OPTION PRICING IN NON-COMPETITIVE MARKETS

HAI ZHANG

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Abstract

In the classic option pricing theory, the market is assumed to be competitive. The relaxation of the competitive market assumption introduces two features: liquidity cost and feedback effect. In our study, investors in non-competitive markets are divided into two categories: small investors and large investors. Small investors encounter liquidity cost while large investors face both liquidity cost and feedback effect. For small investors, liquidity cost could be modelled by a supply curve function. For large investors, liquidity cost could be modelled via trading speed and a trading action is assumed to have a feedback effect on underlying asset price. Chapter 2 and chapter 3 are dedicated to investigate the option pricing for small investors. In chapter 2, how to perfectly hedge options (including vanilla options and exotic options) under the supply curve model in a geometric Brownian motion model was studied. In Chapter 3,local risk minimization method was used to price European options with liquidity cost in a jump-diffusion model. In chapter 4, the utility indifference pricing method was applied to price European options for large investors.

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1 Introduction

1.1 Dissertation Objective

The objective of this dissertation is to study how to price options in non-competitive markets. Based on new features of non-competitive markets, liquidity cost and feedback effects, market participants are divided into two categories: small investors and large investors. Different pricing models are proposed for both small and large investors.

1.2 Background

As financial markets grow, derivatives have become more and more important for speculating and hedging purposes. Derivatives are financial contracts whose value depends on underlying variables. Futures, options, swaps, and forwards are the main categories of derivatives. The valuation of derivatives poses one of the most important challenges in mathematical finance. The Black-Scholes option pricing model proved a breakthrough in pricing derivatives. Its main insight is that options can be replicated by two primary assets:

the underlying stock and the bank account. The option's price is simply the value of the portfolio consisting of the underlying asset and the bank account.

It is well known, however, that the Black-Scholes option pricing model was built on excessively restrictive assumptions on market conditions and asset processes. The following assumptions provide an ideal world for deriving the Black-Scholes equation:

1. There are no transaction costs (including taxes) and no restrictions on trading (e.g. short sale constraints). These two conditions contribute to a frictionless market.

2. An investor can buy or sell unlimited quantities of the stock without changing the stock price. This assumption contributes to a competitive market.

3. The interest rate is constant, and the stock price follows a geometric Brownian motion with constant drift and volatility. These conditions forms a complete market.

A market satisfying both assumptions (1) and (2) is considered to be a perfect market. It is important to distinguish complete market from perfect market. Note that a market could be complete but imperfect. If there are transaction costs in the market, but the interest rate is constant and the stock price follows a geometric Brownian motion, the market is imperfect but complete. A market could also be incomplete but perfect, for instance, a jump-diffusion model without transaction cost. In a Black-Scholes world, it satisfies all the three assumptions above, so the market is both perfect and complete. The options become redundant securities and can be replicated by the stock and the bank account (Merton (1976)). Therefore, the replication cost is the unique option price.

1.3 Option pricing in a perfect market

In a perfect market, when the stock price follows a geometric Brownian motion and the interest rate is constant, the market is complete and the Black-Scholes price is the unique arbitrage-free price for options. However, when the stock price follows other processes, such as a jump-diffusion model, replicating the option's payoff with a stock and bank account becomes impossible, and the Black-Scholes theory is no longer applicable. Harrison and Kreps (1979), Harrison and Pliska (1981), and Harrison and Pliska (1983) developed the risk neutral pricing theory, which provides a framework to pricing options for general stock price processes. They also introduces two fundamental theorems of asset pricing.

Theorem 1.3.1. ***(The First Fundamental Theorem of Asset Pricing)*** *If* (S, B) *models the stock and bank account on a probability space* $(\Omega, \mathcal{F}, P)$*, then the market is arbitrage free if and only if there exists a risk neutral measure that makes the discounted asset process a martingale.*

Theorem 1.3.2. ***(The Second Fundamental Theorem of Asset Pricing)*** *The market is complete if and only if there exists a unique risk neutral measure for the asset price process* (S, B).

Risk neutral pricing is a general pricing method and can be used to price options in general asset price models. Let $V(t, S_t)$ denote the option price at time t with the stock price S_t, and the payoff of the option $(S_T - K)^+$ at the time of maturity T. In a perfect and arbitrage free market, the value of a European option is the discounted expectation of its payoff under a risk neutral measure Q:

$$V(t, S_t) = E_t^Q \left[e^{-r(T-t)} (S_T - K)^+ \right]$$

where r is the constant interest rate. If the stock price S_t follows a geometric Brownian motion model with constant volatility, the risk neutral measure is unique. However, in more general cases, such as a jump-diffusion model (Kou (2002)) or a stochastic volatility model (Heston (1993)), more than one risk neutral measure exists. A particular risk neutral measure is chosen to price options in general asset price models.

1.4 Option pricing in non-competitive markets

When the market is perfect, risk neutral valuation provides a general approach to value options. Pricing options is simplified to the calculation of expectations of discounted options payoff under the risk neutral measure. The market, however, is imperfect; more precisely, it is neither frictionless nor fully competitive. Risk neutral valuation collapses when the market is imperfect. Much research has been devoted to extending the option pricing theory to imperfect markets, including markets with friction (transaction costs or short sell constraints) and non-competitive markets (markets with liquidity risk or feedback effects).

When the frictionless market assumption was relaxed, transaction costs were introduced and option pricing with transaction costs was extensively studied. Leland (1985) proposed a Black-Scholes type equation with modified volatility to characterize the option price, which shows that transaction costs add extra cost to the option writer, resulting in higher prices for options. Boyle and Vorst (1992) investigated the option pricing with transaction costs in a binomial model, and a simple Black-Scholes approximation formula was derived for the option prices. Unlike the perfect continuous delta hedging with finite initial cost in the Black-Scholes model, replication of options with transaction costs in continuous time will incur infinite transaction costs. By replacing the perfect hedging with a super hedging strategy, Edirisinghe et al. (1993) and Bensaid et al. (1992) showed that it is cheaper to dominate a contingent claim than to replicate it. Later, Soner et al. (1995)proved that the minimal cost to hedge a European call option with transactions costs is just trivial hedging. Also, there is considerable literature focusing on options pricing with short sell constraints. Related results can be found in Cvitanić and Karatzas (1993), Jouini and Kallal (1995), and Pham (2000).

Compared to markets with friction, there has been much less attention paid to investigating options pricing in non-competitive markets. The relaxation of the competitive market assumption has a twofold impact on the market. First, it brings liquidity risk to the market. Liquidity risk is the risk that due to the timing and size of a trade, a given security or asset cannot be traded quickly enough to meet the short term financial demands of the holder. The source of liquidity risk is demand pressure. Demand pressure arises because not all investors are present in the market at the same time, meaning that if an investor needs to sell a security quickly, then bid limit orders will be consumed by the investor's sell market order, forcing the price the investor receives to be less than the market price. In order words, liquidity risk leads quick selling at a price less than the market price and quick buying at a price higher than the market price. Liquidity risk is considered to be the most significant risk in addition to market risk and credit risk. In a market with liquidity risk, investors cannot buy or sell large quantities of security at the given market price. As the market for a security becomes less liquid, investors are more likely to take losses because of the bigger Bid-Ask spread. Liquidity risk results an extra cost associated with buying or selling a given security. We regard this newly incurred cost as liquidity cost. The average liquidity cost is dependent upon both the securities market price and the trading volume or trading speed.

Liquidity risk is a critical consideration in derivative pricing. When the market is liquid for a derivative, the trader has no difficulty in doing the daily hedging to maintain the delta neutrality. However, for some securities, the market is not liquid, which means the liquidity risk needs to be considered when pricing and hedging derivatives on this underlying security. One approach, proposed by Cetin et al. (2004), is to introduce a supply curve to model the security price as a function of market price and trading volume. This supply curve function is a non-decreasing function of trading volume: buying more shares of stock means paying higher price per share, which is natural. In Cetin et al. (2004), the option price under the supply curve function is the same as the Black-Scholes price, which means that there is no liquidity premium. On the other hand, the option pricing model with liquidity cost in Cetin and Rogers (2007) produces a nonzero liquidity premium for options when considered in discrete time. Motivated by the lack of liquidity premium in the continuous time model, a super hedging European option under the supply curve function in continuous time was studied by Çetin et al. (2010). They studied the super replication problem under the supply curve function with the additional constraint on the boundedness of the quadratic variation and the absolute continuous parts of the portfolio processes. A dynamic programming equation is used to characterize the minimal hedging cost of European options with liquidity risk. The equations shows that a nonzero liquidity premium in continuous-time for a set of appropriately defined admissible strategies could be generated. Gökay and Soner (2012) considered the super hedging of European options in a binomial model, and it led the same liquidity premium as the continuous time limit mentioned in Çetin et al. (2010). Also, Ku et al. (2012) derived a partial differential equation that provided discrete time delta hedging strategies, concluding that the expected hedging errors approach zero almost surely as the length of the revision interval goes to zero. All these approaches provided us with new insights on European option pricing with liquidity risk, but it is difficult to apply them to pricing American options and exotic options. A general method for pricing different options with liquidity risk is still lacking.

In addition, the relaxation of the competitive market assumption raises another problem: feedback effects. In non-competitive markets, feedback effects refer to the price effects that trading actions by investors place on the security's future price evolution. A security's future price becomes dependent on an investor's trading action. Some investors could take advantage of making a profit by choosing an optimal trading strategy to influence a security's future price. Investors whose trading has a feedback effect on a security's price evolution are considered to be large investors. Regarding large investors' hedging strategies in asset pricing, Frey and Stremme (1997) , Platen and Schweizer (1998), and Schied and Schöneborn (2009) followed a microeconomic equilibrium approach to study the feedback effects from such hedging strategies. Frey and Stremme (1997) investigated the impact of dynamic hedging on the price process in a general discrete time economy with the equilibrium model. Ronnie Sircar and Papanicolaou (1998) analysed the increases in market volatility of asset prices. Following an equilibrium analysis, they derived a nonlinear partial differential equation for the derivative price and the hedging strategy. They observed that the increase in volatility can be attributed to the feedback effect of

	Liquidity cost	*No liquidity cost*
Feedback effects	Large investor model	Not investigated
No feedback effects	Small investor model	Black-Scholes model

Table 1.1: Different models with respect to liquidity cost and feedback effects

Black-Scholes hedging strategies.

Another approach to investigating the feedback effects is to study the coefficients of the price process relying exogenously on the large trader's trading strategy. Kraft and Kühn (2011) modelled the permanent price impact by making the expected returns dependent on the stock position of a large investor. Jarrow (1994) studied option pricing when large investors are manipulating the market through their trading strategies. Cvitanić and Ma (1996) and Cuoco and Cvitanić (1998) assumed that the large trader has a price impact on the expected return through the investor's stock holdings. Almgren (2003), Schied and Schöneborn (2009) and Forsyth (2011) modelled the permanent price impact of the stock price from the size of the transaction and the speed of change of the position in the stock. However, how to price and hedge the option for large investors considering both liquidity cost and feedback effects is still not answered.

In this dissertation, I addressed the option pricing problem with liquidity cost and feedback effects in a unified framework. In non-competitive markets, the new features—liquidity cost and feedback effects—violate the perfect market assumption. When considering the market participants in a non-competitive market, the market participants are divided into two categories: small investors and large investors. Small investors are associated with liquidity cost while large investors in a market are associated with both liquidity cost and feedback effects. The criteria for characterizing an investor are not only determined by the investor's wealthy but also on the security the investor is trading. A specific investor who owns $10,000$ shares in Apple might not be able to influence Apple's stock price. Some small companies, trading $10,000$ shares, however, might influence and even manipulate the small company's stock price. Therefore, for big companies, this investor is a relatively small investor, while for small companies, this investor becomes a large investor. Table 1.1 provides a big picture of different models for pricing and hedging options in different market assumptions. In this study, I propose different pricing methods for two types of investors.

Small investors do not have the market power to change the security's future price. But liquidity cost is unavoidable, and it will add extra cost to hedging options. In Çetin et al. (2010) and Ku et al. (2012), investors are assumed to be small investors, and their hedging strategy does not affect the price evolution. Only liquidity cost needs to be considered when studying option pricing for small investors, and feedback effects are not taken into consideration. As for large investors, their market power to influence security price evolution could be a great advantage to the large investors and cannot be ignored. Both liquidity cost and feedback effects need to be considered when pricing options for large investors.

1.5 Chapter Breakdown

Chapter 2 and Chapter 3 are devoted to the study of options pricing for small investors. Small investors in non-competitive markets face a liquidity cost, which is modelled by a supply curve function. Chapter 2 will show the existence of a perfect hedging of options for small investors when the stock price follows a geometric Brownian motion. There are perfect hedging strategies for the party writing the options and the party buying the options. Partial differential equations used to characterize the perfect hedging cost for vanilla and exotic options are presented. The chapter will also show that the hedging cost for the party writing the options forms an upper bound for the option price and the hedging cost for the party buying the options forms a lower bound.

Chapter 3 will show how to apply local risk minimization to price European options in a jump-diffusion model for small investors. The jump-diffusion model is approximated by discrete time models, and local risk minimization is used to price and hedge European options in the discrete time model. When the time interval in the discrete time model goes to zero, the option price obtained from the discrete time model converges to the option price in a jump-diffusion model.

Chapter 4 will study option pricing for European options for large investors. Large investors face both liquidity cost and feedback effects in the non-competitive market. In this chapter, the utility indifference price method will be applied to price options for large investors in a non-competitive market, since the utility indifference pricing approach has been proven to be a good pricing methodology to price options for large investors. HJB equations to characterize the value function will be derived. The existence and uniqueness of viscosity solution of HJB equations will also be proved.

2 Options Pricing and Hedging for Small Investors

2.1 Introduction

In a perfect market, risk neutral valuation provides a general framework for pricing options. Liquidity risk and feedback effects exist in a non-competitive market, causing the market to be imperfect, and risk neutral valuation is no longer applicable. This dissertation will attempt to develop new methods for pricing and hedging options in a non-competitive market. Generally, the market participants could be divided into two categories: small investors and large investors. Small investors are defined as investors who do not have the market power to change a security's future price; feedback effects are not taken into consideration when pricing and hedging options for small investors. Liquidity risk is unavoidable for small investors, however, adding a liquidity cost for hedging options. The question is how to price and hedge options for small investors with liquidity cost. The first step toward an answer involves modelling liquidity risk and characterizing the liquidity cost. Cetin et al. (2004) introduced a supply curve to model the security price as a function of market price and trading volume. Based on the supply curve model, Çetin et al. (2010) studied a super hedging European option in continuous time. Gökay and Soner (2012) considered the super hedging problem in a binomial model. Ku et al. (2012) derived a partial differential equation that provided discrete time delta hedging strategies whose expected hedging errors approach zero almost as surely as the length of the revision interval goes to zero. All these approaches are limited to pricing European options with liquidity risk, but it seems quite difficult to generalize them to pricing American options and exotic options.

This chapter proposes a general method for pricing different options with liquidity risk. Adapting the Black-Scholes' replication idea, this chapter will show the existence of perfect replication for European options with liquidity risk, and will derive a partial differential equation to characterize the replication cost. Perfect replication of American options and exotic options (Barrier options and Asian options) will then be presented and the corresponding partial differential equations to characterize the replication cost will be derived. My approach could be applied to pricing other exotic options and early exercise options,e.g. Lookback options and Bermudan options. For simplicity, this chapter will limit its coverage to European options, American options, Barrier options, and Asian options. For each kind of option, there exists a buyer's replication cost and a seller's replication cost. The buyer's and the seller's replication costs can then be considered the lower bound and the upper bound, respectively, for the option price for small investors in a non-competitive market.

2.2 The supply curve model

Let us consider a financial market that consists of a risk-free bank account and a risky stock. The interest rate is r and the bank account B_t is given by:

$$dB_t = rB_t dt, \qquad t \in [0,\ T]. \tag{2.1}$$

The stock price is defined on a probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t : t \geq 0\}$ generated by a one-dimensional Brownian motion W_t. The stock price S_t follows the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad t \in [0,\ T] \tag{2.2}$$

where μ is the drift rate and σ is the volatility.

An investor who writes an option needs to construct a portfolio consisting of the underlying stock and the bank account to hedge the option. During the hedging process, the hedging portfolio needs to be adjusted frequently to reflect the change of the value of the options. In a non-competitive market, the market is not fully liquid and liquidity risk exists. Investors cannot buy or sell a large volume of stock at the given quoted price. Cetin et al. (2004) introduced a supply curve function to model the liquidity risk. A supply curve function $S_t(x)$ represents the stock price per share that the investor pays for an order size of x when the stock price is S_t at time t. A positive x represents a buying of stock and a negative x represents a selling of stock. The supply curve function is determined by the market structure. A single investor's past actions, wealth, and risk attitude therefore have no impact on the supply curve. It is believed that the supply curve satisfies the following assumptions:

(1): $S_t(x)$ is $\mathcal{F}_t$ measurable and non-negative.

(2): $S_t(x)$ is non-decreasing in x.

(3): $S_t(x)$ is continuous for all x.

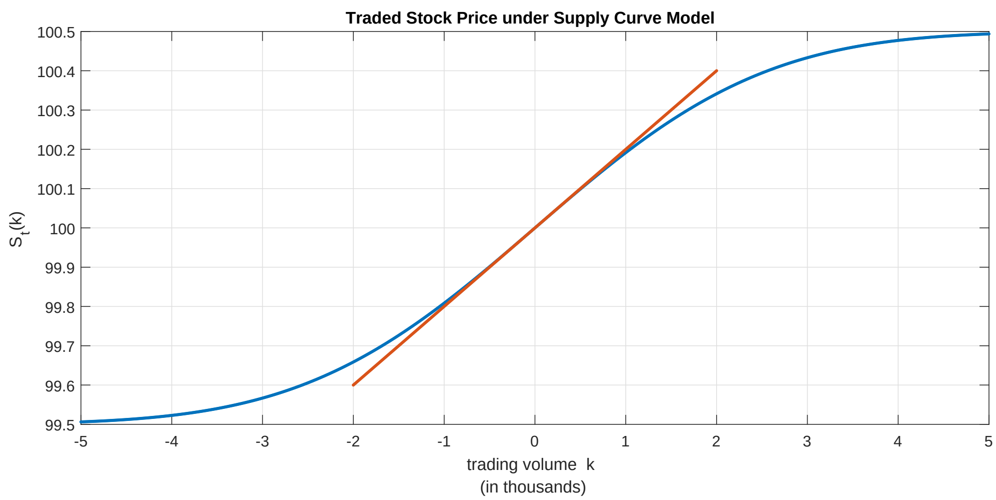

Figure 2.1: Traded stock price under supply curve model

Due to the liquidity risk, investors face the fact of selling at a lower price than the market quoted price and buying at a higher price than the market quoted price; liquidity risk therefore adds extra cost for trading. An example could be found in Figure 2.1. In respect to the general form of a supply curve function, Ku et al. (2012) applied a separable form of supply curve function, which is given by:

$$S_t(x) = f(x)S_t, \tag{2.3}$$

where $f(\cdot)$ is a twice differential non-decreasing function with $f(0) = 1$. This chapter will use the above separable form of supply curve function.

A trading strategy is defined by a pair of (B_t, P_t), where B_t denotes the wealth in the bank account and P_t is the number of stock at time t. We restrict P_t to the form:

$$P_t = P_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s, \tag{2.4}$$

where α_s and β_s are two progressively $\mathcal{F}_s$ measurable processes and both $E\left[\int_0^t |\alpha_s| ds\right]$ and $E\left[\int_0^t \beta_s^2 ds\right]$ are finite for every $t \in [0, T]$. P_t is a continuous process, which has finite quadratic variation and infinite variation. The differential form of P_t will be

$$dP_t = \alpha_t dt + \beta_t dW_t. \tag{2.5}$$

The quadratic term of P_t is

$$(dP_t)^2 = \beta_t^2 dt. \tag{2.6}$$

In a fully liquid market, there is no liquidity risk, and the cost to change the stock position from P_t to P_{t+dt} during $[t, t + dt]$ is $dP_t \times S_t$. When liquidity risk exists and the traded price of the stock is given by a supply curve function $S_t(x) = f(x)S_t$, the cost becomes $dP_t \times S_t(dP_t)$. The liquidity cost incurred from $[t, t+dt]$ is the extra cost introduced by the supply curve function. It is defined as:

$$dP_t \times S_t(dP_t) - dP_t \times S_t. \tag{2.7}$$

From (2.7) and the Taylor expansion of $f(P_t)$, it follows that:

$$\begin{aligned} dP_t \times S_t(dP_t) - dP_t \times S_t &= dP_t(f(dP_t) - 1)S_t \\ &= dP_t\left(f'(0)dP_t + \frac{f''(0)}{2}dP_t^{\,2}\right)S_t. \end{aligned} \tag{2.8}$$

Substituting $(dP_t)^2 = \beta_t^2 dt$ into (2.8), we obtain the result

$$dP_t \times S_t(dP_t) - dP_t \times S_t = f'(0)\beta_t^2 S_t dt. \tag{2.9}$$

The portfolio's value is defined by

$$\Lambda_t = P_t S_t + B_t.$$

Under the supply curve model, a trading strategy (P_t, B_t) is self-financing when

$$P_t S_t + B_t = P_0 S_0 + B_0 + \int_0^t P_u dS_u + \int_0^t rB_u du - \int_0^t f'(0)\beta_u^2 S_u du, \tag{2.10}$$

where $P_0 S_0 + B_0$ is the value of the initial portfolio, $\int_0^t P_u dS_u$ is the capital gain from the stock, $\int_0^t rB_u du$ is the gain from bank account and $\int_0^t f'(0)\beta_u^2 S_u du$ is the accumulated liquidity cost. Under self-financing condition, the differential form of Λ_t is

$$d\Lambda_t = P_t dS_t - f'(0)S_t\beta_t^2 dt + rB_t dt. \tag{2.11}$$

Compared with the self-financing condition in the Black-Scholes model, the self-financing condition with liquidity cost has an extra term $f'(0)S_t\beta_t^2 dt$ to account for the liquidity cost incurred during the trading.

2.3 European options

An investor who writes one call option $(S_T - K)^+$ needs to set a hedging portfolio to hedge the option. It is assumed that the option is covered, which means the option writer already owns P_0 position of the stock. Therefore, there is no liquidity cost for constructing the initial hedging portfolio (P_0, B_0). The investor continuously adjust the hedging position P_t during the hedging, and under the supply curve model, the value of the hedging portfolio at time T is

$$P_T S_T + B_T = P_0 S_0 + B_0 + \int_0^T P_u dS_u + \int_0^t rB_u du - \int_0^T f'(0)\beta_u^2 S_u du.$$

In this study, replicating the option means the market value of the option writer's hedging portfolio at maturity T equals the option's payoff. In other words, this study does not consider the liquidity cost of delivering the option's payoff at maturity. The option could then be replicated by a self-financing portfolio:

$$(S_T - K)^+ = P_T S_T + B_T,$$

and the replication cost for the option seller is $P_0 S_0 + B_0$. Symmetrically, there exists a replication strategy from the option buyer. When an investor buys one option, the investor shorts a portfolio to hedge the option. Assuming the short portfolio is $(-\hat{P}_0, -\hat{B}_0)$ and the option buyer can replicate the option at time T, we have the following equations:

$$-(S_T - K)^+ = -\hat{P}_T S_T - \hat{B}_T = -\hat{P}_0 S_0 - \hat{B}_0 + \int_0^T (-\hat{P}_u) dS_u - \int_0^t r\hat{B}_u du - \int_0^T f'(0)\hat{\beta}_u^2 S_u du$$

and

$$(S_T - K)^+ = \hat{P}_T S_T + \hat{B}_T = \hat{P}_0 S_0 + \hat{B}_0 + \int_0^T \hat{P}_u dS_u + \int_0^t r\hat{B}_u du + \int_0^T f'(0)\hat{\beta}_u^2 S_u du.$$

The replication cost for option buyer is $\hat{P}_0 S_0 + \hat{B}_0$. The replication cost for the option seller and buyer will be different. This chapter shall show that the seller's replication cost will be greater than buyer' replication cost.

