential energy

μ	Ratio between maximum propeller force to minimum propeller force
ω_n	Natural frequency
ω_t	Trajectory's acceleration frequency
ϕ, θ, ψ	Euler angles: roll, pitch and yaw angles, respectively
Π	Positive constant < 1
ρ	Radius of a circle
ρ_1, ρ_2	Semi-major and semi-minor axes of an ellipse, respectively
σ	Standard deviation
θ_1, θ_2	Cable angles
φ_1, φ_2	Positive constants ≥ 1
a	Acceleration profile
a_0, a_k, b_k	Fourier coefficients
a_t	Tangential acceleration
C^p	p continuity of a spline
d	Dimension of the space
E	Energy consumption
E_Q	Energy quotient
$e_{(\cdot)}$	Error of the quantity in the subscript
F_i	i th propeller force

g	Acceleration of gravity
h_i	Ratio between the i th propeller force to the first propeller force
j	Degree of a polynomial
l	Cable length
M	Payload-to-quadrotor mass ratio
m	Mass
M_i	i th propeller moment
n	Number of trajectory segments
P	Total power required by a quadrotor
P_i	Power required by the i th propeller
P_Q	Power quotient
r	Position profile
r_t	Frequency ratio
T	Relative time
t	Time
T_R	Transmissibility ratio
V, V_1, V_2	Lyapunov functions
v	Velocity profile
$v_1, v_2, \dots, v_n$	Elements of an arbitrary vector $\mathbf{v}$

x, y, z 3D coordinates

Vectors

$\hat{\mathbf{L}}$ Unit vector from the center of mass of the quadrotor to the payload

$\mathcal{C}$ Coefficient vector

$\boldsymbol{\eta}$ Vector-like format of cable angles

$\boldsymbol{\omega}$ Angular velocity

$\boldsymbol{\Phi}$ Vector-like format of roll, pitch and yaw angles

$\boldsymbol{\tau}$ Moment vector in the body-fixed frame

$\boldsymbol{\theta}_{M_i}$ Angular displacement vector of the i th propeller

$\boldsymbol{\xi}$ Vibration mode

$\mathbf{e}$ Unit vector equal to $[0, 0, 1]^\top$

$\mathbf{e}(\cdot)$ Error vector of the subscript, e.g., $\mathbf{e}_p$ is a position error vector

$\mathbf{F}$ Total thrust vector in the inertial frame

$\mathbf{F}_i$ i th propeller force vector in the inertial frame

$\mathbf{G}$ Gravity forces

$\mathbf{g}$ Gravity acceleration vector

$\mathbf{M}_i$ i th propeller moment vector in the inertial frame

$\mathbf{Q}$ Generalized forces

$\mathbf{q}$ Generalized coordinates

$\mathbf{r}$	Position vector
$\mathbf{r}_{\mathcal{B}_i}$	Position vector from quadrotor's center of mass to the center of the i th propeller in the body-fixed frame
$\mathbf{r}_{F_i}$	Position vector of the center of a propeller in the inertial frame
$\mathbf{r}_{q_{1,2}}$	A vector contains the first two elements of quadrotor's position vector
$\mathbf{S}$	Sliding surface
$\mathbf{S}_k$	Kinematic constraints vector
$\mathbf{v}$	Arbitrary vector

Matrices

$\boldsymbol{\eta}_A$	Cable angle amplitude matrix
$\boldsymbol{\kappa}$	Gain matrix
$\boldsymbol{\Xi}$	Vibration modal matrix
$\boldsymbol{\zeta}$	Viscous damping ratio matrix
$\mathbf{0}_{m \times n}$	Null matrix (or vector) of the dimensions indicated by the subscript
$\mathbf{A}, \mathbf{B}$	Auxiliary matrices
$\mathbf{A}_1, \mathbf{A}_2$	Arbitrary square matrices
$\mathbf{B}_k$	Trajectory's acceleration amplitude matrix
$\mathbf{C}$	Coriolis forces matrix
$\mathbf{I}$	Inertia matrix

$\mathbf{I}_n$	Square identity matrix of dimension n
$\mathbf{k}$	Gain matrix
$\mathbf{K}_s$	Simplified stiffness matrix
$\mathbf{k}_s$	Gain matrix
$\mathbf{K}_{(\cdot)}$	Gain matrix
$\mathbf{M}$	Mass matrix
$\mathbf{M}_s$	Simplified inertia matrix
$\mathbf{R}$	Rotation matrix
$\mathbf{R}_t$	Frequency ratio matrix
$\mathbf{R}_v$	Matrix mapping Euler rates to angular velocity
$\mathbf{T}$	Time matrix
$\mathbf{T}_R$	Transmissibility matrix

List of Acronyms

UAV	Unmanned aerial vehicle
DoF	Degree of freedom
COM	Center of mass
PIV	Proportional-Integral-Velocity
PID	Proportional-Integral-Derivative
PV	Proportional-Velocity
PD	Proportional-Derivative
SDCNLab	Spacecraft Dynamics, Control and Navigation Laboratory
YU-AUV	York University's Autonomous Unmanned Vehicle facility
KCCP	Kinematically-constraint continuous-path
MAT	Minimum acceleration trajectory
MJT	Minimum jerk trajectory
MST	Minimum snap trajectory
MCT	Minimum crackle trajectory
AKC	Arbitrary kinematic constraints
PKP	Piecewise kinematic profile
LTI	Linear time-invariant
HTAV	Heavier-than-air vehicle

MPC	Model predictive controller
DNN	Deep neural network
iLQR	Iterative linear quadratic regulator
NMPC	Nonlinear model predictive controller
CNC	Computer numerical control

1 Introduction

1.1 Overview and Motivation

Unmanned aerial vehicles (UAVs) are widely used in a variety of applications that range from entertainment [5, 6] to military applications [7]. UAVs have huge potential to play an important role in many industrial applications such as search and rescue [8, 9], inspection [10, 11], mapping [12, 13], manipulation [14, 15], agriculture [16, 17], and payload and package delivery [18, 19] to name a few applications [20]. UAVs, or drones, cover a broad range of vehicles such as rotorcrafts [21], fixed-wing vehicles [22], and flapping-wing vehicles [23].

Rotorcrafts could have one or more rotors [24, 25]. However, multirotors are commonly used because they allow for more actuation and higher level of controllability over the flight of UAVs [26]. Among multirotors, quadrotors are commonly used in practical applications because of their advantages such as maneuverability, autonomy and size [27]. Therefore, this dissertation considers quadrotors.

Among quadrotors' applications, payload transportation tasks such as package delivery is a promising application for quadrotors [28, 29]. Quadrotors allow for safe, fast, accurate, and environmentally friendly transportation tasks [30]. Transporting a payload with a quadrotor or multiple quadrotors can be done by attaching the payload to the quadro-

tor/quadrotors either rigidly [31,32] or via cables [33,34]. Transporting a payload by picking it with a robotic manipulator, e.g., robotic arm, is also an option in practice [35,36].

Dynamically, a rigidly attached payload transportation system can be dealt with as a new UAV with a larger mass, different dimensions, and a different center of mass [29,37]. On the other hand, a cable-suspended payload (or slung payload) transportation system adds underactuated degrees of freedom (DoFs) to the dynamics model, which makes the dynamics of the transportation system more complex [29,37]. Additionally, the vibration of the slung payload is a big challenge that requires a lot of effort to resolve. In addition to the danger of carrying a vibrating payload, the vibration may add excessive force components and moments to the system, which can lead to undesired motions and energy consumption increase. Slung payload transportation systems provide more agility to the transportation task compared to rigidly-attached payload systems. Additionally, slung payload transportation systems have some advantages such as the ability to pick up and/or deliver payloads without landing, which can be vital in many scenarios, e.g., search and rescue. Therefore, this system grasped our attention and it is the focus of this dissertation.

There are many challenges that block the way of implementing payload transportation using UAVs in our daily life such as safe navigation and collision avoidance [38,39], slung payload vibrations [40,41], and limited flight times due to limited onboard energy [42] to name a few. Despite the importance of the aforementioned challenges, and others, energy limitation is considered a major challenge since most quadrotors, and multirotors in general, are powered by energy-limited batteries. This limits quadrotor's flight time, especially for small and mid-sized quadrotors, to about 30 minutes [43]. Payload vibrations affect the energy consumption since they add excessive forces and moments. Thus, they are vital when addressing quadrotors' energy consumption.

Energy saving is important to prolong flight times and/or flight distances. Although,

energy saving is not the only approach to do so, it is an effective approach with many advantages compared to the alternatives. For example, increasing the energy capacity of the batteries by increasing the number or the size of the batteries can be helpful up to a certain limit, after which, increasing the energy capacity negatively affects the flight time due to the extra weight [44]. Additionally, adding more batteries or using larger batteries reduces the payload mass that can be transported. Flying faster reduces hovering time, which may result in longer flight distances [30]. However, flying faster increases payload vibration, which is a dangerous motion that can lead to instability of the transportation system. Saving energy, on the other hand, is helpful to increase the flight time and/or the flight distance, while avoiding such disadvantages, although it may not result in the maximum flight time or distance.

In fact, the energy consumption challenges affect a range of UAVs wider than just multirotors. Moreover, this challenge faces all types of payload transportation, since UAVs consume energy all the time, in contrast to other challenges that do not affect all transportation tasks. For example, rigidly-attached payloads do not suffer from payload vibration problems. Also, repeated payload transportation tasks in well-defined environments do not require complicated navigation approaches since trajectories can be predefined. Moreover, transporting a payload in a free-of-obstacles environment or a well-defined cluttered environment can be done without the need of a collision avoidance technique. However, all these tasks require reducing energy consumption to reduce the cost and save battery replacing/charging times. For all of these reasons, this dissertation focuses on addressing two related challenges: payload vibrations and energy consumption for quadrotors with slung payloads.

1.2 Related Work

The energy consumption challenge is a general challenge and it is not limited to quadrotors with slung payloads. Therefore, its literature review is wider than energy analysis and energy-saving strategies for quadrotors with slung payloads. However, it all contributes to this dissertation. Additionally, quadrotor's trajectory affects energy consumption both directly and indirectly by affecting payload vibrations. Therefore, a literature review of payload vibration reduction and polynomial trajectories, especially in quadrotor applications, is presented.

Some researchers worked on enhancing the battery charging process by either implementing wireless charging [45,46] or considering energy harvesting from high-voltage power lines [47,48]. To reduce UAVs' energy consumption, changing UAVs' conventional configuration is an option [49–51]. In [49], energy consumption was optimized by rotating the quadrotor's arms which reduced the overall energy consumption by 13%. In [50], a quadrotor with a large rotor in its center and three small rotors in a triangular configuration was designed. The designed quadrotor required 15% less power compared to a corresponding conventional quadrotor. In [51], a quadrotor with titled propellers was proposed to overcome quadrotor's underactuation as well as enhance energy consumption.

Typically, there are some assumptions in most of the literature such as ignoring cables' flexibility and extension, and assuming that the attachment point on the quadrotor is its center of mass (COM). However, some of the literature has not made these assumptions.

The flexibility and extensibility of the cables are considered in the literature [52–54]. In [52], the flexible cable was modeled as a system of serially-connected links with spherical joints. In [53], the same concept was extended to consider a payload carried by multiple UAVs with flexible cables. This approach is beneficial especially when considering aggres-

sive flights¹. However, increasing the number of cable links is computationally expensive. In [54], a time-varying cable length was considered by mounting an actuator beneath the quadrotor to lower the payload, i.e., extend the length of the cable. This is to overcome the assumption of inextensible cables, which can be helpful in package delivery applications.

The offset of payload attachment point is considered in [55,56]. The dynamics model in such a case becomes more complicated. However, in practice, it is acceptable to ignore this offset if it is small as stated in [57], who are the authors of the aforementioned paper [55].

A heavy payload can be transported using a larger UAV or cooperatively using multiple smaller UAVs. In [58], the payload trajectory was optimized based on the cable tension distribution, while considering the limits of the cable tension and the UAV thrust. They designed a decentralized controller for each UAV, which will passively control the payload. In [59], a bar-shaped payload was transported considering collision avoidance of dynamic obstacles, wind perturbations, and landing on a formation of moving UGVs, which move according to the payload position. In [60], a vision-based controller was developed for a system of a rigid payload carried by multiple quadrotors. In [61], payload inertial parameters, i.e., mass, COM, and moments of inertia, were estimated using multiple quadrotors and without force sensors on the cables. In [62], a payload trajectory planning and control strategy for a multilift system was proposed to minimize the differences in cable forces. Flying with a near-equal cable force is intuitively safe, energy-efficient and enhances the flight performance for cooperative transportation systems [63]. In [64], Kane approach was used to derive the dynamics model of a payload transported by multiple quadrotors. Then, a path-following controller was designed, while maintaining a certain payload attitude.

¹Aggressive flights are flights with large roll and pitch angles and angle rates, which come from large differences between propeller forces. Although there is no rigorous definition for aggressive flights in the literature, this dissertation defines aggressive flights as the flights with the maximum propeller force being more than the double of the minimum propeller force.

A main strategy to save UAVs' energy consumption is by designing energy-saving trajectories. UAVs are heavier-than-air vehicles (HTAVs), which implies that they must hover all the time, so they consume energy during the whole flight time. One can think of flight energy consumption to be split into two main components: a) hovering energy consumption; and b) trajectory-following energy consumption. Fast flights require larger attitude angles, i.e., more trajectory-following energy consumption, and less flight time, i.e., less hovering and energy consumption. Slow flights behave oppositely. This contradictory effect of fast flights on energy consumption was studied in [65]. Then, they proposed a dynamic quadrotor's speed profile that varies with the geometry of the desired path and integrated this profile into a path-following controller to save energy.

Quadrotor's energy consumption can be related to its propellers' rotational speed and acceleration. This relation was used to solve the optimal control problem to find the energy-minimized trajectory of a quadrotor between two points in [43] and for a hexarotor with tilting rotors in [66]. This approach requires prior knowledge of many UAV-related parameters such as motor and propeller parameters, i.e., aerodynamic and electrical specifications of the UAV. Changing the UAV may require solving the optimization problem again using the new specifications. This is a limitation of this approach since the designed energy-minimized trajectory may not be as efficient for other UAVs.

Increasing quadrotor's flight distance or flight time with the same amount of energy was achieved by utilizing a model-free online adaptive technique based on Extremum Seeking control to find the velocity in [67]. In [68] an energy-efficient path planning genetic algorithm was proposed. It was highlighted that trajectories with repeated acceleration and deceleration are not energy-efficient especially at high speeds. Following that, a Q-learning-based path planning was developed in [69]. The Q-learning policy weighted path length, obstacle avoidance, and energy consumption. For the energy consumption part,

they depended on the acceleration/deceleration point mentioned in [68] and penalized heavy turns. Machine learning techniques were used to design energy-efficient trajectories, e.g., in [70,71] for a single UAV and multiple UAVs, respectively. The approaches presented in [67–71] depend, in a way, on the velocity and acceleration profiles of the trajectory to reduce UAVs’ energy consumption. However, the literature lacks a comprehensive analysis of how these profiles affect UAVs’ energy consumption.

Polynomial trajectories are one of the important trajectory families. In the context of UAVs, the energy performance of three different 1D polynomial trajectories at fixed elevation were studied in [72]. For a short straight line, minimum snap trajectories (MSTs) were seemed to be energy-efficient. To reduce quadrotor’s power, a minimum jerk trajectory (MJT) along with terminal sliding mode control was designed in [73]. The controller was replaced with a proportional-integral-derivative (PID) controller in [74]. In spite of that, the work reported in [72–74] did not provide analytical analyses of the effect of polynomial trajectories on UAVs’ energy consumption.

Controller design significantly affects how quadrotors perform. Therefore, designing such controllers in a way that guarantees saving energy is one of the energy saving approaches. It is worth recalling that quadrotors are underactuated systems, and when a slung payload is added, the level of underactuation rises. This increases the difficulty of controller design of quadrotors. Additionally, there may be a trade off between energy consumption and controller performance. Therefore, two main paths are considered in the literature: a) saving control computation energy; b) constraining control command, i.e., the thrust and moments, or propeller forces.

The first approach does not sacrifice the performance, but focuses on reducing the control computation energy. Model predictive controllers (MPCs) are known for their good performance; however, they suffer from the expensiveness of the required online compu-

tation. Reducing the control computation of MPCs to obtaining an explicitly defined piecewise function, i.e., explicit MPC, saves online computation energy [75, 76]. However, with many states, input, and time steps, the computation time increases exponentially [77]. In the context of quadrotors, a deep neural network (DNN) was trained to approximate the behaviour of an MPC for a quadrotor in [78]. The proposed DNN reduced the computation energy, so quadrotor’s flight time was increased by 8.75 to 12.5% compared to a corresponding MPC. A drawback here is that the performance of the DNN heavily depends on the training data and the trained DNN is limited to a specific quadrotor since changing the quadrotor results in changing the MPC and the collected data.

The second approach sacrifices a portion of the performance to save some energy by constraining the control command within an energy-efficient threshold. However, how reducing the control command is dealt with in the literature of quadrotors is crucial. For example, in [79] a sliding mode controller was used with a nonlinear sliding surface. The time constant of conventional sliding mode controllers was replaced with a nonlinear function in the proposed controller. This nonlinear function changes the time constant as needed during a flight to save energy. This approach was able to save about 4.1 to 4.9% of energy under different conditions. However, this comes at the cost of the convergence speed. In terms of application, understanding how such a nonlinear function affects both the controller performance and the energy consumption is not straightforward since this nonlinear function is for defining the sliding surface, not the control command directly. A close approach is proposed in [80] by reducing the chattering of a sliding mode controller by smoothing the control discontinuity in a thin neighbourhood of the sliding surface. This chattering reduction resulted in energy saving of about 1.48 to 1.61% compared to a corresponding sliding mode controller without chattering reduction. Similar to [79], reducing the chattering around the sliding surface is not a straightforward approach to get

intuitively. Another example is the integral sliding mode controller proposed in [4]. The control command, and consequently the energy consumption, was reduced by bounding roll and pitch angles within energy-efficient ranges. These ranges are specified experimentally. Despite the effectiveness of the proposed controller in saving about 3.5 to 6% of the energy consumption under different conditions, many practical limitations exist. The energy-efficient ranges are experimentally-obtained, which is a subjective range that may not work for all tasks. Also, bounding roll and pitch angles only, not their rates, may not result in an energy-efficient performance in all cases, e.g., trajectories with high roll and pitch rate within the energy-efficient roll and pitch angles.

Payload vibrations are a critical aspect when it comes to transporting slung payloads. It is dangerous and energy consuming. Typically, payload vibrations are reduced by either anti-swing controllers or trajectory design. Anti-swing controllers are the controllers that aim at reducing or eliminating payload vibrations. Several controllers have been designed for this purpose, such as an iterative linear quadratic regulator (iLQR) [81] and an energy-based adaptive nonlinear anti-swing controllers [82, 83]. In [84], a dynamic programming approach was used to generate a trajectory that is swing-free. In [85], an online trajectory generation approach was used to design an acceleration trajectory, which eliminated the payload vibration. Reinforcement learning was applied to generate trajectories that reduce the cable vibration in [86, 87].

The differential flatness of the quadrotor was used in trajectory generation, e.g., in [88], where the authors found that part of the attitude control inputs are functions of the snap, i.e., the 4th time derivative of the position. Consequently, they designed a trajectory that minimized the snap assuming that this may minimize the control input. This approach was extended in [89] to consider the collision avoidance in cluttered environments. In [90], an optimal model-free back stepping controller has been developed for the quadcopter. Imple-

menting this control approach with a cuckoo search algorithm dealt with the uncertainties of the system dynamics and the external disturbances. A very similar dynamics system is the gantry crane. One of the popular ways to reduce the residual payload vibration is to apply the input shaping techniques, which was used for gantry cranes in [91, 92].

To the best of our knowledge, the literature of payload vibrations lacks comprehensive investigations of how transportation system parameters, e.g., cable length and payload mass, affects payload vibrations. Despite the significance of payload vibrations on UAVs' energy consumption, studies on the effect of payload vibrations on UAVs' energy consumption are needed. Additionally, in the context of polynomial trajectories, the effect of different polynomial trajectories on payload vibrations requires deep investigations.

Trajectories can be categorized based on many criteria. In this dissertation, we categorize trajectories into waypoint trajectories and continuous-path trajectories. A waypoint trajectory is required to pass through certain waypoints without any constraints on the path between the waypoints. So, the path between waypoints could be a straight line, a sinusoidal curve or a polynomial. A waypoint trajectory may also be required to satisfy certain kinematic constraints at the waypoints. On the other hand, a continuous-path trajectory is a trajectory that has to follow a certain path at all times such as circular, elliptical and helical trajectories. Many indoor experiments, especially for validation purposes, consider continuous-path trajectories. For example, [58, 93] considered a circular trajectory, and [58] considered an lemniscate trajectory to validate their proposed controllers. On the other hand, outdoor applications, especially package delivery, tend to utilize waypoint trajectories [94]. Waypoint trajectories can also be used to get the most out of UAVs [95]. A UAV takes off at an initial waypoint, then flies to its destination, i.e., a final waypoint, via some intermediate waypoint, which are chosen for specific reasons, e.g., obstacle avoidance.

1.3 Research Gaps and Challenges

Despite the rich literature presented in Section 1.2, there are many research gaps to explore. A major research gap is that the energy challenge is not considered in the literature for the transportation system consisting of a quadrotor with a slung payload. More specifically:

I For energy modeling:

Energy models for a system consisting of a quadrotor with a slung payload are lacked in the literature.

II For energy analysis and energy-saving strategies:

- (a) A comprehensive energy analysis of the effect of parameters like cable length, payload mass, and/or quadrotor's trajectory on the energy consumption of the transportation system is lacked in the literature.
- (b) Most of the trajectory-based energy-saving strategy in the literature are energy-minimized trajectories. This requires knowing many quadrotor-related parameters in addition to solving optimization problems. This may be computationally inefficient and may not run online. Simpler, but effective, alternatives that are more general and computationally efficient are lacked in the literature.
- (c) Energy-saving controllers for the system are lacked in the literature.

III For payload vibrations and trajectory design:

- (a) A comprehensive vibration analysis of the effect of parameters like cable length and/or quadrotor's trajectory on payload vibrations of the transportation system is lacked in the literature.
- (b) A trajectory-based payload vibration reduction is lacked in the literature.

- (c) A systematic computationally efficient framework to design polynomial trajectories of any degree with any number of waypoints is lacked in the literature.

1.4 Contributions

This dissertation aims to address the research gaps reported in Section 1.3 to help quadrotors and quadrotors with slug payloads fly longer, and eventually be used in daily life package delivery. Specifically, the main contributions of this dissertation are as follows:

I) **For energy modeling:**

Developing the first energy model for quadrotors with slung payloads. This model does not depend on quadrotor-related parameters (Research Gap I - Chapter 2).

II) **For energy analysis and energy-saving strategies:**

- (a) Conduction the first mathematical analysis of the effect of polynomial trajectories, cable length, and payload masses on the energy consumption of the transportation system (Research Gap IIa - Chapter 5).
- (b) Proposing a trajectory-based energy-saving strategy that does not require solving optimization problems and depends only on polynomial trajectories. (Research Gap IIb - Chapter 5).
- (c) Designing the first robust energy-saving integral sliding mode controller with input saturation for the transportation system. (Research Gap IIc - Chapter 6).

III) **For payload vibrations and trajectory design:**

- (a) Conducting the first comprehensive vibration analysis of the transportation system, which resulted in a novel analogy between the transportation system and

car suspension systems. This led to better understanding of the system in addition to providing a novel two-stage payload vibration reduction strategy (Research Gap IIIa - Chapter 4).

- (b) Proposing a novel two-stage trajectory-based payload vibration reduction strategy based on the analogy with car suspension systems (Research Gap IIIb - Chapter 4)
- (c) Developing a generic framework to generate polynomial trajectories of an arbitrary degree with a large number of waypoints. (Research Gap IIIc - Chapter 3).

1.5 Assumptions

The following assumptions are made throughout this dissertation:

- I) Point mass payloads. This excludes:
 - (a) The rotational motion of the payload.
 - (b) The aerodynamics effect on the payload.
- II) Massless taut cables. This implies:
 - (a) No dynamics model for the cables. Instead, they kinematically constrain the relative motion of the payload with respect to the quadrotor.
 - (b) Non-extendable non-deformable cables, i.e., rigid cables in the modeling.
- III) Flight altitude does not change significantly, which means that the air density can be considered constant.
- IV) Non-aggressive flights, which means that there is no large roll and pitch angles and/or angle rates. A rigorous definition of non-aggressiveness flights is presented in Section 2.3.2.

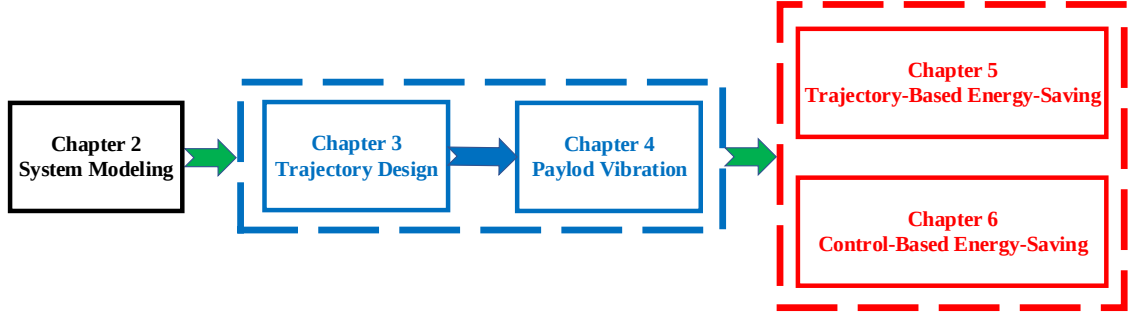


Figure 1.1: Dissertation Structure

V) Real-time position information is known for both the quadrotor and the payload.

VI) Communication time delay is negligible.

1.6 Dissertation Organization

This dissertation is organized in six chapters in addition to the Introduction. The structure of the dissertation is shown in Fig. 1.1. For the remainder of the dissertation, Chapter 2 is dedicated for system modeling. Specifically, Section 2.1 presents the kinematics modeling, whereas the dynamics models are derived in Section 2.2 using the Lagrangian approach. In Section 2.3, a novel energy model is proposed. The experimental setup and the main controller used in most of the dissertation are introduced in Section 2.4.

Chapter 3 focuses on polynomial trajectories design. In Section 3.1 a general framework to generate polynomial trajectories is developed, then the kinematic properties of polynomial trajectories are investigated. After that, the kinematically-constrained continuous-path (KCCP) polynomial trajectory is introduced. The kinematic and dynamic feasibility of polynomial trajectories are discussed in Section 3.2. Simulation and experimental results are presented in Sections 3.3 and 3.4, respectively, to show the effectiveness of the proposed KCCP polynomial trajectory.

Chapter 4 discusses payload vibrations. A comprehensive vibration analysis of the transportation system is conducted in Section 4.1. Then, a frequency analysis of polynomial trajectories is presented in Section 4.2. After that, a novel two-stage payload vibrations reduction approach is introduced in Section 4.3. Finally, simulations and experiments to validate the results and conclusions of this chapter are presented in Sections 4.4 and 4.5, respectively.

Chapter 5 analyzes quadrotors' energy consumption with and without payloads and investigates the effect of different aspects on energy consumption. Section 5.1 shows an extensive mathematical analysis of the effect of polynomial trajectories, cable length, and payload mass on quadrotor's energy consumption under different kinematic boundary conditions. It also provides a trajectory-based energy-saving strategy. The proposed energy-saving strategy is validated experimentally in Section 5.2.

Chapter 6 introduces an energy-saving control strategy for quadrotors with slung payloads. The idea of the proposed integral sliding mode controller with input saturation is introduced in Section 6.1, whereas the controller is designed in Section 6.2. The experimental validation of the proposed controller is demonstrated in Section 6.3.

Finally, the conclusions and the future work are presented in Chapter 7, followed by the references and appendices.

2 System Modeling

In this chapter, kinematics, dynamics and energy models for a single quadrotor and a quadrotor with a slung payload are derived. Then, the main controller utilized in most of this dissertation is presented. For the convenience of the reader, some equations may be repeated at subsequent chapters without derivations.

2.1 Kinematics Modeling

Quadrotors have different geometric and/or flight configurations [96]. A conventional cross configuration quadrotor is considered in this dissertation. Fig. 2.1 shows the geometrical parameters of the QDrone used to validate the results of this dissertation as well as the cable angles in case of attaching a slung payload.

2.1.1 Single Quadrotor

As shown in Fig. 2.1a, two main frames of reference are considered: a) a west-south-up inertial frame, $\mathcal{I}$; and b) a body-fixed frame, $\mathcal{B}$. The inertial frame is a time-invariant non-rotating non-moving frame, whereas the body-fixed frame is attached to quadrotor's center of mass (COM) and rotates with it.

Position vectors, and their time derivatives, are expressed in the inertial frame, $\mathcal{I}$, unless otherwise stated. Therefore, $\mathbf{r} = [x, y, z]^T \in \mathbb{R}^3$, with different subscripts, denotes

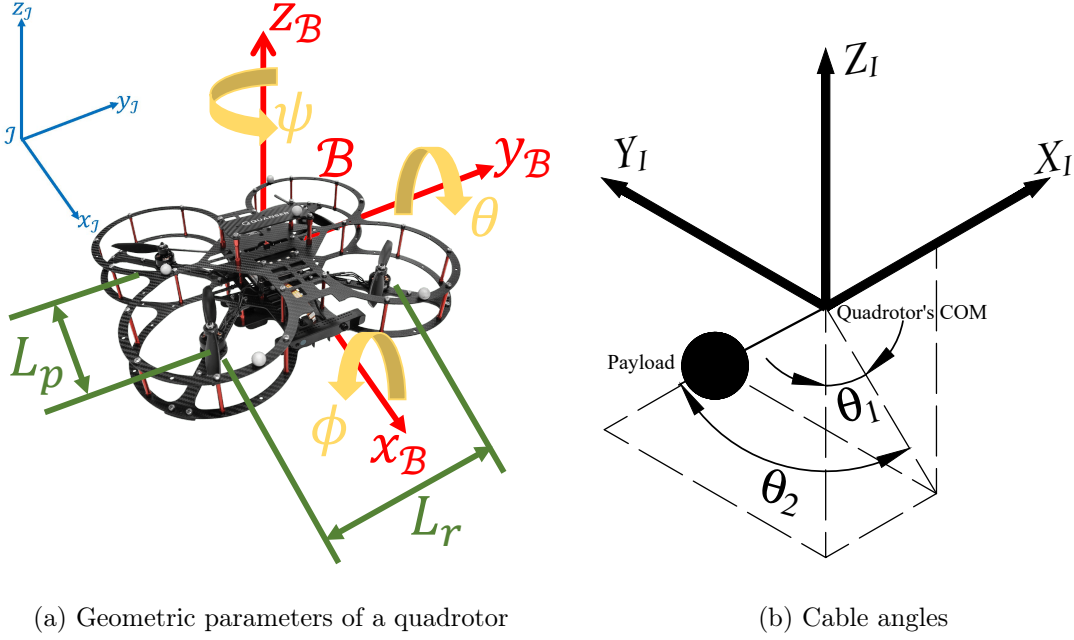


Figure 2.1: Geometric parameters for a quadrotor and cable angles

position vectors in 3D, unless otherwise stated. The subscript q denotes quadrotor-related parameters/quantities, e.g., $\mathbf{r}_q$ and $\dot{\mathbf{r}}_q$ denote quadrotor position and velocity vectors in the inertial frame, respectively.

On the other hand, rotations are expressed in the body-fixed frame, $\mathcal{B}$, unless otherwise stated. Euler angles are utilized in this dissertations, which can be stacked in a vector-like format as $\Phi = [\phi, \theta, \psi]^T \in \mathbb{R}^3$ with ϕ , θ , and ψ being the roll, pitch, and yaw angles, respectively. A rotation matrix that transforms a vector expressed in the body-fixed frame, $\mathcal{B}$ to be expressed in the inertial frame, $\mathcal{I}$, is $\mathbf{R} \in SO(3)$. This dissertation considers intrinsic rotations with the order yaw-pitch-roll, which results in the following rotation matrix:

$$\mathbf{R} = \begin{bmatrix} \cos \theta \cos \psi & \sin \theta \sin \phi \cos \psi - \cos \phi \sin \psi & \sin \theta \cos \phi \cos \psi + \sin \phi \sin \psi \\ \cos \theta \sin \psi & \sin \theta \sin \phi \sin \psi + \cos \phi \cos \psi & \sin \theta \cos \phi \sin \psi - \sin \phi \cos \psi \\ -\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \end{bmatrix} \quad (2.1)$$

which yields the following relation between the angular velocity, $\omega \in \mathbb{R}^3$, and the rates of Euler angles, $\dot{\Phi}$:

$$\boldsymbol{\omega} = \mathbf{R}_v \dot{\boldsymbol{\Phi}}$$

$$\mathbf{R}_v = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix} \in \mathbb{R}^{3 \times 3} \quad (2.2)$$

2.1.2 Quadrotor with a Slung Payload

Similar to the definitions in Section 2.1.1, the subscript p denotes payload-related parameters/quantities, e.g., $\mathbf{r}_p$ and $\dot{\mathbf{r}}_p$ denote payload position and velocity vectors in the inertial frame, respectively.

This dissertation focuses on point mass payloads, i.e., payload motion is a pure translation. However, the presence of the cable adds a kinematic constraint to the motion. Since the attachment point on the quadrotor is close to quadrotor's COM, it is practically acceptable to assume that the attachment point is the COM [57]. So, when the cable is taut, the kinematic constraint is represented mathematically as:

$$\mathbf{L} = \mathbf{r}_p - \mathbf{r}_q \in \mathbb{S}^2 \quad (2.3)$$

where $\mathbf{L} = l\hat{\mathbf{L}}$ with $l \in \mathbb{R}_{>0}$ being the cable length and $\hat{\mathbf{L}}$ is a unit vector in the direction of the cable from the attachment point on the quadrotor to the attachment point on the payload as shown in Fig. 2.1b.

To fully capture the motion of the payload, two cable angles are defined as θ_1 and θ_2 , which can be stacked in a vector-like format as $\boldsymbol{\eta} = [\theta_1, \theta_2]^\top \in \mathbb{R}^2$ (see Fig. 2.1b). The cable angle θ_2 is chosen to project the length of the cable onto the inertial xz plane and provides the inertial y -component of the cable length. The cable angle θ_1 decomposes this projection into two components in the inertial x - and z -directions. Consequently, the unit vector $\hat{\mathbf{L}}$ can be defined as:

$$\hat{\mathbf{L}} = \begin{bmatrix} \sin \theta_1 \cos \theta_2 \\ \sin \theta_2 \\ -\cos \theta_1 \cos \theta_2 \end{bmatrix} \in \mathbb{S}^2 \quad (2.4)$$

which leads to defining the time derivatives of $\hat{\mathbf{L}}$ as follows:

$$\begin{aligned}\dot{\hat{\mathbf{L}}} &= \mathbf{A}\dot{\boldsymbol{\eta}} \\ \ddot{\hat{\mathbf{L}}} &= \mathbf{A}\ddot{\boldsymbol{\eta}} + \mathbf{B}\dot{\boldsymbol{\eta}}\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} \cos \theta_1 \cos \theta_2 & -\sin \theta_1 \sin \theta_2 \\ 0 & \cos \theta_2 \\ \sin \theta_1 \cos \theta_2 & \cos \theta_1 \sin \theta_2 \end{bmatrix} = \frac{\partial \hat{\mathbf{L}}}{\partial \boldsymbol{\eta}} = \frac{\partial \dot{\hat{\mathbf{L}}}}{\partial \dot{\boldsymbol{\eta}}} \in \mathbb{R}^{3 \times 2} \\ \mathbf{B} &= \dot{\mathbf{A}} = \frac{\partial \dot{\hat{\mathbf{L}}}}{\partial \boldsymbol{\eta}}\end{aligned}\tag{2.6}$$

To measure the cable angles, θ_1 and θ_2 , the position of both the quadrotor and the payload are used to obtain $\mathbf{L}$ from Eq. (2.3). Then, $\mathbf{L}$ along with the cable length, l , are used to define the cable angles, θ_1 and θ_2 , from Eq. (2.4)

2.2 Dynamics Modeling

There is a large body of dynamics modeling work presented in classical mechanics textbooks, e.g., Newtonian approach, Lagrangian approach, and Kane approach to name a few. This dissertation utilizes the Lagrangian approach to get rid of the existence of cable tensions, i.e., constraint forces, in the dynamics model, contrarily to the Newtonian approach.

2.2.1 Single Quadrotor

The Lagrangian $\mathcal{L}$ is defined as:

$$\mathcal{L} = \mathcal{T} - \mathcal{U}\tag{2.7}$$

where $\mathcal{T} \in \mathbb{R}_{\geq 0}$ is the kinetic energy and $\mathcal{U} \in \mathbb{R}$ is the potential energy.

For a single quadrotor, the Lagrangian is expressed as:

$$\mathcal{L} = \frac{1}{2}m_q\dot{\mathbf{r}}_q^\top\dot{\mathbf{r}}_q + \frac{1}{2}\boldsymbol{\omega}^\top\mathbf{I}\boldsymbol{\omega} - m_qg\mathbf{r}_q^\top\mathbf{e} \quad (2.8)$$

where $m_q \in \mathbb{R}_{\geq 0}$ is quadrotor's mass, g is the acceleration of gravity, $\mathbf{e} = [0, 0, 1]^\top$ is a unit vector in the inertial z -direction, and $\mathbf{I} \in \mathbb{R}^{3 \times 3}$ is the inertia matrix of the quadrotor around the body-fixed frame, $\mathcal{B}$. If the body-fixed frame axes are the principal axes, then $\mathbf{I} \in \mathbb{R}_{>0}^{3 \times 3}$ is a diagonal matrix with diagonal elements of quadrotor's moments of inertia.

The generalized coordinates for a single quadrotor is chosen to be as follow:

$$\mathbf{q} = \begin{bmatrix} \mathbf{r}_q \\ \boldsymbol{\Phi} \end{bmatrix} \quad (2.9)$$

The following derivatives are needed to derive the dynamics model:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_q} = m_q \dot{\mathbf{r}}_q \quad (2.10)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_q} = m_q \ddot{\mathbf{r}}_q \quad (2.11)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}_q} = -m_q g \mathbf{e} \quad (2.12)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\Phi}}} &= \left[\frac{\partial \boldsymbol{\omega}}{\partial \dot{\boldsymbol{\Phi}}} \right]^\top \frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}} \\ &= \mathbf{R}_v^\top \mathbf{I} \boldsymbol{\omega} \end{aligned} \quad (2.13)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\Phi}}} = \dot{\mathbf{R}}_v^\top \mathbf{I} \boldsymbol{\omega} + \mathbf{R}_v^\top \mathbf{I} \dot{\boldsymbol{\omega}} \quad (2.14)$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Phi}} = \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top \mathbf{I} \boldsymbol{\omega} \quad (2.15)$$

Note that $\dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top \in \mathbb{R}^{1 \times 3 \times 3}$ is a 3D tensor. This tensor is unfolded before being plugged into Eq. (2.15) as a 2D matrix as follows:

$$\underbrace{\dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top}_{1 \times 3 \times 3 \text{ tensor}} = \underbrace{\begin{bmatrix} \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \phi} \right]^\top \\ \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \theta} \right]^\top \\ \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \psi} \right]^\top \end{bmatrix}}_{3 \times 3 \text{ matrix}} \in \mathbb{R}^{3 \times 3} \quad (2.16)$$

Quadrotors have four propellers, which generate four propeller forces F_i , where $i \in \{1, 2, 3, 4\}$. Since all propeller forces are in the z -direction of the body-fixed frame, $\mathcal{B}$, they can be expressed in the inertial frame as:

$$\mathbf{F}_i = \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ F_i \end{bmatrix} \quad (2.17)$$

Similarly, the four propellers generate four moments around the z -direction of the body-fixed frame, i.e., M_i , where $i \in \{1, 2, 3, 4\}$. Therefore, propeller moments in the inertial frame are expressed as:

$$\mathbf{M}_i = \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ M_i \end{bmatrix} \quad (2.18)$$

The position vector of the center of each propeller in the inertial frame is defined as:

$$\mathbf{r}_{F_i} = \mathbf{r}_q + \mathbf{R}\mathbf{r}_{\mathcal{B}_i} \quad (2.19)$$

where $\mathbf{r}_{\mathcal{B}_i}$ is a position vector from quadrotor's COM to the center of the i th propeller expressed in the body-fixed frame, $\mathcal{B}$. Similarly, the angles corresponding to the moments $\mathbf{M}_i$ are expressed in the inertial frame as:

$$\boldsymbol{\theta}_{M_i} = \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix} \quad (2.20)$$

From the principal of virtual work, the generalized forces can be found as follows:

$$\mathbf{F}_{\mathbf{r}_q} = \sum_{i=1}^4 \underbrace{\left[\frac{\partial \mathbf{r}_{F_i}}{\partial \mathbf{r}_q} \right]^\top}_{=\mathbf{I}_{3 \times 3}} \mathbf{F}_i + \sum_{i=1}^4 \underbrace{\left[\frac{\partial \boldsymbol{\theta}_{M_i}}{\partial \mathbf{r}_q} \right]^\top}_{=0} \mathbf{M}_i = \sum_{i=1}^4 \mathbf{F}_i = \mathbf{F} \quad (2.21)$$

$$\boldsymbol{\tau}_{\boldsymbol{\Phi}} = \sum_{i=1}^4 \left[\frac{\partial \mathbf{r}_{F_i}}{\partial \boldsymbol{\Phi}} \right]^\top \mathbf{F}_i + \sum_{i=1}^4 \left[\frac{\partial \boldsymbol{\theta}_{M_i}}{\partial \boldsymbol{\Phi}} \right]^\top \mathbf{M}_i \quad (2.22)$$

where $\mathbf{F} \in \mathbb{R}^3$ is the total thrust vector expressed in the inertial frame, and $\mathbf{I}_{3 \times 3}$ is an identity matrix with dimensions expressed in the subscript, i.e., 3×3 . The partial derivatives in Eq. (2.22) are defined as:

$$\frac{\partial \mathbf{r}_{F_i}}{\partial \Phi} = \begin{bmatrix} \underbrace{\frac{\partial \mathbf{r}_{F_i}}{\partial \phi}}_{3 \times 1} & \underbrace{\frac{\partial \mathbf{r}_{F_i}}{\partial \theta}}_{3 \times 1} & \underbrace{\frac{\partial \mathbf{r}_{F_i}}{\partial \psi}}_{3 \times 1} \end{bmatrix} \quad (2.23)$$

$$\frac{\partial \theta_{M_i}}{\partial \Phi} = \begin{bmatrix} \underbrace{\frac{\partial \theta_{M_i}}{\partial \phi}}_{3 \times 1} & \underbrace{\frac{\partial \theta_{M_i}}{\partial \theta}}_{3 \times 1} & \underbrace{\frac{\partial \theta_{M_i}}{\partial \psi}}_{3 \times 1} \end{bmatrix} \quad (2.24)$$

Finally, the dynamics model of a single quadrotor can be expressed as follows:

$$m_q \ddot{\mathbf{r}}_q + m_q g \mathbf{e} = \mathbf{F} \quad (2.25)$$

$$\mathbf{R}_v^\top \mathbf{I} \dot{\boldsymbol{\omega}} + \left[\dot{\mathbf{R}}_v^\top - \dot{\Phi}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \Phi} \right]^\top \right] \mathbf{I} \boldsymbol{\omega} = \boldsymbol{\tau}_\Phi \quad (2.26)$$

where Eq. (2.25) represents quadrotor's translational dynamics, and Eq. (2.26) represents quadrotor's rotational (attitude) dynamics. This model agrees with what exists in the literature, e.g., [97]. It is worth mentioning that Eq. (2.26) is equivalent to Euler's equation for rigidbody attitude dynamics [98] (see Appendix A):

$$\mathbf{I} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I} \boldsymbol{\omega} = \boldsymbol{\tau} \quad (2.27)$$

For the sake of completeness, the external forces and moments, $F = \|\mathbf{F}\| \in \mathbb{R}$ and $\boldsymbol{\tau} \in \mathbb{R}^3$, can be mapped to propeller force F_i as follows:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2L_r} & \frac{1}{2L_p} & \frac{k_\tau}{4} \\ \frac{1}{4} & \frac{1}{2L_r} & -\frac{1}{2L_p} & -\frac{k_\tau}{4} \\ \frac{1}{4} & \frac{1}{2L_r} & \frac{1}{2L_p} & -\frac{k_\tau}{4} \\ \frac{1}{4} & -\frac{1}{2L_r} & -\frac{1}{2L_p} & \frac{k_\tau}{4} \end{bmatrix} \begin{bmatrix} F \\ \boldsymbol{\tau} \end{bmatrix} \quad (2.28)$$

where k_τ is the motor thrust-torque constant, and L_r and L_p are the roll and pitch motor-to-motor distances, respectively (see Fig. 2.1a).

2.2.2 Quadrotor with a Slung Payload

When a slung payload is considered, the dynamics model will change. Since calculating the magnitude of the cable tension is out of this dissertation's focus, an embedding technique is utilized [99]. In this embedding technique, the dynamics of the system is represented in terms of the minimum number of coordinates, i.e., the DoFs, by imposing the kinematic constraint of the cable length to relate the position of the quadrotor to the position of the payload (see Eq. (2.3)). This eliminates the constraint force, i.e., the cable tension, from the dynamics model. Therefore, the generalized coordinates in this case will be:

$$\mathbf{q} = \begin{bmatrix} \mathbf{r}_q \\ \boldsymbol{\eta} \\ \boldsymbol{\Phi} \end{bmatrix} \quad (2.29)$$

and the Lagrangian is defined as:

$$\mathcal{L} = \frac{1}{2} (m_q + m_p) \dot{\mathbf{r}}_q^T \dot{\mathbf{r}}_q + \frac{1}{2} m_p l^2 \dot{\mathbf{L}}^T \dot{\mathbf{L}} + m_p l \dot{\mathbf{r}}_q^T \dot{\mathbf{L}} + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} - (m_q + m_p) g \mathbf{r}_q^T \mathbf{e} - m_p g l \hat{\mathbf{L}}^T \mathbf{e} \quad (2.30)$$

where m_p is the mass of the payload. Under the assumption of attaching the cable to quadrotor's COM, the attitude dynamics in Eq. (2.26) and (2.27) remains the same. This would affect the translational dynamics. Thus, further equations should be derived to represent the payload vibrations. The following derivatives are used in deriving the dynamics model:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_q} = (m_q + m_p) \dot{\mathbf{r}}_q + m_p l \dot{\mathbf{L}} \quad (2.31)$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_q} &= (m_q + m_p) \ddot{\mathbf{r}}_q + m_p l \ddot{\mathbf{L}} \\ &= (m_q + m_p) \ddot{\mathbf{r}}_q + m_p l \mathbf{A} \ddot{\boldsymbol{\eta}} + m_p l \mathbf{B} \dot{\boldsymbol{\eta}} \end{aligned} \quad (2.32)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}_q} = - (m_q + m_p) g \mathbf{e} \quad (2.33)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\eta}}} = \left[\frac{\partial \dot{\hat{\mathbf{L}}}}{\partial \dot{\boldsymbol{\eta}}} \right]^\top \frac{\partial \mathcal{L}}{\partial \dot{\hat{\mathbf{L}}}} \quad (2.34)$$

$$= m_p l^2 \mathbf{A}^\top \dot{\hat{\mathbf{L}}} + m_p l \mathbf{A}^\top \dot{\mathbf{r}}_q$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\eta}}} &= m_p l^2 \left(\mathbf{B}^\top \dot{\hat{\mathbf{L}}} + \mathbf{A}^\top \ddot{\hat{\mathbf{L}}} \right) + m_p l \left(\mathbf{B}^\top \dot{\mathbf{r}}_q + \mathbf{A}^\top \ddot{\mathbf{r}}_q \right) \\ &= m_p l \mathbf{A}^\top \ddot{\mathbf{r}}_q + m_p l^2 \mathbf{A}^\top \mathbf{A} \ddot{\boldsymbol{\eta}} + m_p l \mathbf{B}^\top \dot{\mathbf{r}}_q + m_p l^2 \left(\mathbf{B}^\top \mathbf{A} + \mathbf{A}^\top \mathbf{B} \right) \dot{\boldsymbol{\eta}} \end{aligned} \quad (2.35)$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\eta}} = \left[\frac{\partial \dot{\hat{\mathbf{L}}}}{\partial \boldsymbol{\eta}} \right]^\top \frac{\partial \mathcal{L}}{\partial \dot{\hat{\mathbf{L}}}} + \left[\frac{\partial \hat{\mathbf{L}}}{\partial \boldsymbol{\eta}} \right]^\top \frac{\partial \mathcal{L}}{\partial \hat{\mathbf{L}}} \quad (2.36)$$

$$= m_p l^2 \mathbf{B}^\top \mathbf{A} \dot{\boldsymbol{\eta}} + m_p l \mathbf{B}^\top \dot{\mathbf{r}}_q - m_p g l \mathbf{A}^\top \mathbf{e}$$

The generalized forces $\mathbf{F}_{\mathbf{r}_q}$, $\boldsymbol{\tau}_\Phi$, and $\boldsymbol{\tau}$ remain the same as defined in Section 2.2.1; however, a generalized force vector for the generalized coordinates $\boldsymbol{\eta}$ should be added. Following the principal of virtual work, similar to Section 2.2.1, one can obviously find that:

$$\mathbf{F}_\boldsymbol{\eta} = \mathbf{0}_{2 \times 1} \quad (2.37)$$

where $\mathbf{0}$ is a null vector (or matrix) with the dimensions indicated in its subscript.

The translational dynamics and payload vibrations of a quadrotor with a slung payload can be expressed as:

$$(m_q + m_p) \ddot{\mathbf{r}}_q + m_p l \mathbf{A} \ddot{\boldsymbol{\eta}} + m_p l \mathbf{B} \dot{\boldsymbol{\eta}} + (m_q + m_p) g \mathbf{e} = \mathbf{F} \quad (2.38)$$

$$m_p l \mathbf{A}^\top \ddot{\mathbf{r}}_q + m_p l^2 \mathbf{A}^\top \mathbf{A} \ddot{\boldsymbol{\eta}} + m_p l^2 \mathbf{A}^\top \mathbf{B} \dot{\boldsymbol{\eta}} + m_p g l \mathbf{A}^\top \mathbf{e} = \mathbf{0}_{2 \times 1} \quad (2.39)$$

which can also be combined in one matrix equation as follows:

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} + \mathbf{G} = \mathbf{Q} \quad (2.40)$$

where

$$\mathbf{M} = \begin{bmatrix} (m_q + m_p) \mathbf{I}_{3 \times 3} & m_p l \mathbf{A} \\ m_p l \mathbf{A}^\top & m_p l^2 \mathbf{A}^\top \mathbf{A} \end{bmatrix} \in \mathbb{R}^{5 \times 5} \quad (2.41)$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{0}_{3 \times 3} & m_p l \mathbf{B} \\ \mathbf{0}_{2 \times 3} & m_p l^2 \mathbf{A}^\top \mathbf{B} \end{bmatrix} \in \mathbb{R}^{5 \times 5} \quad (2.42)$$

$$\mathbf{G} = \begin{bmatrix} (m_q + m_p) g \mathbf{e} \\ m_p g l \mathbf{A}^\top \mathbf{e} \end{bmatrix} \in \mathbb{R}^5 \quad (2.43)$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0}_{2 \times 1} \end{bmatrix} \in \mathbb{R}^5 \quad (2.44)$$

Recall that the rotational dynamics remains the same as expressed in Section 2.2.1. It is worth mentioning that this dynamics model agrees with what exists in the literature, e.g., [83].

2.3 Energy Modeling

As shown in the Section 1.2, many energy models have been developed for quadrotors. However, these models depend on quadrotor-related parameters, e.g., propeller and motor parameters. This has some drawbacks and adds limitations as discussed in Section 1.2. Therefore, a new energy model that eliminates quadrotor-related parameters and facilitates the energy analysis process is developed in this section.

2.3.1 Power and Energy Quotients

As discussed in [49, 72], and based on the aerodynamics principles, the propeller force, F_i , can be calculated as:

$$F_i = C_F \rho_a \Omega^2 D^4 \quad (2.45)$$

where C_F is the thrust coefficient, ρ_a is the air density, Ω is the angular velocity of the propeller, and D is the diameter of the propeller. Also, the power of a propeller, P_i , can be expressed as:

$$P_i = C_P \rho_a \Omega^3 D^5 \quad (2.46)$$

where C_P is the power coefficient. From Eqs. (2.45) and (2.46), propeller's power can be expressed in terms of the propeller's force as follows:

$$P_i = \frac{C_P}{C_F^{1.5} \rho_a^{0.5} D} F_i^{1.5} \quad (2.47)$$

which can be rewritten as:

$$P_i = \alpha F_i^{1.5} \quad (2.48)$$

where $\alpha = \frac{C_P}{C_F^{1.5} \rho_a^{0.5} D} > 0$ is a power-thrust constant depends on air density, and motor and propeller parameters. This relation has been validated experimentally in [49].

Therefore, the total power, P , can be expressed as follows:

$$P = \alpha \sum_{i=1}^4 F_i^{1.5} \quad (2.49)$$

Consequently, quadrotor's energy consumption can be expressed as follows:

$$E = \int_{t_0}^{t_f} P dt = \alpha \int_{t_0}^{t_f} \sum_{i=1}^4 F_i^{1.5} dt \quad (2.50)$$

where t_0 and t_f are the initial and final times, respectively. Both α and F_i depend on motor and propeller properties, so they need to be eliminated.

First, to eliminate propeller forces, F_i , the force ratio, β , is introduced as follows:

$$\beta = \frac{F^{1.5}}{\sum_{i=1}^4 F_i^{1.5}} \quad (2.51)$$

where $F = \|\mathbf{F}\|$ is the total thrust, which can be found from dynamics model without knowing any quadrotor-related parameters.

From Eqs. (2.49) and (2.51), quadrotor's power consumption can be expressed in terms of the thrust F and $\gamma = \alpha/\beta$ as:

$$P = \gamma F^{1.5} \quad (2.52)$$

Substituting in Eq. (2.50), the energy consumption over the course of flight can be found as:

$$E = \int_{t_0}^{t_f} \gamma F^{1.5} dt \quad (2.53)$$

Second, to eliminate α , which is already included in γ , the concept of power and energy quotients, P_Q and E_Q , is introduced as follows:

$$\begin{aligned} P_Q &= \frac{P}{\gamma} = F^{1.5} \\ E_Q &= \frac{E}{\gamma} = \int_{t_0}^{t_f} F^{1.5} dt \end{aligned} \quad (2.54)$$

The proposed power and energy quotients in Eq. (2.54) are proportional to the actual power and energy consumption; however, they can be computed without knowing any of the motor and propeller parameters. However, Eq. (2.54) is valid and accurate when γ is constant. Since α is constant as indicated in [49] as long as the air density does not vary significantly, e.g., no significant change in quadrotor's altitude, the focus becomes on β , which relates F_i to F as presented in Eq. (2.51). This is discussed in the following subsection.

