

DEVELOPMENT OF SOFTWARE TO
MEASURE THE GRAVITATIONAL REDSHIFT
WITH RADIOASTRON

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Abstract

The incompatibility of General Relativity (GR) and quantum theory continues to be a stumbling block in the unification of our understanding of the physical world. Tests of the Einstein Equivalence Principle, lying at the core of GR, are vital to constrain new theories that require departures from GR's predictions. One such test is measuring the gravitational redshift using RadioAstron – a space-VLBI mission launched in 2011 into a highly elliptical orbit. Utilizing the onboard H-maser frequency standard along with a Doppler compensation scheme that minimizes systematics, the gravitational redshift is anticipated to be determined with a relative accuracy of 10^{-5} . A third-order dynamical model has been implemented to predict frequency shifts in the spacecraft's carrier signal needed to conduct the experiment. The theory behind this model is developed, the software implementation described and third-order effects necessary to achieve the desired measurement accuracy are discussed.

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Chapter 1

Background

1.1 Einstein Equivalence Principle

In 1905, Albert Einstein laid to rest the incompatibilities between Newtonian mechanics and Maxwell’s equations by publishing the theory of Special Relativity (Einstein 1905). The insight that all observers should agree on the laws of physics within inertial frames of reference, now known as the Special Principle of Relativity, was key to explaining away the existence of a luminiferous ether. Over the years that followed, as he sought to include acceleration and gravitation in a more general theory, he expanded the notion of relativity, writing in a 1907 paper that: “As far as we know, the physical laws with respect to Σ_1 (an accelerated system) do not differ from those with respect to Σ_2 (an equivalent gravitational system); that is based on the fact that all bodies are equally accelerated in the gravitational field. At our present state of experience, we have thus no reason to assume that the systems Σ_1 and Σ_2 differ from each other in any respect, and in the discussion that follows, we shall therefore assume the *complete physical equivalence* of a gravitational field and a corresponding acceleration of the reference frames” (Einstein 1907). As the importance of this hypothesis grew, he elevated it to a foundational principle in his developing theory: “However, for our conception to acquire

deeper significance, the systems K (gravitational) and K' (accelerated) must be equivalent with respect to all physical processes, i.e., the natural laws with respect to K must coincide completely with those with respect to K'. If we accept this assumption, we obtain a principle that possess great heuristic significance, provided that it is really correct" (Einstein 1911). This seemingly simple idea, that the laws of nature should remain the same in all frames of reference, came to be known as the Einstein Equivalence Principle (EEP) and led to General Relativity (GR).

It is now common to divide the EEP into phenomenologically distinct sub-principles that can be separately tested through experiment, for example as laid out in Will (2014):

- i. Weak Equivalence Principle (WEP): the trajectories of uncharged, freely falling test bodies are independent of their internal structure or composition
- ii. Local Lorentz Invariance (LLI): the outcome of any local non-gravitational experiment in a freely falling frame is independent of the velocity of that frame
- iii. Local Position Invariance (LPI): the outcome of any local non-gravitational experiment in a freely falling frame is independent of where or when it is performed

It is important to note that tests that focus on only one of these sub-principles cannot be considered independent. Indeed, it is conjectured that a violation of one would naturally lead to the violation of all three (Schiff 1960).

The theory of GR that Einstein eventually developed is based on the interpretation of gravity as a warping of space and time due to the presence of mass and energy. Metric theories of gravitation, like GR, are based on a geometric interpretation of this warping and include the following basic postulates:

- i. Spacetime is endowed with a metric, $(g_{\alpha\beta})$, that defines a line element that can be used to measure the space-time separation, (ds) , between events: $-ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$

- ii. The trajectories of freely falling test bodies are independent of their structure or composition being geodesics of extremal length
- iii. Clocks measure proper time, (τ) , along their trajectory, given by: $cd\tau = ds$
- iv. In local freely falling frames, the non-gravitational laws of physics are those of special relativity

From these postulates, we see how the EEP lies at the heart of any metric theory of gravity. GR, however, goes further and embodies a generalization of the EEP called the Strong Equivalence Principle (SEP) that includes test bodies with non-negligible self-gravitation together with experiments involving gravitational forces. Therefore, EEP being a true property of nature, implies that a metric theory is the correct description of gravity. However, GR can only be the correct metric theory if SEP is also found to be true.

But why should we expect EEP, or SEP for that matter, to be satisfied in nature? Our understanding of the physical Universe at the most fundamental level is based not just on GR but also on quantum theory. The latter having been extremely successful in explaining all observed particle phenomena and having been experimentally verified with a precision significantly exceeding that of GR. In particular, Quantum Field Theory and the Standard Model (SM) of particle physics provide a unified view of the fundamental forces, other than gravity, by describing their interactions through vector fields (electromagnetic, weak, colour) and a scalar field (Higgs) that permeate space. However, GR is fundamentally different from these types of theories. Attempts to formulate a quantum theory of gravity and unify it into a common framework with the other fundamental forces has ultimately led to a violation of EEP (Will 2014). Thus, to many, it would seem more natural that EEP should be violated at some level rather than it being a true property of the Universe and implying that a metric theory is not the correct fundamental description of gravity.

Based on Schiff’s conjecture that a violation of any aspect of EEP leads to a violation of all others, one might conclude that searching for a violation of EEP should focus on whichever of the sub-principles is most accessible to experiment. However, it is not known *a priori* what the relative size of such violations may be. As a result, it is important that empirical tests of all three sub-principles proceed in tandem in the hopes of finding violations that will shed light on the nature of a possible non-metric theory of gravity. Even the non-observation of a violation helps to constrain non-metric theories as these typically require much larger violations than the limits currently set by experiment.

Beyond just a better understanding of gravity, a violation of EEP could also shed light on other mysteries in our understanding of nature. By combining GR at cosmological scales with the SM and an assumption of the homogeneity and isotropy of the Universe, the concordance model of cosmology, known as Λ -CDM, is able to describe present-day observations of the anisotropies of the cosmic microwave background (Ade et al. 2014). However, the Λ -CDM model suggests that 26.8% of the mass-energy of the Universe is made of non-baryonic dark matter particles currently unexplained by the SM. A further 68.3% is dark energy in the form of a cosmological constant accelerating the expansion of the Universe. Theories attempting to explain these dark constituents often involve long-range scalar fields that may also give rise to violations of EEP (Khoury & Weltman 2004; Carroll et al. 2009). Solving these cosmological mysteries, together with the pursuit of unifying gravity with the other forces, makes experimental searches for violations of EEP one of the most important areas of research in fundamental physics today.

1.2 Gravitational Redshift

The first test of the EEP was suggested by Einstein himself (Einstein 1907). In the paper in which EEP was first proposed, he points out that a clock within a gravitational potential

should experience a form of time dilation. We can see how this arises as a direct consequence of EEP as follows. Consider two systems: S sitting on Earth's surface subjected to a gravitational acceleration $\vec{g}$ and S' which is far from any gravitating body, but which is accelerated uniformly through space at a rate $\vec{a} = -\vec{g}$. Both consist of a laboratory in which a radio transmitter is attached to the floor with a receiver attached to the ceiling, a distance h away. Signals will take a time $\Delta t = h/c$ to cross the laboratory where c is the speed of light. If S' is initially at rest, in the time Δt , it will accelerate to a velocity $v = a\Delta t = ah/c$. The signal reaching the receiver will thus be Doppler shifted by a fractional amount $\Delta\nu/\nu = v/c = ah/c^2$. By the EEP, an observer should not be able to distinguish between the two systems. So, we would expect an equivalent frequency shift in the signal received in S , namely:

$$y_{grav} \equiv \frac{\Delta\nu}{\nu} = \frac{\Delta U}{c^2} \quad (1.1)$$

where gh has been replaced with the difference in Newtonian gravitational potentials, ΔU , between the floor and ceiling. And so we conclude that the signal in both systems will be shifted towards longer wavelengths. Einstein suggested that this effect would be apparent in atomic emission lines in the Solar spectrum and so this effect became known as gravitational redshift (y_{grav}). However, since the gravitational redshift is a consequence of the EEP, experiments that look for a violation of Eq. (1.1) test not just GR but whether a metric theory is, in fact, the correct description of gravity.

Violations of Eq. (1.1) may arise due to couplings between gravity and the composition of gravitating bodies or experimental apparatuses beyond those described in GR. The simplest form for a violation is usually parameterized as follows:

$$\frac{\Delta\nu}{\nu} = (1 + \varepsilon) y_{grav} \quad (1.2)$$

where ε is called a violation parameter. Measurements of the gravitational redshift that seek

to constrain ε are considered tests of the LPI sub-principle of the EEP and fall into two broad categories:

- i. Global clock experiments in which identical frequency standards, relying on the same atomic transitions, are compared at different locations within a gravitational field by the exchange of electromagnetic radiation; or
- ii. Local clock experiments, or null redshift experiments, where frequency standards based on different atomic transitions moving together through a gravitational potential are compared to search for a differential signal

While many high-precision tests of the gravitational redshift have been performed since the 1960's, LPI is still the least well constrained aspect of the EEP.

1.3 Tests of the Gravitational Redshift

Measuring the gravitational redshift in the Solar spectrum, as proposed by Einstein, turned out to be complicated due to systematic effects from Doppler shifts due to the Sun's turbulent photosphere. These were eventually overcome setting a limit of $|\varepsilon| < 2\%$ (Brault 1962; Snider 1972; LoPresto et al. 1991). The first high precision tests exceeding the 1% level, were performed in 1960-1965 using the Mössbauer effect to accurately measure the frequency shift of gamma-ray photons from ^{57}Fe as they traversed the 22 m Jefferson Physical Laboratory tower at Harvard University (Pound & Rebka 1960; Pound & Snider 1964). Since then, numerous other experiments have found no violation at the percent level employing a variety of different techniques including: flying atomic clocks on commercial aircraft (Hafele & Keating 1972), using the highly stable millisecond pulsar PSR 1937+21 (Taylor 1987), using spacecraft in the Solar System (Krisner et al. 1990, 1993) and using the star S2 in orbit around the supermassive black hole at the centre of our galaxy (Amorim et al. 2019). The

most accurate measurements of the gravitational redshift are null redshift experiments which have reached the level of $|\varepsilon| < 2.5 \times 10^{-7}$ by comparing different types of atomic clocks (Ashby et al. 2018). However, these local tests do not eliminate the need for global tests, even with lower accuracy, as the two classes of tests are potentially sensitive to different types of violations.

Until recently, the most accurate global test of LPI within the Earth’s gravitational field was the well-known Gravity Probe A (GP-A) experiment that launched a hydrogen maser (H-maser) onboard a Scout rocket to an altitude of 10,000 km (Vessot & Levine 1979). By employing a sophisticated combination of up and down microwave links, the systematic effects of the first-order Doppler shift due to the rocket’s motion and Earth’s rotation, along with effects due to the Earth’s atmosphere could be eliminated. By comparing to a H-maser on the ground, a limit of $|\varepsilon| < 1.4 \times 10^{-4}$ was measured. More recently, spacecraft in Earth orbit are being used to improve on this measurement by an order of magnitude. Due to a fortunate accident, from the perspective of gravitational research, the Galileo 5 and 6 satellites from the European global navigation satellite system (GNSS) ended up in slightly elliptical orbits allowing two independent groups to make measurements several times more accurate than GP-A: $\varepsilon = (4.5 \pm 3.1) \times 10^{-5}$ (Herrmann et al. 2018) and $\varepsilon = (0.19 \pm 2.48) \times 10^{-5}$ (Delva et al. 2018). The space-VLBI mission RadioAstron, with a spacecraft in a highly elliptical orbit reaching distances as far as the Moon, is also being used by York and its international collaborators to measure the gravitational redshift over much larger distances with hopes of reaching an accuracy of $\delta\varepsilon \sim 10^{-5}$ (Litvinov et al. 2018).

Global measurements of the gravitational redshift will continue to increase in accuracy over the coming years with additional space missions planned and due to the ongoing improvements in the stability of frequency standards. Due to be launched later this year after numerous delays, the Atomic Clock Ensemble in Space (ACES) project will place advanced atomic clocks on the International Space Station enabling another order of magnitude improvement

in accuracy over the best measurements to date (Savalle et al. 2019). The STE-QUEST mission that may fly later this decade would also search for violations as small as $\delta\epsilon \sim 10^{-7}$ in both the gravitational fields of the Earth and the Sun (Altschul et al. 2015). Finally, the ongoing development of more robust optical lattice clocks will enable tests of LPI with even greater precision. In fact, Earth-based experiments with these frequency standards are already approaching the accuracy of space-based measurements, albeit over much smaller differences in gravitational potential (Takamoto et al. 2020).

The model development project detailed in this report relates to the measurement of the gravitational redshift using RadioAstron. The next section includes a description of the mission in general followed by details on the parts of the spacecraft particularly important to this project and how they were used to perform a gravity experiment between 2015 and 2017 whose data are now being analysed.

1.4 The RadioAstron Mission

It’s important to note from the outset that the RadioAstron mission was not designed for a gravitational experiment. Rather, in the 1970s the Space Research Institute of the USSR Academy of Sciences (IKI) was tasked with designing a 10 m radio telescope for the purpose of space interferometry. The first prototype was deployed to the Salyut 6 space station in 1979 helping to finalize the technical specifications of the telescope by the end of that decade. Between December 17-18, 1985, a conference was held in Moscow at which an international group of scientists and their institutions signed up to build the onboard radio astronomy receivers for the mission. Thus, turning what had been a purely Soviet endeavour into an international collaboration. With the breakdown of the Soviet Union, development slowed but the project survived. Leadership passed to the Astro Space Center of the Lebedev Physical Institute (ASC) formed in 1990 from the Astrophysics Division of IKI



Figure 1.1: Rendering of RadioAstron in orbit. From RadioAstron (2019).

and the Radio Astronomy Station of the Lebedev Physical Institute in Pushchino. By this point, the technique of Very Long Baseline Interferometry (VLBI), where radio telescopes work together to achieve higher angular resolutions, had been demonstrated in space using NASA's TDRSS geostationary satellite (Levy et al. 1986, 1989 and Linfield et al. 1990). In 1997, Japan launched the first dedicated space-VLBI mission, VSOP, with an 8 m diameter antenna mounted on the HALCA satellite orbiting the Earth in an elliptical orbit with a maximum geocentric distance of 28,000 km (Hirabayashi et al. 2000). The mission successfully functioned until 2003. Both the TDRSS experiment and VSOP confirmed the potential for space-VLBI and provided important experience that helped finalize the RadioAstron mission, in particular, establishing the need for a larger orbit.

After many delays, the RadioAstron spacecraft, named Spektr-R (see Figure 1.1), was launched on July 18, 2011, from the Baikonur cosmodrome aboard a Zenit-3F rocket, and placed into a highly elliptical orbit (Kardashev et al. 2013). The orbit evolves significantly

over time due to the gravitational effects of the Moon and Sun with apogee distances varying between 270,000 km and 370,000 km, perigee heights between 7,000 km and 80,000 km, eccentricity between 0.96 to 0.59 and orbital period between 8.1 and 10.2 days (see Figure 1.2). The variation in the orbit allowed observations with a range of angular resolutions and provided the ability to observe radio sources distributed all over the sky. The spacecraft is made up of the scientific payload, a 10 m parabolic reflector or space radio telescope (SRT) equipped with 1.35, 6.2, 18 and 92 cm receivers, and a Navigator service module, or spacecraft bus, common to other scientific missions but with the addition of an onboard H-maser clock (space H-maser or SHM) that provided an ultra-stable frequency standard for VLBI (see Figure 1.3). To enable communication with the ground at the largest distances, the spacecraft is also equipped with a 1.5 m diameter steerable high-gain communication antenna (HGCA). Command and telemetry sessions were performed every 2 – 3 days using the 64 m antenna at Bear Lakes and the 70 m antenna at Ussuriysk, both in Russia. While scientific data was downlinked and recorded during communication sessions using two ground stations: the 22 m radio telescope of the Pushchino Radio Astronomical Observatory (PU, see Figure 1.4a) near Moscow, Russia (RadioAstron 2019) and the 43 m telescope of the Green Bank Observatory (GB, see Figure 1.4b) in West Virginia, USA (Ford et al. 2014). The mission was able to continue after the failure of its SHM in June 2017 until communication with the spacecraft was lost on January 11, 2019, effectively ending the mission. Over an operational lifetime that significantly exceeded the planned 5 years and working with some of the largest ground-based radio telescopes (GRTs), RadioAstron performed the highest angular resolution observations of celestial objects ever made contributing to our understanding of supermassive black holes, accretion disks and relativistic jets in active galactic nuclei, stellar-mass black holes, neutron stars, interplanetary and interstellar plasma and regions with active formation of stars and planetary systems in our and other galaxies (Gurvits 2020).