Theorem 2.3.1. *(**European Options**) Under the supply curve model, option sellers can construct a portfolio replicating the option's payoff. The replication cost $C(x,t)$ satisfies the following equation:*

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 = rC \tag{2.12}$$

with the terminal condition

$$C(x,T) = (x-K)^+. \tag{2.13}$$

Proof. Let Y_t be the value of the option at time t. Y_t is a function of S_t and t, and could be written as $Y_t = C(S_t, t), t < T$. Assume $C(S_t, t)$ is twice differentiable on $(0, \infty) \times [0, T)$. From Ito's Formula, it is obvious that

$$dY_t = \left(\frac{\partial C}{\partial S}(S_t, t)uS_t + \frac{\partial C}{\partial t}(S_t, t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, t)\sigma^2 {S_t}^2\right)dt + \frac{\partial C}{\partial S}(S_t, t)\sigma S_t dW_t. \tag{2.14}$$

The option seller who writes an option needs to construct a self-financing portfolio (B_t, P_t) to hedge the option. The option seller's portfolio then consists of -1 option, B_t bank account and P_t stock. The dynamic hedging position P_t has finite quadratic variation and infinite variation. It can be written in the following form

$$P_t = P_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s, \tag{2.15}$$

where α_s and β_s are two progressively $\mathcal{F}_s$ measurable processes. The value of the portfolio is

$$\Pi_t = P_t S_t + B_t - Y_t = \Lambda_t - Y_t. \tag{2.16}$$

(2.11), (2.14) and (2.16) imply that

$$\begin{aligned}
d\Pi_t =& P_t dS_t + rB_t dt - f'(0)S_t\beta_t^2 dt - dY_t \\
=& P_t\left(\mu S_t dt + \sigma S_t dW_t\right) + rB_t dt - \left(\frac{\partial C}{\partial S}(S_t,t)uS_t + \frac{\partial C}{\partial t}(S_t,t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t,t)\sigma^2 {S_t}^2\right)dt \\
&- \frac{\partial C}{\partial S}(S_t,t)\sigma S_t dW_t - f'(0)S_t\beta_t^2 dt \\
=& \left(P_t\sigma S_t - \frac{\partial C}{\partial S}(S_t,t)\sigma S_t\right)dW_t \\
&+ \left(P_t\mu S_t - \frac{\partial C}{\partial S}(S_t,t)uS + rB_t - \frac{\partial C}{\partial t}(S_t,t) - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t,t)\sigma^2 {S_t}^2 - f'(0)\beta_t^2 S_t\right)dt.
\end{aligned} \tag{2.17}$$

In order to perfectly hedge the option, the option writer needs to make $d\Pi_t = 0$. The unique decomposition property of the Ito process implies that the only way to make $d\Pi_t = 0$ is to make both the dW_t and dt term in (2.17) zero. The first step is to make the dW_t term 0:

$$P_t\sigma S_t - \frac{\partial C}{\partial S}(S_t,t)\sigma S_t = 0. \tag{2.18}$$

The result is:

$$P_t = P_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s = \frac{\partial C}{\partial S}(S_t,t). \tag{2.19}$$

Applying Ito Lemma to both sides of (2.19) , gives the result:

$$\beta_t = \sigma S_t \frac{\partial^2 C}{\partial S^2}(S_t,t). \tag{2.20}$$

We also know:

$$B_t = C(S_t,t) - S_t P_t = C(S_t,t) - S_t\frac{\partial C}{\partial S}(S_t,t).$$

Then, we make the dt term of (2.17) to be 0:

$$P_t\mu S_t - \frac{\partial C}{\partial S}(S_t, t)uS_t + rB_t - \frac{\partial C}{\partial t}(S_t, t) - \frac{1}{2}\sigma^2 S_t{}^2\frac{\partial^2 C}{\partial S^2}(S_t, t) - f'(0)S_t\beta_t^2 = 0. \quad (2.21)$$

Substituting $\beta_t = \sigma S_t \frac{\partial^2 C}{\partial S^2}(S_t, t)$, $B_t = C(S_t, t) - S_t \frac{\partial C}{\partial S}(S_t, t)$ and $P_t = \frac{\partial C}{\partial S}(S_t, t)$ into (2.21), we obtain

$$\frac{\partial C}{\partial t}(S_t, t) + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t, t) + f'(0)\sigma^2 S_t^3\left(\frac{\partial^2 C}{\partial S_t^2}(S_t, t)\right)^2 = rC(S_t, t). \quad (2.22)$$

Replacing S_t with dummy variable x, the replication cost of European options satisfies the following equation

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 C}{\partial x^2} + f'(0)x\left(\sigma x\frac{\partial^2 C}{\partial x^2}\right)^2 = rC, \quad (2.23)$$

with the terminal condition:

$$C(x, T) = (x - K)^+. \quad (2.24)$$

□

Similarly, we can characterize the replication cost for option buyers.

Theorem 2.3.2. *Under the supply curve model, option buyers can construct a portfolio replicating the option's payoff. The replication cost $C(x, t)$ satisfies the following equation:*

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 C}{\partial x^2} - f'(0)x\left(\sigma x\frac{\partial^2 C}{\partial x^2}\right)^2 = rC \quad (2.25)$$

with the terminal condition:

$$C(x, T) = (x - K)^{+}. \tag{2.26}$$

From the fundamental theorem of asset pricing, we can perfectly hedge derivatives when the market is perfect and complete. In a market with transaction costs or short selling constraints, perfect hedging does not exist in a continuous time model. People tend to agree that perfect hedging is impossible in an imperfect market in a continuous time model. Surprisingly, we achieve continuous perfect hedging in a market with liquidity risk. In other words, we have found an example where continuous perfect hedging exists in an imperfect market. What's the difference between our liquidity risk model and a transaction costs model? Why is perfect hedging possible in our model when it is impossible in a transaction costs model?

In both the proportional transaction costs model and our liquidity cost model, we need to adopt a dynamic hedging strategy to hedge the option to replicate the option's payoff. When the stock price follows a geometric Brownian motion, the dynamic hedging position P_t usually has the following form:

$$P_t = P_0 + \int_0^t \alpha_u du + \int_0^t \beta_u dW_u, \tag{2.27}$$

which has finite quadratic variation and infinite variation. The incurred proportional transaction costs during $[0,\ T]$ is

$$\int_0^T kS_t\,|dP_t| = \int_0^T kS_t\,|\alpha_t dt + \beta_t dW_t|\,,$$

where k is the parameter of transaction costs proportion. When β_t is not 0,

$$\int_0^T kS_t\,|\alpha_t dt + \beta_t dW_t|$$

will be infinite because of the infinite variation of the Brownian motion. This means that under a continuous hedging strategy, the incurred transaction costs will go to infinity if we adopt a continuous time hedging strategy. This is the reason we cannot replicate an option's payoff with finite initial cost in the transaction costs model.

In the liquidity cost model, however, the liquidity cost is:

$$\int_0^T f'(0) S_t \beta_t^2 dt.$$

The fundamental difference between the transaction costs model and our liquidity cost model is that under a continuous hedging strategy, the transaction costs will go to infinity while the liquidity cost will be finite, which is why we can replicate options in the liquidity cost model.

2.4 Upper bound and lower bound of option prices

We denote the option seller's replication cost by $C^+(x,t)$. From the option seller's side, the replication cost $C^+(x,t)$ is determined by

$$\frac{\partial C^+}{\partial t}(x,t) + rx\frac{\partial C^+}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C^+}{\partial x^2}(x,t) + f'(0)x\left(\sigma x \frac{\partial^2 C^+}{\partial x^2}(x,t)\right)^2 = rC^+(x,t) \tag{2.28}$$

with $C^+(x,T) = (x-K)^+$, $C^+(0,t) = 0$, and $\lim_{x\to+\infty} C(x,t) = +\infty$. The Black-Scholes price $C(x,t)$ satisfies

$$\frac{\partial C}{\partial t}(x,t) + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x,t) = rC(x,t) \tag{2.29}$$

with $C(x,T) = (x-K)^+$, $C(0,t) = 0$, and $\lim_{x\to+\infty} C(x,t) = +\infty$.

Theorem 2.4.1. *When $f'(0) \geq 0$, suppose $C^+(x,t)$ is a classical solution to*

$$\frac{\partial C^+}{\partial t}(x,t) + rx\frac{\partial C^+}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C^+}{\partial x^2}(x,t) + f'(0)x\left(\sigma x \frac{\partial^2 C^+}{\partial x^2}(x,t)\right)^2 = rC^+(x,t) \tag{2.30}$$

and $C(x,t)$ is a classical solution to

$$\frac{\partial C}{\partial t}(x,t) + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x,t) = rC(x,t) \tag{2.31}$$

on $(0,+\infty)\times[0,T)$. If we have $C^+(x,T) = C(x,T)$, $C^+(0,t) = C(0,t)$ and $\lim_{x\to+\infty} C^+(x,t) = \lim_{x\to+\infty} C(x,t)$, then $C^+(x,t) \geq C(x,t)$ on $(0,+\infty)\times[0,T]$.

Proof. Denote $D(x,t) = C^+(x,t) - C(x,t)$, then we have

$$D(x,T) = 0, D(0,t) = 0 \text{ and } \lim_{x\to+\infty} D(x,t) = 0.$$

Differentiating $D(x,t) = C^+(x,t) - C(x,t)$ w.r.t t and x, we have:

$$\frac{\partial D}{\partial t}(x,t) = \frac{\partial C^+}{\partial t}(x,t) - \frac{\partial C}{\partial t}(x,t) \tag{2.32}$$

$$\frac{\partial D}{\partial x}(x,t) = \frac{\partial C^+}{\partial x}(x,t) - \frac{\partial C}{\partial x}(x,t) \tag{2.33}$$

and

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 D}{\partial x^2}(x,t) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C^+}{\partial x^2}(x,t) - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x,t). \tag{2.34}$$

Subtracting (2.31) from (2.30), we obtain

$$\frac{\partial C^+}{\partial t}(x,t) - \frac{\partial C}{\partial t}(x,t) + rx\frac{\partial C^+}{\partial t}(x,t) - rx\frac{\partial C}{\partial t}(x,t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C^+}{\partial x^2}(x,t) \tag{2.35}$$

$$- \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x,t) + f'(0)x\left(\frac{\partial^2 C^+}{\partial x^2}(x,t)\sigma x\right)^2 = rC^+(x,t) - rC(x,t). \tag{2.36}$$

Substituting (2.33) and (2.34) into (2.35), we obtain

$$\frac{\partial D}{\partial t}(x,t) + rx\frac{\partial D}{\partial x}(x,t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 D}{\partial x^2}(x,t) - rD(x,t) = -f'(0)x\left(\frac{\partial^2 C^+}{\partial x^2}(x,t)\sigma x\right)^2. \tag{2.37}$$

We denote

$$F(x,t) = -f'(0)x\left(\frac{\partial^2 C^+}{\partial x^2}(x,t)\sigma x\right)^2,$$

so we have $F(x,t) \leq 0$ and

$$\frac{\partial D}{\partial t}(x,t) + rx\frac{\partial D}{\partial x}(x,t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 D}{\partial x^2}(x,t) - rD(x,t) = F(x,t). \tag{2.38}$$

Suppose $D(x, t)$ has a negative local minimum at some point (x^*, t^*) in $(0, \infty) \times (0, T]$, then we have

$$D(x^*, t^*) < 0. \tag{2.39}$$

The necessary condition for local minimum implies

$$\frac{\partial D}{\partial t}(x^*, t^*) = \frac{\partial D}{\partial x}(x^*, t^*) = 0. \tag{2.40}$$

The scale function $h(x) = D(x, t^*)$ has its minimum at x^*, thus

$$h''(x^*) = \frac{\partial^2 D}{\partial x^2}(x^*, t^*) \geq 0. \tag{2.41}$$

From (2.38), we know

$$\frac{\partial D}{\partial t}(x^*, t^*) + rx^* \frac{\partial D}{\partial x}(x^*, t^*) + \frac{1}{2}\sigma^2 x^{*2} \frac{\partial^2 D}{\partial x^2}(x^*, t^*) - rD(x^*, t^*) = -f'(0)x^* \left(\frac{\partial^2 C^+}{\partial x^2}(x^*, t^*)\sigma x^* \right)^2 \tag{2.42}$$

From (2.40), we have

$$\frac{1}{2} \frac{\partial^2 D}{\partial x^2}(x^*, t^*)\sigma^2 x^{*2} - rD(x^*, t^*) = -f'(0)x^* \left(\frac{\partial^2 C^+}{\partial x^2}(x^*, t^*)\sigma x^* \right)^2 \tag{2.43}$$

which implies

$$rD(x^*, t^*) \geq 0 \tag{2.44}$$

Because $r \geq 0$, (2.44) is contrast to

$$D(x^*, t^*) < 0.$$

By now, we can conclude

$$D(x,t) \geq 0$$

and

$$C^+(x,t) \geq C(x,t), \quad \text{when} \ \ (t,x) \in [0,+\infty) \times [0,T].$$

$\square$

We denote the option buyer's replication cost by $C^-(x,t)$. From the option buyer's side, the replication cost $C^-(x,t)$ is determined by

$$\frac{\partial C^-}{\partial t} + rx\frac{\partial C^-}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C^-}{\partial x^2} - f'(0)x\left(\sigma x \frac{\partial^2 C^-}{\partial x^2}\right)^2 = rC^- \tag{2.45}$$

with $C^-(x,T) = (x-K)^+$, $C^-(0,t) = 0$, and $\lim_{x\to+\infty} C^-(x,t) = +\infty$. Applying the same argument, we can prove

$$C^-(x,t) \leq C(x,t), \quad \text{when} \ \ (t,x) \in [0,+\infty) \times [0,T].$$

So, we can conclude that

$$C^-(x,t) \leq C^+(x,t).$$

Under the supply curve model, any price above $C^+(x,t)$ will lead to arbitrage for the option seller and below $C^-(x,t)$ will lead to arbitrage for the option buyer. We can consider $C^+(x,t)$ as the upper bound for option price, and $C^-(x,t)$ as lower bound. So the quoted option price C^p in the market with liquidity risk should satisfy

$$C^-(x,t) \leq C^p \leq C^+(x,t).$$

The price of the option satisfies the differential equation

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + f'(0)x\left(\frac{\partial^2 C}{\partial x^2}\sigma x\right)^2 = rC \tag{2.46}$$

Since $\Theta = \frac{\partial C}{\partial t}$, $\Delta = \frac{\partial C}{\partial x}$, $\Gamma = \frac{\partial^2 C}{\partial x^2}$ it follows that

$$\Theta + rx\Delta + \frac{1}{2}\sigma^2 x^2 \Gamma + f'(0)\sigma^2 x^3 \Gamma^2 = rC \tag{2.47}$$

Hedging options with a large absolute value of Γ will lead to a large liquidity cost, which is reflected in the term $f'(0)\sigma^2 x^3 \Gamma^2$. Intuitively, a large Γ means frequent trading of stocks, and frequent trading leads to a large liquidity cost. Imagine that if one option has $\Gamma = 0$, which means there is no need to change the stock position in the hedging portfolio, then the liquidity risk will be zero. The option price in the market with liquidity risk will be the Black-Scholes price. The option price is also reflected in the equation (2.47).

2.4.1 Asymptotic Expansion

In this section, we analyze the solution of equation (2.46) with an asymptotic expansion method, the idea that can be tracked to Ku et al. (2012). In this section, we present an approximation formula for equation (2.46).

When $f'(0)$ is sufficiently small, the solution can be approximated by the form

$$C(x,t) = C_0(x,t) + f'(0)C_1(x,t) + O(f'(0)^2).$$

The $C_0(x,t)$ term is determined by the Black-Scholes equation:

$$\frac{\partial C_0}{\partial t} + rx\frac{\partial C_0}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_0}{\partial x^2} = rC_0 \tag{2.48}$$

with Boundary condition $C_0(x,T) = (x-K)^+$. In the order of $O(f'(0))$, we have $C_1(x,t)$ determined by

$$\frac{\partial C_1}{\partial t} + rx\frac{\partial C_1}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_1}{\partial x^2} + \sigma^2 x^3 \left(\frac{\partial^2 C_0}{\partial x^2} + f'(0)\frac{\partial^2 C_1}{\partial x^2}\right)^2 = rC_1 \tag{2.49}$$

with $C_1(x,T) = 0$.

The explicit solution is the Black-Scholes formula

$$C_0(x,t) = xN(d_1) - Ke^{-rT}N(d_2) \tag{2.50}$$

where

$$d_1 = \frac{\ln(x/k) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

For the European call option, the gamma, $\frac{\partial^2 C_0}{\partial x^2}$ is given by

$$\frac{\partial^2 C_0}{\partial x^2} = \frac{1}{x\sigma\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}.$$

Substituting the formula of $\frac{\partial^2 C_0}{\partial x^2}$ into equation (2.49) and simplifying it, we obtain

$$\frac{\partial C_1}{\partial t} + rx\frac{\partial C_1}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_1}{\partial x^2} + \frac{x}{2\pi(T-t)}e^{-d_1^2} = rC_1 \tag{2.51}$$

Please note that the $f'(0)\frac{\partial^2 C_1}{\partial x^2}$ in equation(2.49) is ignored in deriving equation(2.51) because it is in the order term of $O(f'(0))$. In order to derive the explicit solution of equation

(2.51), we make the following variables transformations to transform equation (2.51) into a standard boundary value problem for the heat equation:

$$x = e^y, t = T - \frac{2\tau}{\sigma^2}$$
$$C_1(x,t) = v(y,\tau) = v\left(\ln(x), \frac{\sigma^2(T-t)}{2}\right)$$

The partial derivative of $C_1(x,t)$ with respect to x and t expressed in terms of partial derivatives of v in terms of y and τ are:

$$\frac{\partial C_1}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} \tag{2.52}$$
$$\frac{\partial C_1}{\partial x} = \frac{1}{x}\frac{\partial v}{\partial y} \tag{2.53}$$
$$\frac{\partial^2 C_1}{\partial x^2} = -\frac{1}{x^2}\frac{\partial v}{\partial y} + \frac{1}{x^2}\frac{\partial^2 v}{\partial y^2} \tag{2.54}$$

Substituting (2.52), (2.53) and (2.54) into equation (2.51), we obtain:

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau} + rx\frac{1}{x}\frac{\partial v}{\partial y} + \frac{1}{2}\sigma^2 x^2(-\frac{1}{x^2}\frac{\partial v}{\partial y} + \frac{1}{x^2}\frac{\partial^2 v}{\partial y^2}) + \frac{x}{2\pi(T-t)}e^{-d_1^2} = rv.$$

After we rearrange the equation of $v(y,\tau)$ and simplify it, we get:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial y} - \frac{2r}{\sigma^2}v + \frac{2}{\sigma^2}g(y,\tau) \tag{2.55}$$

where

$$g(y,\tau) = \frac{x}{2\pi(T-t)}e^{-d_1^2} = \frac{e^y}{2\pi(\frac{2\tau}{\sigma^2})}e^{-d_1^2} = \frac{\sigma^2}{4\pi\tau}e^{y-\frac{\left(\ln(e^y/K)+(r+\frac{\sigma^2}{2})\frac{2\tau}{\sigma^2}\right)^2}{2\tau}}.$$

For further reference, we denote

$$a = \frac{\sigma^2 - 2r}{2\sigma^2}, \quad b = -\left(\frac{\sigma^2 + 2r}{2\sigma^2}\right)^2.$$

We set $v(y, \tau) = e^{ay+b\tau} w(y, \tau)$. Computing the partials of v in terms of y and τ, we have

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= b e^{ay+b\tau} w + e^{ay+b\tau} \frac{\partial w}{\partial \tau} \\
\frac{\partial v}{\partial y} &= a e^{ay+b\tau} w + e^{ay+b\tau} \frac{\partial w}{\partial y} \\
\frac{\partial^2 v}{\partial y^2} &= a^2 e^{ay+b\tau} w + 2a e^{ay+b\tau} \frac{\partial w}{\partial y} + e^{ay+b\tau} \frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$

Substituting them into equation (2.55) and simplifying it, we obtain

$$bw + \frac{\partial w}{\partial \tau} = a^2 w + 2a \frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} + \left(\frac{2r}{\sigma^2} - 1 \right) \left(aw + \frac{\partial w}{\partial y} \right) - \frac{2r}{\sigma^2} w + \frac{2}{\sigma^2 e^{ay+b\tau}} g(y, \tau). \quad (2.56)$$

We denote

$$a = \frac{\sigma^2 - 2r}{2\sigma^2}, \quad b = -\left(\frac{\sigma^2 + 2r}{2\sigma^2} \right)^2$$

and simplifying equation (2.56) by substituting a, b and $g(y, \tau)$, we have:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial y^2} + \frac{1}{2\pi\tau} e^{y - ay - b\tau - \frac{\left(\ln(e^y/K) + (r + \frac{\sigma^2}{2}) \frac{2\tau}{\sigma^2} \right)^2}{2\tau}} \quad (2.57)$$

The initial condition for (2.57) is $w(y, 0) = 0$. The solution $w(y, \tau)$ is solved using Duhamel's principle:

$$w(y, \tau) = \int_0^\tau \int_{-\infty}^\infty \frac{1}{2\pi u} e^{\xi - a\xi - bu - \frac{\left(\ln(e^\xi/K) + (r + \frac{\sigma^2}{2}) \frac{2u}{\sigma^2} \right)^2}{2u}} \frac{1}{2\sqrt{\pi(\tau - u)}} e^{-\frac{(y-\xi)^2}{4(\tau-u)}} d\xi du. \quad (2.58)$$

Equation (2.58) is a double integration with respect to ξ and u.

$$\int \frac{1}{2\pi u} e^{\xi - a\xi - bu - \frac{\left(\ln(e^{\xi}/K)+(r+\frac{\sigma^2}{2})\frac{2u}{\sigma^2}\right)^2}{2u}} \frac{1}{2\sqrt{\pi(\tau-u)}} e^{-\frac{(y-\xi)^2}{4(\tau-u)}} d\xi$$

$$= \frac{K^{\frac{2r}{\sigma^2}+1} \mathrm{e}^{\frac{\left(4\tau u-4u^2\right)\left(\sigma^2\left(uy+u^2-\tau u\right)+r\left(2u^2-2\tau u\right)+\ln(K)\sigma^2(2\tau-2u)\right)^2}{4\sigma^4(2\tau-u)\left(2\tau u-2u^2\right)^2}+\frac{y^2}{4u-4\tau}-\frac{ru}{\sigma^2}-\frac{r^2 u}{\sigma^4}-\frac{u}{4}-\frac{\ln^2(K)}{2u}}}{4\pi\sqrt{\tau-u}\,u\sqrt{\frac{u-2\tau}{u(u-\tau)}}} \cdot$$

$$\operatorname{erf}\left(\frac{\sqrt{\frac{u-2\tau}{u(u-\tau)}}\xi}{2} - \frac{\sigma^2\left(uy+u^2-\tau u\right)+r\left(2u^2-2\tau u\right)+\ln(K)\sigma^2(2\tau-2u)}{\sigma^2\left(2\tau u-2u^2\right)\sqrt{\frac{u-2\tau}{u(u-\tau)}}}\right)$$

where

$$\operatorname{erf}(x) = \int_{-x}^{x} \frac{\mathrm{e}^{-v^2}}{\sqrt{\pi}} \mathrm{d}v$$