2.3.2 Force Ratio Analysis

Mathematically, $F^n > \sum_{i=1}^4 F_i^n \forall n > 1$. So, in this case $\beta > 1$. Additionally, β is not constant and its value depends on the values of F_i . Relating all propeller forces to the first propeller force F_1 results in $F_2 = h_2 F_1$, $F_3 = h_3 F_1$ and $F_4 = h_4 F_1$. Therefore, β can be expressed as:

$$\beta = \frac{(1 + h_2 + h_3 + h_4)^{1.5}}{1 + h_2^{1.5} + h_3^{1.5} + h_4^{1.5}} \quad (2.55)$$

where $h_i = F_i/F_1$ is the propeller force ratio and $i = \{2, 3, 4\}$.

At hovering, all propeller forces are equal, so $\beta = 2$. For aggressive flights, i.e., large differences between propeller forces F_i , the value of β will be significantly different from

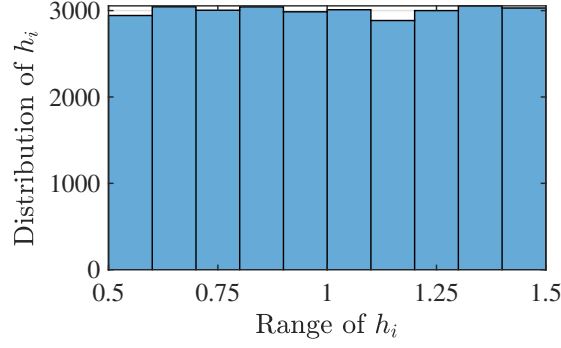


Figure 2.2: Distribution of propeller force ratios h_i in the sample

$\beta = 2$. However, for non-aggressive flights, where the differences between propeller forces are considerably small, the force ratio β can be considered constant and close to $\beta = 2$.

This dissertation defines non-aggressive flights to be flights with the ratio between the maximum propeller force, $F_{i_{max}}$, to the minimum propeller force, $F_{j_{min}}$, being in the range of $\mu \leq 2$, where $\mu = F_{i_{max}}/F_{j_{min}}$ and $i \neq j$. However, in the current analysis, a more conservative case is considered and non-aggressive flights are defined to be flights with $\mu \leq 3$. That is, $F_{i_{max}}$ is allowed to be triple $F_{j_{min}}$ at extreme cases, e.g., $0.5 \leq h_i \leq 1.5$.

To better understand the behaviour of β , 10,000 simulations are run using a sample of 30,000 arbitrary values of propeller force ratios in the range of $0.5 \leq h_i \leq 1.5$. The distribution of the arbitrary values of h_i in the sample is shown in Fig. 2.2, which shows that all arbitrary values of h_i in the range of interest are represented in the sample almost equally.

At each simulation run, three values of h_i are arbitrarily picked from the sample to calculate β from Eq. (2.55). Fig. 2.3a shows the distribution of β over the 10,000 runs. The results show that about 75% of the 10,000 runs has $1.95 \leq \beta \leq 2$ and about 24% of the 10,000 runs has $1.9 \leq \beta \leq 1.95$. That is about 99% of the 10,000 runs in the range of $1.9 \leq \beta \leq 2$. Note that these results include the extreme cases when $F_{i_{max}}$ is triple, or

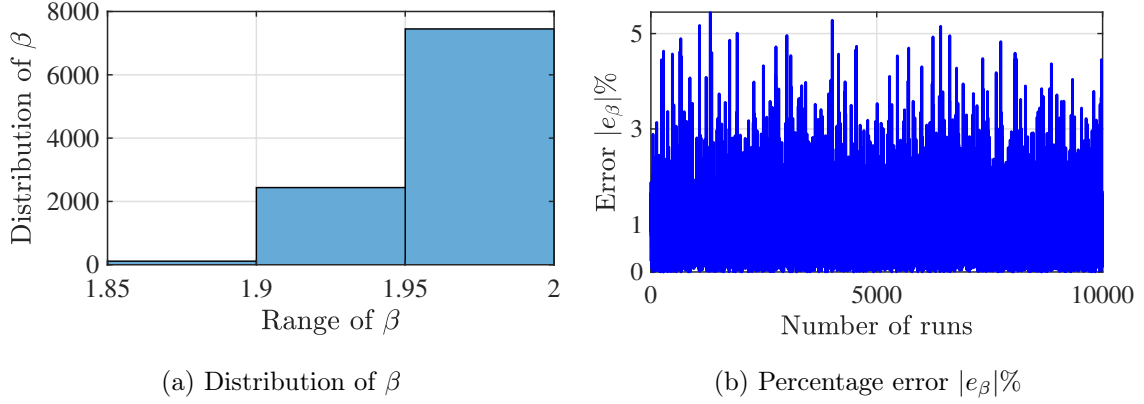


Figure 2.3: Statistical analysis of force ratio β

close to triple, F_{jmin} , i.e., $\mu \approx 3$. Despite considering these extreme cases, the results in Fig. 2.3a show a mean of $\beta_m \approx 1.964$ with a standard deviation of $\sigma \approx 0.023$. If the force ratio, β , is considered constant with a value of $\beta = \beta_m$, the absolute percentage error of the force ratio, $|e_\beta|/\%$, throughout most of the 10,000 runs will be less than 3% as shown in Fig.2.3b. Some error spikes up to about 5.5% are found, which represent the extreme cases, i.e., $\mu \approx 3$.

Eq. (2.55) and Fig. 2.3 show mathematically and statistically that the force ratio β can be considered constant for non-aggressive flights. This assumption is practically accepted as shown by experimental results, for example in [49]. Since this dissertation considers non-aggressive flights, the assumption of constant β is made throughout this dissertation. This assumption leads to considering γ as a constant.

It is worth mentioning that, although Eqs. (2.52) and (2.53) are expressed as functions of the thrust, F , which is calculated from the translational dynamics as shown in Section 2.2, they consider both translational and rotational dynamics. That is because the rotational dynamics are considered in β , which relates the thrust, F , to propeller forces, F_i .

Table 2.1: QDrone’s physical and electrical parameters

Physical Parameters			
L_r	Roll motor-to-motor distance	0.2136	m
L_p	Pitch motor-to-motor distance	0.1758	m
m_q	QDrone mass (with battery)	1.121	kg
I_x	Roll Moment of Inertia	0.01	kg m ²
I_y	Pitch Moment of Inertia	0.0082	kg m ²
I_z	Yaw Moment of Inertia	0.0148	kg m ²
Electrical Parameters			
K_{eff}	Effective motor speed constant	1295.4	rpm/V
ω_c	Voltage to angular velocity offset	2132.6	rpm
ω_f	Angular velocity to force offset	1004.5	rpm
C_t	Motor force constant	2.0784×10^{-8}	N/rpm ²
F_b	Motor force offset	-0.2046	N
k_τ	Motor thrust-torque constant	81.0363	N/Nm

2.4 Experimental Setup

All experiments have been conducted at York University’s Autonomous Unmanned Vehicle (YU-AUV) facility, associated with the Spacecraft Dynamics, Control and Navigation Laboratory (SDCNLab). QDrones and the indoor OptiTrack system were used to validate the results and concepts presented in this dissertation. Some experiments involved a single QDrone, whereas others involved a QDrone with a slung payload. The following subsections provide more details about this experimental setup.

Qdrone, shown in Fig. 2.4, is a quadrotor manufactured by Quanser equipped with many devices and sensors as listed in [1]. In the context of this dissertation, the following devices and sensors are important:

- Onboard Intel[®] Aero Compute Board with an Intel Atom[®] x7-Z8750 Quad-core 64-bit 2.56 GHz processor and 4 GB LPDDR3 – 1600 RAM.
- Intel[®] RealSense[™] (R200) RGBD forward camera. that provides vision with either



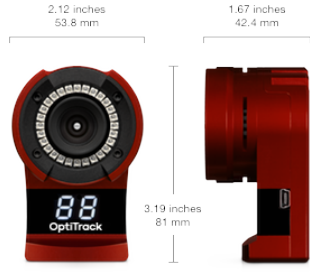
Figure 2.4: QDrone quadrotor [1]

640×480 at 60 fps or 1080p at 30 fps and depth sensing in the range of $0.5 - 3.5$ m.

- Omnivision OV7251 downward optical flow grayscale camera. that provides vision with 640×480 at 120 fps.
- BMI160 IMU with a 6-DOF 16-bit triaxial accelerometer and gyroscope.
- BMM150 Magnetometer with a 3-axis geomagnetic sensor.
- MS5611 Barometer with a 24-bit pressure and temperature sensor.
- IEEE 802.11 b,g,n,ac - Intel Dual Band Wireless - AC 8620 2×2 MIMO for WiFi.
- 3S 11.1V LiPo (3300 mAh) battery.

Additionally, QDrones also have some physical and electrical parameters that are shown in Table 2.1 [1]. It is worth mentioning that the electrical parameters in Table 2.1 are estimated based on a nonlinear model of the QDrone.

The YU-AUV lab is equipped with 16 Flex 13 cameras to capture the motion of objects inside the flight space. Figs. 2.5a and 2.5b show one of the Flex 13 cameras and the flight space with the OptiTrack cameras, respectively.

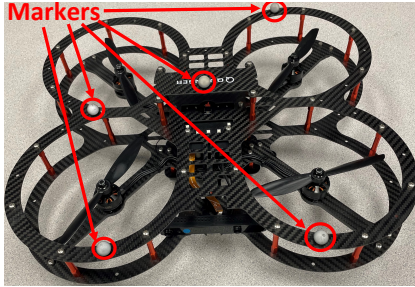


(a) Flex 13 camera [100]



(b) Flight space

Figure 2.5: OptiTrack system in the flight space at YU-AUV



(a) QDrone with five markers



(b) Blue payload



(c) Black payload

Figure 2.6: QDrone and different payloads with markers

In order to capture objects' motions by the OptiTrack system, markers need to be attached to these objects. When motions of multiple objects to be tracked in a single experiment, markers with different patterns must be assigned to different objects, e.g., the QDrone and the payload. Fig. 2.6 shows a QDrone with five markers and different payloads used in this dissertations.

The desired trajectory is generated on the workstation. The controller on the work station computes the error between the desired states and the actual states feedbacked from the OptiTrack system, then generate control commands. The feedback is sent to the workstation via USB cables, whereas, the control commands are sent to the QDrone via WiFi. The workstation is equipped with MATLAB, SIMULINK and QUARC to handle

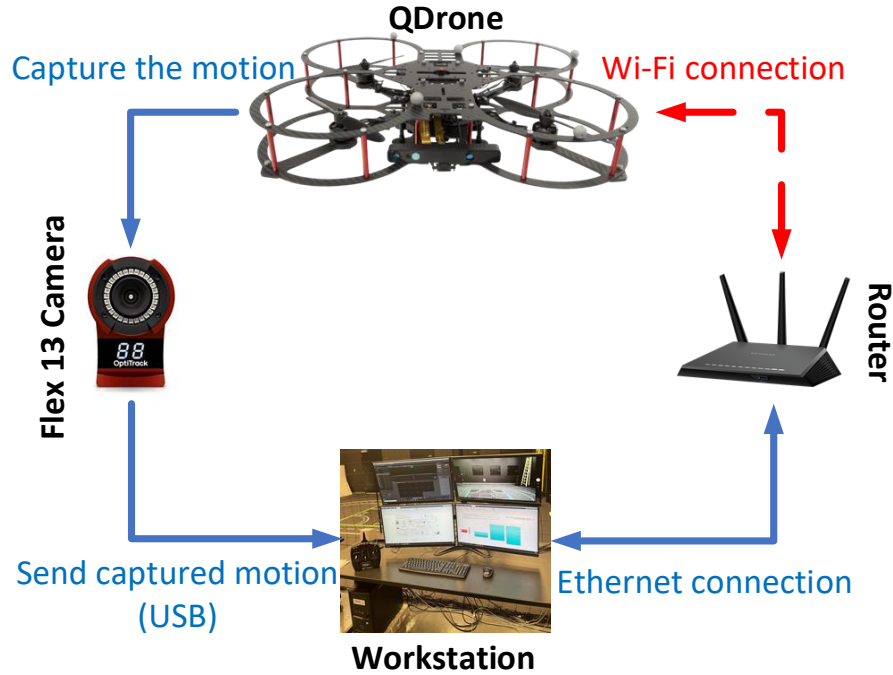


Figure 2.7: Communication chart

this communication process. Fig. 2.7 shows the communication chart used in this dissertation.

Some experiments are conducted in the presence of disturbances. For this purpose, a Lasko box fan model 3733C was used to generate wind disturbances. The fan size is 20 inches with voltage and current of 120 V and 0.8 A, respectively (see Fig. 2.8).

To control a quadrotor, a cascade control structure is considered. This control structure consists of two control loops: a) an outer Proportional-Integral-Velocity (PIV) control loop for position control (relatively slow); b) an inner cascade Proportional-Velocity (PV) control loop for attitude control (relatively fast). This control structure is designed by the manufacturer, and is used in this dissertation without change, unless otherwise stated.

The main difference between PIV controllers and conventional Proportional-Integral-Derivative (PID) controllers comes from the definition of the error. In the PID, there is

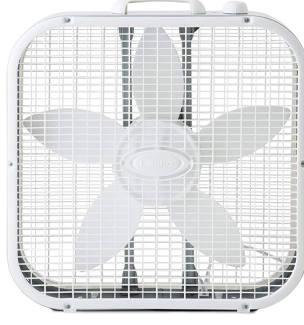


Figure 2.8: Lasko box fan model 3733C to generate wind disturbances [2]

only a position error $\mathbf{e}_p$. In the PIV, however, there are position and velocity errors, $\mathbf{e}_p$ and $\mathbf{e}_v$. The velocity error $\mathbf{e}_v$ is calculated as the difference between the desired velocity and the actual velocity (not the time derivative of $\mathbf{e}_p$). This reduces the effect of noisy measurements; therefore, PIV controllers, generally, perform better than PID controllers.

The inputs of the outer loop are quadrotor's desired position, velocity, yaw angle, and yaw rate, i.e., $\mathbf{r}_{qd}$, $\dot{\mathbf{r}}_{qd}$, ψ_d , and $\dot{\psi}_d$, where the subscript d denotes a desired value. The outputs of the outer loop are the thrust and the commanded roll and pitch angles, and yaw rate, i.e., F , ϕ_c , θ_c , and $\dot{\psi}_c$. The inputs of the inner loop are ϕ_c , θ_c , and $\dot{\psi}_c$ in addition to the angular velocity and acceleration, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$. The output of the inner loop is the moment vector, $\boldsymbol{\tau}$. This control structure is illustrated in Fig. 2.9.

The position controller, i.e., outer loop PIV controller, is designed as follows:

$$\begin{bmatrix} F \\ \phi_c \\ \theta_c \\ \dot{\psi}_c \end{bmatrix} = \mathbf{K}_p \mathbf{e}_p + \mathbf{K}_v \mathbf{e}_v + \mathbf{K}_i \mathbf{e}_i \quad (2.56)$$

where the control gains matrices are defined as follows: $\mathbf{K}_p = \text{diag}(\frac{\pi}{2}, \frac{\pi}{2}, 35, 15) \in \mathbb{R}^{4 \times 4}$, $\mathbf{K}_v = \text{diag}(\frac{\pi}{4}, \frac{\pi}{4}, 14, 2) \in \mathbb{R}^{4 \times 4}$, and $\mathbf{K}_i = \text{diag}(0, 0, 4, 0) \in \mathbb{R}^{4 \times 4}$. The function $\text{diag} : \mathbb{R}^j \mapsto \mathbb{R}^{j \times j}$ returns a diagonal matrix with diagonal elements being the elements of its argument.

The errors can be defined as $\mathbf{e}_p = \begin{bmatrix} \mathbf{r}_{qd} - \mathbf{r}_q \\ \psi_d - \psi \end{bmatrix} \in \mathbb{R}^4$, $\mathbf{e}_v = \begin{bmatrix} \dot{\mathbf{r}}_{qd} - \dot{\mathbf{r}}_q \\ \dot{\psi}_d - \dot{\psi} \end{bmatrix} \in \mathbb{R}^4$, and $\dot{\mathbf{e}}_i = \mathbf{e}_p$. A

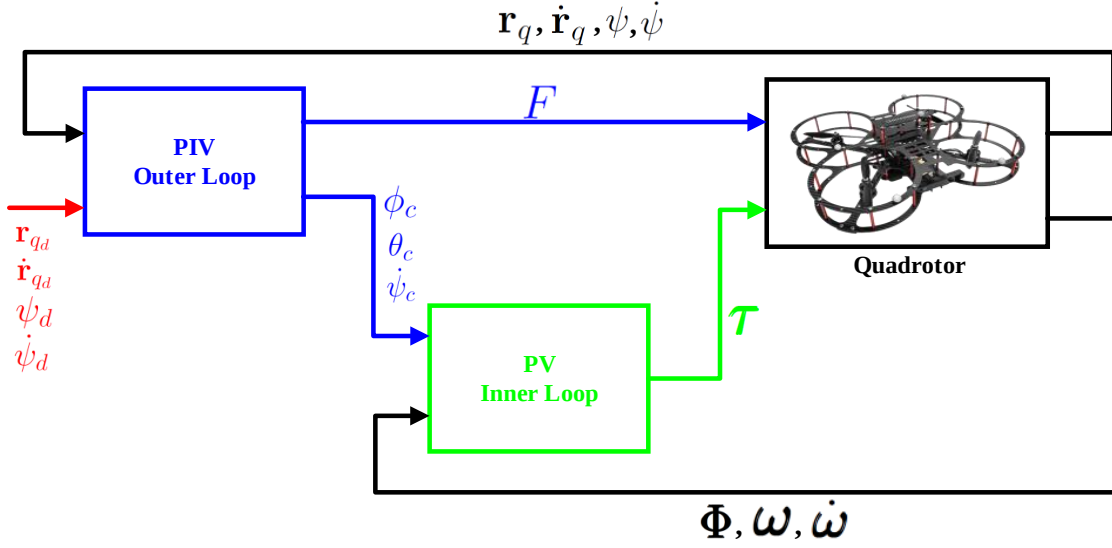


Figure 2.9: Control structure

block diagram of this PIV controller is shown in Fig. 2.10.

The attitude controller, i.e., inner loop cascade PV controller, is designed with the assumption that the desired angular velocity and acceleration, ω_d and $\dot{\omega}_d$, are zeros except for the yaw angular velocity. Desired ω_d and $\dot{\omega}_d$ can be considered easily using the same control structure; however, for non-aggressive flights it is acceptable to assume that ω_d and $\dot{\omega}_d$ are zeros. Therefore, the inner loop controller is designed as follows (see Fig. 2.11):

$$\tau = \mathbf{K}_{p\tau} \left(\mathbf{K}_{p\phi,\theta} \mathbf{e}_{p\phi,\theta} + \mathbf{K}_{v\phi,\theta} \underbrace{\mathbf{e}_{v\phi,\theta}}_{=-\omega} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_c \end{bmatrix} \right) + \mathbf{K}_{a\tau} \underbrace{\mathbf{e}_{a\tau}}_{=-\dot{\omega}} \quad (2.57)$$

where $\mathbf{K}_{p\tau} = \text{diag}(0.1876, 0.1544, 0.0395) \in \mathbb{R}^{3 \times 3}$, $\mathbf{K}_{a\tau} = \text{diag}(0.0032, 0.0026, 0) \in \mathbb{R}^{3 \times 3}$, $\mathbf{K}_{p\phi,\theta} = \text{diag}(12, 12, 1) \in \mathbb{R}^{3 \times 3}$, $\mathbf{K}_{v\phi,\theta} = \text{diag}(0.1, 0.1, 0) \in \mathbb{R}^{3 \times 3}$, $\mathbf{e}_{p\phi,\theta} = \begin{bmatrix} \phi_c - \phi \\ \theta_c - \theta \\ 0 \end{bmatrix} \in \mathbb{R}^3$, $\mathbf{e}_{v\phi,\theta} = \underbrace{\omega_d}_{=0_{3 \times 1}} - \omega = -\omega \in \mathbb{R}^3$, $\mathbf{e}_{a\phi,\theta} = \underbrace{\dot{\omega}_d}_{=0_{3 \times 1}} - \dot{\omega} = -\dot{\omega} \in \mathbb{R}^3$.

Finally, it is worth mentioning that when mentioning this controller in the dissertation for a quadrotor with a slung payload, payload measurements are not used in the control,

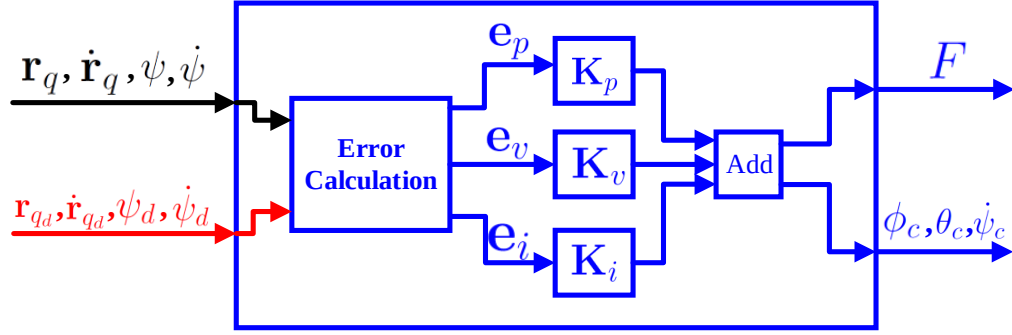


Figure 2.10: Position controller (outer loop PIV controller)

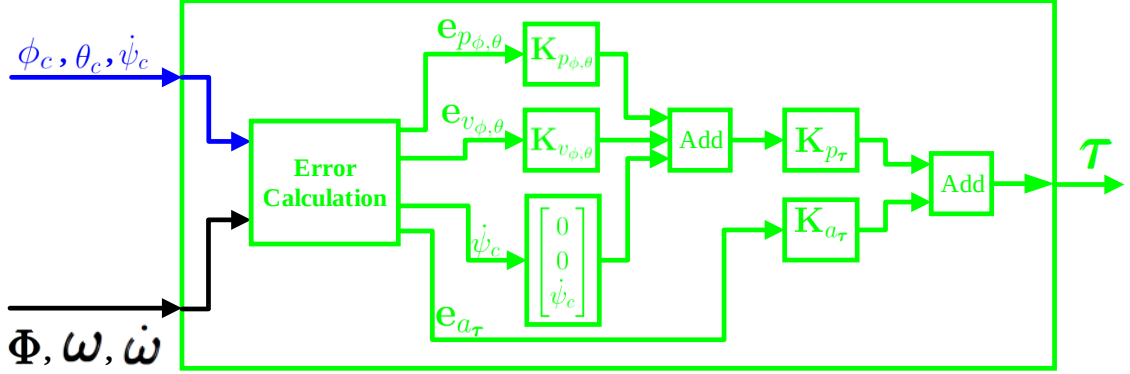


Figure 2.11: Attitude controller (inner loop cascade PV controller)

i.e., they are not feedbacked. However, they are used in the energy and payload vibration analysis.

2.5 Summary

In this chapter, the kinematics and dynamics model for both a single quadrotor and a quadrotor with a slung payload have been derived. Then, a novel energy model is developed in the form of power and energy quotients, which do not depend on quadrotor-related pa-

rameters, e.g., motor and propeller parameters. These quotients have been mathematically proved to be proportional to the actual power and energy consumption for non-aggressive flights. Lastly, the experimental setup and the main controller used in this dissertation has been presented.

3 Trajectory Design

Trajectory design is a key step in any successful UAV mission. As presented in Section 1.2, the literature covers many methods to design trajectories. Trajectories themselves can be categorized in many different ways. As shown in Section 1.2, this dissertation categorizes trajectories as: (a) waypoint trajectories; and (b) continuous-path trajectories [101].

If the path between the waypoints of a waypoint trajectory is a polynomial function, such a trajectory is called a waypoint polynomial trajectory or a polynomial trajectory for short. Polynomial trajectories are widely-used in practice because they are smooth and easy to implement, e.g., [102, 103]. They are also used to reduce the cutting tool vibration in computer numerical control (CNC) machines [104] and reduce the cycle time for the laser drilling [105]. The MJT, as one of the polynomial trajectory, was found to be a approximation of human arm manipulation motions [106].

Polynomial trajectories allow for a huge flexibility in design. Therefore, they are widely used in the field of quadrotor applications and mechatronics in general, e.g., [88, 93, 107–111]. In [110], a polynomial trajectory that minimized the pop, i.e., the 6th time derivative of the position was generated. A minimum time trajectory was designed, while considering reducing the swing motion in [112]. In [113], a real-time predictive control trajectory considering the shortest route with reduced payload vibration and obstacle avoidance was proposed. The differential flatness of the quadrotor was used in trajectory generation, e.g.,

in [88, 114, 115].

On the other hand, continuous-path trajectories impose limitations on adding kinematic constraints to the trajectory other than the ones defined by standard equations of the trajectory. Standard equations are the mathematical parametric equations that define a trajectory. For example, a circular trajectory in 2D can have a standard equation in the form of:

$$\mathbf{r} = [\rho \cos(\omega t), \rho \sin(\omega t)]^\top \in \mathbb{R}^2 \quad (3.1)$$

where ρ is the radius of the circle, ω is its angular speed, and t is the time. Such a standard equation adds limitations on setting some kinematic constraints to this circular trajectory. For example, a rest-to-rest circular trajectory cannot be generated since the velocity and/or the acceleration cannot be set to zero at any point on the trajectory. So, a control command may be applied to start/stop the motion, which may not be desirable at high speeds and/or large inertia. In fact, this rest-to-rest example is a special case of a larger problem with continuous-path trajectories. That is, generating continuous-path trajectories with desired arbitrary kinematic constraints, e.g., velocities and accelerations. Also, ω is typically constant for circular trajectories; however, if its profile is complex, e.g., piecewise complex function, generating such a circular trajectory becomes difficult. The following sections propose a generic framework to design trajectories to overcome these limitations, then the proposed approach is validated by both simulations and experiments.

3.1 Polynomial Trajectory

As mentioned in Section 1.2, polynomial trajectories form a special family of waypoint trajectories. Trajectories with n trajectory segments, i.e., $(n + 1)$ waypoints, are said to be splines [116]. A general spline is shown in Fig. 3.1.

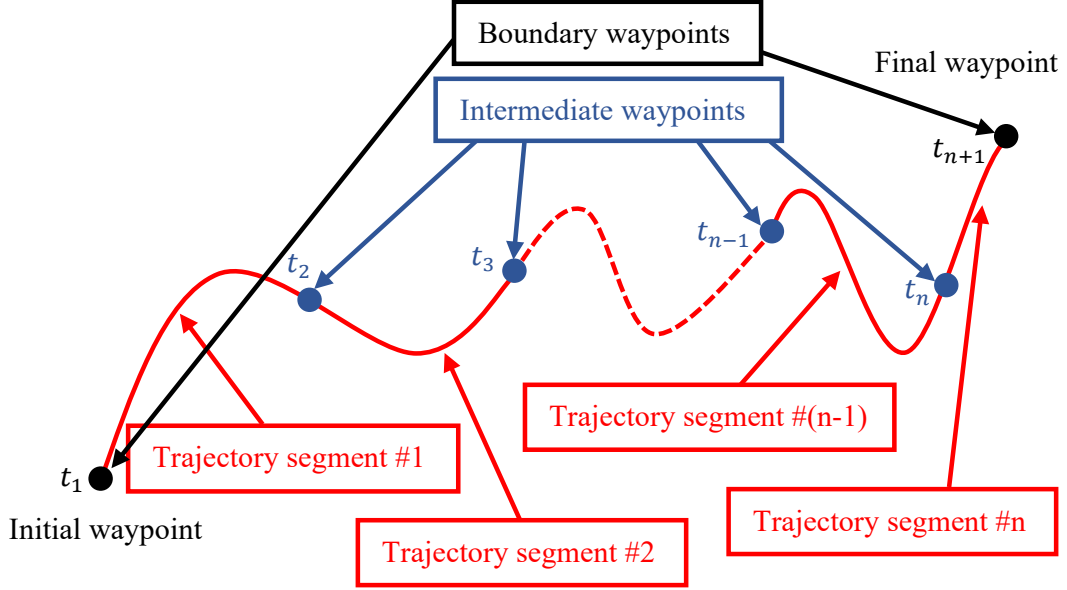


Figure 3.1: A general spline

First, a polynomial trajectory of degree j in d -dimensional space between two waypoints is defined as:

$$\mathbf{r} = \mathbf{c}\mathbf{t} \in \mathbb{R}^d \quad (3.2)$$

where $\mathbf{t} = [t^j, t^{j-1}, \dots, 1]^\top \in \mathbb{R}_{\geq 0}^{j+1}$ is the time vector and $\mathbf{c} \in \mathbb{R}^{d \times (j+1)}$ is the coefficient matrix.

Generalizing Eq. (3.2) to a multi-segment trajectory with n segments, i.e., a spline, results in the following mathematical definition:

$$\mathbf{r} = \begin{cases} \mathbf{c}_1 \mathbf{t}, & t_1 \leq t \leq t_2, \\ \mathbf{c}_2 \mathbf{t}, & t_2 < t \leq t_3, \\ \mathbf{c}_3 \mathbf{t}, & t_3 < t \leq t_4, \\ \vdots & \\ \mathbf{c}_n \mathbf{t}, & t_n < t \leq t_{n+1} \end{cases} \quad (3.3)$$

where $\mathbf{t}_w = [t_1, t_2, \dots, t_{n+1}]^\top \in \mathbb{R}_{\geq 0}^{n+1}$ is the waypoints time vector, which includes the times of the $(n+1)$ waypoints, and $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n \in \mathbb{R}^{d \times (j+1)}$ are the coefficient matrices for each trajectory segment. These coefficient matrices can be stacked into a single matrix as follows:

$$\mathfrak{C} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n] \in \mathbb{R}^{d \times n(j+1)} \quad (3.4)$$

then, a coefficient vector $\mathcal{C} \in \mathbb{R}^{nd(j+1)}$ can be defined as the vectorization of $\mathfrak{C}$ as follows:

$$\mathcal{C} = \text{vec}(\mathfrak{C}) \quad (3.5)$$

where $\text{vec}(\cdot)$ is a vectorization function that converts a matrix into a column vector as follows:

$$\text{vec}(\mathfrak{C}) = \left[\mathfrak{C}_{1,1}, \mathfrak{C}_{2,1}, \dots, \mathfrak{C}_{d,1}, \mathfrak{C}_{1,2}, \dots, \mathfrak{C}_{d,2}, \dots, \mathfrak{C}_{1,n(j+1)}, \mathfrak{C}_{2,n(j+1)}, \dots, \mathfrak{C}_{d,n(j+1)} \right]^\top \quad (3.6)$$

Computationally, the spline in Eq. (3.3) can be computed by specifying the coefficient vector $\mathcal{C}$ as follows:

$$\mathcal{C} = \mathbf{T}^{-1} \mathbf{S}_k \quad (3.7)$$

where $\mathbf{T} \in \mathbb{R}^{nd(j+1) \times nd(j+1)}$ is the time matrix and $\mathbf{S}_k \in \mathbb{R}^{nd(j+1)}$ is the kinematic constraints vector, i.e., the desired kinematic conditions at the waypoints. Specifying $\mathbf{T}$ and $\mathbf{S}_k$ depends on the required smoothness of polynomial trajectory, i.e., the degree of the polynomial and the continuity of the spline. Therefore, we introduce these concepts first, then the proposed approach to mathematically and systematically specify $\mathbf{T}$ and $\mathbf{S}_k$ is presented.

The polynomial trajectory defined in Eq. (3.2) can be designed to minimize a kinematic quantities of interest, e.g., minimizing the acceleration or any other time derivative of the position. One can define the functional $\mathcal{K} \in \mathbb{R}^d$ as follows:

$$\mathcal{K} = \left[\frac{d^k \mathbf{r}}{dt^k} \right]^{\circ 2} \quad (3.8)$$

where k is the order of the time derivative of the kinematic quantity that is required to be minimized by the trajectory, e.g., $k = 3$ for a minimum jerk trajectory, and $[\cdot]^{\circ 2}$ is a function returns the Hadamard square power of its argument, i.e., element-wise square. Therefore, the polynomial trajectory that can minimize this kinematic quantity can be expressed as follows [110]:

$$\mathbf{r} = \arg \min_{\mathbf{r}} \int_0^t \mathcal{K} dt \quad (3.9)$$

The Euler–Lagrange solution to the optimization problem in Eq. (3.9) can be obtained as follows [110]:

$$\text{di} \left(\frac{\partial \mathcal{K}}{\partial \mathbf{r}} \right) - \text{di} \left(\frac{d}{dt} \left(\frac{\partial \mathcal{K}}{\partial \dot{\mathbf{r}}} \right) \right) + \cdots + (-1)^k \text{di} \left(\frac{d^k}{dt^k} \left(\frac{\partial \mathcal{K}}{\partial \mathbf{r}^{(k)}} \right) \right) = \mathbf{0}_{d \times 1} \quad (3.10)$$

where, $\mathbf{r}^{(k)}$ is the k th time derivative of $\mathbf{r}$ and $\text{di} : \mathbb{R}^{d \times d} \mapsto \mathbb{R}^d$ is a function returns the diagonal elements of its argument.

Mathematically, the solution to Eq. (3.10) is always a polynomial trajectory of odd degree, j , where $j = 2k - 1$ with k being defined as in Eq. (3.8) [88, 110]. Therefore, this dissertation focuses on polynomial trajectories of odd degrees.

An important definition associated with splines is the continuity of a spline, which describes how smoothly spline segments join each other at intermediate waypoints. The continuity of a spline is said to be C^p if all spline segments' derivatives up to the p th derivative match across the spline, where $p \leq (j - 1)$. Obviously, increasing C^p increases the smoothness of the trajectory, e.g., [88, 110]. However, increasing C^p limits the capability of setting more kinematic constraints at intermediate waypoints, i.e., it may not possible to set desired velocities, accelerations or any other kinematic constraints at the intermediate waypoints except for the position. On the other hand, decreasing C^p allows for setting more

desired kinematic constraints at intermediate waypoints; however, trajectory smoothness decreases. For example, consider a spline in 1D of a degree j with three waypoints. The unknown coefficient vector $\mathcal{C} \in \mathbb{R}^{2(j+1)}$ requires $2(j+1)$ equations to be solved. The boundary conditions form $(j+1)$ of the required equations, i.e., the boundary positions and their time derivatives up to the $\left(\frac{j-1}{2}\right)$ th time derivative. The rest is $(j+1)$ equations. A spline with a continuity $C^{(j-1)}$ means that the rest of the equations are as follows:

$$\begin{aligned}
r_{2_1} &= r_{2_2} = r_{2_{des}} \\
\frac{dr_{2_1}}{dt} &= \frac{dr_{2_2}}{dt} \\
&\vdots \\
\frac{d^{(j-1)} r_{2_1}}{dt^{(j-1)}} &= \frac{d^{(j-1)} r_{2_2}}{dt^{(j-1)}}
\end{aligned} \tag{3.11}$$

where the subscripts 2_1 and 2_2 denote the kinematic constraints of the intermediate waypoint (the second waypoint) from the first and second segments, respectively, and $r_{2_{des}}$ denotes the desired position of the intermediate waypoints (the second waypoint). On the other hand, considering a spline of continuity $C^{\left(\frac{j-1}{2}\right)}$ means that the rest of the equations are as follows:

$$\begin{aligned}
r_{2_1} &= r_{2_2} = r_{2_{des}} \\
\frac{dr_{2_1}}{dt} &= \frac{dr_{2_2}}{dt} = \left(\frac{dr_2}{dt}\right)_{des} \\
&\vdots \\
\frac{d^{\left(\frac{j-1}{2}\right)} r_{2_1}}{dt^{\left(\frac{j-1}{2}\right)}} &= \frac{d^{\left(\frac{j-1}{2}\right)} r_{2_2}}{dt^{\left(\frac{j-1}{2}\right)}} = \left(\frac{d^{\left(\frac{j-1}{2}\right)} r_2}{dt^{\left(\frac{j-1}{2}\right)}}\right)_{des}
\end{aligned} \tag{3.12}$$

Therefore, choosing C^p requires always a trade off and this dissertation considers trajectories with continuity of $C^{\frac{j-1}{2}}$, which allows for setting kinematic constraints at intermediate waypoints up to the $\left(\frac{j-1}{2}\right)$ th time derivative as well as providing practically accepted level of smoothness.

For splines of odd degrees, j , with $C^{\frac{j-1}{2}}$, the matrices $\mathbf{T}$ and $\mathbf{S}_k$ in Eq. (3.7) can be computed systematically for any d and n as follows:

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}^{(1)} \\ \vdots \\ \mathbf{A}^{\left(\frac{j-1}{2}\right)} \end{bmatrix}, \quad \mathbf{A}^{(\cdot)} = \text{diag} \left(\mathbf{A}_k^{(\cdot)} \right), \quad k = \{1, 2, \dots, n\} \quad (3.13)$$

where the superscript of $\mathbf{A}^{(\cdot)}$ indicates the number of time derivatives, which can be expressed as:

$$\begin{aligned} \mathbf{A}_k &= \begin{bmatrix} t_k^j \mathbf{I}_d & t_k^{j-1} \mathbf{I}_d & \cdots & t_k \mathbf{I}_d & \mathbf{I}_d \\ t_{k+1}^j \mathbf{I}_d & t_{k+1}^{j-1} \mathbf{I}_d & \cdots & t_{k+1} \mathbf{I}_d & \mathbf{I}_d \end{bmatrix} \\ \mathbf{A}_k^{(1)} &= \begin{bmatrix} j t_k^{j-1} \mathbf{I}_d & (j-1) t_k^{j-2} \mathbf{I}_d & \cdots & \mathbf{I}_d & \mathbf{0}_{d \times d} \\ j t_{k+1}^{j-1} \mathbf{I}_d & (j-1) t_{k+1}^{j-2} \mathbf{I}_d & \cdots & \mathbf{I}_d & \mathbf{0}_{d \times d} \end{bmatrix} \\ \mathbf{A}_k^{\left(\frac{j-1}{2}\right)} &= \begin{bmatrix} j(j-1) \cdots \left(\frac{j+3}{2}\right) t_k^{\frac{j+1}{2}} \mathbf{I}_d & \cdots & \mathbf{0}_{d \times d} & \mathbf{0}_{d \times d} \\ j(j-1) \cdots \left(\frac{j+3}{2}\right) t_{k+1}^{\frac{j+1}{2}} \mathbf{I}_d & \cdots & \mathbf{0}_{d \times d} & \mathbf{0}_{d \times d} \end{bmatrix} \end{aligned} \quad (3.14)$$

where $\mathbf{I}_d \in \mathbb{R}^{d \times d}$ is the identity matrix and $\mathbf{0}_{d \times d}$ is a null matrix of dimension $d \times d$. The kinematic constraints vector $\mathbf{S}_k$ can be defined as:

$$\mathbf{S}_k = \text{vec} \left(\begin{bmatrix} \mathbf{r}_s, \mathbf{r}_s^{(1)}, \dots, \mathbf{r}_s^{\left(\frac{j-1}{2}\right)} \end{bmatrix} \right) \quad (3.15)$$

where

$$\begin{aligned} \mathbf{r}_s &= \left[\mathbf{r}_1^\top, \mathbf{r}_2^\top, \mathbf{r}_2^\top, \mathbf{r}_3^\top, \mathbf{r}_3^\top, \dots, \mathbf{r}_n^\top, \mathbf{r}_n^\top, \mathbf{r}_{n+1}^\top \right]^\top \\ \mathbf{r}_s^{(1)} &= \left[\mathbf{r}_1^{(1)\top}, \mathbf{r}_2^{(1)\top}, \mathbf{r}_2^{(1)\top}, \mathbf{r}_3^{(1)\top}, \mathbf{r}_3^{(1)\top}, \dots, \mathbf{r}_n^{(1)\top}, \mathbf{r}_n^{(1)\top}, \mathbf{r}_{n+1}^{(1)\top} \right]^\top \\ \mathbf{r}_s^{\left(\frac{j-1}{2}\right)} &= \left[\mathbf{r}_1^{\left(\frac{j-1}{2}\right)\top}, \mathbf{r}_2^{\left(\frac{j-1}{2}\right)\top}, \mathbf{r}_2^{\left(\frac{j-1}{2}\right)\top}, \mathbf{r}_3^{\left(\frac{j-1}{2}\right)\top}, \mathbf{r}_3^{\left(\frac{j-1}{2}\right)\top}, \dots, \mathbf{r}_n^{\left(\frac{j-1}{2}\right)\top}, \mathbf{r}_n^{\left(\frac{j-1}{2}\right)\top}, \mathbf{r}_{n+1}^{\left(\frac{j-1}{2}\right)\top} \right]^\top \end{aligned} \quad (3.16)$$

where the subscript indicates the number of the waypoint at which a kinematic constraint is set. Eqs. (3.15) and (3.16) provide a means for setting kinematic constraints independently and straightforwardly at any waypoint.

Kinematic properties of polynomial trajectories, e.g., velocity and acceleration profiles, have a significant effect on energy performance of the moving object, e.g., a quadrotor. Therefore, the relation between the degree of a polynomial trajectory, j , and its kinematics properties is investigated in this subsection.

Consider a rest-to-rest polynomial trajectory between two waypoints. The velocity profile of such polynomial trajectories is found to be [117]:

$$v = v_{max} (4T(1-T))^{\frac{j-1}{2}} \quad (3.17)$$

where $T = \frac{t}{t_f}$ is the relative time, t_f is the total time of the trajectory, and the peak velocity, v_{max} , can be expressed in terms of the average speed, v_{av} , as follows:

$$v_{max} = v_{av} \frac{1}{4 \binom{\frac{j-1}{2}}{\frac{j-1}{2}}} \frac{j!}{\left(\frac{j-1}{2}\right)!^2} \quad (3.18)$$

Taking the time derivative of Eq. (3.17) results in the following acceleration profile:

$$a = 4^{\frac{j-3}{2}} \left(\frac{j-1}{2}\right) \left(\frac{v_{max}}{t_f^{\frac{j-3}{2}}}\right) ((1-2T))^{\frac{j-3}{2}} \quad (3.19)$$

Eqs. (3.17) - (3.19) show that increasing the degree of the polynomial increases trajectory's velocity and acceleration. To graphically show these kinematic properties of polynomial trajectories, a parametric study is conducted considering odd degrees of polynomial $j = \{1, 3, 5, \dots, 100\}$. For fair comparison, the total distance and total time remain the same for all cases, i.e., v_{av} is constant. Fig. 3.2 shows that increasing the degree of the polynomial makes the position trajectory steeper and approaches a step function. Increasing the degree of the polynomial, also, increases the peak velocity and acceleration monotonically. Please note that the z -axes of the graphs in Fig. 3.2 are a scaled version of the actual position, r , velocity, v , and acceleration, a , where u is the scaling constant that depends on the total distance and time, with $u = v_{av}$. Therefore, the units are still

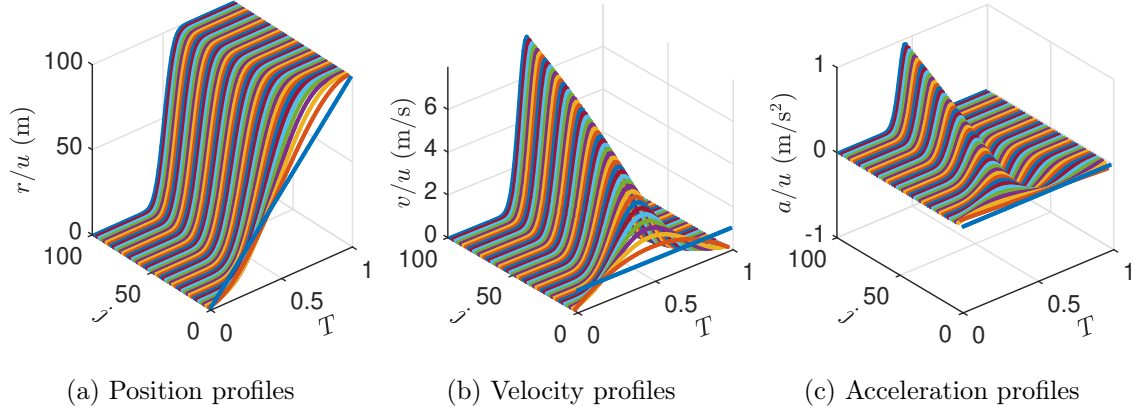


Figure 3.2: Position, velocity, and acceleration profiles for $j = \{1, 3, 5, \dots, 100\}$

m, m/s, and m/s^2 , respectively. Additionally, the accelerations in Fig. 3.2c have peaks and valleys because the velocity starts to decrease, i.e., deceleration motion, at $T = 0.5$.

Now, let us focus on one scenario from Fig. 3.2 to provide a detailed explanation. Consider a scenario of a motion between an initial position $\mathbf{r}_0 = [0, 0, 0]^T$ and a final position $\mathbf{r}_f = [1, 1, 1]^T$ with a total time $t_f = 3$ s. Also, consider three different polynomial trajectories: a minimum jerk trajectory (MJT) with $j = 5$, a minimum snap trajectory (MST) with $j = 7$, and a minimum crackle trajectory (MCT) with $j = 9$. Fig. 3.3 shows that increasing the degree of the polynomial trajectory leads the position profile to become steeper, and approaches the behavior of the step function as shown in the left graph of Fig. 3.3. The increase in the slope of the position profile, which is associated with the increase in polynomial degree, greatly affects the velocity profile. It increases the peak velocity and make the velocity profile narrower and steeper as shown in the middle graph in Fig. 3.3. Similarly, the peak value of the acceleration increases with the increase in the degree of the polynomial trajectory, and it reaches zero at the middle of the time span, which is corresponding to the turning point in the velocity profile.

Polynomial trajectories, generated using the framework presented in this section, allow

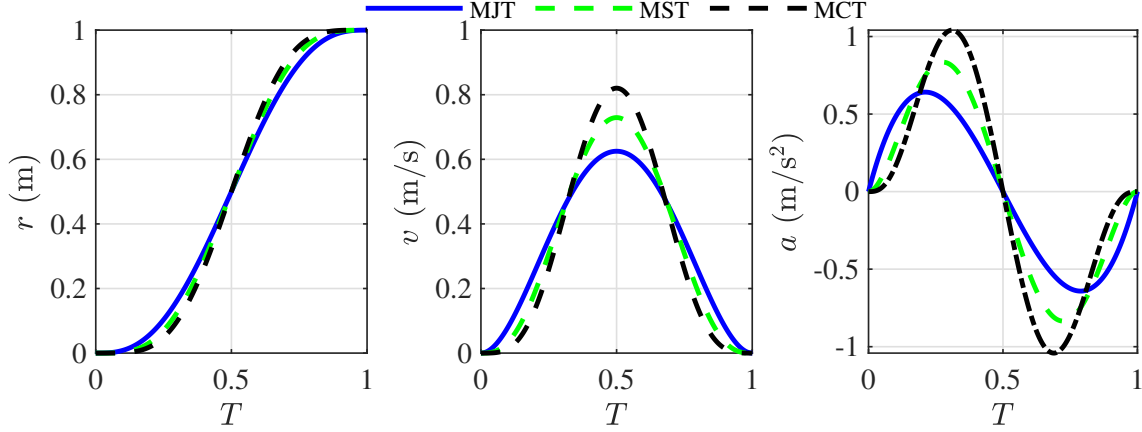


Figure 3.3: Position, velocity, and acceleration profiles for MJT, MST, and MCT

to set desired kinematic constraints at waypoints freely and straightforwardly as presented in Eqs. (3.15) and (3.16). Therefore, a simple, but effective, approach to address the limitations associated with setting kinematic constraints on continuous-path trajectories, e.g., the example presented in Eq. (3.1), is proposed by introducing the kinematically-constrained continuous-path (KCCP) polynomial trajectory.

The idea of the KCCP polynomial trajectory is to adopt a polynomial trajectory with a large number of waypoints to well-approximate the corresponding continuous-path trajectory. Then, implementing the proposed framework to set the desired kinematic constraints at all waypoints. This way, the accuracy of the path is achieved by the large number of waypoints, whereas the desired kinematic constraints are satisfied straightforwardly as shown in Eqs. (3.15) and (3.16).

3.2 Trajectory Feasibility

An important step in any trajectory design task is to check the feasibility of the designed trajectory. This includes two main checks: a) kinematic feasibility; and dynamics feasibility. A kinematically feasible trajectory is a trajectory that does not violate any kinematic

constraints, e.g., maximum angles and singularities, throughout the entire motion. A dynamically feasible trajectory, on the other hand, is a trajectory that has enough actuation to follow the desired trajectory, e.g., required torques and forces do not exceed the maximum limits of actuators. The following subsections investigate the feasibility of the proposed KCCP polynomial trajectories.

3.2.1 Kinematic Feasibility

Consider a general nonlinear dynamics systems defined as follows:

$$\dot{\mathbf{q}} = f(\mathbf{q}, \mathbf{u}) \quad (3.20)$$

where $\mathbf{q} \in \mathbb{R}^n$ is the states and $\mathbf{u} \in \mathbb{R}^m$ is the inputs. Such a system is said to be differentially flat if there exists an output $\mathbf{y} \in \mathbb{R}^m$ such that both the states and the inputs can be expressed as $\mathbf{q} = \mathbf{q}(\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \dots, \mathbf{y}^{(\nu)})$ and $\mathbf{u} = \mathbf{u}(\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \dots, \mathbf{y}^{(\nu)})$ with ν being a finite integer [118]. In other words, for a quadrotor, if the states $[\mathbf{r}_q, \Phi]^\top$ and the input $[F, \boldsymbol{\tau}]^\top$ can be expressed as function of some outputs and its finite time derivatives, the dynamics system of a quadrotor is said to be differentially flat. Similarly, one can apply the same definition to a quadrotor with a slung payload.

Quadrotors have been proven to be differentially flat with four flat outputs $\{\mathbf{r}_q, \psi\}$ [88, 118]. Additionally, quadrotors with slung payloads have been also proven to be differentially flat with four flat outputs $\{\mathbf{r}_p, \psi\}$ [110, 114]. Recall that $\mathbf{r}_p$ can be related to $\mathbf{r}_q$ as shown in Eq. (2.3). This flatness property has been even further extended to multiple quadrotors carrying a slung payload [119].

The differential flatness property guarantees that if the trajectory of the flat outputs is differentiable, it is kinematically feasible. Since polynomial trajectories are differentiable, they are kinematically feasible.

3.2.2 Dynamics Feasibility

Although polynomial trajectories are kinematically feasible, the generated trajectory may require thrust beyond the capabilities of quadrotor motors. For differentially flat systems, mapping actuators constraints into the flat output space, results in a non-convex problem, which is hard to solve [89,120]. For quadrotor applications, actuator saturation is a problem in aggressive flights. So, instead of dealing with difficult non-convex problems, a feasibility condition is proposed that is suitable for quadrotor applications with non-aggressive flights, e.g., package delivery.

Since the maximum thrust, F_{max} , is known for a certain quadrotor, then from Eq. (2.25) a dynamic feasibility condition is:

$$F_{max} \geq m_q \sqrt{\|\ddot{\mathbf{r}}_{q_{max}}\|^2 + g^2 + 2g\ddot{z}_{max}} \quad (3.21)$$

where $\|\ddot{\mathbf{r}}_{q_{max}}\|$ is the maximum acceleration and $\ddot{z}_{max}$ is the maximum acceleration in the inertial z -direction. If the condition in Eq. (3.21) is not satisfied, the generated thrust will not be able to drive the quadrotor on the desired trajectory according to the dynamics model in Eq. (2.25). In other words, Eq. (3.21) is a necessary dynamic feasibility condition.

If the condition in Eq. (3.21) is not satisfied for a segment in the polynomial trajectory, one can increase the segment flight time of this trajectory segment. Defining the average acceleration of a trajectory segment as the segment length divided by the square of the segment flight time leads to an estimate of a new segment flight time, $t_{s_{new}}$, from the infeasible old flight time, $t_{s_{old}}$, for the same segment distance as follows:

$$t_{s_{new}} = \sqrt{\frac{a_{in}}{a_{max}}} t_{s_{old}} \quad (3.22)$$

where a_{in} is the maximum acceleration of the infeasible trajectory and a_{max} is the maximum acceleration calculated from Eq. (3.21) by substituting with F_{max} , and $t_{s_{old}}$ is the old

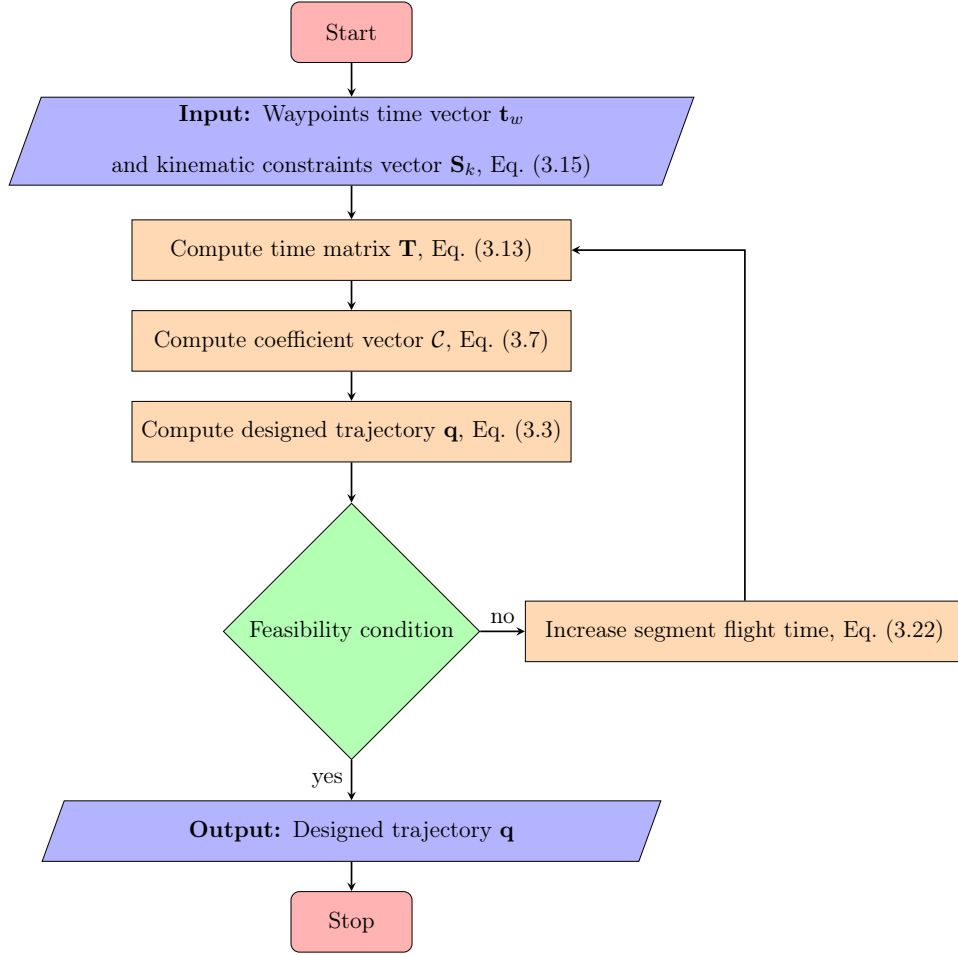


Figure 3.4: Trajectory design flowchart

segment flight time. This results in scaling the infeasible segment flight time by $\sqrt{\frac{a_{in}}{a_{max}}}$. A similar concept of feasibility check was successfully implemented experimentally in [89]. Fig. 3.4 presents a flowchart of the trajectory design process.