During each ground station communication session, the spacecraft broadcast a tone at

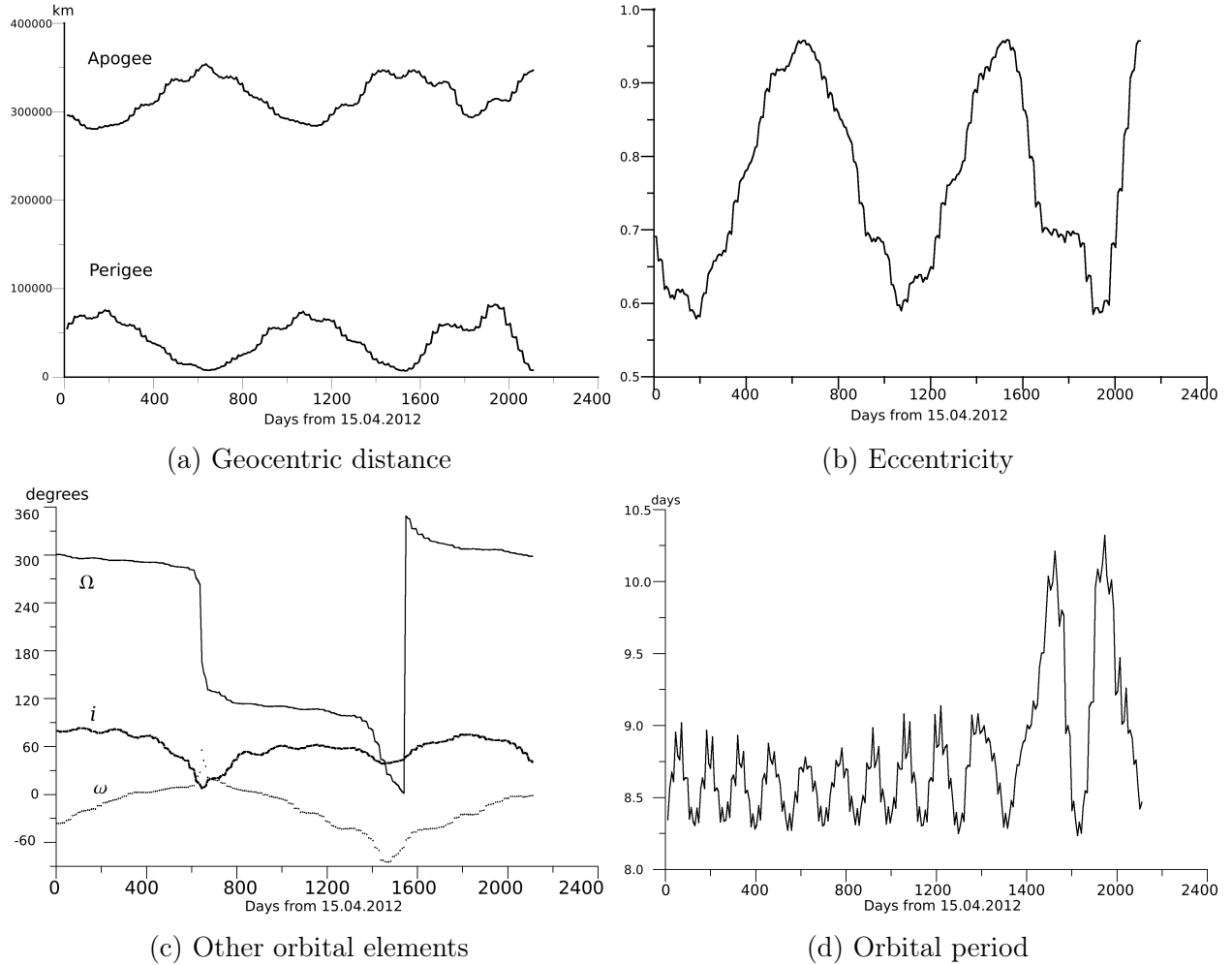


Figure 1.2: Orbital elements for RadioAstron over time. Note how the orbit evolved significantly over time mainly due to gravitational perturbations by the Moon and Sun with a characteristic period of ~ 3 years. Orbit correction maneuvers were performed in March 2012 and July 2017 to extend the lifetime of the spacecraft that would otherwise have been shortened by extended periods in Earth's shadow or premature re-entry. Plots from RadioAstron (2012).

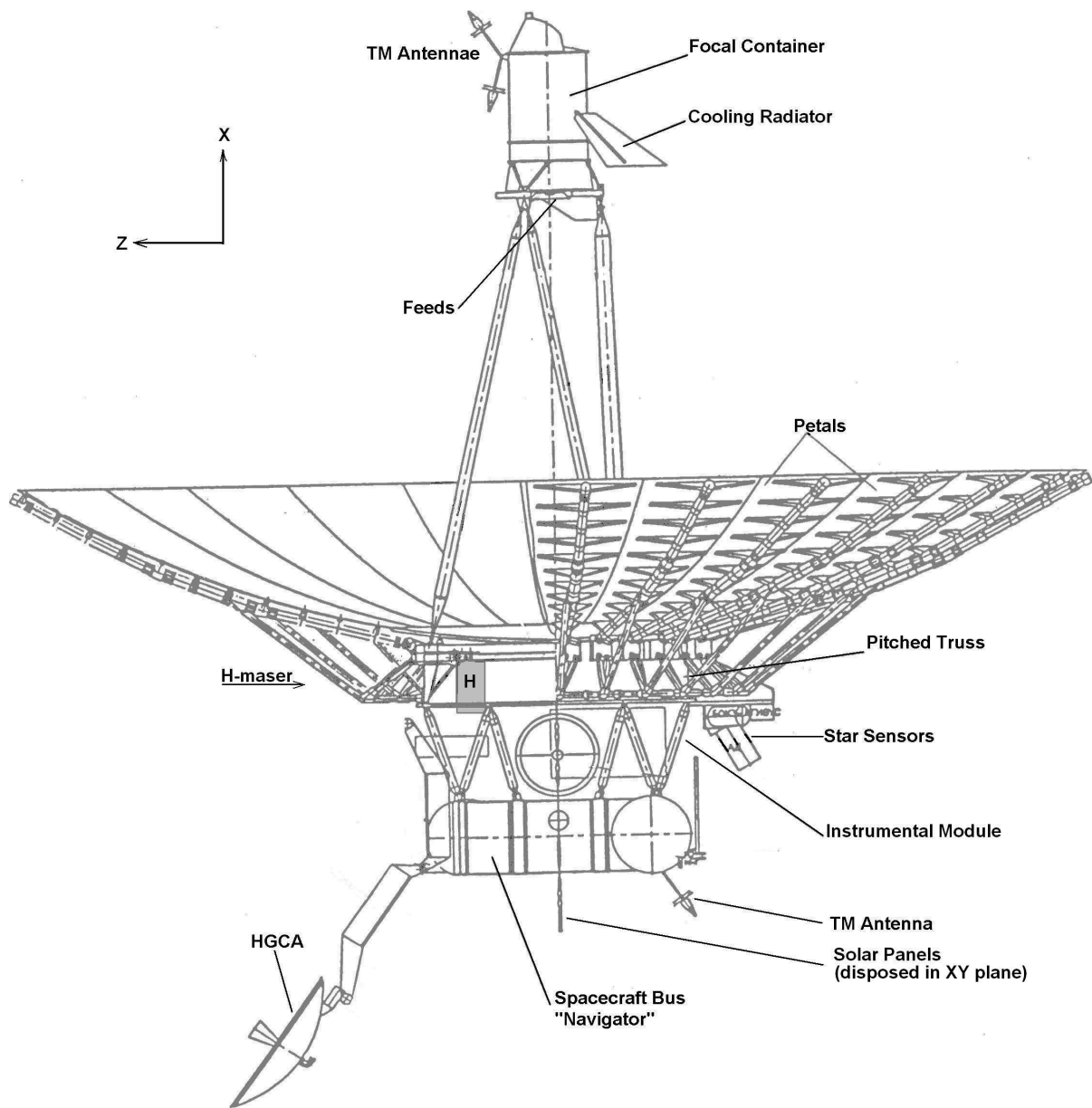


Figure 1.3: Schematic showing the main components of the RadioAstron spacecraft including the main antenna or SRT, the SHM and the HGCA. Additional omnidirectional TM antennae were used during command and control sessions to upload commands to the spacecraft and download telemetry. From RadioAstron (2019).



(a) Pushchino



(b) Green Bank

Figure 1.4: Ground stations for the RadioAstron mission: (a) 22 m RT-22 telescope at the Pushchino Radio Astronomical Observatory near Moscow, Russia © PRAO and (b) 43 m telescope at the Green Bank Observatory in West Virginia, USA © Green Bank Observatory.

8.4 GHz and a Quadrature Phase Shift Keying (QPSK) modulated 144 Mbps data stream on a 15 GHz carrier. The frequency of these signals was referenced to either the onboard SHM (in the H-maser or 1way mode) or to a tone uplinked from the ground station with a nominal frequency of 7.207,5 GHz using a phase-synchronization loop (the Coherent or 2way mode). The latter mode was also referenced to an H-maser, albeit one at the ground station. The two communication modes are illustrated in Figure 1.5. An H-maser is a frequency standard based on the hyperfine transition of atomic hydrogen with the characteristic frequency of 1,420 MHz (for example, see Vessot (2005)). Of particular interest to a gravity experiment, was the stability of the SHM — the first active H-maser flown onboard a spacecraft. Frequency stability is usually characterized by an Allan Deviation (ADEV or σ_y), which is, the square root of the relative variance of a standard’s frequency over a given averaging interval (Allan 1966). While ground testing found RadioAstron’s VCH-1010 H-maser, built by Vremya-Ch JSC of Russia, to have an ADEV of 7×10^{-15} over 100s, once in orbit destabilizing factors due to launch and the space environment limited stability, as measured by ADEV, to 2×10^{-14} over the same averaging interval. This was still better than the stability of the phase-synchronization loop, and so the 1way mode became the default mode of operation until the SHM failed. The post-launch stability is also comparable to that of the clock used on GP-A, being only a factor of two less stable (see Vessot & Levine 1979). However, with the ability to repeat a gravity experiment numerous times over the course of the mission, the random contribution of this instability to the uncertainty could be diminished by building up statistics. As a result, it was proposed that RadioAstron be used to test EEP by measuring the change in the gravitational redshift as the spacecraft orbited the Earth.

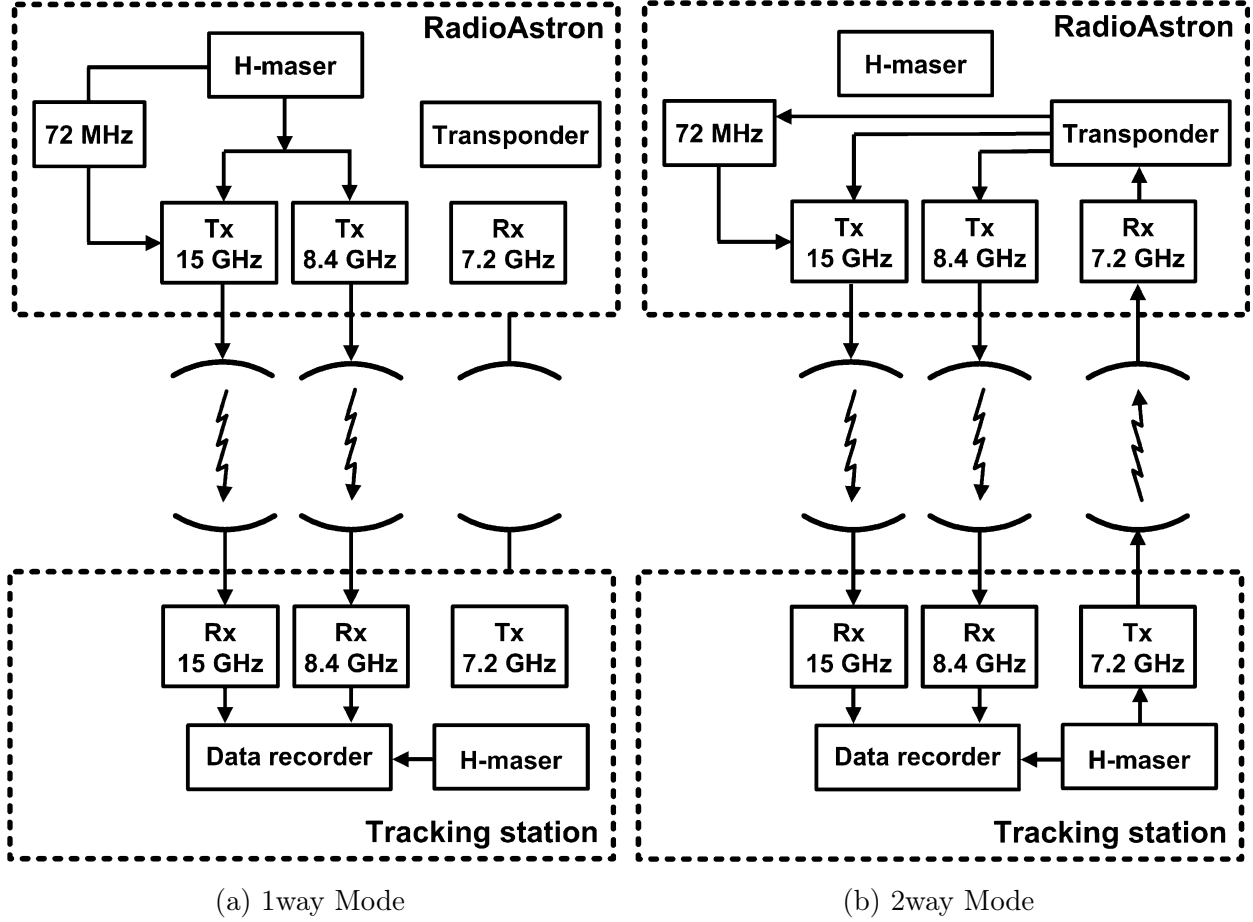


Figure 1.5: RadioAstron's communication modes. In the 1way mode, the frequency standard used to generate the tone at 8.4 GHz and the carrier at 15 GHz was the SHM. In contrast, in the 2way mode, a tone with a nominal frequency of 7.207,5 GHz was transmitted by the ground station and used in a phase-synchronization loop to provide the onboard frequency reference. From Litvinov et al. (2018).