When $\xi \to +\infty$,

$$\operatorname{erf}\left(\frac{\sqrt{\frac{u-2\tau}{u(u-\tau)}}\xi}{2} - \frac{\sigma^2\left(uy+u^2-\tau u\right)+r\left(2u^2-2\tau u\right)+\ln(K)\sigma^2(2\tau-2u)}{\sigma^2\left(2\tau u-2u^2\right)\sqrt{\frac{u-2\tau}{u(u-\tau)}}}\right) \to 1.$$

and when $\xi \to -\infty$,

$$\operatorname{erf}\left(\frac{\sqrt{\frac{u-2\tau}{u(u-\tau)}}\xi}{2} - \frac{\sigma^2\left(uy+u^2-\tau u\right)+r\left(2u^2-2\tau u\right)+\ln(K)\sigma^2(2\tau-2u)}{\sigma^2\left(2\tau u-2u^2\right)\sqrt{\frac{u-2\tau}{u(u-\tau)}}}\right) \to -1.$$

Therefore, the integration with respect to ξ could be written in an explicit form:

$$\int_{-\infty}^{\infty} \frac{1}{2\pi u} e^{\xi - a\xi - bu - \frac{\left(\ln(e^{\xi}/K)+(r+\frac{\sigma^2}{2})\frac{2u}{\sigma^2}\right)^2}{2u}} \frac{1}{2\sqrt{\pi(\tau-u)}} e^{-\frac{(y-\xi)^2}{4(\tau-u)}} d\xi$$

$$= \frac{K^{\frac{2r}{\sigma^2}+1} \mathrm{e}^{\frac{\left(4\tau u-4u^2\right)\left(\sigma^2\left(uy+u^2-\tau u\right)+r\left(2u^2-2\tau u\right)+\ln(K)\sigma^2(2\tau-2u)\right)^2}{4\sigma^4(2\tau-u)\left(2\tau u-2u^2\right)^2}+\frac{y^2}{4u-4\tau}-\frac{ru}{\sigma^2}-\frac{r^2 u}{\sigma^4}-\frac{u}{4}-\frac{\ln^2(K)}{2u}}}{2\pi\sqrt{\tau-u}\,u\sqrt{\frac{u-2\tau}{u(u-\tau)}}}. \tag{2.59}$$

Then $w(y, \tau)$ could be written as:

$$w(y,\tau) = \int_0^{\tau} \frac{K^{\frac{2r}{\sigma^2}+1} \mathrm{e}^{\frac{\left(4\tau u-4u^2\right)\left(\sigma^2\left(uy+u^2-\tau u\right)+r\left(2u^2-2\tau u\right)+\ln(K)\sigma^2(2\tau-2u)\right)^2}{4\sigma^4(2\tau-u)\left(2\tau u-2u^2\right)^2}+\frac{y^2}{4u-4\tau}-\frac{ru}{\sigma^2}-\frac{r^2u}{\sigma^4}-\frac{u}{4}-\frac{\ln^2(K)}{2u}}}{2\pi\sqrt{\tau-u}\,u\sqrt{\frac{u-2\tau}{u(u-\tau)}}} du.$$

We know

$$C_1(x,t) = v(y,\tau) = e^{ay+b\tau} w(y,\tau) \tag{2.60}$$

Substituting

$$y = \ln(x), \tau = \frac{\sigma^2}{2}(T-t)$$

into equation (2.60), we have

$$C_1(x,t) = e^{\frac{\sigma^2-2r}{2\sigma^2}\ln(x)-\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^2 \frac{1}{2}\sigma^2(T-t)} \times$$

$$\int_0^{\frac{1}{2}\sigma^2(T-t)} \frac{K^{\frac{2r}{\sigma^2}+1} e^{\frac{\left(2\sigma^2(T-t)u-4u^2\right)\left(\sigma^2\left(u\ln(x)+u^2-(\frac{1}{2}\sigma^2(T-t))u\right)+r\left(2u^2-\sigma^2(T-t)u\right)+\ln(K)\sigma^2\left(\sigma^2(T-t)-2u\right)\right)^2}{4\sigma^4\left(\sigma^2(T-t)-u\right)\left(\sigma^2(T-t)u-2u^2\right)^2}+\frac{\ln^2(x)}{4u-2\sigma^2(T-t)}-\frac{ru}{\sigma^2}-\frac{r^2u}{\sigma^4}-\frac{u}{4}-\frac{\ln^2(K)}{2u}}}{2\pi u\sqrt{\frac{1}{2}\sigma^2(T-t)-u}\sqrt{\frac{u-\sigma^2(T-t)}{u\left(u-\frac{1}{2}\sigma^2(T-t)\right)}}} du.$$

The solution $C(x,t)$ is approximated by

$$C(x,t) \approx C_0(x,t) + f'(0)C_1(x,t). \tag{2.61}$$

Therefore, the European call option price with liquidity cost could be approximated by

$$C(x,t) = xN(d_1) - Ke^{-rT}N(d_2) + f'(0)e^{\frac{\sigma^2-2r}{2\sigma^2}\ln(x)-\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^2 \frac{1}{2}\sigma^2(T-t)} \times$$

$$\int_0^{\frac{1}{2}\sigma^2(T-t)} \frac{K^{\frac{2r}{\sigma^2}+1} e^{\frac{\left(2\sigma^2(T-t)u-4u^2\right)\left(\sigma^2\left(u\ln(x)+u^2-(\frac{1}{2}\sigma^2(T-t))u\right)+r\left(2u^2-\sigma^2(T-t)u\right)+\ln(K)\sigma^2\left(\sigma^2(T-t)-2u\right)\right)^2}{4\sigma^4\left(\sigma^2(T-t)-u\right)\left(\sigma^2(T-t)u-2u^2\right)^2}+\frac{\ln^2(x)}{4u-2\sigma^2(T-t)}-\frac{ru}{\sigma^2}-\frac{r^2u}{\sigma^4}-\frac{u}{4}-\frac{\ln^2(K)}{2u}}}{2\pi u\sqrt{\frac{1}{2}\sigma^2(T-t)-u}\sqrt{\frac{u-\sigma^2(T-t)}{u\left(u-\frac{1}{2}\sigma^2(T-t)\right)}}} du$$

where

$$d_1 = \frac{\ln(x/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t}.$$

The liquidity premium is

$$f'(0)e^{\frac{\sigma^2-2r}{2\sigma^2}\ln(x)-\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^2\frac{1}{2}\sigma^2(T-t)}\times$$

$$\int_0^{\frac{1}{2}\sigma^2(T-t)} \frac{K^{\frac{2r}{\sigma^2}+1} e^{\frac{\left(2\sigma^2(T-t)u-4u^2\right)\left(\sigma^2\left(u\ln(x)+u^2-(\frac{1}{2}\sigma^2(T-t))u\right)+r\left(2u^2-\sigma^2(T-t)u\right)+\ln(K)\sigma^2\left(\sigma^2(T-t)-2u\right)\right)^2}{4\sigma^4\left(\sigma^2(T-t)-u\right)\left(\sigma^2(T-t)u-2u^2\right)^2}+\frac{\ln^2(x)}{4u-2\sigma^2(T-t)}-\frac{ru}{\sigma^2}-\frac{r^2u}{\sigma^4}-\frac{u}{4}-\frac{\ln^2(K)}{2u}}}{2\pi u\sqrt{\frac{1}{2}\sigma^2(T-t)-u}\sqrt{\frac{u-\sigma^2(T-t)}{u\left(u-\frac{1}{2}\sigma^2(T-t)\right)}}}du.$$

The liquidity premium is positive and is a linear function of liquidity parameter $f'(0)$. When the liquidity parameter $f'(0)$ is sufficiently small, the liquidity premium increases linearly with respect to $f'(0)$. In the next section, we will present the numerical results of option prices with the approximation formula.

2.4.2 Numerical results of European options

In this section, we present some numerical results of European options. There are two ways to calculate the option prices: using the finite difference method to solve the PDE numerically and by using the approximation formula. We will present and compare the option prices using the two methods. Also, numerical simulation of the hedging strategy and hedging error will be shown to illustrate the perfect hedging of the option with liquidity cost.

Compared to the Black-Scholes equation, the PDE of the option price with liquidity risk

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} \pm f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 = rC, \tag{2.62}$$

has a nonlinear term $\frac{\partial^2 C}{\partial S^2}(S,t)^2$, which makes the PDE fully nonlinear.

The numerical results of the fully nonlinear partial differential equation are presented in Table 2.2; parameter values are $S_0 = 100$, $\sigma = 0.2$, $T = 1$, $r = 0$ with strike K and liquidity parameter $f'(0)$ varying as shown in the table. Table 2.3 shows the option prices

	$f'(0) = 0.000$	$f'(0) = 0.0005$		$f'(0) = 0.001$		$f'(0) = 0.002$		$f'(0) = 0.005$	
K	BS Price	Seller	Buyer	Seller	Buyer	Seller	Buyer	Seller	Buyer
90	13.587	13.596	13.577	13.606	13.566	13.625	13.542	13.676	13.518
95	10.516	10.528	10.504	10.539	10.492	10.561	10.463	10.625	10.424
100	7.9616	7.9741	7.9487	7.9864	7.9352	8.0102	7.9030	8.0784	7.8563
105	5.9019	5.9143	5.8891	5.9265	5.8756	5.9501	5.8424	6.0179	5.7941
110	4.2891	4.3006	4.2772	4.3118	4.2646	4.3337	4.2321	4.3965	4.1877

Table 2.1: Seller's and Buyer's replication costs with different Strikes and liquidity parameters when $T = 1$.

when parameter values are $S_0 = 100$, $\sigma = 0.2$, $T = 0.5$, $r = 0$ with strike K and liquidity parameter $f'(0)$ varying as shown in the table.

The first column gives the Black-Scholes values for the corresponding European call option. The Black-Scholes price is a special case in our model when $f'(0) = 0$. When in the case of $f'(0) = 0$, the buyer's price equals the seller's price, and the Black-Scholes

K	$f'(0) = 0.000$ BS Price	$f'(0) = 0.0005$ Seller	Buyer	$f'(0) = 0.001$ Seller	Buyer	$f'(0) = 0.002$ Seller	Buyer	$f'(0) = 0.005$ Seller	Buyer
90	11.7704	11.7787	11.7617	11.7868	11.7526	11.8024	11.7309	11.8466	11.7243
95	8.3486	8.3599	8.3369	8.3708	8.3245	8.3918	8.2944	8.4513	8.2700
100	5.6316	5.6442	5.6185	5.6564	5.6044	5.6801	5.5693	5.7466	5.5346
105	3.6132	3.6252	3.6006	3.6369	3.5871	3.6595	3.5517	3.7232	3.5165
110	2.2085	2.2186	2.1980	2.2283	2.1867	2.2470	2.1548	2.3002	2.1268

Table 2.2: Buyer's and Seller's replication costs with different Strikes and liquidity parameters when $T = 0.5$.

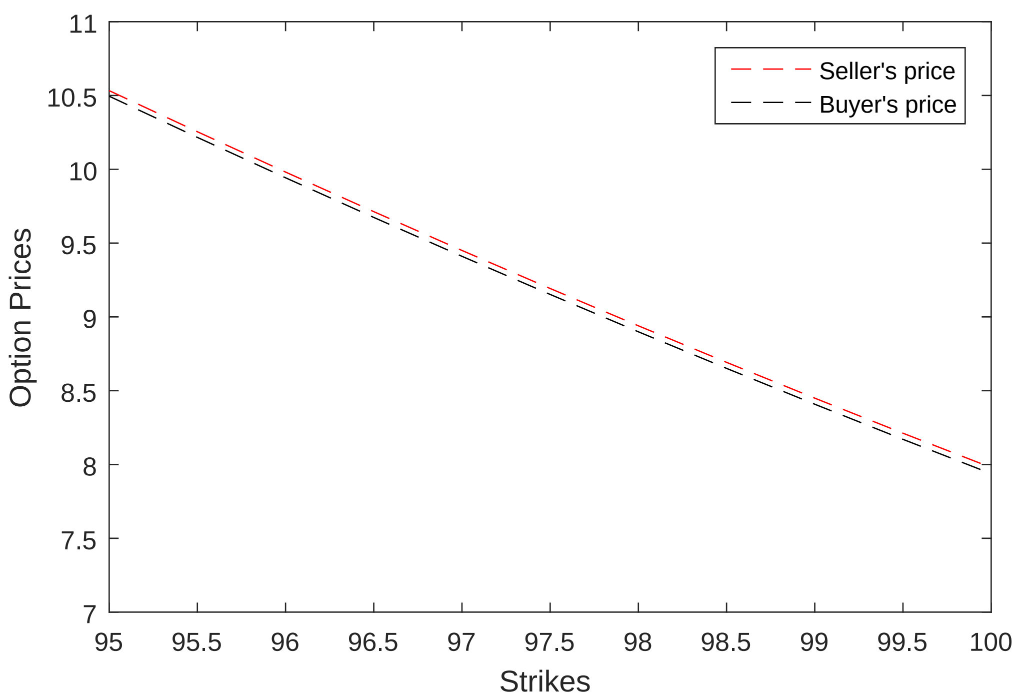

Figure 2.2: Buyer's and seller's prices with varying Strikes

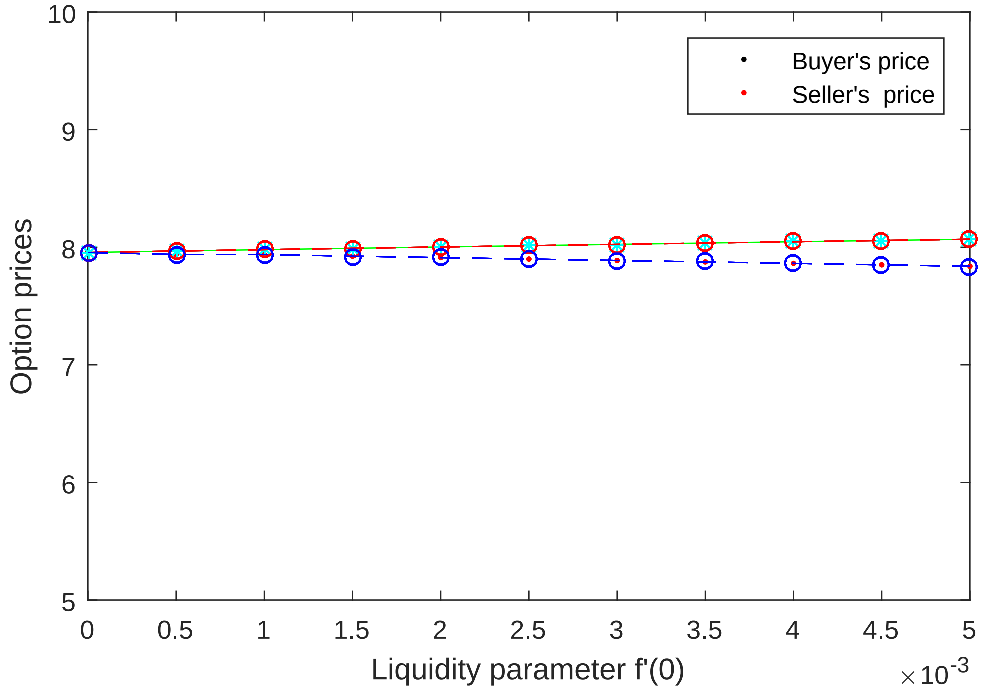

Figure 2.3: Buyer's and seller's prices with varying f'(0)

price is the unique price for the option. The column of $f'(0) = 0.0005$ gives the buyer's and seller's prices when the liquidity parameter $f'(0) = 0.0005$. In this case, liquidity cost is non-zero, and the seller's price is larger than the buyer's price (Figure 2.3). We can regard the seller's price as an ask price and the buyer's price as a bid price. Because any price that is higher than the seller's price or lower than the buyer's price will lead to arbitrage, the quoted option price in a market with liquidity risk should lie between the seller's price and the buyer's price. Next, we give the buyer's and seller's prices for different liquidity parameters. With the increase of $f'(0)$, the seller's prices increase and

	$\alpha = 0$	$\alpha = 0.0005$	$\alpha = 0.001$	$\alpha = 0.002$	$\alpha = 0.005$
Approximation	8.9260	8.9378	8.9497	8.9734	9.0445
PDE method	8.9240	8.9360	8.9478	8.9707	9.0373

Table 2.3: Option seller' prices comparison of PDE method and approximation formula

buyer's prices decrease. Correspondingly, the difference between the seller's and buyer's prices increases (Figure 2.3). We can conclude that if the liquidity depth increases the bid-ask spread increases.

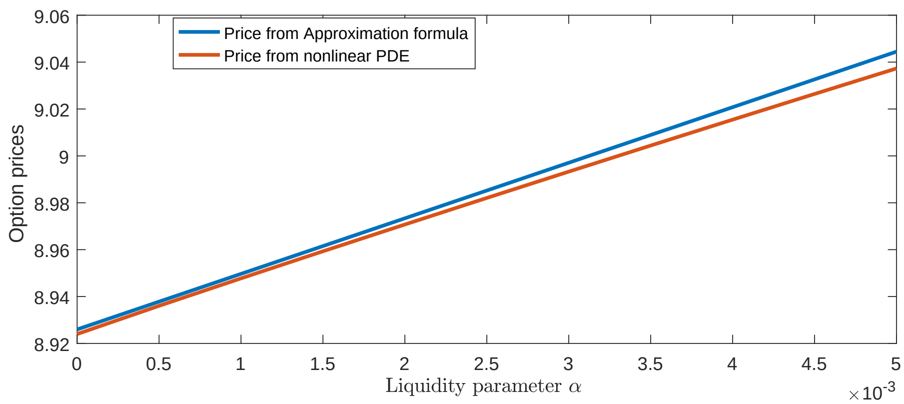

Figure 2.4: Option seller' prices comparison of PDE method and approximation formula

In Table 2.3 and Figure 2.4, we compare the European call option prices obtained from the PDE method and the approximation method. The Strike K is 98. The liquidity parameter $f'(0)$ increases from 0.000 to 0.005, and the option prices increase with respect to the liquidity parameter. From Figure 2.4, we can see that the difference of the option prices from the two methods are extremely small. We can conclude that when liquidity parameter $f'(0)$ is sufficiently small, the asymptotic approximation method is quite accurate.

We analyse the hedging error of our model. The hedging error H_T is defined to be

$$H_T = (P_T S_T + B_T) - (S_T - K)^+.$$

In theory, we can perfectly hedge the option with a self-financing portfolio in continuous time, which means the hedging error is zero almost surely. In practice, we can apply discrete time hedging, so we can not perfectly replicate the option. When our hedging period goes to zero, however, the hedging error will converge to zero. We did the Monte Carlo simulation to compute the mean and variance of the hedging error. Table 2.4 presents the Monte Carlo simulation results for the option seller with Strike $K = 100$ and varying liquidity parameter $f'(0)$. There are $10,000$ paths used in the simulation, with 100 time steps in each simulation. The mean row shows the mean hedging error. The mean and variance of the hedging error do not vary too much with different liquidity parameter $f'(0)$. Moreover, the mean and variance of the hedging error with $f'(0) > 0$ are almost the same as the mean and variance of the hedging error of the Black-Scholes case (when $f'(0) = 0$).

	$f'(0)$=0.000	$f'(0)$=0.0005	$f'(0)$=0.001	$f'(0)$=0.002	$f'(0)$= 0.005
$C^+(S_0, 0)$	7.9616	7.9741	7.9864	8.0102	8.0784
$E(H_T)$	-0.1093	-0.1106	-0.0905	-0.1025	-0.0921
$Var(H_T)$	0.0072	0.0073	0.0074	0.0074	0.0074

Table 2.4: Monte Carlo simulation of hedging error.

2.5 American options

In this section, we consider finite expiration American put options. Our argument can be easily extended to other kinds of early exercise options. The value of the American option is $Y_t = C(S_t, t)$. For each $t \in [0, T]$ we want to split the S axis into two subintervals. Doing so will divide the cylinder into two subregions. The boundary between the regions will be given by a function $S_f(t)$. Appropriate boundary conditions will hold on each of the subregions and the boundary between them. Since the location of the boundary between the two subregions is not known in advance, we have what is called a free boundary problem. As it happens, this free boundary problem is interpreted as a differential inequality problem.

(1): First subregion: $S_f(t) < S < \infty$.

For these values of S , early exercise is not optimal, and the option holder should hold the American option. The option seller constructs a portfolio consisting of -1 American put options and a number P_t of the underlying asset. The value of this portfolio is:

$$\Pi_t = P_t S_t + B_t - Y_t. \tag{2.63}$$

From the self-financing condition (2.10) and Ito Lemma, the change of the value of the portfolio is

$$\begin{aligned}
d\Pi_t =& P_t dS_t + dB_t - dY_t - dP_t[S_t(dP_t) - S_t] \\
=& P_t(uS_t dt + \sigma S_t dW_t) + rB_t dt - \left(\frac{\partial C}{\partial S}(S_t,t)uS_t + \frac{\partial C}{\partial t}(S_t,t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t,t)\sigma^2 {S_t}^2\right)dt \\
& - \frac{\partial C}{\partial S}(S_t,t)\sigma S_t dW_t - f'(0)dP_t dP_t S(t) \\
=& \left(P_t\sigma S_t - \frac{\partial C}{\partial S}(S_t,t)\sigma S_t\right)dW_t + rB_t dt + \Bigg[P_t u S_t - \frac{\partial C}{\partial S_t}(S_t,t)uS - \frac{\partial C}{\partial t}(S_t,t) \\
& - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t,t)\sigma^2 {S_t}^2 - f'(0)\beta_t^2 S_t\Bigg]dt.
\end{aligned} \tag{2.64}$$

Because stock price S_t has not reached optimal exercise boundary S_f, for the option holder, it is optimal to continue holding the option. The option writer will make $P_t = \frac{\partial C}{\partial S}(S_t, t)$. Applying Ito Lemma to $P_t = \frac{\partial C}{\partial S}(S_t, t)$ gives us

$$\beta_t = \sigma S_t \frac{\partial^2 C}{\partial S^2}(S_t, t). \tag{2.65}$$

The change in the hedging portfolio will be $d\Pi_t = 0$. Substituting $P_t = \frac{\partial C}{\partial S}(S_t, t)$, $B_t = C(S_t, t) - P_t S_t$ and (2.65) into (2.64), we have

$$\begin{cases}
\frac{\partial C}{\partial t}(S_t,t) + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}(S_t,t) + f'(0)S_t\left(\sigma S_t \frac{\partial^2 C}{\partial S_t^2}(S_t,t)\right)^2 = rC(S_t,t) \\
C(S_t,t) > K - S_t
\end{cases}$$

(2): Second subregion: $0 \leq S < S_f(t)$.

For these values of S, early exercise is optimal. The option writer constructs a portfolio consisting of -1 American put option and a number P_t of the underlying asset and B_t units of the bank account. Therefore, the value of the portfolio is

$$\Pi_t = P_t S_t + B_t - Y_t. \tag{2.66}$$

From the self-financing condition equation (2.10) and Ito Lemma, the change of the value of the portfolio is

$$\begin{aligned}
d\Pi_t =& P_t dS_t + dB_t - dY_t - dP_t[S_t(dP_t) - S_t] \\
=& P_t(uS_t dt + \sigma S_t dW_t) + rB_t dt - \left(\frac{\partial C}{\partial S}(S_t, t)uS_t + \frac{\partial C}{\partial t}(S_t, t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, t)\sigma^2 {S_t}^2\right) dt \\
& - \frac{\partial C}{\partial S}(S_t, t)\sigma S_t dW_t - f'(0) dP_t dP_t S(t) \\
=& (P_t \sigma S_t - \frac{\partial C}{\partial S}(S_t, t)\sigma S_t) dW_t + \left[P_t u S_t + rB_t - \frac{\partial C}{\partial S}(S_t, t)uS - \frac{\partial C}{\partial t}(S_t, t)\right. \\
& \left. -\frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, t)\sigma^2 {S_t}^2 - f'(0)\beta_t^2 S_t\right] dt.
\end{aligned} \tag{2.67}$$

Because stock price S_t has already entered the optimal exercise boundary S_f, for the option holder, it is optimal to exercise the option immediately. If the option holder did not exercise the option, it would give the option writer an arbitrage opportunity. The option writer will make $P_t = \frac{\partial C}{\partial S}(S_t, t)$, $B_t = C(S_t, t) - P_t S_t$ and $\beta_t = \sigma S_t \frac{\partial^2 C}{\partial S^2}(S_t, t)$. The hedging portfolio will give the option writer a return of more than the risk-free rate return, i.e., $d\Pi_t > 0$. So, we have:

$$\begin{cases}
\frac{\partial C}{\partial t}(S_t, t) + rS_t \frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}(S_t, t) + f'(0) S_t \left(\sigma S_t \frac{\partial^2 C}{\partial S_t^2}(S_t, t)\right)^2 < rC(S_t, t) \\
C(S_t, t) = K - S_t
\end{cases}$$

(3): On the free boundary: $S = S_f(t)$.