It is worth mentioning that the dynamic feasibility condition in Eq. (3.21) considers only F_{max} . While considering only F_{max} to check the feasibility is valid for non-aggressive flights, it is not sufficient for aggressive flights. Instead, all propeller forces must be within the operating range, i.e., $0 \leq F_i \leq F_{i_{max}}$.

3.3 Simulation Results

Without loss of generality, a KCCP MJT is considered for the simulation presented in this section. Three cases are considered to validate the effectiveness of the proposed KCCP polynomial trajectory:

- I) Comparison with standard continuous-path trajectories from an energy consumption perspective.
- II) Generation of a continuous-path trajectory with arbitrary kinematic constraints (AKC) using KCCP polynomial trajectories.
- III) Generation of a continuous-path trajectory with a complex kinematic profile, i.e., piecewise kinematic profile (PKP).

3.3.1 Comparison with Standard Continuous-Path Trajectories

Quadrotor's energy consumption is a critical aspect when designing trajectories because of the limited energy resources onboard. Therefore, this subsection compares the energy performance of the proposed KCCP polynomial trajectories with standard continuous-path trajectories.

Recall the following:

- The idea of the proposed KCCP polynomial trajectory is to well-approximate the desired path of a standard continuous-path trajectory using a large number of waypoints, and the kinematic constraints can be set freely to match the kinematic constraints of the standard continuous-path trajectory.
- The energy consumption depends on propeller and thrust forces, F_i and F , as shown in Eq. (2.54).

- Quadrotors, and quadrotors with slung payloads, are differentially flat, i.e., the thrust depends of the trajectory and its derivatives as explained in Sec. 3.2.1.

Thus, one can expect that a reasonable number of waypoints that can well-approximate the standard trajectory will result in similar propeller and thrust forces, and similar energy performance.

To verify that, a circular trajectory in the inertial horizontal $x - y$ plane with a total flight time $t_f = 8$ s is considered. The standard equation of such a trajectory is as follows:

$$\mathbf{r} = [\rho \cos(\omega t), \rho \sin(\omega t)] \in \mathbb{R}^2 \quad (3.23)$$

where ρ is the radius of the circular trajectory and ω is the angular speed. This simulation considers $r = 1$ m and $\omega = \pi/4$. A proposed KCCP MJT corresponding to this continuous-path circular trajectory is generated considering 33 waypoints distributed evenly over the flight time, i.e., one waypoint every 0.25 s.

Fig. 3.5 shows that the position, velocity and acceleration of the KCCP MJT can track the continuous-path circular trajectory well as proposed. Therefore, similar propeller and thrust forces are expected, which in turn means similar energy performance. Although for non-aggressive flights, such as this simulation, the thrust is enough to represent energy consumption (as also discussed in Section 2.3), aggressive flights require knowing all propeller forces, F_i . Additionally, the dynamics feasibility of non-aggressive flights is guaranteed if the thrust force, F_i , is within the limits of the actuators. This is not enough for non-aggressive flights as explained in Section 3.2.2. Instead, one needs to check either all propeller forces, F_i , or the total control command $[F, \boldsymbol{\tau}]^\top = [F, \tau_{roll}, \tau_{pitch}, \tau_{yaw}]^\top$ (see Figs. 3.6 and 3.7).

Fig. 3.6 shows propeller forces of the KCCP MJT and the corresponding standard continuous-path circular trajectory. This figure serves two purposes: a) showing that

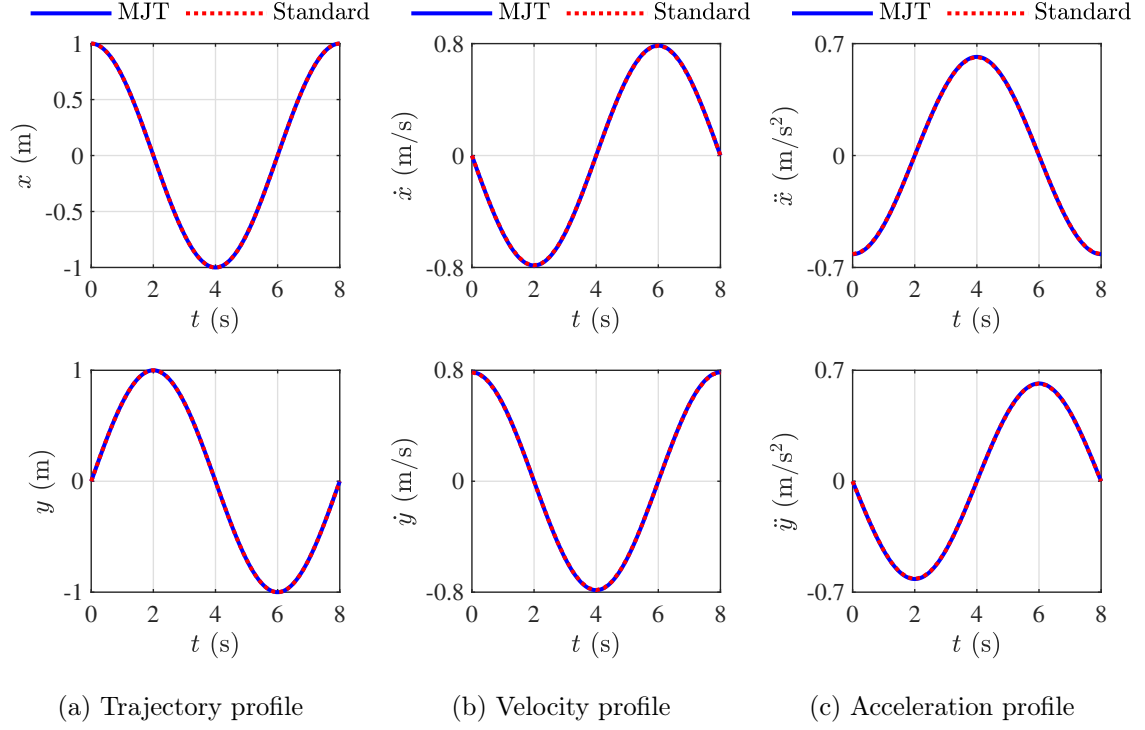


Figure 3.5: Simulation results of trajectory, velocity, and acceleration profiles of standard continuous-path circular trajectory and its corresponding KCCP MJT

KCCP MJT propeller forces match standard continuous-path circular trajectory's propeller forces; b) showing that all propeller forces are within the actuator limits, which is $0 \leq F_i \leq 5$ N for the QDrone, i.e., the KCCP MJT is dynamically feasible.

Alternatively, Fig. 3.7 shows the total thrust and roll, pitch, and yaw moment for both the KCCP MJT and the corresponding standard continuous-path circular trajectory. It also shows the match between the KCCP MJT and the continuous-path trajectory control commands as well as the dynamics feasibility of the trajectory. For the considered QDrone, actuator limits are $0 \leq F \leq 20$ N and $|\tau_{roll}, \tau_{pitch}, \tau_{yaw}| \leq [1.09, 0.9, 0.1]$ N·m.

Figs. 3.5 - 3.7 show that the proposed KCCP trajectories can well-approximate standard continuous-path trajectories and match their kinematic, dynamics, and energy be-

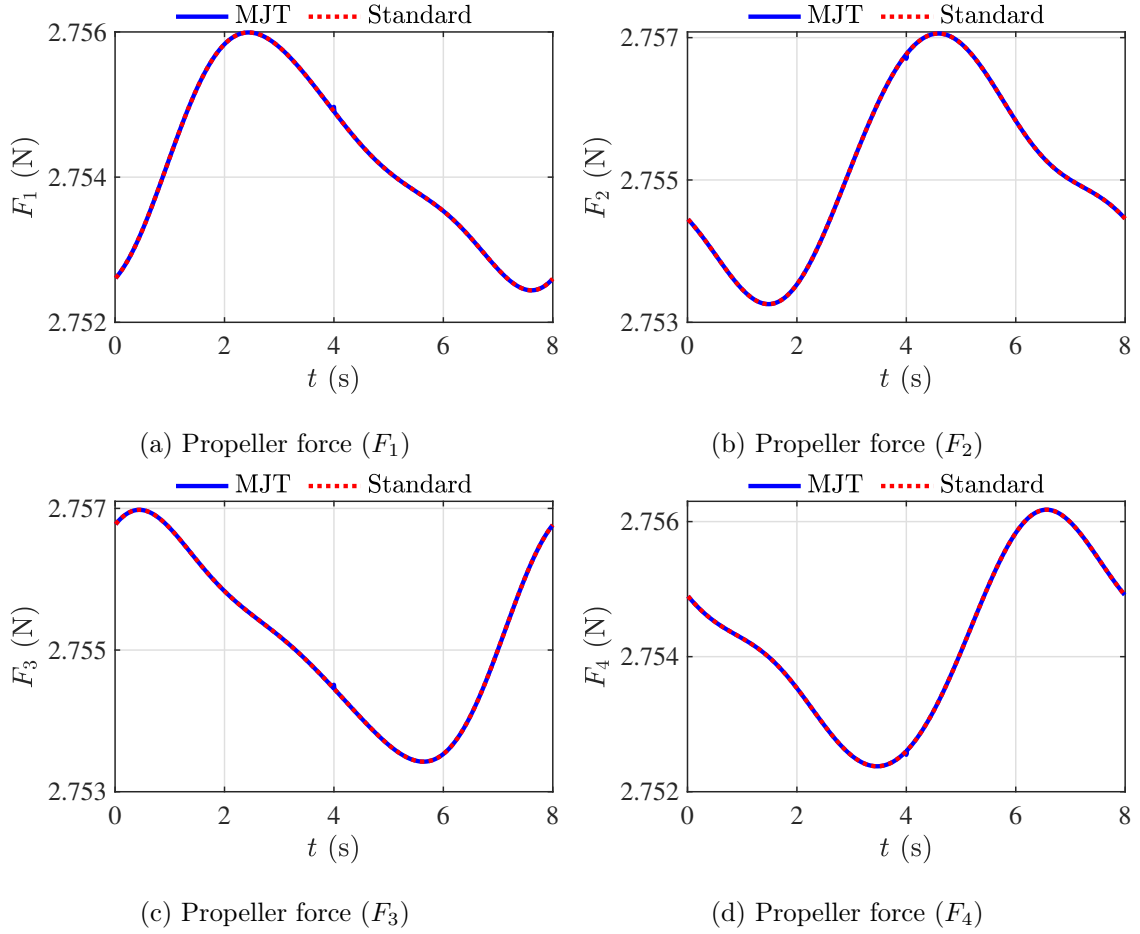


Figure 3.6: Simulation results of propeller forces for the standard circular trajectory and its corresponding KCCP MJT

haviour. However, KCCP polynomial trajectories can overcome the limitations associated with standard continuous-path trajectories (see the following subsections).

3.3.2 Arbitrary Kinematic Constraints (AKC)

The subsection considers an elliptical trajectory in the inertial vertical $y-z$ plane described by the standard equation:

$$\mathbf{r} = [\rho_1 \cos(\omega t), \rho_2 \sin(\omega t)] \in \mathbb{R}^2 \quad (3.24)$$

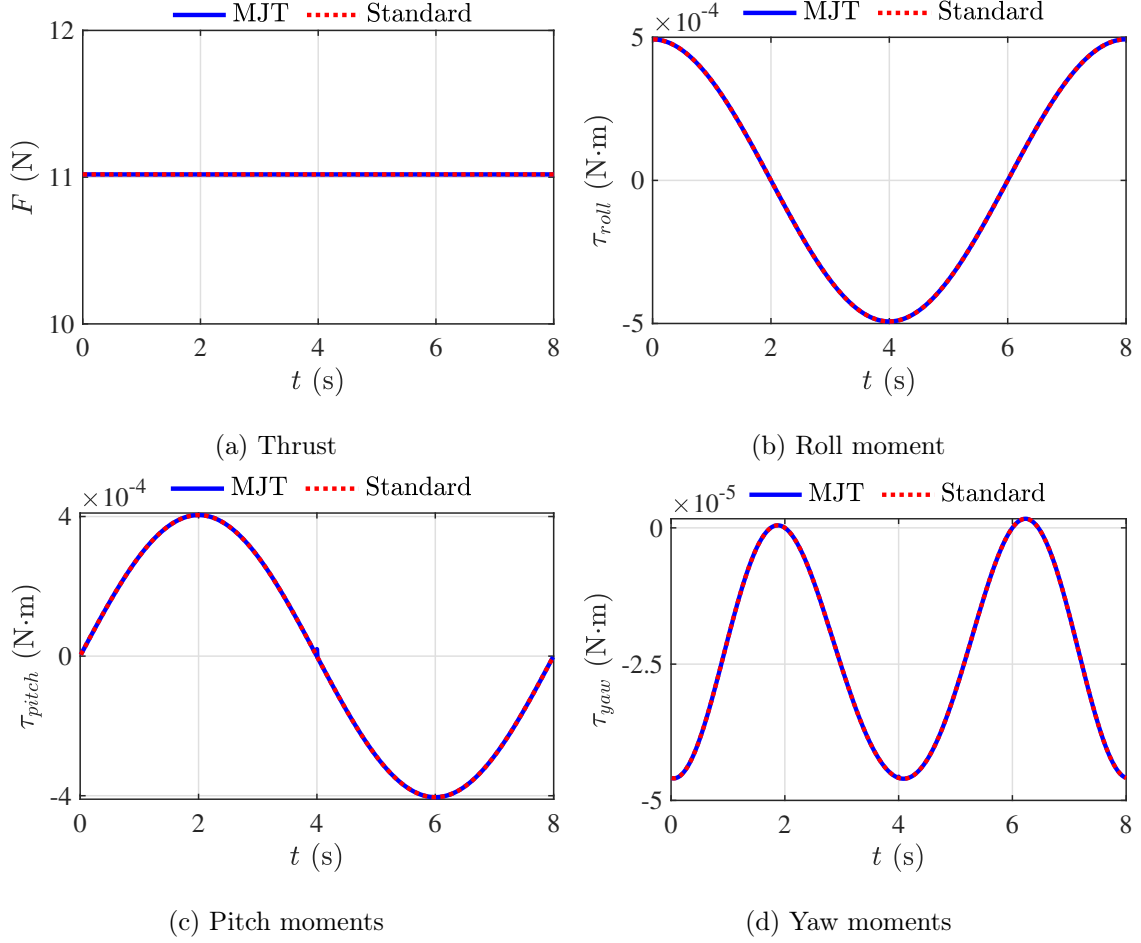


Figure 3.7: Simulation results of thrust force and roll, pitch, and yaw moments for the standard circular trajectory and its corresponding KCCP MJT

where ρ_1 and ρ_2 are the semi-major and semi-minor axes, respectively, and ω is the angular speed.

The objective is to generate a trajectory that can follow the path of the upper half of the elliptical trajectory in Eq. (3.24) in 6 s, while satisfying some arbitrary kinematic constraints on velocities and accelerations at 25 different positions evenly distributed over the flight time. These 25 waypoints are distributed evenly over the flight time, i.e., one waypoint every 0.25 s.

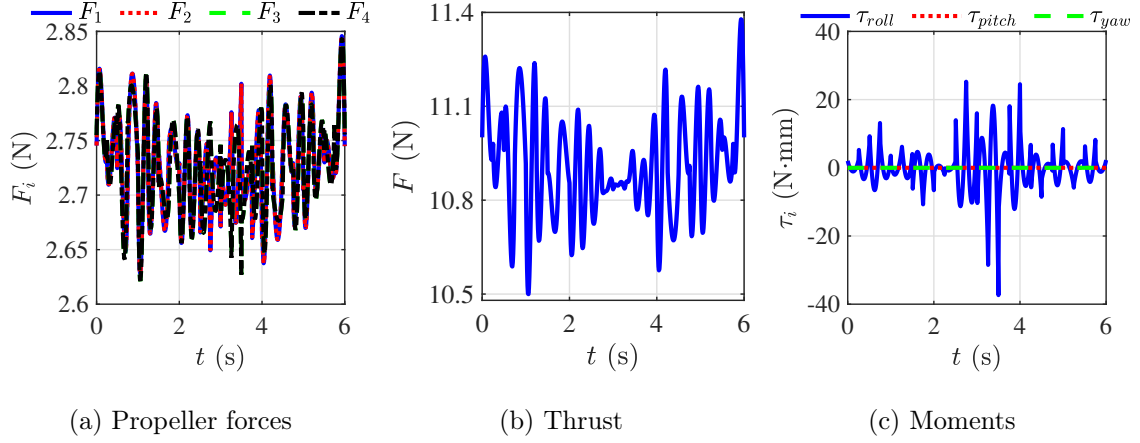


Figure 3.8: Simulation results of propeller forces, thrust force, and moments of the KCCP MJT (AKC)

To the best of our knowledge, this trajectory cannot be generated by standard methods provided in the literature. So, the path is split into the desired 25 waypoints and the desired arbitrary kinematic constraints are assigned to these waypoints. Then, a KCCP MJT is designed as proposed in Fig. 3.4.

The standard velocity and acceleration, $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, can be obtained by taking the time derivative of Eq. (3.24). The desired arbitrary kinematic constraints, $\dot{\mathbf{r}}_{per}$ and $\ddot{\mathbf{r}}_{per}$, are found by arbitrarily perturbing the standard velocities and accelerations at the 25 waypoints by $\pm 10\%$. That is:

$$\begin{aligned} 0.9\dot{\mathbf{r}} &\leq \dot{\mathbf{r}}_{per} \leq 1.1\dot{\mathbf{r}} \\ 0.9\ddot{\mathbf{r}} &\leq \ddot{\mathbf{r}}_{per} \leq 1.1\ddot{\mathbf{r}} \end{aligned} \tag{3.25}$$

First, the dynamic feasibility of the proposed KCCP MJT must be checked. Since the trajectory is non-aggressive, checking the maximum thrust F_{max} is enough. The generated KCCP MJT has $\|\ddot{\mathbf{r}}_{max}\| = 0.912 \text{ m/s}^2$ with $\ddot{z}_{max} = 0.335 \text{ m/s}^2$. From Eq. (3.21), one can see that the necessary dynamic feasibility condition is satisfied and the trajectory is dynamically feasible. Alternatively, the dynamic feasibility check can be done by either

confirming that all propeller forces are within the operating range or both the thrust and the moments are within the operating range. Fig. 3.8 shows that the proposed KCCP MJT is dynamically feasible. Fig. 3.8a shows that all propeller forces are within the range, i.e., $0 \leq F_i \leq 5$ N. Figs. 3.8b and 3.8c show that both the thrust and the moments are within the operating range, i.e., $0 \leq F \leq 20$ N and $|\tau_{roll}, \tau_{pitch}, \tau_{yaw}| \leq [1.09, 0.9, 0.1] \times 10^3$ N-mm. Hence, the KCCP MJT is dynamically feasible.

Second, the effectiveness of the proposed KCCP polynomial trajectory is analyzed in Fig. 3.9. Figs. 3.9a - 3.9c show the position-related results. Fig. 3.9a shows that the proposed KCCP MJT can follow the desired elliptical path successfully. The trajectory of the KCCP MJT is compared to the standard elliptical trajectory in Fig. 3.9b. The trajectory tracking error is plotted in Fig. 3.9c, which shows a maximum of about 2 mm tracking error in both y and z directions. This shows the accuracy of the proposed approach in approximating the desired continuous-path trajectory.

Figs. 3.9d - 3.9f show the velocity-related results. Fig. 3.9d shows the desired values of the velocity and how they are arbitrarily perturbed within the range of $\pm 10\%$ of the standard velocity. Fig. 3.9e shows that the proposed KCCP MJT can satisfy all the desired arbitrary kinematic constraints. The error at the 25 waypoints of interest is almost zero as plotted in Fig. 3.9f.

Similarly, Figs. 3.9g - 3.9i show the acceleration-related results. The arbitrariness of the kinematic constraints on the acceleration is shown in Fig. 3.9g. The acceleration profile of the proposed KCCP MJT can satisfy all the desired arbitrary kinematic constraints as shown in Fig. 3.9h with almost zero error as shown in Fig. 3.9i.

The results in Figs. 3.8 and 3.9 show that the proposed KCCP polynomial trajectories can well-approximate continuous-path trajectories, while satisfying arbitrary kinematic constraints, which cannot be satisfied by standard continuous-path trajectories.

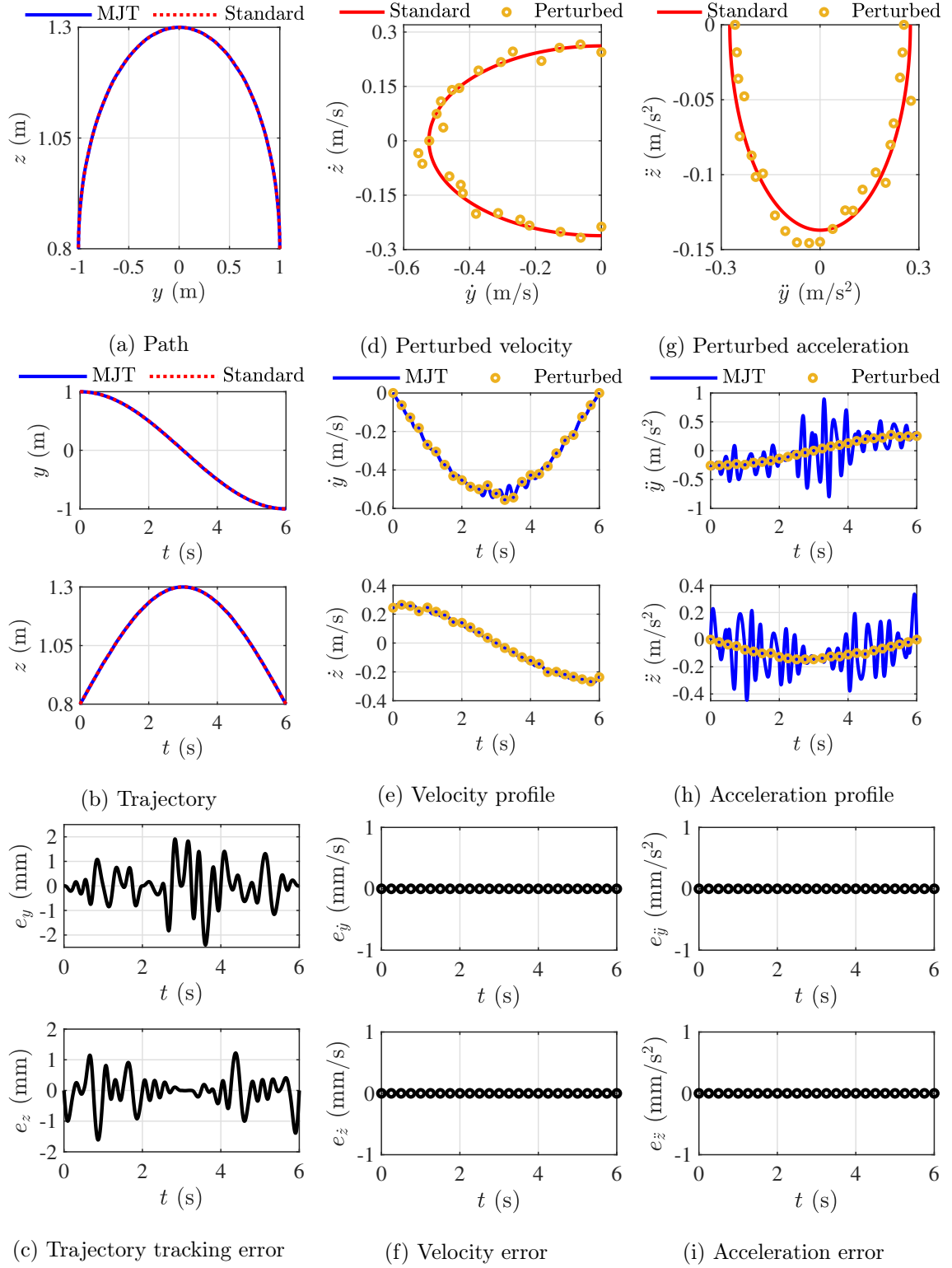


Figure 3.9: Simulation results of the KCCP MJT (AKC)

3.3.3 Piecewise Kinematic Profile (PKP)

The subsection considers a helical trajectory with radius $\rho = 1$ m for its inertial $x - y$ circular projection and a rise in the inertial z -direction $\Delta z = 1$ m with initial height $z_0 = 0.7$ m. The velocity in the inertial z -direction is kept constant and equal to $\dot{z} = 0.0229$ m/s. The standard equation of such a helix is:

$$\begin{aligned} \mathbf{r} &= [\rho \cos(\omega t), \rho \sin(\omega t), \dot{z}t]^\top \in \mathbb{R}^3 \\ \theta_h &= \omega t \in \mathbb{R} \end{aligned} \tag{3.26}$$

where ω is the angular speed. In most cases in the literature, ω is constant; however, it can be time varying in practice. For example, building inspection tasks, in which a quadrotor may be required to scan a building from all sides with different speeds based on the sensitivity of the collected data from each side [121]. If ω is a time varying piecewise function, dealing with Eq. (3.26) will be challenging. So, considering KCCP polynomial trajectories in such cases facilitates trajectory design process.

Consider a helical trajectory split into four segments, each has its own kinematic constraints in the horizontal inertial $x - y$ plane, as follows (reported concisely in Table 3.1):

- (a) For $0^\circ \leq \theta_h \leq 90^\circ$, the horizontal velocity in $x - y$ plane, v , is required to increase exponentially with t from $v = 0$ to $v = 0.2$ m/s.
- (b) For $90^\circ < \theta_h \leq 180^\circ$, v is required to increase linearly with θ_h from $v = 0.2$ m/s to $v = 0.4$ m/s.
- (c) Then, the tangential acceleration, a_t , is required to decrease parabolically with t from the final value of the second segment $a_{t_{f_2}}$ to $-a_{t_{f_2}}$ in 4 seconds. In the meanwhile, v is required to decrease to $v = 0.2$ m/s.
- (d) Finally, v is required to decrease linearly with t to $v = 0$ m/s at $\theta_h = 360^\circ$.

Table 3.1: Kinematic constraints for the piecewise kinematic profile

Segment	v	a_t	t_f	v_f	θ_{h_f}	a_{t_f}
a	$e^{0.0113t} - 1$	—	—	0.2 m/s	90°	—
b	$0.1273\theta_h$	—	—	0.4 m/s	180°	—
c	—	$0.0188t^2 - 0.9121t + 11$	25.65 s	0.2 m/s	—	-0.0509 m/s ²
d	$-0.0111t + 0.484$	—	—	0 m/s	360°	—

Designing such a trajectory using standard equations is difficult, but doable. First, the standard equations with time varying ω are:

$$\begin{aligned}
 \mathbf{r} &= [\rho \cos(\omega t), \rho \sin(\omega t), ht]^\top \in \mathbb{R}^3 \\
 \dot{\mathbf{r}} &= [-\rho(\omega + \dot{\omega}t) \sin \omega t, \rho(\omega + \dot{\omega}t) \cos \omega t, h]^\top \\
 \ddot{\mathbf{r}} &= [-\rho(\omega + \dot{\omega}t)^2 \cos \omega t - \rho(2\dot{\omega} + \ddot{\omega}t) \sin \omega t, -\rho(\omega + \dot{\omega}t)^2 \sin \omega t + \rho(2\dot{\omega} + \ddot{\omega}t) \cos \omega t, 0]^\top \\
 \omega &= \frac{v}{\rho} \in \mathbb{R}
 \end{aligned} \tag{3.27}$$

Second, to satisfy the kinematic constraints in Table 3.1, ω must be:

$$\omega = \begin{cases} \frac{e^{0.0113t} - 1}{\rho}, & t_0 \leq t < t_a, \\ \frac{0.2e^{0.1273t}}{\rho}, & t_a \leq t < t_b, \\ \frac{0.00627t^3 - 0.45605t^2 + 11t + 0.4}{\rho}, & t_b \leq t < t_c, \\ \frac{-0.0111t + 0.484}{\rho}, & t_c \leq t \leq t_f \end{cases} \tag{3.28}$$

where the horizontal velocity of Segment (b) is converted from being in terms of θ_h to be in terms of t to fit in Eq. (3.28). Also, the tangential acceleration of Segment (c) is integrated with respect to time to obtain the corresponding horizontal velocity. Third, the kinematic constraints of the four segments can be used to calculate $[t_0, t_a, t_b, t_c, t_f]$ to be equal to $[0, 16.2, 21.65, 25.65, 43.7]$ s, respectively, which are referred to as transition times in this chapter.

One can see that generating such continuous-path trajectories is doable this way, but it is difficult and tedious. The situation deteriorates when the number of segments increases and/or the kinematic constraints become more complex, e.g., varying r and/or more complicated ω .

Alternatively, when the proposed KCCP polynomial trajectory is utilized, the problem becomes much easier. It is converted from a piecewise function with possibly complex kinematic functions to a set of waypoints with desired kinematic constraints assigned to these waypoints. In the present example, a KCCP MJT is chosen to have 30, 10, 20 and 20 waypoints at Segments (a), (b), (c) and (d), respectively. To ensure continuity, the last waypoint of a segment has the same allocated time as the first waypoint of the next segment. This results in 77 waypoints in total and a total trajectory time of 43.7 s. At transition between segments, the initial conditions of the next waypoint overwrites the final conditions of the previous waypoints as introduced in Eq. (3.28), which can be expressed for the KCCP MJT as:

$$\mathbf{r}_{af}^{(k)} = \mathbf{r}_{b0}^{(k)}, \quad \mathbf{r}_{bf}^{(k)} = \mathbf{r}_{c0}^{(k)}, \quad \mathbf{r}_{cf}^{(k)} = \mathbf{r}_{d0}^{(k)} \quad (3.29)$$

where $k = \{0, 1, 2\}$ is the number of time derivatives taken, the subscripts 0 and f denote initial and final boundary conditions, respectively, and a, b, c and d are the four trajectory segments. Therefore, nonsmooth transitions due to the difference in boundary conditions of the four segments is expected.

First, the dynamic feasibility of the KCCP MJT is checked using the maximum thrust method. The generated KCCP MJT has $\|\ddot{\mathbf{r}}_{max}\| = 0.171 \text{ m/s}^2$ with $\ddot{z} = 0$ at all times. From Eq. (3.21), one can see that the necessary dynamic feasibility condition is satisfied and the trajectory is dynamically feasible. Alternatively, the dynamic feasibility check can be done by either confirming that all propeller forces are within the operating range or both the thrust and the moments are within the operating range. Fig. 3.10 shows

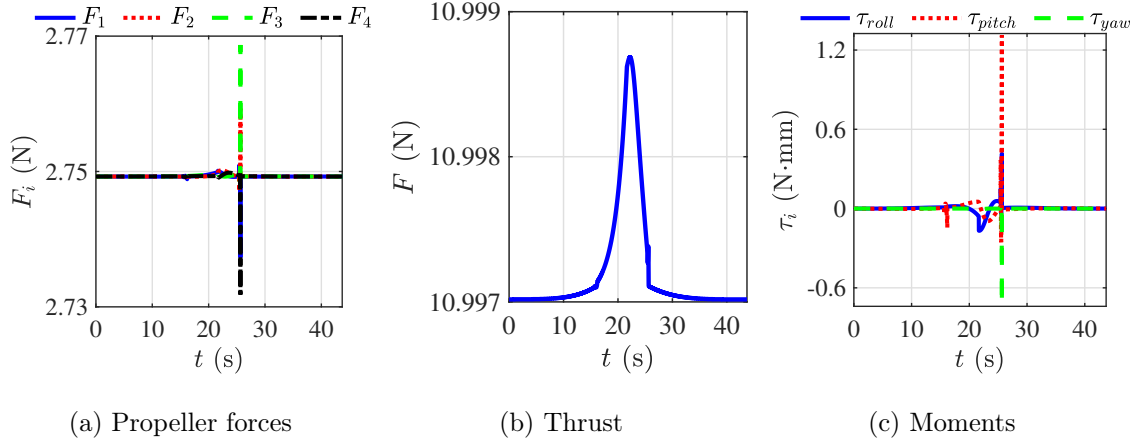


Figure 3.10: Simulation results of propeller forces, thrust force, and moments of the KCCP MJT (PKP)

that the proposed KCCP MJT is dynamically feasible. Fig. 3.10a shows that all propeller forces are within the range, i.e., $0 \leq F_i \leq 5$ N. Figs. 3.10b and 3.10c show that both the thrust and the moments are within the operating range, i.e., $0 \leq F \leq 20$ N and $|\tau_{roll}, \tau_{pitch}, \tau_{yaw}| \leq [1.09, 0.9, 0.1] \times 10^3$ N·mm. Hence, the KCCP MJT is dynamically feasible.

Second, the effectiveness of the proposed KCCP polynomial trajectory is analyzed in Figs. 3.11 and 3.12. Since the desired kinematic profile of the four segments, i.e, Segments (a), (b), (c), and (d), reported in Table. 3.1 are related to the path, horizontal velocity and tangential acceleration, Fig. 3.11 presents how the proposed KCCP MJT satisfy these desired kinematic constraints for each segment. Segments are referred to as S_a , S_b , S_c , and S_d in Figs. 3.11 and 3.12.

Fig. 3.12 presents the detailed position, velocity and acceleration tracking performance. Figs. 3.12a and 3.12b show that the proposed KCCP MJT can follow the four segments of the desired trajectory with almost zero error except some small spikes at transition times, i.e., at $t \approx \{16.2, 21.65, 25.65\}$ s. Similarly, Figs. 3.12c and 3.12d show that the proposed

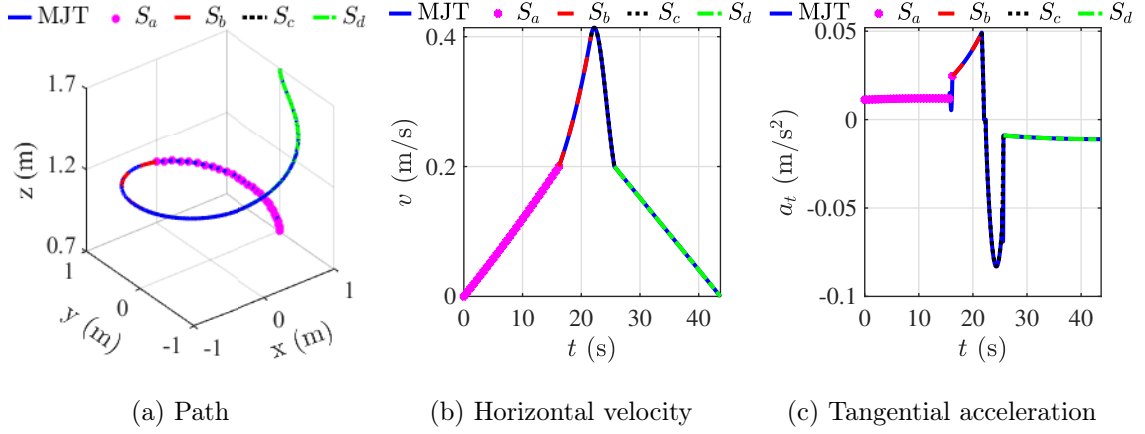


Figure 3.11: Simulation results of path, horizontal velocity, and tangential acceleration of the KCCP MJT (PKP)

KCCP MJT can track the desired velocity profile in 3D with almost zero error for small spikes at transition times. The desired acceleration profile is also tracked with a small error except for small spikes at transition times as shown in Figs. 3.12e and 3.12f.

The results in Figs. 3.11 and 3.12 show that the proposed KCCP polynomial trajectories can be used to well-approximate continuous-path trajectories, while satisfying a piecewise kinematic profile, which is difficult to be satisfied by standard continuous-path trajectory methods. As expected, the results show spikes at transition times, i.e., $t = \{16.2, 21.65, 25.65\}$ s due to the nonsmooth transition (see Eq. (3.29)).

3.4 Experimental Results

The main objective of this section is to experimentally validate the effectiveness of the KCCP polynomial trajectories proposed in Sections 3.1 and 3.2. This is conducted by experimentally validate the results of the three cases of simulations considered in Section 3.3. The test videos can be found at <https://youtu.be/yG0-KFKxyp8>.

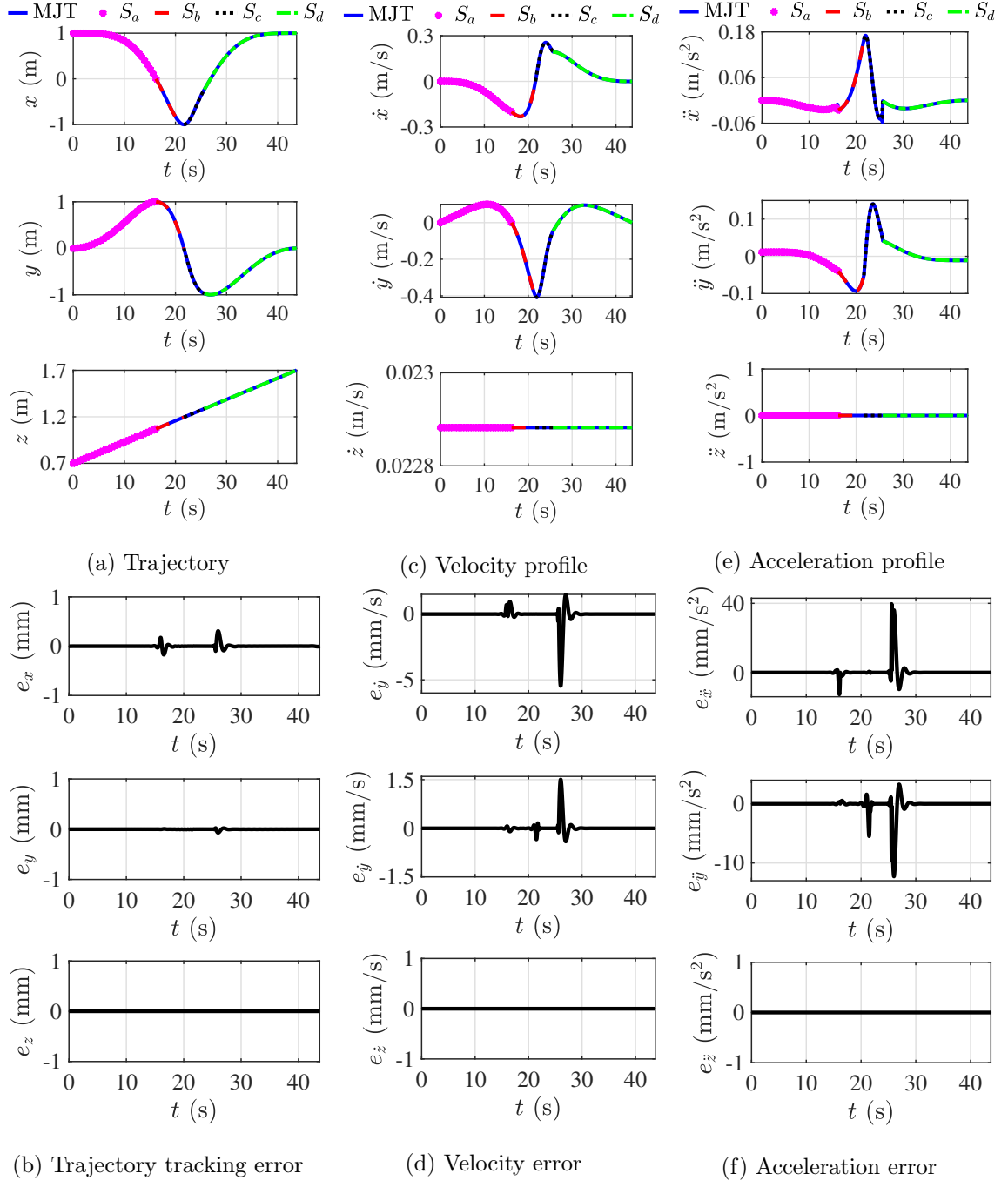


Figure 3.12: Simulation results of trajectory, velocity, and acceleration tracking performance of the KCCP MJT (PKP)

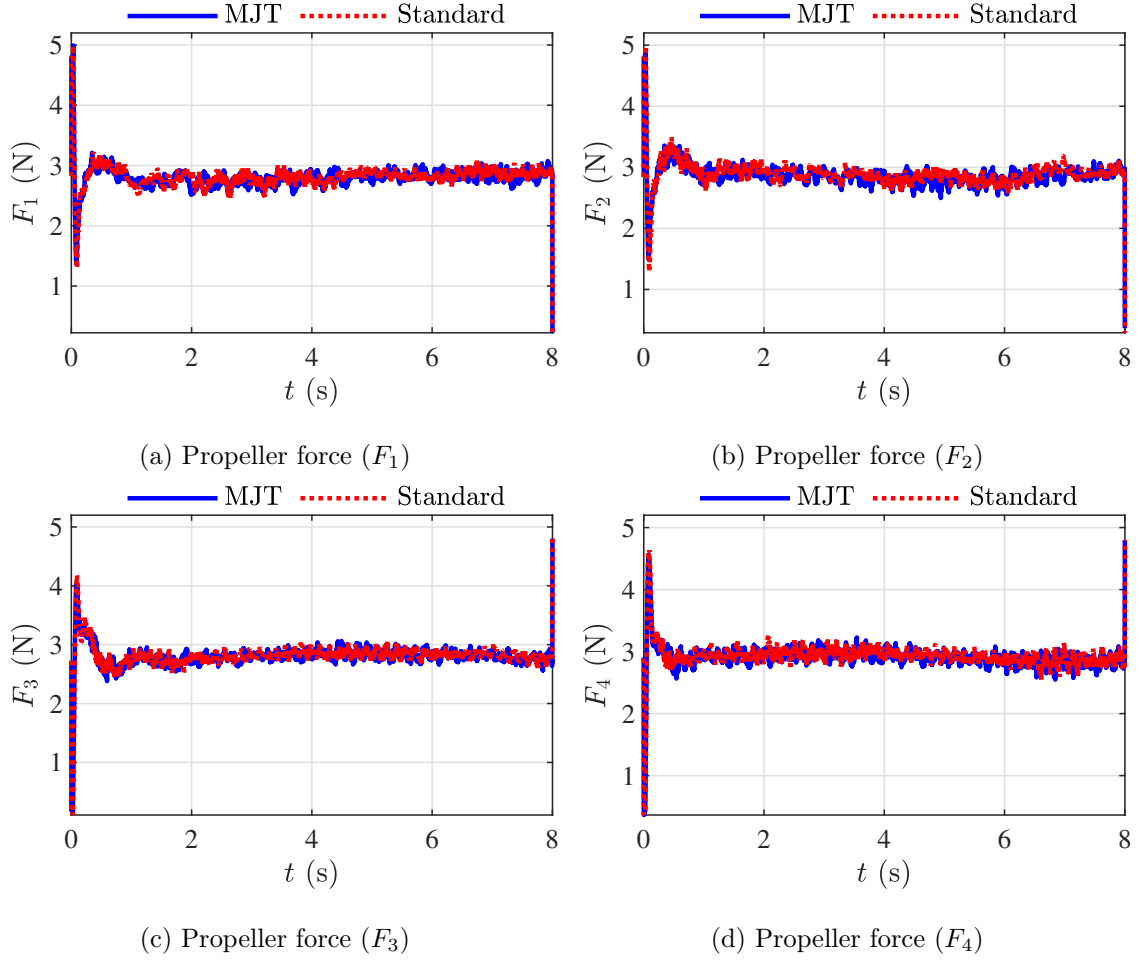


Figure 3.13: Experimental results of propeller forces for the standard circular trajectory and its corresponding KCCP MJT

3.4.1 Comparison with Standard Continuous-Path Trajectories

This experiment validates that the proposed KCCP MJTs and standard continuous-path trajectories require similar propeller forces, thus consumes equivalent energy. Considering the same circular trajectory used in Section 3.3.1, Fig. 3.13 validates that the KCCP MJT and the standard circular trajectory require similar propeller forces.

The spikes at the beginning and at the end of the motion are due to the sudden change

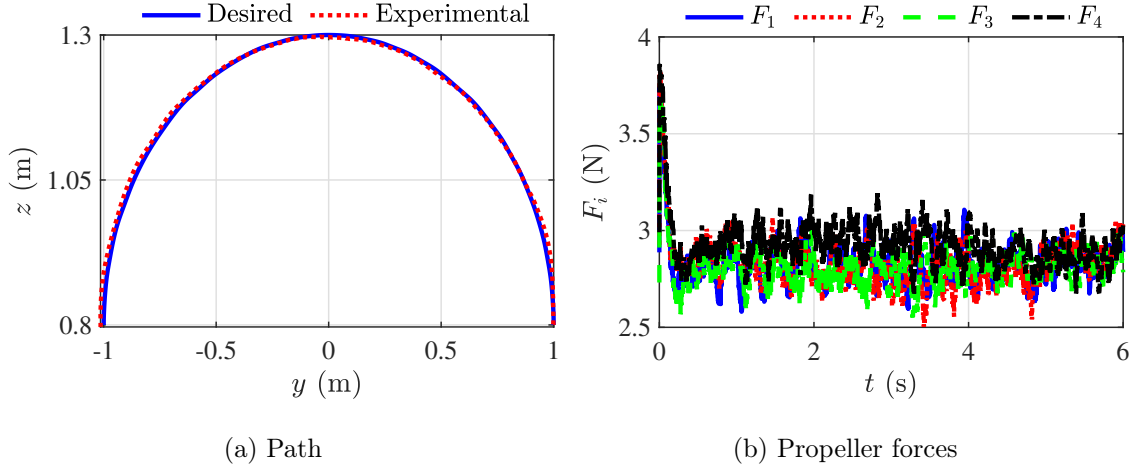


Figure 3.14: Experimental results of path and propeller forces of the KCCP MJT (AKC)

in velocities and acceleration in x - and y -directions at the boundary waypoints. That is, at the beginning of the trajectory velocity in y is $\dot{y} = \omega \cos(0) = \pi/4$ m/s, where as acceleration at the end of the trajectory in x is $\ddot{x} = -\omega^2 \cos 2\pi = -\pi^2/16$ m/s². Since quadrotor's flight is a rest-to-rest motion, such sudden increase and drop must occur at the start and the end of the motion.

3.4.2 Arbitrary Kinematic Constraints (AKC)

This experiment validates the results of the arbitrary kinematic constraints case presented in Section 3.3.2. The experimental results are presented in Figs. 3.14 and 3.15. In Fig. 3.14, the results of Fig. 3.14a show that the proposed KCCP MJT can follow the desired elliptical path successfully. Additionally, Fig. 3.14b shows that all propeller forces are within the range $0 \leq F_i \leq 5$ N, thus the KCCP MJT is dynamically feasible.

Fig. 3.15 shows the trajectory, velocity and acceleration tracking performance in y and z directions. Figs. 3.15a and 3.15b show that the quadrotor can follow the desired trajectory with a small error, i.e., less than 2 cm in y and less than 1 cm in z with an initial spike at

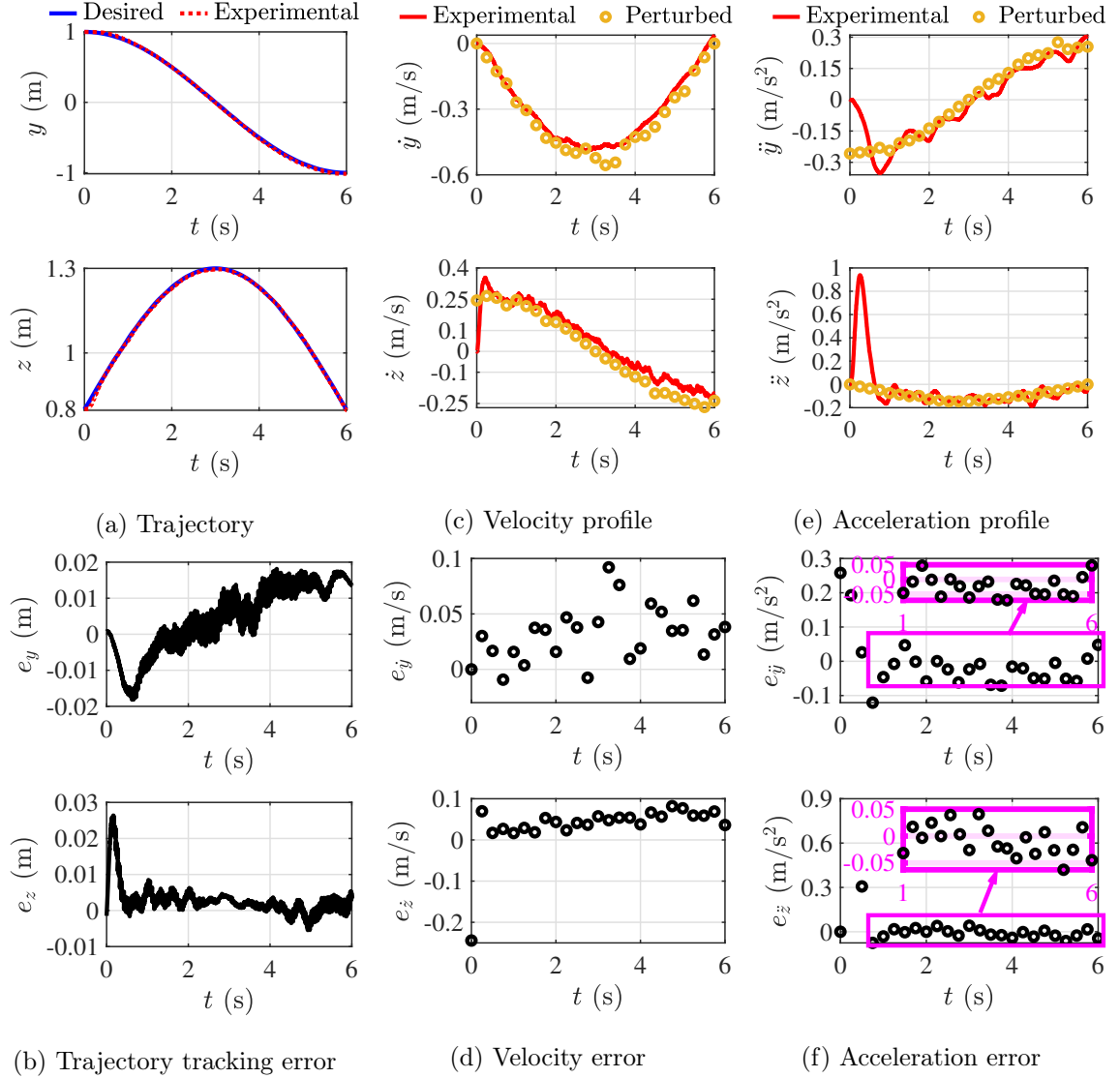


Figure 3.15: Experimental results of the trajectory, velocity, and acceleration tracking performance of the KCCP MJT (AKC)

the beginning of the trajectory. The spike at the beginning of the trajectory tracking error in z is due to having an arbitrary initial vertical velocity $\dot{z}_0 = 0.244$ m/s. Since the quadrotor starts from rest, the controller must command a sudden initial acceleration to compensate for this velocity error, i.e. $\Delta\dot{z}_0 = 0.244 - 0$ m/s. This sudden control command

results in this spike.

Fig. 3.15c and 3.15d show that the KCCP MJT can satisfy the arbitrary velocity constraints with a small error. The error is less than 5 cm/s at most of the waypoints in y direction. The velocity error in z direction is less than 7 cm/s at most of the waypoints. One can see the overshoot in $\dot{z}$ due to initial velocity error $\Delta\dot{z}_0$ as explained earlier.

Similarly, Figs. 3.15e and 3.15f show that the quadrotor can satisfy the arbitrary acceleration constraints with small error. The acceleration error is less than 5 cm/s² for most of the waypoints both in y and z . One can see the resonance of the mentioned initial velocity error, $\Delta\dot{z}_0$, in the acceleration in z . In addition to that, there is an initial difference in the acceleration in y , $\Delta\ddot{y}_0$, which leads to a larger tracking error at the first four waypoints.

In conclusion, the experimental results in Figs. 3.14 and 3.15 validate that the proposed KCCP MJT can be used to design a dynamically feasible trajectory that can follow a continuous-path, such as this ellipse, while satisfying arbitrary kinematic constraints. Such trajectories cannot be generating using standard continuous-path trajectory generation methods.

3.4.3 Piecewise Kinematic Profile (PKP)

This experiment validates the results of the piecewise kinematic profile case presented in Section 3.3.3. The experimental results are presented in Fig. 3.16 and 3.17.

In Fig. 3.16, the results of Fig. 3.16a show that the proposed KCCP MJT can follow the desired helical path successfully. Additionally, Fig. 3.16b shows that all propeller forces are within the range $0 \leq F_i \leq 5$ N, thus the KCCP MJT is dynamically feasible.

Fig. 3.17 shows the trajectory, velocity and acceleration tracking performance in 3D. Figs. 3.17a and 3.17b show that the quadrotor can follow the desired trajectory with a small error, i.e., less than 3 cm in x , less than 2 cm in y , and less than 1 cm in z except for

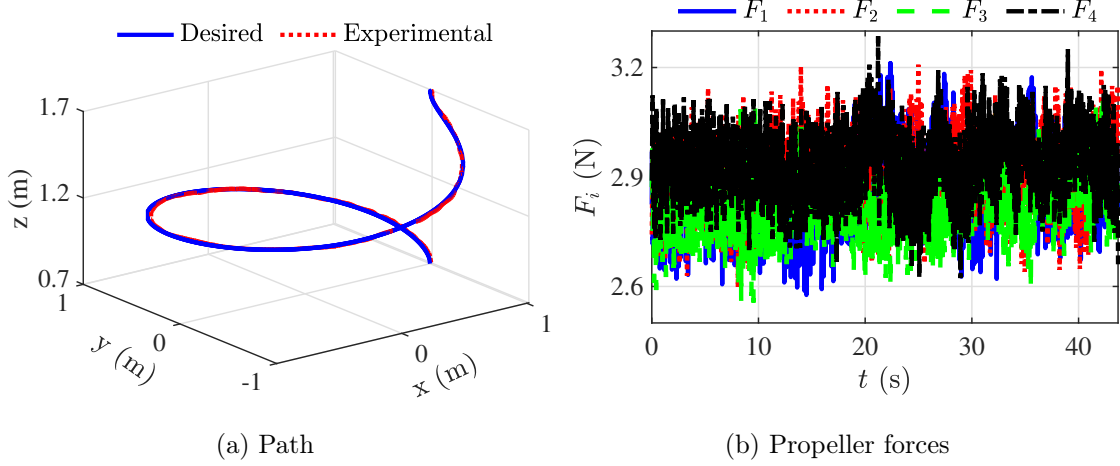


Figure 3.16: Experimental results of path and propeller forces of the KCCP MJT (PKP)

an initial spike.