1.5 Gravity Experiment with RadioAstron

A gravity experiment using RadioAstron was first suggested following the addition of the SHM to the design. The Cronos hydrogen maser clock redshift experiment (Busca et al. 2003), called for a number of additional changes to the design, particularly to the onboard communication system to enable a Doppler compensation scheme similar to GP-A. These changes never materialized leaving the spacecraft without a way to simultaneously transmit signals synchronized to the SHM and the ground station H-maser. However, a different approach was later proposed based on the ability to synchronize the science instruments to the SHM while leaving the downlink transponders locked to the uplink tone (Biriukov et al. 2014). This was called the semi-coherent mode and would have allowed the SHM’s frequency to be extracted from the science data modulated on the 15 GHz carrier which itself would have been locked to the phase synchronization loop and thus referenced to the ground H-maser. It was shown that the signals could be combined to achieve a Doppler compensation scheme. To further improve statistics, it was proposed that additional GRTs also observe and record the spacecraft’s communication signal using their VLBI backend systems and data recorders. If the semi-coherent mode were used, the number of available telescopes would be restricted by the need to observe the spacecraft at 15 GHz. Ultimately, using the semi-coherent mode was abandoned in favour of supporting additional GRTs only able to observe at 8.4 GHz. Instead of using simultaneous signals, the spacecraft communication system would instead be switched between 1way and 2way communication modes every few minutes in dedicated interleaved sessions for which an interpolated Doppler compensation scheme could be used (Litvinov et al. 2018). In total, 164 of these interleaved sessions were performed prior to the failure of the SHM and these form the basis of the RadioAstron gravity experiment.

Having a Doppler compensation scheme is a critical component of a redshift experiment in space. As an example, the reconstructed orbit for RadioAstron has a velocity uncertainty of

$\delta v \sim 1.4 \text{ mm/s}$ (Zakhvatkin et al. 2020) which corresponds to an uncertainty in the first-order Doppler shift of $\delta v/c = 5 \times 10^{-12}$. Although the gravitational redshift of the spacecraft compared to the ground is relatively large, on the order of $\Delta f/f \sim 10^{-9}$, without removing the first-order Doppler shift, such a measurement would be limited to 1%. Using a Doppler compensation scheme, it has been estimated (see Litvinov et al. (2017)) that a measurement with uncertainty of $\delta \varepsilon \sim 2.5 \times 10^{-5}$ may be possible with RadioAstron. This will require systematic effects on the order of 10^{-15} and larger, affecting the observed relative frequency shift in the spacecraft’s communication signal, to be calculated and subtracted.

1.6 Thesis Project Overview

The purpose of the project described in the remainder of this document was to develop and implement a dynamical model accurate enough to support the RadioAstron gravity experiments. To reach an uncertainty of $\sigma_{sys} \sim 10^{-15}$, effects down to third order in v/c must be considered. These include effects related to:

- Gravitational fields of the Earth, Moon and Sun
- Earth’s rotation and orientation in space including precession, nutation, oceanic tides, solid earth tides, pole tides and other small effects
- Motion over the signal propagation time
- Doppler shifts between spacecraft and ground station over both the uplink and downlink signal paths including relativistic or transverse Doppler shifts caused by time dilation
- Antenna phase-centre motion caused by an offset between the antenna phase-centre and the state vector reference point
- Refraction due to the non-dispersive dry and wet components of the troposphere

- Refraction due to the dispersive ionosphere
- H-maser frequency bias

Note that frequency bias refers to the difference in the rate at which a particular clock ticks relative to others due to internal effects. In the case of an H-maser, the frequency of the standard is very sensitive to the physical properties of the cavity in which hydrogen gas is undergoing hyperfine transitions. Over time, as the cavity ages and physically degrades, the H-maser will drift in frequency. So, at any given moment, the H-maser will have a frequency offset, or frequency bias, relative to other co-located clocks. To perform a gravity experiment using this type of clock, it is essential that this drift in frequency be taken into account. RadioAstron unfortunately lacked a method to independently measure the frequency bias of the SHM once in orbit. Instead, the bias will need to be determined as part of the analysis but is beyond the scope of this project. The frequency bias of the H-masers at the ground stations, however, could be determined using measurements relative to the GPS GNSS and will be described herein.

The remainder of this report is organized as follows. Chapter 2 derives a theoretical model of the relativistic Doppler effect using GR with care taken to keep all effects down to third order. Additional effects that are important at this level are also presented in the form they were used. Chapter 3 describes the software that was written to implement this model and includes sample output that showcases the various effects that are relevant to the gravity experiment. In Chapter 4, the magnitude of the various effects appearing after applying the Doppler compensation scheme is discussed in the context of the interleaved sessions. Chapter 5 discusses the impact this work will have on measuring the gravitational redshift. Finally, we conclude in Chapter 6 with a summary of the conclusions reached during this work.

Chapter 2

Theoretical Model

In this chapter we will develop mathematical expressions for the frequency shift of RadioAs-
tron's carrier signals as observed by the ground stations during communication sessions. We
will consider the following events:

1. Emission of the uplink phase synchronization tone by the ground station at time t_1
2. Reception of the uplink phase synchronization tone by the spacecraft at time t_2
3. Emission of the downlink signals by the spacecraft at time t_2 under the assumption
that instrumental delays onboard the spacecraft are negligible
4. Reception of the downlink signals by the ground station at time t_3

The state vectors related to these events are illustrated in Figure 2.1. These include the
ground (Earth) station position and velocity vectors ($\vec{r}_e$ and $\vec{v}_e$), the satellite position and
velocity vectors ($\vec{r}_s$ and $\vec{v}_s$) and the signal anti-propagation vectors ($\vec{s}_{21}$ and $\vec{s}_{32}$).

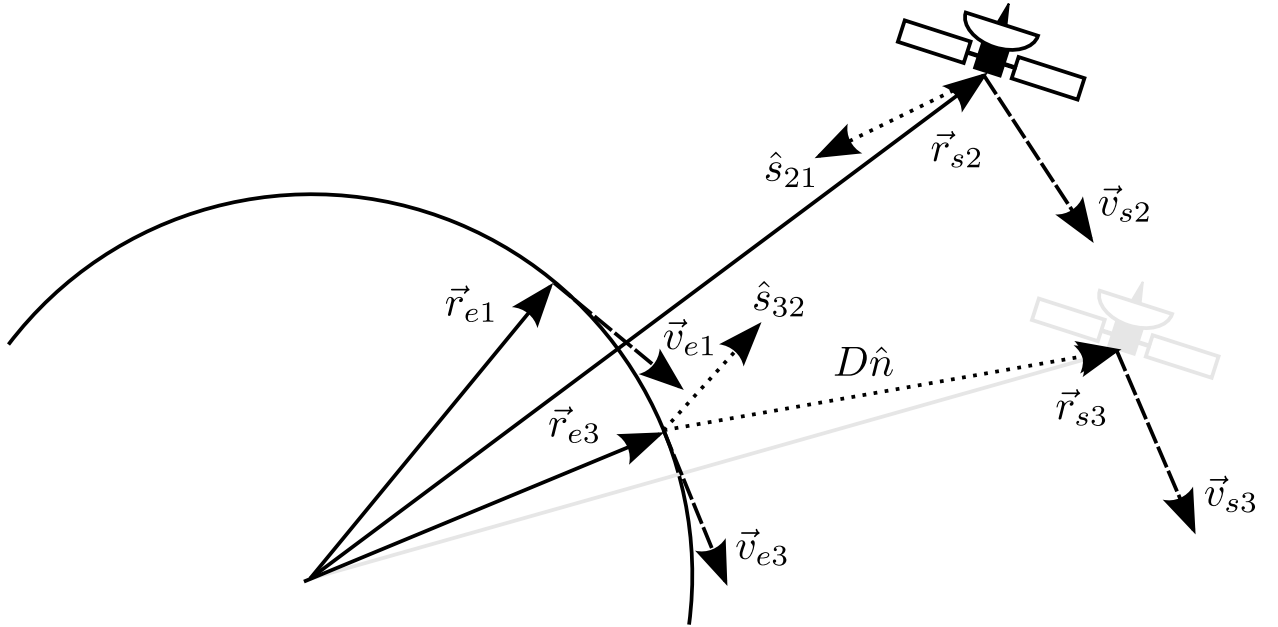


Figure 2.1: Key state vectors for the three events: (i) emission of uplink phase synchronization tone by ground station at time t_1 , (ii) reception of uplink tone by spacecraft and emission of downlink signal at time t_2 and (iii) reception of downlink signal at ground station at time t_3 . The vectors include the ground station state vectors at t_1 ($\vec{r}_{e1}$, $\vec{v}_{e1}$) and t_3 ($\vec{r}_{e3}$, $\vec{v}_{e3}$), spacecraft state vectors at t_2 ($\vec{r}_{s2}$, $\vec{v}_{s2}$) and t_3 ($\vec{r}_{s3}$, $\vec{v}_{s3}$), signal anti-propagation vectors ($\hat{s}_{21}$ and $\hat{s}_{32}$) and the range vector ($\vec{D} = D\hat{n}$) at t_3 .

2.1 Geometry

2.1.1 Frames of Reference

When working in the near Earth environment, it is convenient to use Earth-centered reference frames whose origin is the Earth’s centre-of-mass. These can be Earth-Centred Inertial (ECI) reference frames whose axes are fixed in space or Earth-Centred Earth-Fixed (ECEF) in which the coordinates of points on Earth’s surface do not change due to Earth’s rotation. Note that ECEF frames are only quasi-inertial as they are accelerated, along with Earth, around the Sun. To transform between these two types of reference frames, a model of the Earth’s rotation that includes the effects of precession and nutation as well as smaller corrections to Earth orientation in space, called Earth Orientation Parameters (EOP), is required.

We will mostly work in the Geocentric Celestial Reference Frame (GCRF) which is an ECI frame whose axes are aligned with those of the International Celestial Reference Frame (ICRF) which in turn is aligned relative to extragalactic radio sources with no apparent proper motion as seen from Earth. However, the spacecraft’s orbit is given in the J2000 reference frame, also an ECI frame, whose axes are aligned with the mean equinox and Earth’s rotation axis as at January 1, 2000 at 12:00. As the GCRF and J2000 frames are both ECI, they are related by a rotation called a frame bias or frame tie. For an example of early attempts at measuring this rotation, see Bartel et al. (1996) and Nunes & Bartel (1998).

When working with ECEF coordinates, we will use the International Terrestrial Reference Frame (ITRF) from the International Earth Rotation and Reference Systems Service (IERS). IERS also publishes a time series of EOP that will be used when transforming between the ITRF and the GCRF along with a detailed model of Earth’s rotation.

2.1.2 Range and Range Rate

The direction of propagation for the uplink signal will be opposite to the unit vector pointing from the spacecraft (observer) at t_2 to the ground station (emitter) at t_1 :

$$\hat{s}_{21} = \frac{\vec{r}_e(t_1) - \vec{r}_s(t_2)}{|\vec{r}_e(t_1) - \vec{r}_s(t_2)|} \quad (2.1)$$

While the downlink signal will propagate in the direction opposite to the unit vector pointing from the ground station (observer) at t_3 to the spacecraft (emitter) at t_2 :

$$\hat{s}_{32} = \frac{\vec{r}_s(t_2) - \vec{r}_e(t_3)}{|\vec{r}_s(t_2) - \vec{r}_e(t_3)|} \quad (2.2)$$

These signal anti-propagation vectors are closely related, but not identical, to the range vector which is defined using state vectors at the same instant in time:

$$\vec{D}(t) \equiv \vec{r}_s(t) - \vec{r}_e(t) \quad (2.3)$$

This vector's magnitude is called the range:

$$D(t) = |\vec{D}(t)| \quad (2.4)$$

The unit vector pointing in the direction of the range vector is then:

$$\hat{n}(t) = \frac{\vec{D}(t)}{D(t)} \quad (2.5)$$

The rate of change of the range, called the range rate, is given by:

$$\dot{D}(t) = \frac{d}{dt}D(t) = \hat{n}(t) \cdot [\vec{v}_s(t) - \vec{v}_e(t)] \quad (2.6)$$

The rate of change of the unit range vector is given by:

$$\dot{\hat{n}}(t) = \frac{d}{dt} \left(\frac{\vec{D}(t)}{D(t)} \right) = \frac{\vec{v}_s(t) - \vec{v}_e(t) - \dot{D}(t) \hat{n}(t)}{D(t)} \quad (2.7)$$

2.1.3 Declination

The direction to the spacecraft relative to the ground station can be given within an Earth-centred frame in terms of angles, including the declination angle given by:

$$\begin{aligned} \theta(t) &= \frac{\pi}{2} - \angle \hat{n}(t), \hat{i} \\ \therefore \sin \theta(t) &= \hat{n}(t) \cdot \hat{i} \end{aligned} \quad (2.8)$$

where $\hat{i}$ points along the polar axis:

$$\hat{i} \equiv [0, 0, 1] \quad (2.9)$$

and $\hat{n}(t)$ is the vector pointing from the ground station to the spacecraft. The rate of change of the declination is given by:

$$\dot{\theta}(t) = \frac{d}{dt} \left[\sin^{-1} \left(\hat{n}(t) \cdot \hat{i} \right) \right] = \sec \theta(t) \dot{\hat{n}}(t) \cdot \hat{i} \quad (2.10)$$

2.1.4 Elevation and Azimuth

Elevation and azimuth are defined within a local coordinate system whose axes are aligned with East, North and Up (ENU). The direction vectors for these axes are related to the local

latitude (ϕ) and longitude (λ) defined within an ECEF frame as follows:

$$\begin{aligned}\hat{e} &= \hat{n} \times \hat{u} \\ \hat{n} &= [-\sin \phi \cos \lambda, -\sin \phi \sin \lambda, \cos \phi] \\ \hat{u} &= [\cos \phi \cos \lambda, \cos \phi \sin \lambda, \sin \phi]\end{aligned}\tag{2.11}$$

The rotation that transforms coordinates between the frames, R , will apply the following transformation:

$$\begin{bmatrix} -\sin \lambda & -\sin \phi \cos \lambda & \cos \phi \cos \lambda \\ \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \sin \lambda \\ 0 & \cos \phi & \sin \phi \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\tag{2.12}$$

The direction to the spacecraft is given within the ECEF frame by:

$$\hat{s}(t) = \frac{\vec{r}_s(t) - \vec{r}_e(t)}{|\vec{r}_s(t) - \vec{r}_e(t)|}\tag{2.13}$$

and so within ENU coordinates this will be given by:

$$\hat{\rho}(t) = R \hat{s}(t)\tag{2.14}$$

Finally, elevation (E) and azimuth (A) are given by transforming from Cartesian to spherical coordinates:

$$\begin{aligned}E &= \sin^{-1} \hat{\rho}_z \\ A &= \frac{\pi}{2} - \tan^{-1} \frac{\hat{\rho}_y}{\hat{\rho}_x}\end{aligned}\tag{2.15}$$

where azimuth is reversed relative to the Cartesian right-handed system as it is conventionally measured from North-to-East.

2.1.5 Notational Conventions

We will use the convention that if a quantity x_α appears without an explicit time, then we assume it is as measured at time t_3 , referred to as the observation time:

$$x_\alpha \equiv x_\alpha(t_3) \quad (2.16)$$

For example, $\vec{D} \equiv \vec{D}(t_3) = \vec{r}_s(t_3) - \vec{r}_e(t_3)$. In general, we will work in terms of the observation time. When this is not the case, we may indicate the time using a subscript as follows:

$$x_{\alpha i} \equiv x_\alpha(t_i) \quad (2.17)$$

So for example, $\vec{v}_{s2}$ means the velocity of the spacecraft at time t_2 or $\vec{v}_s(t_2)$. When declination is given without reference to time, we assume the direction vector used is $\hat{n}$ at t_3 :

$$\sin \theta \equiv \hat{n} \cdot \hat{i} = \hat{n}(t_3) \cdot \hat{i} \quad (2.18)$$

and similarly for its rate of change:

$$\dot{\theta} \equiv \sec \theta \dot{\hat{n}} \cdot \hat{i} = \sec \theta(t_3) \dot{\hat{n}}(t_3) \cdot \hat{i} \quad (2.19)$$

Finally, we will use the same notation for the time associated with the position vector used to calculate the gravitational potential:

$$U_{\alpha i} \equiv U(\vec{r}_\alpha(t_i)) \quad (2.20)$$

and:

$$U_\alpha \equiv U(\vec{r}_\alpha(t_3)) \quad (2.21)$$

2.2 Metric Tensor and Spacetime Interval

In GR, the interval between two events is calculated using the infinitesimal line element, ds , defined as:

$$-ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta \quad (2.22)$$

where $g_{\alpha\beta}$ is called the metric tensor and describes the curvature of spacetime. This form of the line element is based on the metric signature $[+, -, -, -]$ which we will use. We will adopt International Astronomical Union (IAU) 2000 Conventions (in particular, resolutions B1.3 and B1.4 described in IAU (2000)) that solve the Einstein field equations in the weak gravitational field of the near-Earth environment assuming the Earth rotates at a constant rate and has a static mass distribution. The resulting metric is given in terms of what is called a Post-Newtonian expansion that treats spacetime curvature as a second order correction to a flat Minkowski spacetime. This metric takes on the following form:

$$g_{\alpha\beta} \equiv \begin{bmatrix} g_{00} & h_{01} & h_{02} & h_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \quad (2.23)$$

with the matrix elements being:

$$\begin{aligned} g_{00} &= -(1 - h_{00}) + o(c^{-4}) & h_{00} &= -\frac{2U}{c^2} + o(c^{-4}) \\ g_{11} = g_{22} = g_{33} &= (1 + h_{00}) + o(c^{-4}) & h_{01} = h_{02} = h_{03} &= o(c^{-3}) \end{aligned} \quad (2.24)$$

where U is the total gravitational potential and c the speed of light. In special relativity and

GR, it is natural to expand in terms of $1/c$, so the notation $o(c^{-i})$ is used to refer to the sum of terms including the i -th power of $1/c$ or higher where $i - 1$ is the order of the calculation. For example, in the case of a third order model, like the one we are developing here, we can discard terms of $o(c^{-4})$.