The boundary conditions on $S = S_f(t)$ are that $C(S_f(t), t) = (K - S_f(t))^+$ and its slope is continuous. So, we have

$$C(S_f(t), t) = (K - S_f(t))^+ \quad \text{and} \quad \frac{\partial C}{\partial S}(S_f(t), t) = -1. \tag{2.68}$$

Theorem 2.5.1. *(**American Options**) The replication cost for the option buyer and seller is characterized by this free boundary problem:*

***(1)**: First subregion, $S_f(t) < x < \infty$:*

$$C(x,t) > K - x, \qquad \frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} \pm f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 = rC. \tag{2.69}$$

***(2)**: Second subregion, $0 \le x < S_f(t)$:*

$$C(x,t) = K - x, \qquad \frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} \pm f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 < rC. \tag{2.70}$$

***(3)**: On the free boundary, $x = S_f(t)$:*

$$C(x,t) = (K - x)^+ \quad \textit{and} \quad \frac{\partial C}{\partial x}(x,t) = -1. \tag{2.71}$$

2.5.1 Numerical results

The numerical results is obtained by solving the equation for American options by finite difference method. The results are presented in Table 2.5; parameter values are $S_0 = 100$, $\sigma = 0.2$, $T = 1$, $r = 0$ with strike K and liquidity parameter $f'(0)$ varying as shown in the table. The first column gives the Black-Scholes values for the corresponding American options. When $f'(0) = 0$, the buyer's price equals the seller's price and the Black-Scholes price is the unique price for the option. When $f'(0) = 0.001$ and $K = 100$, the upper bound of American option price is 6.0972 and the lower bound is 6.0485. The quoted American option price in market with the liquidity risk should be in the interval [6.0485 6.0972].

	$f'(0) = 0.000$	$f'(0) = 0.0005$		$f'(0) = 0.001$		$f'(0) = 0.002$		$f'(0) = 0.005$	
K	BS Price	Seller	Buyer	Seller	Buyer	Seller	Buyer	Seller	Buyer
95	3.9968	4.0073	3.9861	4.0178	3.9753	4.0384	3.9534	4.0984	3.8840
100	6.0731	6.0853	6.0609	6.0972	6.0485	6.1209	6.0234	6.1894	5.9435
105	8.7225	8.7350	8.7098	8.7474	8.6971	8.7719	8.6711	8.8430	8.5892

Table 2.5: American option's replication costs with different Strikes and liquidity parameters.

2.6 Exotic options

With a similar approach, we can generalize our pricing method to price Exotic options. We will present our generalization of Barrier options and Asian options. Numerical results of Barrier options and Asian options will be provided. It is easy to generalize our pricing method to other exotic options, such as lookback options, roll-down options, and rainbow options.

2.6.1 Barrier options

In this section, we consider the case of a European style down and out call option, with payoff $(S-K)^+$ at expiration, where K is the strike price, provided S never reaches barrier B during the lifetime of the option. If S ever reaches B, the option becomes worthless. Our analysis can be easily extended to other barrier options.

Suppose that we are above the barrier, i.e., $S > B$ at time t. The next time step, being infinitesimal, will not take us to the barrier. We can apply our continuous hedging analysis in European options to show that the option seller's replication cost of the option $C(S,t)$ satisfies the equation

$$\frac{\partial C}{\partial t}(S,t) + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) + f'(0)S\left(\sigma S\frac{\partial^2 C}{\partial S^2}(S,t)\right)^2 = rC(S,t). \tag{2.72}$$

As usual, the final condition for (2.72) is

$$C(S,t) = (S-K)^+. \tag{2.73}$$

If S ever reaches B then the option becomes worthless; this condition translates into the mathematical condition that on $S = B$ the value of the option is zero:

$$C(B,t) = 0. \tag{2.74}$$

The replication cost for the option buyer and seller can be summarized by the following theorem:

Theorem 2.6.1. ***(Barrier Call Options)*** *Let $C(x,t)$ denote the option replication cost at time t under the assumption that $S_t = x$, then $C(S,t)$ satisfies the equation*

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} \pm f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 = rC. \tag{2.75}$$

As usual, the final condition for (2.75) is

$$C(x,t) = (x-K)^+. \tag{2.76}$$

If x ever reaches B, which is the lower barrier, then the option becomes worthless; on $x = B$ the value of the option is 0:

$$C(B,t) = 0. \tag{2.77}$$

The option price with different initial spot and liquidation parameters, i.e., $f'(0)$, are given in Table 2.6 below. Parameter values are $S_0 = 100$, $S_{down} = 80$, $\sigma = 0.2$, $r = 0$, $T = 1$, $K = 100$. When $f'(0) = 0.001$ and $K = 100$, the upper bound of the option price is 7.8940 and the lower bound is 7.8474, and the option price should be in the interval [7.8474 7.8940].

2.6.2 Asian options

In this section, we consider how to price continuous sampled average strike Asian options, whose payoff includes a time average of the underlying asset price. Like the standard

K	$f'(0) = 0.000$ BS Price	$f'(0) = 0.0005$ Seller	Buyer	$f'(0) = 0.001$ Seller	Buyer	$f'(0) = 0.002$ Seller	Buyer	$f'(0) = 0.005$ Seller	Buyer
90	13.243	13.250	13.237	13.256	13.230	13.268	13.214	13.302	13.201
95	10.338	10.348	10.329	10.357	10.318	10.375	10.294	10.427	10.267
100	7.8715	7.8829	7.8597	7.8940	7.8474	7.9156	7.8180	7.9773	7.7754
105	5.8569	5.8688	5.8447	5.8803	5.8318	5.9028	5.8001	5.9672	5.7539
110	4.2669	4.2782	4.2553	4.2891	4.2431	4.3104	4.2114	4.3715	4.1679

Table 2.6: Barrier option's replication costs with different Strikes and liquidity parameters.

argument, the stock price follows a geometric Brownian motion:

$$dS_t = uS_t dt + \sigma S_t dW_t, \quad 0 \leq t \leq T. \tag{2.78}$$

We define a new process as:

$$Z_t = \int_0^t S_u \, du. \tag{2.79}$$

The stochastic differential equation for $Z(t)$ is

$$dZ_t = S_t \, dt. \tag{2.80}$$

The payoff of the Asian option at expiration is

$$\left(K - \frac{1}{T} \int_0^t S_u \, du\right)^+, \tag{2.81}$$

where T is the expiration time, and K is the strike price. The value of the Asian option Y_t depends on S_t, Z_t and t. Thus, we can denote $Y_t = C(S_t, Z_t, t)$. Applying Ito Lemma, we have

$$dY_t = \left(\frac{\partial C}{\partial S}(S_t, Z_t, t) u S_t + \frac{\partial C}{\partial t}(S_t, Z_t, t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, Z_t, t)\sigma^2 {S_t}^2 + \frac{\partial C}{\partial Z}(S_t, Z_t, t) S_t \right) dt$$
$$+ \frac{\partial C}{\partial S}(S_t, Z_t, t)\sigma S_t dW_t. \tag{2.82}$$

Now, denote the stock position at time t as P_t, and $P_t = X(S_t, Z_t, t)$, where $X(S_t, Z_t, t)$ is twice continuously differentiable on $(0, \infty) \times (0, \infty) \times [0, T]$. By Ito's formula,

$$dP_t = \left(\frac{\partial X}{\partial S}(S_t, Z_t, t) u S_t + \frac{\partial X}{\partial t}(S_t, Z_t, t) + \frac{1}{2}\frac{\partial^2 X}{\partial S^2}(S_t, Z_t, t)\sigma^2 {S_t}^2 + \frac{\partial X}{\partial Z}(S_t, Z_t, t) S_t \right) dt$$
$$+ \frac{\partial X}{\partial S}(S_t, Z_t, t)\sigma S_t dW_t \tag{2.83}$$

$$(dP_t)^2 = \left(\frac{\partial X}{\partial S}(S_t, Z_t, t)\sigma S_t \right)^2 dt \tag{2.84}$$

$$(dP_t)^3 = 0. \tag{2.85}$$

Based on the supply curve function $S_t(x) = f(x)S_t$, we have the liquidity cost term,

$$\begin{aligned} dP_t\,[S_t(dP_t) - S_t] &= dP_t(f(dP_t) - 1)S_t \\ &= dP_t \left(f'(0)dP_t + \frac{f''(0)}{2}(dP_t)^2 \right) S_t \\ &= f'(0)(dP_t)^2 S_t. \end{aligned} \tag{2.86}$$

The option writer constructs a portfolio consisting of -1 option and a number P_t of the underlying asset and B_t units of the bank account. The value of this portfolio is

$$\Pi_t = P_t S_t + B_t - Y_t. \tag{2.87}$$

The change of the value of the portfolio in the time t to $t + dt$ is

$$
\begin{aligned}
d\Pi_t =& P_t dS_t + rB_t dt - dY_t - dP_t\,[S_t(dP_t) - S_t] \\
=& P_t dS_t + rB_t dt - \left(\frac{\partial C}{\partial S}(S_t, t)uS_t + \frac{\partial C}{\partial t}(S_t, t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, t)\sigma^2 {S_t}^2\right) dt \\
&+ \frac{\partial C}{\partial S}(S_t, t)\sigma S_t dW_t + rB_t dt - dP_t[S(t, dP_t) - S(t, 0)] \\
=& P_t\,(uS_t dt + \sigma S_t dW_t) - \frac{\partial C}{\partial S}(S_t, Z_t, t)\sigma S_t dW_t - f'(0)(dP_t)^2 S(t, 0) \\
&- \left(\frac{\partial C}{\partial S}(S_t, Z_t, t)uS_t + \frac{\partial C}{\partial t}(S_t, Z_t, t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, Z_t, t)\sigma^2 {S_t}^2 + \frac{\partial C}{\partial Z}(S_t, Z_t, t)S_t\right) dt \\
=& \left(P_t\sigma S_t - \frac{\partial C}{\partial S}(S_t, Z_t, t)\sigma S_t\right) dW_t + rB_t dt + \Bigg[P_t uS_t - \frac{\partial C}{\partial S}(S_t, Z_t, t)uS_t - \frac{\partial C}{\partial t}(S_t, Z_t, t) \\
&- \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, Z_t, t)\sigma^2 {S_t}^2 - \frac{\partial C}{\partial Z}(S_t, Z_t, t)S_t - f'(0)S_t\left(\frac{\partial X}{\partial S}(S_t, Z_t, t)\sigma S_t\right)^2\Bigg] dt. \quad (2.88)
\end{aligned}
$$

In order to fully hedge the option, the option writer needs to make $d\Pi_t = 0$. By making dB_t and dt terms to 0, we have $P_t = \frac{\partial C}{\partial S}(S_t, Z_t, t)$, $B_t = C(S_t, Z_t, t) - P_t S_t$ and $\frac{\partial X}{\partial S}(S_t, Z_t, t) = \frac{\partial^2 C}{\partial S^2}(S_t, Z_t, t)$. Substituting them into (2.88), we deduce from $d\Pi_t = 0$ that

$$
\begin{aligned}
&\frac{\partial C}{\partial t}(S_t, Z_t, t) + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_t, Z_t, t)\sigma^2 {S_t}^2 + \frac{\partial C}{\partial Z}(S_t, Z_t, t)S_t \quad (2.89) \\
&+ f'(0)S_t\left(\frac{\partial^2 C}{\partial S^2}(S_t, Z_t, t)\sigma S_t\right)^2 = rC(S_t, Z_t, t). \quad (2.90)
\end{aligned}
$$

Replacing S_t with dummy variable x, and Z_t by the dummy variable y, we obtain

$$
\begin{aligned}
&\frac{\partial C}{\partial t}(x, y, t) + rx\frac{\partial C}{\partial x}(x, y, t) + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(x, y, t)\sigma^2 x^2 + \frac{\partial C}{\partial y}(x, y, t)x \\
&+ f'(0)x\left(\sigma x\frac{\partial^2 C}{\partial x^2}(x, y, t)\right)^2 = rC(x, y, t). \quad (2.91)
\end{aligned}
$$

The boundary conditions for continuous average Asian put options are

$$
\begin{aligned}
&C(x, KT, t) = 0, \quad 0 \le t \le T, x \ge 0,\\
&C(x, y, T) = (K - \frac{y}{T})^+, \quad x \ge 0,\ 0 \le y \le KT,\\
&C(0, y, t) = (K - \frac{y}{T})^+, \quad 0 \le t \le T,\ 0 \le y \le KT,\\
&C(x_{max}, y, t) = (K - \frac{y + (T-t)x_{max}}{T})^+, \quad 0 \le t \le T,\ 0 \le y \le KT.
\end{aligned}
$$

For other kinds of Asian options, we have different boundary conditions and the PDE remains the same. For the option buyer's side, following the same analysis, it can be easily known that the replication cost is characterized by:

$$
\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + x\frac{\partial C}{\partial y} - f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 = rC. \tag{2.92}
$$

Theorem 2.6.2. *(**Asian Options**) Let $C(x, y, t)$ denote the option replication cost at time t under the assumption that $S_t = x$ and $Y_t = y$, such that $Y_t = \int_0^t S_u\, du$, we have*

$$
\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + x\frac{\partial C}{\partial y} \pm f'(0)x\left(\sigma x \frac{\partial^2 C}{\partial x^2}\right)^2 = rC, \tag{2.93}
$$

and the boundary conditions

$$
\begin{aligned}
&C(x, KT, t) = 0, \quad 0 \le t \le T, x \ge 0,\\
&C(x, y, T) = (K - \frac{y}{T})^+, \quad x \ge 0,\ 0 \le y \le KT,\\
&C(0, y, t) = (K - \frac{y}{T})^+, \quad 0 \le t \le T,\ 0 \le y \le KT,\\
&C(x_{max}, y, t) = (K - \frac{y + (T-t)x_{max}}{T})^+, \quad 0 \le t \le T,\ 0 \le y \le KT.
\end{aligned}
$$

K	$f'(0) = 0.000$ BS Price	$f'(0) = 0.0005$ Seller	Buyer	$f'(0) = 0.001$ Seller	Buyer	$f'(0) = 0.002$ Seller	Buyer	$f'(0) = 0.005$ Seller	Buyer
95	2.4967	2.5011	2.4922	2.5056	2.4876	2.5145	2.4786	2.5409	2.4510
100	4.7046	4.7100	4.6993	4.7153	4.6939	4.7259	4.6830	4.7573	4.6500

Table 2.7: Asian option's replication costs with different Strikes and liquidity parameters.

We provide some numerical results for Asian options. Parameter values are $S_0 = 100$, $r = 0$, $\sigma = 0.2$, $T = 1$ with varying strike K in Table 2.7. When $f'(0) = 0.001$ and $K = 100$, the upper bound of the option price is 4.7153 and the lower bound is 4.6939. The quoted Asian option price in the market with liquidity risk should be in the interval $[4.6939,\ 4.7153]$.

3 Option Pricing with Liquidity Risk in a Jump-diffusion Model

3.1 Introduction

In this chapter, we will investigate option pricing with liquidity risk in a jump-diffusion model. in Chapter 2, we showed the existence of a perfect hedging of vanilla and exotic options in a non-competitive market for small investors when stock price follows a geometric Brownian motion. However, empirical studies (Jorion (1988), Andersen et al. (2002) and Bates (2000)) suggest that there are jumps in the stock price. Jumps in the stock price is modeled by a jump-diffusion model. Option pricing in a jump-diffusion model was first considered in Merton (1976). Numerous pricing approaches have since been proposed for pricing derivatives in a jump-diffusion model: super hedging, mean variance hedging (Lim (2005)), and local risk minimization hedging (Follmer and Schweizer (1991)). Jumps in stock price bring jump risk, and it is known that liquidity risk and jump risk are not independent but correlated. Specifically, the liquidity risk for options becomes much more critical when there are jumps in the underlying security. For example, in a financial crisis, it is common that an underlying asset price exhibits jumps, leading investors in the market to change their position on the underlying asset quickly to hedge derivatives, which causes a significant liquidity problem. This motivates us to study the pricing and hedging of options in a jump-diffusion model with liquidity risk.

When the underlying stock price follows a jump-diffusion model, the market becomes incomplete, which makes perfect hedging impossible. Other pricing methods have been developed for hedging options. Local risk minimization has been proven to be an easily applicable pricing method to price options in incomplete markets. Options can be priced by the local risk minimization method for a jump-diffusion model in the continuous time setting (without liquidity risk), giving us a partial differential equation to characterize the hedging cost. It is natural to ask whether a modified partial differential equation can be derived to describe the local risk minimization hedging cost of options in a market with liquidity risk. It doesn't seem possible to derive such a partial differential equation due to the complexity introduced by liquidity risk. In order to value options with liquidity risk in a jump-diffusion model, we turn to a discrete-time model.

A jump-diffusion model could be approximated by a discrete-time process (Amin (1993)). Local risk minimization is easily applicable in discrete-time (see Coleman et al.

(2007)). In this chapter, we apply local risk minimization to price options with liquidity risk for a discrete-time Markov process. The discrete-time Markov process converges to a continuous jump-diffusion process as the time step goes to zero. By letting the length of the time intervals go to zero, the option price obtained from the discrete-time model approaches the option price in the jump-diffusion model. Therefore, the method we suggest is useful for pricing and hedging options in a jump-diffusion model with the presence of liquidity costs.

3.2 Local risk minimization in a jump-diffusion model

We consider a financial market that consists of one bank account and one stock. The interest rate is r and the bank account B_t is given by:

$$dB_t = rB_t dt, \qquad t \in [0,\ T]. \tag{3.1}$$

Without loss of generality, it is assumed that $r = 0$. The stock price is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t : t \geq 0\}$ generated by a one-dimensional Brownian motion W_t and a Poisson process N_t with intensity λ. The stock price S_t is modeled by a jump-diffusion process and follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t + (V_i - 1)S_t dN_t, \qquad t \in [0,\ T] \tag{3.2}$$

where σ is the volatility, μ is the drift term of the stock, and V_i is the jump size where

$$\mathbb{P}\{V_i = e^{q_j}\} = p_j \quad 1 \le j \le m \tag{3.3}$$

and

$$p_1 + p_1 + ... + p_m = 1.$$

The solution of the stochastic differential equation (3.2) is written as

$$S_t = S_0 \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right\} \prod_{i=1}^{N(t)} V_i. \tag{3.4}$$

Assume that an investor writes an option with maturity T. In order to hedge the option, the investor needs to construct a portfolio (P_t, B_t) to hedge both the diffusion and the jump risk, where P_t stands for the number of shares of the stock and B_t is the amount of the bank account. Let $C(t, S_t)$ denote the option price at time t when the stock price is S_t. Since the interest rate $r = 0$, the change in the portfolio value during time interval $[t, t + dt]$ will be

$$P_t dS_t = P_t \left[\mu S_t dt + \sigma S_t dW_t + (V - 1) S_t d N_t\right]. \tag{3.5}$$

The change of the option price will be

$$\begin{aligned} dC(t, S_t) = & \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S}\mu S_t dt + \frac{\partial C}{\partial S}\sigma S_t dW_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 dt \\ & + [C(t, V \times S_t) - C(t, S_t)] dN_t. \end{aligned} \tag{3.6}$$

In order to make the model simple, we assume $\mathbb{P}(V = e^{\log(1+q)} = 1+q) = 1$ and $E(V) = 1+q$. We can extend the above analysis to cases with multiple jumps. The market is incomplete in a jump-diffusion model, since there are multiple sources of randomness, a derivative cannot be fully hedged with the underlying stock and bank account. If we try to hedge the jump risk and make dN_t terms in both (3.4) and (3.6), we get

$$qS_tP_t = C(t,(1+q)S_t) - C(t,S_t), \tag{3.7}$$

which means

$$P_t = \frac{C(t,(1+q)S_t) - C(t,S_t)}{qS_t}.$$

On the other hand, if we try to hedge the diffusion risk and make dW_t terms in both (3.4) and (3.6), we get

$$\mu S_tP_t = \frac{\partial C}{\partial S}\mu S_t, \tag{3.8}$$

which leads to

$$P_t = \frac{\partial C}{\partial S}.$$

As expected, we cannot eliminate the jump risk and diffusion risk simultaneously. Therefore, we need to find a compromise in hedging the two risks.

When the two risks cannot be hedged completely and the hedging error is not zero, the local risk minimization approach could be used to minimize the variance of the hedging error caused by the diffusion risk and jump risk. This hedging strategy is a compromise between hedging the jump risk and hedging diffusion risk. In the local risk minimization approach, we make the expected hedging error equal zero, and minimize the variance of the hedging error. Let $M_t = N_t - \lambda t$, then M_t is a martingale. The change of the portfolio becomes

$$P_t dS_t = P_t \left[(\mu + \lambda q) S_t dt + \sigma S_t dW_t + qS_t d M_t \right]. \tag{3.9}$$

From Ito's formula of jump-diffusion processes, it can be shown

$$\begin{aligned} dC(t, S_t) =& \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} \mu S_t dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S_t^2 dt + \lambda \left[C(t, (1+q) S_t) - C(t, S_t) \right] dt \\ &+ \frac{\partial C}{\partial S} \sigma S_t dW_t + \left[C(t, (1+q) S_t) - C(t, S_t) \right] dM_t. \end{aligned} \tag{3.10}$$

The hedging error is defied as $P_t dS_t - dC(t, S_t)$. Notice that in order to make the expected hedging error equal zero, we must set the dt term of (3.9) and (3.10) to equal

$$\frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} \mu S_t dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S_t^2 dt + \lambda \left[C(t, (q+1) S_t) - C(t, S_t) \right] dt = P_t (\mu + \lambda q) S_t dt. \tag{3.11}$$

The variance of the hedging error is

$$\begin{aligned} \operatorname{Var}\left[P_t dS_t - dC(t, S_t) \right] =& \left\{ P_t q S_t - \left[C(t, (q+1) S_t) - C(t, S_t) \right] \right\}^2 \lambda dt, \\ &+ \left(P_t \sigma S_t - \frac{\partial C}{\partial S} \sigma S_t \right)^2 dt, \end{aligned} \tag{3.12}$$

which is minimized by

$$P_t = \frac{\partial C}{\partial S} \frac{\sigma^2}{\sigma^2 + \lambda q^2} + \frac{C(t, (q+1) S_t) - C(t, S_t)}{q S_t} \frac{\lambda q^2}{\sigma^2 + \lambda q^2}.$$

Then P_t represents the stock position of the local risk minimization hedging strategy. Substituting P_t into (3.11), we have the pricing equation for $C(t,S)$

$$\frac{\partial C}{\partial t} - q\left(\lambda\frac{\sigma^2 - qu}{\sigma^2 + \lambda q^2}\right)\frac{\partial C}{\partial S}S + \left(\lambda\frac{\sigma^2 - qu}{\sigma^2 + \lambda q^2}\right)[C(t,(q+1)S) - C(t,S_t)]$$
$$+ \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 = 0. \tag{3.13}$$

for $0 \le t < T$. Equation (3.13) characterizes the option's hedging cost under local minimization hedging strategy.