The quadrotor can also follow the desired velocity profile with a small error as shown in Figs. 3.17c and 3.17d. The velocity error is approximately 2 cm/s for all the components almost everywhere except for the range of $21.6 \leq t \leq 25.65$ s, i.e., Segment (c). This is mainly due to the relatively rapid change in the desired tangential acceleration as seen in Table 3.1 and Fig. 3.11c.

Similarly, Fig. 3.17e shows that the quadrotor can track the desired acceleration profile with a small error. The errors of all components are plotted in Fig. 3.17f, which is less than 3 cm/s² in x , less than 4 cm/s² in y and less than 3 cm/s² in z . One can see larger errors at the same range, i.e., $21.6 \leq t \leq 25.65$ s, for the same reason explained earlier.

In conclusion, the experimental results in Figs. 3.16 and 3.17 validate that the proposed KCCP polynomial trajectory can be used to design a dynamically feasible trajectory that can follow a continuous-path, such as this helix, while satisfying a piecewise kinematic profile. Such trajectories can be generating using standard trajectory generation methods. However, standard methods are more difficult than the proposed one (see Section 3.3.3).

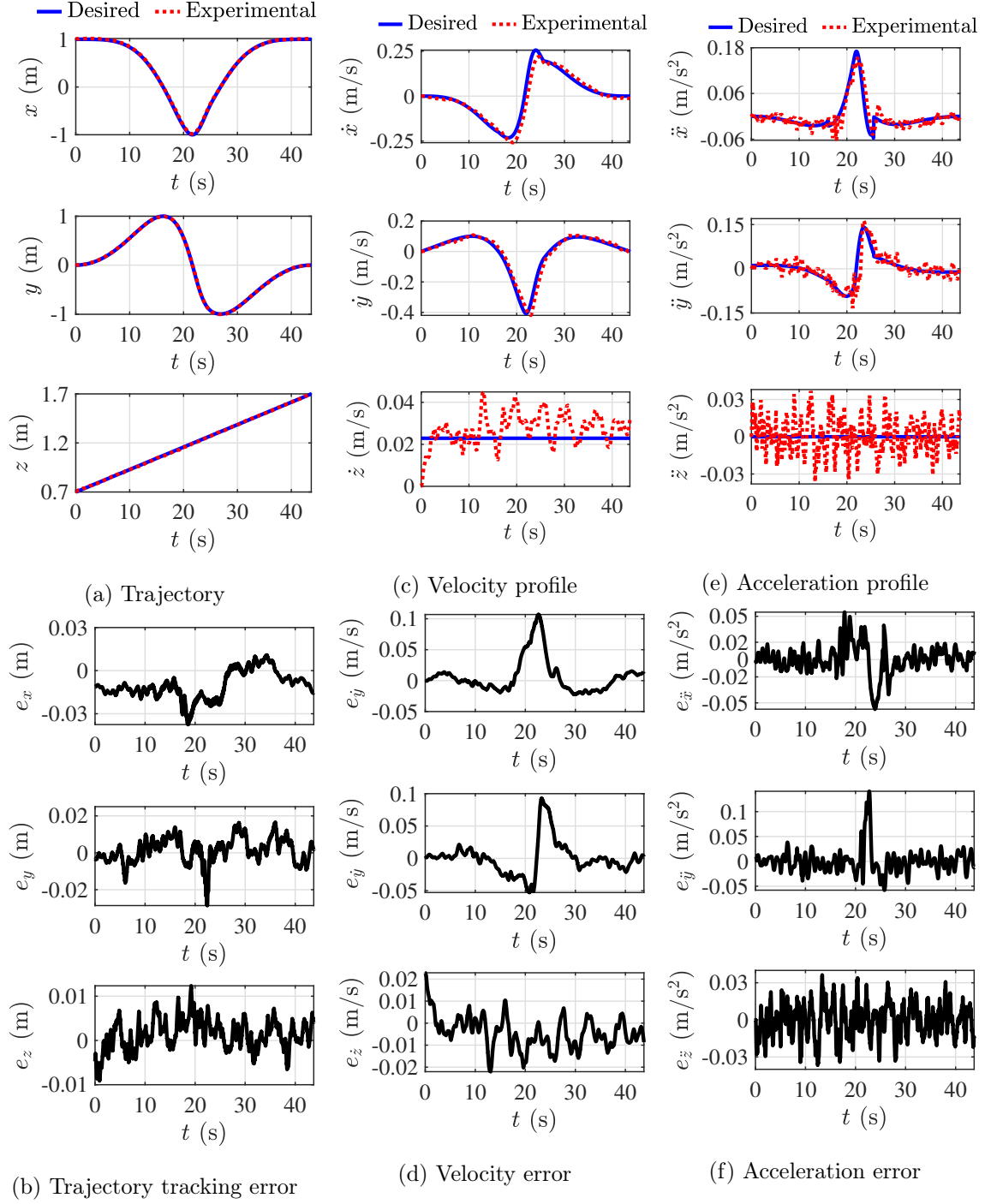


Figure 3.17: Experimental results of the trajectory, velocity, and acceleration tracking performance of the KCCP MJT (PKP)

3.5 Summary

This chapter addresses one of the main limitations of generating continuous-path trajectories, which is the inability to set desired arbitrary kinematic constraints on the trajectory. The chapter proposes the kinematically-constrained continuous-path (KCCP) polynomial trajectories as a solution to this problem. The proposed trajectory have similar energy performance as standard continuous-path trajectories; however, it overcomes their kinematic limitations.

The chapter provides a framework to generate polynomial trajectories of any degree with any arbitrary number of waypoints in any dimensional space. This frame work is used to generate KCCP polynomial trajectories with a large number of waypoints to well-approximate the desired continuous-path, then any desired arbitrary kinematic constraints can be set freely and straightforwardly at the waypoints. Additionally, a dynamic feasibility condition for non-aggressive flights is introduced.

A standard circular continuous-path trajectory was compared to its corresponding KCCP polynomial trajectory to investigate the difference in energy consumption. Also, a trajectory with arbitrary kinematic constraints and another one with piecewise kinematic profile were considered to show the effectiveness of the proposed KCCP polynomial trajectory.

Simulation and experimental results showed that the proposed KCCP trajectory require similar propeller forces as standard continuous-path trajectories, thus have similar energy consumption. However, the results showed that the proposed KCCP trajectories can overcome the standard continuous-path kinematic limitation successfully and generate a dynamically feasible trajectory that can follow the desired path, while satisfying the desired kinematic constraints.

4 Payload Vibration

Payload vibration is a major practical challenge for quadrotors with cable-suspended payloads. In addition to the danger of carrying a vibrating payload, these vibrations may add some excessive force components and moments to the system, which can lead to undesired motions. More importantly, in the context of this dissertation, payload vibrations significantly affect quadrotor’s energy consumption. Therefore, this chapter discusses how to reduce payload vibrations.

Typically, payload vibrations are reduced or eliminated in the literature using anti-swing controllers as reported in Section 1.2. Many controllers have been developed in the literature such as a nonlinear model predictive controller (NMPC) [122], input shaping [123], sliding mode controller [41], a PID position controller with feedforward input shaping [124], and many other controllers [125–127].

Alternatively, this chapter proposes a trajectory-based payload vibration reduction strategy, where polynomial trajectories along with other system parameters, e.g., cable length, are used. First, this chapter presents a comprehensive investigation of payload vibrations, then studies the relation between polynomial trajectories and payload vibrations. Finally, a novel trajectory-based two-stage payload vibration reduction strategy and does not depend on the controller is introduced. The system consisting of a quadrotor with a slung payload is called the “*transportation system*” throughout this chapter.

4.1 Vibration Analysis of the Transportation System

As presented in Section 2.2.2, payload vibrations of the transportation system of interest can be concisely expressed as follows:

$$\underbrace{l\mathbf{A}^\top\mathbf{A}\ddot{\boldsymbol{\eta}}}_{\text{inertia}} + \underbrace{l\mathbf{A}^\top\mathbf{B}\dot{\boldsymbol{\eta}}}_{\text{damping}} + \underbrace{g\mathbf{A}^\top\mathbf{e}}_{\text{stiffness}} = \underbrace{-\mathbf{A}^\top\ddot{\mathbf{r}}_q}_{\text{excitation}} \quad (4.1)$$

As shown in Eq. (4.1), the transportation system undergoes nonlinear damped vibrations. In pursuance of determining the natural frequencies and vibration modes of such a transportation system, it is acceptable to simplify the system in Eq. (4.1) by linearizing the system around the slung payload equilibrium state. Under this assumption, Eq. (4.1) is linearized as:

$$\mathbf{M}_s\ddot{\boldsymbol{\eta}} + \mathbf{K}_s\boldsymbol{\eta} = -\ddot{\mathbf{r}}_{q1,2} \quad (4.2)$$

where $\mathbf{M}_s = \text{diag}(l, l) \in \mathbb{R}^{2 \times 2}$ is the simplified mass matrix, $\mathbf{K}_s = \text{diag}(g + \ddot{z}_q, g + \ddot{z}_q) \in \mathbb{R}^{2 \times 2}$ is the simplified stiffness matrix, and $\mathbf{r}_{q1,2} = [x_q, y_q]^\top \in \mathbb{R}^2$ is a vector contains the first two elements of quadrotor's position vector.

Consequently, the natural frequencies and the mode shapes can be found by solving the following eigenvalue problem:

$$\left[\mathbf{M}_s^{-1}\mathbf{K}_s \right] \boldsymbol{\xi} = \omega_n^2 \boldsymbol{\xi} \quad (4.3)$$

where ω_n is the natural frequency of the system and $\boldsymbol{\xi} \in \mathbb{R}^2$ is its corresponding mode shape. For the system in Eq. (4.2), the two natural frequencies are:

$$\omega_{n1} = \omega_{n2} = \sqrt{\frac{g + \ddot{z}_q}{l}} \quad (4.4)$$

and the two vibration mode shapes are:

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4.5)$$

which form a unit modal matrix, Ξ , as follows:

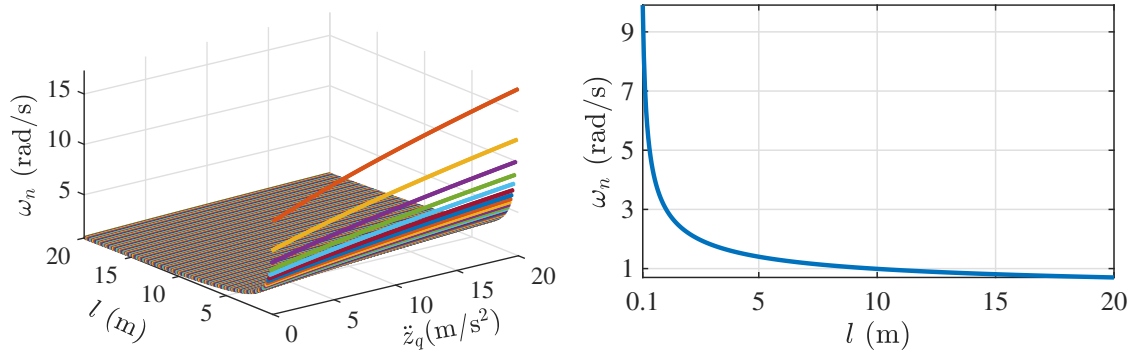
$$\Xi = [\xi_1, \xi_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.6)$$

This unit modal matrix shows that the simplified dynamics model in Eq. (4.2) is decoupled in the states. In fact, this is due to the way of defining the cable angles, θ_1 and θ_2 , as shown in Fig. 2.1b. Since $\mathbf{K}_s$ depends on $\ddot{z}_q$, the system in Eq. (4.2) is neither completely linear nor completely decoupled. the reason is that Eq. (4.1) was not linearized about the input operating point, i.e., the operating point of $\ddot{\mathbf{r}}_q$. Instead of linearizing the system about the input operating point, the system will be completely linear and decoupled in the states and the inputs if quadrotor's motion is constrained to have a constant acceleration in the inertial vertical direction z , e.g., $\ddot{z}_q = 0$. In such a case, the system will be a linear time-invariant (LTI) vibratory system.

Eq. (4.4) shows that the natural frequency, ω_n , increases with the increase in $\ddot{z}_q$, and it is inversely proportional to the square root of the cable length, $\sqrt{l}$. Fig. 4.1a illustrates the effect of the trajectory's vertical acceleration, $\ddot{z}_q$, on the natural frequency, ω_n , at different cable lengths, l . It is obvious that the effect of $\ddot{z}_q$ becomes less important when the cable length gets longer. Throughout this chapter, the vertical acceleration will be set to $\ddot{z}_q = 0$, so Fig. 4.1b shows the effect of the cable length on the natural frequency for the case of $\ddot{z}_q = 0$. Fig. 4.1b starts with $l = 0.1$ m because shorter cable lengths will results in higher frequencies, which approaches infinity at $l = 0$ based on Eq. (4.4).

Before proceeding with the analysis, it is worth emphasizing two points:

1. Mathematically, the input of the system in Eq. (4.2) is $\ddot{\mathbf{r}}_{q1,2}$, whereas the state is $\boldsymbol{\eta}$.
2. Dynamically, the dynamics system in Eq. (4.2) can be thought of as either a forced vibratory system or a vibratory system with base excitation. In the former description, the excitation force is $\ddot{\mathbf{r}}_{q1,2}$, whereas in the latter description, the base motion follows



(a) Variation of the natural frequency, ω_n , for different acceleration, $\ddot{z}_q$, and cable length, l (b) Variation of the natural frequency, ω_n , with respect to the cable length, l , at $\ddot{z}_q = 0$

Figure 4.1: Variation of the natural frequency, ω_n , of the transportation system

the trajectory $\mathbf{r}_{q1,2}$. Although there is no mathematical difference between either description, thinking of the transportation system as a base excitation model makes more sense because, practically, the payload vibration comes from the motion of the quadrotor, which is the base in the latter description. Thus, throughout this chapter, payload vibrations will be dealt with as base excited vibrations. More details about vibratory systems with base excitation can be found, for example, in [129,130]. Additionally, this description allows to make analogy between the transportation system of interest and the suspension system of a car as shown in Fig. 4.2. In this analogy, quadrotor's acceleration is analogous to the profile of a road and the cable length is analogous to the stiffness of the spring, and the payload is analogous to car body. This analogy provides a better view to see and suppress payload vibrations.

4.2 Frequency Analysis of Polynomial Trajectories

As shown in Eq. (4.2), the base excitation frequency is the frequency of the acceleration of the polynomial trajectories, $\ddot{\mathbf{r}}_{q1,2}$. Recall that this dissertation considers quadrotor's

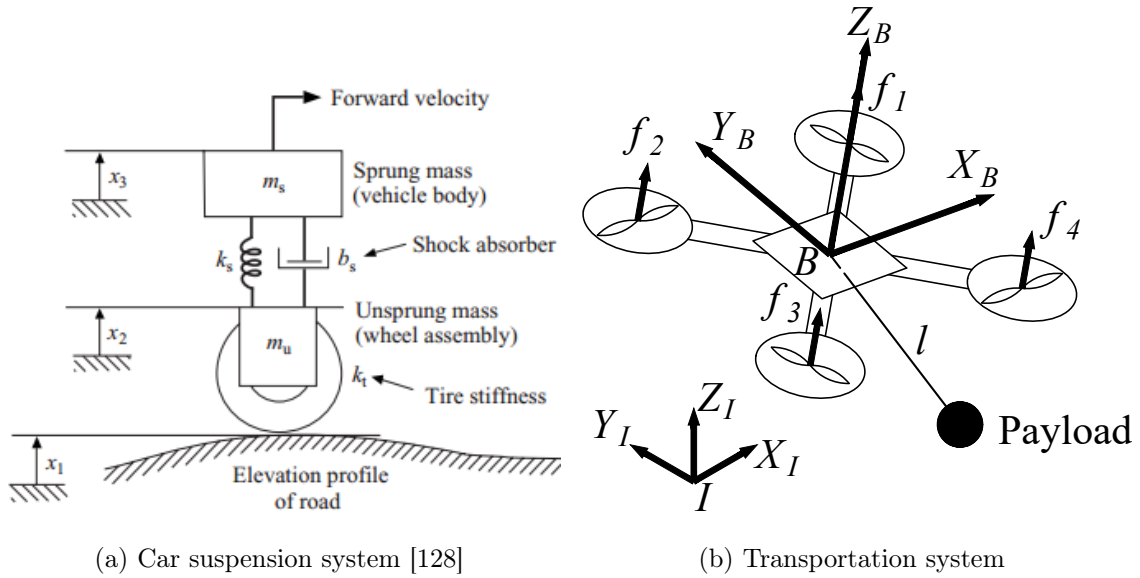


Figure 4.2: Analogy between a car suspension system and the transportation system of interest

trajectories to be polynomial trajectories as introduced in Chapter 3. Consequently, a deep frequency analysis of polynomial trajectories is required. Then, investigating the effect of their frequencies and the amplitudes at these frequencies on the natural frequencies of the transportation system indicated in Eq. (4.4) is needed.

In order to provide a rigours and analytical analysis, the frequency analysis will be conducted by obtaining Fourier series of the accelerations of the trajectories, which represent the base excitation. For the sake of clearly conveying the fundamentals and the logic of the frequency analysis of the polynomial trajectories without getting involved in unnecessary complex mathematics, rest-to-rest scenarios are considered. However, the main equations used in this frequency analysis are presented in Appendix B for general boundary conditions. From there, it can be found that polynomial trajectories have similar vibration behavior under any arbitrary boundary conditions.

It is worth mentioning that this chapter considers MJT, MST, and MCT. However, the

results, as this chapter will show, generalize and apply to any other polynomial trajectory. Additionally, without loss of generality, the initial time and position are set to zero for the rest-to-rest scenarios.

Fourier series of a general function, $f(t)$, which is periodic over the period δ can be found as an infinite sum of sines and cosines as follows [131]:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \left(\frac{2\pi kt}{\delta} \right) + b_k \sin \left(\frac{2\pi kt}{\delta} \right) \right) \quad (4.7)$$

where $f(t)$ has infinite frequencies:

$$\omega_k = \left(\frac{2\pi k}{\delta} \right), \quad k \in \{1, 2, 3, \dots, \infty\} \quad (4.8)$$

and Fourier coefficients are defined as:

$$a_k = \frac{2}{\delta} \int_0^{\delta} f(t) \cos \left(\frac{2\pi kt}{\delta} \right) dt \quad (4.9)$$

$$b_k = \frac{2}{\delta} \int_0^{\delta} f(t) \sin \left(\frac{2\pi kt}{\delta} \right) dt \quad (4.10)$$

$$a_0 = \frac{2}{\delta} \int_0^{\delta} f(t) dt \quad (4.11)$$

For rest-to-rest scenarios, the values of a_k and a_{k0} vanish for the three trajectories of interest, i.e., MJT, MST, and MCT. The coefficient b_k varies according to the trajectory as follows:

$$\begin{aligned} b_{k_j} &= \frac{180}{\pi^3 k^3} \frac{x_f}{t_f^2} \\ b_{k_s} &= - \frac{420 (\pi^2 k^2 - 15)}{\pi^5 k^5} \frac{x_f}{t_f^2} \\ b_{k_c} &= - \frac{18900 (2\pi^2 k^2 - 21)}{\pi^7 k^7} \frac{x_f}{t_f^2} \end{aligned} \quad (4.12)$$

where the subscripts j , s , and c refer to the MJT, MST, and MCT, respectively. Note that $f(t)$ is replaced with the acceleration $\ddot{x}$, whereas δ is replaced with the flight time t_f .

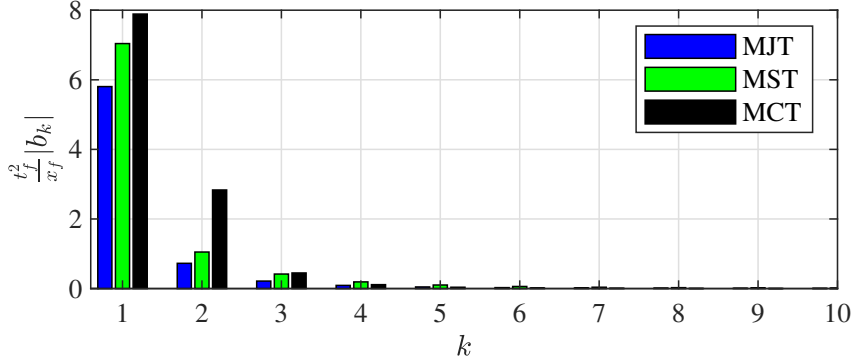


Figure 4.3: Variation of the coefficient b_k at each iteration k for MJT, MST and MCT

It is worth recalling that the coefficients b_k are the amplitudes of the sines of quadrotor's trajectory acceleration $\ddot{x}$ as expressed by $f(t)$ in Eq. (4.7). The analysis of Eq. (4.12) shows that $|b_{k_s}| > |b_{k_j}|$ for all $k \geq 1$. It also shows that $|b_{k_c}| > |b_{k_j}|$ for $k < 5$. Fig. 4.3 is the graphical representation of Eq. (4.12), and it shows that the largest magnitudes are associated with the first few frequencies, i.e., $k < 5$, which means that the acceleration $\ddot{x}$ can be well-approximated by the first four terms of its Fourier series, $\ddot{x}^{1 \rightarrow 4}$.

To be more rigorous, Fig. 4.4 shows the exact acceleration trajectory, $\ddot{x}$, along with its first four Fourier series terms, $\ddot{x}^{1 \rightarrow 4}$, and the remainder of the Fourier series terms up to the 100th term, $\ddot{x}^{5 \rightarrow 100}$. Fig. 4.4 also shows the percentage of the error, $\%e_{\ddot{x}}$, between the exact acceleration, $\ddot{x}$, and its first four-term approximation, $\ddot{x}^{1 \rightarrow 4}$. This percentage is calculated according to the following relation:

$$\%e_{\ddot{x}} = \frac{\ddot{x} - \ddot{x}^{1 \rightarrow 4}(t)}{\ddot{x}} \times 100 \quad (4.13)$$

For the sake of showing a numerical example, t_f and x_f were chosen, arbitrarily, to be equal to one in Fig. 4.4. However, similar behavior is found under any other arbitrary values. Figs. 4.4a and 4.4b show the results of the MJT, Figs. 4.4c and 4.4d are for the MST, and lastly Figs. 4.4e and 4.4f, are for the MCT. It is obvious that due to the rest-to-rest boundary conditions, Eq. (4.13) has infinite or undefined value at the beginning and the

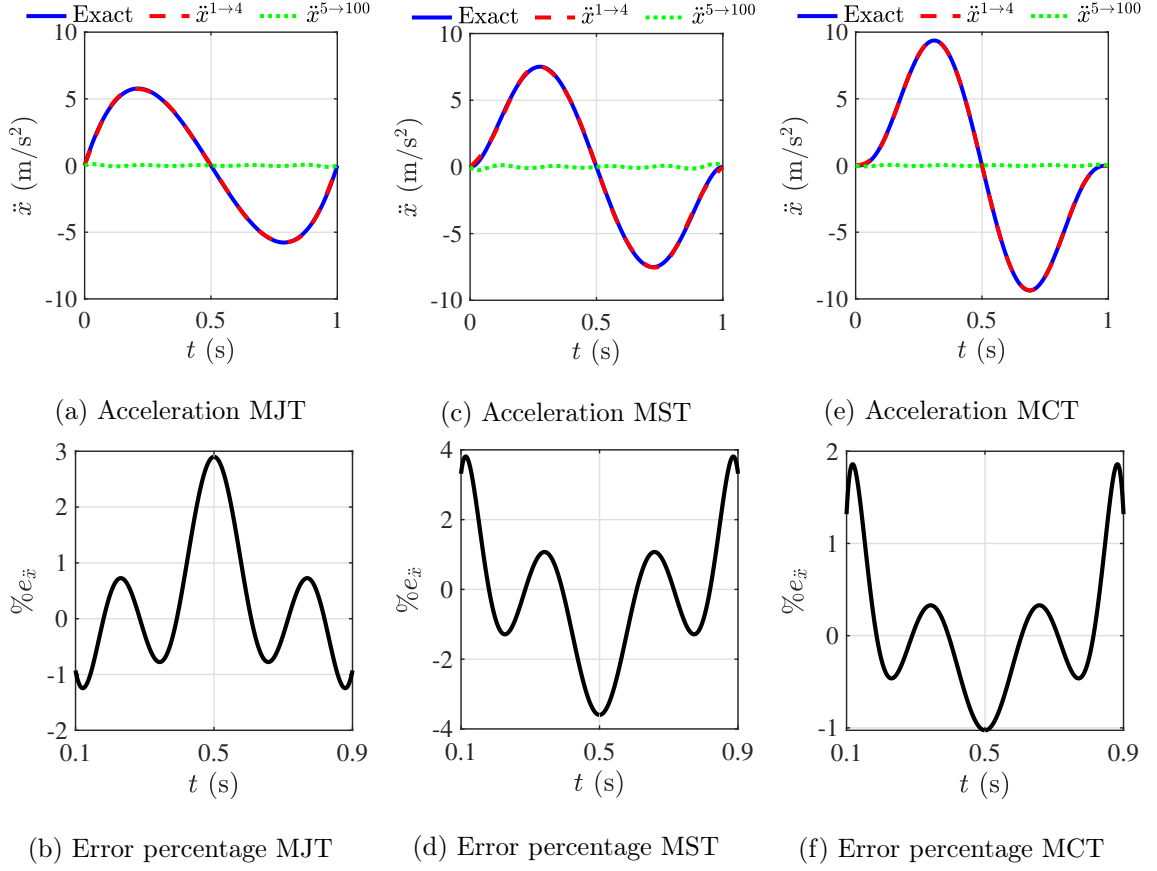


Figure 4.4: Efficiency of the first four-term approximation of acceleration for the MJT, MST and MCT

end of the trajectory since $\ddot{x}_0 = \ddot{x}_f = 0$ and $\ddot{x}^{1 \rightarrow 4}$ can be equal to zero or very small value due to approximation. Consequently, Figs. 4.4b, 4.4d and 4.4f are plotted for $0.1 \leq t \leq 0.9$ to avoid the infinite or undefined values at the beginning and the end of the motion. Obviously, $\ddot{x}^{1 \rightarrow 4}$ is a good approximation of $\ddot{x}$. The main conclusion from Eqs. (4.7) - (4.13), and Figs. 4.3 and 4.4 is that increasing the degree of the polynomial increases b_k , which will excite more payload vibration. In the considered example in Fig. 4.4, the four-term approximation of the amplitudes $b_k^{1 \rightarrow 4} \approx \{6.8367, 8.6996, 11.2838\}$ m/s² for the MJT, MST and MCT, respectively, whereas the corresponding $b_k^{1 \rightarrow 100} \approx \{6.9780, 9.0174, 11.3538\}$.

4.3 Effect of the Polynomial Trajectory on Slung Payload Vibrations

The ratio between the frequency of a polynomial trajectory, ω_t , and the natural frequency of the transportation system, ω_n , can be found from Eqs. (4.4) and (4.8) as follows:

$$r_t = \frac{\omega_t}{\omega_n} = \frac{2\pi k}{t_f} \sqrt{\frac{l}{g}} \quad (4.14)$$

and the frequency ratio, r_t , is always required to be far from $r_t = 1$ to avoid the resonance.

Recall that l is the cable length and t_f is the flight time. From Eq. (4.14), the values of l and t_f can be chosen, almost freely, to satisfy the desired frequency ratio, r_t . However, a smart choice of l and/or t_f can reduce payload vibrations.

Recalling the base excitation description in Section 4.1 helps borrow the concept of the transmissibility ratio, T_R , from base excitation vibration models. The transmissibility ratio, T_R , is defined as the ratio between the transmitted vibration to the slung payload and the base excitation [129, 130]. It is worth mentioning that the concept of the transmissibility ratio is presented, in most cases in the literature, as a dimensionless scalar quantity, T_R . This is because, in most cases, the concept of the transmissibility ratio is presented for single degree of freedom (DoF) systems with the same dimensions and units for both the excitation and the response, e.g., distances in meter [129, 130]. Recall that the transportation system of interest in this dissertation is not a single DoF system. Additionally, the excitation and the response have different dimensions and units, i.e., $\ddot{\mathbf{r}}_{q1,2}$ in m/s^2 is the excitation and $\boldsymbol{\eta}$ in rad is the response. Consequently, the transmissibility ratio is extended in this chapter to consider multiple DoFs systems, therefore, it will be a matrix, $\mathbf{T}_R$. Moreover, $\mathbf{T}_R$ will have units of $\text{rad}\cdot\text{s}^2/\text{m}$. This extension has been conducted considering the viscous damping, which results in the following dynamics model:

$$\mathbf{M}_s \ddot{\boldsymbol{\eta}} + \mathbf{C}_s \dot{\boldsymbol{\eta}} + \mathbf{K}_s \boldsymbol{\eta} = \mathbf{B}_k e^{i\omega_t t} \quad (4.15)$$

where $\mathbf{B}_k = \text{diag}(b_{k_1}, b_{k_2}) \in \mathbb{R}^{2 \times 2}$ is the acceleration amplitude matrix. The subscripts 1 and 2 in the definition of the matrices refer to the variables and the parameters related to the first and the second DoFs in Eq. (4.2), respectively. The particular solution to Eq. (4.15) is:

$$\boldsymbol{\eta} = \boldsymbol{\eta}_{max} e^{i\omega_t t} \quad (4.16)$$

where the maximum cable angle $\boldsymbol{\eta}_{max}$ can be found from Eqs. (4.15) and (4.16) as follows:

$$\boldsymbol{\eta}_{max} = \mathbf{B}_k \left[\left(\mathbf{K}_s - \mathbf{M}_s \boldsymbol{\omega}_t^2 \right) + i \boldsymbol{\omega}_t \mathbf{C}_s \right]^{-1} \quad (4.17)$$

where $\boldsymbol{\omega}_t = \text{diag}(\omega_t, \omega_t) \in \mathbb{R}^{2 \times 2}$. By multiplying the denominator and the numerator of the right hand side of Eq. (4.17) by the conjugate $\left[\left(\mathbf{K}_s - \mathbf{M}_s \boldsymbol{\omega}_t^2 \right) - i \boldsymbol{\omega}_t \mathbf{C}_s \right]$, then substitute with the frequency ration from Eq. (4.14), one can find the extended transmissibility matrix as follows:

$$\begin{aligned} \mathbf{T}_R &= \mathbf{B}_k^{-1} \boldsymbol{\eta}_{max} \\ &= \mathbf{K}_s^{-1} \left(\left[\left(\mathbf{I}_2 - \mathbf{R}_t^2 \right)^2 + (2\boldsymbol{\zeta} \mathbf{R}_t)^2 \right]^{-1} \right)^{0.5} \end{aligned} \quad (4.18)$$

where $\mathbf{T}_R = \text{diag}(T_{R_1}, T_{R_2}) \in \mathbb{R}^{2 \times 2}$ is the transmissibility matrix, $\mathbf{R}_t = \text{diag}(r_{t_1}, r_{t_2}) \in \mathbb{R}^{2 \times 2}$ is the frequency ratio matrix, and $\boldsymbol{\zeta} = \text{diag}(\zeta_1, \zeta_2) \in \mathbb{R}^{2 \times 2}$ is the viscose damping ratio matrix.

Although it is possible to have a dimensionless form of the transmissibility ratio, the form in Eq. (4.18) is found to be more suitable for the considered application because it relates the amplitudes of quadrotor's trajectory to the cable angles, $\boldsymbol{\eta}$, straightforwardly.

Considering the transportation system of interest, where there is no viscous damping, Fig. 4.5 is the graphical representation of Eq. (4.18), which can be used to specify the

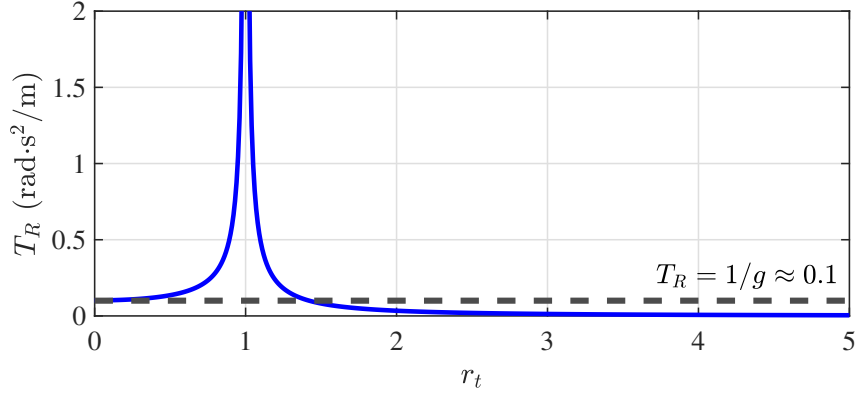


Figure 4.5: Transmissibility ratio of a transportation system at different frequency ratios

frequency ratio required to achieve a desired transmissibility ratio. It shows that if the system is close to the resonance, i.e., $0.95 \leq r_t \leq 1.05$, the transmissibility ratio, T_R , will be more than 1, which means that the transmitted vibration is more than the excitation. If the system has a frequency ratio $r_t < 1$, i.e., before the resonance, one can reduce the cable length to reduce the frequency ratio, and consequently reduce the transmissibility ratio. This will decrease the payload vibrations. Contrarily, if the system has a frequency ratio $r_t > 1$, i.e., after the resonance, one can increase the cable length to increase the frequency ratio, and consequently reduce the transmissibility ratio. This will decrease the payload vibrations. It is guaranteed to get $T_R < 1/g \approx 0.1 \text{ rad}\cdot\text{s}^2/\text{m}$, if $r_t > \sqrt{2} \approx 1.421$. In two DoFs, this can be interpreted as follows:

$$\mathbf{T}_R < 0.1\mathbf{I} \quad \forall \mathbf{R}_t > \text{diag}(\sqrt{2}, \sqrt{2}) \quad (4.19)$$

From Eqs. (4.14), (4.18), and (4.19), two conditions can be helpful to choose l and t_f to reduce the transmitted vibration from the base, i.e., the quadroto, to the slung payload. These conditions can be expressed mathematically as follows:

$$\begin{aligned} \frac{\sqrt{l_1}}{t_{f1}} &> \frac{\sqrt{g}}{2\pi k} r_{t1} \\ \frac{\sqrt{l_2}}{t_{f2}} &> \frac{\sqrt{g}}{2\pi k} r_{t2} \end{aligned} \quad (4.20)$$

where l_1 and l_2 are the required cable length to satisfy the desired transmissibility ratios in the first and the second DoFs, respectively, t_{f_1} and t_{f_2} are the required trajectory flight time to satisfy the desired transmissibility ratios in the first and the second DoFs, respectively, and r_{t_1} and r_{t_2} are the frequency ratios corresponding to the desired transmissibility ratios.

There are a few points, which need to be emphasized before proceeding with the discussion.

1. Mathematically, any combination of l and t_f that satisfies Eq. (4.20) is acceptable.
2. Practically, it may not be allowed to freely choose both l and t_f . For example, the transportation system may be constrained with a certain cable length l or a certain range of l , which means that it is not allowed to choose l freely. On the other hand, the transportation task may be required to be completed in a certain amount of time, which means that t_f is constrained and cannot be chosen freely. Consequently, it is important to investigate the cases, in which, only one of the two parameters, l and t_f , can be chosen freely.
3. For a certain transportation system, there is only one cable with a certain cable length, so l_1 and l_2 must be equal. Consequently, there is a need to find a way to choose if Eq. (4.20) results in different values of l_1 and l_2 . The same concept applies to flight times t_{f_1} and t_{f_2} .
4. The trajectory is represented as an infinite sum of linear combinations of sines and cosines based on Eq. (4.7), so changing k in Eq. (4.20) has a great effect on the way of choosing l and t_f , since their values will change at each time k changes.

To tackle these points, first, a clear method to identify how to address the effect of different values of k on specifying l and t_f is proposed. Second, this should be done within

the assumption that only one parameter can be changed; either l or t_f . Third, in case of two different values of l_1 and l_2 , or t_{f1} and t_{f2} , it is important to propose a method to choose which value should be used. A key point here is to rearrange Eq. (4.20) as follows:

$$\begin{aligned} l_1 &> \frac{w_1^2}{k^2} \\ l_2 &> \frac{w_2^2}{k^2} \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} t_{f1} &< v_1 k \\ t_{f2} &< v_2 k \end{aligned} \tag{4.22}$$

where $w_1 = \frac{r_{t1} t_{f1}}{2\pi} \sqrt{g}$, $w_2 = \frac{r_{t2} t_{f2}}{2\pi} \sqrt{g}$, $v_1 = \frac{2\pi}{r_{t1}} \sqrt{\frac{l_1}{g}}$, and $v_2 = \frac{2\pi}{r_{t2}} \sqrt{\frac{l_2}{g}}$ are all constants. Therefore, the flight time and frequency ratio are constrained in Eq. (4.21), i.e., t_f and r_t are given. On the other hand, the cable length and frequency ratio are constrained in Eq. (4.22), i.e., l and r_t are given.

From Eq. (4.21), specifying l using $k = 1$ guarantees that this l satisfies the transmissibility requirements for all k . Similarly, from Eq. (4.22), the value of t_f at $k = 1$ guarantees that this t_f satisfies the transmissibility requirements for all k .

If it happened that $l_1 \neq l_2$, the largest of them should be chosen as it satisfies the transmissibility requirements for both DoFs. Contrarily, if it happened that $t_{f1} \neq t_{f2}$, the smallest of them should be chosen.

It is worth noting that based on Eq. (4.12), decreasing t_f leads to increasing the amplitude, b_k , which leads to more payload vibrations. So, it is important to highlight the difference between the effect of the amplitude, b_k , and the transmissibility ratio, T_R . The amplitude, b_k , is the amplitude of the sinusoidal wave itself, which represents quadrotor's acceleration. This means that increasing this amplitude will increase the base excitation. The transmissibility ratio, T_R , on the other hand, indicates how much of the base excitation

will transfer to the slung payload, regardless of the amount of the base excitation itself. It is also important to notice that t_f in Eq. (4.12) is squared, whereas it is just t_f in Eq. (4.14). This means that decreasing t_f is not a suggested method to reduce payload vibration since the effect of increasing the amplitude, b_k , will be more than the effect of decreasing the transmissibility, T_R , which will lead, in most cases, to more payload vibrations.

At the end of this discussion it is important to sum up some points. First, the transportation task can be designed so that it can satisfy a required vibration level. To achieve that, a clear and straightforward design method was proposed and explained. Second, if the flight time, t_f , is not specified as a requirement, Eq. (4.12) can be used to determine t_f that can achieve certain amplitude b_k . Third, it is possible to achieve a two-stage vibration reduction. The first stage is achieved by choosing the a polynomial trajectory with low b_k , e.g., a MJT, as a desired trajectory. The second stage can be achieved, if possible, by decreasing the transmissibility ratio, T_R . The best way to decrease the transmissibility ratio is by changing the cable length, l , not the flight time t_f . Fourth, although the analysis in Section 4.2 have been conducted for a 1D trajectory, the results are still applicable to any 3D trajectory, which has a constant vertical acceleration $\ddot{z}_q$. This is because the analysis in Section 4.1 showed that the system has a unit modal matrix, which means that the vibratory dynamics model in Eq. (4.2) is decoupled. For the t is when the cable angles are small.

4.4 Simulation Results

This section presents and discusses the simulation results to illustrate the effect of the degree of the polynomial trajectory and the transmissibility ratio on quadrotor's payload vibration.

Two rest-to-rest trajectories were considered in this study: a two-point planar trajectory

and a multi-segment five-point 3D trajectory with sharp corners.

For the two-point trajectory, the quadrotor has to move from an initial position $\mathbf{r}_{q_0} = [0, 0, 1.2]^\top$ to a final position $\mathbf{r}_{q_f} = [2, 2, 1.2]^\top$. The trajectory flight time is chosen to be $t_f = 4$ seconds. This results in the following first dominating amplitudes, i.e., $k = 1$, in x and y directions, according to Eq. (4.12):

$$\begin{aligned} b_{1_j} &= 0.7257 \quad \text{m/s}^2 \\ b_{1_s} &= 0.8802 \quad \text{m/s}^2 \\ b_{1_c} &= 0.9862 \quad \text{m/s}^2 \end{aligned} \tag{4.23}$$

For the five-point trajectory, the quadrotor is required to pass through the following five waypoints in this order: $\mathbf{r}_{q_0} = [0, 0, 1.1]^\top$, $\mathbf{r}_{q_1} = [2, 0, 1.25]^\top$, $\mathbf{r}_{q_2} = [2, 2, 1.4]^\top$, $\mathbf{r}_{q_3} = [0, 2, 1.55]^\top$, and $\mathbf{r}_{q_f} = [0, 0, 1.7]^\top$. All of the four trajectory segments are required to have stationary boundary conditions in the horizontal inertial plane, i.e., $x - y$. For the vertical motion in z -direction, a constant velocity $\dot{z}_q = 0.0429$ m/s is utilized. The flight time of each of the four trajectory segments is 3.5 seconds. Using Eq. (4.12), the following first dominating amplitudes, i.e., $k = 1$, in x and y directions are obtained as follows:

$$\begin{aligned} b_{1_j} &= 0.9478 \quad \text{m/s}^2 \\ b_{1_s} &= 1.1496 \quad \text{m/s}^2 \\ b_{1_c} &= 1.2881 \quad \text{m/s}^2 \end{aligned} \tag{4.24}$$

and the first dominating amplitudes in z direction are as follows:

$$\begin{aligned} b_{1_j} &= 0.0711 \quad \text{m/s}^2 \\ b_{1_s} &= 0.0862 \quad \text{m/s}^2 \\ b_{1_c} &= 0.0966 \quad \text{m/s}^2 \end{aligned} \tag{4.25}$$

It is worth noting that Eqs. (4.23) - (4.25) show that increasing the degree of the polynomial increases the amplitude. This agrees with the discussions in Section 4.2 and Fig. 4.3.

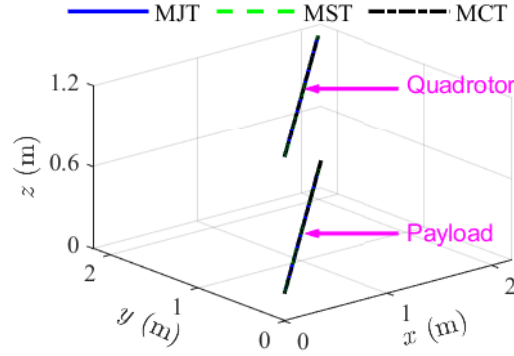


Figure 4.6: Simulation results of quadrotor and payload paths (two-point, $l = 1$ m)

This multi-segment 3D five-point trajectory was chosen with sharp corners on purpose because these sharp corners and the stationary boundary conditions of all trajectory segments in the horizontal inertial plane will break the smoothness between the trajectory segments and increase payload vibrations. This helps demonstrate the effectiveness of the proposed methods in reducing payload vibration.

Slung payload vibrations are quantified by the two cable angles, θ_1 and θ_2 , i.e., $\boldsymbol{\eta}$. The trajectory that has the lowest maximum cable angles, θ_1 and θ_2 , is said to be the trajectory that results in the lowest payload vibration. In all simulations, a payload of mass $m_p = 0.2$ kg is considered.

4.4.1 Effect of the Degree of the Polynomial Trajectory

A cable length of $l = 1$ m is considered to examine the effect of the degree of the polynomial on payload vibrations, while a MJT, a MST, and a MCT are considered. The simulation results of the two-point planar trajectory is presented followed by the results of the multi-segment five-point 3D trajectory.

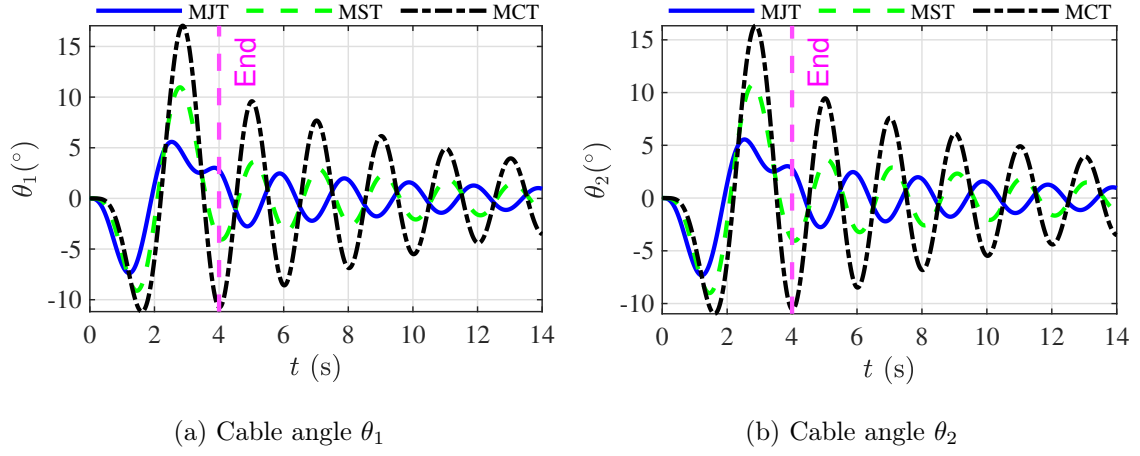


Figure 4.7: Simulation results of cable angles θ_1 and θ_2 (two-point, $l = 1$ m)

4.4.1.1 Two-Point Planar Trajectory

The path of the quadrotor and the slung payload are presented in Fig. 4.6, while the cable angles, θ_1 and θ_2 , are shown in Fig. 4.7. These results show that increasing the degree of the polynomial increases payload vibrations monotonically. Note that the vertical dashed magenta line in Fig. 4.7 indicates the end of the trajectory. The vibrations after this line are naturally damped, while the quadrotor is stationary.

4.4.1.2 Five-Point 3D Trajectory

The 3D paths of the quadrotor and the slung payload are shown in Fig. 4.8. The cable angles, θ_1 and θ_2 , are shown in Fig. 4.9. These results show that increasing the degree of the polynomial increases payload vibrations. The vertical dashed magenta lines in Fig. 4.9, denote the end of each trajectory segment. The vibrations after the last line represents the naturally damped vibrations, while the quadrotor is stationary.

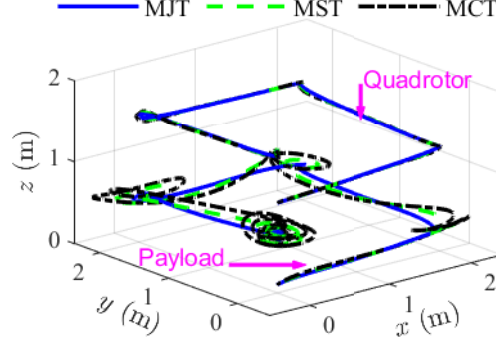


Figure 4.8: Simulation results of quadrotor and payload paths (five-point, $l = 1$ m)

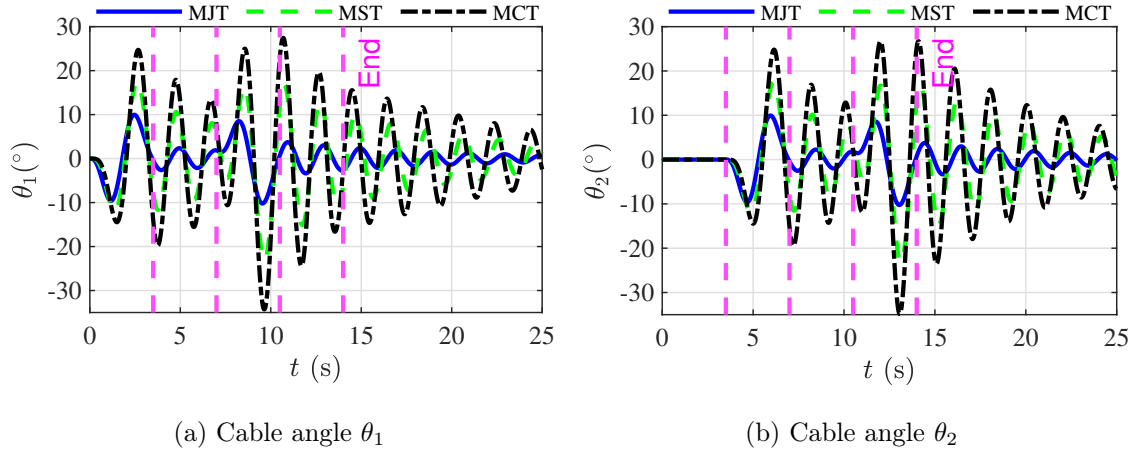


Figure 4.9: Simulation results of cable angles θ_1 and θ_2 (five-point, $l = 1$ m)

4.4.2 Effect of the Transmissibility Ratio

In this subsection, the effectiveness of the transmissibility ratio in reducing payload vibrations as a second stage of payload vibration reduction process is investigated. Recall that the first stage is by choosing a polynomial trajectory of a low degree, e.g., MJT.

In our case, Fig 4.5 and Eq. (4.18) tell that it is possible to reduce the slung payload vibration level by reducing the transmissibility ratio, $\mathbf{T}_R$, through two different methods. The first method is by reducing the frequency ratio, $\mathbf{R}_t$, to be less than $\text{diag}(0.5015, 0.5015)$

Table 4.1: Diagonal elements of the frequency and transmissibility ratio matrices for different cable lengths for both two-point and five-point trajectories

Trajectory	$l = 0.5$ m		$l = 1$ m		$l = 16$ m	
	$\mathbf{R}_t$	$\mathbf{T}_R$ (rad·s ² /m)	$\mathbf{R}_t$	$\mathbf{T}_R$ (rad·s ² /m)	$\mathbf{R}_t$	$\mathbf{T}_R$ (rad·s ² /m)
Two-point	(0.3546, 0.3546)	(0.1166, 0.1166)	(0.5015, 0.5015)	(0.1362, 0.1362)	(2, 2)	(0.034, 0.034)
Five-point	(0.4053, 0.4053)	(0.122, 0.122)	(0.5732, 0.5732)	(0.1518, 0.1518)	(2.3, 2.3)	(0.024, 0.024)

in case of the two-point trajectory, and less than $\text{diag}(0.5732, 0.5732)$ in case of the five-point trajectory. Alternatively, the payload vibration level can be reduced by increasing the frequency ratio, $\mathbf{R}_t$, to be more than $\text{diag}(1.3223, 1.3223)$ in case of the two-point trajectory, and more than $\text{diag}(1.2929, 1.2929)$ in case of the five-point trajectory. The case where $\mathbf{T}_R < 0.1\mathbf{I}$ rad·s²/m is worth demonstration. That is when $\mathbf{R}_t > \text{diag}(\sqrt{2}, \sqrt{2})$ is considered for both trajectories. As discussed in Section 4.3, the selected method to achieve that is to change the cable length. Consequently, there are two different ways to provide a second stage of vibration reduction. For the two-point trajectory, these two ways are either $l < 1$ m or $l > 7.95$ m, while for the five-point trajectories, the two ways are either $l < 1$ m or $l > 6.088$ m. For consistency, a *short cable* of length $l = 0.5$ m and a *long cable* of length $l = 16$ m are chosen to investigate the effect of reducing the transmissibility ratio on payload vibrations for both trajectories. The terms *short cable* and *long cable* are used in the rest of this chapter for simplicity. The corresponding frequency and transmissibility ratios of these cable lengths for the two trajectories are shown in Table 4.1.

4.4.2.1 Two-Point Planar Trajectory

The simulation results of the cable angles, θ_1 and θ_2 , are plotted in Fig. 4.10. Figs. 4.10a and 4.10b show the results of the *short cable*, whereas Figs. 4.10c and 4.10d show the results of the *long cable*. First, these results confirm that the increasing the degree of the

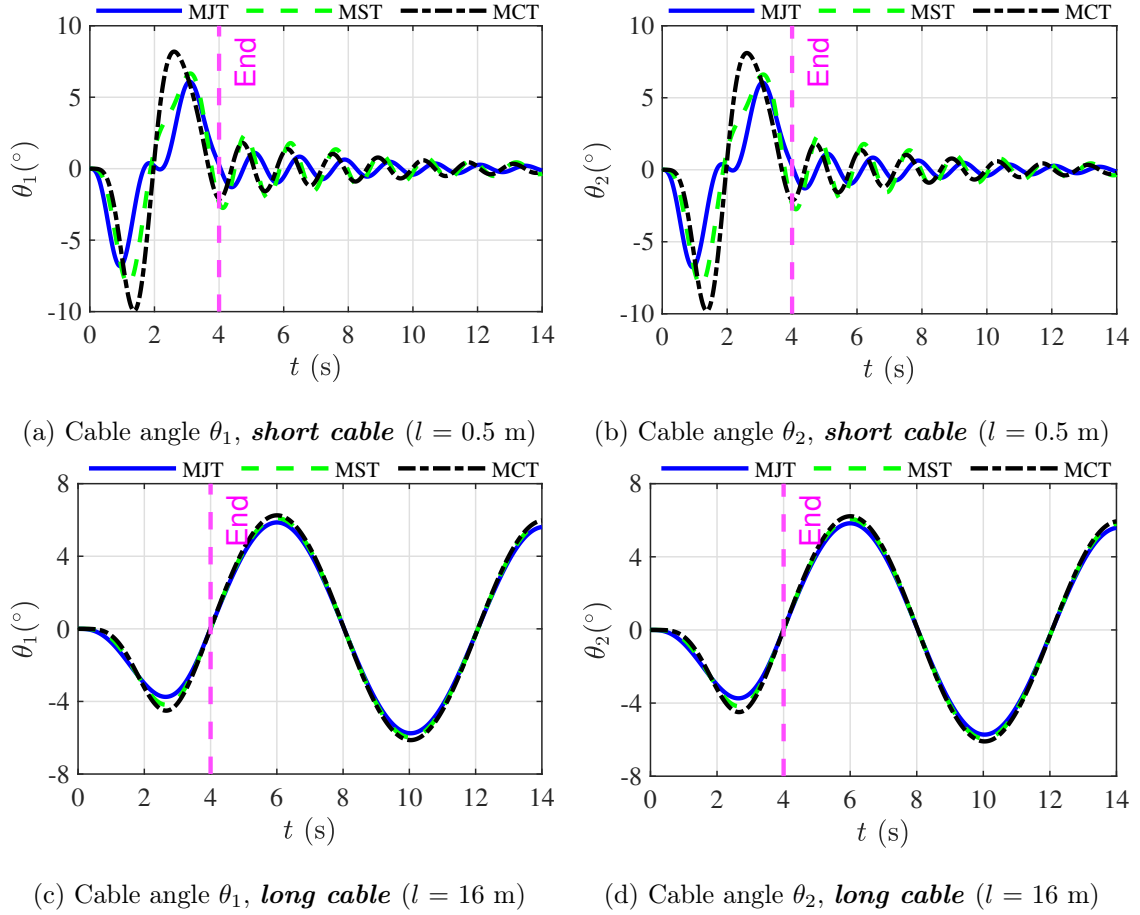


Figure 4.10: Simulation results of cable angles θ_1 and θ_2 (two-point, *short* and *long*)

polynomial increases payload vibrations, which agrees with the conclusions of Section 4.4.1. Second, the results show that the transmissibility ratio can be used to provide a second stage of vibration reduction since all cable angles in Fig. 4.10 are less than their analogs in Fig. 4.7. The main conclusion here is that decreasing the transmissibility ratio can provide a second stage of vibration reduction.