For a locally inertial, non-rotating and freely falling coordinate system ¹ with origin at the Earth's centre-of-mass, such as the GCRF, the line element from Eq. (2.22) is given by:

$$-ds^2 = -\left(1 + \frac{2U}{c^2}\right) (cdt)^2 + \left(1 - \frac{2U}{c^2}\right) d\sigma^2 + o(c^{-3}) \quad (2.25)$$

Where in isotropic coordinates aligned with the Earth's rotation axis:

$$d\sigma^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (2.26)$$

and where r is the radial direction, θ is the polar angle measured from the rotation axis towards the equatorial plane in a right-handed sense with negative angles being in the lower hemisphere of the system and ϕ is the azimuthal angle measured from a given zero point.

2.3 Gravitational Potential

The total gravitational potential at a given point in an ECEF frame, referred to by $\vec{r}$ measured from the Earth's centre-of-mass, is the sum of the Earth's gravitational potential and the tidal potentials of third-bodies:

$$U(\vec{r}) = U_{\oplus}(\vec{r}) + \sum_i U_{T,i}(\vec{r}, \vec{r}_i) \quad (2.27)$$

To first order the Earth's gravitational field is spherical, so it is natural to describe it using a

¹Freely falling indicates the origin of the coordinate system moves freely within the gravitational field of third-bodies including the Sun and Moon.

multipole expansion using spherical harmonics as follows:

$$U_{\oplus}(r, \theta, \phi) = -\frac{\mu_{\oplus}}{r} \sum_{n=0}^N \left(\frac{a_{\oplus}}{r}\right)^n \sum_{m=0}^n (C_{nm} \cos m\phi + S_{nm} \sin m\phi) P_n^m(\cos \theta) \quad (2.28)$$

where C_{nm} and S_{nm} are coefficients, P_n^m are Legendre polynomials and N is the degree to which the expansion is done. An example of this is the Earth Gravitational Model (EGM) 2008 model (Pavlis et al. 2012) from the National Geospatial Intelligence Agency (NGA). EGM 2008 uses the following values for the constants in this equation:

$$\begin{aligned} \mu_{\oplus} &= 3.986,004,415 \times 10^{14} \text{ m}^3\text{s}^{-2} \\ a_{\oplus} &= 6,378,136.3 \text{ m} \end{aligned} \quad (2.29)$$

For a satellite orbiting at the distances of RadioAstron, it is sufficient to retain only the monopole (radial) and quadrupole components of Earth's gravitational field ², and so we can use the following approximation:

$$U_{\oplus}(r, \theta) \approx -\frac{\mu_{\oplus}}{r} \left[1 - J_2 \left(\frac{a_{\oplus}}{r}\right)^2 \frac{1}{2} (3 \cos^2 \theta - 1) \right] \quad (2.30)$$

The substitution $\cos \theta = r_z/r$ is convenient when working with a Cartesian state vector and so we arrive at:

$$U_{\oplus}(\vec{r}) \approx -\frac{\mu_{\oplus}}{r} \left[1 - J_2 \left(\frac{a_{\oplus}}{r}\right)^2 \frac{1}{2} \left(3 \frac{r_z^2}{r^2} - 1 \right) \right] \quad (2.31)$$

The appropriate values for the constants in this equation is taken from IERS Conventions

²The Earth's gravitational field, as defined herein, does not have a dipole moment as this would require the existence of negative mass.

2010 (Petit & Luzum 2010):

$$\begin{aligned}\mu_{\oplus} &= 3.986,004,418 \times 10^{14} \text{ m}^3\text{s}^{-2} \\ a_{\oplus} &= 6,378,136.6 \text{ m} \\ J_2 &= 1.0826300 \times 10^{-3}\end{aligned}\tag{2.32}$$

As our coordinate system is freely falling within the gravitational potential of third-bodies, only tidal effects contribute to the total gravitational potential in the near-Earth environment (for example, see Ashby & Weiss (2013)). The expression for the tidal gravitational potential of the i -th body is given by:

$$U_{T,i}(\vec{r}, \vec{r}_i) = -\frac{GM_i}{|\vec{r} - \vec{r}_i|} + \frac{GM_i}{r_i} + \frac{GM_i \vec{r} \cdot \vec{r}_i}{r_i^3}\tag{2.33}$$

where $\vec{r}_i$ is the position for the third-body relative to the centre-of-mass of the Earth. In our case, it is sufficient to consider only the fields of the Moon and Sun. Note that it is important when explicitly adding tidal gravitational potentials that the tide-free coefficients³ of the EGM 2008 model be used so that tidal effects are not duplicated. Smaller tidal effects caused by the Moon and Sun that affect the Earth's gravitational field, such as solid earth and oceanic tides, as well as other small effects like atmospheric loading, polar motion and crustal rebound should also be added explicitly when using the tide-free coefficients.

2.4 Proper Time

In GR, proper time as measured by a local clock, is defined as:

$$d\tau \equiv \frac{ds}{c}\tag{2.34}$$

³Tidal effects are cyclical so mean tide coefficients are also available and include the time averaged value of these effects.

The rate of proper time will be different from point to point. To facilitate comparisons, we define coordinate time (t) as being the time at the centre-of-mass of the Earth. This is known as Geocentric Coordinate Time (TCG). We can then define the rate at which proper time passes relative to TCG as:

$$\frac{d\tau}{dt} = \frac{1}{c} \frac{ds}{dt} \quad (2.35)$$

From Eq. (2.25) we have:

$$\begin{aligned} -ds^2 &= -\left(1 + \frac{2U}{c^2}\right) (cdt)^2 + \left(1 - \frac{2U}{c^2}\right) d\sigma^2 + o(c^{-3}) \\ &= -\left[1 + \frac{2U}{c^2} - \left(1 - \frac{2U}{c^2}\right) \frac{d\sigma^2}{(cdt)^2}\right] (cdt)^2 + o(c^{-3}) \\ &= -\left[1 + \frac{2U}{c^2} - \frac{v^2}{c^2}\right] (cdt)^2 + o(c^{-3}) \end{aligned} \quad (2.36)$$

where $v = d\sigma/dt$ is the velocity through space as measured in the inertial coordinate system.

For a clock moving at $v \ll c$ and using the following Taylor expansion at $x = 0$:

$$\sqrt{1+x} = 1 + \frac{x}{2} + \dots \quad (2.37)$$

we have:

$$\frac{d\tau}{dt} = \frac{1}{c} \sqrt{\frac{ds^2}{dt^2}} = 1 + \frac{U}{c^2} - \frac{v^2}{2c^2} + o(c^{-4}) \quad (2.38)$$

This equation matches Eq. 28 in Ashby (2003) and Eq. 10 in Sazhin et al. (2010). In practice, a local clock may not be tuned so precisely that it follows the ideal time at a given location within the gravitational field, so we must add in a clock-specific offset, called the frequency

bias h_α , and so:

$$\frac{d\tau_\alpha}{dt} = 1 + \frac{U(\vec{r}_\alpha)}{c^2} - \frac{v_\alpha^2}{2c^2} + h_\alpha + o(c^{-4}) \quad (2.39)$$

where α refers to the specific clock with state vectors $\vec{r}_\alpha$ and $\vec{v}_\alpha$.

2.5 Terrestrial Time

Let us now consider a clock at rest on the surface of the Earth, such as the H-maser at the ground station. The spatial differentials will be given by:

$$dr_\alpha = 0, \quad d\theta_\alpha = 0, \quad d\phi_\alpha = \omega_\oplus dt \quad (2.40)$$

where $\omega_\oplus$ is the sidereal rotation rate of the Earth:

$$\omega_\oplus = 7.292,115,146,7 \times 10^{-5} \text{ rad/s} \quad (2.41)$$

Substituting these differentials into Eq. (2.26) we have:

$$d\sigma_\alpha^2 = \omega_\oplus^2 r_\alpha^2 \sin^2 \theta_\alpha dt^2 \quad (2.42)$$

and:

$$v_\alpha^2 = \frac{d\sigma_\alpha^2}{dt^2} = \omega_\oplus^2 r_\alpha^2 \sin^2 \theta_\alpha \quad (2.43)$$

and so Eq. (2.39) becomes:

$$\frac{d\tau_\alpha}{dt} = 1 + \frac{U(\vec{r}_\alpha)}{c^2} - \frac{\omega_\oplus^2 r_\alpha^2 \sin^2 \theta_\alpha}{2c^2} + h_\alpha + o(c^{-4}) \quad (2.44)$$

or alternately:

$$\frac{d\tau_\alpha}{dt} = 1 + \frac{\phi(\vec{r}_\alpha)}{c^2} + h_\alpha + o(c^{-4}) \quad (2.45)$$

where the effective gravitational potential, including the effect of centripetal motion, is given by:

$$\phi(\vec{r}) \equiv U(\vec{r}) - \frac{1}{2}\omega_\oplus^2 r^2 \sin^2 \theta \quad (2.46)$$

Terrestrial Time (TT) is a timescale that approximates an ideal clock at rest on Earth's equator at a distance equal to the mean radius of the Earth's surface from its centre-of-mass. Therefore:

$$r_{TT} = a_\oplus, \quad \theta_{TT} = \frac{\pi}{2}, \quad h_{TT} = 0 \quad (2.47)$$

and so:

$$\frac{dTT}{dt} = 1 + \frac{\phi_0}{c^2} + o(c^{-4}) \quad (2.48)$$

where:

$$\phi_0 = U\left(a_\oplus, \frac{\pi}{2}\right) - \frac{\omega_\oplus^2 a_\oplus^2}{2} \equiv -c^2 L_G \quad (2.49)$$

The constant L_G is an auxiliary defining constant in the IERS Conventions 2010 (Petit & Luzum 2010) and has the value:

$$L_G = 6.969290134 \times 10^{-10} \quad (2.50)$$

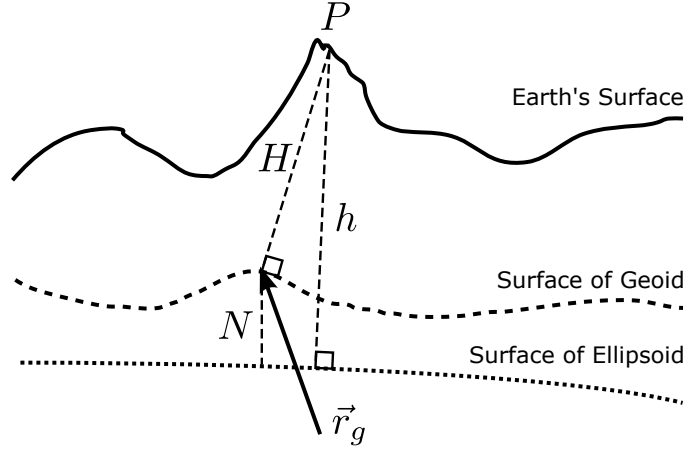


Figure 2.2: Each point (P) on Earth's irregular surface can be described by an elevation (h) above a reference ellipsoid and by an orthometric height (H) above the surface of gravitational equipotential called the geoid. The point on the geoid closest to P has the geocentric position $\vec{r}_g$. The distance between the ellipsoid and geoid is called the geoid height ($N = h - H$).

Using this, the rate of TT is then:

$$\frac{dTT}{dt} = 1 - L_G \quad (2.51)$$

The surface of equipotential where $\phi(\vec{r}_g(\theta, \phi)) = \phi_0$ is called the geoid with $\vec{r}_g(\theta, \phi)$ being the position of the points on this surface at which an ideal clock will tick in sync with TT.

Now returning to the clock α , it will be at a point on Earth's surface a distance H , called the orthometric height, above or below the geoid (see Figure 2.2). Orthometric height is measured perpendicularly to the geoid which will not necessarily be the radial direction, however, the following is a sufficiently accurate approximation for our purposes:

$$r_\alpha \approx r_{g,\alpha} + H_\alpha \quad (2.52)$$

where $r_{g,\alpha}$ is given for the point on the geoid closest to the clock. For orthometric heights $H < 1$ km, as is the case with our ground stations, the effective potential can be approximated

using Eq. (2.30) as follows:

$$\begin{aligned}\phi(\vec{r}_\alpha) &\approx -\left(\frac{\mu_\oplus}{r_{g,\alpha} + H_\alpha}\right) \left[1 - J_2 \left(\frac{a_\oplus}{r_{g,\alpha} + H_\alpha}\right)^2 \frac{1}{2} (3 \cos^2 \theta_\alpha - 1)\right] - \frac{1}{2} \omega_\oplus^2 (r_{g,\alpha} + H_\alpha)^2 \sin^2 \theta_\alpha \\ &\approx \phi_0 + \mu_\oplus \frac{H_\alpha}{r_{g,\alpha}^2}\end{aligned}\tag{2.53}$$

where in the second line we have omitted terms smaller than 10^2 which cause changes in the rate of proper time $< 10^{-15}$. Substituting this result into Eq. (2.45) and using the definition of ϕ_0 from Eq. (2.49) we find:

$$\frac{d\tau_\alpha}{dt} \approx 1 - L_G + \frac{\mu_\oplus}{c^2} \frac{H_\alpha}{r_{g,\alpha}^2} + h_\alpha\tag{2.54}$$

Note that h_α is the offset between the clock α and TCG. If the offset is instead measured relative to Terrestrial Time, for example as realized by the Global Positioning System (GPS), then the frequency bias will have a contribution due to the clock not being on the geoid:

$$h_\alpha^{TT} = h_\alpha + \frac{\mu_\oplus}{c^2} \frac{H_\alpha}{r_{g,\alpha}^2}\tag{2.55}$$

Substituting this into Eq. (2.54) we arrive at:

$$\frac{d\tau_\alpha}{dt} \approx 1 - L_G + h_\alpha^{TT}\tag{2.56}$$

2.6 Gravitational Redshift

Now let us consider the simple case of a geostationary satellite for which there is no relative motion between the satellite and the ground station on Earth. The frequency measured onboard the satellite, ν_s , compared to that on Earth, ν_e , is related to the rate of proper time

as follows:

$$\frac{\nu_e}{\nu_s} = \frac{d\tau_s}{d\tau_e} \quad (2.57)$$

where we note the reciprocal relationship between frequency and the rate of proper time.