3.3 Approximate a jump-diffusion process with a discrete-time model

It is well known that a geometric Brownian motion can be approximated by a binomial model. If the jump size takes finite values, a jump-diffusion process could be approximated by a discrete-time process (Figure 3.1). We can approximate the jump-diffusion process (3.2) in the following way. For any $t \in [0,\ T]$, we have n stages over time horizon $[0,\ t]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = t$ with $\Delta t = \frac{t}{N}$. Given S_k the stock price at time t_k and time step Δt, there are $m+2$ possible values for S_{k+1} at time $k+1$:

$$S_{k+1} = \begin{cases} S_k e^{(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}}, & \text{if } S_k \text{ goes up;} \\ S_k e^{(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}, & \text{if } S_k \text{ goes down;} \\ e^{q_1} S_k, & \text{if } S_k \text{ jumps to } e^{q_1} S_k; \\ \ldots & \\ e^{q_m} S_k, & \text{if } S_k \text{ jumps to } e^{q_m} S_k. \end{cases}$$

The relation of S_{k+1} and S_k is $S_{k+1} = S_k \xi_{k+1}$, where

$$\xi_{k+1} = \begin{cases} e^{(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}}, & \text{with probability } \frac{1-\lambda\Delta t}{2}; \\ e^{(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}, & \text{with probability } \frac{1-\lambda\Delta tt}{2}; \\ e^{q_1}, & \text{with probability } p_1\lambda\Delta t; \\ \ldots & \\ e^{q_m}, & \text{with probability } p_m\lambda\Delta t. \end{cases}$$

Theorem 3.3.1. *As $N \to \infty$, $(S_k)_{k=0,1,\ldots,N}$ is convergent to the following jump-diffusion process in distribution*

$$S_t = S_0 \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\} \prod_{i=1}^{N(t)} V_i, \tag{3.14}$$

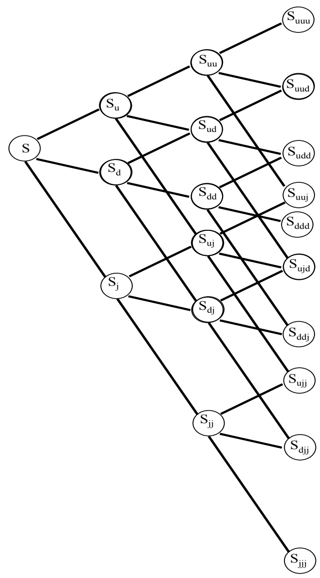

Figure 3.1: 3 periods trinomial model to approximate a jump-diffusion model

where $\mathbb{P}\{V_i = e^{q_j}\} = p_j$ *for* $1 \leq i \leq N(t)$ *and* $p_1 + p_1 + ... + p_m = 1$.

Proof. Notice that $S_{k+1} = S_k \xi_{k+1}$, then S_N could be written as

$$S_N = S_0 \xi_1 \xi_2 ... \xi_N. \tag{3.15}$$

Denoting $\eta_k = \ln(\xi_k)$ and $X_N = \ln\left(\frac{S_N}{S_0}\right)$, we have

$$S_N = S_0 e^{\eta_1 + \eta_2 + ... + \eta_N}. \tag{3.16}$$

For the discrete-time model, the log return X_N has the form

$$X_N = \eta_1 + \eta_2 + ... + \eta_N, \tag{3.17}$$

where $\eta_1, \eta_2, ...\eta_N$ are independent and identically distributed.

For the continuous time jump-diffusion model, the log return $X_t = \ln\left(\frac{S_t}{S_0}\right)$ is expressed as

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{i=1}^{N(t)} U_i, \tag{3.18}$$

where $U_i = \ln(V_i)$. The generating function of X_t is

$$\begin{aligned} G_{X_t}(\theta) &= E\left[e^{\theta X_t}\right] \\ &= E\left[e^{\theta[(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t+\sum_{i=1}^{N(t)} U_i]}\right] \\ &= e^{\theta(\mu-\frac{1}{2}\sigma^2)t} E\left[e^{\theta\sigma W_t}\right] E\left[e^{\theta \sum_{i=1}^{N(t)} U_i}\right]. \end{aligned} \tag{3.19}$$

From iterated conditional expectation, we have

$$\begin{aligned} E\left[e^{\theta \sum_{i=1}^{N(t)} U_i}\right] &= E\left\{E\left[e^{\theta \sum_{i=1}^{N(t)} U_i} | N(t)\right]\right\} \\ &= E\left[\left(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta}\right)^{N(t)}\right] \\ &= \sum_{k=1}^{\infty} \left(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta}\right)^k \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ &= \exp\left\{\lambda\left(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta} - 1\right)t\right\}. \end{aligned} \tag{3.20}$$

Also, we know

$$E\left[e^{\theta\sigma W_t}\right] = \exp\left(\frac{1}{2}\sigma^2\theta^2 t\right). \tag{3.21}$$

Together with (3.20) and (3.21), the moment generating function of $G_{X_t}(\theta)$ could be expressed as

$$G_{X_t}(\theta) = \exp\left\{\theta\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2\theta^2 t + \lambda\left(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta} - 1\right)t\right\}. \quad (3.22)$$

The moment generating function of X_N is $G_{X_N}(\theta) = E[e^{\theta X_N}]$ and it could be written as

$$\begin{aligned}
& G_{X_N}(\theta) \\
&= \left[G_{\eta_k}(\theta)\right]^N \\
&= \left\{\frac{1-\lambda\Delta t}{2}e^{\theta[(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}]} + \frac{1-\lambda\Delta t}{2}e^{\theta[(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}]} + p_1\lambda\Delta t e^{\theta q_1} + ... + p_m\lambda\Delta t e^{\theta q_m}\right\}^N \\
&= \left\{\frac{1-\lambda\Delta t}{2}\left[1 + (\mu - \frac{1}{2}\sigma^2)\Delta t\theta + \sigma\sqrt{\Delta t}\theta + \frac{1}{2}\sigma^2\Delta t\theta^2 + O(\Delta t)^{3/2}\right]\right. \\
&\quad \left. + \frac{1-\lambda\Delta t}{2}\left[1 + (\mu - \frac{1}{2}\sigma^2)\Delta t\theta - \sigma\sqrt{\Delta t}\theta + \frac{1}{2}\sigma^2\Delta t\theta^2 + O(\Delta t)\right] + p_1\lambda\Delta t e^{\theta q_1} + ... + p_m\lambda\Delta t e^{\theta q_m}\right\}^N \\
&= \left\{1 + (\mu - \frac{1}{2}\sigma^2)\theta\Delta t + \frac{1}{2}\sigma^2\theta^2\Delta t + \lambda(p_1 e^{\theta q_1} + p_2 e^{\theta q_2} + ... p_m e^{\theta q_m} - 1)\theta\Delta t + O(\Delta t)^{3/2}\right\}^N \\
&= \left\{1 + \left[\theta(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\theta^2 + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta} - 1)\right]\Delta t + O(\Delta t)^{3/2}\right\}^N.
\end{aligned}$$

We know $N = \frac{t}{\Delta t}$, and as $N \to \infty$ we have

$$\begin{aligned}
& \lim_{N\to\infty}\left\{1 + \left[\theta(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\theta^2 + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta} - 1)\right]\Delta t + O(\Delta t)^{3/2}\right\}^N \\
&= \lim_{\Delta t\to 0}\left\{1 + \left[\theta(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\theta^2 + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta} - 1)\right]\Delta t + O(\Delta t)^{3/2}\right\}^{\frac{t}{\Delta t}} \\
&= \exp\left\{\theta(\mu - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2\theta^2 t + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + ... + p_m e^{q_m\theta} - 1)t\right\},
\end{aligned}$$

which is exactly the generating function of X_t. We proved the moment generating function of X_N converges to the moment generating function of X_t. From the convergence in distribution theorem, we have X_N is convergent to X_t in distribution and $S_N = S_0 e^{X_N}$ converges to $S_t = S_0 e^{X_t}$ in distribution. □

3.4 Supply curve model and Local risk minimization considering liquidity risk

Liquidity risk is modeled by the supply curve model. A supply curve function $S_t(z)$ represents the stock price per share that the investor pays for an order size of z given the stock price is S_t at time t. A positive z represents a buying of stock and a negative x represents a selling of stock. In the discrete-time model, separable form of supply curve function has the following form:

$$S_k(z) = f(z) S_k. \tag{3.23}$$

Since we can approximate a jump-diffusion process by using a discrete-time model, pricing and hedging options in a jump-diffusion process could de addressed in the discrete-time model that approximates the jump-diffusion model. When the time interval $\frac{t}{N}$ goes to zero, the option price obtained from the discrete-time model will converge to the option price with liquidity risk in the jump-diffusion model.

Assume that we are going to hedge a European call option with maturity t_N and payoff $H_N = (S_N - K)^+$ that is $\mathcal{F}_{t_N}$ measurable. A trading strategy is given by two stochastic processes as $(x_k)_{k=0,1,...,N}$ and $(y_k)_{k=0,1,...,N}$, where x_k stands for the number of shares and y_k is the bank account amount at time t_k. Both x_k and y_k are $\mathcal{F}_{t_k}$ measurable for $0 \leq k \leq N$. The portfolio is the combination of the stock and the bank account given by the trading strategy. The value of portfolio at time t_k is given by:

$$V_k = x_k S_k + y_k. \tag{3.24}$$

The liquidity cost incurred from t_1 to t_k is defined by

$$L_k = \sum_{i=1}^{k-1} [f(x_{i+1} - x_i) - 1] S_{i+1}(x_{i+1} - x_i). \tag{3.25}$$

Since the interest rate r is assumed to be 0, the accumulated gain G_k is given by

$$G_k = \sum_{i=0}^{k-1} x_i(S_{i+1} - S_i) - \sum_{i=0}^{k-1} [f(x_{i+1} - x_i) - 1] S_{i+1}(x_{i+1} - x_i), \tag{3.26}$$

and $G_0 = 0$. Indeed, the accumulated gain in the market with liquidity cost equals the accumulated gain from the stock minus the liquidity cost of dynamic hedging.

The accumulated cost at time t_k is defined by

$$C_k = V_k - G_k. \tag{3.27}$$

A strategy is self-financing if the accumulated cost process $(C_k)_{k=0,1,...,N}$ is a constant over time. The self-financing strategy means:

$$C_{k+1} - C_k = (V_{k+1} - G_{k+1}) - (V_k - G_k) \tag{3.28}$$

$$= x_{k+1} S_{k+1} + y_{k+1} + [f(x_{i+1} - x_i) - 1] S_{i+1}(x_{i+1} - x_i) - x_k S_{k+1} - y_k = 0. \tag{3.29}$$

The value of a self-financing portfolio is given by $V_k = V_0 + G_k$ for $0 \le k \le N$. If the market is complete and perfect, such as in the binomial model, there is a self-financing strategy with $V_N = H_N$ a.s. But if the market is incomplete, for example, if there is a jump risk of the stock price and a contingent claim is non-attainable, the cost process $(C_k)_{k=1,2,\ldots,N}$ cannot be constant and a hedging strategy has to be chosen based on some optimality criterion.

We apply a local risk minimization hedging method to hedge options in the discrete-time model. First, we impose $V_N = H_N$. Local risk minimization requires the cost process $(C_k)_{k=1,2,\ldots,N}$ be a martingale and the variance of incremental cost process $(C_{k+1} - C_k)_{k=0,1,\ldots,N-1}$ be minimal. The traditional criterion for local risk minimization is the quadratic criterion, i.e.,

$$\min \mathrm{Var}[(C_{k+1} - C_k)|\mathcal{F}_k] \tag{3.30}$$

$$\text{Subject to: } E[C_{k+1} - C_k|\mathcal{F}_k] = 0. \tag{3.31}$$

It is equivalent to minimize

$$E[(C_{k+1} - C_k)^2|\mathcal{F}_k]. \tag{3.32}$$

In our discrete-time model, given the payoff at maturity of the option H_N, we set $V_N = x_N S_N + y_N = H_N$. By the local risk minimization method, the trading strategy (x^*_{N-1}, y^*_{N-1})

at t_{N-1} is

$$(x_{N-1}^*, y_{N-1}^*) = \arg \min_{x_{N-1}, y_{N-1}} E[(H_N - x_{N-1}S_N - y_{N-1})^2 | \mathcal{F}_k]. \tag{3.33}$$

For $0 \le k < N$, when we know (x_{k+1}^*, y_{k+1}^*), we want to minimize $E[(C_{k+1} - C_k)^2 | \mathcal{F}_k]$ to deduce (x_k^*, y_k^*). It can be done by minimizing the following optimization problem:

$$(x_k^*, y_k^*) = \arg \min_{x_k, y_k} E[(x_{k+1}^* S_{k+1} + y_{k+1}^* + [f(x_{k+1}^* - x_k) - 1] S_{k+1}(x_{k+1}^* - x_k) - x_k S_{k+1} - y_k)^2 | \mathcal{F}_k]. \tag{3.34}$$

By the backward deduction, accordingly we can have (x_{N-1}^*, y_{N-1}^*), (x_{N-2}^*, y_{N-2}^*)...,(x_1^*, y_1^*), (x_0^*, y_0^*). Then the initial option price at time t_0 is determined is $x_0^* S_0 + y_0^*$, and (x_{N-1}^*, y_{N-1}^*), (x_{N-2}^*, y_{N-2}^*)...,(x_1^*, y_1^*), (x_0^*, y_0^*) provide the local risk minimization hedging strategies. As N goes to infinity, the discrete-time model is convergent to the jump-diffusion model. The option price and hedging strategy obtained from the discrete time model give a good approximation of the corresponding price and hedging strategy in the jump-diffusion model.

The discrete-time model we just presented can be viewed as an extension of the classic binomial model. When the liquidity parameter is 0 (the supply curve function is flat everywhere), our approach coincides with the discrete-time model of a jump-diffusion process with local risk minimization hedging. Also, when the jump parameter $\lambda = 0$ (there are no jumps), our model is reduced to the binomial model with liquidity costs, which is a discrete-time version of a continuous perfect replication model. It is obvious that our model becomes the classic binomial model when both parameters are zero.

3.5 Numerical results

In this section, we present compare results of numerical studies of three hedging strategies: delta hedging, traditional local risk minimization hedging (local risk minimization without liquidity risk), and modified local risk minimization hedging (local risk minimization considering liquidity risk). First we describe the discrete model used to approximate the jump-diffusion model.

The jump-diffusion model we are going to approximate is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t + (V_i - 1)S_t dN_t, \qquad t \in [0,\ T] \tag{3.35}$$

where $\sigma = 0.2$, $\mu = 0.2$, N_t is a Poisson process with intensity $\lambda_1 + \lambda_2$ and V_i is the jump size with

$$\mathbb{P}\{V_i = 0.9\} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \text{ and } \mathbb{P}\{V_i = 1.12\} = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

The discrete-time model used to approximate the jump-diffusion model has N periods with time interval $\Delta t = \frac{T}{N}$. If the stock price is S_k at period k in the discrete-time model, then the stock price at period $k+1$ has four scenarios: goes up, goes down, jumps down, and jumps up. The probability distribution for S_{k+1} are:

$$S_{k+1} = \begin{cases} S_k(1 + \mu\Delta t + \sigma\sqrt{\Delta t}), & \text{with probability } \frac{1-\lambda_1\Delta t-\lambda_2\Delta t}{2}; \\ S_k(1 + \mu\Delta t - \sigma\sqrt{\Delta t}), & \text{with probability } \frac{1-\lambda_1\Delta t-\lambda_2\Delta t}{2}; \\ 0.9S_k, & \text{with probability } \lambda_1\Delta t; \\ 1.12S_k, & \text{with probability } \lambda_2\Delta t. \end{cases}$$

As the time interval $\Delta t \to 0$, the discrete-time model convergences to the jump-diffusion model. We assume the supply curve function $f(\cdot)$ is linear and has the following form:

$$S_k(z) = (1 + \alpha z)S_k, \tag{3.36}$$

where α is nonnegative and represents the liquidity parameter. The hedging error is defined as:

$$x_N S_N + y_N - H_N.$$

Table 3.1 presents European call option prices with different Strikes and volatilities. The parameter values are: $S_0 = 100$, $T = 1$, $r = 0$, $K = 100$, $\alpha = 0.1$, $\lambda_1 = 1$, $\lambda_2 = 1$ and $N = 50$. Table 3.2 shows the option prices with different λ_1 and λ_2. Table 3.3 and Table 3.4 present results of three hedging strategies to hedge European call options and with varying λ_1, λ_2, σ and T. *Delta* refers to the Delta hedging; *LRM* refers to the traditional local risk minimization hedging, and *MLRM* refers to the modified local risk minimization hedging. *Cost* refers to the mean cost if we need to make the hedging error having zero

	Strike								
Volatility	*95*	*96*	*97*	*98*	*99*	*100*	*101*	*102*	*103*
0.10	10.8885	10.3317	9.7946	9.2755	8.7754	8.2944	7.8312	7.3869	6.9605
0.15	12.2087	11.6831	11.1751	10.6824	10.2043	9.7432	9.2979	8.8699	8.4532
0.20	13.7460	13.2437	12.7540	12.2798	11.8187	11.3702	10.9373	10.5153	10.1065
0.25	15.3913	14.9088	14.4325	13.9696	13.5198	13.0828	12.6587	12.2483	11.8482
0.30	17.0910	16.6219	16.1636	15.7159	15.2788	14.8515	14.4344	14.0270	13.6295

Table 3.1: Option prices with different Strikes and volatility

expectation, *Std* is the standard deviation of hedging error, and *Liq Cost* stands for the mean liquidity cost. Figure 3.3 presents the Monte Carlo simulation of the hedging error of *MLRM*. When we compare the hedging cost of the three different hedging methods, the mean hedging cost of the modified hedging strategy is less than those of the Delta hedging strategy and the traditional local risk minimization hedging strategy. Also, compared with Delta hedging and traditional local risk minimization, the hedging strategy under the modified local risk minimization reduces the standard deviation of the hedging error significantly. We thus conclude that among the three hedging strategies, our modified local risk minimization method outperforms the other two hedging methods.

λ_1 \ λ_2	0	0.25	0.50	0.75	1.00
0	9.5957	9.6390	9.6990	9.7583	9.8134
0.25	9.9215	10.0109	10.1081	10.1981	10.2795
0.50	10.2177	10.3445	10.4710	10.5861	10.6897
0.75	10.4771	10.6361	10.7890	10.9266	11.0504
1.00	10.7055	10.8932	11.0694	11.2279	11.3702

Table 3.2: Option prices with different values of λ_1 and λ_2

		Delta			*LRM*			*MLRM*		
λ_1	λ_2	*Cost*	*Std*	*Liq Cost*	*Cost*	*Std*	*Liq Cost*	*Cost*	*Std*	*Liq Cost*
0.0	0.0	9.8149	1.4773	1.9347	9.7508	1.4291	1.8706	9.5957	0	1.5119
0.5	0.5	11.104	2.6048	2.2733	10.7442	1.9669	1.8768	10.4719	1.3408	1.4221
1.0	1.0	11.864	2.3914	1.9873	11.7065	2.2253	1.9024	11.3645	1.6480	1.4713

Table 3.3: Comparison of results of three hedging strategies when $\sigma = 0.2$ and $T = 1$

		Delta			*LRM*			*MLRM*		
λ_1	λ_2	*Cost*	*Std*	*Liq Cost*	*Cost*	*Std*	*Liq Cost*	*Cost*	*Std*	*Liq Cost*
0.0	0.0	14.0274	1.5385	2.1693	14.0610	1.5608	2.2029	13.6866	0	1.7411
0.5	0.5	14.8156	2.1349	2.3289	14.6390	1.7918	2.1548	14.2763	0.8923	1.6970
1.0	1.0	15.5659	2.6193	2.4833	15.2275	1.9397	2.1357	14.8698	1.1849	1.7019

Table 3.4: Comparison of results of three hedging strategies when $\sigma = 0.3$ and $T = 1$

4 Utility Indifference Pricing for Large Investors

4.1 Introduction

We considered option pricing with liquidity risk for small investors in Chapters 2 and 3. For large investors, liquidity risk is extremely important and cannot be neglected. Large investors will face liquidity costs when trying to trade especially fast. Rogers and Singh (2010), and Forsyth (2011) assumed that the effect of liquidity costs is dependent on the speed of trades. In optimal liquidation of a large position of assets (see Almgren and Chriss (2001), and Almgren (2003)), the liquidity cost is also modelled through the trading speed. Generally, it is assumed that the liquidity cost depends on the rate of change of holding, and that the faster the large investors trade, the more liquidity costs they will incur.

Liquidity risk adds liquidity cost for options hedging for large investors. However, their market power to influence security price evolution could be a great advantage. The price impact could be regarded as feedback effects of large investors. Many reserchers have studited the feedback effects of large investors. In Frey and Stremme (1997), nonlinear partial differential equations are derived for the replication prices of path-independent European contingent claims.. Also, see Jarrow (1994), Frey (1998), Platen and Schweizer (1998), and Bank and Baum (2004). These papers assume trading actions have a lasting effect on the stock price evolution. The feedback effects of large investors are often considered in optimal liquidation of a large position of assets. For the literature on optimal liquidation in which the aim is to unwind an initial position by some fixed time horizon, we refer to Almgren and Chriss (2001), Almgren (2003), and Forsyth (2011). These papers try to liquidate a given initial position optimally by some fixed time. Longstaff (2001) considered the optimal portfolio choices in an illiquid market, where the trading strategies were assumed to be of bounded variation. The paper by Avellaneda et al. (2003) discussed stock pinning on option expiration date and the price impact of delta-hedging. In a non-competitive market, the large investor is defined as the investor who can influence or manipulate an underlying asset's price. Therefore, we assume that the drift term of the underlying asset's price depends on the large investor's trading speed.

In this chapter, we investigate the option pricing and hedging problem for a large investor considering both liquidity risk and feedback effects. Specifically, we assume illiquidity will pose some kind of nonlinear transaction cost on trading and a trading action will have a lasting impact on the stock price evolution. An investor will face costs in trying to trade rapidly. Thus, the effect of illiquidity costs depends on the rate of the change of holding, rather than the size of the change of holding. We use the utility based approach to price European options in a market with liquidity risk. Utility indifference pricing has been proven to be a powerful method in pricing options in markets with frictions, such as markets with transaction costs in Hodges and Neuberger (1989) and Davis et al. (1993), or markets with non-traded assets in Henderson (2002). We apply the utility indifference approach to price European options for large investors in non-competitive markets. We study the large investor's utility maximization problems with writing options and without writing options, and derive two Hamilton-Jacobi-Bellman equations to characterize the value functions for two optimal control problems. The option price is defined as the difference between the initial wealth of two utility maximization problems achieving the same expected utility. We use viscosity solutions to characterize HJB equations and prove the existence and uniqueness of solutions of the HJB equations. An example incorporating liquidity risk and feedback effects is presented to illustrate our model.