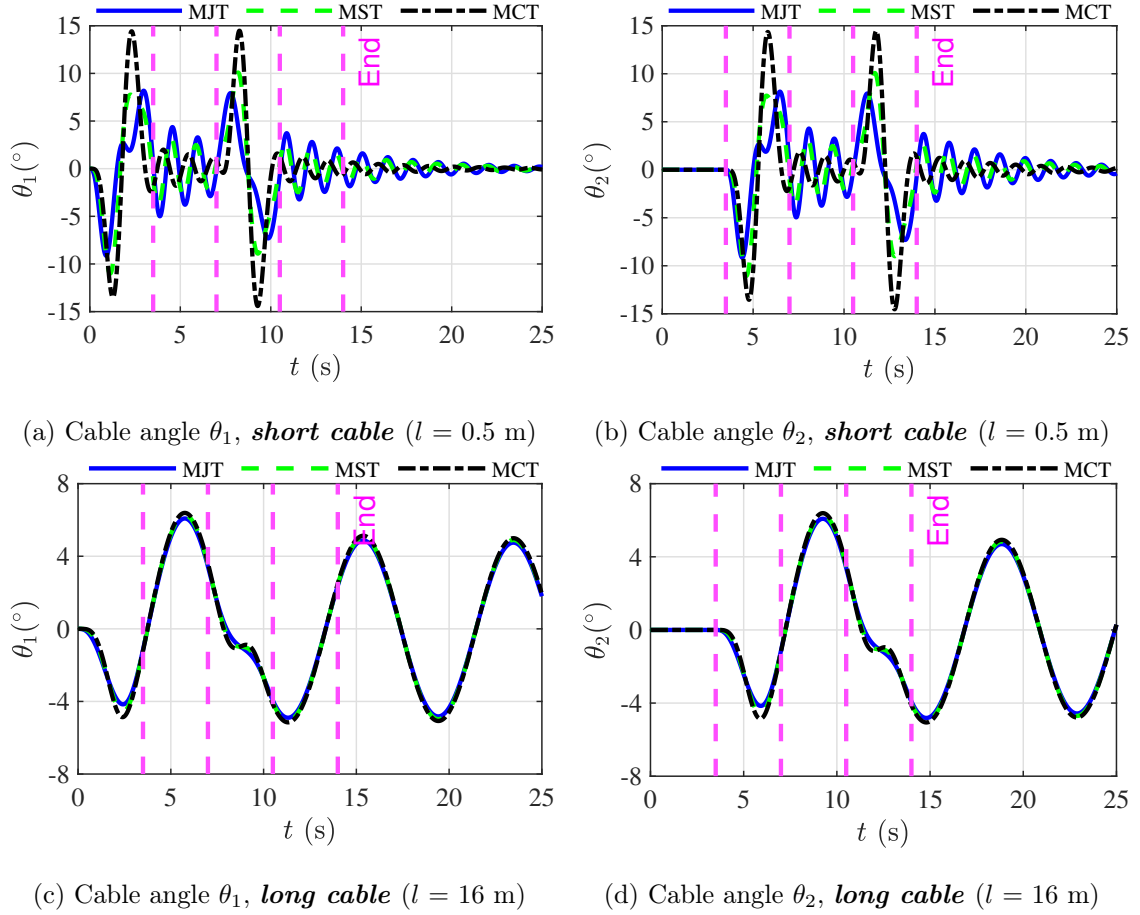


Figure 4.11: Simulation results of cable angles θ_1 and θ_2 (five-point, *short* and *long*)

4.4.2.2 Five-point 3D Trajectory

The simulation results of the cable angles, θ_1 and θ_2 , are plotted in Fig. 4.11. Figs. 4.11a and 4.11b show the results of the *short cable*, whereas Figs. 4.11c and 4.11d show the results of the *long cable*.

Similar to the two-point trajectory results in Section 4.4.2.1, in addition to confirming that the increasing the degree of the polynomial increases payload vibrations, the results show that the transmissibility ratio can be used to provide a second stage of vibration reduction. All cable angles in Fig. 4.11 are less than their analogs in Fig. 4.9. The main

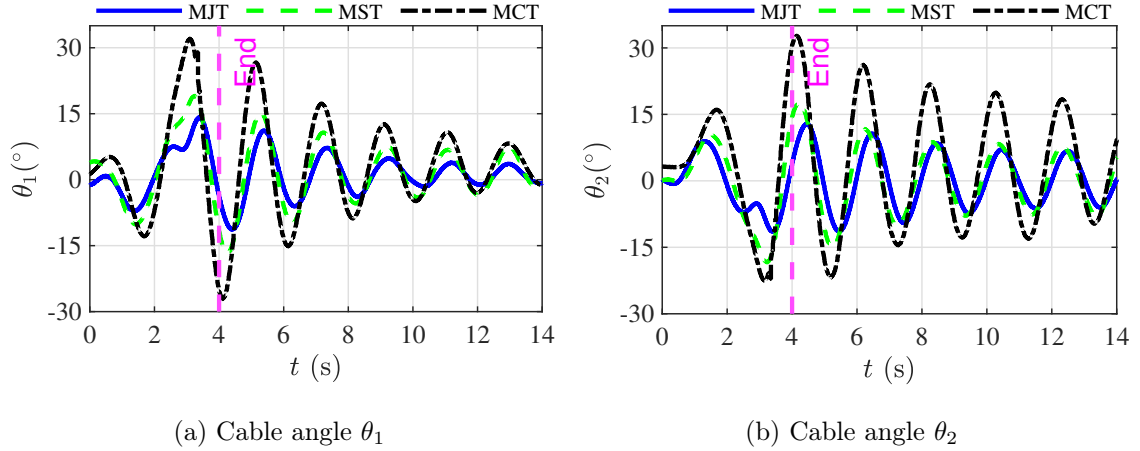


Figure 4.12: Experimental results of cable angles θ_1 and θ_2 (two-point, $l = 1$ m)

conclusion here, again, is that decreasing the transmissibility ratio can provide a second stage of vibration reduction.

4.5 Experimental Results

The objective here is to experimentally validate the effect of the degree of the polynomial trajectory on payload vibrations and to show that the transmissibility ratio can provide a second stage of payload vibration reduction. The blue payload shown in Fig. 2.6b, which has a mass of $m_p = 0.2$ kg, was used. This section follows the same trajectories presented in Section 4.4. However, due to space limitations, the experiments for the *long cable* case were not conducted. The test videos can be found at <https://youtu.be/6umbq97AK00>.

4.5.1 Effect of the Degree of the Polynomial Trajectory

A cable length of $l = 1$ m is considered for the experiments of both the two-point planar trajectory and the five-point 3D trajectory.

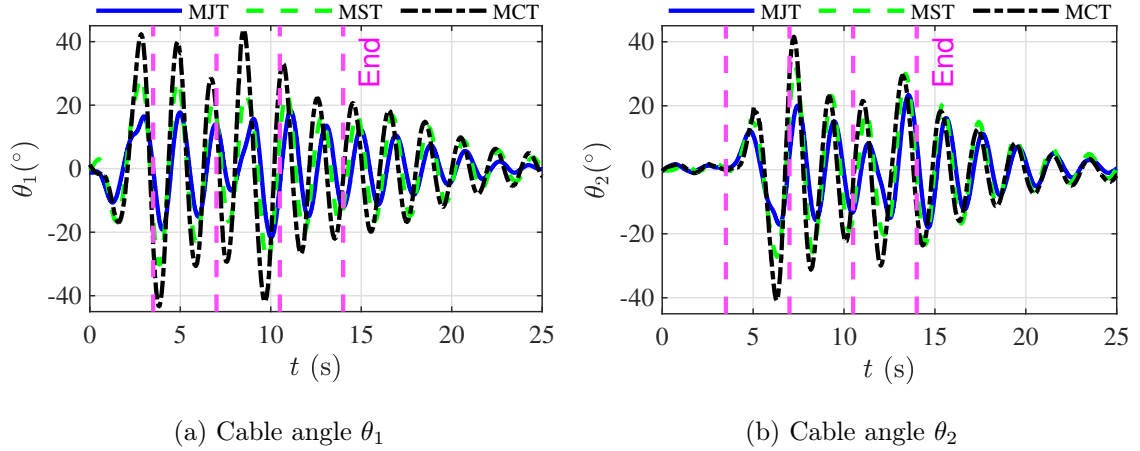


Figure 4.13: Experimental results of cable angles θ_1 and θ_2 (five-point, $l = 1$ m)

4.5.1.1 Two-point Planar Trajectory

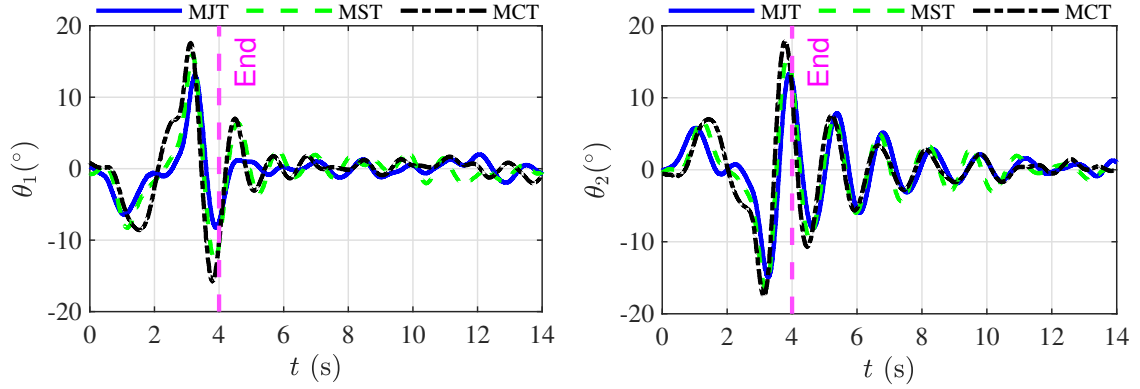
The experimental results of the cable angles, θ_1 and θ_2 , for the two-point trajectory are shown in Fig. 4.12. These experimental results confirm that increasing the degree of the polynomial increases payload vibrations.

4.5.1.2 Five-point 3D Trajectory

The experimental results of the cable angles, θ_1 and θ_2 , for the five-point trajectory are shown in Fig. 4.13. Similar to Section 4.5.1.1, the experimental results confirm that increasing the degree of the polynomial increases payload vibrations.

4.5.2 Effect of the Transmissibility ratio

A cable length of $l = 0.5$ m is considered for the experiments of both the two-point planar trajectory and the five-point 3D trajectory. Recall that the experimental validation is provided only for the *short cable* case.



(a) Cable angle θ_1 , *short cable* ($l = 0.5$ m) (b) Cable angle θ_2 , *short cable* ($l = 0.5$ m)

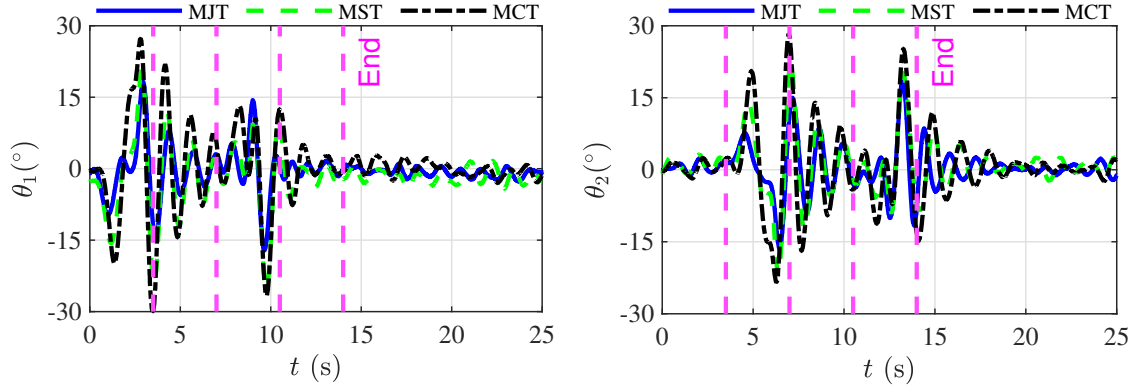
Figure 4.14: Experimental results of cable angles θ_1 and θ_2 (two-point, *short cable*)

4.5.2.1 Two-point Planar Trajectory

The experimental results of the cable angles, θ_1 and θ_2 , for the two-point trajectory are shown in Fig. 4.14. First, the results confirm that increasing the degree of the polynomial increases payload vibrations. Second, it shows, experimentally, that the transmissibility ratio can be used to provide a second stage of vibration reduction. This can be seen by comparing the results in Fig. 4.14 to Fig. 4.12.

4.5.2.2 Five-point 3D Trajectory

The experimental results of the cable angles, θ_1 and θ_2 , for the two-point trajectory are shown in Fig. 4.15. Similar to Section 4.5.2.1, the results confirm that increasing the degree of the polynomial increases payload vibrations. Second, it shows, experimentally, that the transmissibility ratio can be used to provide a second stage of vibration reduction. This can be seen by comparing the results in Fig. 4.15 to Fig. 4.13.



(a) Cable angle θ_1 , *short cable* ($l = 0.5$ m) (b) Cable angle θ_2 , *short cable* ($l = 0.5$ m)

Figure 4.15: Experimental results of cable angles θ_1 and θ_2 (five-point, *short cable*)

4.5.3 Remarks on Simulation and Experimental Results

Using the four-term approximation approach to calculate trajectories' amplitudes as introduced in Section 4.2, and the transmissibility ratio matrix presented in Table 4.1, the maximum cable angles can be calculated. A summary of the maximum calculated, simulated, and experimental cable angles for $l = 1$ m cases is presented in Table 4.2. The results in Table 4.2 show that, for the two-point trajectory, the calculated results are very close to the simulation results. As for the experimental results, they are not as close to the calculated results as simulations. A main reason for that is that the calculated cable angles are based on a linearized model, which is not the case especially for large cable angles. Additionally, uncertainties and disturbances in real experiments have an effect.

For the five-point trajectory, there is a larger difference between the calculated angles and the simulation results. The main reason for this difference, which does not appear in the two-point trajectory case, is that the calculated angles were calculated for each trajectory segment separately. This means assuming zero initial condition at each way point, i.e., zero cable angles and zero cable angles rates. Contrarily, in the actual case, each

Table 4.2: Maximum cable angles ($l = 1$ m)

<i>Two-point Trajectory</i>						
	Calculated		Simulation		Experimental	
	$\theta_{1_{max}}$	$\theta_{2_{max}}$	$\theta_{1_{max}}$	$\theta_{2_{max}}$	$\theta_{1_{max}}$	$\theta_{2_{max}}$
MJT	9.4°	9.4°	7.3°	7.3°	14.1°	12.7°
MST	12°	12°	10.7°	10.7°	19°	17.5°
MCT	15.6°	15.6°	17°	16.3°	32°	32.7°
<i>Five-point Trajectory</i>						
	Calculated		Simulation		Experimental	
	$\theta_{1_{max}}$	$\theta_{2_{max}}$	$\theta_{1_{max}}$	$\theta_{2_{max}}$	$\theta_{1_{max}}$	$\theta_{2_{max}}$
MJT	9.7°	9.7°	9.9°	9.9°	21.5°	23.2°
MST	12.3°	12.3°	23°	23.2°	30.2°	32.1°
MCT	16°	16°	34°	34.8°	43°	41.5°

trajectory segment starts with a non-zero initial conditions, which are equal to the actual final conditions of the previous segment because the slung payload reaches the intermediate waypoints with oscillations, i.e., non-zero cable angles and cable angles rates. Fig. 4.16 shows the cable angles and the cable angles' rates at each of the intermediate waypoints, i.e., $t = \{3.5, 7, 10.5\}$ s, which are indicated by the vertical dashed magenta lines. Fig. 4.16 also explains why the MJT simulation results are close to the calculated results since the MJT reaches the intermediate waypoints with cable angles and cable angles' rates close to zero. These non-zero cable angles rates in addition to nonlinearity, uncertainties, and disturbances contribute to increase the difference between the calculated cable angles and the experimental ones.

Similarly, a summary of the maximum calculated, simulated, and experimental cable angles for both the *short cable* cases is reported in Table 4.3, whereas Table 4.4 presents a summary of the maximum calculated and simulated cable angles for both the *long cable* cases. The differences between calculated, simulated, and experimental maximum cable angles are due to the same reasons explained in Table 4.2.

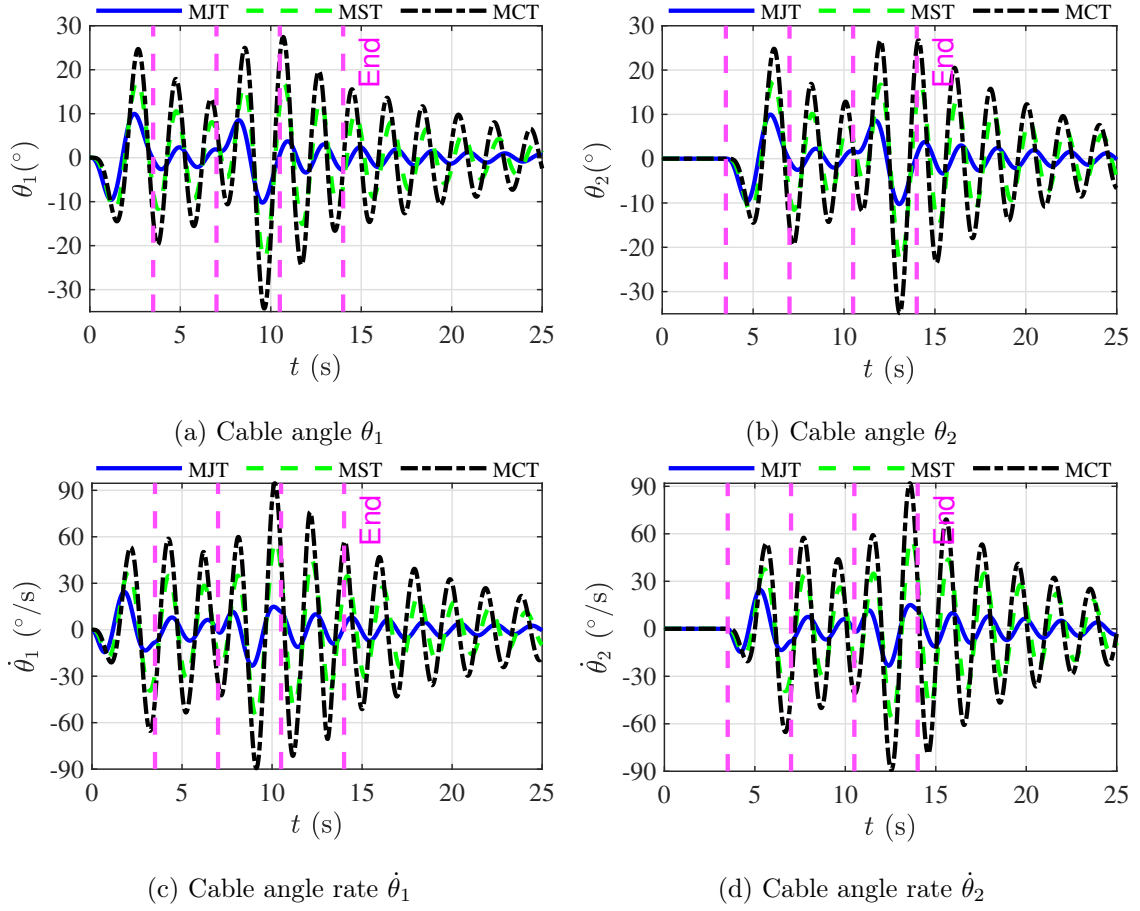


Figure 4.16: Simulation results of cable angles and cable angles rates θ_1 , θ_2 , $\dot{\theta}_1$, and $\dot{\theta}_2$ (five-point, $l = 1$ m)

More importantly, the results summarized in Tables 4.2 - 4.4, confirm the main two conclusions of this chapter:

- I) Increasing the degree of the polynomial increases payload vibrations.
- II) The transmissibility ratio can be used to provide a second stage of payload vibration reduction.

These two conclusions are validated for the calculated, simulated, and experimental results.

Table 4.3: Maximum cable angles (*short cable*)

<i>Two-point Trajectory</i>						
	Calculated		Simulation		Experimental	
	θ_{1max}	θ_{2max}	θ_{1max}	θ_{2max}	θ_{1max}	θ_{2max}
MJT	8.1°	8.1°	6.8°	6.7°	12.9°	13°
MST	10.3°	10.3°	8°	7.9°	16°	16.5°
MCT	13.3°	13.3°	9.9°	9.7°	17.5°	17.5°
<i>Five-point Trajectory</i>						
	Calculated		Simulation		Experimental	
	θ_{1max}	θ_{2max}	θ_{1max}	θ_{2max}	θ_{1max}	θ_{2max}
MJT	7.8°	7.8°	9.1°	9.2°	18.2°	18.2°
MST	9.9°	9.9°	11°	11°	25°	22.1°
MCT	12.9°	12.9°	14.4°	14.5°	30°	28.2°

4.6 Summary

This chapter considers a transportation system consisting of a quadrotor with a slung payload. The main objective is to reduce payload vibrations, which will reduce quadrotor's energy consumption. A mathematical vibration analysis of the transportation system as well as a frequency analysis of polynomial trajectories are conducted. These analyses led to investigate the effect of polynomial trajectories on payload vibrations.

More importantly, these analyses led to proposing a two-stage approach to reduce payload vibrations. The first stage is by choosing a polynomial degree of the lowest possible degree based on the application. The second stage is by choosing a transmissibility ratio that can further reduce payload vibrations. The proposed approach is a trajectory-based approach; therefore, it can be used with any controller.

The results and conclusions of this chapter are validated by simulations and experiments. Two different trajectories are considered: a two-point planar trajectory, and a multi-segment five-point 3D trajectory with sharp corners.

Table 4.4: Maximum cable angles (*long cable*)

<i>Two-point Trajectory</i>				
	Calculated		Simulation	
	$\theta_{1_{max}}$	$\theta_{2_{max}}$	$\theta_{1_{max}}$	$\theta_{2_{max}}$
MJT	2.4°	2.4°	5.8°	5.8°
MST	3°	3°	6.1°	6°
MCT	3.9°	3.9°	6.2°	6.2°
<i>Five-point Trajectory</i>				
	Calculated		Simulation	
	$\theta_{1_{max}}$	$\theta_{2_{max}}$	$\theta_{1_{max}}$	$\theta_{2_{max}}$
MJT	1.5°	1.5°	6°	6°
MST	2°	2°	6.2°	6.2°
MCT	2.5°	2.5°	6.3°	6.3°

The main conclusions of this chapters are that increasing the degree of the polynomial increases payload vibrations in addition to borrowing the concept of transmissibility ratio from car suspension systems and apply it to the transportation system of interest to provide a second stage of payload vibration reduction.

5 Trajectory-Based Energy-Saving

Quadrotor's energy consumption is a major limitation that needs to be overcome before package delivery using quadrotors becomes widely used. This is a challenging objective with a considerable amount of effort made in the literature to overcome this limitation (see Section 1.2). As reported in Section 1.2, there are mainly four approaches to reduce quadrotors' energy consumption: a) enhancing/increasing battery performance/capacity; b) modifying UAV's configuration; c) controller design; and d) trajectory design. This chapter focuses on the latter approach, whereas the Chapter 6 focuses on the controller design approach.

Design energy-minimized trajectories by minimizing a cost function of quadrotor's energy is the most common approach in the literature. This has been conducted for a quadrotor flying between two points without wind in [3, 43], and in the presence of wind in [97]. The same concept was applied to a hexarotor with tilting rotors in [66]. A constrained multi-objective minimization problem was solved to find an energy-minimized trajectory for a hexarotor, while considering the constraints associated with the state of health of LiPo battery in [132]. As stated in Section 1.2, in addition to the computational cost associated with solving this optimization problem to find an energy-minimized trajectory for each task, a main drawback of this method is that it depends on many quadrotor-related parameters, e.g., motor and propeller parameters. This limits the designed trajectories

to certain quadrotors with same quadrotor-related parameters. In contrast, this chapter approaches trajectory-based energy-saving strategies differently. Since polynomial trajectories are known for their smoothness and ease of implement as presented in Section 1.2, this chapter investigates their energy performance, then discusses the possibility of replacing energy-minimized trajectories with polynomial trajectories. This provides a more general approach that does not depend on quadrotor-related parameters in addition to the simplicity of implementing such trajectories in practice.

Focusing on investigating the effect of polynomial trajectories on energy consumption has many reasons. First, they are widely used in quadrotor applications. Second, they are easy to implement in real life. Third, they are computationally efficient. Forth, our proposed approach in Section 3.1 extends the capabilities of polynomial trajectories and enhances their computational efficiency. Fifth, polynomial trajectories are known for their smoothness, so they have the potential to be energy-efficient. Last, but not least, human manipulation motion was found to be very close to the MJT [106]. Recall that, human manipulation motion, like most of things in nature, tends to follow trajectories that optimize a certain cost function, which is now known as the principle of least action motivated by Maupertuis and then formally formulated by Hamilton [133, 134].

Despite all the potential of polynomial trajectories to provide a practical, efficient, and easy-to-implement trajectory-based energy-saving approach for quadrotor applications, a comprehensive investigation of such trajectories has not been conducted. Therefore, Section 5.1 mathematically investigates the energy performance of polynomial trajectories for quadrotors and quadrotors with slung payloads based on the models of power and energy quotients proposed in Section 2.3. A comparative study with energy-minimized trajectories for quadrotors is also conducted in Section 5.1. Next, extensive experimental validation is presented in Section 5.2, where a wide variety of scenarios are considered for

both a single quadrotor and a quadrotor with a slung payload.

5.1 Energy Analysis of Polynomial Trajectories

We proposed the concepts of power and energy quotients in Section 2.3 to provide an energy model that eliminates the existence of quadrotor-related parameters, e.g., motor and propeller parameters, from the energy expression. Recall that the power and energy quotients are introduced in Eq. (2.54), which can be summarized as follows:

$$\begin{aligned} P_Q &= \frac{P}{\gamma} = F^{1.5} \\ E_Q &= \frac{E}{\gamma} = \int_{t_0}^{t_f} F^{1.5} dt \end{aligned} \tag{5.1}$$

where γ is constant for non-aggressive flights as defined and discussed in Section 2.3.

The power and energy quotients depend on the total thrust, which depend on quadrotor's trajectory as shown in Eqs. (2.25) and (2.38). Precisely, it depends on the acceleration of the trajectory, which is well-known from Newtonian mechanics, i.e., Newton's second law. So, we analyze the acceleration of polynomial trajectories to provide intuition about the energy behaviour of polynomial trajectories. We start by a mathematical analysis of rest-to-rest polynomial trajectories in Section 5.1.1, then a comprehensive statistical analysis via Monte Carlo simulations is conducted in Section 5.1.2 to consider arbitrary boundary conditions scenarios. The last part of this section presents a comparative study with an energy-minimized trajectory in Section 5.1.3

5.1.1 Rest-to-Rest Scenario

Consider a rest-to-rest polynomial trajectory starts at $t_0 = 0$ and ends at t_f such as the one expressed in Eq. (3.2) and its acceleration:

$$\begin{aligned}\mathbf{r}_k &= \mathbf{c}\mathbf{t} \in \mathbb{R}^d \\ \mathbf{a}_k &= \ddot{\mathbf{r}}_k = \mathbf{c} \frac{d^2 \mathbf{t}}{dt^2} \in \mathbb{R}^d\end{aligned}\tag{5.2}$$

where the subscript k denotes the degree of the polynomial trajectory, $k = \{a, j, s, c\}$, which refer to MAT, MJT, MST, and MCT, respectively. Additionally, the time vector is $\mathbf{t} = [t^j, t^{j-1}, \dots, 1]^\top \in \mathbb{R}_{\geq 0}^{j+1}$ and the coefficient matrix is $\mathbf{c} \in \mathbb{R}^{d \times (j+1)}$ as defined in Section 3.1.

Without loss of generality, the motion starts from the origin and has a total travelled distance $\mathbf{D} = [D_1, D_2, \dots, D_d]^\top \in \mathbb{R}^d$. To help generalize the conclusions, consider the relative time, $T = t/t_f$, $0 \leq T \leq 1 \forall t$. From Eq. (5.2), the acceleration of different polynomial trajectories can be expressed as follows:

$$\begin{aligned}\mathbf{a}_a &= -6\mathbf{T}_a \mathbf{a}_{av} \\ \mathbf{a}_j &= 60\mathbf{T}_j \mathbf{a}_{av} \\ \mathbf{a}_s &= -420\mathbf{T}_s \mathbf{a}_{av} \\ \mathbf{a}_c &= 2520\mathbf{T}_c \mathbf{a}_{av}\end{aligned}\tag{5.3}$$

where $\mathbf{a}_{av} = \mathbf{D}/t_f^2$ can be called the average acceleration and

$$\begin{aligned}\mathbf{T}_a &= \text{diag}(2T - 1) \in \mathbb{R}^{d \times d} \\ \mathbf{T}_j &= \text{diag}(2T^3 - 3T^2 + T) \in \mathbb{R}^{d \times d} \\ \mathbf{T}_s &= \text{diag}(2T^5 - 5T^4 + 4T^3 - T^2) \in \mathbb{R}^{d \times d} \\ \mathbf{T}_c &= \text{diag}(2T^7 - 7T^6 + 9T^5 - 5T^4 + T^3) \in \mathbb{R}^{d \times d}\end{aligned}\tag{5.4}$$

To further generalize the conclusions, the accelerations in Eq. (5.3) can be normalized using the scale $\mathbf{a}_{av}$. This results in a normalized acceleration $\mathbf{a}_{k_n} = \mathbf{a}_k / \mathbf{a}_{av}$. An example of the normalized acceleration in any arbitrary direction, a_{k_n} , is plotted in Fig. 5.1. It can be seen that increasing the degree of the polynomial increases the peak value of the

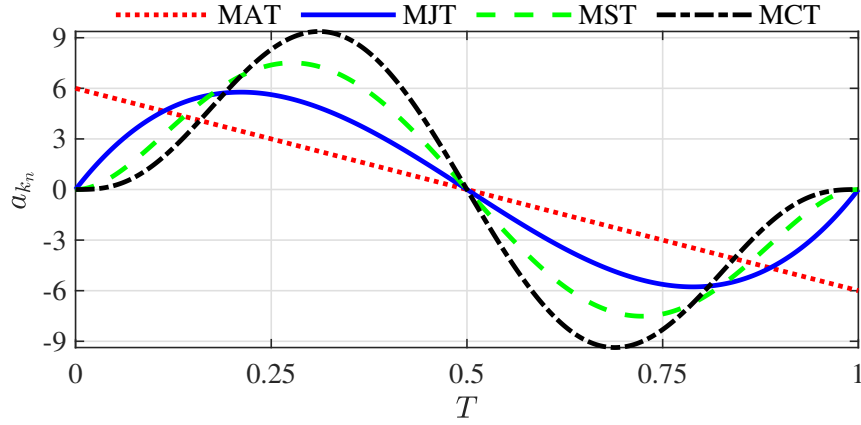


Figure 5.1: Normalized acceleration profile for a rest-to-rest scenario

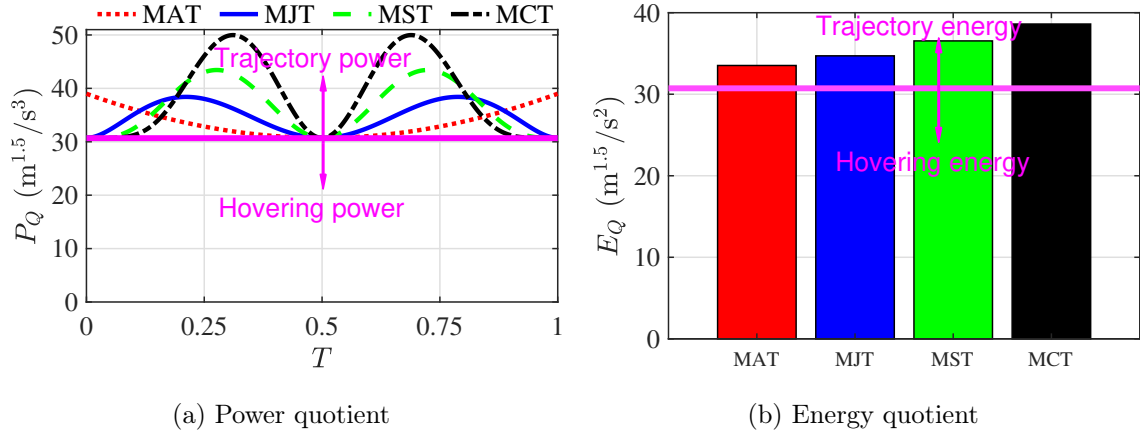


Figure 5.2: Power and energy quotients for a rest-to-rest scenario

acceleration and the area under the acceleration curve. Recalling the dynamics model in Eqs. (2.25) and (2.38) gives an intuition that this may increase the energy consumption.

5.1.1.1 Single Quadrotor

This section analyzes the effect of polynomial trajectories on a single quadrotor. A specific version of P_Q and E_Q quotients for a single quadrotor can be expressed as:

$$\begin{aligned}
P_Q &= \frac{P}{\gamma_m} = \|\ddot{\mathbf{r}}_q + \mathbf{g}\|^{1.5} \\
E_Q &= \frac{E}{\gamma_m} = \int_{t_0}^{t_f} \|\ddot{\mathbf{r}}_q + \mathbf{g}\|^{1.5} dt
\end{aligned} \tag{5.5}$$

where $\mathbf{g} = [0, 0, g]^\top$ and γ_m is constant for non-aggressive flights, which can be defined as $\gamma_m = \gamma m_q^{1.5}$ based on the definitions in Section 2.3.

To simplify the mathematical analysis, consider a rest-to-rest scenario in 1D with an average acceleration $a_{av} = 1$ and no change in elevation, i.e., no motion in the inertial z -direction. The results of the power and energy quotients, P_Q and E_Q , respectively, are presented in Fig. 5.2. It can be found that increasing the degree of the polynomial increases quadrotor's power and its energy consumption as expected from the analysis in Section 5.1.1. The horizontal straight magenta line in Fig. 5.2 indicates the hovering component of P_Q and E_Q . At an average acceleration $a_{av} = 1$, most of P_Q and E_Q is due to hovering. This is a well-known property for quadrotors as heavier-than-air vehicles (HTAVs). The rest of P_Q and E_Q is due to trajectory-following.

Finally, it is worth noting that the hovering component is constant and represented by the gravity vector $\mathbf{g}$ in Eq. (5.5), whereas the trajectory-following component is time-varying and represented by the acceleration $\ddot{\mathbf{r}}_q$ in Eq. (5.5).

For the sake of completeness, the effect of the average acceleration a_{av} on quadrotor's E_Q is illustrated in Fig. 5.3, where the average acceleration a_{av} ranges from 0 to 100 m/s². It can be seen that increasing the average acceleration a_{av} increases the quadrotor's energy consumption for the same polynomial trajectory type, i.e., the same polynomial degree. Additionally, increasing the average acceleration increases the energy difference between different polynomial trajectories. For small values of a_{av} , the difference between different polynomial trajectories is relatively small compared to larger values of a_{av} .

It is worth mentioning that MATs are energy efficient; however, they have some drawbacks. MATs allow only for setting kinematic constraints on positions and velocities as

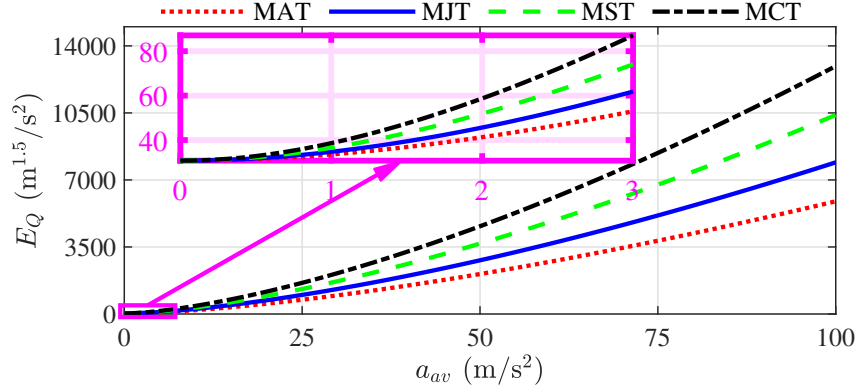


Figure 5.3: Effect of the average acceleration a_{av} for a rest-to-rest scenario

seen in Eq. (5.3). So, sudden changes in acceleration at the beginning and the end of flight are expected, which are seen in Fig. 5.1. This may be undesirable in many applications. In such cases, MJTs can be considered as the second best option from the perspective of quadrotor's energy consumption as shown in Fig. 5.2.

5.1.1.2 Quadrotor with a Slung Payload

This section considers energy analysis of a quadrotor with a slung payload following a rest-to-rest trajectory. Recall that the dynamics model in Eq. (2.40) consists of two equations, Eqs. (2.38) and (2.39). The translational dynamics in Eq. (2.38) shows that the thrust vector, $\mathbf{F}$, depends on quadrotor and payload masses, m_q and m_p , cable length, l , quadrotor's trajectory represented by its acceleration, $\ddot{\mathbf{r}}_q$, and payload vibration dynamics, $\ddot{\boldsymbol{\eta}}$ and $\dot{\boldsymbol{\eta}}$. Payload vibrations in Eq. (2.39), on the other hand, shows that payload vibration dynamics, $\ddot{\boldsymbol{\eta}}$, depends on quadrotor's trajectory, $\ddot{\mathbf{r}}_q$, and cable length, l . Therefore, the thrust vector, $\mathbf{F}$, and consequently quadrotor's power and energy quotients, P_Q and E_Q , depend on quadrotor and payload masses, cable length and quadrotor's trajectory.

This section analyzes the effect of the aforementioned three elements on quadrotor's energy consumption. Dynamically, quadrotor's motion in z -direction is decoupled from

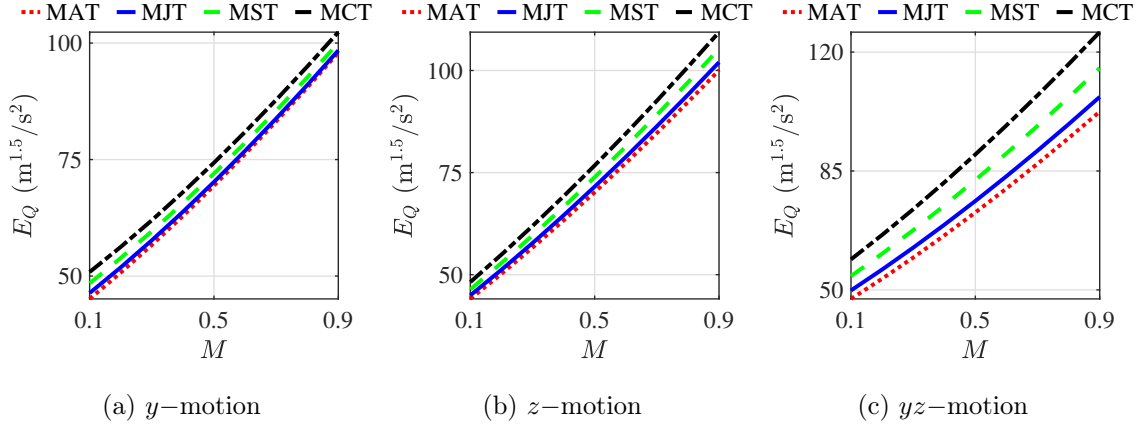


Figure 5.4: Effect of payload-to-quadrotor mass ratio, M , on quadrotor's energy quotient, E_Q , ($l = 1$ m and $\mathbf{a}_{av} = [1, 1]^\top$ m/s²)

the motion in the rest of the directions. Similarly, the rotation in the yaw direction, i.e., ψ , is decoupled from the rest motions. Contrarily, the motion in x - and y -directions are coupled with pitch and roll motions, θ and ϕ , respectively. However, and without loss of generality, this mathematical analysis considers the motion in the inertial $y - z$ plane. This simplifies the mathematical analysis, while capturing the dynamical behaviour of 3D flights since the motion in either x - or y -direction is mathematically the same; however, they are coupled with different angles, i.e., roll and pitch angles.

Payload Mass Effect

Recall that a concise version of payload vibrations in Eq. (2.39) is presented in Eq. (4.1), where payload mass does not affect payload vibration dynamics since it is similar to a simple pendulum with a moving pivot or a vibratory system with base excitation as analogized in Section 4.1.

Intuitively and according to Eq. (2.38), increasing quadrotor and/or payload mass, m_q and m_p , increases the thrust, F , which will increase quadrotor's energy consumption as

shown in Eqs. (2.53) and (2.54). A payload-to-quadrotor mass ratio, $M = m_p/m_q$, is defined to normalize the mass and help generalize the results and conclusions.

Fig. 5.4 shows the effect of payload mass on quadrotor's energy quotient, E_Q . A cable length equal to $l = 1$ m and an average acceleration equal to $\mathbf{a}_{av} = [1, 1]^T$ m/s² were used in this simulation. The payload-to-quadrotor mass ratio was examined in the range $0.1 \leq M \leq 0.9$.

The three motion cases in Fig. 5.4 show that increasing payload mass increases E_Q as expected. At small M , quadrotor's energy consumption for y -motion is a little bit larger than z -motion. However, it is the opposite at larger M . These results are applicable to the four polynomial trajectories of interest in this paper, i.e., MAT, MJT, MST and MCT.

To explain that, one should notice that y -motion results in payload vibration, which adds an extra moment to the system, thus more energy consumption. Contrarily, z -motion does not result in payload vibration as shown in Eq. (2.39). On the other hand, a z -motion with a certain acceleration, $\ddot{z}$, results in a larger thrust, F , than a y -motion with the same acceleration $\ddot{y} = \ddot{z}$. Recall the dynamics model in Eq. (2.38), where the thrust vector, $\mathbf{F}$, can be rewritten as follows:

$$(1 + M) \underbrace{(\ddot{\mathbf{r}}_q + g\mathbf{e})}_{\mathbf{a}_{eq}} + m_p l \mathbf{A} \ddot{\boldsymbol{\eta}} + m_p l \mathbf{B} \dot{\boldsymbol{\eta}} = \mathbf{F} \quad (5.6)$$

and the thrust, F , is proportional to, what we call, the equivalent acceleration $\|\mathbf{a}_{eq}\| = \|(\ddot{\mathbf{r}}_q + g\mathbf{e})\|$. The variation of $\|\mathbf{a}_{eq}\|$ in y and z motions can be presented as follows:

$$\begin{aligned} \|\mathbf{a}_{eq_y}\| &= \|\ddot{y}, g\|^T = \sqrt{\ddot{y}^2 + g^2} \\ \|\mathbf{a}_{eq_z}\| &= \|[0, (\ddot{z} + g)]^T\| = \sqrt{(\ddot{z} + g)^2} = \sqrt{\ddot{z}^2 + g^2 + 2\ddot{z}g} \end{aligned} \quad (5.7)$$

At small M , the effect of the extra vibration moment due to payload vibration exceeds the effect of the increased equivalent acceleration, i.e., increased inertia. Therefore, y -motion consumes more energy at small M . In contrast, at large M , the effect of the

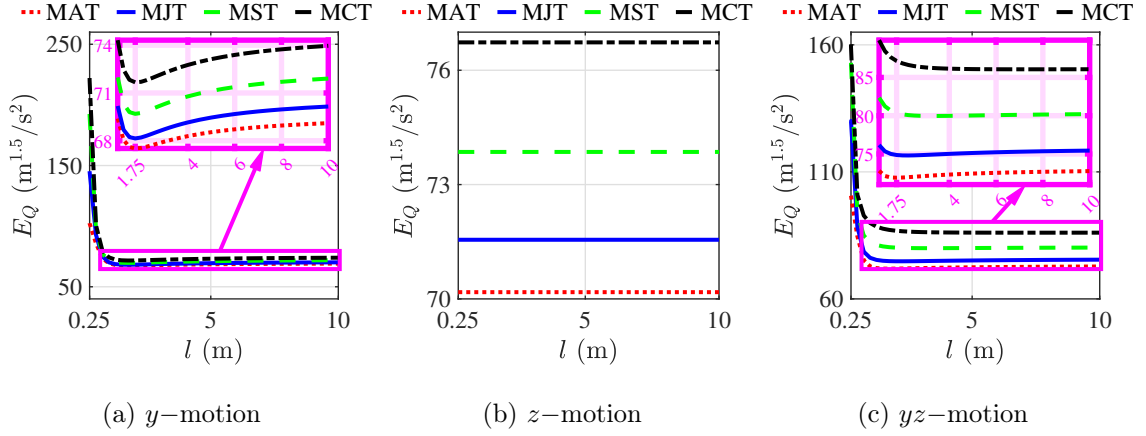


Figure 5.5: Effect of cable length, l , on energy quotient, E_Q , ($M = 0.5$ and $\mathbf{a}_{av} = [1, 1]^\top$ m/s²)

increased equivalent acceleration increases the inertia, which means more energy consumption. This effect is more than the payload vibration's effect at large M .

For yz -motion, the energy consumption is larger than any of the 1D motions as expected. However, it is not a linear superposition as Eqs. (2.38) and (2.39) suggest.

This energy analysis shows that increasing payload mass increases the energy consumption monotonically for all polynomial trajectories. Note that these calculations are very dependent on l and $\mathbf{a}_{av}$. Since these two parameters are not normalized in this analysis, changing either one may change the magnitudes of E_Q ; however, the conclusions hold.

Cable Length Effect

The cable length, l , affects payload vibration dynamics in Eq. (2.39). Increasing l will decrease payload vibration's natural frequency, ω_n , as shown in Eq. (4.4) in addition to increasing the frequency ratio, r_t , as shown in Eq. (4.14). Note that larger frequency ratios for $r_t > 1$ leads to reducing the transmissibility ratio as shown in Fig. 4.5 and expressed in Eq. 4.18. From this point of view, increasing l decreases quadrotor's energy consumption.

On the other hand, the cable length affects the thrust as shown in Eq. (2.38), where it can be seen that increasing the cable length, l , increases the thrust, and consequently quadrotor's energy consumption. Such a contradictory effect of cable length on quadrotor's energy consumption suggests the existence of an energy optimal cable length, at which, the energy consumption is minimal. This energy optimal cable length depends on quadrotor and payload masses, and quadrotor's trajectory, i.e., M, n and $\mathbf{a}_{av}$.

Fig. 5.5 shows the simulation results of the effect of cable length on quadrotor's energy quotient, E_Q . This simulation considers $M = 0.5$, $\mathbf{a}_{av} = [1, 1]^T$ m/s² and $0.25 \leq l \leq 10$ m.

Fig. 5.5a shows the results of y -motion. As expected, there is an optimal cable length regardless the degree of the polynomial trajectory. Less than this optimal cable length, increasing the length of the cable decreases quadrotor's energy consumption. It is the opposite when the cable length is longer than this optimal cable length.

Fig. 5.5b shows the results of z -motion. Since no payload vibration is induced by z -motion as seen in Eq. (2.39), the cable length has no effect on energy consumption at all. That is, when there is no payload vibration, the cable length vanishes from Eq. (2.38) since $\ddot{\eta}$ and $\dot{\eta}$ vanish.

Fig. 5.5c shows the results of yz -motion. The combined effect of the two 1D motions results in increasing quadrotor's energy consumption. Interestingly, this increase is not only due to the nonlinear superposition of the two 1D motions energy consumption, but Eq. (4.4) indicates another reason. Since payload vibration's natural frequency, ω_n , in y -motion is less than its value in yz -motion because $\ddot{z} = 0$ in y -motion, yz -motion's vibration has larger natural frequency than y -motion, and less frequency ratio, r_t , thus consumes more energy, especially for longer cable lengths.

In conclusion, the energy analysis and the simulation results of the cable length effect on quadrotor's energy consumption show that there is an energy optimal cable length

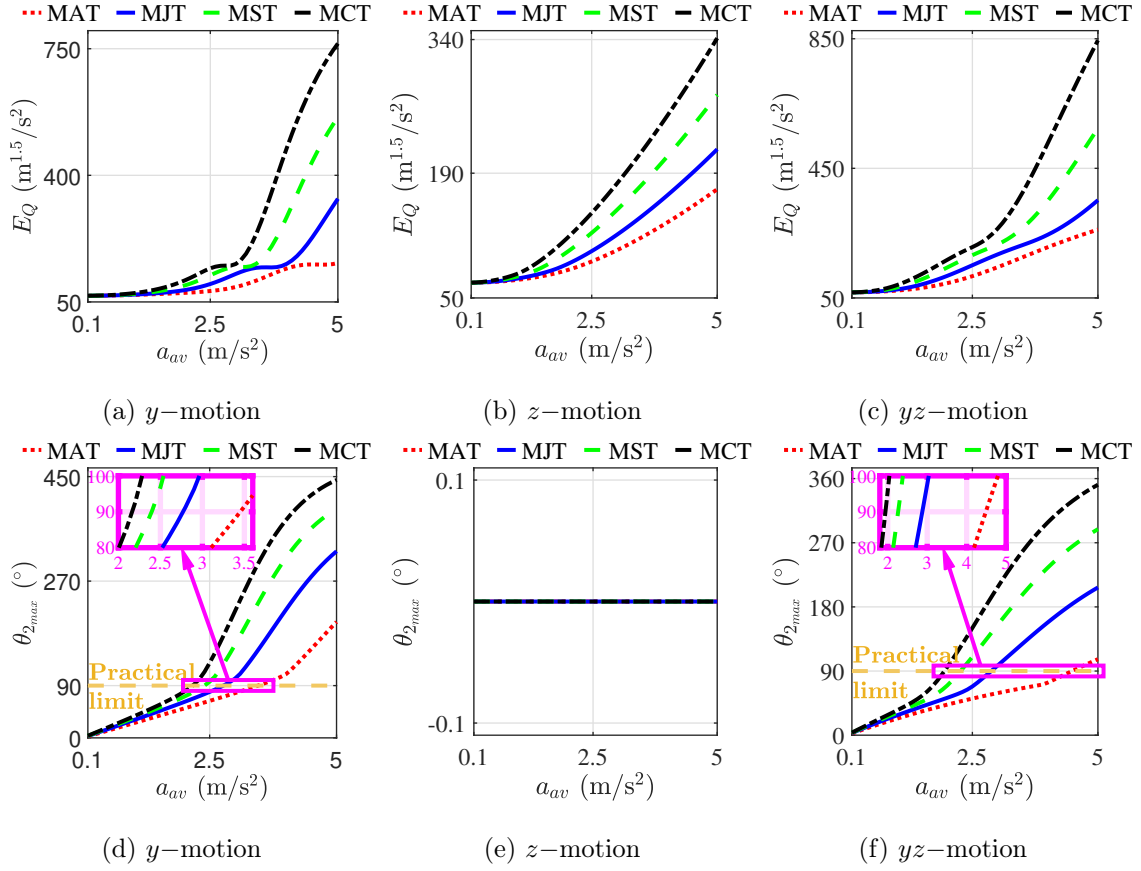


Figure 5.6: Effect of trajectory's average acceleration, $\mathbf{a}_{av}$, on energy quotient, E_Q , and maximum cable angle, $\theta_{2_{max}}$, ($M = 0.5$ and $l = 1$ m)

that results in minimum energy consumption. This energy optimal cable length exists for all polynomial trajectories, and it depends on payload-to-quadrotor mass ratio and quadrotor's trajectory.

Trajectory Effect

In the dynamics model, quadrotor's trajectory is represented by its acceleration, $\ddot{\mathbf{r}}_q$. For the considered rest-to-rest polynomial trajectories, the acceleration in Eq. (5.3) depends on two main parameters: the average acceleration, $\mathbf{a}_{av}$, and the trajectory type, i.e., the

degree of the polynomial.

First, we analyze the average acceleration, $\mathbf{a}_{av}$. Eq. (2.39) shows that the acceleration of the trajectory works as a base excitation as introduced in Section 4.1. Therefore, increasing $\mathbf{a}_{av}$ will increase the acceleration, i.e., the base excitation, and this will increase payload vibrations and consequently energy consumption. Additionally, from Eq. (2.38), increasing the $\mathbf{a}_{av}$ will increase the acceleration $\ddot{\mathbf{r}}_q$, which will increase the thrust F , and consequently quadrotor's energy consumption. Therefore, increasing $\mathbf{a}_{av}$ will increase quadrotor's energy consumption. One should recall that the major part of quadrotor's energy is consumed just because of hovering. Therefore, flight time has a significant impact on energy consumption than flight distance [65]. So, to ensure a fair comparison, the flight time for different experiments must be the same. This will affect the flight distance. That is, a smaller $\mathbf{a}_{av}$ with the same flight time yields a shorter flight distance.

Fig. 5.6 shows the simulation results. The effect of the average acceleration, $\mathbf{a}_{av}$, on E_Q is shown in Figs. 5.6a - 5.6c, whereas the effect on the maximum cable angle, $\theta_{2_{max}}$, is shown in Figs. 5.6d - 5.6f. Recall that a yz -motion results in payload vibrations with $\theta_1 = 0$ based on cable angles definition in Fig. 2.1b. A payload-to-quadrotor mass ratio $M = 0.5$ and a cable length $l = 1$ m were used in this simulation. The average acceleration has the range $0.1 \leq \mathbf{a}_{av} \leq 5$ m/s². It can be seen from Figs. 5.6a - 5.6c that increasing $\mathbf{a}_{av}$ increases quadrotor's energy consumption for all motions monotonically as expected.

Increasing the average acceleration, $\mathbf{a}_{av}$, significantly affects payload vibration. Large values of $\mathbf{a}_{av}$ can lead to large cable angles, that could exceed 90° after a critical average acceleration $a_{av_{cr}}$. Figs. 5.6d - 5.6f show $\theta_{2_{max}}$ at different $\mathbf{a}_{av}$ for all motions. As explained, z -motion does not induce any payload vibrations. However, the maximum cable angle $\theta_{2_{max}}$ can exceed 90° in y - and yz -motions. Two main points to emphasize from this discussion. First, although z -motion does not induce payload vibration, the

critical average acceleration $a_{av_{cr}}$ in case of y -motion is different from the yz -motion. This is because adding z -motion will change the natural frequency of payload vibrations as Eq. (4.4) suggests, which will change the vibration behaviour and where $\theta_{2_{max}}$ occurs, i.e., $a_{av_{cr}}$. Second, when $\theta_{2_{max}}$ exceeds 90° , the rate of change of E_Q with respect to $\mathbf{a}_{av}$ increases. This cannot happen in reality because:

- If the maximum cable angle $\theta_{2_{max}}$ exceeds 90° , the cable will, based on quadrotor's dimensions and cable length, intersect with the physical body of the quadrotor and could damage it, add a huge disturbance to the motion, and/or forcibly stop at 90° and rotate with slackness in the other direction, which is dangerous and trajectories should be designed to avoid such scenarios.
- The simulation results in Figs. 5.6a - 5.6c assume a constant cable length at all times, which means a taut cable at all times. If $\theta_{2_{max}}$ exceeds 90° , there is a possibility that the cable force becomes negative, mathematically, which means that the slackness in reality and a compression force in simulation, which is not practical.