Next, we define the frequency shift:

$$\Delta\nu \equiv \nu_e - \nu_s \quad (2.58)$$

and fractional or relative frequency shift:

$$y \equiv \frac{\Delta\nu}{\nu_s} = \frac{\nu_e}{\nu_s} - 1 \quad (2.59)$$

Using the relative frequency shift, we can write the rate of proper time onboard the satellite as:

$$\frac{d\tau_s}{dt} = \frac{d\tau_s}{d\tau_e} \cdot \frac{d\tau_e}{dt} = (1 + y) \cdot \frac{d\tau_e}{dt} \quad (2.60)$$

Substituting in Eq. (2.39) we get:

$$1 + \frac{U(\vec{r}_s)}{c^2} - \frac{v_s^2}{2c^2} + h_s \approx (1 + y) \left(1 + \frac{U(\vec{r}_e)}{c^2} - \frac{v_e^2}{2c^2} + h_e \right) \quad (2.61)$$

which after some rearranging becomes:

$$y \approx \frac{\frac{U(\vec{r}_s)}{c^2} - \frac{U(\vec{r}_e)}{c^2} - \frac{v_s^2}{2c^2} + \frac{v_e^2}{2c^2} + h_s - h_e}{1 + \frac{U(\vec{r}_e)}{c^2} - \frac{v_e^2}{2c^2} + h_e} \quad (2.62)$$

Note that the largest of the non-unity terms is the gravitational term of the ground station and is of order L_G (10^{-9}), so the product of any pair of these terms will fall below our

threshold of 10^{-15} . Using the following Taylor expansion at $x = 0$:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad (2.63)$$

and discarding insignificant terms, the expression becomes:

$$y \approx \frac{\Delta U}{c^2} - \frac{v_s^2 - v_e^2}{2c^2} + \Delta h \quad (2.64)$$

where:

$$\Delta U \equiv U(\vec{r}_s) - U(\vec{r}_e) \quad (2.65)$$

and:

$$\Delta h \equiv h_s - h_e \quad (2.66)$$

The first term is the gravitational redshift from Eq. (1.1), the second term is due to time dilation and the third is the difference in frequency bias between the clocks. However, we are not concerned with a geostationary satellite and must take into account the relative motion of the satellite and ground station which will introduce Doppler shifts that are several orders of magnitude larger than the effect due to gravity.

2.7 Frequency Bias

A frequency bias in an H-maser is caused by numerous systematic effects (for example, see Vessot (2005)). Ground station H-masers are routinely tuned to keep this bias within an acceptable range relative to TT as realized by GPS. This is done by comparing time on the local clock, T_e , that uses an H-maser as a frequency standard to time on a GPS receiver,

T_{GPS} , locked to the timing signals broadcast by the constellation of GPS satellites. The clocks onboard these satellites are also atomic clocks and are steered to follow TT. The difference, usually measured in nanoseconds, is given by:

$$\Delta T = T_e - T_{GPS} \quad (2.67)$$

The rate at which this difference changes relative to TT will be given by:

$$\frac{d\Delta T}{dT} = \frac{dT_e}{dT} - \frac{dT_{GPS}}{dT} \quad (2.68)$$

Since GPS time is steered to follow TT, this can be written as:

$$\frac{d\Delta T}{dT} = \frac{d\tau_e/dt}{dT/dt} - 1 \quad (2.69)$$

which can then be rearranged as follows:

$$\frac{d\tau_e}{dt} = \left(\frac{d\Delta T}{dT} + 1 \right) \frac{dT}{dt} \quad (2.70)$$

The left side is given by Eq. (2.54) while the rate of TT is given by Eq. (2.51). And so:

$$\begin{aligned} 1 - L_G + \frac{\mu_{\oplus}}{c^2} \frac{H}{r_g} + h_e &= \left(1 + \frac{d\Delta T}{dT} \right) (1 - L_G) \\ h_e &= \frac{d\Delta T}{dT} - \frac{\mu_{\oplus}}{c^2} \frac{H_e}{r_{g,e}^2} - L_G \frac{d\Delta T}{dT} \end{aligned} \quad (2.71)$$

As the H-maser is steered to stay close to TT, the product of $\frac{d\Delta T}{dT}$ and L_G will be sufficiently small to neglect leaving:

$$h_e = \frac{d\Delta T}{dT} - \frac{\mu_{\oplus}}{c^2} \frac{H_e}{r_{g,e}^2} \quad (2.72)$$

Therefore, by keeping track of the difference between time measured by a local clock at the

ground station and GPS time, we can determine the frequency bias relative to TCG.

2.8 Propagation Time

Before considering the relativistic Doppler effect, we must derive expressions for the signal propagation times between three key events in 4D spacetime: $P_1 = (ct_1, \vec{r}_e(t_1))$ when the uplink signal is transmitted by the ground station, $P_2 = (ct_2, \vec{r}_s(t_2))$ when the spacecraft receives this signal and instantaneously transmits the downlink signal and $P_3 = (ct_3, \vec{r}_e(t_3))$ when the ground station receives the downlink signal.

In GR, electromagnetic signals travel along null geodesics connecting events such that the spacetime interval between them is zero:

$$-ds = g_{\alpha\beta}dx^\alpha dx^\beta = 0 \quad (2.73)$$

And so using Eq. (2.25) we have the condition that:

$$0 = -\left(1 + \frac{2U}{c^2}\right)(cdt)^2 + \left(1 - \frac{2U}{c^2}\right)d\sigma^2 + o(c^{-3}) \quad (2.74)$$

which can be rearranged to solve for dt :

$$dt = \frac{1}{c} \sqrt{\frac{1 - \frac{2U}{c^2}}{1 + \frac{2U}{c^2}}} d\sigma + o(c^{-4}) \quad (2.75)$$

Using the following Taylor expansion at $x = 0$:

$$\sqrt{\frac{1-x}{1+x}} = 1 - x + \frac{x^2}{2} - \dots \quad (2.76)$$

we can simplify this expression to the following:

$$dt = \frac{d\sigma}{c} - \frac{2U}{c^3}d\sigma + o(c^{-4}) \quad (2.77)$$

By integrating this expression, we can derive the propagation time over the one way and two way signal paths used by the communication modes of RadioAstron.

2.8.1 One Way Propagation Time

First, let us integrate Eq. (2.77) from P_2 to P_3 to get the one way (downlink) propagation time:

$$\Delta t_{1w} = \int_{t_2}^{t_3} dt = \int_{\vec{r}_s(t_2)}^{\vec{r}_e(t_3)} \frac{d\sigma}{c} - \int_{\vec{r}_s(t_2)}^{\vec{r}_e(t_3)} \frac{2U}{c^3} d\sigma \quad (2.78)$$

The first term is the propagation time as described by special relativity (Δt_{1w}^{SR}) and the second is the additional time caused by traveling through the gravitational field called the Shapiro delay (Δt_{1w}^{SH}). Focusing on the first term:

$$\Delta t_{1w}^{SR} = \int_{\vec{r}_s(t_2)}^{\vec{r}_e(t_3)} \frac{d\sigma}{c} = \frac{|\vec{r}_e(t_3) - \vec{r}_s(t_2)|}{c} = \frac{|\vec{r}_e(t_2) - \vec{r}_s(t_3)|}{c} \quad (2.79)$$

We would like to write this in terms of D , the range, but to do so we need to replace $\vec{r}_s(t_2)$ with quantities at time t_3 . Let us start by assuming that over Δt_{1w} the acceleration of the spacecraft is approximately constant, that is, $\vec{a}_s(t_2) \approx \vec{a}_s(t_3)$. This is a reasonable assumption as the spacecraft is held at a fixed attitude during communication sessions and the change in

gravitational acceleration over this short time period is negligible. So we can write:

$$\begin{aligned}
\vec{r}_s(t_3) &\approx \vec{r}_s(t_2) + \vec{v}_s(t_2) \Delta t_{1w} + \frac{1}{2} \vec{a}_s(t_2) \Delta t_{1w}^2 \\
&\approx \vec{r}_s(t_2) + [\vec{v}_s(t_3) - \vec{a}_s(t_3) \Delta t] \Delta t_{1w} + \frac{1}{2} \vec{a}_s(t_3) \Delta t_{1w}^2 \\
\therefore \vec{r}_s(t_2) &\approx \vec{r}_s - \vec{v}_s \Delta t_{1w} + \frac{1}{2} \vec{a}_s \Delta t_{1w}^2
\end{aligned} \tag{2.80}$$

where in the final line we have used the notation described earlier where the absence of a time implies a quantity at t_3 . Next, we must deal with the problem that $\vec{r}_s(t_2)$ is dependent on Δt_{1w} which we want to calculate using $\vec{r}_s(t_2)$. So, we must iterate to find relevant terms up to third order in $1/c$. We bootstrap this process using $\Delta t_{1w} \approx D/c$ to get:

$$\vec{r}_s(t_2) \approx \vec{r}_s - \frac{D}{c} \vec{v}_s + \frac{D^2}{2c^2} \vec{a}_s \tag{2.81}$$

Substituting this approximation into Eq. (2.79) and working to second order we have:

$$\begin{aligned}
\Delta t_{1w}^{SR} &\approx \left| \frac{\vec{D}}{c} - \frac{D}{c^2} \vec{v}_s + \frac{D^2}{2c^3} \vec{a}_s \right| \\
&\approx \sqrt{\left(\frac{\vec{D}}{c} - \frac{D}{c^2} \vec{v}_s + \frac{D^2}{2c^3} \vec{a}_s \right) \cdot \left(\frac{\vec{D}}{c} - \frac{D}{c^2} \vec{v}_s + \frac{D^2}{2c^3} \vec{a}_s \right)} \\
&\approx \sqrt{\frac{D^2}{c^2} - 2 \frac{D \vec{D} \cdot \vec{v}_s}{c^3} + o(c^{-4})} \\
&\approx \frac{D}{c} \sqrt{1 - \left[2 \frac{\hat{n} \cdot \vec{v}_s}{c} + o(c^{-2}) \right]}
\end{aligned} \tag{2.82}$$

Using the following Taylor expansion at $x = 0$:

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \dots \tag{2.83}$$

we can simplify this expression to:

$$\begin{aligned}\Delta t_{1w}^{SR} &\approx \frac{D}{c} \left[1 - \frac{\hat{n} \cdot \vec{v}_s}{c} + o(c^{-2}) \right] \\ &\approx \frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + o(c^{-3})\end{aligned}\tag{2.84}$$

Notice the second order term that we did not include in our initial approximation for Δt_{1w} .

So, we revise our expression for $\vec{r}_s(t_2)$ by adding the additional term:

$$\vec{r}_s(t_2) = \vec{r}_s - \frac{D}{c} \vec{v}_s + \frac{\vec{D} \cdot \vec{v}_s}{c^2} \vec{v}_s + \frac{D^2}{2c^2} \vec{a}_s + o(c^{-3})\tag{2.85}$$

Substituting this into Eq. (2.79) and this time working to third order:

$$\begin{aligned}\Delta t_{1w}^{SR} &= \left| \frac{\vec{D}}{c} - \frac{D}{c^2} \vec{v}_s + \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{v}_s + \frac{D^2}{2c^3} \vec{a}_s + o(c^{-4}) \right| \\ &= \sqrt{\frac{D^2}{c^2} - 2 \frac{D \vec{D} \cdot \vec{v}_s}{c^3} + 2 \frac{(\vec{D} \cdot \vec{v}_s)^2}{c^4} + \frac{D^2 \vec{D} \cdot \vec{a}_s}{c^4} + \frac{D^2 v_s^2}{c^4} + o(c^{-5})} \\ &= \frac{D}{c} \sqrt{1 - \left[2 \frac{\hat{n} \cdot \vec{v}_s}{c} - \frac{v_s^2}{c^2} - 2 \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} - \frac{\vec{D} \cdot \vec{a}_s}{c^2} + o(c^{-3}) \right]} \\ &= \frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + \frac{D v_s^2}{2c^3} + \frac{D (\hat{n} \cdot \vec{v}_s)^2}{2c^3} + \frac{D \vec{D} \cdot \vec{a}_s}{2c^3} + o(c^{-4})\end{aligned}\tag{2.86}$$

This is a sufficiently accurate expression for our purposes so we will stop here.

Now we consider the Shapiro delay. It is a relatively small effect and so we only need to consider the monopole component of Earth's gravitational field. We can also disregard tidal effects due to third bodies when working to third order in $1/c$. Thus:

$$\Delta t_{1w}^{SH} = - \int_{\vec{r}_s(t_2)}^{\vec{r}_e(t_3)} \frac{2}{c^3} \left(-\frac{\mu_{\oplus}}{r} \right) d\sigma = \frac{2\mu_{\oplus}}{c^3} \int_{\vec{r}_s(t_2)}^{\vec{r}_e(t_3)} \frac{d\sigma}{r}\tag{2.87}$$

As the effect is small and the spacecraft is moving slowly compared to c , it is sufficient to approximate $\vec{r}_s(t_2) \approx \vec{r}_s(t_3)$. We now define a parameter l , ranging from 0 to $|\vec{r}_e - \vec{r}_s|$, that

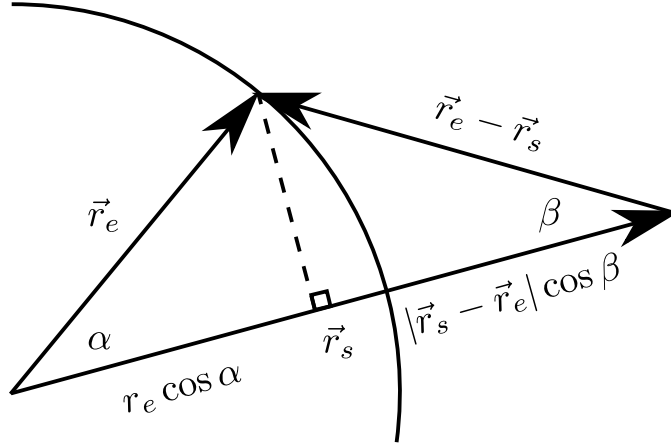


Figure 2.3: The relationship between the spacecraft ($\vec{r}_s$) and ground station ($\vec{r}_e$) position vectors and the angles between them (α and β).

is used to define a vector, $\vec{r}$, ranging from $\vec{r}_s$ to $\vec{r}_e$ (see Figure 2.3):

$$\vec{r} = \vec{r}_s + l \frac{\vec{r}_e - \vec{r}_s}{|\vec{r}_e - \vec{r}_s|} \quad (2.88)$$

Using this vector, we can rewrite the integral as follows:

$$\begin{aligned} \Delta t_{1w}^{SH} &= \frac{2\mu_{\oplus}}{c^3} \int_0^{|\vec{r}_e - \vec{r}_s|} \frac{dl}{\left| \vec{r}_s + l \frac{\vec{r}_e - \vec{r}_s}{|\vec{r}_e - \vec{r}_s|} \right|} \\ &= \frac{2\mu_{\oplus}}{c^3} \int_0^D \frac{dl}{\sqrt{r_s^2 + 2lr_s \cos \beta + l^2}} \end{aligned} \quad (2.89)$$

where β is the angle between $\vec{r}_s$ and $(\vec{r}_e - \vec{r}_s)$. An integral of this form has the following general solution:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \ln \left(2\sqrt{a}\sqrt{ax^2 + bx + c} + 2ax + b \right) \quad (2.90)$$

In our case:

$$dx = dl, \quad x = l, \quad a = 1, \quad b = 2r_s \cos \beta, \quad c = r_s^2 \quad (2.91)$$

And so the solution to our integral is:

$$\begin{aligned}
\Delta t_{1w}^{SH} &= \frac{2\mu_{\oplus}}{c^3} \ln (2 |\vec{r}_s + l\hat{n}| + 2r_s \cos \beta + 2l) \Bigg|_{l=0}^{l=D} \\
&= \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e + r_s \cos \beta + D}{r_s + r_s \cos \beta} \\
&= \frac{2\mu_{\oplus}}{c^3} \ln \left(\frac{r_e + r_s - D}{r_e + r_s - D} \cdot \frac{r_e + r_s \cos \beta + D}{r_s + r_s \cos \beta} \right) \\
&= \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e^2 + r_e r_s \cos \beta + r_s r_e + r_s^2 \cos \beta + r_s D - r_s D \cos \beta - D^2}{(r_s + r_s \cos \beta) (r_e + r_s - D)}
\end{aligned} \tag{2.92}$$