4.2 The Model

4.2.1 Permanent price impact modelling

We consider a financial market that consists of one bank account and one stock on a given probability space $(\Omega, \mathcal{F}, \mathrm{F}, P)$ where $\mathrm{F} = \{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by one-dimensional Brownian motion W_t. The interest rate is r and the bank account B_t is given by

$$dB_t = rB_t dt, \qquad t \in [0,\ T]. \tag{4.1}$$

We define the set of trading strategies to be the set of all $\mathcal{F}_t$ adapted processes with left continuous paths that have right limits. We let P_t be the number of shares of stock held at time t. We shall assume that P_t to be a finite-variation process, where $P_t = \int_0^t v_\xi \, d\xi$ and v_ξ is uniformly bounded by $M < \infty$. Trading speed can be defined by

$$v_t = \frac{dP_t}{dt}.$$

We restrict the set of trading strategies available to the investor with one condition: that the changes in the number of shares of stock held over any time interval never exceed M-multiple of the length of the time interval. We note that M might be determined by market conditions such as the daily trading volume of the asset. We also assume that a trading strategy is allowed if it keeps the wealth (mark-to-market value) bounded below, which ensures that an investor cannot take advantage of certain pathological varieties of arbitrage, such as doubling strategies. We denote by $\{\Gamma = v_t : 0 \leq t \leq T\}$ the set of admissible trading strategies available to the investor.

The feedback effects can be modelled by imposing a function of trading speed into the drift term of the stock price. Then, with trading speed v_t, the stock price evolves in the following way

$$dS_t = (\mu + g(v_t))S_t dt + \sigma S_t dW_t, \qquad t \in [0,\ T] \tag{4.2}$$

where $g(\cdot)$ is the function describing feedback effects and it is a nondecreasing function with $g(0) = 0$. The symbol $v_t > 0$ means the large investor is buying, and this buying action will drive the stock price up. So, the drift term will increase and $g(v_t) > 0$. The same idea applies to the case when $v_t < 0$.

Another approach to modelling the permanent price impact is by making the expected returns dependent on the stock position of such a large investor (Kraft and Kühn (2011)):

$$dS_t = (\mu + \mu_1\theta_t)S_t dt + \sigma S_t dW_t, \qquad t \in [0, T] \tag{4.3}$$

where μ_1 is the parameter to characterize feedback effects and θ_t represents the value of the stock position.

This model falls into the class of models suggested by Cuoco and Cvitanić (1998) who studied stock dynamics of

$$dS_t = \mu(\theta_t)S_t dt + \sigma(\theta_t)S_t dW_t, \qquad t \in [0, T] \tag{4.4}$$

where μ and σ are functions of θ_t. In those two models, the large investor's position will pose lasting feedback effects on the stock price evolution, even thought the large investor may not change the stock position in the future.

4.2.2 Liquidity risk modelling

The non-competitive market provides different prices for buying and selling stock, depending on how many shares a large investor wants to trade, or how rapidly the investor wants to change the position. Let $S(t, v_t, \omega)$ be the stock price per share at time $t \in [0, T]$ that an investor pays/receives for a trading speed $v_t \in R$. The actual execution price of the stock to be paid/received is different from the price initially quoted. In practice, if an investor wants to change the holding with speed v_t the actual traded price $S(t, v_t, \omega)$ will not be equal to the market price S_t due to the effect of illiquidity. More specifically, when $v_t > 0$, the stock is purchased and the buying price will be greater than S_t. When $v_t < 0$, the stock is sold and the selling price will be less than S_t. We assume the (stochastic) traded price of stock is given by

$$S(t, v_t, \omega) = f(v_t)S_t, \quad -M \leq v_t \leq M \tag{4.5}$$

where $f(\cdot)$ is a positive and nondecreasing function with $f(0) = 1$. We assume $S(t, v_t, \omega)$ increases as v_t increases, which is consistent with intuition. The faster the buying speed, the higher the average paid price per share. The quicker the selling speed, the lower the average received price per share of stock.

4.2.3 No arbitrage under large investor model

In a non-competitive market, the large investor will incur liquidity cost during the trading, and at the same time, the trading action will place a feedback effects on stock price evolution. We consider the model combining liquidity risk and feedback effects as large investor model. Before we apply a utility indifference pricing method to price European options, we need to verify that the large investor model is arbitrage free. Assume that at time 0, the stock price is S_0 and a large investor holds B_0 units of bank account and P_0 shares of stock. The large investor tries to make an arbitrage at time T. In our large investor model, if we can prove that in order to have $P_T S_T + B_T \geq 0 \ \ \mathbb{P} - a.s.$ at time T, where S_T is the stock price at time T, P_T is the stock position and B_T is the unit of bank account, we must have $P_0 S_0 + B_0 \geq 0$ at time 0, then we can conclude the large investor model is arbitrage free.

We know P_t is the stock position and S_t is the stock price. Denote the portfolio value by Y_t and is defined as

$$Y_t = P_t S_t + B_t.$$

Let us first investigate the evolution of the portfolio's value. If the trading speed between $[t, \ t+h]$ is v, the the stock position at time $t+h$ will be $P_{t+h} = P_t + vh$. By the principle of a self-financing condition, the stock purchase must be financed by the sale of the bank account. Therefore, the bank account at time $t+h$ will be $B_{t+h} = B_t(1+rh) - f(v)S_t vh$. The portfolio value at time $t+h$ will be:

$$Y_{t+h} = P_{t+h} S_{t+h} + B_{t+h}.$$

The increase of the the portfolio's value is

$$Y_{t+h} - Y_t = (P_{t+h}S_{t+h} + B_{t+h}) - (P_tS_t + B_t) \tag{4.6}$$

$$= (P_t + vh)(S_t + S_{t+h} - S_t) + \left[B_t(1 + rh) - f(v)S_t vh\right] - P_tS_t - B_t \tag{4.7}$$

$$= vS_t h + P_t(S_{t+h} - S_t) + rB_t h - f(v_t)S_t v_t h \tag{4.8}$$

$$= P_t(S_{t+h} - S_t) + rB_t h + [1 - f(v_t)]S_t v_t h. \tag{4.9}$$

In the large investor model, when the trading strategy is self-financing, the portfolio value Y_t satisfies

$$dY_t = P_t dS_t + rB_t dt + [1 - f(v_t)]S_t v_t dt. \tag{4.10}$$

Under self-financing condition, the dynamics of (P_t, S_t, B_t) are

$$dP_t = v_t dt \tag{4.11}$$

$$dS_t = (\mu + g(v_t))S_t dt + \sigma S_t dW_t \tag{4.12}$$

$$dB_t = rB_t dt - f(v_t)S_t v_t dt. \tag{4.13}$$

Theorem 4.2.1. ***(No arbitrage in the large trader model)*** *When S_t is given as $dS_t = (\mu + g(v_t))S_t dt + \sigma S_t dW_t$ and Y_t is given as $dY_t = P_t dS_t + rB_t dt + [1 - f(v_t)]S_t v_t dt$, the model is arbitrage-free.*

Proof. When S_t is given as

$$dS_t = (\mu + g(v_t))S_t dt + \sigma S_t dW_t$$

and Y_t is given by

$$dY_t = P_t dS_t + rB_t dt + [1 - f(v_t)]S_t v_t dt,$$

the evolution of the discounted stock price and portfolio's value are

$$d(e^{-rt}S_t) = (\mu + g(v_t) - r)e^{-rt}S_t dt + \sigma e^{-rt}S_t dd W_t,$$

and

$$d(e^{-rt}Y_t) = P_t d(e^{-rt}S_t) + [1 - f(v_t)]e^{-rt}S_t v_t dt.$$

Let v_t be the trading speed corresponding to an admissible trading strategy (P_t, B_t). Since $g(v_t)$ is bounded, it is obvious that

$$\mathrm{E}\left[\exp\left(\int_0^T \left(\frac{\mu + g(v_t) - r}{\sigma}\right)^2 dt\right)\right] < \infty,$$

therefore the Novikov condition is satisfied. By Girsanov's theorem, there exists an equivalent probability measure Q under which the process

$$\hat{W}_t = W_t + \int_0^t \frac{\mu + g(v_s) - r}{\sigma} dt, \quad t \in [0, T]$$

is a Brownian motion. The Radon-Nikodym derivative reads

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \frac{\mu + g(v_t) - r}{\sigma} dW_t - \frac{1}{2}\int_0^T \left(\frac{\mu + g(v_t) - r}{\sigma}\right)^2 dt\right).$$

Under the probability measure Q, we have

$$d(e^{-rt}S_t) = \sigma e^{-rt}S_t d\hat{W}_t$$

and

$$d(e^{-rt}Y_t) = P_t\sigma e^{-rt}S_t d\hat{W}_t + [1 - f(v_t)]e^{-rt}S_t v_t dt.$$

We therefore further have:

$$e^{-rT}Y_T = Y_0 + \int_0^T P_t\sigma e^{-rt}S_t d\hat{W}_t + \int_0^T [1 - f(v_t)]e^{-rt}S_t v_t dt.$$

Since $f(\cdot)$ is a nondecreasing function with $f(0) = 1$, we have

$$[1 - f(v_t)]e^{-rt}S_t v_t \leq 0.$$

Suppose there exits an arbitrage strategy with $Y_0 \leq 0$ and

$$P(Y_T \geq 0) = 1 \text{ and } P(Y_T > 0) > 0 \tag{4.14}$$

then

$$P(Y_T < 0) = 0.$$

By the equivalence of P and Q, we have

$$P^Q(Y_T < 0) = 0. \tag{4.15}$$

We also have

$$\begin{aligned}
\mathrm{E}^Q\left[\int_0^T (P_t\sigma e^{-rt}S_t)^2\, dt\right] &\leq \mathrm{E}^Q\left[\int_0^T \sigma^2 \sup\{P_t^2\}\, (e^{-rt}S_t)^2\, dt\right] \\
&\leq \sigma^2 \sup\{P_t^2\} \int_0^T \mathrm{E}^Q\left[(e^{-rt}S_t)^2\right] dt \\
&= \sup\{P_t^2\}S_0^2(e^{\sigma^2 T} - 1).
\end{aligned}$$

and

$$\mathrm{E}^Q\left[\int_0^T \left|[1-f(v_t)]e^{-rt}S_t v_t\right| dt\right] \leq \mathrm{E}^Q\left[\int_0^T M \times \sup|[1-f(v_t)]v_t|e^{-rt}S_t dt\right]$$
$$\leq M \times \sup|1-f(v_t)|\left[\int_0^T \mathrm{E}^Q(e^{-rt}S_t)dt\right]$$
$$= M \times \sup\{|1-f(v_t)|\}S_0 T$$

From the above conditions, we know

$$\int_0^T P_t \sigma e^{-rt} S_t d\hat{W}_t$$

is a martingale and

$$\int_0^T [1-f(v_t)]e^{-rt}S_t v_t dt$$

is well defined. Taking the expectation of $e^{-rT}Y_T$, from

$$[1-f(v_t)]e^{-rt}S_t v_t \leq 0,$$

we then have

$$\mathrm{E}^Q\left[e^{-rT}Y_T\right] = Y_0 + \mathrm{E}^Q\left[\int_0^T [1-f(v_t)]e^{-rt}S_t v_t dt\right] \leq Y_0 \leq 0 \tag{4.16}$$

From (4.15) and (4.16), we have

$$P^Q(Y_T > 0) = 0.$$

By the equivalence of P and Q, it implies

$$P(Y_T > 0) = 0,$$

which contradicts (4.14). Therefore, we can conclude that the model is arbitrage-free. □

4.3 Utility indifference price for European options

4.3.1 An example of utility indifference pricing

The idea of applying the utility maximization approach to pricing European options is as follows. The utility indifference price for a European option is the price at which the large investor's utility is indifferent between paying nothing at time T, and receiving v to pay the option's payoff $C(S_T)$ at time T.

Given a utility function U and an option C_T with known payoffs at some terminal time T, then we let the function $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$V(x, k) = \sup_{X_T \in \mathcal{A}(x)} \mathrm{E}\left[U\left(X_T + kC_T\right)\right]$$

where x is the initial endowment, $\mathcal{A}(x)$ is the set of all self-financing portfolios at time T starting with endowment x, and k is the number of options to be sold. The indifference price $v(k)$ for k units of C_T is the solution of $V(x + v(k), k) = V(x, 0)$.

Consider a market with a risk free asset B with $B_0 = 100$ and $B_T = 110$, and a risky asset S with $S_0 = 100$ and $S_T \in \{90, 110, 130\}$ each with probability 1/3. Let the utility function be given by $U(x) = 1 - \exp(-x/10)$. To find the indifference price for a single

European call option with strike 110, first calculate $V(x,0)$.

$$V(x,0) = \max_{\alpha B_0+\beta S_0=x} \mathrm{E}[1-\exp(-.1\times(\alpha B_T+\beta S_T))] \tag{4.17}$$

$$= \max_{\beta}\left\{1-\frac{1}{3}\left[\exp\left(-\frac{1.10x-20\beta}{10}\right)+\exp\left(-\frac{1.10x}{10}\right)+\exp\left(-\frac{1.10x+20\beta}{10}\right)\right]\right\}$$

which is maximized when $\beta = 0$, therefore $V(x,0) = 1-\exp\left(-\frac{1.10x}{10}\right)$. To find the indifference bid price, we now need to calculate $V(x+v(1),1)$.

$$\begin{aligned} V(x+v(1),1) &= \max_{\alpha B_0+\beta S_0=x-v(1)} \mathrm{E}[1-\exp(-.1\times(\alpha B_T+\beta S_T-C_T))] \\ &= \max_{\beta}\left[1-\frac{1}{3}\exp\left(-\frac{1.10(x-v(1))-20\beta}{10}\right)-\frac{1}{3}\exp\left(-\frac{1.10(x-v(1))}{10}\right)\right. \\ &\quad \left.-\frac{1}{3}\exp\left(-\frac{1.10(x-v(1))+20\beta-20}{10}\right)\right] \end{aligned} \tag{4.18}$$

which is maximized when $\beta = -\frac{1}{2}$, therefore,

$$V(x+v(1),1) = 1-\frac{1}{3}\exp(-1.10x/10)\exp(1.10v(1)/10)\left[1+2\exp(-1)\right].$$

Therefore, $V(x,0) = V(x+v(1),1)$ when $v(1) = \frac{10}{1.1}\log\left(\frac{3}{1+2\exp(-1)}\right) \approx 4.97$.

4.3.2 Utility indifference pricing for European options

Let us consider how to apply the utility indifference method to pricing European options in a large investor model. Assume that at time t, the stock price is S and a large investor holds B units of bank account and P shares of stock. The large investor with concave utility function U tries to maximize the expected utility at time T. The large investor uses a trading strategy (P_t, B_t) which is denoted by π, by choosing trading speed v_t, to adjust the stock position. When the investor starts in the state (t, S, P, B) and uses the trading strategy π, the value of the portfolio dynamically traded by the large investor at time T is Z_T^π . The portfolio Z_T^π consists of B_T^π units of bank account and P_T^π shares of stock

$$Z_T^\pi = P_T^\pi S_T^\pi + B_T^\pi.$$

The maximum utility the large investor can achieve is given by

$$V(t, P, S, B) = \max_\pi \mathrm{E}_t \left[U \left(Z_T^\pi \right) \right],$$

where E_t denotes the expectation operator conditional on the information at time t .

We consider the expected utility maximization of final wealth when the price for n units of options are sold with price $p(n)$ at time t. The large investor's bank account B_t will be increased to $B + p(n)$. When the investor starts in state $(t, S, P, B + p(n))$ and uses the trading strategy π, the value of the portfolio dynamically traded by the large investor at time T is Z_T^π . The portfolio Z_T^π consists of B_T^π units of bank account and P_T^π shares of stock.

$$Z_T^\pi = P_T^\pi S_T^\pi + B_T^\pi$$

The maximum utility the large investor can obtain is

$$\hat{V}(t, P, S, B + n \times p_w) = \max_\pi \mathrm{E}_t \left[U \left(Z_T^\pi - nC(S_T) \right) \right],$$

where $C(S_T)$ represents the European option's payoff at maturity T and n is the number of options. When the large investor has (P, B) at time t, the utility indifference price of option $p(n)$ is defined as the value at which the maximum utility that the large investor can achieve, is no different than the maximum utility that the large investor can achieve without selling options:

$$\hat{V}(t, P, S, B + p(n)) = V(t, P, S, B).$$

We have the dynamic of (P_t, S_t, B_t) under self-financing conditions:

$$dP_t = v_t dt, \tag{4.19}$$

$$dS_t = (\mu + g(v_t))S_t dt + \sigma S_t dW_t, \tag{4.20}$$

$$dB_t = rB_t dt - f(v_t)S_t v_t dt. \tag{4.21}$$

The control variable is the trading speed v_t. From the definitions of value functions and the dynamic programming principle, the value function $\hat{V}(t, P, S, B)$ and $V(t, P, S, B)$ satisfies the following HJB equation:

$$0 = \max_{v \in K} \left\{ v\frac{\partial V}{\partial P} - f(v)S v\frac{\partial V}{\partial B} + (\mu + g(v_t))S\frac{\partial V}{\partial S} \right\} + \frac{\partial V}{\partial t} + rB\frac{\partial V}{\partial B} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}. \tag{4.22}$$

The to HJB equations have different terminal conditions. For $\hat{V}(t, P, S, B)$, the terminal condition is

$$\hat{V}(T, P, S, B) = U\left(S \times P + B - n \times C(S)\right).$$

And the terminal condition for $V(t, P, S, B)$ is

$$V(T, P, S, B) = U(S \times P + B).$$

4.4 Existence and uniqueness of the solution of the HJB equation

In order to study existence and uniqueness of the solution for (4.22), we use the notion of viscosity solutions, introduced by Crandal and Lions. For a general view of the theory, we refer to the *user's guide* by Crandall, Ishii, and Lions Crandall et al. (1992). In this chapter, we consider a nonlinear second-order PDE of the form

$$-\frac{\partial W(t,x)}{\partial t} + H(x, D_x W(t,x), D_x^2 W(t,x)) = 0 \tag{4.23}$$

where$(t, x) \in [0,\ T] \times Q$ and $H(x, p, M)$ is a continuous mapping from $Q \times R^N \times\ S_N \rightarrow R$, and where S_N denotes the set of symmetric $N \times N$ matrices.

Definition 4.4.1. *(**Touzi (2012)**) W is a continuous function from* $[0,\ T] \times Q\ \rightarrow R$

1. *We say W is a viscosity subsolution of* (4.23) *if*

$$-\frac{\partial W(t_0,x_0)}{\partial t} + H(x_0, D_x\phi(t_0,x_0), D_x^2\phi(t_0,x_0)) \leq 0; \tag{4.24}$$

for all pair $(t_0, x_0, \phi) \in [0,\ T] \times Q \times C^2([0,\ T] \times Q)$ *such that* (t_0, x_0) *is a maximizer of the difference* $(W - \phi)$ *on* $[0,\ T] \times Q$*.*

2. We say W is a viscosity supersolution of (4.23) *if*

$$-\frac{\partial W(t_0, x_0)}{\partial t} + H(x_0, D_x\phi(t_0, x_0), D_x^2\phi(t_0, x_0)) \geq 0. \tag{4.25}$$

for all pair $(t_0, x_0, \phi) \in [0,\ T] \times Q \times C^2([0,\ T] \times Q)$ *such that* (t_0, x_0) *is a minimizer of the difference* $(W - \phi)$ *on* $[0,\ T] \times Q$.

3. We say W is a viscosity solution of (4.23) *if it is both a viscosity supersolution and subsolution of* (4.23).

For the PDE in this chapter, $H(x, p, X)$ has the following specific form:

$$\begin{aligned} H(x, p, M) = &-\max_{v \in K} \left\{ [v, (\mu + g(v))x_2, -vf(v)x_2 - rx_3] \cdot [p_1, p_2, p_3]^T \right\} \\ &- \frac{\sigma^2}{2}(0, x_2, 0)M(0, x_2, 0)^T \end{aligned} \tag{4.26}$$

where $x = (x_1, x_2, x_3) \in Q$, $p = (p_1, p_2, p_3) \in R^3$, and $M \in S_3$.

Theorem 4.4.1. *The value function* $W(t, P, S, B)$ *is a viscosity solution of*

$$-\frac{\partial W}{\partial t} - \max_{v \in K} \left\{ v\frac{\partial W}{\partial P} - f(v)Sv\frac{\partial W}{\partial B} - rB\frac{\partial W}{\partial B} + (\mu + g(v_t))S\frac{\partial W}{\partial S} \right\} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 W}{\partial S^2} = 0 \tag{4.27}$$

on $(0\ \ T) \times Q$.

Proof. (1): Let $X = (P, S, B)$. We first need to prove $W(t, X)$ is a viscosity subsolution of (4.27) on $(0,\ T) \times Q$. To do so, we need to show that for all smooth function $\phi(t, X)$, such that $W(t, X) - \phi(t, X)$ has a local maximum at $(t_0, X_0) \in (0,\ T) \times Q$, the following inequality holds:

$$-\frac{\partial \phi(t_0, X_0)}{\partial t} - \max_{v \in K}\left\{v\frac{\partial \phi(t_0, X_0)}{\partial P} - (f(v)S_{t_0}v + rB_{t_0})\frac{\partial \phi(t_0, X_0)}{\partial B} + (\mu + g(v))S_{t_0}\frac{\partial \phi(t_0, X_0)}{\partial S}\right\}$$
$$- \frac{\sigma^2 S_{t_0}^2}{2}\frac{\partial^2 \phi(t_0, X_0)}{\partial S^2} \leq 0. \tag{4.28}$$

Without loss of generality, we assume that $W(t_0, X_0) = \phi(t_0, X_0)$, and $W \leq \phi$ on $(0,\ T) \times Q$. Suppose that, on the contrary, there exists a function ϕ and a control variable $v_0 \in \Gamma$, where Γ is the set of all the admissible controls, satisfying the property that there exists an open set $B(t_0, X_0)$ containing (t_0, X_0) such that $\phi(t_0, X_0) = W(t_0, X_0)$ and $\phi(t, X) \geq W(t, X)$ for all $(t, X) \in B(t_0, X_0)$. Then there exits $\theta > 0$ such that

$$-\frac{\partial \phi(t, X)}{\partial t} - \max_{v \in K}\left\{v\frac{\partial \phi(t, X)}{\partial P} - (f(v)Sv + rB)\frac{\partial \phi(t, X)}{\partial B} + (\mu + g(v))S\frac{\partial \phi(t, X)}{\partial S}\right\}$$
$$- \frac{\sigma^2 S^2}{2}\frac{\partial^2 \phi(t, X)}{\partial S^2} > \theta. \tag{4.29}$$

for all $(t, X) \in B(t_0, X_0)$. Let τ be the stopping time

$$\tau = \inf\{t \in [t_0,\ T], (t, X) \notin B(t_0, X_0)\}$$

such that for $t_0 \leq t \leq \tau$, $(t, X) \in B(t_0, X_0)$. For $t_0 \leq t \leq \tau$ and fixed $v \in \Gamma$, when the value function is $J(t_0, P_{t_0}, S_{t_0}, B_{t_0}, v)$, we have

$$J(t_0, P_{t_0}, S_{t_0}, B_{t_0}, v) \leq \mathrm{E}_{t_0}\left[W(\tau, S_\tau, P_\tau, B_\tau)\right] \leq \mathrm{E}_{t_0}\left[\phi(t_\tau, S_\tau, P_\tau, B_\tau)\right].$$

By Dynkin's formula (see Øksendal (2003)), we have

$$\begin{aligned}
\mathrm{E}\left[\phi(t_\tau, P_\tau, S_\tau, B_\tau)\right] =&\phi(t_0, P_{t_0}, S_{t_0}, B_{t_0}) + \mathrm{E}_{t_0}\left[\int_{t_0}^{\tau} \frac{\partial\phi(t,X)}{\partial t} + v\frac{\partial\phi(t,X)}{\partial P}\right.\\
&\left. -(f(v)Sv + rB)\frac{\partial\phi(t,X)}{\partial B} + (\mu + g(v))S\frac{\partial\phi(t,X)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2\phi(t,X)}{\partial S^2}dt\right]\\
&\leq \phi(t_0, P_{t_0}, S_{t_0}, B_{t_0}) - \mathrm{E}_{t_0}\left[\int_{t_0}^{\tau}\theta dt\right]. \qquad (4.30)
\end{aligned}$$

Taking the supremum over all admissible control $v \in \Gamma$, we have

$$\begin{aligned}
\phi(t_0, P_{t_0}, S_{t_0}, B_{t_0}) =&W(t_0, P_{t_0}, S_{t_0}, B_{t_0}) \qquad (4.31)\\
=&\max_{v\in\Gamma} J(t_0, P_{t_0}, S_{t_0}, B_{t_0}, v) \qquad (4.32)\\
\leq&\phi(t_0, P_{t_0}, S_{t_0}, B_{t_0}) - \mathrm{E}_{t_0}\left[\int_{t_0}^{\tau}\theta dt\right]. \qquad (4.33)
\end{aligned}$$

This contradicts the fact that $\theta > 0$. Therefore, $W(t, P, S, B)$ is a viscosity subsolution.