Therefore, the values of E_Q beyond the critical average acceleration $a_{av_{cr}}$ cannot be trusted, and it is presented only for the sake of completeness.

Second, we investigate the effect of the degree of the polynomial on quadrotor's energy consumption. For all simulation results in Figs. 5.4 - 5.6, increasing the degree of the polynomial increases the energy consumption regardless of the payload-to-quadrotor mass ratio, cable length, and average acceleration. This is due to two main reasons: (a) increasing the polynomial degree increases payload vibration as discussed in Section 4.2; and (b) increasing the polynomial degree increases the peak acceleration of the trajectory as discussed in Section 3.1. This will increase the thrust, F , and consequently the energy consumption as seen in Eq. (2.38).

To conclude, the energy analysis and the simulation results of the effect of quadrotor's trajectory on quadrotor's energy consumption show that increasing the average acceleration and/or the degree of the polynomial results in increasing the energy consumption monotonically.

5.1.2 Arbitrary Boundary Conditions Scenario

Considering arbitrary boundary conditions, in contrast to rest-to-rest scenarios presented in Section 5.1.1, is more complicated and worth investigation. From Eq. (5.2), an analogous equation to Eq. (5.3) can be found for arbitrary boundary conditions as:

$$a_a = \left(\frac{6\dot{\mathbf{r}}_0}{t_f} + \frac{6\dot{\mathbf{r}}_f}{t_f} - \frac{12\mathbf{r}_f}{t_f^2} \right) T - \frac{4\dot{\mathbf{r}}_0}{t_f} - \frac{2\dot{\mathbf{r}}_f}{t_f} + \frac{6\mathbf{r}_f}{t_f^2} \quad (5.8)$$

$$\begin{aligned} a_j = & \left(10\ddot{\mathbf{r}}_f - 10\ddot{\mathbf{r}}_0 - \frac{60\dot{\mathbf{r}}_0}{t_f} - \frac{60\dot{\mathbf{r}}_f}{t_f} + \frac{120\mathbf{r}_f}{t_f^2} \right) T^3 \\ & + \left(18\ddot{\mathbf{r}}_0 - 12\ddot{\mathbf{r}}_f + \frac{96\dot{\mathbf{r}}_0}{t_f} + \frac{84\dot{\mathbf{r}}_f}{t_f} - \frac{180\mathbf{r}_f}{t_f^2} \right) T^2 \\ & + \left(3\ddot{\mathbf{r}}_f - 9\ddot{\mathbf{r}}_0 - \frac{36\dot{\mathbf{r}}_0}{t_f} - \frac{24\dot{\mathbf{r}}_f}{t_f} + \frac{60\mathbf{r}_f}{t_f^2} \right) T + \ddot{\mathbf{r}}_0 \end{aligned} \quad (5.9)$$

$$\begin{aligned} a_s = & \left(84\ddot{\mathbf{r}}_0 + 7\ddot{\mathbf{r}}_0 t_f - 84\ddot{\mathbf{r}}_f + 7\ddot{\mathbf{r}}_f t_f + \frac{420\dot{\mathbf{r}}_0}{t_f} + \frac{420\dot{\mathbf{r}}_f}{t_f} - \frac{840\mathbf{r}_f}{t_f^2} \right) T^5 \\ & + \left(195\ddot{\mathbf{r}}_f - 20\ddot{\mathbf{r}}_0 t_f - 225\ddot{\mathbf{r}}_0 - 15\ddot{\mathbf{r}}_f t_f - \frac{1080\dot{\mathbf{r}}_0}{t_f} - \frac{1020\dot{\mathbf{r}}_f}{t_f} + \frac{2100\mathbf{r}_f}{t_f^2} \right) T^4 \\ & + \left(200\ddot{\mathbf{r}}_0 + 20\ddot{\mathbf{r}}_0 t_f - 140\ddot{\mathbf{r}}_f + 10\ddot{\mathbf{r}}_f t_f + \frac{900\dot{\mathbf{r}}_0}{t_f} + \frac{780\dot{\mathbf{r}}_f}{t_f} - \frac{1680\mathbf{r}_f}{t_f^2} \right) T^3 \\ & + \left(30\ddot{\mathbf{r}}_f - 8\ddot{\mathbf{r}}_0 t_f - 60\ddot{\mathbf{r}}_0 - 2\ddot{\mathbf{r}}_f t_f - \frac{240\dot{\mathbf{r}}_0}{t_f} - \frac{180\dot{\mathbf{r}}_f}{t_f} + \frac{420\mathbf{r}_f}{t_f^2} \right) T^2 + \ddot{\mathbf{r}}_0 t + \ddot{\mathbf{r}}_0 \end{aligned} \quad (5.10)$$

Table 5.1: Monte Carlo simulation parameters

Parameter	Range	Unit
Flight time (t_f)	0.1 : 10	s
Quadrotor mass (m_q)	0.5 : 20	kg
Payload-to-quadrotor mass ratio (M)	0.05 : 0.95	-
Cable length (l)	0.01 : 20	m
Initial & final positions ($x_0, y_0, z_0, x_f, y_f, z_f$)	-10 : 10	m
Initial & final velocities ($\dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{x}_f, \dot{y}_f, \dot{z}_f$)	-5 : 5	m/s
Initial & final accelerations ($\ddot{x}_0, \ddot{y}_0, \ddot{z}_0, \ddot{x}_f, \ddot{y}_f, \ddot{z}_f$)	-5 : 5	m/s ²
Initial & final jerks ($\dddot{x}_0, \dddot{y}_0, \dddot{z}_0, \dddot{x}_f, \dddot{y}_f, \dddot{z}_f$)	-5 : 5	m/s ³
Initial & final snaps ($\ddddot{x}_0, \ddddot{y}_0, \ddddot{z}_0, \ddddot{x}_f, \ddddot{y}_f, \ddddot{z}_f$)	-5 : 5	m/s ⁴

$$\begin{aligned}
 a_c = & \left(540\ddot{\mathbf{r}}_f - 60\ddot{\mathbf{r}}_0 t_f - 540\ddot{\mathbf{r}}_0 - 3\dddot{\mathbf{r}}_0 t_f^2 - 60\dddot{\mathbf{r}}_f t_f + 3\dddot{\mathbf{r}}_f t_f^2 - \frac{2520\ddot{\mathbf{r}}_0}{t_f} - \frac{2520\ddot{\mathbf{r}}_f}{t_f} \right. \\
 & + \frac{5040\ddot{\mathbf{r}}_f}{t_f^2} \Big) T^7 + \left(1960\ddot{\mathbf{r}}_0 + 224\ddot{\mathbf{r}}_0 t_f - 1820\ddot{\mathbf{r}}_f + \frac{35}{3}\dddot{\mathbf{r}}_0 t_f^2 + 196\dddot{\mathbf{r}}_f t_f - \frac{28}{3}\dddot{\mathbf{r}}_f t_f^2 \right. \\
 & + \frac{8960\ddot{\mathbf{r}}_0}{t_f} + \frac{8680\ddot{\mathbf{r}}_f}{t_f} - \frac{17640\ddot{\mathbf{r}}_f}{t_f^2} \Big) T^6 + \left(2226\ddot{\mathbf{r}}_f - 315\ddot{\mathbf{r}}_0 t_f - 2646\ddot{\mathbf{r}}_0 - \frac{35}{2}\dddot{\mathbf{r}}_0 t_f^2 \right. \\
 & - 231\dddot{\mathbf{r}}_f t_f + \frac{21}{2}\dddot{\mathbf{r}}_f t_f^2 - \frac{11760\ddot{\mathbf{r}}_0}{t_f} - \frac{10920\ddot{\mathbf{r}}_f}{t_f} + \frac{22680\ddot{\mathbf{r}}_f}{t_f^2} \Big) T^5 + (1575\ddot{\mathbf{r}}_0 \\
 & + 200\ddot{\mathbf{r}}_0 t_f - 1155\ddot{\mathbf{r}}_f + \frac{25}{2}\ddot{\mathbf{r}}_0 t_f^2 + 115\ddot{\mathbf{r}}_f t_f - 5\ddot{\mathbf{r}}_f t_f^2 + \frac{6720\ddot{\mathbf{r}}_0}{t_f} + \frac{5880\ddot{\mathbf{r}}_f}{t_f} \\
 & - \frac{12600\ddot{\mathbf{r}}_f}{t_f^2} \Big) T^4 + \left(210\ddot{\mathbf{r}}_f - 50\ddot{\mathbf{r}}_0 t_f - 350\ddot{\mathbf{r}}_0 - \frac{25}{6}\dddot{\mathbf{r}}_0 t_f^2 - 20\dddot{\mathbf{r}}_f t_f + \frac{5}{6}\dddot{\mathbf{r}}_f t_f^2 \right. \\
 & - \frac{1400\ddot{\mathbf{r}}_0}{t_f} - \frac{1120\ddot{\mathbf{r}}_f}{t_f} + \frac{2520\ddot{\mathbf{r}}_f}{t_f^2} \Big) T^3 + \frac{\ddot{\mathbf{r}}_0 t_f^2}{2} + \ddot{\mathbf{r}}_0 t + \ddot{\mathbf{r}}_0
 \end{aligned} \tag{5.11}$$

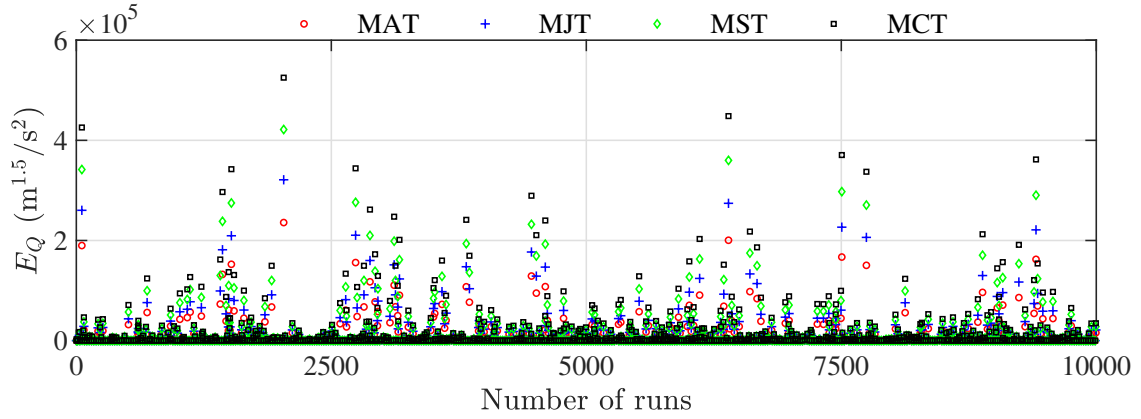
Eqs. (5.8) - (5.11) show that the trajectory's boundary conditions are complicatedly involved in acceleration calculations and, consequently, energy consumption analysis. Although analyzing the individual effect of each boundary condition on quadrotor's energy consumption can be performed analytically, the collective effect of varying multiple bound-

ary conditions simultaneously is too complex to be investigated analytically. So, we run extensive Monte Carlo simulations to examine such collective effects. Similar to rest-to-rest analysis, this energy analysis considers MAT, MJT, MST, and MCT. For each trajectory, we run 10,000 random simulations in 3D with different boundary conditions that are arbitrarily picked from bounded sets as shown in Table 5.1.

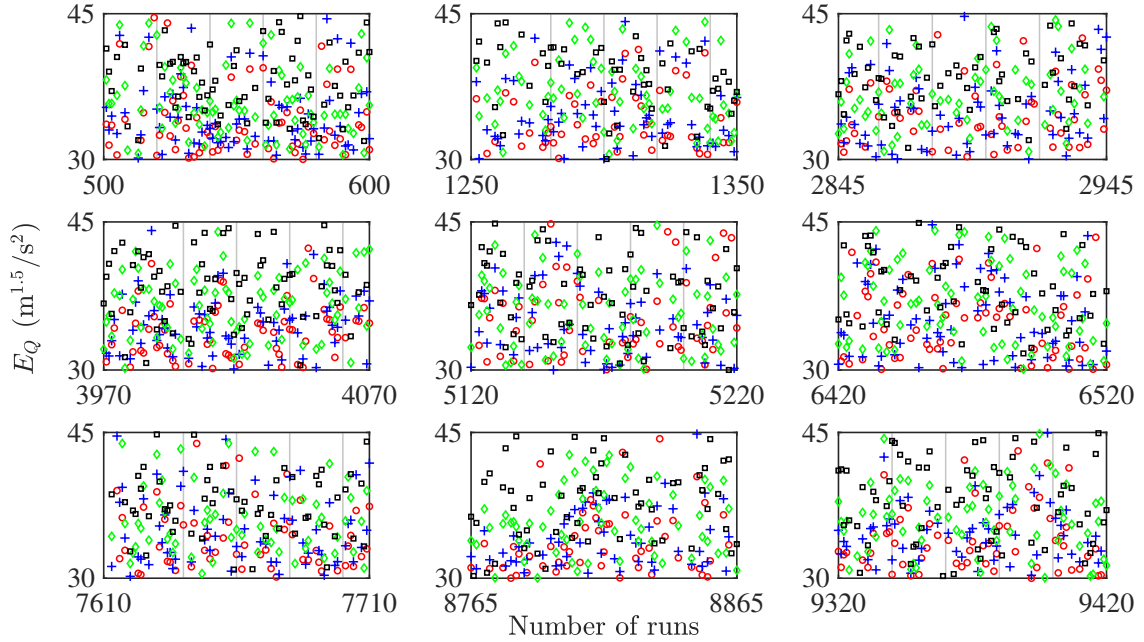
5.1.2.1 Single Quadrotor

The complete results of the energy quotient of the four considered polynomial trajectories for the entire 10,000 runs are plotted in Fig. 5.7. Since Fig. 5.7a is dense with 40,000 points, we zoomed in at nine random segments of the 10,000 runs to facilitate the investigation process for the reader (see Fig. 5.7b). The reader can examine any vertical line in Fig. 5.7, i.e., a certain run, by following either the color code (red, blue, green and black) or the symbol code (circle, cross, diamond and square) for the MAT, MJT, MST, and MCT, respectively, to compare the energy quotient of the different trajectories. Additionally, four sample results of the power and energy quotients at arbitrary runs are shown in Figs. 5.8 and 5.9, respectively.

The results show that increasing the degree of the polynomial trajectory increases the power and the energy consumption regardless of the boundary conditions. Statistically, we found that increasing the degree of the polynomial trajectory increases the minimum, median, mean, and maximum values of the energy quotient throughout the 10,000 runs. The minimum is (19.79, 20.87, 21.83, 22.86) $\text{m}^{1.5}/\text{s}^2$, the median is (33.56, 35.06, 37.65, 41.37) $\text{m}^{1.5}/\text{s}^2$, the mean is (900.21, 1212.71, 1584.22, 1968.14) $\text{m}^{1.5}/\text{s}^2$, and the maximum is (235794.65, 321228.45, 421606.07, 525157.16) $\text{m}^{1.5}/\text{s}^2$ for the MAT, MJT, MST and MCT, respectively.



(a) Energy quotient (complete)



(b) Energy quotient (zoomed-in)

Figure 5.7: Energy quotient for 3D motions with different boundary conditions (complete results - single quadrotor)

5.1.2.2 Quadrotor with a Slung Payload

Similar to the Monte Carlo simulation in Section 5.1.2.1, but while considering the existence of the slung payload, the results of the 10,000 runs are plotted in Fig. 5.10. While Fig. 5.10a

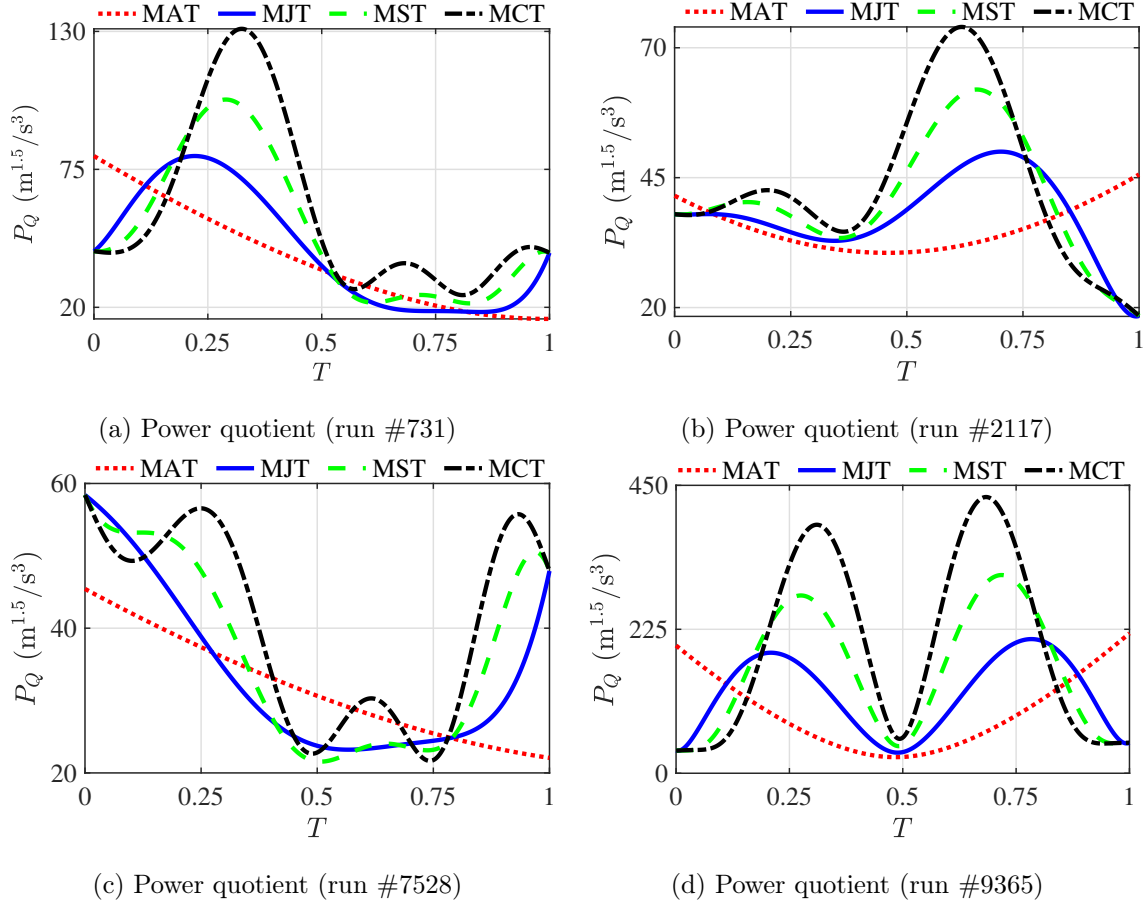


Figure 5.8: Power quotient samples for 3D motions with different boundary conditions (single quadrotor)

shows the complete results, we zoomed in at nine random segments of the 10,000 runs to facilitate the investigation process for the reader (see Fig. 5.10b). The reader can examine any vertical line in Fig. 5.10, i.e., a certain run, by following either the color code or the symbol code for the MAT, MJT, MST, and MCT, respectively, to compare the energy quotient of the different trajectories.

The results show that increasing the degree of the polynomial trajectory increases the power and the energy consumption regardless of the boundary conditions, payload mass, and cable length. Statistically, we found that increasing the degree of the polynomial tra-

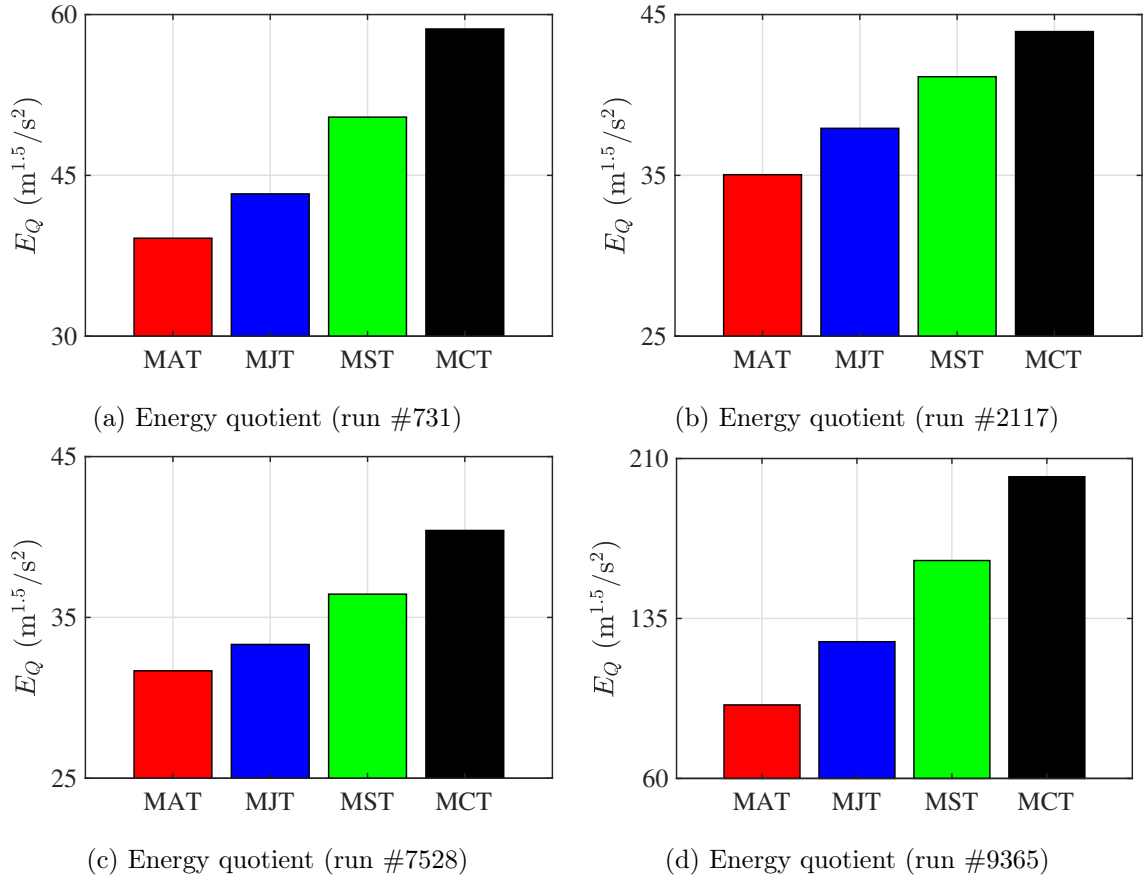
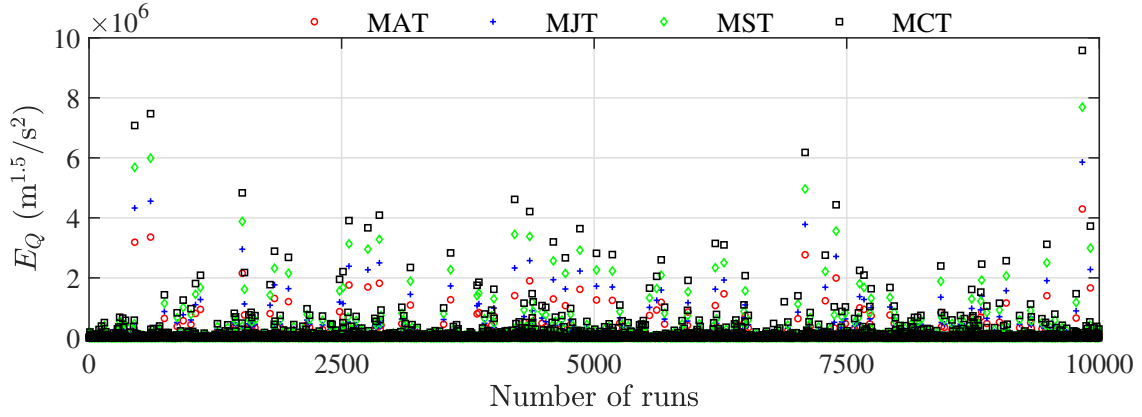
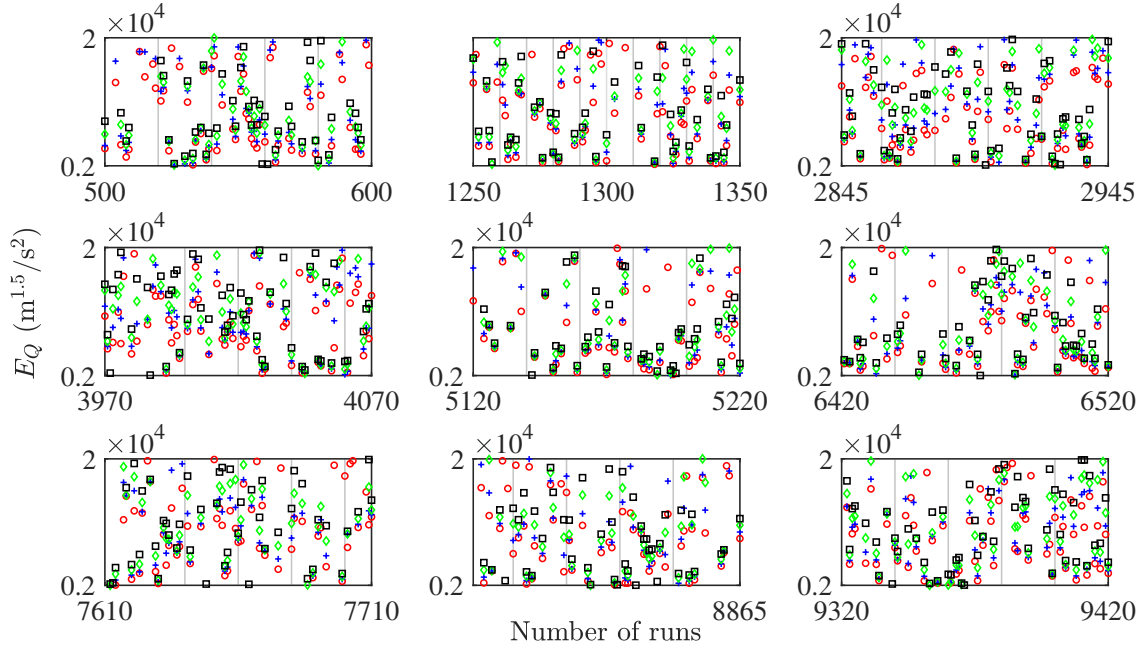


Figure 5.9: Energy quotient samples for 3D motions with different boundary conditions (single quadrotor)

jectory increases the minimum, median, mean, and maximum values of the energy quotient throughout the 10,000 runs. The minimum is (36.86, 42.29, 48.93, 57.89) $\text{m}^{1.5}/\text{s}^2$, the median is (9776.98, 11098.89, 12802.4, 14784.36) $\text{m}^{1.5}/\text{s}^2$, the mean is (25570.94, 32310.51, 40580.81, 50098.04) $\text{m}^{1.5}/\text{s}^2$, and the maximum is (4295802.49, 5856464.40, 7688914.34, 9580182.51) $\text{m}^{1.5}/\text{s}^2$ for the MAT, MJT, MST and MCT, respectively.



(a) Energy quotient (complete)



(b) Energy quotient (zoomed-in)

Figure 5.10: Energy quotient for 3D motions with different boundary conditions (complete results - quadrotor with a slung payload)

Another Monte Carlo simulation is conducted for the case of a quadrotor with a slung payload to examine the effect of the cable length under different boundary conditions. At each run, all parameters are picked arbitrarily from Table 5.1 except the cable length.

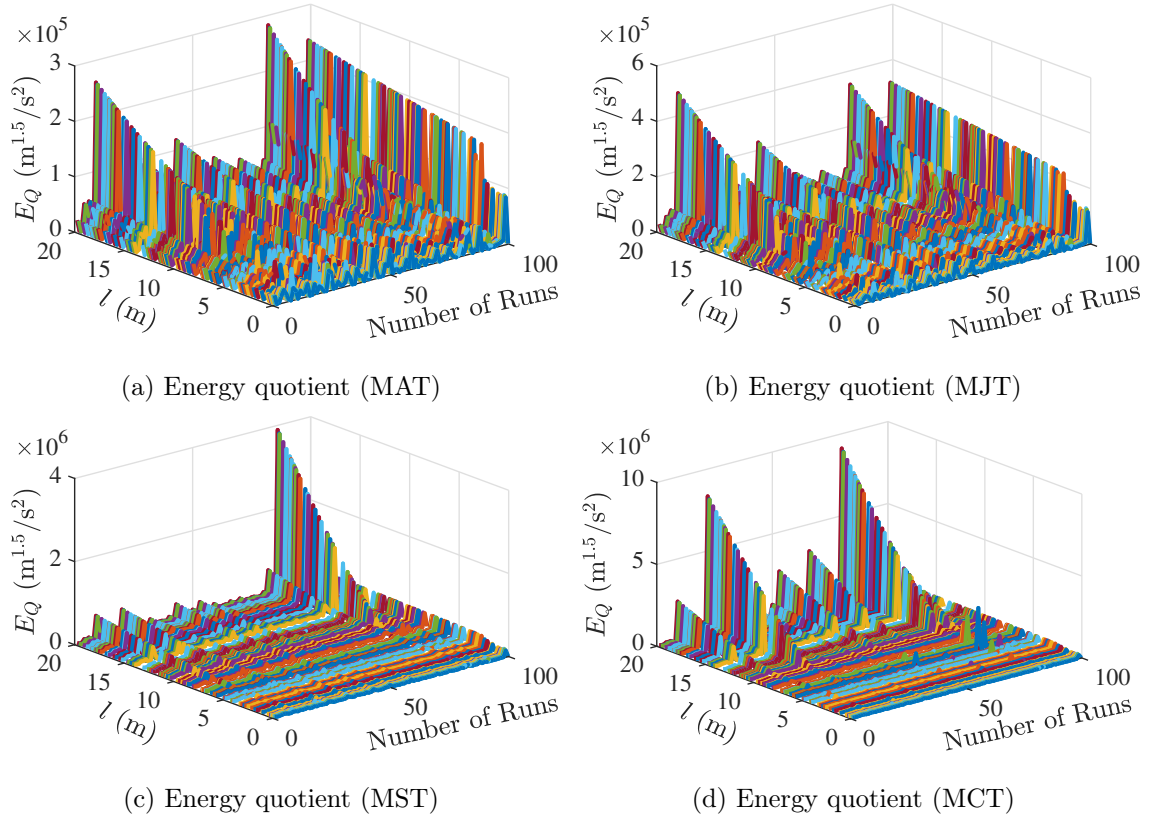


Figure 5.11: Effect of cable length on energy consumption with different boundary conditions (quadrotor with a slung payload)

These picked parameters are used to obtain the energy quotients at 99 different values of the cable length, which are picked arbitrarily from Table 5.1. This is one run, and the simulation has 100 of these runs, which means a total of 9,900 simulations. The complete results of this simulation for MAT, MJT, MST, and MCT are shown in Fig. 5.11.

The results in Fig. 5.11 show that a considerable increase of the cable length increases the energy consumption. However, this is not a monotonic relationship due to the contradictory effect of the cable length as explained in Section 5.1.1.2. The existence of an energy optimal cable length, at which, the energy consumption is minimal is valid under different boundary conditions as well.

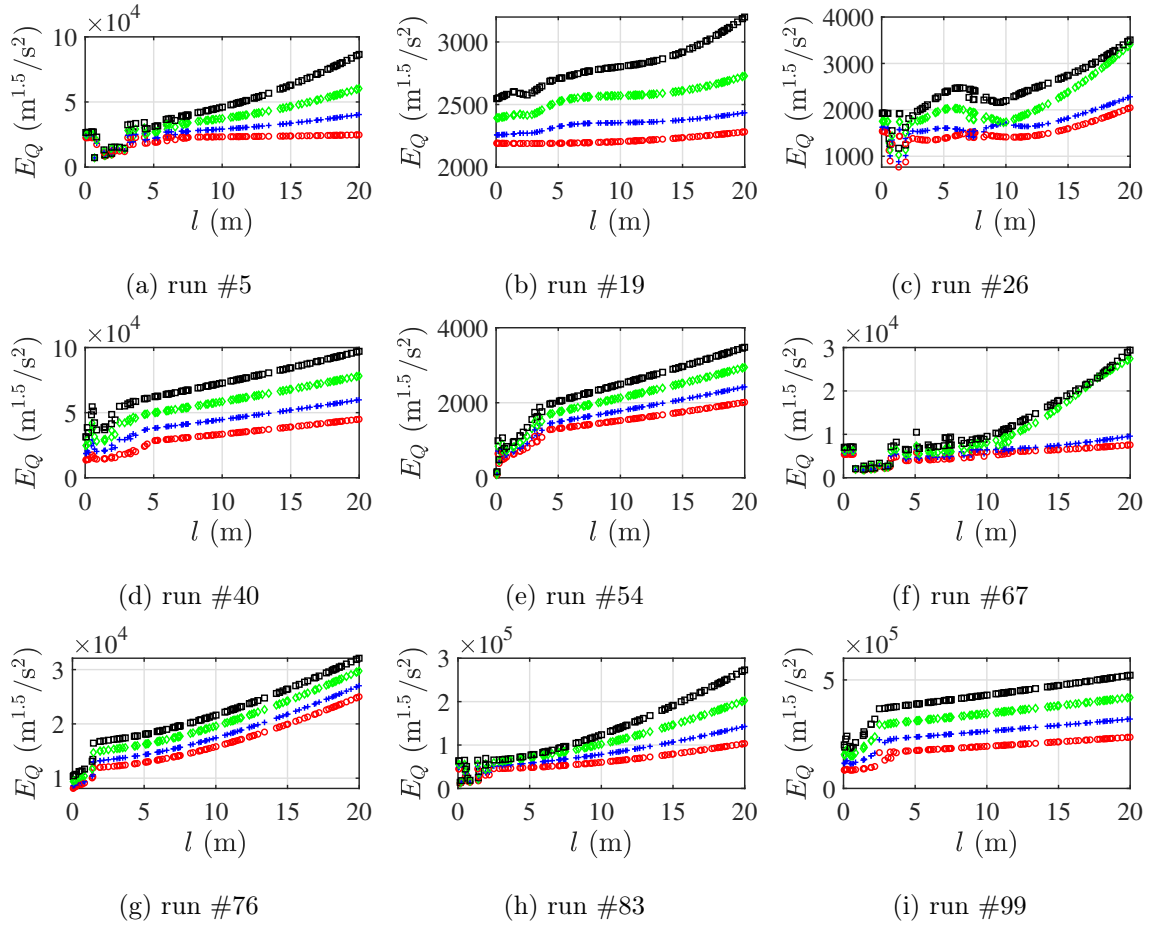


Figure 5.12: Effect of cable length on energy quotient with different boundary conditions (samples - quadrotor with a slung payload). Red circles for MAT, blue crosses for MJT, green diamonds for MST, and black squares for MST.

To better present the effect of the cable length, nine samples of Fig. 5.11 are chosen arbitrarily at different runs and plotted in Fig. 5.12. The results show the existence of the energy optimal cable length. It is worth mentioning that the energy optimal cable length is not constant and it depends on the boundary conditions and the degree of the polynomial trajectory. However, for all cases, a considerably large cable length results in more energy consumption because the direct effect of the cable length on the thrust

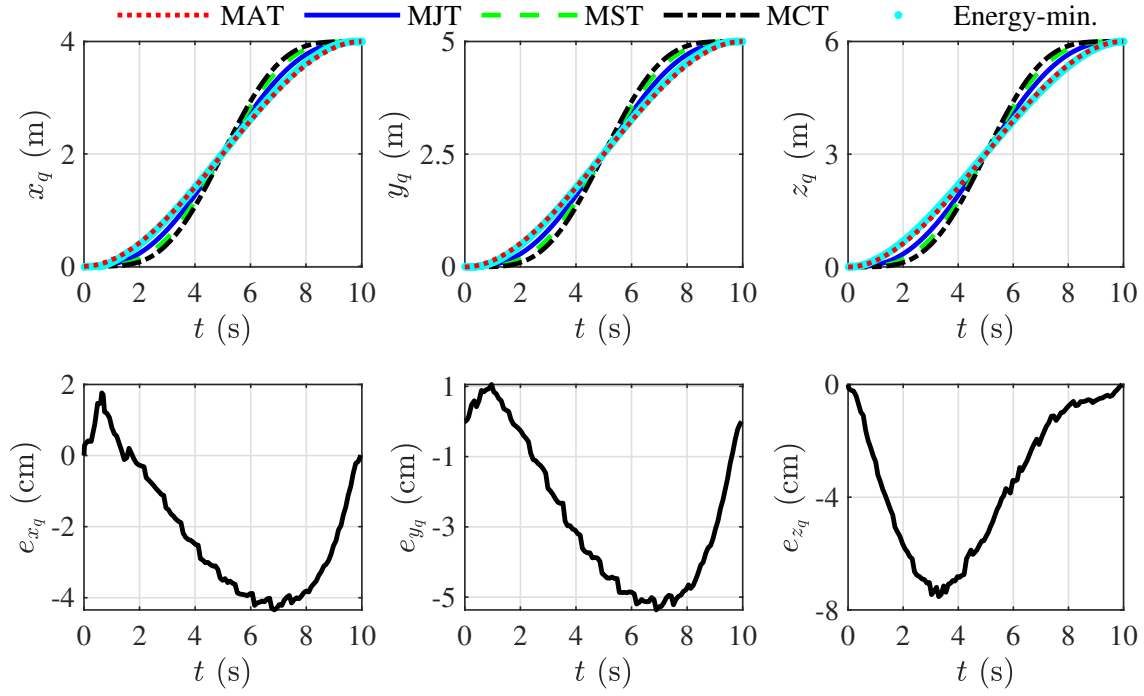
force is more significant than its effect on payload vibrations at longer cable lengths (see Eqs. (4.4) and (2.38) - (2.39) and Fig. 4.1). Specifying this energy optimal cable length analytically is tedious; however, it can be specified easily by simulations for a certain transportation system with certain parameters as shown in Fig. 5.5.

Additionally, Figs. 5.11 and 5.12 enforce the conclusion that increasing the degree of the polynomial increases the energy consumption regardless of the boundary conditions, payload mass, and cable length.

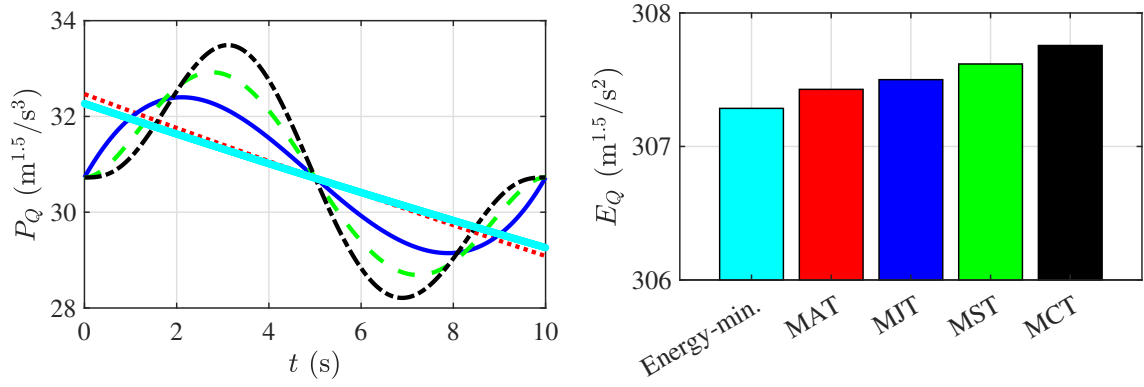
5.1.3 Comparative Study

This section discusses the interesting question of how polynomial trajectories' energy performance is compared to energy-minimized trajectories in the literature. Therefore, we compare the energy-minimized trajectory in [3] to the corresponding polynomial trajectories. In [3], the objective was to design a 3D rest-to-rest energy-minimized trajectory starts from the origin to the position $\mathbf{r}_f = [4, 5, 6]^\top$ m in the inertial frame with a flight time of 10 s.

Fig. 5.13 presents the comparison results. Fig. 5.13a shows that the MAT is close to the energy-minimized trajectory. So, it is expected that they result in close energy consumption. Note that the error plotted in Fig. 5.13a is the error between the MAT and the energy-minimized trajectory. Figs. 5.13b and 5.13c show that the power and energy quotients of the MAT and the energy-minimized trajectory are close as expected. Also, since the scale $\mathbf{a}_{av}$ is small in this case, the difference in energy consumption between different polynomial trajectories is small, which agrees with the analysis in Section 5.1.1.1 and Fig. 5.3. The energy consumption of the MAT, MJT, MST, and MCT is about 0.05%, 0.07%, 0.11%, and 0.15% more than the energy-minimized trajectory, respectively.



(a) Trajectory and position error



(b) Power quotient

(c) Energy quotient

Figure 5.13: Comparison between the energy-minimized trajectory in [3] and polynomial trajectories

The close performance of the MAT and the energy-minimized trajectories can be expected by comparing the coefficients of the approximate 3^{rd} degree polynomial fitting of the

energy-minimized trajectory (with subscript e) and the corresponding MAT as presented in Eqs. (5.12) and (5.14).

$$\begin{aligned}\mathbf{r}_e &= \mathbf{c}_e \mathbf{t} \\ \mathbf{r}_a &= \mathbf{c}_a \mathbf{t}\end{aligned}\tag{5.12}$$

where $\mathbf{t} = [t^3, t^2, t]^\top$, and

$$\mathbf{c}_e = \begin{bmatrix} -0.0086 & 0.128 & -0.02 \\ -0.0108 & 0.162 & -0.04 \\ -0.011 & 0.16 & 0.09 \end{bmatrix}\tag{5.13}$$

and

$$\mathbf{c}_a = \begin{bmatrix} -0.008 & 0.12 & 0 \\ -0.01 & 0.15 & 0 \\ -0.012 & 0.18 & 0 \end{bmatrix}\tag{5.14}$$

The results in Fig. 5.13 and Eqs. (5.12) and (5.14) suggest that polynomial trajectories can be used as reasonable alternatives to energy-minimized trajectories. Specifically, MATs can be used instead of energy-minimized trajectories. If the average acceleration $\mathbf{a}_{av}$ is small, any polynomial trajectory can be used. So, MJTs can be considered in such cases to avoid sudden changes in acceleration.

Replacing energy-minimized trajectories with polynomial trajectories allows to design energy-efficient trajectories straightforwardly using only the kinematic boundary conditions, as introduced in Chapter 3, and avoid solving optimization problems that depend on quadrotor's physical parameters to obtain the desired trajectory, e.g., [3, 4, 43, 49, 50, 65, 66, 97, 132]. For example, the MAT in this comparative study is designed without knowing any of the physical parameters of the quadrotor, e.g., mass, dimensions, propeller aerodynamic parameters and motor specifications. Conversely, the energy-minimized trajectory in the same comparative study, i.e., in [3], requires knowing all of these physical parameters to solve the optimization problem. Eventually, the MAT and the energy-minimized trajectory

Table 5.2: Waypoints' times, positions, velocities, accelerations, jerks, and snaps for the experimental *Set 3*

	Position (m)			Velocity (m/s)			Acceleration (m/s ²)			Jerk (m/s ³)			Snap (m/s ⁴)			Time (s)
	x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$	$\ddot{x}$	$\ddot{y}$	$\ddot{z}$	$\dddot{x}$	$\dddot{y}$	$\dddot{z}$	$\ddot{\ddot{x}}$	$\ddot{\ddot{y}}$	$\ddot{\ddot{z}}$	t
Waypoint #1	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
Waypoint #2	1	1	1	-0.2	-0.2	0.1	-0.5	-0.5	0.2	-0.1	-0.3	0	-0.2	-0.2	0	2
Waypoint #3	-1	-1	1.5	0	-0.5	0	0	-0.6	0.1	0	-0.2	0	0	-0.5	0	6
Waypoint #4	-1	-2	1.5	0	0	0	0	0	0	0	0	0	0	0	0	8

resulted in a close trajectory, power, and energy consumption, which is an advantage for adopting MATs.

5.2 Experimental Results

The main objective of this section is to experimentally validate the assumptions, concepts, and results presented in this chapter. Specifically, we validate:

- I) The assumption of a constant force ratio β for non-aggressive flights.
- II) The concept of power and energy quotients, i.e., $P_Q \propto P$ and $E_Q \propto E$.
- III) The relationship between the degree of the polynomial trajectory and quadrotor's energy consumption for both a single quadrotor and quadrotor with a slung payload.
- IV) The effects of payload mass and cable length on quadrotor's energy consumption for a quadrotor with a slung payload.

To this end, eight sets of experiments have been conducted to validate the aforementioned points. Each set includes the four trajectories of interest, i.e., MAT, MJT, MST, and MCT. These sets are as follows:

Table 5.3: Waypoints' positions of the experimental **Set 4**

Waypoint number	#1	#2	#3	#4	#5	#6
x (m)	-1	-1	0	1	0	-1
y (m)	1	-1	-1	0	1	1
z (m)	1	1.15	1.3	1.45	1.6	1.75

I) **Single Quadrotor:**

- (a) **Set 1** considers a 3D rest-to-rest motion from the position $\mathbf{r}_0 = [0, 0, 1]^\top$ m to $\mathbf{r}_f = [1.5, -2, 1.5]^\top$ m with a flight time $t_f = 4$ s. The traveled distances is $\mathbf{D} = [1.5, 2, 0.5]^\top$ m. We refer to the trajectories of this set as **slow** trajectories.
- (b) **Set 2** is similar to **Set 1**, but the flight time is $t_f = 2$ s. So, trajectories of this set are referred to as **fast** trajectories.
- (c) **Set 3** considers four-waypoint trajectories with the kinematic constraints reported in Table 5.2. **Set 3** is split into two subsets:
 - i. **Set 3a:** The motion between the two middle waypoints, i.e., $t = [2, 6]$ s, is used to validate the results of different boundary conditions scenarios presented in Section 5.1.2 since it is physically impossible to start and/or end the flight at non-zero boundary conditions except for position. Trajectories of this case are referred to as **boundary**.
 - ii. **Set 3b:** The entire motion of this set, i.e., $t = [0, 8]$ s, is used to address trajectories with multiple waypoints. Because of the sharp corners at the intermediate waypoints, the trajectories of this case are referred to as **sharp corners** in this chapter.
- (d) **Set 4** considers rest-to-rest smooth six-waypoint trajectories between the waypoints reported in Table 5.3 with a flight time $t_f = 12$ s. This set has smooth

trajectories since all polynomial trajectories have a C^{k-1} continuity, where k is the degree of the polynomial as presented in Section 3.1. Therefore, they are referred to as *smooth* trajectories in this chapter.

II) Quadrotor with a Slung Payload

Rest-to-rest flights between $\mathbf{r}_0 = [1.5, 1.2, 1]^\top$ m to $\mathbf{r}_f = [-1.5, 1.9, 1]^\top$ m are considered in $t_f = 4$ s. Three parameters vary throughout the experiments:

- Two payload masses are considered: a *heavy* payload of mass 0.2 kg and a *light* one of mass 0.1 kg. The black payload shown in Fig. 2.6c is used.
- Two cable lengths are considered: a *long* cable of length 1 m and a *short* one of length 0.5 m.
- Two average accelerations are considered: $\mathbf{a}_{av} = [0.1875, 0.04375, 0]^\top$ m/s² for a *fast* trajectory and $\mathbf{a}_{av} = [0.047, 0.011, 0]^\top$ m/s² for a *slow* one. Note that we overload the terms *fast* and *slow*, but this will not cause any confusion.

Based on these variations, four experimental sets are defined as follows:

- (a) **Set 5** considers the combination of *Light - Short - Fast*.
- (b) **Set 6** considers the combination of *Heavy - Short - Fast*.
- (c) **Set 7** considers the combination of *Heavy - Long - Fast*.
- (d) **Set 8** considers the combination of *Heavy - Long - Slow*.

To maintain the consistency of results and eliminate performance variations, all experiments were conducted with a fully-charged LiPo battery with energy capacity of 3,300 mAh. The test videos of the single quadrotor are available at <https://youtu.be/HtGvK6vBuQY>, whereas test videos of a quadrotor with a slung payload can be found at https://youtu.be/doJK_rkFmDc.

5.2.1 Single Quadrotor

The results of *Sets: 1, 2, 3a, 3b* and *4* are shown in Figs. 5.14 - 5.17. These are the results of the *slow, fast, boundary, sharp corners*, and *smooth* experiments, respectively.

5.2.1.1 Constant Force Ratio Assumption

To validate our assumption that it is reasonable to assume constant force ratio, β , for non-aggressive flights we need to: a) show that the assumption of non-aggressive flights in Section 2.3.2 is reasonable. That is, for non-aggressive flights, $\mu = F_{i_{max}}/F_{j_{min}} \leq 2$ or conservatively $\mu \leq 3$, where $i \neq j$; and b) show that β can be considered constant for non-aggressive flights.

Figs. 5.14a - 5.14c and 5.15a - 5.15b show the propeller forces, F_i , whereas Figs. 5.14d - 5.14f and 5.15c - 5.15d show the total thrust, F , for the five cases considered in this validation. These results show that the trajectories are dynamically feasible, which indirectly validate the conclusions of Section 3.2.

From these force results, the ratios μ and β can be calculated. Figs. 5.14g - 5.14i and 5.15e - 5.15f show the experimental results of μ . The experimental results show that considering non-aggressive flights to be flights with $\mu \leq 2$, or conservatively $\mu \leq 3$, is reasonable. The large spikes of μ in the *fast* and *sharp corners* trajectories are due to a sudden aggressive flights as seen in the test videos. We found that these spikes are beneficial to enforce our assumption. This can be seen in Figs. 5.14h and 5.15e, where values of $\mu > 3$ exist. These results show that the definition of non-aggressive flights in Section 2.3.2 is reasonable.

Also, based on the force results, the force ratio, β , is presented in Figs. 5.14j - 5.14l and 5.15g - 5.15h. The results show that β can be assumed constant for non-aggressive

Table 5.4: Analysis of the experimental values of γ_m for the *slow*, *fast* and *boundary* sets of experiments

	Slow				Fast				Boundary			
	MAT	MJT	MST	MCT	MAT	MJT	MST	MCT	MAT	MJT	MST	MCT
$\gamma_{m_{max}}$	48.34	48.32	48.80	48.35	51.65	49.31	51.51	53.45	49.04	48.88	48.98	48.65
$\gamma_{m_{min}}$	48.22	48.22	48.22	48.22	48.21	48.21	48.22	48.22	48.22	48.23	48.23	48.23
$\gamma_{m_{mean}}$	48.25	48.24	48.27	48.26	48.45	48.34	48.49	48.60	48.27	48.28	48.29	48.28
$\gamma_{m_{energy}}$	48.25	48.24	48.26	48.26	48.45	48.34	48.48	48.60	48.27	48.28	48.29	48.28

Table 5.5: Analysis of the experimental values of γ_m in case of multiple waypoints trajectories (*sharp corners* and *smooth* sets of experiments)

	Sharp corners				Smooth			
	MAT	MJT	MST	MCT	MAT	MJT	MST	MCT
$\gamma_{m_{max}}$	52.23	59.06	58.36	59.02	48.35	48.35	48.95	50.09
$\gamma_{m_{min}}$	48.22	48.23	48.22	48.22	48.21	48.22	48.21	48.21
$\gamma_{m_{mean}}$	48.28	48.33	48.36	48.33	48.25	48.25	48.26	48.27
$\gamma_{m_{energy}}$	48.28	48.33	48.35	48.34	48.25	48.25	48.27	48.26

flights. The mean values of β are indicated in Figs. 5.14j - 5.14l and 5.15g - 5.15h. The large spikes seen in the *fast* and *sharp corners* trajectories are corresponding to aggressive flights, i.e., $\mu \geq 3$. It is worth noting that these spikes do not last for long time, thus they are not expected to affect the results significantly. This can be seen in the mean values of β , which are very close in all cases, i.e., $\beta_{m_{min}} = 1.987$ and $\beta_{m_{max}} = 1.999$. This validates our assumption of a constant β for non-aggressive flights in Section 2.3.2 experimentally.