Now, using the following result:

$$D^2 = |\vec{r}_s - \vec{r}_e|^2 = r_s^2 - 2r_s r_e \cos \alpha + r_e^2 \tag{2.93}$$

where α is the angle between $\vec{r}_s$ and $\vec{r}_e$, we get:

$$\begin{aligned}
\Delta t_{1w}^{SH} &= \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e r_s \cos \beta + r_s r_e + r_s^2 \cos \beta + r_s D - r_s D \cos \beta - r_s^2 + 2r_s r_e \cos \alpha}{(r_s + r_s \cos \beta) (r_e + r_s - D)} \\
&= \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e (r_s + r_s \cos \beta) + 2r_s (r_e \cos \alpha - D \cos \beta) - r_s^2 + r_s^2 \cos \beta + D (r_s + r_s \cos \beta)}{(r_s + r_s \cos \beta) (r_e + r_s - D)}
\end{aligned} \tag{2.94}$$

From Figure 2.3, we see that:

$$r_s = r_e \cos \alpha + D \cos (\pi - \beta) = r_e \cos \alpha - D \cos \beta \tag{2.95}$$

And so, substituting in r_s we find:

$$\begin{aligned}
\Delta t_{1w}^{SH} &= \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e (r_s + r_s \cos \beta) + r_s^2 + r_s^2 \cos \beta + D (r_s + r_s \cos \beta)}{(r_s + r_s \cos \beta) (r_e + r_s - D)} \\
&= \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e + r_s + D}{r_e + r_s - D}
\end{aligned} \tag{2.96}$$

Combining the two components of Δt_{1w} from Eqs. (2.86) and (2.96), we arrive at the following third order expression for the one way propagation time:

$$\begin{aligned}\Delta t_{1w} &= \Delta t_{1w}^{SR} + \Delta t_{1w}^{SH} \\ &= \frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + \frac{2\mu_{\oplus}}{c^3} \ln \frac{r_e + r_s + D}{r_e + r_s - D} \\ &\quad + \frac{Dv_s^2}{2c^3} + \frac{D(\hat{n} \cdot \vec{v}_s)^2}{2c^3} + \frac{D(\vec{D} \cdot \vec{a}_s)}{2c^3} + o(c^{-4})\end{aligned}\tag{2.97}$$

This expression is equivalent to Eq. 25 in Sazhin et al. (2010) by approximating the spacecraft acceleration with $\vec{a}_s = -\frac{\mu_{\oplus}\vec{r}_s}{r_s^3}$ which is valid at third order.

2.8.2 Two Way Propagation Time

Next, we will calculate the propagation time, Δt_{2w} , between events P_1 and P_3 . We can separate this time into two components:

$$\Delta t_{2w} = \Delta t_{up} + \Delta t_{1w}\tag{2.98}$$

where Δt_{up} is the time between P_1 and P_2 which, similarly to Eq. (2.78), will be given by:

$$\Delta t_{up} = \int_{\vec{r}_e(t_1)}^{\vec{r}_s(t_2)} \frac{d\sigma}{c} - \int_{\vec{r}_e(t_1)}^{\vec{r}_s(t_2)} \frac{2U}{c^3} d\sigma\tag{2.99}$$

In the same way as before, we will work on each term individually, the first being a special relativistic term (Δt_{2w}^{SR}) and the second the Shapiro delay (Δt_{2w}^{SH}). As previously noted, $\vec{r}_s(t_2) \approx \vec{r}_s(t_3)$ at third order. Similarly, we can use $\vec{r}_e(t_1) \approx \vec{r}_e(t_3)$ at third order. Applying these approximations to the Shapiro delay term, we find:

$$\Delta t_{up}^{SH} = - \int_{\vec{r}_e(t_1)}^{\vec{r}_s(t_2)} \frac{2U}{c^3} d\sigma \approx - \int_{\vec{r}_e(t_3)}^{\vec{r}_s(t_3)} \frac{2U}{c^3} d\sigma = -\Delta t_{1w}^{SH}\tag{2.100}$$

Thus the Shapiro delay will be approximately equal and opposite over the two signal paths and therefore will cancel out in the two way propagation time. The remainder of the uplink propagation time is given by:

$$\Delta t_{up}^{SR} = \int_{\vec{r}_e(t_1)}^{\vec{r}_s(t_2)} \frac{d\sigma}{c} = \frac{|\vec{r}_s(t_2) - \vec{r}_e(t_1)|}{c} \quad (2.101)$$

While we have an expression for $\vec{r}_s(t_2)$ in terms of quantities at t_3 , we now need to develop an expression for $\vec{r}_e(t_1)$. We begin with the following, similar to Eq. (2.80):

$$\vec{r}_e(t_1) \approx \vec{r}_e(t_3) - \vec{v}_e \Delta t_{2w} + \frac{1}{2} \vec{a}_e \Delta t_{2w}^2 \quad (2.102)$$

As before, we will iterate to find the appropriate terms starting with the approximation $\Delta t_{up} \approx \frac{D}{c}$ which, when combined with the first and second order terms from Δt_{1w} , provides the initial approximation of $\Delta t_{2w} \approx 2\frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2}$. Substituting this into Eq. (2.102) we get:

$$\vec{r}_e(t_1) \approx \vec{r}_e(t_3) - 2\frac{D}{c} \vec{v}_e + \frac{\vec{D} \cdot \vec{v}_s}{c^2} \vec{v}_e + 2\frac{D^2}{c^2} \vec{a}_e \quad (2.103)$$

Using this approximation in Eq. (2.101) and working to second order we get:

$$\begin{aligned} \Delta t_{up}^{SR} &\approx \left| \frac{\vec{D}}{c} - \frac{D}{c^2} \vec{v}_s + 2\frac{D}{c^2} \vec{v}_e + \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{v}_s - 2\frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{v}_e + \frac{D^2}{2c^3} \vec{a}_s - 2\frac{D^2}{c^3} \vec{a}_e \right| \\ &\approx \sqrt{\frac{D^2}{c^2} - 2\frac{D\vec{D} \cdot \vec{v}_s}{c^3} + 4\frac{D\vec{D} \cdot \vec{v}_e}{c^3} + o(c^{-4})} \\ &\approx \frac{D}{c} \sqrt{1 - \left[2\frac{\hat{n} \cdot \vec{v}_s}{c} - 4\frac{\hat{n} \cdot \vec{v}_e}{c} + o(c^{-2}) \right]} \\ &\approx \frac{D}{c} \left[1 - \frac{\hat{n} \cdot \vec{v}_s}{c} + 2\frac{\hat{n} \cdot \vec{v}_e}{c} + o(c^{-2}) \right] \\ &\approx \frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + 2\frac{\vec{D} \cdot \vec{v}_e}{c^2} + o(c^{-3}) \end{aligned} \quad (2.104)$$

Notice the additional second order terms that we need to add to Δt_{2w} as follows:

$$\begin{aligned}
\Delta t_{2w} &= \Delta t_{up} + \Delta t_{1w} \\
&\approx \left(\frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + 2 \frac{\vec{D} \cdot \vec{v}_e}{c^2} \right) + \left(\frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} \right) + o(c^{-3}) \\
&\approx 2 \frac{D}{c} - 2 \frac{\vec{D} \cdot \vec{v}_s}{c^2} + 2 \frac{\vec{D} \cdot \vec{v}_e}{c^2} + o(c^{-3})
\end{aligned} \tag{2.105}$$

We use this to revise our expression for $\vec{r}_e(t_1)$ using Eq. (2.102):

$$\vec{r}_e(t_1) = \vec{r}_e - 2 \frac{D}{c} \vec{v}_e + 2 \frac{\vec{D} \cdot \vec{v}_s}{c^2} \vec{v}_e - 2 \frac{\vec{D} \cdot \vec{v}_e}{c^2} \vec{v}_e + 2 \frac{D^2}{c^2} \vec{a}_e + o(c^{-3}) \tag{2.106}$$

Now using this updated expression, we revise Δt_{up} :

$$\begin{aligned}
\Delta t_{up}^{SR} &= \left| \frac{\vec{D}}{c} - \frac{D}{c^2} \vec{v}_s + 2 \frac{D}{c^2} \vec{v}_e + \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{v}_s - 2 \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{v}_e + 2 \frac{\vec{D} \cdot \vec{v}_e}{c^3} \vec{v}_e \right. \\
&\quad \left. + \frac{D^2}{2c^3} \vec{a}_s - 2 \frac{D^2}{c^3} \vec{a}_e + o(c^{-4}) \right| \\
&= \sqrt{\frac{D^2}{c^2} - 2 \frac{D \vec{D} \cdot \vec{v}_s}{c^3} + 4 \frac{D \vec{D} \cdot \vec{v}_e}{c^3} + 2 \frac{(\vec{D} \cdot \vec{v}_s)^2}{c^4} - 4 \frac{(\vec{D} \cdot \vec{v}_s)(\vec{D} \cdot \vec{v}_e)}{c^4} + 4 \frac{(\vec{D} \cdot \vec{v}_e)^2}{c^4} \\
&\quad + \frac{D^2 \vec{D} \cdot \vec{a}_s}{c^4} - 4 \frac{D^2 \vec{D} \cdot \vec{a}_e}{c^4} + \frac{D^2 v_s^2}{c^4} - 4 \frac{D^2 \vec{v}_s \cdot \vec{v}_e}{c^4} + 4 \frac{D^2 v_e^2}{c^4} + o(c^{-5})} \\
&= \frac{D}{c} \sqrt{1 - \left[\begin{aligned} &2 \frac{\hat{n} \cdot \vec{v}_s}{c} - 4 \frac{\hat{n} \cdot \vec{v}_e}{c} - 2 \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} + 4 \frac{(\hat{n} \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e)}{c^2} - 4 \frac{(\hat{n} \cdot \vec{v}_e)^2}{c^2} \\ &- \frac{\vec{D} \cdot \vec{a}_s}{c^2} + 4 \frac{\vec{D} \cdot \vec{a}_e}{c^2} - \frac{v_s^2}{c^2} + 4 \frac{\vec{v}_s \cdot \vec{v}_e}{c^2} - 4 \frac{v_e^2}{c^2} + o(c^{-3}) \end{aligned} \right]} \\
&= \frac{D}{c} \left[\begin{aligned} &1 - \frac{\hat{n} \cdot \vec{v}_s}{c} + 2 \frac{\hat{n} \cdot \vec{v}_e}{c} + \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} - 2 \frac{(\hat{n} \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e)}{c^2} + 2 \frac{(\hat{n} \cdot \vec{v}_e)^2}{c^2} \\ &+ \frac{\vec{D} \cdot \vec{a}_s}{2c^2} - 2 \frac{\vec{D} \cdot \vec{a}_e}{c^2} + \frac{v_s^2}{2c^2} - 2 \frac{\vec{v}_s \cdot \vec{v}_e}{c^2} + 2 \frac{v_e^2}{c^2} \\ &- \frac{(\hat{n} \cdot \vec{v}_s)^2}{2c^2} - 2 \frac{(\hat{n} \cdot \vec{v}_e)^2}{c^2} + 2 \frac{(\hat{n} \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e)}{c^2} + o(c^{-3}) \end{aligned} \right] \\
&= \frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + 2 \frac{\vec{D} \cdot \vec{v}_e}{c^2} + 2 \frac{D v_e^2}{c^3} + \frac{D v_s^2}{2c^3} - 2 \frac{D \vec{v}_s \cdot \vec{v}_e}{c^3} \\
&\quad + \frac{D (\hat{n} \cdot \vec{v}_s)^2}{2c^3} + \frac{D \vec{D} \cdot \vec{a}_s}{2c^3} - 2 \frac{D \vec{D} \cdot \vec{a}_e}{c^3} + o(c^{-4})
\end{aligned} \tag{2.107}$$

Finally, combining Eqs. (2.107) and (2.86) we have the two way propagation time to third order:

$$\begin{aligned}
\Delta t_{2w} &= \Delta t_{up}^{SR} + \Delta t_{1w}^{SR} \\
&= 2\frac{D}{c} - 2\frac{\vec{D} \cdot \vec{v}_s}{c^2} + 2\frac{\vec{D} \cdot \vec{v}_e}{c^2} + 2\frac{Dv_e^2}{c^3} + \frac{Dv_s^2}{c^3} - 2\frac{D\vec{v}_s \cdot \vec{v}_e}{c^3} \\
&\quad + \frac{D(\hat{n} \cdot \vec{v}_s)^2}{c^3} + \frac{D\vec{D} \cdot \vec{a}_s}{c^3} - 2\frac{D\vec{D} \cdot \vec{a}_e}{c^3} + o(c^{-4})
\end{aligned} \tag{2.108}$$

This expression is equivalent to Eq. 28 in Sazhin et al. (2010).

2.9 Earlier State Using Observation Time State

2.9.1 Position Vectors

In the previous section, we developed expressions for the position vectors using only the observation time state. For ease of reading, these are restated here with new numbers starting with the ground station at t_1 (Eq. (2.106)):

$$\vec{r}_{e1} = \vec{r}_e - 2\frac{D}{c}\vec{v}_e + 2\frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_e - 2\frac{\vec{D} \cdot \vec{v}_e}{c^2}\vec{v}_e + 2\frac{D^2}{c^2}\vec{a}_e + o(c^{-3}) \tag{2.109}$$

and next the spacecraft at t_2 (Eq. (2.85)):

$$\vec{r}_{s2} = \vec{r}_s - \frac{D}{c}\vec{v}_s + \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3}) \tag{2.110}$$

2.9.2 Velocity Vectors

Next, we derive expressions for the velocity vectors starting with the ground station at t_1 using Eq. (2.108):

$$\begin{aligned}
\vec{v}_{e1} &\approx \vec{v}_e - \vec{a}_e \Delta t_{2w} + \frac{1}{2} \vec{j}_e \Delta t_{2w}^2 \\
&= \vec{v}_e - \vec{a}_e \left[2 \frac{D}{c} + 2 \frac{\vec{D} \cdot (\vec{v}_e - \vec{v}_s)}{c^2} + o(c^{-3}) \right] + \frac{1}{2} \vec{j}_e \left[4 \frac{D^2}{c^2} + o(c^{-3}) \right] \\
&= \vec{v}_e - 2 \frac{D \vec{a}_e}{c} + 2 \frac{\vec{D} \cdot (\vec{v}_s - \vec{v}_e)}{c^2} \vec{a}_e + 2 \frac{D^2}{c^2} \vec{j}_e + o(c^{-3})
\end{aligned} \tag{2.111}$$

And the spacecraft at t_2 using Eq. (2.97):

$$\begin{aligned}
\vec{v}_{s2} &\approx \vec{v}_s - \vec{a}_s \Delta t_{1w} + \frac{1}{2} \vec{j}_s \Delta t_{1w}^2 \\
&= \vec{v}_s - \vec{a}_s \left[\frac{D}{c} - \frac{\vec{D} \cdot \vec{v}_s}{c^2} + o(c^{-3}) \right] + \frac{1}{2} \vec{j}_s \left[\frac{D^2}{c^2} + o(c^{-3}) \right] \\
&= \vec{v}_s - \frac{D \vec{a}_s}{c} + \frac{\vec{D} \cdot \vec{v}_s}{c^2} \vec{a}_s + \frac{D^2}{2c^2} \vec{j}_s + o(c^{-3})
\end{aligned} \tag{2.112}$$

2.9.3 Signal Anti-Propagation Vectors

Next, we find expressions for the signal anti-propagation vectors solely using state at t_3 , starting with $\hat{s}_{32}$ from Eq. (2.2) and using Eq. (2.110):

$$\begin{aligned}
\hat{s}_{32} &= \frac{\vec{r}_{s2} - \vec{r}_e}{|\vec{r}_{s2} - \vec{r}_e|} \\
&= \frac{\vec{D} - \frac{D}{c}\vec{v}_s + \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3})}{\left| \vec{D} - \frac{D}{c}\vec{v}_s + \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3}) \right|} \\
&= \frac{\vec{D} - \frac{D}{c}\vec{v}_s + \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3})}{\sqrt{D^2 - 2\frac{D\vec{D} \cdot \vec{v}_s}{c} + 2\frac{(\vec{D} \cdot \vec{v}_s)^2}{c^2} + \frac{D^2\vec{D} \cdot \vec{a}_s}{c^2} + \frac{D^2v_s^2}{c^2} + o(c^{-3})}} \\
&= \frac{\hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c^2}\vec{v}_s + \frac{D}{2c^2}\vec{a}_s + o(c^{-3})}{\sqrt{1 - 2\frac{\hat{n} \cdot \vec{v}_s}{c} + 2\frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} + \frac{v_s^2}{c^2} + o(c^{-3})}}
\end{aligned} \tag{2.113}$$

With the following Taylor expansion at $x = 0$:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \dots \tag{2.114}$$

we find:

$$\begin{aligned}
\hat{s}_{32} &= \left[\hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c^2}\vec{v}_s + \frac{D}{2c^2}\vec{a}_s + o(c^{-3}) \right] \\
&\quad \left[1 + \frac{\hat{n} \cdot \vec{v}_s}{c} - \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} - \frac{\vec{D} \cdot \vec{a}_s}{2c^2} - \frac{v_s^2}{2c^2} + \frac{3}{2} \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} + o(c^{-3}) \right] \\
&= \hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c}\hat{n} + \frac{1}{2c^2} \left[(\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D\vec{a}_s - (\vec{D} \cdot \vec{a}_s) \hat{n} \right] + o(c^{-3})
\end{aligned} \tag{2.115}$$

This expression is equivalent to Eq. 57 in Sazhin et al. (2010). The uplink signal anti-propagation vector is derived in the same way though the number of relevant terms is larger.