(2): Next, we prove $W(t, X)$ is a viscosity supersolution of (4.27) on $(0\ \ T) \times Q$. Given $(t_0, P_{t_0}, S_{t_0}, B_{t_0}) \in (0,\ T) \times Q$, let $\phi(t, X) \in C^{1,2}((0,\ T) \times Q)$ such that $W(t, X) - \phi(t, X)$ has a local minimum in $B(t_0, X_0)$. Without loss of generality, we assume that $W(t_0, X_0) = \phi(t_0, X_0)$ and $W(t, X) \geq \phi(t, X)$ on $B(t_0, X_0)$. Let τ be the stopping time

$$\tau = \inf\{t \in [t_0,\ T], (t, X) \notin B(t_0, X_0)\}.$$

Given $t_0 < t_1 < \tau$, consider the control variable $v_t = v \in \Gamma$, where v is a constant for $t \in [t_0,\ t_1]$. From the dynamic programming principle, we have

$$W(t_0, P_{t_0}, S_{t_0}, B_{t_0}) \geq \mathrm{E}_{t_0}\left[W(t_1, P_{t_1}, S_{t_1}, B_{t_1})\right].$$

Also, we know

$$W(t_1, P_{t_1}, S_{t_1}, B_{t_1}) \geq \phi(t_1, P_{t_1}, S_{t_1}, B_{t_1}).$$

From Dynkin's formula

$$\begin{aligned} \mathrm{E}\left[\phi(t_1, P_{t_1}, S_{t_1}, B_{t_1})\right] =& \phi(t_0, P_{t_0}, S_{t_0}, B_{t_0}) + \mathrm{E}_{t_0}\Bigg[\int_{t_0}^{t_1} \frac{\partial\phi(t,X)}{\partial t} + v\frac{\partial\phi(t,X)}{\partial P} \\ & -(f(v)Sv + rB)\frac{\partial\phi(t,X)}{\partial B} + (\mu + g(v))S\frac{\partial\phi(t,X)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2\phi(t,X)}{\partial S^2}dt\Bigg]. \end{aligned} \tag{4.34}$$

From the fact that

$$W(t_0, P_{t_0}, S_{t_0}, B_{t_0}) \geq \mathrm{E}_{t_0}\left[W(t_1, P_{t_1}, S_{t_1}, B_{t_1})\right] \geq \mathrm{E}_{t_0}\left[\phi(t_1, P_{t_1}, S_{t_1}, B_{t_1})\right] \tag{4.35}$$

and $W(t_0, P_{t_0}, S_{t_0}, B_{t_0}) = \phi(t_0, P_{t_0}, S_{t_0}, B_{t_0})$, we obtain

$$\begin{aligned} \mathrm{E}_{t_0}\Bigg[\int_{t_0}^{t_1} & \frac{\partial\phi(t,X)}{\partial t} + v\frac{\partial\phi(t,X)}{\partial P} - (f(v)Sv + rB)\frac{\partial\phi(t,X)}{\partial B} + (\mu + g(v))S\frac{\partial\phi(t,X)}{\partial S} \\ & + \frac{\sigma^2 S^2}{2}\frac{\partial^2\phi(t,X)}{\partial S^2}dt\Bigg] \leq 0. \end{aligned} \tag{4.36}$$

Letting $t_1 \to t_0$, we have

$$\begin{aligned} & \frac{\partial\phi(t_0,X_0)}{\partial t} + v\frac{\partial\phi(t_0,X_0)}{\partial P} - (f(v)Sv + rB_0)\frac{\partial\phi(t_0,X_0)}{\partial B} \\ & + (\mu + g(v))S\frac{\partial\phi(t_0,X_0)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2\phi(t_0,X_0)}{\partial S^2} \leq 0. \end{aligned} \tag{4.37}$$

Taking the supremum over $v \in K$, we can conclude

$$-\frac{\partial\phi(t_0, X_0)}{\partial t} - \max_{v\in K}\left\{v\frac{\partial\phi(t_0, X_0)}{\partial P} - (f(v)S\,v + rB_0)\frac{\partial\phi(t_0, X_0)}{\partial B} + (\mu + g(v))S\,\frac{\partial\phi(t_0, X_0)}{\partial S}\right\} - \frac{\sigma^2 S^2}{2}\frac{\partial^2\phi(t_0, X_0)}{\partial S^2} \geq 0. \tag{4.38}$$

Thus, $W(t, P, S, B)$ is a viscosity supersolution of (4.27).

From (1) and (2), $W(t, P, S, B)$ is both a viscosity supersolution and subsolution of (4.27) and the proof is completed. □

By now, we can conclude $\hat{V}(t, P, S, B)$ is a viscosity solution of

$$0 = \max_{v\in K}\left\{v\frac{\partial W}{\partial P} - f(v)S\,v\frac{\partial W}{\partial B} + (\mu + g(v_t))S\,\frac{\partial W}{\partial S}\right\} + \frac{\partial W}{\partial t} + rB\frac{\partial W}{\partial B} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 W}{\partial S^2}, \tag{4.39}$$

with the terminal condition

$$W(t = T, P, S, B) = U\left(S \times P + B - N \times C_T\right)$$

and $V(t, P, S, B)$ is a viscosity solution of

$$0 = \max_{v\in K}\left\{v\frac{\partial W}{\partial P} - f(v)S\,v\frac{\partial W}{\partial B} + (\mu + g(v_t))S\,\frac{\partial W}{\partial S}\right\} + \frac{\partial W}{\partial t} + rB\frac{\partial W}{\partial B} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 W}{\partial S^2}, \tag{4.40}$$

with the terminal condition

$$W(t = T, P, S, B) = U\left(S \times P + B\right).$$

Next, we show the value function is the unique viscosity solution of (4.27).

In order to prove uniqueness of the viscosity solution, we need the following maximum principle for semicontinuous function, which is stated in a suitable form for our application. For the convenience of the reader we restate the relevant theorem from Crandall et al. (1992) without proofs, thus making our exposition self-contained.

Theorem 4.4.2. ***(Crandall, Lions and Ishii)*** *For* $i = 1, 2$*, let* Q_i *be locally compact subsets of* $\mathbb{R}^N$*, and* $Q = Q_1 \times Q_2$*, let* u_i *be upper semicontinuous in* $(0, T) \times Q_i$*, and* $J^{2,+}_{(0,T)\times Q_i} u_i(t, x)$ *the parabolic superjet of* $u_i(t, x)$*, and* ϕ *be twice continuously differentiable in a neighborhood of* $(0, T) \times Q$*.*

Set

$$\omega(t, x_1, x_2) = u_1(t, x_1) + u_2(t, x_2)$$

for $(t, x_1, x_2) \in (0, T) \times Q$*, and suppose* $(\hat{t}, \hat{x}_1, \hat{x}_2)$ *is a local maximum of* $\omega - \phi$ *relative to* $(0, T) \times Q$*. Moreover let us assume that, there is an* $r > 0$ *such that for every* $M > 0$ *there exists a* C *such that for* $i = 1, 2$

$$b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \bar{J}^{2,+}_{[0,T]\times Q_i} u_i(t, x)$$

$$|x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |u_i(t, x_i)| + |q_i| + \|X_i\| \leq M$$

Then for each $\varepsilon > 0$ *there exists* $X_i \in S(N)$ *such that*

1.

$$(b_i, D_{x_i}\phi(\hat{t}, \hat{x}), X_i) \in \bar{J}^{2,+}_{(0,T)\times Q_i} u_i(\hat{t}, \hat{x}) \text{ for } i = 1, 2$$

2.

$$-(\frac{1}{\varepsilon} + \|D^2\phi(\hat{x})\|)I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq D^2\phi(\hat{x}) + \varepsilon(D^2\phi(\hat{x}))^2$$

3.

$$b_1 + b_2 = \frac{\partial \phi(\hat{t}, \hat{x}, \hat{y})}{\partial t}$$

where for a symmetric matrix A, $\|A\| := \sup\{\xi^T A \xi : |\xi| \leq 1\}$.

Now we present the following comparison principle in our case.

Theorem 4.4.3. *(**Comparison Principle**) Suppose* $V_1(t,x)$ *and* $V_2(t,x)$ *are continuous in* (t,x) *and are respectively viscosity subsolution and supersolution of* (4.64)*, with at most a linear growth* $V_i(t,x) \leq K(1+|x|)$ $i = 1,2$*. If* $V_1(t,x) \leq V_2(t,x)$ *for* $x \in Q$ *and* $V_1(T,x) \leq V_2(T,x)$ *for* $0 < t \leq T$ *and* $x \in \partial Q$*, then* $V_1(t,x) \leq V_2(t,x)$ *for all* $(t,x) \in (0,T] \times Q$*.*

Proof. (1): We can rewrite the equation in the following form:

$$-\frac{\partial W(t,x)}{\partial t} + H(x, D_x W(t,x), D_x^2 W(t,x)) = 0 \tag{4.41}$$

where

$$H(x,p,M) = -\max_{v \in K}\left\{[v, (\mu + g(v))x_2, -vf(v)x_2 - rx_3] \cdot [p_1, p_2, p_3]^T\right\} - \frac{\sigma^2}{2}(0, x_2, 0)M(0, x_2, 0)^T. \tag{4.42}$$

Denote

$$\hat{H}(x,p) = -\max_{v\in K}\left\{[v, (\mu + g(v))x_2, -vf(v)x_2 - rx_3]\cdot[p_1, p_2, p_3]^T\right\}. \tag{4.43}$$

Let $V_1(t,x)$ be a viscosity subsolution of (4.41). For $\rho > 0$, define

$$V_1^\rho(t,x) = V_1(t,x) - \frac{\rho}{t+T},\ (t,x)\in(0,T]\times Q.$$

We have

$$\frac{d}{dt}(-\frac{\rho}{t+T}) = \frac{\rho}{(t+T)^2} > 0.$$

We can therefore claim that $V_1^\rho(t,x)$ is a viscosity subsolution of (4.41). In fact,

$$-\frac{\partial V_1^\rho(t,x)}{\partial t} + H(x, D_x V_1^\rho(t,x), D_x^2 V_1^\rho(t,x)) \le -\frac{\rho}{(t+T)^2} \le -\frac{\rho}{4T^2}. \tag{4.44}$$

(2): For any $0<\delta<1$ and $0<\gamma<1$, define

$$\Phi(t,x,y) = V_1^\rho(t,x) - V_2(t,y) - \frac{1}{\delta}|x-y|^2 - \gamma e^{T-t}(x^2+y^2) \tag{4.45}$$

and

$$\phi(t,x,y) = \frac{1}{\delta}|x-y|^2 + \gamma e^{T-t}(x^2+y^2).$$

We know $V_1(t,x)$ and $V_2(t,x)$ satisfy the linear growth. We the have

$$\lim_{|x|+|y|\to\infty}\Phi(t,x,y) = -\infty$$

and $\Phi(t,x,y)$ is continuous in (t,x,y). Therefore, $\Phi(t,x,y)$ has a global maximum at a point $(t_\delta, x_\delta, y_\delta)$. Note that

$$\Phi(t_\delta,x_\delta,y_\delta) = V_1^\rho(t_\delta,x_\delta) - V_2(t_\delta,y_\delta) - \frac{1}{\delta}|x_\delta-y_\delta|^2 - \gamma e^{T-t_\delta}(x_\delta^2+y_\delta^2).$$

In particular,

$$\Phi(t_\delta, x_\delta, x_\delta) + \Phi(t_\delta, y_\delta, y_\delta) \leq 2\Phi(t_\delta, x_\delta, y_\delta)$$

which means

$$\begin{aligned}
&V_1^\rho(t_\delta, x_\delta) - V_2(t_\delta, x_\delta) - \gamma e^{T-t_\delta}(x_\delta^2 + x_\delta^2)\\
&+V_1^\rho(t_\delta, y_\delta) - V_2(t_\delta, y_\delta) - \gamma e^{T-t_\delta}(y_\delta^2 + y_\delta^2)\\
&\leq 2V_1^\rho(t_\delta, x_\delta) - 2V_2(t_\delta, y_\delta) - \frac{2}{\delta}|x_\delta - y_\delta|^2 - 2\gamma e^{T-t_\delta}(x_\delta^2 + y_\delta^2).
\end{aligned}$$

Thus, we have

$$\frac{2}{\delta}|x_\delta - y_\delta|^2 \leq \left[V_1^\rho(t_\delta, x_\delta) - V_1^\rho(t_\delta, y_\delta)\right] + [V_2(t_\delta, y_\delta) - V_2(t_\delta, y_\delta)]. \tag{4.46}$$

By the linear growth condition, there exists K_1, K_2 such that $V_1^\rho(t, x) \leq K_1(1 + |x|)$ and $V_2(t, x) \leq K_2(1 + |x|)$. So, there exists C such that

$$\frac{2}{\delta}|x_\delta - y_\delta|^2 \leq C(1 + |x_\delta| + |y_\delta|). \tag{4.47}$$

We know $\Phi(t_\delta, 0, 0) \leq \Phi(t_\delta, x_\delta, x_\delta)$, which implies

$$\Phi(t_\delta, 0, 0) \leq V_1^\rho(t_\delta, x_\delta) - V_2(t_\delta, y_\delta) - \frac{1}{\delta}|x_\delta - y_\delta|^2 - \gamma e^{T-t_\delta}(x_\delta^2 + y_\delta^2).$$

So, we have

$$\gamma e^{T-t_\delta}(x_\delta^2 + y_\delta^2) \leq V_1^\rho(t_\delta, x_\delta) - V_2(t_\delta, y_\delta) - \frac{1}{\delta}|x_\delta + y_\delta|^2 - \Phi(t_\delta, 0, 0) \leq 3C(1 + |x_\delta| + |y_\delta|)$$

and

$$\frac{\gamma e^{T-t_\delta}(x_\delta^2 + y_\delta^2)}{1 + |x_\delta| + |y_\delta|} \leq 3C.$$

Therefore, there exists C_γ such that

$$|x_\delta| + |y_\delta| \leq C_\gamma.$$

The result implies that (x_δ, y_δ) is bounded by C_γ and there exists subsequence $(t_\delta, x_\delta, y_\delta)$ that are convergent to some (t_0, x_0, y_0). From (4.47), we can conclude

$$\lim_{\delta\to 0} x_\delta = x_0 = y_0 = \lim_{\delta\to 0} y_\delta \text{ and } \lim_{\delta\to 0} t_\delta = t_0.$$

(3): Suppose that there exists $(\hat{t}, \hat{x}) \in (0, T] \times Q$ satisfying

$$V_1(\hat{t}, \hat{x}) \geq V_2(\hat{t}, \hat{x}).$$

Then $\exists\, \tau > 0$

$$V_1(\hat{t}, \hat{x}) - V_2(\hat{t}, \hat{x}) = 2\tau.$$

Equation (4.46) and the semicontinuities of $U^\rho(t, x)$ and $V_2(t, x)$ give us

$$\lim_{\delta\to 0} \frac{2}{\delta}|x_\delta - y_\delta|^2 = 0.$$

Letting $\delta \to 0$, we have

$$\begin{aligned}
\lim_{\delta\to 0} \Phi(t_\delta, x_\delta, y_\delta) &\leq \lim_{\delta\to 0}(V_1^\rho(t_\delta, x_\delta) - V_2(t_\delta, y_\delta)) \\
&\leq \lim_{\delta\to 0} \sup(V_1^\rho(t_\delta, x_\delta) - \lim_{\delta\to 0} \inf(V_2(t_\delta, y_\delta)) \\
&\leq V_1^\rho(t_0, x_0) - V_2(t_0, x_0) \qquad (4.48)
\end{aligned}$$

also

$$\begin{aligned}\Phi(t_\delta, x_\delta, y_\delta) &\geq \Phi(\hat{t}, \hat{x}, \hat{x}) \\ &\geq V_1^\rho(\hat{t}, \hat{x}) - V_2(\hat{t}, \hat{x}) - \gamma e^{T-\hat{t}}(\hat{x}^2 + \hat{x}^2) \\ &\geq V_1(\hat{t}, \hat{x}) - V_2(\hat{t}, \hat{x}) - \frac{\rho}{\hat{t}+T} - \gamma e^{T-\hat{t}}(\hat{x}^2 + \hat{x}^2) \\ &\geq 2\tau - \frac{\rho}{\hat{t}+T} - \gamma e^{T-\hat{t}}(\hat{x}^2 + \hat{x}^2).\end{aligned}$$

When γ and ρ are small enough, we have

$$2\tau - \frac{\rho}{\hat{t}+T} - \gamma e^{T-\hat{t}}(\hat{x}^2 + \hat{x}^2) \geq \tau. \tag{4.49}$$

So, we can claim

$$\tau \leq \Phi(t_\delta, x_\delta, y_\delta)$$

and

$$\tau \leq \lim_{\delta\to 0} \Phi(t_\delta, x_\delta, y_\delta) \leq V_1^\rho(t_0, x_0) - V_2(t_0, x_0).$$

From $V_1 \leq V_2$ on $\partial([0,T] \times Q)$, we have

$$V_1^\rho = V_1 - \frac{\rho}{t+T} \leq V_1 \;\; on \;\; (\{T\} \times Q) \cup ((0,T] \times \partial Q).$$

So $(t_0, x_0, y_0) \notin (\{T\} \times Q) \cup ((0,T] \times \partial Q)$ and $(t_\delta, x_\delta, y_\delta)$ is a local maximizer of $\Phi(t, x, y)$.

(4): By Theorem 4.4.2, for $\epsilon > 0$ there exists $b_{1\delta}, b_{2\delta}, X_\delta, Y_\delta$ such that

$$(b_{1\delta}, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{T-t_\delta} x_\delta, X_\delta) \in \bar{J}^{2,+}_{(0,T)\times Q} V_1^\rho(t_\delta, x_\delta), \tag{4.50}$$

$$(b_{2\delta}, \frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma e^{T-t_\delta} y_\delta, Y_\delta) \in \bar{J}^{2,-}_{(0,T)\times Q} V_2(t_\delta, y_\delta), \tag{4.51}$$

and

$$b_{1\delta} - b_{2\delta} = \frac{\partial \phi(t_\delta, x_\delta, y_\delta)}{\partial t} = -\gamma e^{T-t_\delta}(x_\delta^2 + y_\delta^2).$$

Equations (4.44) and (4.50) imply that there exists $c > 0$ making

$$-b_{1\delta} + H(x_\delta, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{T-t_\delta} x_\delta, X_\delta) \le -c \tag{4.52}$$

and equation (4.51) implies

$$-b_{2\delta} + H(y_\delta, \frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma e^{T-t_\delta} y_\delta, Y_\delta) \ge 0. \tag{4.53}$$

From equations (4.52) and (4.53)

$$b_{1\delta} - b_{2\delta} + H(y_\delta, \frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma e^{T-t_\delta} y_\delta, Y_\delta) - H(x_\delta, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{T-t_\delta} x_\delta, X_\delta) \ge c. \tag{4.54}$$

By the maximum principle, we have

$$-\left(\frac{1}{\varepsilon} + \|D^2\phi(t_\delta, x_\delta, y_\delta)\|\right) I \le \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \le D^2\phi(t_\delta, x_\delta, y_\delta) + \varepsilon(D^2\phi(t_\delta, x_\delta, y_\delta))^2$$

$$D^2\phi(t_\delta, x_\delta, y_\delta) = \frac{2}{\delta}\begin{pmatrix} I_3 & -I_3 \\ -I_3 & I_3 \end{pmatrix} + 2\gamma e^{T-t_\delta}\begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}$$

and

$$(D^2\phi(\hat{x}))^2 = \frac{8}{\delta^2}\begin{pmatrix} I_3 & -I_3 \\ -I_3 & I_3 \end{pmatrix} + \frac{8\gamma e^{T-t_\delta}}{\delta}\begin{pmatrix} I_3 & -I_3 \\ -I_3 & I_3 \end{pmatrix} + 4\gamma e^{2(T-t_\delta)}\begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}.$$

We can rewrite

$$x_\delta X_\delta x_\delta^T - y_\delta Y_\delta y_\delta^T = (x_\delta, y_\delta)\begin{pmatrix} X_\delta & 0 \\ \\ 0 & -Y_\delta \end{pmatrix}\begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} \tag{4.55}$$

$$\le (x_\delta, y_\delta)\left[\frac{2}{\delta}\begin{pmatrix} I_3 & -I_3 \\ \\ -I_3 & I_3 \end{pmatrix} + (2\gamma e^{T-t_\delta} + 4\epsilon\gamma^2 e^{2(T-t_\delta)})\begin{pmatrix} I_3 & 0 \\ \\ 0 & I_3 \end{pmatrix}\right.$$

$$\left. + \varepsilon\frac{8 + 8\gamma\delta e^{T-t_\delta}}{\delta^2}\begin{pmatrix} I_3 & -I_3 \\ \\ -I_3 & I_3 \end{pmatrix}\right]\begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix}. \tag{4.56}$$

Letting $\gamma \to 0$ and $\epsilon = \frac{\delta}{4}$, we have

$$x_\delta X_\delta x_\delta^T - y_\delta Y_\delta y_\delta^T \le \frac{\delta}{4}(x_\delta - y_\delta)^2, \tag{4.57}$$

$$y_\delta Y_\delta y_\delta^T - x_\delta X_\delta x_\delta^T \ge -\frac{4}{\delta}(x_\delta - y_\delta)^2. \tag{4.58}$$

We can have the following (4.54)

$$H(y_\delta, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{T-t_\delta} y_\delta, Y_\delta) - H(x_\delta, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{T-t_\delta} x_\delta, X_\delta) \ge b_{2\delta} - b_{1\delta} + c \tag{4.59}$$

We can deduce from (4.59) that

$$\begin{aligned}
&\hat{H}(y_\delta, \frac{2}{\delta}(x_\delta - y_\delta) - 2\gamma e^{T-t_\delta} y_\delta) - \hat{H}(x_\delta, \frac{2}{\delta}(x_\delta - y_\delta) + 2\gamma e^{T-t_\delta} x_\delta) \\
&\ge (b_{2\delta} - b_{1\delta}) + \frac{\sigma^2}{2}(y_\delta Y_\delta y_\delta^T - x_\delta X_\delta x_\delta^T) + c \\
&\ge \gamma e^{T-t_\delta}(x_\delta^2 + y_\delta^2) - \frac{4\sigma^2}{2\delta}(x_\delta - y_\delta)^2 + c
\end{aligned} \tag{4.60}$$

Letting $\gamma \longrightarrow 0$

$$\hat{H}(y_\delta, \frac{2}{\delta}(x_\delta - y_\delta)) - \hat{H}(x_\delta, \frac{2}{\delta}(x_\delta - y_\delta)) \geq -\frac{4\sigma^2}{2\delta}(x_\delta - y_\delta)^2 + c. \tag{4.61}$$

We have $\lim_{\delta\to 0} \frac{2}{\delta}|x_\delta - y_\delta|^2 = 0$ and from the continuity of $\hat{H}$, and $\lim_{\delta\to 0} x_\delta = x_0 = \lim_{\delta\to 0} y_\delta$, we have

$$0 = \lim_{\delta\to 0}\left[\hat{H}(y_\delta, \frac{2}{\delta}(x_\delta - y_\delta)) - \hat{H}(x_\delta, \frac{2}{\delta}(x_\delta - y_\delta))\right] \geq c \tag{4.62}$$

which is a contradiction. □

The uniqueness of the viscosity solution follows from the comparison theorem because any viscosity solution is both supersolution and subsolution.