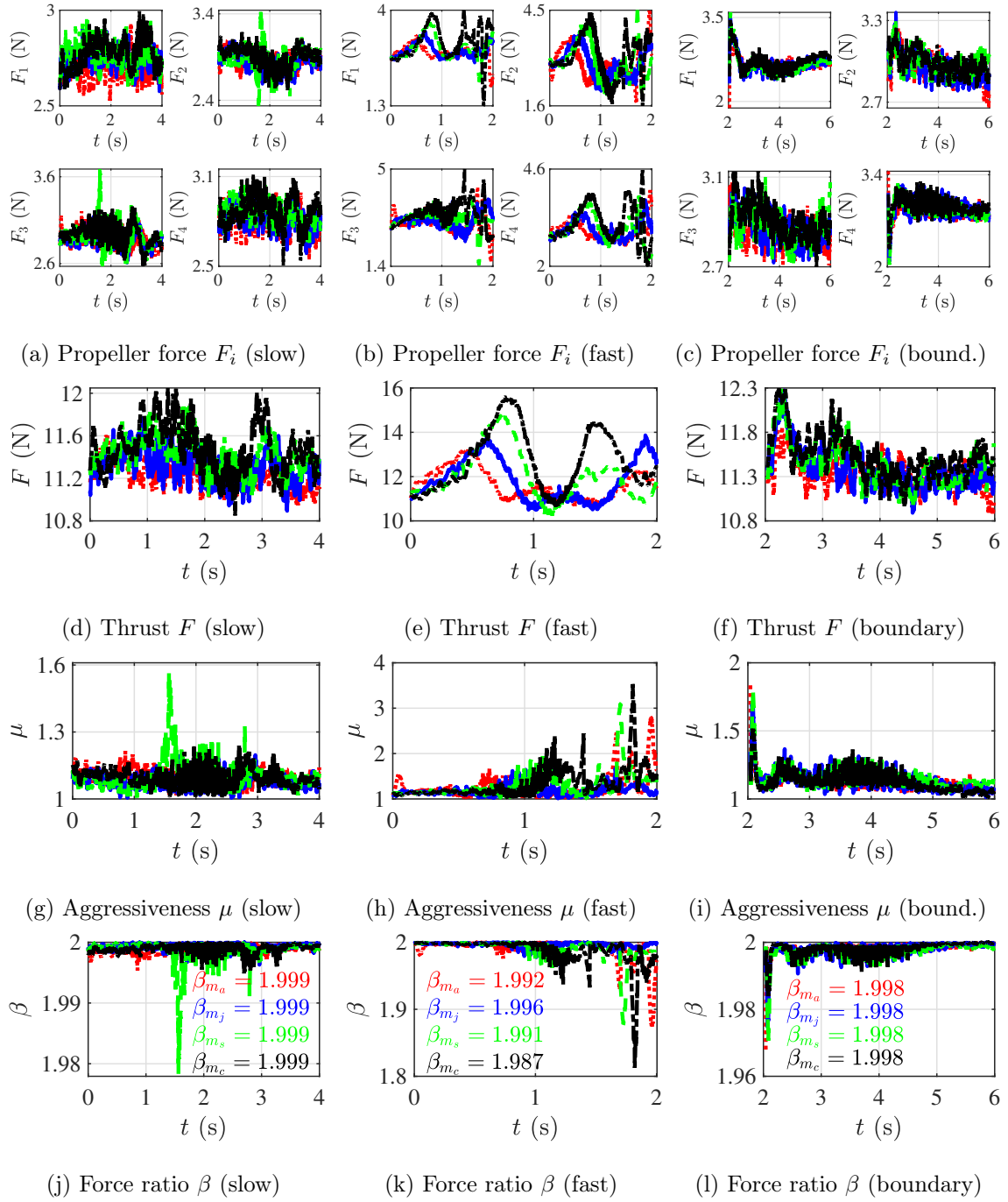


Figure 5.14: Experimental results of F_i , F , μ and β for *slow*, *fast*, and *boundary* cases. MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

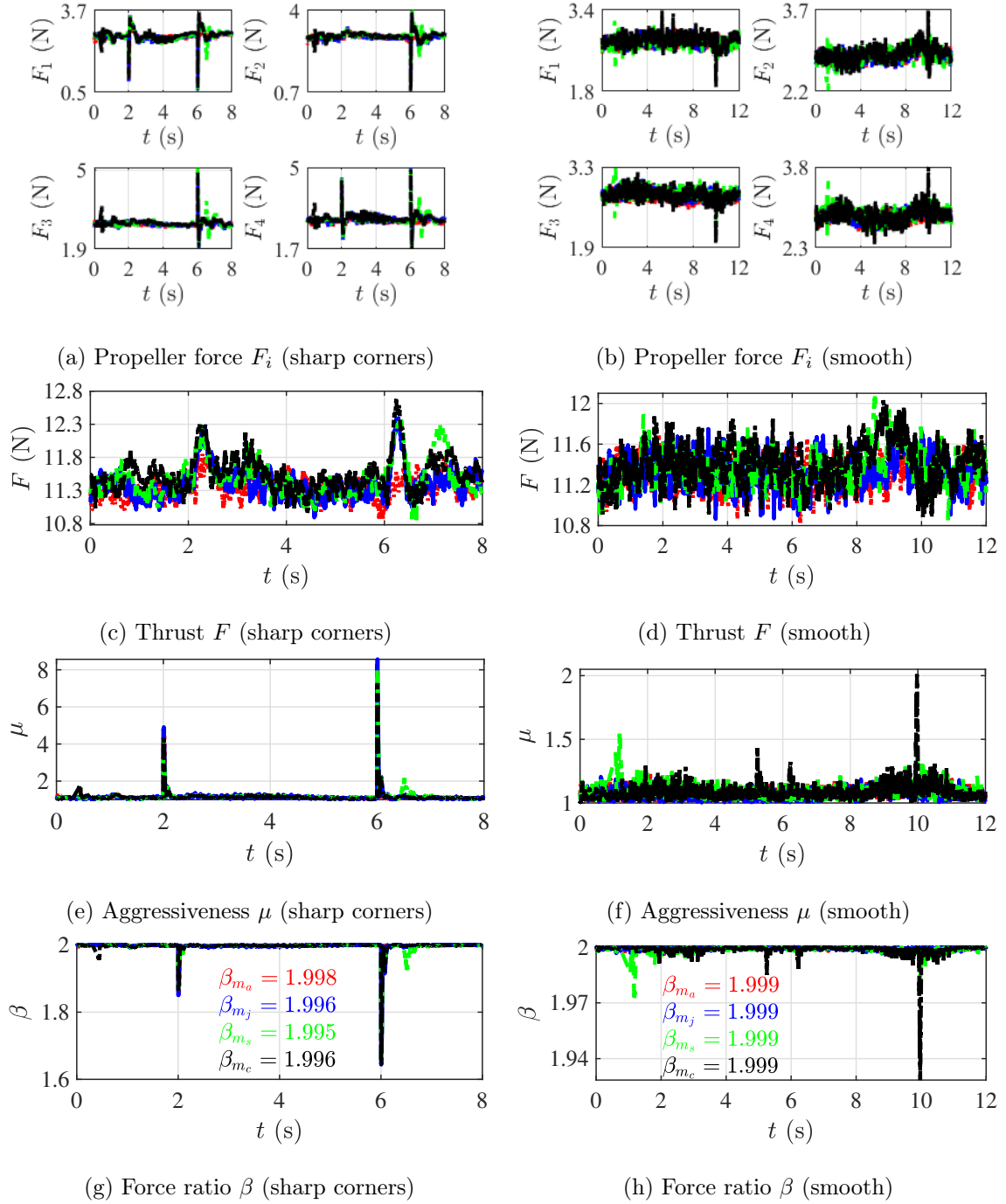


Figure 5.15: Experimental results of F_i , F , μ and β for multiple waypoints trajectories (*sharp corners* and *smooth*). MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

5.2.1.2 Power and Energy Quotients

The assumptions used to derive the power and energy quotients, P_Q and E_Q , are validated experimentally in Section 5.2.1.1. This section shows, experimentally, that the concepts of power and energy quotients can be used to represent quadrotor's power and energy consumption. This is done by comparing the power and energy quotients, P_Q and E_Q , to the actual power and energy consumption, P and E , for the *slow*, *fast* and *boundary*, *sharp corners*, and *smooth* sets of experiments.

Figs. 5.16a - 5.16c and 5.17a - 5.17b show the power quotients, P_Q , for the sets of experiments of interest, whereas Figs. 5.16d - 5.16f and 5.17c - 5.17d show the actual power required by the QDrone, P , for the same experiments. It can be seen that P_Q and P are related and proportional to each other through the ratio γ_m as shown in Eq. (5.5). Tables 5.4 and 5.5 show the experimental results of the maximum, minimum, and mean values of $\gamma_m = P/P_Q$. It can be seen that it is acceptable to assume γ_m as a constant value especially for non-aggressive flights. For aggressive flights, γ_m may be different, e.g., some parts of the *fast* and *sharp corners* trajectories.

Similarly, Figs. 5.16g - 5.16i and 5.17e - 5.17f show the energy quotients, E_Q , whereas Figs. 5.16j - 5.16l and 5.17g - 5.17h show the actual energy consumption, E . Calculating γ_m from the energy consumption as $\gamma_{m_{energy}} = E/E_Q$ is reported in the last row of Tables 5.4 and 5.5. Note that these results show that it is acceptable to deal with $\gamma_{m_{energy}}$ and γ_m interchangeably. That is because the energy consumption is the integration of the power over the flight time.

These results show that the concepts of power and energy quotients can be used to estimate quadrotor's power and energy consumption for non-aggressive flights as introduced in Section 2.3.

5.2.1.3 Effect of Polynomial Degree

Figs. 5.16 and 5.17 show that for the *slow*, *fast*, *boundary*, *sharp corners*, and *smooth* sets of experiments, increasing the degree of the polynomial increases the required power and the energy consumption. This is validated regardless of the kinematic boundary conditions of the trajectory. This validates the results in Sections 5.1.1.1 and 5.1.2.

It is worth noting that the *fast* trajectory is more aggressive than the *slow*, *boundary*, *sharp corners*, and *smooth* trajectories, so the QDrone requires more power to follow such a trajectory. Meanwhile, the flight time of the *fast* trajectory is shorter than the rest of the considered trajectories, so the *fast* trajectory consumes less energy due to less hovering time.

According to the energy consumption in Figs. 5.16j - 5.16l, the MAT saves approximately 3.2%, 12.6%, and 2.5% of the energy consumption compared to the MCT for the *slow*, *fast*, and *boundary* cases, respectively. Similarly, the MJT saves approximately 2.2%, 10.4%, and 1.8% of the energy consumption compared to the MCT for the sets. The MST saves approximately 0.7%, 7.9%, and 1.2% of the energy consumption compared to the MCT for the sets. Recall that the energy consumption has two main components: the hovering component and the trajectory-following component as explained in Section 5.1.1.1. The hovering component forms the major portion of the energy consumption as shown in Fig. 5.2. The hovering component is dominated by the mass of the quadrotor and is not affected by the trajectory. Therefore, the trajectory-following component plays an important role in the *fast* case because the flight time is short and the trajectory is fast. Contrarily, when the trajectory is slow and the flight time is larger, the effect of the trajectory-following component becomes less significant compared to the hovering component. That is why the energy saved in the *fast* case due to changing the trajectory is more significant compared to the *slow* and *boundary* cases.

5.2.1.4 Polynomial Trajectories with Multiple Waypoints

The energy analysis presented in Sections 5.1.1.1 and 5.1.2 is conducted considering two-waypoint trajectories. However, the conclusions can be considered to multiple-waypoint trajectories because a multiple-waypoint trajectory consists of a combination of two-waypoint trajectories with rest-to-rest and/or arbitrary kinematic boundary conditions. This section examines the effect of following a polynomial trajectory with multiple waypoints instead of just two waypoints by considering the *sharp corner* and the *smooth* sets of experiments shown in Fig. 5.15 and 5.17.

As discussed in Sections 5.2.1.1 - 5.2.1.3, the assumption of a constant force ratio, β , for non-aggressive flights, the concept of the power and energy quotients, and the effect of the degree of the polynomial trajectories are validated for multiple-waypoint trajectories. That is, the force ratio, β , can be assumed as a constant value and the power and energy quotients are proportional to the actual power and energy consumption of the quadrotor. Also, increasing the degree of the polynomial increases the energy consumption.

From the energy consumption shown in Figs. 5.17g - 5.17h, the MAT saves approximately 2.9% and 1.6% of the energy consumption compared to the MCT for the *sharp corner* and *smooth* cases, respectively. Similarly, the MJT saves approximately 1.9% and 1.1% of the energy consumption compared to the MCT for the sets. The MST saves approximately 0.8% and 0.6% of the energy consumption compared to the MCT for the sets.

In fact, this shows how powerful the proposed power and energy quotients are because they depend only on quadrotor's thrust as shown in Eq. (2.54), which reduces to quadrotor's acceleration profile in case of a single quadrotor as shown in Eq. (5.5). Therefore, the number of waypoints does not matter.

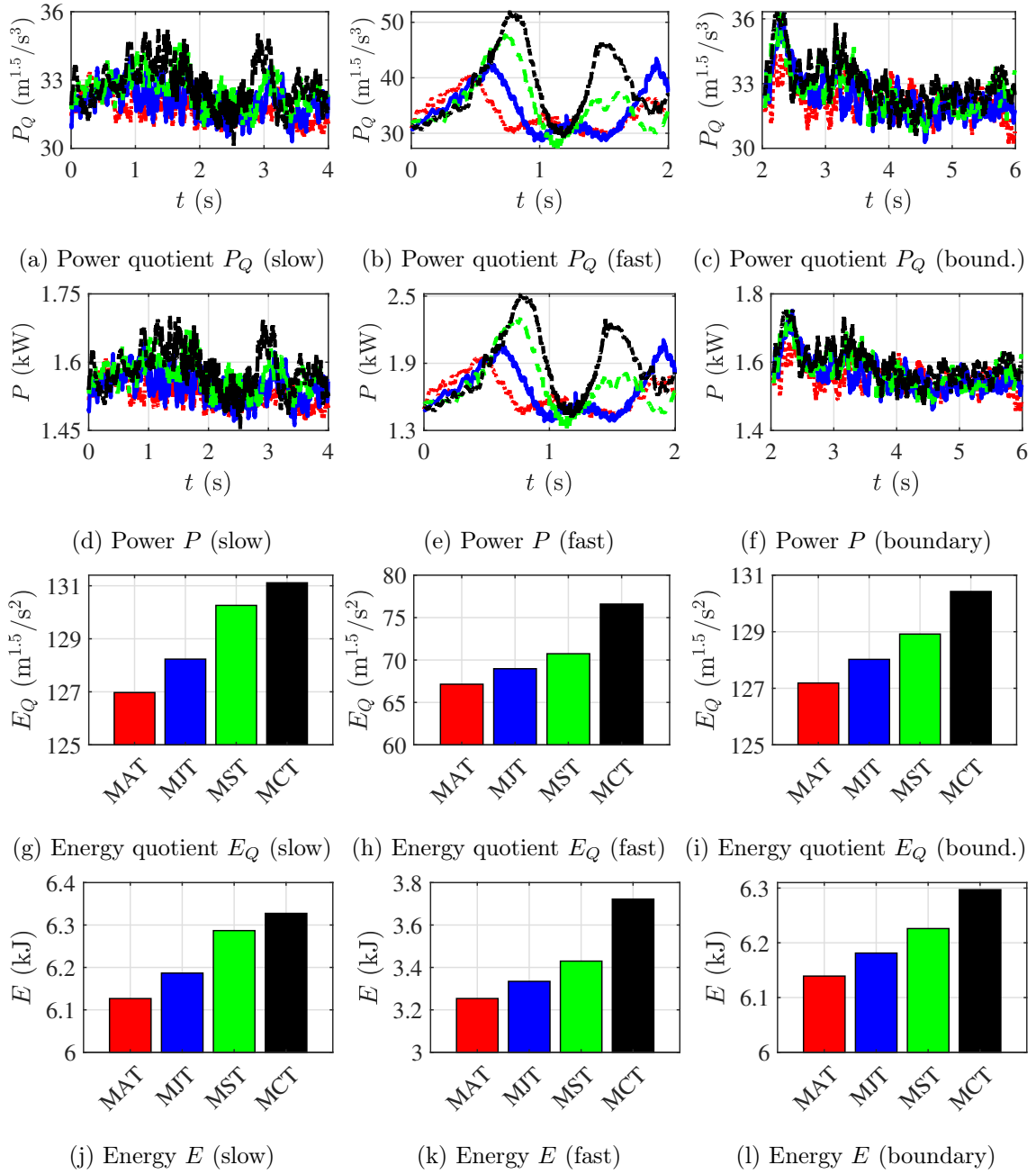


Figure 5.16: Experimental results of P_Q , P , E_Q and E for *slow*, *fast*, and *boundary* cases. MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

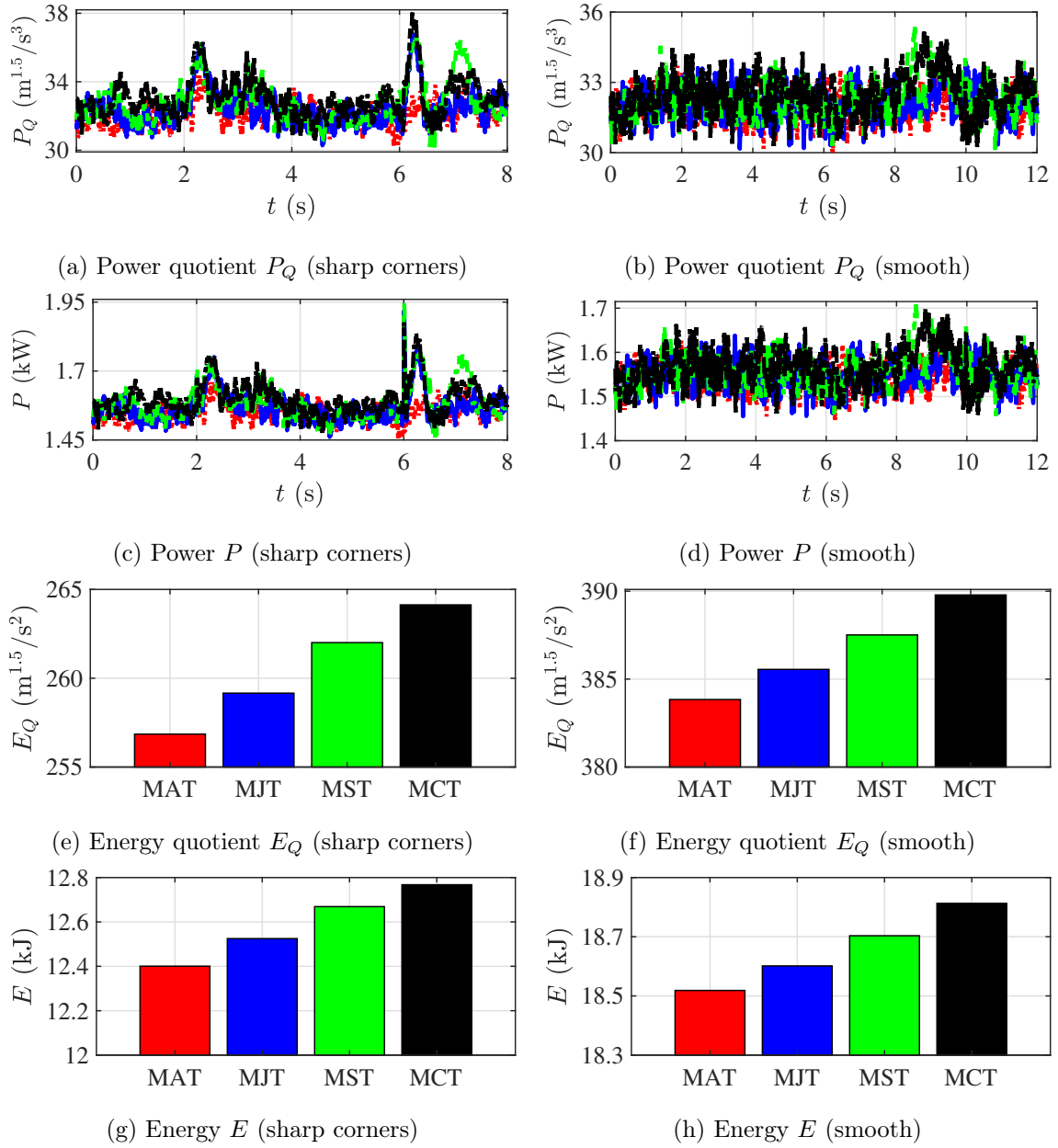


Figure 5.17: Experimental results of P_Q , P , E_Q and E for multiple waypoints trajectories (*sharp corners* and *smooth*). MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

5.2.2 Quadrotor with a Slung Payload

The results of *Sets: 5, 6, 7* and *8* are shown in Figs. 5.18 - 5.21. These are the results of the *Light - Short - Fast*, *Heavy - Short - Fast*, *Heavy - Long - Fast*, and *Heavy - Long - Slow*, respectively.

5.2.2.1 Constant Force Ratio Assumption

Similar to our approach in Section 5.2.1.1, validating our assumption of a constant force ratio, β , for non-aggressive flights requires validating that $\mu \leq 2$ or conservatively $\mu \leq 3$ for non-aggressive flights as well as showing that β can be considered constant for non-aggressive flights experimentally.

Figs. 5.14a and 5.14c, and 5.15a and 5.15b show the propeller forces, F_i , whereas Figs. 5.14d and 5.14f, and 5.15c and 5.15d show the total thrust, F , for the five cases considered in this validation. These results show that the trajectories are dynamically feasible, which indirectly validate the conclusions of Section 3.2.

From these force results, the ratios μ and β can be calculated. Figs. 5.14g and 5.14i, and 5.15e and 5.15f show the experimental results of μ . The experimental results show that considering non-aggressive flights to be flights with $\mu \leq 2$, or conservatively $\mu \leq 3$, is reasonable.

Also, based on the force results, the force ratio, β , is presented in Figs. 5.14j and 5.14l, and 5.15g and 5.15h. The results show that β is almost constant and close to $\beta = 2$ for non-aggressive flights. The mean values of β are indicated in Figs. 5.14j and 5.14l, and 5.15g and 5.15h. This validates our assumption of a constant β for non-aggressive flights in Section 2.3.2 experimentally.

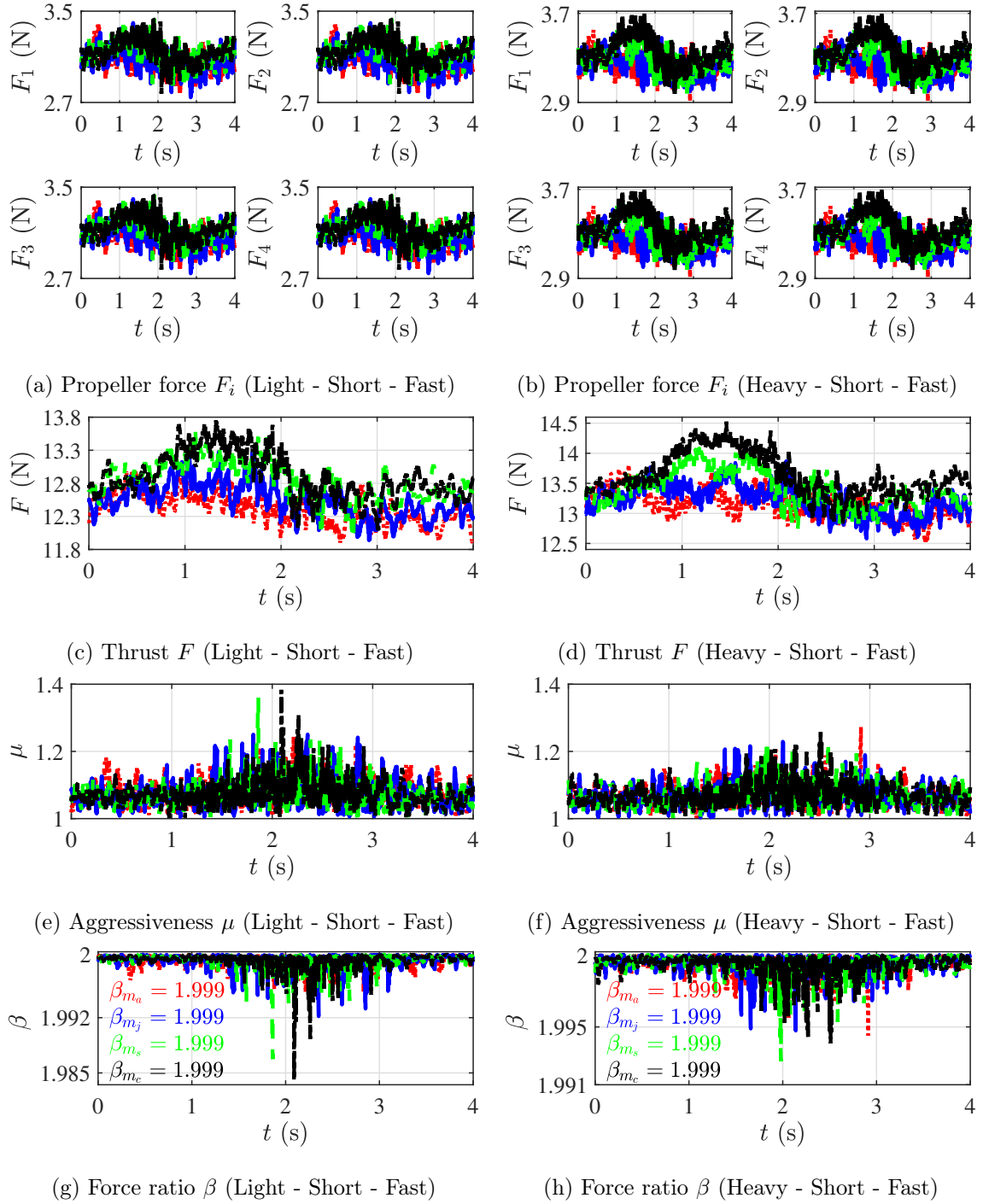


Figure 5.18: Experimental results of F_i , F , μ and β for a quadrotor with a slung payload (*Light - Short - Fast* and *Heavy - Short - Fast*). MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

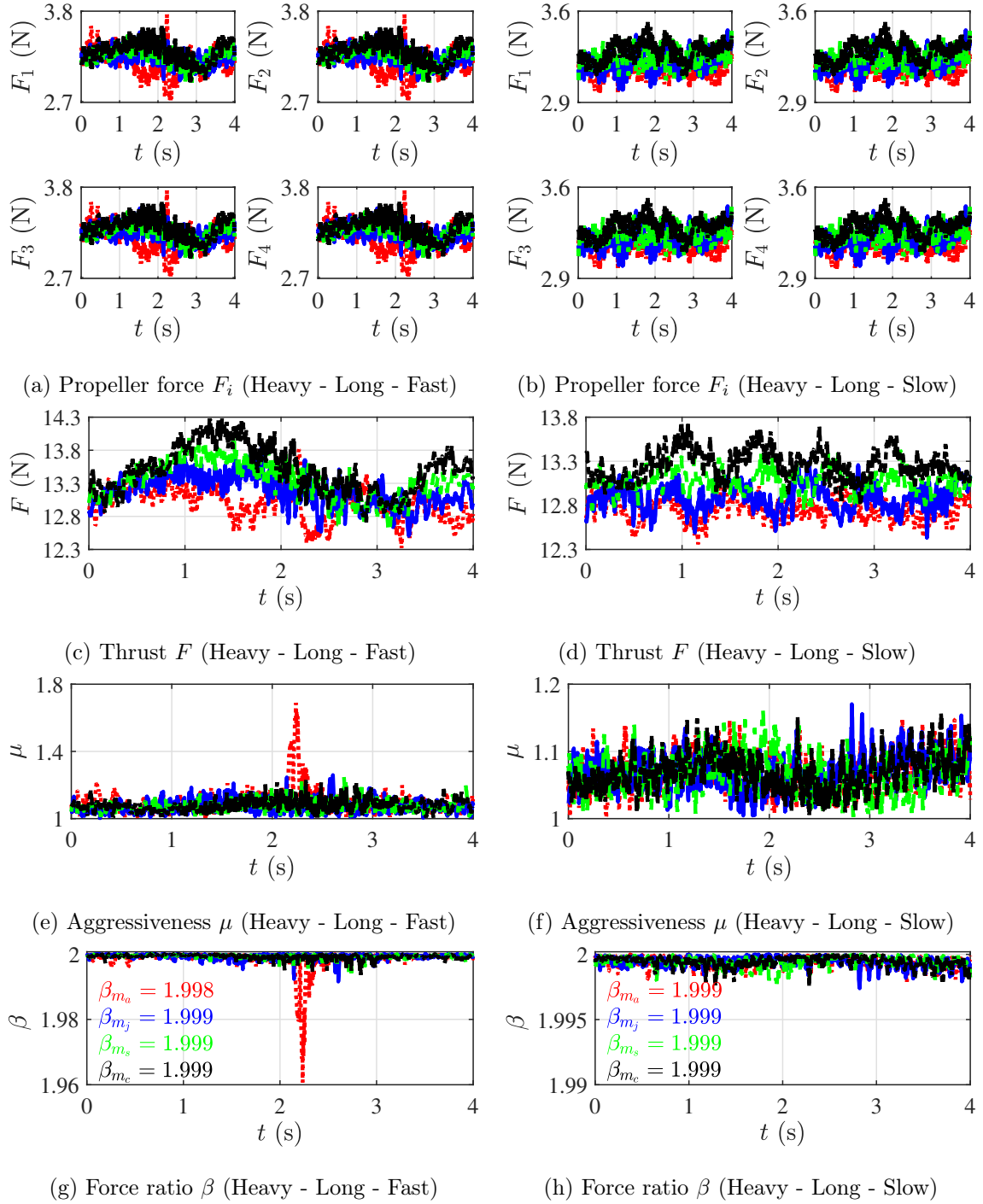


Figure 5.19: Experimental results of F_i , F , μ and β for a quadrotor with a slung payload (**Heavy - Long - Fast** and **Heavy - Long - Slow**). MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

5.2.2.2 Power and Energy Quotients

After validating the assumption of a constant β for non-aggressive flights in Section 5.2.2.1, showing that the proposed concepts of power and energy quotients can be used to represent quadrotor's power and energy consumption is the objective of this section. Specifically, this section shows, experimentally, that for non-aggressive flights $P_Q \propto P$ and $E_Q \propto E$.

Comparing the power and power quotient results for the different cases presented in Figs. 5.20 and 5.21 shows that $P = \gamma P_Q$, where γ is constant as proposed in Eq. (2.54). The maximum, minimum, and mean values of γ are reported in Tables 5.6 and 5.7.

Similarly, comparing the energy consumption and energy quotient results for the different cases presented in Figs. 5.20 and 5.21 shows that $E = \gamma E_Q$. Calculating the proportional constant γ based on energy calculations is shown in the last row of Tables 5.6 and 5.7 as γ_{energy} .

Note that these results also enforce the results in Tables 5.4 and 5.5. For example, the mean value of the all $\gamma_{m_{energy}}$ in these two tables is 48.32. Recall that $\gamma_m = \gamma m_q^{1.5}$ as expressed in Eq. (5.5) and $m_q = 1.121$ kg as reported in Table 2.1. Therefore, the corresponding average $\gamma_{energy} = 40.71$, which is equal to the mean value of all γ_{energy} in Tables 5.6 and 5.7. This shows that γ is constant regardless of the boundary conditions, trajectories, and/or the existence of a slung payload. Therefore, $P_Q \propto P$ and $E_Q \propto E$ regardless of these conditions.

These results show that the concepts of power and energy quotients can be used to estimate quadrotor's power and energy consumption for non-aggressive flights for a quadrotor with a slung payload as introduced in Section 2.3.

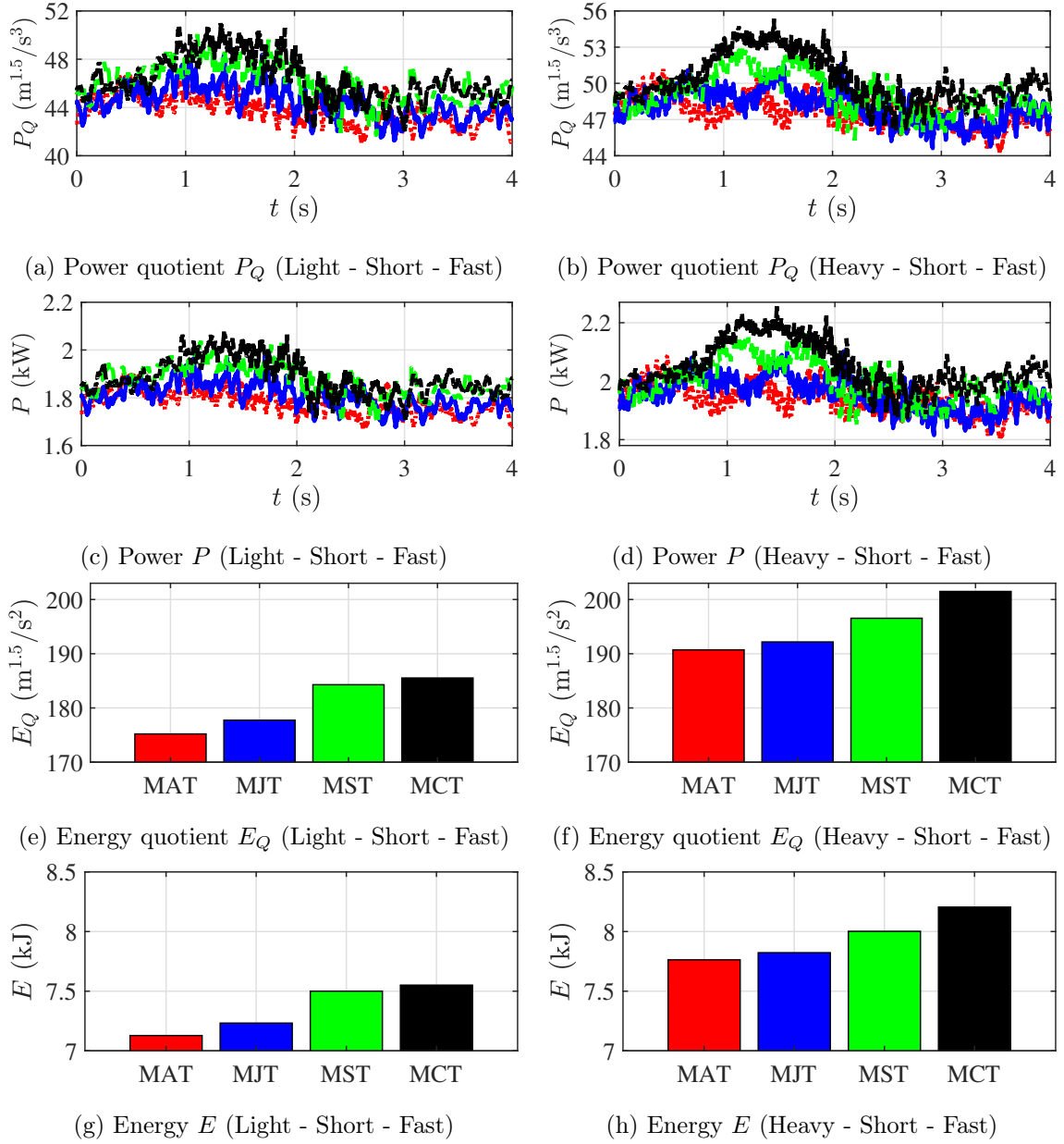


Figure 5.20: Experimental results of P_Q , P , E_Q and E for a quadrotor with a slung payload (*Light - Short - Fast* and *Heavy - Short - Fast*). MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

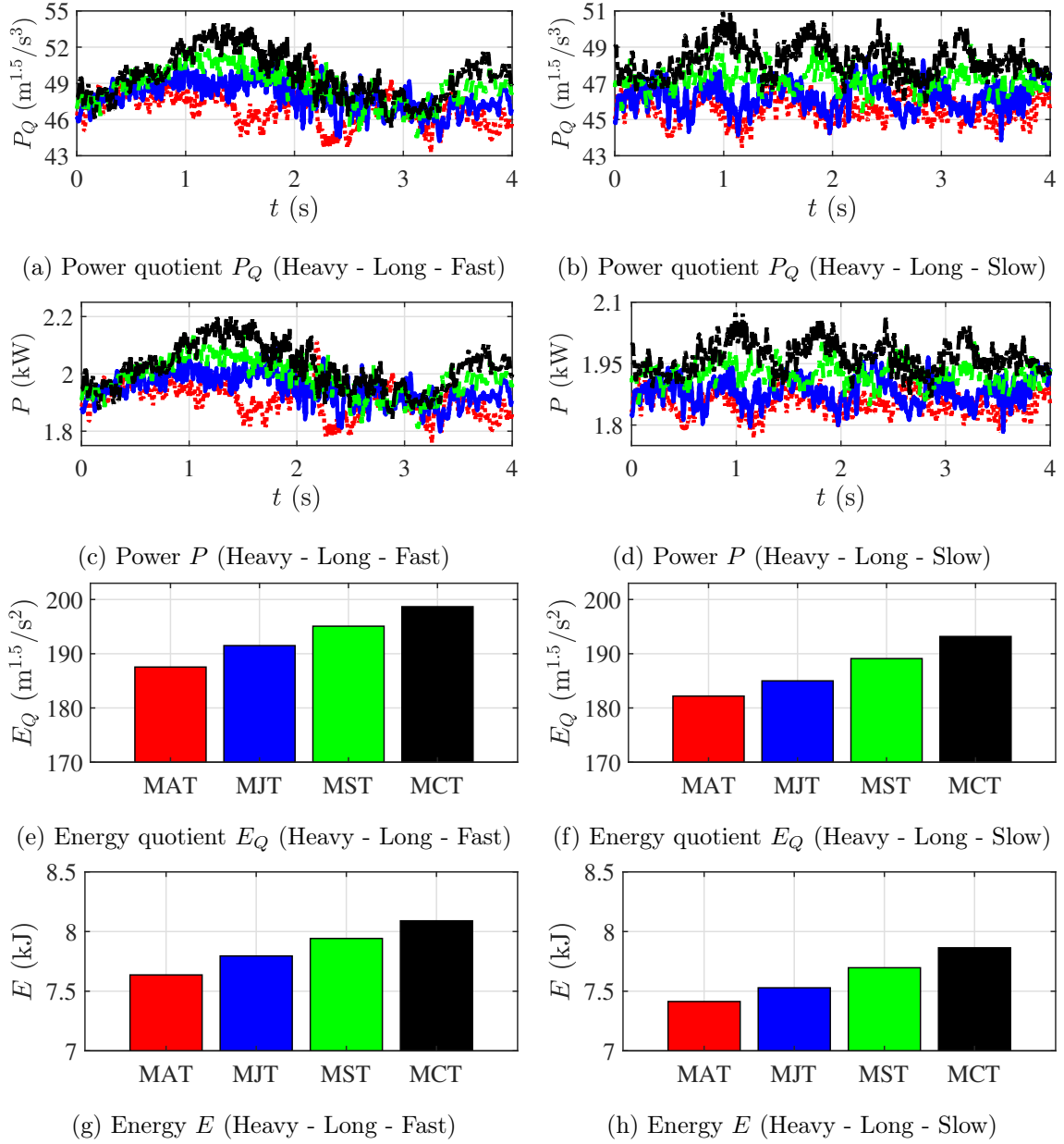


Figure 5.21: Experimental results of P_Q , P , E_Q and E for a quadrotor with a slung payload (**Heavy - Long - Fast** and **Heavy - Long - Slow**). MAT is red dotted, MJT is blue solid, MST is green dashed, MCT is black dash-dotted

Table 5.6: Analysis of the experimental values of γ in case of a quadrotor with a slung payload (*Light - Short - Fast* and *Heavy - Short - Fast*)

	Light - Short - Fast				Heavy - Short - Fast			
	MAT	MJT	MST	MCT	MAT	MJT	MST	MCT
γ_{max}	40.78	40.83	40.98	41.02	40.81	40.82	40.86	40.83
γ_{min}	40.65	40.66	40.66	40.66	40.68	40.67	40.68	40.69
γ_{mean}	40.69	40.69	40.70	40.70	40.70	40.71	40.71	40.72
γ_{energy}	40.69	40.69	40.70	40.70	40.70	40.71	40.71	40.72

Table 5.7: Analysis of the experimental values of γ in case of a quadrotor with a slung payload (*Heavy - Long - Fast* and *Heavy - Long - Slow*)

	Heavy - Long - Fast				Heavy - Long - Slow			
	MAT	MJT	MST	MCT	MAT	MJT	MST	MCT
γ_{max}	41.52	40.88	40.86	40.82	40.73	40.74	40.74	40.75
γ_{min}	40.67	40.67	40.68	40.68	40.67	40.67	40.68	40.68
γ_{mean}	40.72	40.71	40.71	40.72	40.69	40.70	40.70	40.71
γ_{energy}	40.72	40.71	40.71	40.72	40.69	40.70	40.70	40.71

5.2.2.3 Effect of Payload Mass

Fig. 5.20 shows the power and energy results of the *Light - Short - Fast* and the *Heavy - Short - Fast* sets of experiments. The only difference between the two sets is the mass of the payload. One can see that the increasing the mass of the payload increases quadrotor's power and energy consumption.

This validates the energy analysis and the results of the effect of payload mass on quadrotor's energy consumption presented in Section 5.1.1.2.

5.2.2.4 Effect of Cable Length

The right column of results in Fig. 5.20 and the left column of results in Fig. 5.21 show the power and energy results of the *Heavy - Short - Fast* and the *Heavy - Long -*

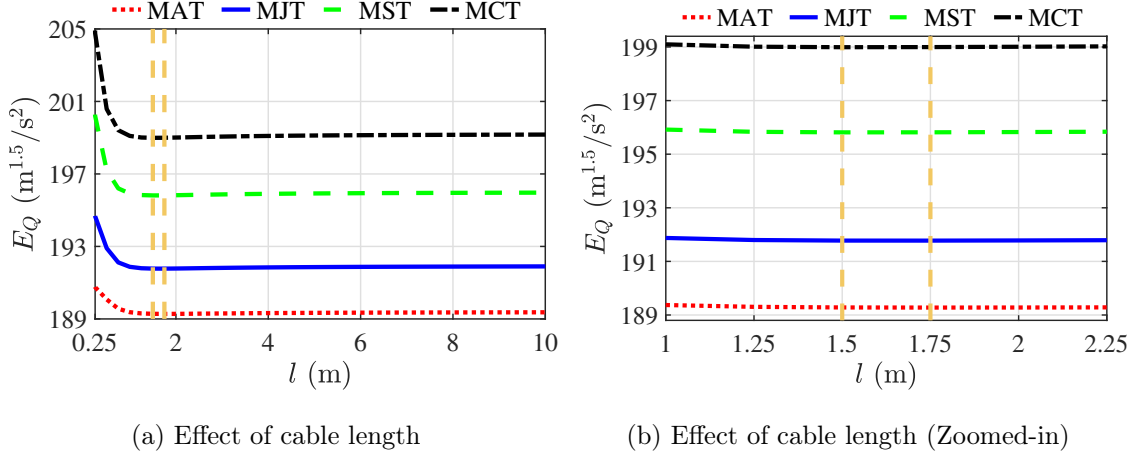


Figure 5.22: Optimal cable length for the **Heavy - Fast** sets

Fast sets of experiments. The only difference between them is the cable length. Fig. 5.22 shows that the optimal cable length is between 1.5 and 1.75 m. One can see that when the cable length is shorter than the optimal cable length, increasing the cable length decreases quadrotor's power and energy consumption. Due to lab space limitation, results of cable lengths longer than the optimal cable length cannot be validated experimentally.

This validates the energy analysis and the results of the effect of cable length on quadrotor's energy consumption presented in Section 5.1.1.2.

5.2.2.5 Effect of Trajectory

This section validates two results: the effect of the degree of the polynomial and the effect of trajectories average acceleration on energy consumption.

First, Figs. 5.20 and 5.21 show that for all sets of experiments, increasing the degree of the polynomial increases the required power and the energy consumption. This is validated regardless of the cable length, payload mass, and/or trajectories average acceleration.

For the **Light - Short - Fast** set, the MAT, MJT, and MST saves about 5.6%, 4.2%, and 0.7% of the energy consumption compared to the MCT as shown in Figs. 5.20g.

Similarly, for the **Heavy - Short - Fast** set, the MAT, MJT, and MST saves about 5.4%, 4.7%, and 2.5% of the energy consumption compared to the MCT as shown in Figs. 5.20h. As for the **Heavy - Long - Fast** set, the MAT, MJT, and MST saves about 5.6%, 3.6%, and 1.8% of the energy consumption compared to the MCT as shown in Figs. 5.21g. For the **Heavy - Long - Slow** set, the MAT, MJT, and MST saves about 5.7%, 4.3%, and 2.1% of the energy consumption compared to the MCT as shown in Figs. 5.21h.

Second, Fig. 5.21 shows the power and energy results of the **Heavy - Long - Fast** and **Heavy - Long - Slow** sets of experiments. The only difference between them is the average acceleration, $\mathbf{a}_{av}$. It can be seen from both the power and energy results that increasing the average acceleration, $\mathbf{a}_{av}$, increases quadrotor's power and energy consumption.

5.3 Summary

This chapter provides a comprehensive energy analysis of both a single quadrotor and a quadrotor with a slung payload, which leads to trajectory-based energy-saving strategies. It analyzes the effects of the polynomial trajectory, cable length, and payload mass on the energy consumption of a single quadrotor and a quadrotor with a slung payload. The proposed trajectory-based energy-saving strategy does not require knowing any of quadrotor-related parameters. Thus, it is simpler than energy-minimized trajectories, yet provide a close energy performance. Additionally, this is the first energy analysis and trajectory-based energy-saving strategy for a quadrotor with a slung payload.

First, an extensive mathematical analysis for the effect of polynomial trajectories on the energy consumption of a quadrotor and a quadrotor with a slung payload under different scenarios, e.g., rest-to-rest and different boundary conditions, was conducted. The main conclusion is that increasing the degree of the polynomial increases quadrotor's energy consumption monotonically and regardless of any other conditions.

Second, a parametric analysis for the case of a quadrotor with a slung payload was performed. The effect of payload mass, cable length, and trajectory's average acceleration on quadrotor's energy consumption were investigated. The results of the existence of an optimal cable length, at which, quadrotor's energy consumption is minimal was presented and explained.

Third, the energy performance of the MAT is compared to a corresponding energy-minimized trajectory from the literature. Interestingly, it was found that the MAT behaves very closely to the energy-minimized trajectory. The results showed that the MAT consumed about 0.05% more energy compared to the energy-minimized trajectory. In fact, the four considered polynomial trajectories showed a close energy consumption with an increase of 0.07%, 0.11%, and 0.15% for the MJT, MST, and MCT, respectively, compared to the energy-minimized trajectory. However, generating a MAT, or any other polynomial trajectory, does not require knowing any of quadrotor-related parameters, e.g., motor and propeller parameters. The energy-minimized trajectory, on the other hand, requires knowing all of these quadrotor-related parameters. So, this chapter suggests utilizing MATs as a practical and more straightforward energy-efficient trajectory.

Finally, all the assumptions, concepts, and results presented in this chapter were validated experimentally. The experiments showed that polynomial trajectories with multiple waypoints have similar energy behaviour as trajectories with two waypoints. Therefore, the power and energy quotients proposed in this chapter are valid for them.

6 Control-Based Energy-Saving

As reported in Section 1.2, designing energy-efficient controllers is one of the approaches to reduce quadrotor's energy consumption. In fact, many energy-efficient controllers depend on constraining the control command in different ways to reduce quadrotor's energy consumption, e.g., [4, 65, 79, 80]. In such cases, there may be a trade off between energy consumption and controller performance [4, 65]. More importantly, it is not straightforward to estimate such a trade off, if it exists, in most of constraining methods in the literature as discussed in Section 1.2.

Additionally, to the best of our knowledge, the literature focuses on quadrotors without payloads. As discussed in the previous chapters, adding a slung payload significantly affects the dynamics and energy consumption of the system.

Therefore, this chapter proposes a controller-based energy-efficient strategy for a transportation system consisting of a quadrotor with a slung payload. Moreover, the proposed controller is robust against disturbances.

6.1 Preliminaries

The dynamics model of the transportation system of interest is expressed in Eq. (2.40), which is repeated here again after considering disturbances, $\mathbf{d}$, for the convenience of the reader:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{G} = \mathbf{Q} + \mathbf{d} \quad (6.1)$$

Consider an Euler-Lagrange system, such as the one presented in Eq. (6.1), with a generalized coordinates vector, $\mathbf{q} \in \mathbb{R}^n$. The following properties are important to design and prove the proposed controller [135–137]:

Property 1. $\frac{1}{2}\dot{\mathbf{M}} - \mathbf{C}$ is a skew-symmetric matrix, which always satisfies the equation for an arbitrary vector, $\mathbf{v} = [v_1, v_2, \dots, v_n]^T \in \mathbb{R}^n$:

$$\mathbf{v}^T \left[\frac{1}{2}\dot{\mathbf{M}} - \mathbf{C} \right] \mathbf{v} = 0 \quad (6.2)$$

Property 2. $\mathbf{M}$ is a symmetric positive definite matrix satisfying the following:

$$\gamma_1 \mathbf{I}_n \leq \mathbf{M} \leq \gamma_2 \mathbf{I}_n \quad (6.3)$$

where γ_1 and γ_2 are positive constants. Recall that for arbitrary square matrices of size $n \times n$, $\mathbf{A}_1$ and $\mathbf{A}_2$:

$$\mathbf{A}_1 > \mathbf{A}_2 \Leftrightarrow \mathbf{v}^T \mathbf{A}_1 \mathbf{v} > \mathbf{v}^T \mathbf{A}_2 \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{R}^n \neq \mathbf{0}_{n \times 1} \quad (6.4)$$

Property 3. $\mathbf{C}$ satisfies the following:

$$\|\mathbf{C}\| \leq \gamma_3 \|\dot{\mathbf{q}}\| \quad (6.5)$$

where γ_3 is a positive constant.

Property 4.

$$\sup \|\mathbf{G}\| \leq \gamma_4 \quad (6.6)$$

where γ_4 is a positive constant and $\sup(\cdot)$ is a functions that returns the supremum of its argument.

Additionally, the following assumptions are made throughout this chapter:

Assumption 1. *The external disturbances, $\mathbf{d}$, satisfy*

$$|\mathbf{d}| \leq \mathbf{d}_{dist} \quad (6.7)$$

with $\mathbf{d}_{dist} \in \mathbb{R}_{\geq 0}^n$ being a non-negative known vector and the inequality is element-wise.

Assumption 2. *The upper bound of the acceleration of the reference exists and is known.*

Lemma 1. *Considering a system consisting of a quadrotor with slung payload, if there exists a Lyapunov function V satisfying $\dot{V} \leq -\gamma_5 V^\Pi$ with $0 < \Pi < 1$ and $\gamma_5 > 0$, it can be concluded that the system is finite-time stable.*

Let us also define the following saturation and signum functions:

$$\text{Sat}(\mathbf{v}) = [\text{sat}(v_1), \text{sat}(v_2), \dots, \text{sat}(v_n)]^\top \quad (6.8)$$

where

$$\text{sat}(v_i) = \begin{cases} \text{sig}(v_i)^{\gamma_6}, & |v_i| \leq 1, \\ \text{sign}(v_i), & |v_i| > 1 \end{cases} \quad (6.9)$$

where $0 < \gamma_6 < 1$, $i = 1, 2, \dots, n$, and

$$\text{sig}(v_i)^{\gamma_6} = \text{sign}(v_i) |v_i|^{\gamma_6} \quad (6.10)$$

and

$$\text{sign}(v_i) = \begin{cases} -1, & v_i < 0, \\ 0, & v_i = 0, \\ 1, & v_i > 0 \end{cases} \quad (6.11)$$

6.2 Controller Design

Sliding mode controllers are robust against disturbances and do not require model linearization [137, 138]. The main drawback of sliding mode controllers is the chattering problem, which may lead to undesired vibrations. These undesired vibrations may affect the tracking performance. However, it was found that a satisfactory performance can be obtained at lower rotor speeds [139, 140]. Since this dissertation focuses on non-aggressive flights, this section proposes an integral sliding mode controller with input saturation for a quadrotor with a slung payload.

As introduced in Section 2.3 and validated in Section 5.2, quadrotor's energy consumption for non-aggressive flights depends on the total thrust, F . Therefore, the proposed controller constrains this input, i.e., saturates F , within a certain limit to ensure energy efficiency. Since the thrust F is the focus of the controller, the proposed controller is a position controller, i.e., outer loop, whereas the attitude controller, i.e., the inner loop, remains the same PV controller introduced in Section 2.4. The proposed controller saturates the thrust by saturating the error to a certain limit. This provides a direct intuition about the performance. An intuition that is not straightforward in the literature.

Inspired by [137], let us define the position and velocity errors of an Euler-Lagrange system with a generalized coordinates vector, $\mathbf{q}$, to be $\mathbf{e}_p = \mathbf{q} - \mathbf{q}_d$ and $\mathbf{e}_v = \dot{\mathbf{q}} - \dot{\mathbf{q}}_d$, respectively.

Under the saturation function shown in Eqs. (6.8) - (6.11), an integral sliding mode surface is developed as:

$$\mathbf{S} = \mathbf{e}_v + \int_0^t \mathbf{M}^{-1} \left(\mathbf{C}\mathbf{e}_v + \mathbf{K}_p \text{Sat}(\mathbf{k}\mathbf{e}_p) + \mathbf{K}_v \text{Sat}(\mathbf{k}\mathbf{e}_v) \right) d\varepsilon \quad (6.12)$$

where ε is the variable of the integration that represents the time, and $\mathbf{K}_p = \text{diag}(k_{p1}, k_{p2}, \dots, k_{pn}) \in \mathbb{R}_{\geq 0}^{n \times n}$ and $\mathbf{K}_v = \text{diag}(k_{v1}, k_{v2}, \dots, k_{vn}) \in \mathbb{R}_{\geq 0}^{n \times n}$ are diagonal non-negative gain matrices.

The gain matrix $\mathbf{k} = \text{diag}(k_1, k_2, \dots, k_n) \in \mathbb{R}_{\geq 0}^{n \times n}$ is a diagonal non-negative matrix.

The control law is designed with the proposed sliding mode manifold as:

$$\mathbf{Q} = \mathbf{G} - (\mathbf{k}_s + \boldsymbol{\kappa}) \text{sign}(\mathbf{M}^{-1}\mathbf{S}) - \mathbf{K}_p \text{Sat}(\mathbf{k}\mathbf{e}_p) - \mathbf{K}_v \text{Sat}(\mathbf{k}\mathbf{e}_v) \quad (6.13)$$

where $\mathbf{k}_s = [k_{s_1}, k_{s_2}, \dots, k_{s_n}]^\top \in \mathbb{R}_{\geq 0}^n$ and $\boldsymbol{\kappa} = [\kappa_1, \kappa_2, \dots, \kappa_n]^\top \in \mathbb{R}_{\geq 0}^n$ are chosen non-negative gain vectors.

Theorem 1. *Under the proposed control law in Eq. (6.13), the finite-time reachability of the sliding mode surface for a quadrotor with a slung payload can be guaranteed under Assumptions 1 and 2 if $\mathbf{k}_s$ and $\boldsymbol{\kappa}$ are selected such that:*

$$\begin{aligned} \mathbf{k}_s &> \mathbf{d}_{dist}, \\ \|\boldsymbol{\kappa}\| &\geq \sup \left\{ \varphi_1 \|\mathbf{M}\ddot{\mathbf{q}}_d\| + \gamma_3 \|\mathbf{e}_v\| \|\dot{\mathbf{q}}_d\| + \varphi_2 \gamma_3 \|\dot{\mathbf{q}}_d\|^2 \right\}, \quad i \in \{1, 2, \dots, n\} \end{aligned} \quad (6.14)$$

where $\varphi_1 \geq 1$ and $\varphi_2 \geq 1$.

Proof: Select a Lyapunov function as follows:

$$V_1 = \frac{1}{2} \mathbf{S}^\top \mathbf{S} \quad (6.15)$$

The time derivative of this Lyapunov function is:

$$\dot{V}_1 = \mathbf{S}^\top \left(\dot{\mathbf{e}}_v + \mathbf{M}^{-1} \left(\mathbf{C}\mathbf{e}_v + \mathbf{K}_p \text{Sat}(\mathbf{k}\mathbf{e}_p) + \mathbf{K}_v \text{Sat}(\mathbf{k}\mathbf{e}_v) \right) \right) \quad (6.16)$$

From Eqs. (6.13) and (6.16), it can be found that:

$$\dot{V}_1 = \mathbf{S}^\top \left(\dot{\mathbf{e}}_v + \mathbf{M}^{-1} \left(\mathbf{C}\mathbf{e}_v + \mathbf{G} - (\mathbf{k}_s + \boldsymbol{\kappa}) \text{sign}(\mathbf{M}^{-1}\mathbf{S}) - \mathbf{Q} \right) \right) \quad (6.17)$$

Recall that $\mathbf{e}_v = \dot{\mathbf{q}} - \dot{\mathbf{q}}_d$ and $\dot{\mathbf{e}}_v = \ddot{\mathbf{q}} - \ddot{\mathbf{q}}_d$, thus utilizing Eq. (6.1) yields:

$$\dot{\mathbf{e}}_v = \mathbf{M}^{-1} (-\mathbf{C}\dot{\mathbf{q}} - \mathbf{G} + \mathbf{Q} + \mathbf{d} - \mathbf{M}\ddot{\mathbf{q}}_d) \quad (6.18)$$

From Eqs. (6.17) and (6.18), it can be found that:

$$\dot{V}_1 = \mathbf{S}^\top \mathbf{M}^{-1} \left(-(\mathbf{k}_s + \boldsymbol{\kappa}) \text{sign}(\mathbf{M}^{-1} \mathbf{S}) + \mathbf{d} - \mathbf{C} \dot{\mathbf{q}}_d - \mathbf{M} \ddot{\mathbf{q}}_d \right) \quad (6.19)$$

which leads to:

$$\begin{aligned} \dot{V}_1 &\leq -\left\| \mathbf{S}^\top \mathbf{M}^{-1} \right\| (\|\mathbf{k}_s\| - \|\mathbf{d}_{dist}\| - \|\mathbf{C} \dot{\mathbf{q}}_d\| - \|\mathbf{M} \ddot{\mathbf{q}}_d\| + \|\boldsymbol{\kappa}\|) \\ &\leq -\frac{\|\mathbf{k}_s\| - \|\mathbf{d}_{dist}\|}{\gamma_2} V_1^{\frac{1}{2}} \end{aligned} \quad (6.20)$$

Based on Lemma 1, the sliding mode surface can be reached in finite time. This completes the proof.