We start with Eq. (2.1) and use Eqs. (2.109) and (2.110):

$$\begin{aligned}
\hat{s}_{21} &= \frac{\vec{r}_{e1} - \vec{r}_{s2}}{|\vec{r}_{e1} - \vec{r}_{s2}|} \\
&= \frac{-\vec{D} - \frac{D}{c}(2\vec{v}_e - \vec{v}_s) + 2\frac{\vec{D} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e - \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + 2\frac{D^2}{c^2}\vec{a}_e - \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3})}{\left| -\vec{D} - \frac{D}{c}(2\vec{v}_e - \vec{v}_s) + 2\frac{\vec{D} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e - \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + 2\frac{D^2}{c^2}\vec{a}_e - \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3}) \right|} \\
&= \frac{-\vec{D} - \frac{D}{c}(2\vec{v}_e - \vec{v}_s) + 2\frac{\vec{D} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e - \frac{\vec{D} \cdot \vec{v}_s}{c^2}\vec{v}_s + 2\frac{D^2}{c^2}\vec{a}_e - \frac{D^2}{2c^2}\vec{a}_s + o(c^{-3})}{\sqrt{D^2 + 2\frac{D}{c}\vec{D} \cdot (2\vec{v}_e - \vec{v}_s) - 4\frac{\vec{D} \cdot (\vec{v}_s - \vec{v}_e)(\vec{D} \cdot \vec{v}_e)}{c^2} + 2\frac{(\vec{D} \cdot \vec{v}_s)^2}{c^2} - 4\frac{D^2}{c^2}\vec{D} \cdot \vec{a}_e + \frac{D^2}{c^2}\vec{D} \cdot \vec{a}_s + \frac{D^2}{c^2}|2\vec{v}_e - \vec{v}_s|^2 + o(c^{-3})}} \\
&= \frac{-\hat{n} - \frac{(2\vec{v}_e - \vec{v}_s)}{c} + 2\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e - \frac{\hat{n} \cdot \vec{v}_s}{c^2}\vec{v}_s + 2\frac{D}{c^2}\vec{a}_e - \frac{D}{2c^2}\vec{a}_s + o(c^{-3})}{\sqrt{1 + 2\frac{\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)}{c} - 4\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)(\hat{n} \cdot \vec{v}_e)}{c^2} + 2\frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} - 4\frac{\vec{D} \cdot \vec{a}_e}{c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} + \frac{|2\vec{v}_e - \vec{v}_s|^2}{c^2} + o(c^{-3})}} \quad (2.116)
\end{aligned}$$

Again using the Taylor expansion from Eq. (2.114), we find:

$$\begin{aligned}
\hat{s}_{21} &= \left[-\hat{n} - \frac{(2\vec{v}_e - \vec{v}_s)}{c} + 2\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e - \frac{\hat{n} \cdot \vec{v}_s}{c^2}\vec{v}_s + 2\frac{D}{c^2}\vec{a}_e - \frac{D}{2c^2}\vec{a}_s + o(c^{-3}) \right] \\
&\quad \left[1 - \frac{\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)}{c} + 2\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)(\hat{n} \cdot \vec{v}_e)}{c^2} - \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} + 2\frac{\vec{D} \cdot \vec{a}_e}{c^2} - \frac{\vec{D} \cdot \vec{a}_s}{2c^2} - \frac{|2\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{3}{2}\frac{[\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)]^2}{c^2} + o(c^{-3}) \right] \\
&= -\hat{n} - 2\frac{\vec{v}_e}{c} + \frac{\vec{v}_s}{c} + 2\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e - \frac{\hat{n} \cdot \vec{v}_s}{c^2}\vec{v}_s + 2\frac{D}{c^2}\vec{a}_e - \frac{D}{2c^2}\vec{a}_s \\
&\quad + \frac{\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)}{c}\hat{n} - 2\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)(\hat{n} \cdot \vec{v}_e)}{c^2}\hat{n} + \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2}\hat{n} - 2\frac{\vec{D} \cdot \vec{a}_e}{c^2}\hat{n} + \frac{\vec{D} \cdot \vec{a}_s}{2c^2}\hat{n} \\
&\quad + \frac{|2\vec{v}_e - \vec{v}_s|^2}{2c^2}\hat{n} - \frac{3}{2}\frac{[\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)]^2}{c^2}\hat{n} + \frac{\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)}{c^2}(2\vec{v}_e - \vec{v}_s) + o(c^{-3}) \quad (2.117)
\end{aligned}$$

We will now reorganize this expression using the following substitutions:

- $-\frac{(\hat{n} \cdot \vec{v}_s)}{c^2}\vec{v}_s + \frac{\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)}{c^2}(2\vec{v}_e - \vec{v}_s) + 2\frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)}{c^2}\vec{v}_e = 4\frac{\hat{n} \cdot \vec{v}_e}{2c^2}\vec{v}_e - 4\frac{\hat{n} \cdot \vec{v}_e}{2c^2}\vec{v}_s$

- $\frac{\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)}{c} \hat{n} = \frac{\hat{n} \cdot \vec{v}_e}{c} \hat{n} - \frac{\dot{D}}{c} \hat{n}$
- $-2 \frac{\hat{n} \cdot (\vec{v}_s - \vec{v}_e)(\hat{n} \cdot \vec{v}_e)}{c^2} \hat{n} + \frac{(\hat{n} \cdot \vec{v}_s)^2}{c^2} \hat{n} - \frac{3}{2} \frac{[\hat{n} \cdot (2\vec{v}_e - \vec{v}_s)]^2}{c^2} \hat{n} = \frac{1}{2c^2} [8(\hat{n} \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e) - 8(\hat{n} \cdot \vec{v}_e)^2 - (\hat{n} \cdot \vec{v}_s)^2] \hat{n}$
- $\frac{8(\hat{n} \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e)}{2c^2} \hat{n} - 4 \frac{(\hat{n} \cdot \vec{v}_e)^2}{2c^2} \hat{n} + 4 \frac{\hat{n} \cdot \vec{v}_e}{2c^2} \vec{v}_e - 4 \frac{\hat{n} \cdot \vec{v}_e}{2c^2} \vec{v}_s = \frac{2(\hat{n} \cdot \vec{v}_e)}{c^2} [\dot{D} \hat{n} + (\hat{n} \cdot \vec{v}_s) \hat{n} + \vec{v}_e - \vec{v}_s]$
- $\frac{|2\vec{v}_e - \vec{v}_s|^2}{2c^2} \hat{n} = 2 \frac{v_e^2}{c^2} \hat{n} - 2 \frac{(\vec{v}_e \cdot \vec{v}_s)}{c^2} \hat{n} + \frac{v_s^2}{2c^2} \hat{n}$

After cancelling terms and factoring we arrive at:

$$\begin{aligned}
\hat{s}_{21} = & -\hat{n} - 2 \frac{\vec{v}_e}{c} + \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_e}{c} \hat{n} - \frac{\dot{D}}{c} \hat{n} \\
& - \frac{2}{c^2} [(\hat{n} \cdot \vec{v}_e)^2 \hat{n} - v_e^2 \hat{n} - D \vec{a}_e + (\vec{D} \cdot \vec{a}_e) \hat{n}] \\
& - \frac{1}{2c^2} [(\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D \vec{a}_s - (\vec{D} \cdot \vec{a}_s) \hat{n}] \\
& - \frac{2(\hat{n} \cdot \vec{v}_e)}{c^2} [\dot{D} \hat{n} + (\hat{n} \cdot \vec{v}_s) \hat{n} + \vec{v}_e - \vec{v}_s] \\
& - 2 \frac{\vec{v}_e \cdot \vec{v}_s}{c^2} \hat{n} + o(c^{-3})
\end{aligned} \tag{2.118}$$

2.9.4 Rate of Proper Time

We will need Eq. (2.39) for the spacecraft at time t_2 :

$$\frac{d\tau_{s2}}{dt} = 1 + \frac{U_{s2}}{c^2} - \frac{v_{s2}^2}{2c^2} + h_{s2} + o(c^{-4}) \tag{2.119}$$

Let us consider each of the non-unity terms, referring to them from left to right as the gravitational term, the relativistic term and the frequency bias term. Starting with the gravitational term, we see from Eq. (2.110) that the leading term will be U_{s3}/c^2 . The only significant correction, at third order in $1/c$, will arise from the monopole component of the

Earth's gravitational potential, and so we can write:

$$\frac{U_{s2}}{c^2} \approx \frac{U_s}{c^2} - \frac{\mu_{\oplus}}{c^2} \left[\frac{1}{r_{s2}} - \frac{1}{r_s} \right] \quad (2.120)$$

Using Eq. (2.110), we can approximate the reciprocal of the geocentric distance as follows:

$$\begin{aligned} \frac{1}{r_{s2}} &= \left[\left(\vec{r}_s - \frac{D}{c} \vec{v}_s + o(c^{-2}) \right) \cdot \left(\vec{r}_s - \frac{D}{c} \vec{v}_s + o(c^{-2}) \right) \right]^{-\frac{1}{2}} \\ &= \left[r_s^2 - 2 \frac{D}{c} \vec{r}_s \cdot \vec{v}_s + o(c^{-2}) \right]^{-\frac{1}{2}} \\ &= \frac{1}{r_s} \left[1 - 2 \frac{D}{c} \frac{\vec{r}_s \cdot \vec{v}_s}{r_s^2} + o(c^{-2}) \right]^{-\frac{1}{2}} \end{aligned} \quad (2.121)$$

With the Taylor expansion from Eq. (2.114), this becomes:

$$\begin{aligned} \frac{1}{r_{s2}} &= \frac{1}{r_s} \left[1 + \frac{D}{c} \frac{\vec{r}_s \cdot \vec{v}_s}{r_s^2} + o(c^{-2}) \right] \\ &= \frac{1}{r_s} + \frac{D}{c} \frac{\vec{r}_s \cdot \vec{v}_s}{r_s^3} + o(c^{-2}) \end{aligned} \quad (2.122)$$

Substituting this result into Eq. (2.120), we arrive at:

$$\frac{U_{s2}}{c^2} = \frac{U_s}{c^2} - \frac{D}{c^3} \frac{\mu_{\oplus} \vec{r}_s \cdot \vec{v}_s}{r_s^3} + o(c^{-4}) \quad (2.123)$$

This can be written in a more compact form by noting that at this order:

$$\nabla U = \frac{d}{dr} \left(-\frac{\mu_{\oplus}}{r} \right) \hat{r} = \frac{\mu_{\oplus}}{r^3} \vec{r} \quad (2.124)$$

and so:

$$\frac{U_{s2}}{c^2} = \frac{U_s}{c^2} - \frac{D}{c^3} \nabla U_s \cdot \vec{v}_s + o(c^{-4}) \quad (2.125)$$

Turning to the relativistic term in Eq. (2.119) and using Eq. (2.112), we note that:

$$\begin{aligned} v_{s2}^2 &= \left[\vec{v}_s - \frac{D\vec{a}_s}{c} + o(c^{-2}) \right] \cdot \left[\vec{v}_s - \frac{D\vec{a}_s}{c} + o(c^{-2}) \right] \\ &= v_s^2 - 2\frac{D}{c}\vec{v}_s \cdot \vec{a}_s + o(c^{-2}) \end{aligned} \quad (2.126)$$

and so:

$$-\frac{v_{s2}^2}{2c^2} = -\frac{v_s^2}{2c^2} + \frac{D}{c^3}\vec{v}_s \cdot \vec{a}_s + o(c^{-4}) \quad (2.127)$$

Finally, we note that the frequency bias of the masers used in this experiment drift at a rate smaller than 1×10^{-13} /day, and so the change in the bias over the propagation time will be insignificant. Thus, we can approximate $h_s(t_2) \approx h_s(t_3)$. Putting all the terms together we arrive at the expression we are looking for:

$$\frac{d\tau_{s2}}{dt} = 1 + \frac{U_s}{c^2} - \frac{v_s^2}{2c^2} + \frac{D}{c^3}\vec{v}_s \cdot \vec{a}_s - \frac{D}{c^3}\nabla U_s \cdot \vec{v}_s + h_s + o(c^{-4}) \quad (2.128)$$

Following the exact same procedure, but using Eq. (2.109) and Eq. (2.111), we arrive at:

$$\frac{d\tau_{e1}}{dt} = 1 + \frac{U_e}{c^2} - \frac{v_e^2}{2c^2} + 2\frac{D}{c^3}\vec{v}_e \cdot \vec{a}_e - 2\frac{D}{c^3}\nabla U_e \cdot \vec{v}_e + h_e + o(c^{-4}) \quad (2.129)$$

2.10 Relativistic Doppler Effect

2.10.1 Doppler Effect in General Relativity

We will now develop the relativistic Doppler effect by considering how frequency is transformed between an emitter and observer in GR. We begin by defining the four-dimensional wave

vector of a beam of light propagating from an emitter (E) to an observer (O):

$$k^\alpha \equiv k^0 (1, -\hat{s}) \quad (2.130)$$

where $k^0 = \frac{2\pi\nu}{c}$ and $\hat{s}$ is the signal anti-propagation vector that points in the direction opposite to propagation. As the wave vector is a null vector, we have the condition that its norm is zero:

$$k^\alpha k_\alpha = 0 \quad (2.131)$$

Now, consider the four-velocity of the emitter and observer moving within a common frame of reference (e.g. the GCRF):

$$u^\alpha \equiv \frac{dx^\alpha}{ds} = \frac{1}{c} \frac{dx^\alpha}{d\tau} \quad (2.132)$$

where $x^\alpha = (ct, \vec{r})$, ds is as defined in Eq. (2.34), $\vec{r}$ is the position vector within the common frame and $d\tau$ is the proper time. Introducing coordinate time (dt), we can write:

$$u^\alpha = \frac{dt}{d\tau} \frac{1}{c} \frac{dx^\alpha}{dt} = \frac{dt}{d\tau} \beta^\alpha = u^0 (1, \vec{\beta}) \quad (2.133)$$

where $\beta^\alpha \equiv (1, \beta^i)$, $\vec{\beta} \equiv \frac{\vec{v}}{c}$, $\vec{v}$ is the velocity vector within the common frame and $u^0 = \frac{dt}{d\tau}$. Since $v < c$, u^α is a timelike vector and so its norm is given by:

$$u^\alpha u_\alpha = -1 \quad (2.134)$$

Next, we define an invariant, a quantity that does not depend on the coordinate system

chosen, by taking the scalar product of the wave vector and four-velocity:

$$K \equiv k^\alpha u_\alpha \quad (2.135)$$

In a frame co-moving with the emitter:

$$u_{EE}^\alpha = (1, 0, 0, 0), \quad k_E^\alpha = \frac{2\pi\nu_E}{c} (1, -\hat{s}_E) \quad (2.136)$$

where the first subscript E on the four-velocity indicates an observable as measured in the frame of the emitter while the second subscript E indicates it is of the emitter. And so the value of the invariant at the location of the emitter will be:

$$K_E = k_E^\alpha u_{\alpha EE} = k_E^\alpha g_{\alpha\beta} u_{EE}^\beta \quad (2.137)$$