Theorem 4.4.4. *The value function $\hat{V}(t, P, S, B)$ is the unique viscosity solution of*

$$0 = \max_{v\in K}\left\{v\frac{\partial W}{\partial P} - f(v)S\,v\frac{\partial W}{\partial B} + (\mu + g(v_t))S\,\frac{\partial W}{\partial S}\right\} + \frac{\partial W}{\partial t} + rB\frac{\partial W}{\partial B} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 W}{\partial S^2}, \tag{4.63}$$

with the terminal condition $W(T, P, S, B) = U(S \times P + B - N \times C_T)$ and the boundary condition $W(t, P, 0, B) = U\left(e^{r(T-t)}B\right)$. The value function $V(t, P, S, B)$ is the unique viscosity solution of

$$0 = \max_{v\in K}\left\{v\frac{\partial W}{\partial P} - f(v)S\,v\frac{\partial W}{\partial B} + (\mu + g(v_t))S\,\frac{\partial W}{\partial S}\right\} + \frac{\partial W}{\partial t} + rB\frac{\partial W}{\partial B} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 W}{\partial S^2}, \tag{4.64}$$

with the terminal condition $W(T, P, S, B) = U(S \times P + B)$ and boundary condition $W(t, P, 0, B) = U\left(e^{r(T-t)}B\right)$.

4.5 Example and numerical experiments

4.5.1 Example of the trading speed model and the feedback effects model

To illustrate our model, we provide a simple example that is interesting enough to give us an explicit solution. Consider the functions of trading speed v_t, $f(v_t) = 1 + \alpha v_t$ and $g(v_t) = \beta v_t$, with $\alpha > 0$ and $\beta > 0$. We then have the following SDE for market price and actual traded price:

$$dS_t = (\mu + \beta v_t)S_t dt + \sigma S_t dB_t,$$

$$S_t(v_t) = (1 + \alpha v_t)S_t, \quad -M \leq v_t \leq M.$$

α is positive and indicates the depth of illiquidity (the parameter for liquidity costs). β is also positive and indicates the feedback effects factor.

Applying Ito's formula to $\log(S_t)$, we have

$$d\log(S_t) = (\mu + g(v_t) - \frac{1}{2}\sigma^2)dt + \sigma dW_t.$$

Integrating both sides, we get

$$\log(S_T) - \log(S_0) = \int_0^T (\mu - \frac{1}{2}\sigma^2)dt + \int_0^T g(v_t)dt + \int_0^T \sigma dW_t$$

A trade is considered to be a round trip trade when $\int_0^T v_t dt = 0$. When $g(v_t) = \beta v_t$, a round trip will not affect the stock price at time T. Indeed,

$$\log(S_T)-\log(S_0) = \int_0^T (\mu-\frac{1}{2}\sigma^2)dt+\int_0^T \beta v_t dt+\int_0^T \sigma dW_t = \int_0^T (\mu-\frac{1}{2}\sigma^2)dt+\int_0^T \sigma dW_t.$$

We substitute $f(v_t) = 1 + \alpha v_t$ and $g(v_t) = \beta v_t$ to (4.64). With a little analysis, we have

$$0 = \max_{v \in K} \left\{ -\alpha S \frac{\partial W}{\partial B} v^2 + \left(\frac{\partial W}{\partial P} + \beta S \frac{\partial W}{\partial S} - S \frac{\partial W}{\partial B} \right) v \right\} + \mu S \frac{\partial W}{\partial S} + \frac{\partial W}{\partial t} + rB \frac{\partial W}{\partial B} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 W}{\partial S^2}. \tag{4.65}$$

Note that $\left\{ -\alpha S \frac{\partial W}{\partial B} v^2 + \left(\frac{\partial W}{\partial P} + \beta S \frac{\partial W}{\partial S} - S \frac{\partial W}{\partial B} \right) v \right\}$ is a quadratic function of v. Since $W(t, P, S, B)$ is monotone increasing with respect to B, $\frac{\partial W}{\partial B} > 0$. Also, the fact that $S > 0$ and $\alpha > 0$ gets

$$-\alpha S \frac{\partial W}{\partial B} < 0.$$

Therefore, the maximum of $\left\{ -\alpha S \frac{\partial W}{\partial B} v^2 + \left(\frac{\partial W}{\partial P} + \beta S \frac{\partial W}{\partial S} - S \frac{\partial W}{\partial B} \right) v \right\}$ is achieved when

$$v^* = \frac{\frac{1}{S} \frac{\partial W}{\partial P} + \beta \frac{\partial W}{\partial S} - \frac{\partial W}{\partial B}}{2\alpha \frac{\partial W}{\partial B}}.$$

Since the term $2\alpha \frac{\partial W}{\partial B}$ is always nonnegative, the sign of v^* is determined by $\frac{1}{S} \frac{\partial W}{\partial P} + \beta \frac{\partial W}{\partial S} - \frac{\partial W}{\partial B}$. There are three possible cases:

Case (1):

$$\frac{\partial W}{\partial B} > \frac{1}{S} \frac{\partial W}{\partial P} + \beta \frac{\partial W}{\partial S};$$

the optimal solution $v^* < 0$ where the maximum is achieved by selling the stock and increasing the holdings on the bank account. Marginal utility per dollar of stock holding plus marginal utility on stock price caused by the feedback effects factor is less than the marginal utility per dollar on the bank account. To maximize the utility, it is recommended that the wealth transferred from from the stocks to the bank account.

Case (2):

$$\frac{\partial W}{\partial B} < \frac{1}{S}\frac{\partial W}{\partial P} + \beta\frac{\partial W}{\partial S};$$

the optimal solution $v^* > 0$ where the maximum is achieved by buying the stock and decreasing our holdings on the bank account. Marginal utility per dollar of the stock holding plus marginal utility on the stock price caused by the permanent price impact factor is greater than the marginal utility per dollar on the bank account. To maximize the utility, it is recommended that wealth be transferred from the bank account to the stocks.

Case (3):

$$\frac{\partial W}{\partial B} = \frac{1}{S}\frac{\partial W}{\partial P} + \beta\frac{\partial W}{\partial S};$$

the optimal solution $v^* = 0$, where the maximum is achieved by doing nothing. Marginal utility per dollar of the stock holding plus marginal utility on the stock price caused by the feedback effects factor is equal to the marginal utility per dollar on the bank account. There is no transaction needed.

At a fixed time t, the above result suggests that the state space can be divided into buying and selling regions by a surface. On the surface, the trading speed is 0 and there is no transaction. The buy region is characterized by

$$\frac{\partial W}{\partial B} < \frac{1}{S}\frac{\partial W}{\partial P} + \beta\frac{\partial W}{\partial S}$$

and the selling region is characterized by

$$\frac{\partial W}{\partial B} > \frac{1}{S}\frac{\partial W}{\partial P} + \beta\frac{\partial W}{\partial S}.$$

We consider the exponential utility function given by

$$U(x) = 1 - \exp(-\lambda x)$$

where the index of risk aversion is $-\frac{U''(x)}{U'(x)} = \lambda$, which is independent of the investors wealth. The integral version of state variable B_T is written as

$$B_T = B_t \exp\Big(r(T-t)\Big) - \int_t^T f(v_u) S_u v_u du$$

Then

$$\begin{aligned}
&V(t, P, S, B) \\
=&\max_{v\in\Gamma} \mathrm{E}_t \left\{1 - \exp\left(-\lambda\left(P_T S_T + B_T\right)\right)\right\} \\
=&1 - \min_{v\in\Gamma} \mathrm{E}_t \left\{\exp\left(-\lambda\left(P_T S_T + B\exp\left(r(T-t)\right) - \int_t^T f(v_u) S_u v_u du\right)\right)\right\} \\
=&1 - \exp\left(-\lambda B \exp\left(r(T-t)\right)\right) \min_{v\in\Gamma} \mathrm{E}_t \left\{\exp\left(-\lambda\left(P_T S_T - \int_t^T f(v_u) S_u v_u du\right)\right)\right\} \\
=&1 - \exp\left(-\lambda B \exp\left(r(T-t)\right)\right) Q(t, P, S)
\end{aligned}$$

where $Q(t, P, S)$ is a continuous function in P and S, and can be defined by $Q(t, P, S) = 1 - V(t, P, S, 0)$. With a little analysis, we have

$$\begin{aligned}
0 = &\max_{v\in K} \left\{-\alpha S \lambda \exp(r(T-t)) Q(t, P, S) v^2 + \left(\frac{\partial Q}{\partial P} + \beta S \frac{\partial Q}{\partial S} - S\lambda \exp(r(T-t)) Q(t, P, S)\right) v\right\} \\
&+ \mu S \frac{\partial Q}{\partial S} + \frac{\partial Q}{\partial t} - \lambda B \exp(r(T-t)) r + rB\lambda \exp(r(T-t)) Q(t, P, S) \frac{\partial Q}{\partial B} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 Q}{\partial S^2}.
\end{aligned}$$

with terminal condition

$$Q(T, P, S) = 1 - \exp(S \times P)$$

Note the term in the above PDE

$$-\alpha S \lambda \exp(r(T-t))Q(t, P, S)v^2 + \left(\frac{\partial Q}{\partial P} + \beta S \frac{\partial Q}{\partial S} - S \lambda \exp(r(T-t))Q(t, P, S)\right)v$$

is a quadratic function of v. The maximum of the quadratic term is achieved when

$$v^* = \frac{\frac{\partial Q}{\partial P} + \beta S \frac{\partial Q}{\partial S} - S \lambda \exp(r(T-t))Q(t, P, S)}{2\alpha S \lambda \exp(r(T-t))Q(t, P, S)}$$

which means the optimal trading speed v^* is independent of B. The utility indifference price for n units of options $p(n)$ at time 0 is defined by the equation

$$\hat{V}(t = 0, P_0, S_0, B + p(n)) = V(t = 0, P_0, S_0, B_0) \tag{4.66}$$

Assuming $P_0 = 0$, we have the following explicit formula for the utility indifference price $p(n)$:

$$p(n) = \frac{1}{-\lambda \exp(rT)} \log \frac{\hat{Q}(0, 0, S)}{Q(0, 0, S)}.$$

At time 0, the large investor's initial wealth is B units of the bank account. The utility indifference price $p(n)$ at time 0 is independent of B, which means the utility indifference price $p(n)$ is independent of the large investor's initial wealth.

4.5.2 Numerical results

In this section, we discuss the numerical solution to the example and present some results. We compute the utility indifference price of a European call option with strike 100 and observe interesting properties. In the numerical experiments, the parameter values that we used are initial stock price $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0$ and $T = 0.1$. We assume $\lambda = 0.00001$ and $n = 1000$. Table 4.1 provides a comparison of the option prices for

	$\alpha = 0.00001$			$\alpha = 0.00005$			$\alpha = 0.00010$		
β	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
0.00000	2.8180	2.8584	2.9452	2.8200	2.8616	2.9484	2.8202	2.8621	2.9488
0.00001	2.8144	2.8580	2.9388	2.8192	2.8615	2.9472	2.8198	2.8620	2.9482
0.00002	2.8112	2.8369	2.8863	2.8188	2.8576	2.9373	2.8196	2.8600	2.9433
0.00005	2.7182	2.6186	2.4406	2.8024	2.8200	2.8553	2.8116	2.8416	2.9028

Table 4.1: Utility indifference price with different α and β.

different α, β and number n of the options written by the investor. We have observed when α increases (the depth of illiquidity increases) for a fixed β, the option price increases. But when β increases (the depth of feedback effects increases) for a fixed α, the option price decreases. When β is large, a large investor has more influence on the stock price evolution. We can conclude that the investor may have the power to manipulate the stock price, to some extent, to maximize the investor's utility.

When α is relatively small compared to β, with the increase of n, the option price

β	$\alpha = 0.00001$			$\alpha = 0.00005$			$\alpha = 0.00010$		
	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
0.00000	2.8240	2.8659	2.9527	2.8240	2.8659	2.9527	2.8240	2.8660	2.9527
0.00001	2.8240	2.8660	2.9526	2.8240	2.8660	2.9526	2.8242	2.8660	2.9526
0.00002	2.8240	2.8657	2.9523	2.8240	2.8657	2.9523	2.8240	2.8657	2.9523
0.00005	2.8232	2.8649	2.9515	2.8232	2.8649	2.9515	2.8232	2.8649	2.9515

Table 4.2: Utility indifference price with different α and β when $M = 5$.

β	$\alpha = 0.00001$			$\alpha = 0.00005$			$\alpha = 0.00010$		
	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
0.00000	2.8200	2.8634	2.9508	2.8230	2.8652	2.9520	2.8236	2.8657	2.9524
0.00001	2.8222	2.8641	2.9498	2.8238	2.8653	2.9510	2.8240	2.8657	2.9518
0.00002	2.8198	2.8592	2.9441	2.8224	2.8619	2.9458	2.8234	2.8637	2.9476
0.00005	2.8040	2.8424	2.9268	2.8102	2.8459	2.9286	2.8154	2.8499	2.9308

Table 4.3: Utility indifference price with different α and β when $M = 100$.

decreases. And when β is relatively small compared to α, with the increase of n, the option price increases. The explanation is that when α is relatively small compared to β, the advantage of feedback effects outweighs the liquidity cost, so the more options the large investor sells, the more benefits the large investor can gain from feedback effects on stock price, and the average hedging cost per unit of an option for the large investor decreases. But when β is relatively small compared to α, the liquidity cost cannot be overturned by the advantage brought by the feedback effects. So, the more options the large investor sells, the larger the liquidity cost the large investor will incur, and the average hedging cost per unit of option for a large investor will increase.

β	$\alpha = 0.00001$			$\alpha = 0.00005$			$\alpha = 0.00010$		
	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
0.00000	2.8180	2.8617	2.9488	2.8230	2.8652	2.9520	2.8236	2.8657	2.9524
0.00001	2.8218	2.8622	2.9437	2.8238	2.8653	2.9508	2.8240	2.8657	2.9518
0.00002	2.8150	2.8444	2.9173	2.8224	2.8613	2.9409	2.8234	2.8637	2.9469
0.00005	2.7490	2.7628	2.8313	2.8062	2.8237	2.8726	2.8154	2.8453	2.9063

Table 4.4: Utility indifference price with different α and β when $M = 500$.

β	$\alpha = 0.00001$			$\alpha = 0.00005$			$\alpha = 0.00010$		
	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
0.00000	2.8180	2.8617	2.9488	2.8230	2.8652	2.9520	2.8236	2.8657	2.9524
0.00001	2.8218	2.8622	2.9424	2.8238	2.8653	2.9508	2.8240	2.8657	2.9518
0.00002	2.8150	2.8407	2.8992	2.8224	2.8613	2.9409	2.8234	2.8637	2.9469
0.00005	2.7222	2.6946	2.7309	2.8062	2.8237	2.8588	2.8154	2.8453	2.9063

Table 4.5: Utility indifference price with different α and β when $M = 1000$.

For a fixed time, we know that the optimal trading speed is independent of B in the case of exponential utility. We have the optimal trading speed at a fixed time in Figure 4.1. Knowing the optimal trading speed, we can define the trading region. The trading region is divided by a smooth surface into a buying region and a selling region. On the curve, the trading speed is 0, and there is no transaction. Above the curve,

$$\frac{\partial Q}{\partial P} + \beta S \frac{\partial Q}{\partial S} < S\lambda Q(t, P, S)$$

it belongs to the buying region. And below the curve

$$\frac{\partial Q}{\partial P} + \beta S \frac{\partial Q}{\partial S} > S\lambda Q(t, P, S)$$

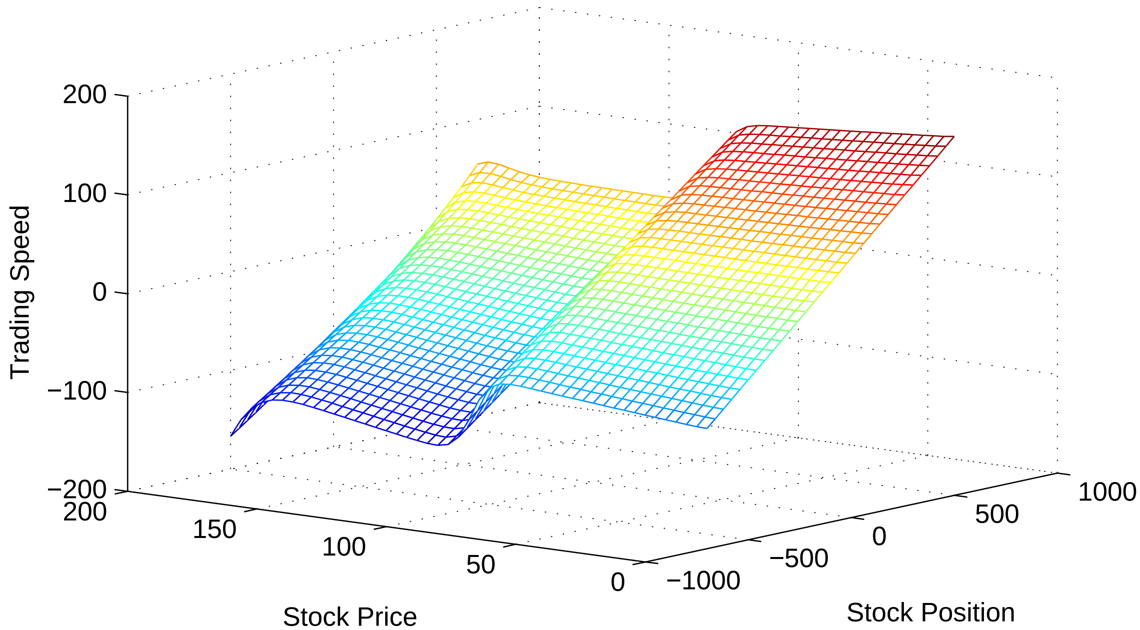

Figure 4.1: Optimal Trading Speed at Fixed Time

it belongs to the selling region.

In our large investor model, a smaller trading speed limit M means a more illiquid market. Comparing numerical results in Table 4.3, Table 4.4 and Table 4.5, we find that when M becomes smaller, the option price becomes larger. The explanation for this is that the more illiquid a market becomes, the harder is is for the option writer to hedge the option. The hedging cost increases, therefore the option price increases.

4.5.3 An extended large investor model

In our large investor model, above, we assumed that the trading speed would have an effect on the drift term of the stock price. We can extend the large investor's model to state that the trading speed places an effect on both the drift term and the volatility term of the stock price evolution. Usually, trading action (including both buying and selling) increases the volatility of trading assets. Volatility of the asset increases with the increase of $|v_t|$. Consider a simple trading speed model $f(v_t) = 1+\alpha v_t$, $g(v_t) = \beta v_t$ and $\sigma(v_t) = \sqrt{\sigma^2 + \gamma v_t^2}$, with $\alpha > 0, \beta > 0$ and $\gamma > 0$. Then

$$dS_t = (\mu + \beta v_t) S_t dt + \sqrt{\sigma^2 + \gamma v_t^2} S_t dB_t,$$

$$S_t(v_t) = (1 + \alpha v_t) S_t, \quad -M \leq v_t \leq M,$$

where $\alpha > 0$ and indicates the depth of illiquidity. $\beta > 0$ and it indicates the feedback effects index. With a little analysis, we have

$$0 = \max_{v \in K} \left\{ -\alpha S \frac{\partial W}{\partial B} v^2 + \left(\frac{\partial W}{\partial P} + \beta S \frac{\partial W}{\partial S} - S \frac{\partial W}{\partial B} \right) v + \frac{\gamma v^2 S^2}{2} \frac{\partial^2 W}{\partial S^2} \right\} + \mu S \frac{\partial W}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 W}{\partial S^2} + \frac{\partial W}{\partial t} + rB \frac{\partial W}{\partial B}. \tag{4.67}$$

Table 4.6 shows the option price when $\gamma = 5 \times 10^{-8}$. Compared with the case a large investor does not influence the stock price's volatility, and option prices are relatively higher when a large investor has an influence on the stock price's volatility. This result is illustrated by a comparison between Table 4.1 and Table 4.6.

β	$\alpha = 0.00001$ $n = 500$	$n = 1000$	$n = 2000$	$\alpha = 0.00005$ $n = 500$	$n = 1000$	$n = 2000$	$\alpha = 0.00010$ $n = 500$	$n = 1000$	$n = 2000$
0.00000	2.8196	2.8619	2.9490	2.8201	2.8620	2.9490	2.8202	2.8621	2.9491
0.00001	2.8190	2.8614	2.9483	2.8198	2.8619	2.9486	2.8200	2.8620	2.9488
0.00002	2.8164	2.8571	2.9443	2.8192	2.8597	2.9455	2.8198	2.8607	2.9464
0.00005	2.7854	2.8219	2.9129	2.8074	2.8404	2.9214	2.8130	2.8480	2.9276

Table 4.6: Utility indifference price when $\gamma = 5 \times 10^{-8}$ with different α and β .

5 Conclusions and Future Work

In this dissertation, we studied option pricing and hedging in a non-competitive market. The non-competitive market is characterized by liquidity risk and feedback effects. Based on an investor's market power to have feedback effects on an underlying security price, market participants in a non-competitive market are divided into small investors and large investors. Different pricing models were proposed for both small and large investors.

In chapter 2, we investigated how to price and hedge options for small investors in a non-competitive market. Based on the supply curve model, partial differential equations were presented to characterize the replication cost for options. We showed that the replication cost from the seller party was greater than the replication cost from the buyer party. In a non-competitive market, we regard the replication cost from the seller party as the upper bound for option price, and the replication cost from buyer's party as a lower bound. Our model boasts several advantages over other models. First, it provides a general framework for pricing different options, including path-dependent options. Second, our model is an extension of the Black-Scholes model; in the case $f'(0) = 0$, our model returns to the Black-Scholes model. Third, we only introduced one parameter $f'(0)$ to incorporate liquidity risk, which makes our model relatively simple.

In chapter 3, we approximated a jump-diffusion process by a discrete-time Markov process and applied the local risk minimization method incorporating liquidity risk to price European options in the discrete-time model with liquidity costs. Numerical results showed that the proposed hedging strategies reduce the standard deviation of the hedging error as well as the mean hedging cost, which confirmed that our modified local risk minimization method performs better than other existing hedging strategies. Therefore, when the underlying asset price is assumed to follow a jump-diffusion process in a market with liquidity risk, our method is useful for option valuation and hedging under liquidity risk.

In chapter 4, we investigated option valuation based on utility maximization for a large investor in a market with liquidity risk. We considered two effects of a non-competitive market: liquidity risk and feedback effects of large investors. In non-competitive markets, trading action will incur liquidity costs, but at the same time, the investor can have an influence on the stock price evolution and gain benefits from the permanent price impact by choosing the optimal strategy. Thus, the option price, in some sense, is determined by these two contradicting phenomena. When both the permanent impact function and the liquidity cost function are linear in the trading speed, the optimal solution is computed explicitly, and the state space can be characterized and divided into the buy and sell regions.

Our work provides a new understanding of option pricing theory in non-competitive markets, yet some interesting and important questions remain. We have only an approximate European option price by approximating a jump diffusion model by a discrete time model. More effort is required to develop a method to determine the precise option price in a jump diffusion model. Another issue is that we modelled the feedback effects of large investors by imposing a function of trading speed to the drift term of underlying stock price without justification from market microstructure. A deep study of limit order book dynamic could help us understand the modelling of feedback effects.

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