Remark 1. From [141], it is obtained that there are bounded scalars $H_v > 0$ and $H_{\dot{v}} > 0$, and $\|v\| \leq H_v$ and $\|\dot{v}\| \leq H_{\dot{v}}$ are constructed. Thus, it is reasonable to select the proper $\boldsymbol{\kappa}$ in Eq. (6.13) to satisfy the condition in Eq. (6.14).

When the states of the quadrotor system reach the sliding mode surface, one has $\mathbf{S} = \dot{\mathbf{S}} = \mathbf{0}_{n \times 1}$. Therefore, the sliding-mode error dynamics satisfy the following forms:

$$\dot{\mathbf{e}}_p = \mathbf{e}_v \quad (6.21)$$

$$\dot{\mathbf{e}}_v = -\mathbf{M}^{-1} \left(\mathbf{C} \mathbf{e}_v + \mathbf{K}_p \text{Sat}(\mathbf{k} \mathbf{e}_p) + \mathbf{K}_v \text{Sat}(\mathbf{k} \mathbf{e}_v) \right) \quad (6.22)$$

Theorem 2. The trajectory tracking is achieved with the proposed controller in Eq. (6.13) for the error dynamics in Eqs. (6.21) and (6.22), and the control input in Eq. (6.13) is bounded under the Assumptions 1 and 2.

Proof: Select a Lyapunov function as follows:

$$V_2 = \frac{1}{2} \mathbf{e}_v^\top \mathbf{M} \mathbf{e}_v + \int_0^{\mathbf{e}_p} \mathbf{K}_p \text{Sat}(\mathbf{k} \boldsymbol{\varsigma}) \, \mathrm{d}\boldsymbol{\varsigma} \quad (6.23)$$

where $\boldsymbol{\varsigma}$ is the variable of the integration that represents the position error and the integral of a vector valued function is the summation of the integrals of each component, i.e., the summation of the function's element-wise integrals. The time derivative of this Lyapunov function is:

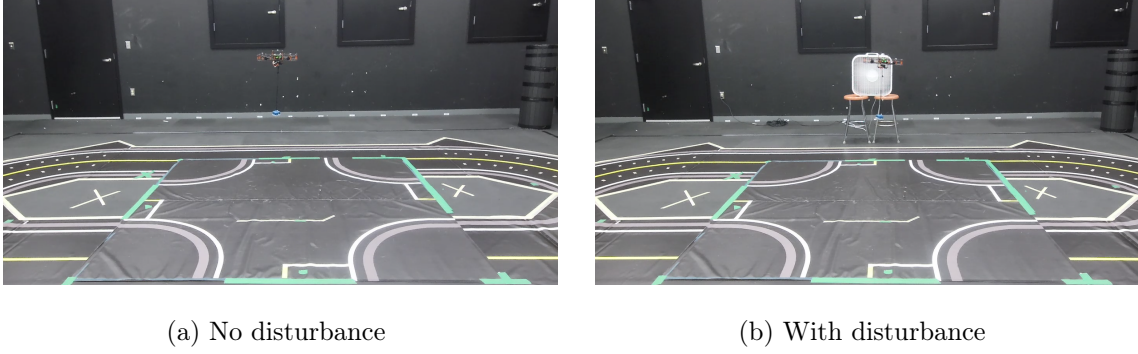


Figure 6.1: Snapshots of the experiments in the presence and absence of disturbances

$$\begin{aligned}
 \dot{V}_2 &= \mathbf{e}_v^T \mathbf{M} \dot{\mathbf{e}}_v + \frac{1}{2} \mathbf{e}_v^T \mathbf{M} \dot{\mathbf{e}}_v + \mathbf{e}_v^T \mathbf{K}_p \text{Sat}(\mathbf{k} \mathbf{e}_p) \\
 &= -\mathbf{e}_v^T \mathbf{K}_v \text{Sat}(\mathbf{k} \mathbf{e}_v) \leq 0
 \end{aligned} \tag{6.24}$$

By utilizing the LaSalle's principle, it is concluded that the trajectory tracking of the system in Eqs. (6.21) and (6.22) is obtained as the quadrotor system reaches the sliding manifold in Eq. (6.12).

6.3 Experimental Results

The experiments conducted in this section is to validate experimentally the effectiveness of the proposed controller in saving energy as well as the robustness against disturbances. Then, a qualitative comparison with literature is presented. Finally, the results of these experiments are used to enforce the experimental validation of proposed concepts of power and energy quotients introduced in Section 2.3. Test videos are available at <https://youtu.be/FmRXXULkQWA>.

For all experiments, the payload shown in Fig. 2.6b is used with a mass of 0.142 kg and the cable length is 0.5 m. The fan shown in Fig. 2.8 was used to generate wind disturbances. Fig. 6.1 shows snapshots of the experiments in the presence and absence of disturbances. The controller gains are chosen as follows:

$$\gamma_6 = 0.5 \quad (6.25)$$

$$\mathbf{K}_p = \begin{bmatrix} 1.5 & 0 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 1.95 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.3 \end{bmatrix} \quad (6.26)$$

$$\mathbf{K}_v = \begin{bmatrix} 0.65 & 0 & 0 & 0 & 0 \\ 0 & 0.65 & 0 & 0 & 0 \\ 0 & 0 & 0.85 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix} \quad (6.27)$$

$$\mathbf{k}_s = \begin{bmatrix} 0.04 & 0 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 & 0 \\ 0 & 0 & 0.06 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & 0 \\ 0 & 0 & 0 & 0 & 0.02 \end{bmatrix} \quad (6.28)$$

$$\boldsymbol{\kappa} = \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0.01 \end{bmatrix} \quad (6.29)$$

The quadrotor is required to follow the following trajectory at an inertial height of $z = 1$ m:

$$[x, y]^T = \begin{cases} [1, 0.2t - 1]^T, & 0 \leq t \leq 10, \\ [-0.2(t - 10) + 1, 1]^T, & 10 < t \leq 20, \\ [-1, -0.2(t - 20) + 1]^T, & 20 < t \leq 30, \\ [0.2(t - 30), -1]^T, & 30 < t \leq 40, \\ [1, -1,]^T, & t > 40 \end{cases} \quad (6.30)$$

The experimental validation considered scenarios with and without wind disturbances. Regarding controller input saturation state, the following sets of experiments were considered:

I) **No saturation:** where the controller in Eq. (6.13) is used without the proposed input saturation strategy. This represents a conventional integral sliding mode controller and is denoted in this chapter as *No Sat*.

II) **Saturation:** where the input saturation is consider in the controller in Eq. (6.13) to save energy. The saturation is specified by the matrix $\mathbf{k}$, where larger $\mathbf{k}$ means lower thrust threshold and less energy consumption. Five levels of saturation are considered in this chapter. The first saturation level is defined by the gain matrix:

$$\mathbf{k} = 7.5\mathbf{I}_5 = \begin{bmatrix} 7.5 & 0 & 0 & 0 & 0 \\ 0 & 7.5 & 0 & 0 & 0 \\ 0 & 0 & 7.5 & 0 & 0 \\ 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 0 & 0 & 7.5 \end{bmatrix} \quad (6.31)$$

which is referred to as $\mathbf{k} = 7.5$ in this chapter. Similarly, four more saturation levels are considered in this chapter: $\mathbf{k} = 8.5$, $\mathbf{k} = 9.5$, $\mathbf{k} = 10$, and $\mathbf{k} = 12.5$. The last two sets are used only in the experiments with wind disturbances.

One of the motivations behind designing this controller is to provide some intuition about the effect of the input saturation and the performance of the quadrotor. This can be explained numerically on these experiments. For the *No Sat* set, the position and velocity errors, $\mathbf{e}_p$ and $\mathbf{e}_v$, are multiplied directly by their corresponding gains regardless of the magnitude of the error. In other words, the control command is, in a sense, proportional to the errors.

On the other hand, the saturated sets are different. We saturate the errors based on Eqs. (6.8) - (6.11). This means that if any element i of $|\mathbf{ke}_p|$ is larger than 1, it will be assumed that $k_i e_{p_i} = \text{sign}(k_i e_{p_i})$. The same concept applies to $\mathbf{ke}_v$. It is worth noting that the gain matrix $\mathbf{k}$ provides control over such a threshold. This implies that the control command is no longer proportional to the error if the error exceeds a certain threshold in

Table 6.1: Error saturation bounds for each state

	$\mathbf{k} = 7.5$	$\mathbf{k} = 8.5$	$\mathbf{k} = 9.5$	$\mathbf{k} = 10$	$\mathbf{k} = 12.5$
$ e_x _{sat}, e_y _{sat}, e_z _{sat}$ (m)	0.13	0.12	0.11	0.10	0.08
$ e_{\theta_1} _{sat}, e_{\theta_2} _{sat}$ ($^\circ$)	7.64	6.74	6.03	5.73	4.58

contrast to the **No Sat** set. For example, for the case of $\mathbf{k} = 7.5$, if the position error of any state $|e_{p_i}| \geq 1/7.5 \approx 0.133$, it will be assumed to be equal to 0.133. This prevents commanding larger thrusts in case of larger errors to save energy. The same conclusion applies to $\mathbf{e}_v$. It is worth mentioning that the gain matrix $\mathbf{k}$ can be different for position and velocity errors if different saturation levels are required; however, it is considered the same in these experiments for simplicity and without loss of generality.

It is crucial to realize that the idea behind input saturation via bounding the errors is to avoid larger thrusts in case of larger errors. However, for a certain error that is within the bounds of two different gain matrices $\mathbf{k}$, the smaller $\mathbf{k}$ will result in less energy consumption. To illustrate this numerically, consider a position error in x direction, $e_x = 0.1$, at a certain instant. This is within the range of $\mathbf{k} = 7.5$ and $\mathbf{k} = 8.5$ as indicated in Table 6.1. Given the gain $\mathbf{K}_p$ in Eq. (6.26) along with Eqs. (6.8) - (6.11), the magnitude of $\mathbf{K}_p \text{Sat}(\mathbf{k}\mathbf{e}_p)$ would be equal to 1.299 for the $\mathbf{k} = 7.5$ case and 1.383 for the $\mathbf{k} = 8.5$ case. The same concept applies, with different values, to all directions and to $\mathbf{e}_v$ as well. Therefore, a considerably larger saturation, i.e., larger $\mathbf{k}$, may result in larger energy consumption because it provides very tight error bounds, which means that the quadrotor's thrust may be saturated for longer time. So, it is crucial to choose the saturation level, i.e., $\mathbf{k}$, smartly according to the flight task of interest. This choice is intuitive and quadrotors' operators can make such a choice based on Table 6.1 straightforwardly. Recall that based on Eqs. (6.9) and (6.10), the power γ_6 can be used to control the speed of convergence. Decreasing γ_6 increases the speed of convergence; however, it increases the energy consumption.

Table 6.2: Statistics of the experimental errors for different saturation levels (No disturbance)

	<i>No Sat</i>			k = 7.5		
	Mean	Median	Maximum	Mean	Median	Maximum
$ e_x $ (cm)	2.56	1.40	12.01	2.77	1.61	11.70
$ e_y $ (cm)	1.91	1.12	6.37	2.41	1.38	13.95
$ e_z $ (cm)	0.44	0.46	1.05	0.26	0.23	0.91
$ e_{\theta_1} $ ($^\circ$)	3.64	2.68	16.51	3.81	2.34	17.69
$ e_{\theta_2} $ ($^\circ$)	3.37	2.32	18.40	4.73	3.62	29.95

6.3.1 Controller Effectiveness

In these experiments we consider the cases without wind disturbances. The power and energy results of *No Sat* set of experiments are compared to the **k = 7.5**, **k = 8.5**, and **k = 9.5** sets to show the effectiveness of the proposed controller in saving energy. Fig. 6.2 shows these results.

It can be seen from Fig. 6.2 that the proposed input saturation approach was able to reduce quadrotor's energy consumption. In these experiments, **k = 7.5** resulted in the least energy consumption with about 2.9% saving of energy consumption. As explained in Section 6.3, increasing the saturation level, i.e., **k**, may increase the energy consumption even more than the *No Sat* level as seen in Fig. 6.2 for cases **k = 8.5**, and **k = 9.5**.

As discussed in Section 1.2, there may be a trade off between input saturation and energy saving. Fig. 6.3 shows the tracking results, whereas Table 6.2 reports the mean, median, and maximum absolute errors of the *No Sat* and **k = 7.5** sets (the minimum error is zero for all cases). Comparing the tracking results of the *No Sat* and **k = 7.5** sets from Fig. 6.3 and Table 6.2 shows that the overall performance of the **k = 7.5** set is worse than the *No Sat* set; however, the difference is not significant.

6.3.2 Controller Robustness

In these experiments, we show the robustness of the proposed controller against wind disturbances. In fact, the proposed integral sliding mode controller is robust against disturbances with and without saturation. The energy and tracking results of the *No Sat* set of experiments and the $\mathbf{k} = 7.5$, $\mathbf{k} = 8.5$, $\mathbf{k} = 9.5$, $\mathbf{k} = 10$, and $\mathbf{k} = 12.5$ sets are shown in Figs. 6.4 and 6.5, respectively. It is worth mentioning that the magnitude of the disturbance depends on the relative position of the quadrotor with respect to the fan; therefore, the disturbance is time-varying throughout the flight.

It can be seen from Fig. 6.4 that the existence of wind disturbances increases the energy consumption compared to the corresponding cases without wind disturbances in Fig. 6.2. However, the proposed controller still can reduce the energy consumption by about 4% and 5.7% for the $\mathbf{k} = 7.5$ and $\mathbf{k} = 8.5$, respectively. The least energy consumption in these experiments is $\mathbf{k} = 8.5$. This depends on tracking errors as explained in Section 6.3. Fig. 6.5 shows that the tracking performance is satisfactory in the presence of wind disturbances.

6.3.3 Qualitative Comparative Study

In [4], the authors proposed an energy-saving controller for a single quadrotor based on constraining the roll and pitch angles. Although they used only a quadrotor (no payload), which is smaller than ours in addition to following shorter trajectories, we still can compare our results to theirs qualitatively.

The experiments in [4] followed a square trajectory like the trajectory followed in this chapter, but shorter (see Fig. 6.6). The perimeter of their square trajectory is 2 m and it was followed in a flight time of 20 s per cycle. This results in an average acceleration in

any direction of about $a_{av} = 0.005 \text{ m/s}^2$ as defined in Section 5.1.1¹. In our experiments, on the other hand, the perimeter of the square trajectory is 8 m and the flight time is 40 s, which means an average acceleration of $a_{av} = 0.005 \text{ m/s}^2$. This is equal to the one in [4], which allows a fair comparison.

The controller designed in [4] bounded the roll and pitch angles to be within $\pm 6^\circ$, as an experimentally-defined energy-efficient range (see Fig. 6.6). Some major differences between our experiments and the experiments in [4] include the size, mass, and velocity. The velocity is important in such a case since it significantly affects the roll and pitch angles as expressed in the dynamics model in Section 2.2. The average velocity in [4] is 0.1 m/s, whereas our average velocity is 0.2 m/s, which is doubled. Even more importantly, the experiments in [4] did not consider any payloads, i.e., just a single quadrotor. Therefore, it is expected to get larger roll and pitch angles. It is worth noting that, according to [4], this energy-efficient range is specified experimentally, so it is different for our QDrone.

We show the roll and pitch angles for our experiments in Fig. 6.7. It can be seen that our proposed controller results in roll and pitch angles that are comparable to the energy-efficient range defined in [4] (despite the aforementioned differences). The maximum angle we got in all cases with and without disturbances is about 10° , which is about 67% more than the energy-efficient range defined in [4]. In [4], they saved about 3.5 – 6% of energy consumption throughout their experiments, whereas we were able to save about 2.9 – 5.7% in our experiments. This is comparable especially when taking the aforementioned differences into consideration. However, this is meant to be a qualitative comparison.

More importantly, our proposed controller is conceptually different from the one pre-

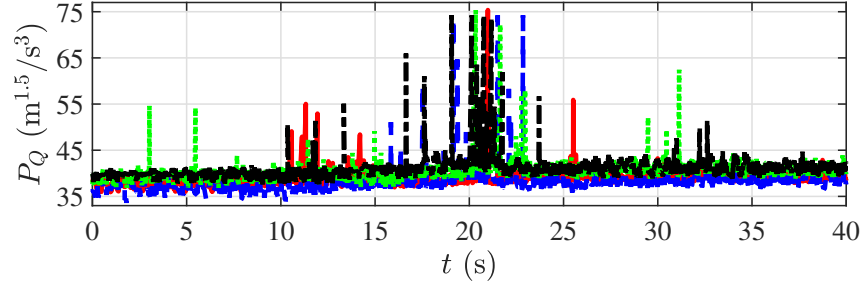
¹The experiments in [4] considered also the time and distance to hover, go to the start point of the square trajectory, landing, and stop at each vortex of the square. This is not relevant in the comparison; however, the average acceleration if we consider all of these motion is $a_{av} \approx 0.0016 \text{ m/s}^2$, which is less than what we consider in our comparison.

sented in [4] and is expected to perform better in some cases such as the cases with higher fluctuation in roll and pitch angles, but the angles themselves are within the energy-efficient range. An example of that could be the KCCP trajectory presented in Sections 3.3.2 and 3.4.2.

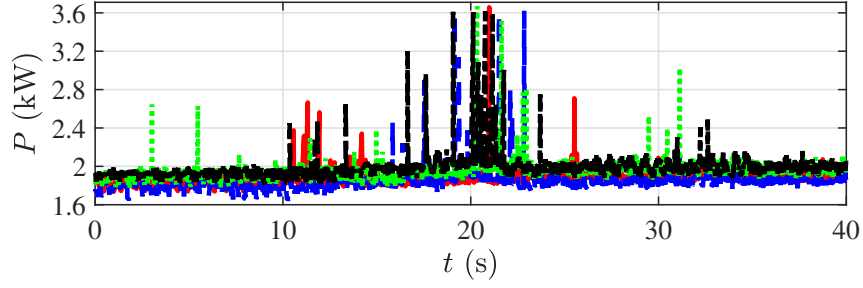
6.3.4 Power and Energy Quotients

We take the advantage of running the *No Sat*, $\mathbf{k} = 7.5$, $\mathbf{k} = 8.5$, $\mathbf{k} = 9.5$, $\mathbf{k} = 10$, and $\mathbf{k} = 12.5$ sets of experiments with the proposed integral sliding mode controller to enforce our conclusions about the validity of the proposed concepts of power and energy quotients presented in Section 2.3.

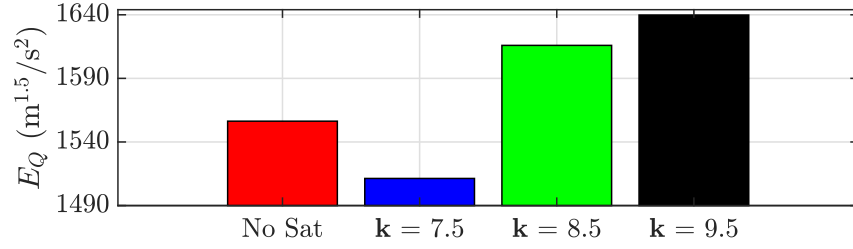
The power and energy results in Figs. 6.2 and 6.4 shows that $P_Q \propto P$ and $E_Q \propto E$, which means that the proposed concepts of power and energy quotients are valid despite the fact that the controller is different from the PIV controller used in Section 5.2, and regardless of the existence or absence of disturbances, e.g., wind disturbances. Additionally, Table 6.3 shows the values of the proportional constant γ . It can be seen that this γ can be considered constant and its value matches with the ones reported in Tables 5.4 - 5.7.



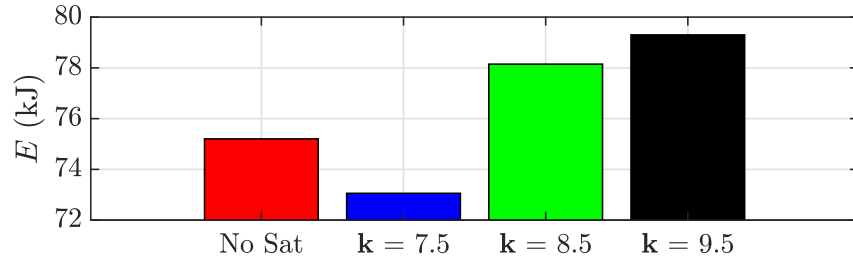
(a) Power quotient P_Q (No disturbance)



(b) Power P (No disturbance)



(c) Energy quotient E_Q (No disturbance)



(d) Energy E (No disturbance)

Figure 6.2: Experimental results of P_Q , P , E_Q and E for a quadrotor with a slung payload (No disturbance). **No Sat** is red solid, $\mathbf{k} = 7.5$ is blue dashed, $\mathbf{k} = 8.5$ is green dotted, and $\mathbf{k} = 9.5$ is black dash-dotted

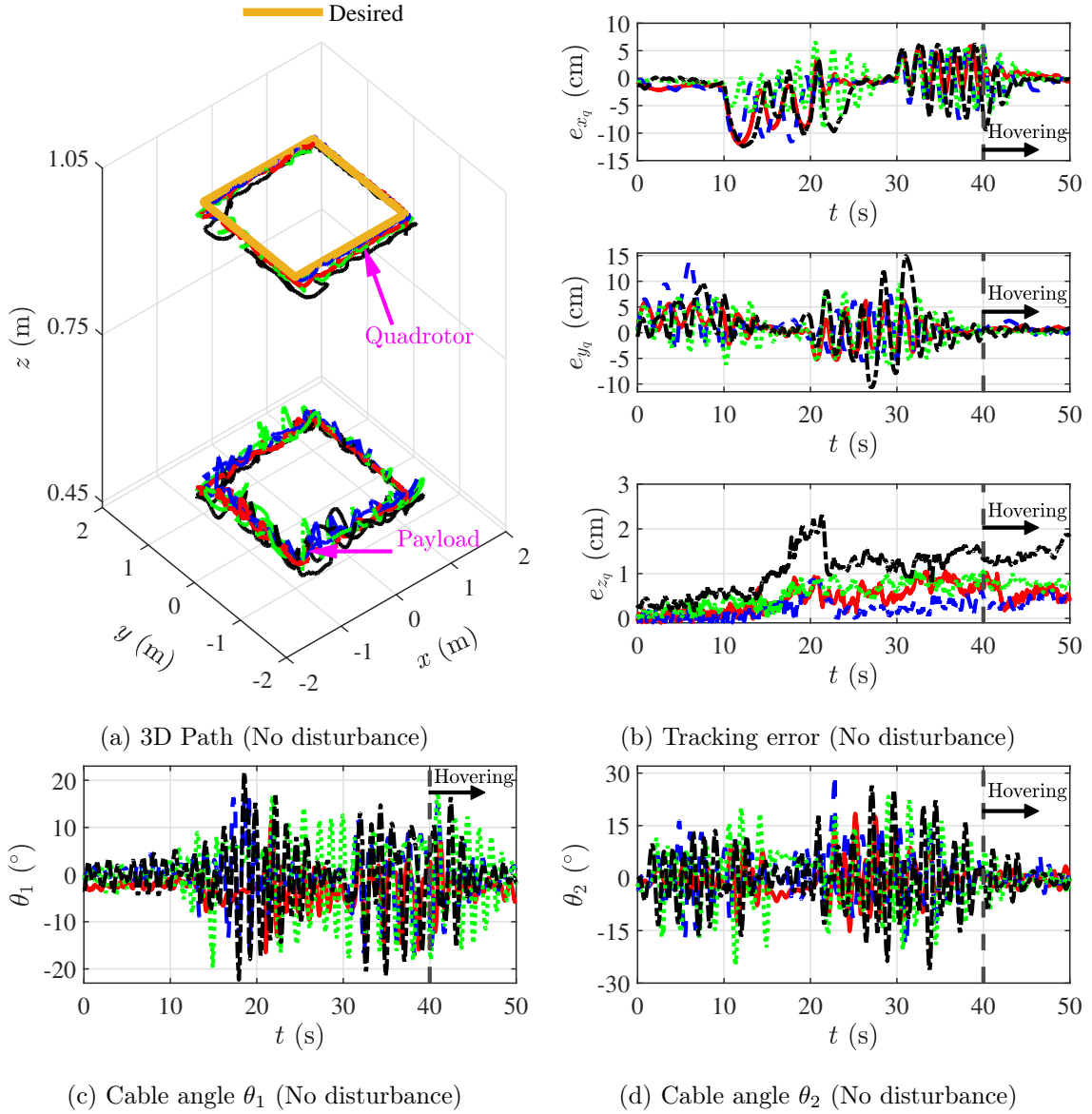


Figure 6.3: Experimental results of the path, tracking errors, and cable angles for a quadrotor with a slung payload (No disturbance). *No Sat* is red solid, $\mathbf{k} = 7.5$ is blue dashed, $\mathbf{k} = 8.5$ is green dotted, $\mathbf{k} = 9.5$ is black dash-dotted

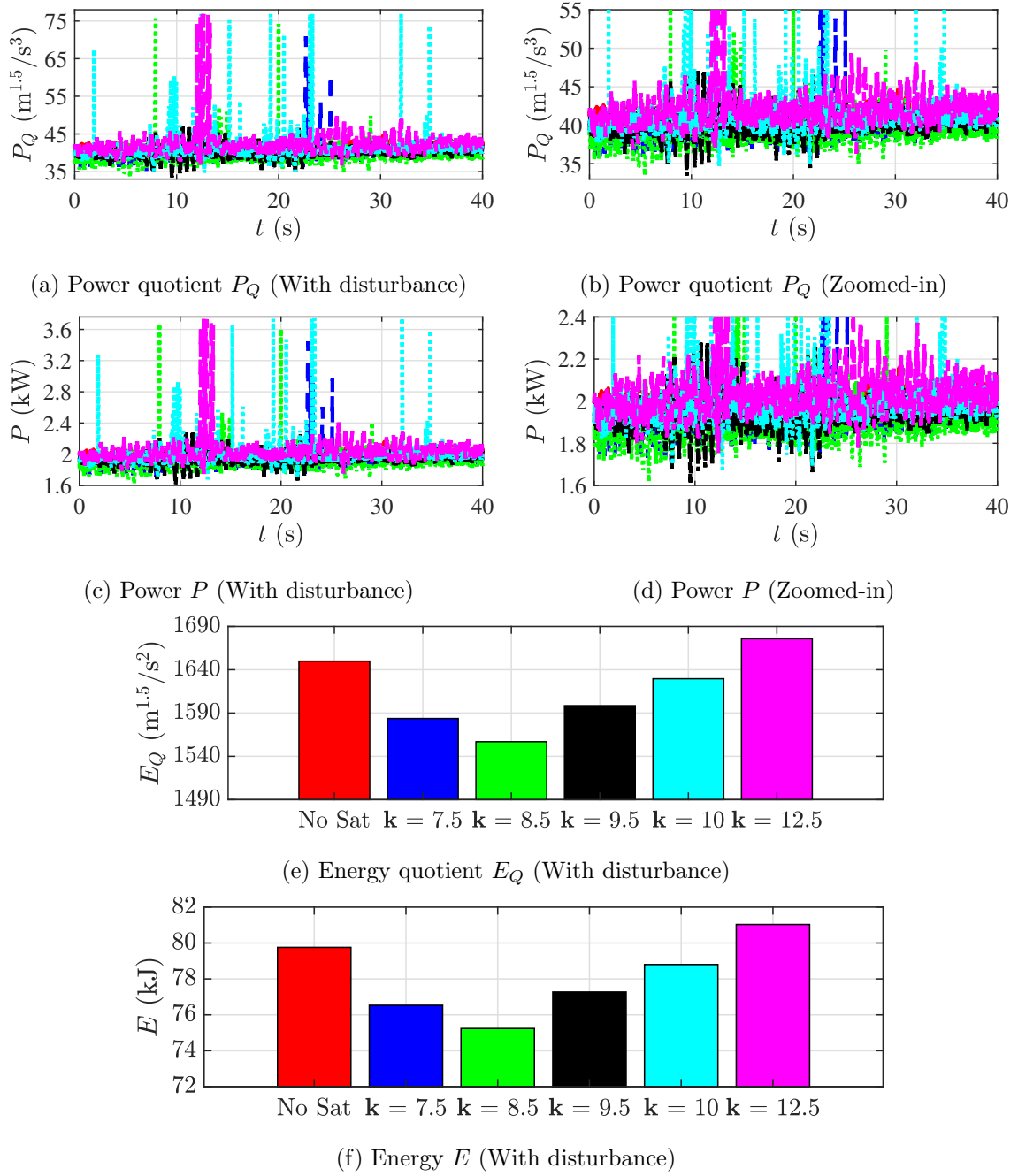


Figure 6.4: Experimental results of P_Q , P , E_Q and E for a quadrotor with a slung payload (With disturbance). **No Sat** is red solid, $k = 7.5$ is blue dashed, $k = 8.5$ is green dotted, $k = 9.5$ is black dash-dotted, $k = 10$ is cyan dotted, and $k = 12.5$ is magenta dashed

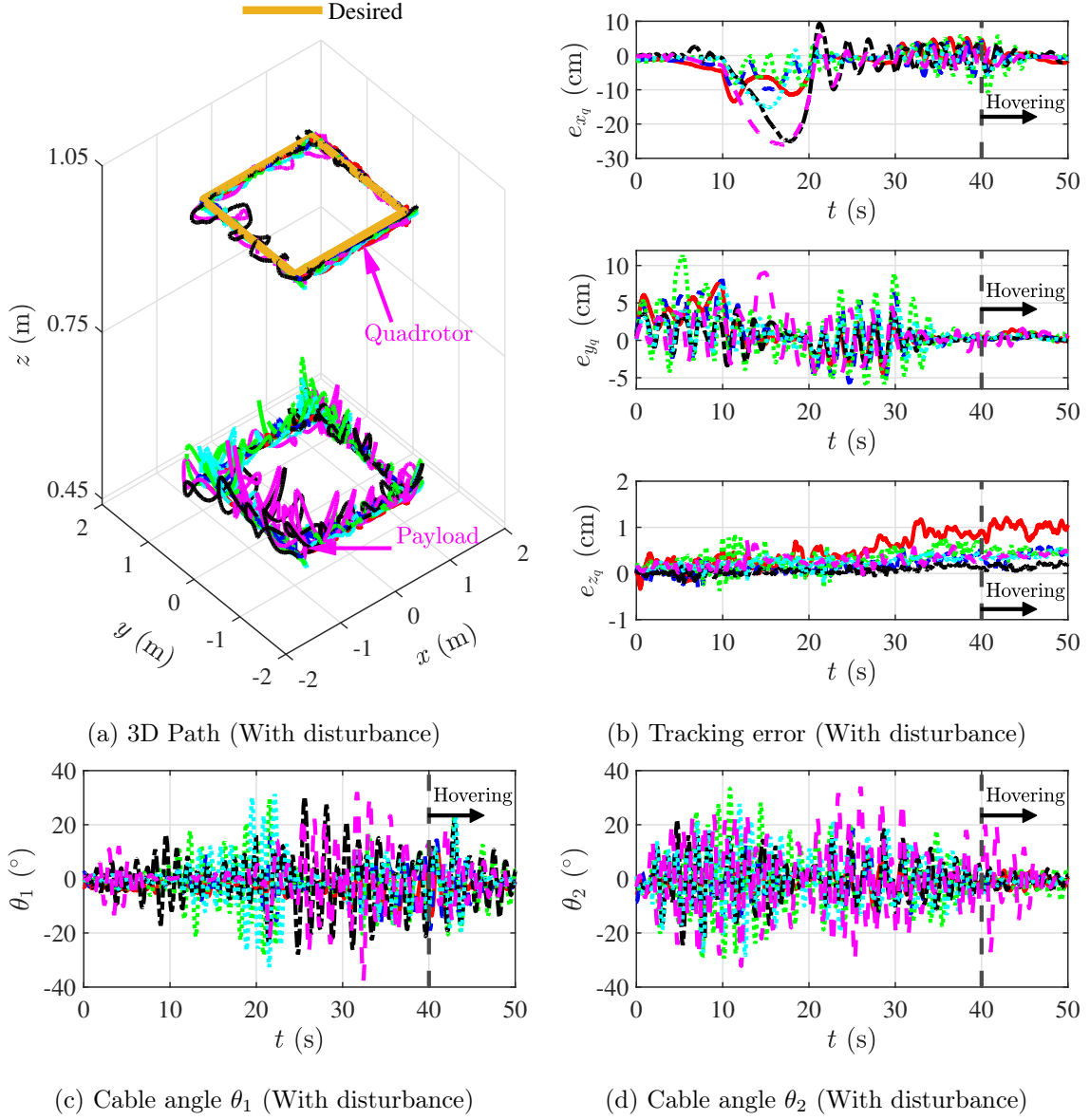


Figure 6.5: Experimental results of the path, tracking errors, and cable angles for a quadrotor with a slung payload (With disturbance). *No Sat* is red solid, $\mathbf{k} = 7.5$ is blue dashed, $\mathbf{k} = 8.5$ is green dotted, $\mathbf{k} = 9.5$ is black dash-dotted, $\mathbf{k} = 10$ is cyan dotted, and $\mathbf{k} = 12.5$ is magenta dashed

Table 6.3: Statistics of the experimental values of γ for different saturation levels

	No Disturbance				With Disturbance					
	<i>No Sat</i>	$k = 7.5$	$k = 8.5$	$k = 9.5$	<i>No Sat</i>	$k = 7.5$	$k = 8.5$	$k = 9.5$	$k = 10$	$k = 12.5$
γ_{max}	40.90	41.25	41.86	41.28	40.82	41.01	41.05	41.05	41.12	41.06
γ_{min}	40.67	40.66	40.67	40.69	40.69	40.66	40.65	40.65	40.67	40.67
γ_{mean}	40.71	40.73	40.74	40.74	40.72	40.71	40.72	40.73	40.74	40.74
γ_{energy}	40.71	40.73	40.75	40.74	40.73	40.72	40.72	40.73	40.74	40.74

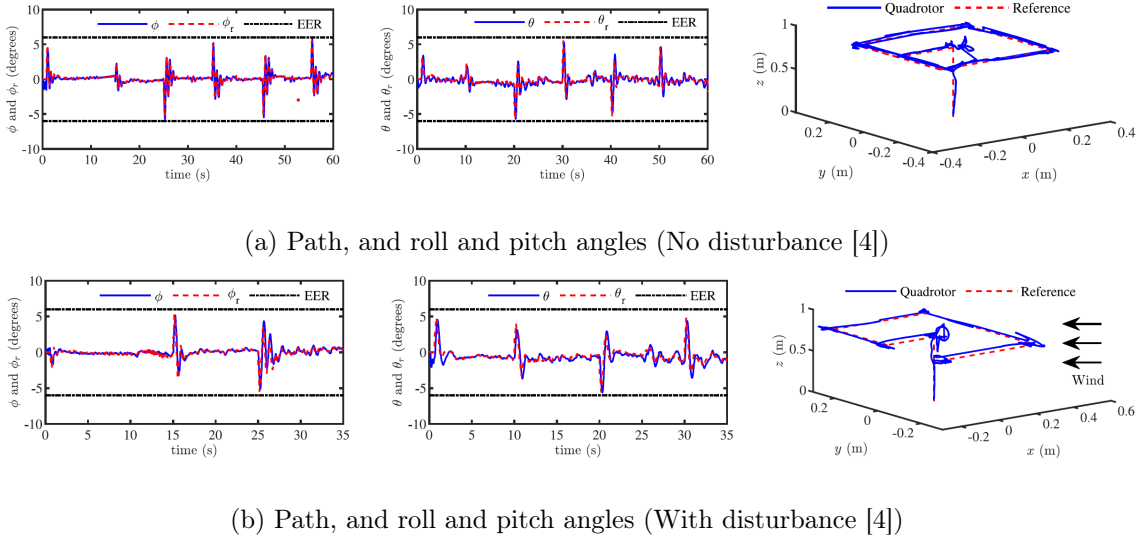


Figure 6.6: Path, and roll and pitch angles for the experiments in [4]

6.4 Summary

This chapter introduces an energy-efficient control strategy for a transportation system consisting of a quadrotor with a slung payload by designing an integral sliding mode controller with input saturation.

First, the controller is introduced and proved for a general Euler-Lagrange system, then its energy efficiency is validated experimentally. The proposed controller is robust against disturbances, so we validated its robustness in case of wind disturbances.

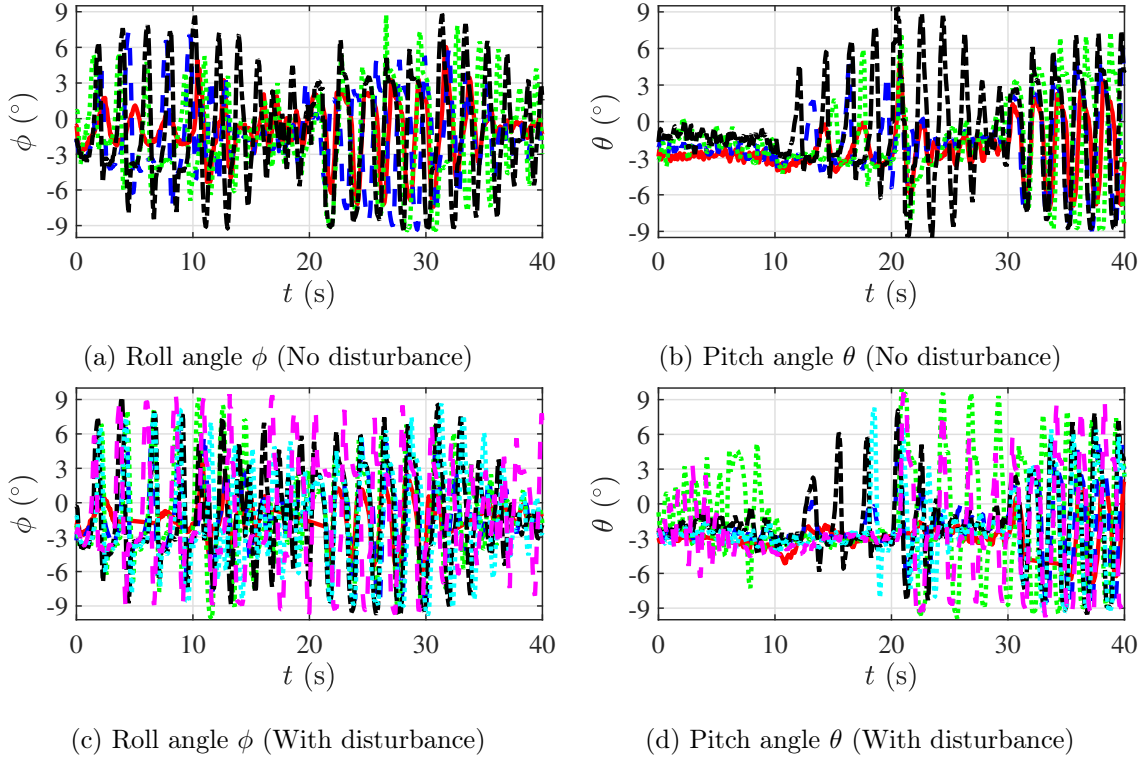


Figure 6.7: Experimental results of roll and pitch angles for a quadrotor with a slung payload (With and without disturbances). *No Sat* is red solid, $\mathbf{k} = 7.5$ is blue dashed, $\mathbf{k} = 8.5$ is green dotted, $\mathbf{k} = 9.5$ is black dash-dotted, $\mathbf{k} = 10$ is cyan dotted, and $\mathbf{k} = 12.5$ is magenta dashed

Second, a qualitative comparison with another energy-saving controller in the literature is conducted. The comparison showed that our proposed controller is qualitatively efficient compared to another controller in the literature despite the differences between ours and theirs, e.g., the existence of a payload, velocity, and the size of the quadrotor.

Third, the results of the experiments in this chapter are used to enforce the validation of the proposed concepts of power and energy quotients that are introduced in Section 2.3. The results showed that regardless of the trajectory, controller, and/or the disturbances, the proposed concepts of power and energy quotients are valid for non-aggressive flights.

7 Conclusions and Future Work

7.1 Conclusions

This dissertation discusses the transportation consisting of a quadrotor with a slung payload to pave the way for package delivery using quadrotors. Two major related challenges are considered in this dissertation: limited energy resources onboard and payload vibrations. Payload vibration is a challenge on its own. Additionally, it affects the energy consumption since it adds excessive forces and moments to the transportation system. Limited energy resources constrain flight times and/or distances. The literature lacks energy models, energy analysis, and energy-saving strategies for such a transportation system. This limits our understanding of the energy consumption of such a system, which may prevent providing practical solutions. Therefore, this dissertation investigated these challenges and provides novel energy-saving strategies based on comprehensive energy analyses.

In this dissertation a novel energy model that does not depend on quadrotor-related parameters such as motor and propeller parameters is developed for both a single quadrotor and the transportation system. This allowed for performing a comprehensive energy analysis for quadrotor's trajectory (focusing on the polynomial trajectories), payload vibrations, payload mass, and cable length. This comprehensive analysis led to proposing novel and effective payload vibration reduction strategies and energy-saving strategies.

A novel two-stage payload vibration reduction inspired by the analogy between the transportation system and the car suspension system resulted in reducing cable angles between 4.5° to 21.5° under different conditions. The results showed that decreasing the degree of the polynomial trajectory is the first payload vibration reduction stage, whereas changing the cable length can provide a second stage of payload vibration reduction.

A novel trajectory-based energy-saving strategy is proposed by, decreasing the degree of the polynomial trajectory. The experimental results showed an energy saving between 1.6 to 12.6% for a single quadrotor, and between 5.4 to 5.7% for a quadrotor with a slung payload under different conditions. For the transportation system, changing the cable length to the optimal cable length, if possible, adds more energy saving. The comparison of the proposed trajectory-based energy-saving strategy with an energy-minimized trajectory from the literature showed that the MAT has a very close energy performance to the energy-minimized trajectory with an increase of about 0.05% in the energy consumption. Even the rest of polynomial trajectories have a close energy performance with a maximum increase of about 0.15% for the MCT. However, designing a MAT, or any other polynomial trajectory, does not require any prior knowledge of the quadrotor-related parameters contrarily to the energy-minimized trajectories in the literature. So, this dissertation suggests utilizing MATs as a practical and more straightforward energy-efficient trajectory.

Moreover, the first energy-saving controller for the transportation system is developed in this dissertation by designing a robust integral sliding mode controller with input saturation. The experimental results showed that the proposed controller can save between 2.9 and 5.7% of the energy consumption under different conditions. The results also showed that the proposed controller is robust against disturbances, which were wind disturbances in this dissertation.

Additionally, a generic framework to generate polynomial trajectories of any degree

with a large number of waypoints was presented, which can be used to overcome some kinematic limitations associated with continuous-path trajectories.

7.2 Future Work

The work of this dissertation can be extended to explore the following potential topics:

- I This dissertation is centered around non-aggressive flights since they form the major part of flights for package delivery applications. Some other applications may require aggressive flights, so the concepts and the analyses presented in this dissertation require further extensions. These extensions can consider obtaining a relation between the aggressiveness, μ , and the force ratio, β , which can help extend the proposed energy model. Once established, this relation between μ and β allows to get all the benefits provided by the proposed model, e.g., independence of quadrotor-related parameters, even for non-aggressive flights. Such a relation can be also used to extend the dynamic feasibility check to non-aggressive flights.
- II The proposed energy-saving controller cannot work with non-aggressive flights since it constrains the total thrust. Designing a new energy-saving controller for aggressive flights is a possible future research work. Unlike Point I, designing an energy-saving controller for aggressive flights cannot be directly extended from our work because constraining the thrust may not be an option for aggressive flights. With that being said, the idea of designing a controller with an intuitive relation between the performance, energy consumption, and the saturation can be extended.
- III This dissertation considers a transportation system consisting of a quadrotor with a slung payload. The case of cooperative transportation, where a payload is transported by multiple quadrotors, is a promising research direction. Energy modeling,

payload vibrations, energy analysis, and different energy-saving strategies are a great deal of extension to be done.

IV One of the assumption in this dissertation is considering a point mass payload. This excludes the attitude dynamics of the payload. With the cooperative transportation presented in Point III, the attitude dynamics can be considered. A full position and attitude control of the payload can be achieved by at least three quadrotors. However, developing an energy-efficient cooperative controllers is an important future direction not only to save energy, but also to ensure that all quadrotors can complete the entire task and now quadrotor will run out of energy before the end of the task due to uneven energy consumption distribution during the task.

V Another assumption that is made in this dissertation is the rigid cable, i.e., no cable flexibility. Considering the effect of the flexibility of the cables on the energy consumption is important for scenarios with long cables in disturbed environments, e.g., flight in a windy weather. This requires developing a new dynamics model with a new energy model, which will results in novel energy-saving strategies based on the elasticity of the cable.

VI In this dissertation, the availability of the real-time position information for both the quadrotor and the payload is assumed. Although this can be available for the quadrotor in real life via the GPS, it is more realistic to consider the absence of such information for the payload. Therefore, different robust payload state estimation techniques is important from a practical point of view.

VII Considering non-rigid payloads, e.g., liquid containers, has a significant importance in applications such as fire fighting, and search and rescue. The sloshing effect of partially-filled containers will significantly affect the dynamics of the system and

the energy consumption. This is a future direction, where developing a new energy model is required. Since the optimal cable length depends on the kinematics and dynamics of the system, the sloshing may affect this length. Therefore, considering the effect of a time-varying cable length, e.g., via dynamically controlled pulleys, is worth investigation.

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A Attitude Dynamics Conversion Between Lagrangian and Eulerian Approaches

The attitude dynamics of a quadrotor derived from the Lagrangian approach is presented in Eq. (2.26) as follows:

$$\mathbf{R}_v^\top \mathbf{I} \dot{\boldsymbol{\omega}} + \left[\dot{\mathbf{R}}_v^\top - \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top \right] \mathbf{I} \boldsymbol{\omega} = \boldsymbol{\tau}_\Phi \quad (\text{A.1})$$

Premultiplying Eq. (A.1) by $\mathbf{R}_v^{\top -1}$ results in:

$$\mathbf{I} \dot{\boldsymbol{\omega}} + \mathbf{R}_v^{\top -1} \left[\dot{\mathbf{R}}_v^\top - \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top \right] \mathbf{I} \boldsymbol{\omega} = \mathbf{R}_v^{\top -1} \boldsymbol{\tau}_\Phi \quad (\text{A.2})$$

where

$$\dot{\mathbf{R}}_v^\top = \begin{bmatrix} 0 & 0 & -\cos \theta \dot{\theta} \\ 0 & \sin \phi \dot{\phi} & -\sin \theta \sin \phi \dot{\theta} + \cos \theta \cos \phi \dot{\phi} \\ 0 & \cos \phi \dot{\phi} & -\sin \theta \cos \phi \dot{\theta} - \cos \theta \sin \phi \dot{\phi} \end{bmatrix} \quad (\text{A.3})$$

and

$$\dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top = \begin{bmatrix} 0 & -\sin \phi \dot{\theta} + \cos \theta \cos \phi \dot{\psi} & \cos \phi \dot{\theta} - \cos \theta \sin \phi \dot{\psi} \\ -\cos \theta \dot{\psi} & -\sin \theta \sin \phi \dot{\psi} & -\sin \theta \cos \phi \dot{\psi} \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.4})$$

Therefore, one can find that:

$$\begin{aligned} \mathbf{R}_v^{\top -1} \left[\dot{\mathbf{R}}_v^\top - \dot{\boldsymbol{\Phi}}^\top \left[\frac{\partial \mathbf{R}_v}{\partial \boldsymbol{\Phi}} \right]^\top \right] = \\ \begin{bmatrix} 0 & \sin \phi \dot{\theta} - \cos \theta \cos \phi \dot{\psi} & \cos \phi \dot{\theta} + \cos \theta \sin \phi \dot{\psi} \\ -\sin \phi \dot{\theta} + \cos \theta \cos \phi \dot{\psi} & 0 & \sin \theta \dot{\psi} - \dot{\phi} \\ -\cos \phi \dot{\theta} - \cos \theta \sin \phi \dot{\psi} & -\sin \theta \dot{\psi} + \dot{\phi} & 0 \end{bmatrix} = \boldsymbol{\omega}^\times \end{aligned} \quad (\text{A.5})$$

where $\boldsymbol{\omega}^\times \in \mathfrak{so3}$ is the skew-symmetric matrix of the angular velocity $\boldsymbol{\omega}$. That is, for $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^\top$, $\boldsymbol{\omega}^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$. This implies that:

$$\boldsymbol{\omega}^\times \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} \quad (\text{A.6})$$

where $\mathbf{v} \in \mathbb{R}^3$ is an arbitrary vector. It can be concluded from Eqs. (A.2) - (A.6) that the attitude dynamics can be expressed as:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \underbrace{\mathbf{R}_v^{\top^{-1}} \boldsymbol{\tau}_\Phi}_{=\boldsymbol{\tau}} \quad (\text{A.7})$$

which leads to the conventional Eulerian form of attitude dynamics presented in Eq (2.27):

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \boldsymbol{\tau} \quad (\text{A.8})$$

B Fourier Coefficients for Polynomial Trajectories with General Boundary Conditions

The cosines coefficients, a_k , for the different polynomial trajectories can be found as follows:

$$a_{k_j} = \frac{1}{\pi^2 k^2 t_f} (3(\ddot{x}_0 + \ddot{x}_f)t_f + 6(\dot{x}_0 - \dot{x}_f)) \quad (\text{B.1})$$

$$a_{k_s} = \frac{1}{2\pi^4 k^4 t_f} \left(-(\pi^2 k^2 - 15)(\ddot{x}_0 - \ddot{x}_f)t_f^2 + 90(\ddot{x}_0 + \ddot{x}_f)t_f + 180(\dot{x}_0 - \dot{x}_f) \right) \quad (\text{B.2})$$

$$\begin{aligned} a_{k_c} = & \frac{1}{4\pi^6 k^6 t_f} \left(-10(\pi^2 k^2 - 10.5)(\ddot{x}_0 + \ddot{x}_f)t_f^3 \right. \\ & -2(\pi^4 k^4 + 45\pi^2 k^2 - 630)(\ddot{x}_0 - \ddot{x}_f)t_f^2 - 420(\pi^2 k^2 - 15)(\ddot{x}_0 + \ddot{x}_f)t_f \\ & \left. -840(\pi^2 k^2 - 15)(\dot{x}_0 - \dot{x}_f) \right) \end{aligned} \quad (\text{B.3})$$

The sines coefficients, b_k , for the different polynomial trajectories can be found as follows:

$$b_{k_j} = \frac{1}{\pi^3 k^3 t_f^2} \left((\pi^2 k^2 - 15)(\ddot{x}_0 - \ddot{x}_f)t_f^2 - 90(\dot{x}_0 + \dot{x}_f)t_f - 180(x_0 - x_f) \right) \quad (\text{B.4})$$

$$\begin{aligned} b_{k_s} = & \frac{1}{2\pi^5 k^5 t_f^2} \left(10(\pi^2 k^2 - 10.5)(\ddot{x}_0 + \ddot{x}_f)t_f^3 + 2(\pi^4 k^4 + 45\pi^2 k^2 - 630)(\ddot{x}_0 - \ddot{x}_f)t_f^2 \right. \\ & \left. + 420(\pi^2 k^2 - 15)(\dot{x}_0 + \dot{x}_f)t_f + 840(\pi^2 k^2 - 15)(x_0 - x_f) \right) \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned}
b_{k_c} = & \frac{1}{4\pi^7 k^7 t_f^2} \left(-(\pi^4 k^4 - 105\pi^2 k^2 + 945)(\ddot{x}_0 - \ddot{x}_f)t_f^4 \right. \\
& + 1890(\pi^2 k^2 - 10)(\ddot{x}_0 + \ddot{x}_f)t_f^3 + 4(\pi^6 k^6 + 4095\pi^2 k^2 - 42525)(\ddot{x}_0 - \ddot{x}_f)t_f^2 \quad (\text{B.6}) \\
& \left. + 75600(\pi^2 k^2 - 10.5)(\dot{x}_0 + \dot{x}_f)t_f + 151200(\pi^2 k^2 - 10.5)(x_0 - x_f) \right)
\end{aligned}$$

The coefficient, a_0 , is the same for all of the different polynomial trajectories, since it is basically the integration of the trajectory's acceleration, i.e., the velocity. The coefficient, a_0 can be found as follows:

$$a_0 = \frac{-2}{t_f} (\dot{x}_0 - \dot{x}_f) \quad (\text{B.7})$$

C List of Publications

1. H. Alkomy and J. Shan, “Investigating the Effects of Polynomial Trajectories on Energy Consumption of Quadrotors,” **IEEE/ASME Transactions on Mechatronics**, 2022 (Early Access) DOI: <https://doi.org/10.1109/TMECH.2022.3220086>
2. H. Alkomy and J. Shan, “Vibration Reduction of a Quadrotor with a Cable-Suspended Payload Using Polynomial Trajectories,” **Nonlinear Dynamics**, vol. 104, no. 4, pp. 3713–3735, 2021. DOI: <https://doi.org/10.1007/s11071-021-06464-6>
3. H. Alkomy and J. Shan, “Modeling and Validation of Reaction Wheel Micro-Vibrations Considering Imbalances and Bearing Disturbances,” **Journal of Sound and Vibration**, vol. 492, p. 115766, 2021. DOI: <https://doi.org/10.1016/j.jsv.2020.115766>
4. H. Alkomy and J. Shan, “Kinematically-Constrained Continuous-Path Polynomial Trajectories for Quadrotors,” in **IEEE 18th International Conference on Automation Science and Engineering (CASE 2022)**, Mexico City, Mexico, 2022. DOI: <https://doi.org/10.1109/CASE49997.2022.9926718>
5. H. Alkomy and J. Shan, “An Energy Analysis of Quadrotors with Cable-Suspended Payloads,” **The International Conference on Unmanned Aircraft Systems (ICUAS)**, Dubrovnik, Croatia, 2022. DOI: <https://doi.org/10.1109/ICUAS54217.2022.9836169>