As the emitter's frame is locally flat, we have:

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2.138)$$

and so:

$$K_E = \frac{2\pi\nu_E}{c} \quad (2.139)$$

Similarly, in a frame co-moving with the observer:

$$u_{OO}^\alpha = (1, 0, 0, 0), \quad k_O^\alpha = \frac{2\pi\nu_O}{c} (1, -\hat{s}_O) \quad (2.140)$$

and so the invariant at the location of the observer will be:

$$K_O = \frac{2\pi\nu_O}{c} \quad (2.141)$$

The ratio of the two invariants is then the ratio of the frequencies as measured by the emitter and observer:

$$\frac{K_O}{K_E} = \frac{\nu_O}{\nu_E} \quad (2.142)$$

Thus:

$$y = \frac{\Delta\nu}{\nu_E} = \frac{\nu_O - \nu_E}{\nu_E} = \frac{K_O}{K_E} - 1 \quad (2.143)$$

This result will not depend on the coordinate system used. It is more convenient to work in terms of a common frame and so next we will calculate the value of the invariants in the GCRF. Using the subscript G to indicate a quantity in the GCRF, we have:

$$u_{GE}^\alpha = \frac{dt}{d\tau_E} \left(1, \vec{\beta}_{GE}\right), \quad u_{GO}^\alpha = \frac{dt}{d\tau_O} \left(1, \vec{\beta}_{GO}\right), \quad k_G^\alpha = \frac{2\pi\nu_G}{c} (1, -\hat{s}_G) \quad (2.144)$$

where the signal frequency ν_G and direction opposite of propagation $\vec{s}_G$, as measured in the GCRF, will be functions of an affine parameter along the path which we will consider to be the coordinate (geocentric) time t . The value of the invariant as measured at the time of emission (t_E) is then given by:

$$\begin{aligned} K_E &= \frac{2\pi\nu_G(t_E)}{c} [1, -\hat{s}_G(t_E)] g_{\alpha\beta} \frac{dt}{d\tau_E}(t_E) [1, \vec{\beta}_{GE}(t_E)] \\ &= \frac{2\pi\nu_G(t_E)}{c} \frac{dt}{d\tau_E}(t_E) \left[1 + \hat{s}_G(t_E) \cdot \vec{\beta}_{GE}(t_E)\right] \end{aligned} \quad (2.145)$$

where we have used the flat metric from Eq. (2.138) which also applies locally to the GCRF.

Similarly, the invariant as measured at the time of observation (t_O) is:

$$K_O = \frac{2\pi\nu_G(t_O)}{c} \frac{dt}{d\tau_O}(t_O) \left[1 + \hat{s}_G(t_O) \cdot \vec{\beta}_{GO}(t_O) \right] \quad (2.146)$$

Using Eq. (2.143) we find:

$$1 + y = \left[\frac{\nu_G(t_O)}{\nu_G(t_E)} \right] \left[\frac{d\tau_E(t_E)}{d\tau_O(t_O)} \right] \left[\frac{1 + \hat{s}_G(t_O) \cdot \vec{\beta}_{GO}(t_O)}{1 + \hat{s}_G(t_E) \cdot \vec{\beta}_{GE}(t_E)} \right] \quad (2.147)$$

While in general $\hat{s}_G$ will change from the time of emission to the time of observation due to the curvature of space near Earth, the curvature in space is very small and so we can consider the path of the beam of light to be a straight line at third order in $1/c$. Thus:

$$\hat{s}_G(t_E) \approx \hat{s}_G(t_O) \approx \hat{s}_G = \frac{\vec{r}_{GE}(t_E) - \vec{r}_{GO}(t_O)}{|\vec{r}_{GE}(t_E) - \vec{r}_{GO}(t_O)|} + o(c^{-4}) \quad (2.148)$$

And so finally, we arrive at the following expression for the general relativistic Doppler shift to third order:

$$1 + y = \left[\frac{\nu_G(t_O)}{\nu_G(t_E)} \right] \left[\frac{d\tau_E(t_E)}{d\tau_O(t_O)} \right] \left[\frac{1 + \hat{s}_G \cdot \vec{\beta}_{GO}(t_O)}{1 + \hat{s}_G \cdot \vec{\beta}_{GE}(t_E)} \right] + o(c^{-4}) \quad (2.149)$$

This shift can be separated into factors:

$$1 + y = (1 + y^E) (1 + y^{SH}) (1 + y^O) + o(c^{-4}) \quad (2.150)$$

where:

$$\begin{aligned} 1 + y^E &= \frac{d\tau_E(t_E)}{dt} \left[1 + \hat{s}_G \cdot \vec{\beta}_E(t_E) \right]^{-1} \\ 1 + y^{SH} &= \frac{\nu_G(t_O)}{\nu_G(t_E)} \\ 1 + y^O &= \frac{dt}{d\tau_O(t_O)} \left[1 + \hat{s}_G \cdot \vec{\beta}_O(t_O) \right] \end{aligned} \quad (2.151)$$

The frequency shifts in each factor can be identified as follows: y^E is the shift from the emitter's co-moving frame to the GCRF, y^O is the shift from the GCRF to the frame co-moving with the observer and y^{SH} is the shift along the path due to gravitational effects (i.e. Shapiro effect). The latter is given by Eq. 60 in Sazhin et al. (2010) as $y^{SH} = -d\Delta t_{1w}^{SH}/dt$ which we can evaluate using Eq. (2.96) in which we replace the subscript s with GE and the subscript e with GO :

$$\begin{aligned}
y^{SH} &= -\frac{d}{dt} \left(\frac{2\mu_{\oplus}}{c^3} \ln \frac{r_{GO} + r_{GE} + D}{r_{GO} + r_{GE} - D} \right) \\
&= -\frac{2\mu_{\oplus}}{c^3} \frac{r_{GO} + r_{GE} - D}{r_{GO} + r_{GE} + D} \frac{d}{dt} \left(\frac{r_{GO} + r_{GE} + D}{r_{GO} + r_{GE} - D} \right) \\
&= -\frac{2\mu_{\oplus}}{c^3} \frac{r_{GO} + r_{GE} - D}{r_{GO} + r_{GE} + D} \\
&\quad \left[\frac{(r_{GO} + r_{GE} - D) \frac{d}{dt} (r_{GO} + r_{GE} + D) - (r_{GO} + r_{GE} + D) \frac{d}{dt} (r_{GO} + r_{GE} - D)}{(r_{GO} + r_{GE} - D)^2} \right]
\end{aligned} \tag{2.152}$$

Note that:

$$\frac{dr}{dt} = \frac{d}{dt} \sqrt{\vec{r} \cdot \vec{r}} = \frac{1}{2} \frac{\vec{r} \cdot \vec{v} + \vec{v} \cdot \vec{r}}{\sqrt{\vec{r} \cdot \vec{r}}} = \frac{\vec{v} \cdot \vec{r}}{r} \tag{2.153}$$

and so:

$$\begin{aligned}
y^{SH} &= -\frac{2\mu_{\oplus}}{c^3} \frac{(r_{GO} + r_{GE} - D) \left(\frac{\vec{v}_{GO} \cdot \vec{r}_{GO}}{r_{GO}} + \frac{\vec{v}_{GE} \cdot \vec{r}_{GE}}{r_{GE}} + \dot{D} \right) - (r_{GO} + r_{GE} + D) \left(\frac{\vec{v}_{GO} \cdot \vec{r}_{GO}}{r_{GO}} + \frac{\vec{v}_{GE} \cdot \vec{r}_{GE}}{r_{GE}} - \dot{D} \right)}{(r_{GO} + r_{GE} + D)(r_{GO} + r_{GE} - D)} \\
&= -\frac{2\mu_{\oplus}}{c^3} \frac{2r_{GO}\dot{D} + 2r_{GE}\dot{D} - 2\frac{\vec{v}_{GO} \cdot \vec{r}_{GO}}{r_{GO}}D - 2\frac{\vec{v}_{GE} \cdot \vec{r}_{GE}}{r_{GE}}D}{r_{GO}^2 + r_{GE}^2 - D^2 + 2r_{GO}r_{GE}}
\end{aligned} \tag{2.154}$$

Using Eq. (2.3) and the subscript substitution noted above, we have:

$$\vec{D} = \vec{r}_{GE} - \vec{r}_{GO} \quad \therefore D^2 = \vec{D} \cdot \vec{D} = r_{GE}^2 - 2\vec{r}_{GE} \cdot \vec{r}_{GO} + r_{GO}^2 \quad (2.155)$$

and so finally:

$$y^{SH} = \frac{2\mu_{\oplus}}{c^3} \frac{\left(\frac{\vec{v}_{GO} \cdot \vec{r}_{GO}}{r_{GO}} + \frac{\vec{v}_{GE} \cdot \vec{r}_{GE}}{r_{GE}} \right) D - (r_{GO} + r_{GE}) \dot{D}}{\vec{r}_{GO} \cdot \vec{r}_{GE} + r_{GO} r_{GE}} \quad (2.156)$$

We now have all the components of the relativistic Doppler shift and can proceed to calculate the frequency shifts over the signal paths depicted in Figure 2.1 and in terms of the state previously developed in this chapter.

2.10.2 One Way Mode (Downlink) Frequency Shift

Let us consider the downlink signal from RadioAstron that is emitted by the spacecraft at t_2 and received by the ground station at time t_3 . We can match the variables in Eq. (2.149) to those developed earlier in the chapter as follows:

$$\begin{aligned} t_E &= t_2, & d\tau_E &= d\tau_{s2}, & \vec{r}_{GE} &= \vec{r}_{s2}, & \vec{\beta}_{GE} &= \vec{\beta}_{s2} = \frac{\vec{v}_{s2}}{c} \\ t_O &= t_3, & d\tau_O &= d\tau_e, & \vec{r}_{GO} &= \vec{r}_e, & \vec{\beta}_{GO} &= \vec{\beta}_e = \frac{\vec{v}_e}{c} \end{aligned} \quad (2.157)$$

The direction opposite propagation will be given by $\hat{s}_G = \hat{s}_{32}$ and so the frequency shift during the downlink is given by:

$$1 + y_{1w} = \frac{\nu_G(t_3)}{\nu_G(t_2)} \frac{d\tau_{s2}}{d\tau_e} \frac{1 + \hat{s}_{32} \cdot \vec{\beta}_e}{1 + \hat{s}_{32} \cdot \vec{\beta}_{s2}} + o(c^{-4}) \quad (2.158)$$

Alternatively, using Eq. (2.150) the factors are:

$$\begin{aligned}
1 + y_{1w}^s &= \frac{d\tau_{s2}}{dt} \left[1 + \hat{s}_{32} \cdot \vec{\beta}_{s2} \right]^{-1} \\
1 + y_{1w}^{SH} &= 1 + \frac{2\mu_{\oplus}}{c^3} \frac{\left(\frac{\vec{v}_e \cdot \vec{r}_e}{r_e} + \frac{\vec{v}_s \cdot \vec{r}_s}{r_s} \right) D - (r_e + r_s) \dot{D}}{\vec{r}_e \cdot \vec{r}_s + r_e r_s} \\
1 + y_{1w}^e &= \frac{dt}{d\tau_e} \left[1 + \hat{s}_{32} \cdot \vec{\beta}_e \right]
\end{aligned} \tag{2.159}$$

Note the absence of explicit times on the state vectors in y_{1w}^{SH} as the difference between the vectors at t_2 and t_3 is not significant at third order so either can be used.

Next, let us develop an expression for $1 + y_{1w}^s$ using Eqs. (2.128), (2.115) and (2.112):

$$\begin{aligned}
1 + y_{1w}^s &= \frac{d\tau_{s2}}{dt} \left[1 + \hat{s}_{32} \cdot \vec{\beta}_{s2} \right]^{-1} \\
&= \left[1 + \frac{U_s}{c^2} - \frac{v_s^2}{2c^2} + \frac{D}{c^3} \vec{v}_s \cdot \vec{a}_s - \frac{D}{c^3} \nabla U_s \cdot \vec{v}_s + h_s + o(c^{-4}) \right] \\
&\quad \left\{ 1 + \left[\hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c} \hat{n} + \frac{1}{2c^2} \left((\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D \vec{a}_s - (\vec{D} \cdot \vec{a}_s) \hat{n} \right) \right] \right. \\
&\quad \left. \cdot \left[\frac{\vec{v}_s}{c} - \frac{D \vec{a}_s}{c^2} + \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{a}_s + \frac{D^2}{2c^3} \vec{j}_s \right] + o(c^{-4}) \right\}^{-1}
\end{aligned} \tag{2.160}$$

After some algebra and using Eq. (2.63) this becomes:

$$\begin{aligned}
1 + y_{1w}^s &= 1 - \frac{\hat{n} \cdot \vec{v}_s}{c} + \frac{U_s}{c^2} + \frac{v_s^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} + h_s \\
&\quad + \frac{(\hat{n} \cdot \vec{v}_s)^3}{2c^3} - \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{D (\vec{v}_s \cdot \vec{a}_s)}{2c^3} - \frac{3 (\hat{n} \cdot \vec{v}_s) (\vec{D} \cdot \vec{a}_s)}{2c^3} - \frac{D (\vec{D} \cdot \vec{j}_s)}{2c^3} \\
&\quad - \frac{U_s (\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{D (\nabla U_s \cdot \vec{v}_s)}{c^3} + o(c^{-4})
\end{aligned} \tag{2.161}$$

Similarly, for $1 + y_{1w}^e$ using Eqs. (2.129), (2.115) and (2.111):

$$\begin{aligned}
1 + y_{1w}^e &= \frac{dt}{d\tau_e} \left[1 + \hat{s}_{32} \cdot \vec{\beta}_e \right] \\
&= \left[1 + \frac{U_e}{c^2} - \frac{v_e^2}{2c^2} + h_e + o(c^{-4}) \right]^{-1} \\
&\quad \left\{ 1 + \left[\hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c} \hat{n} + \frac{1}{2c^2} \left((\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D\vec{a}_s - (\vec{D} \cdot \vec{a}_s) \hat{n} \right) + o(c^{-3}) \right] \cdot \frac{\vec{v}_e}{c} \right\}
\end{aligned} \tag{2.162}$$

Again, after using Eq. (2.63) and some algebra, we find:

$$\begin{aligned}
1 + y_{1w}^e &= 1 + \frac{\hat{n} \cdot \vec{v}_e}{c} - \frac{U_e}{c^2} + \frac{v_e^2}{2c^2} - \frac{\vec{v}_e \cdot \vec{v}_s}{c^2} + \frac{(\hat{n} \cdot \vec{v}_e)(\hat{n} \cdot \vec{v}_s)}{c^2} - h_e \\
&\quad + \frac{(\hat{n} \cdot \vec{v}_e)(\hat{n} \cdot \vec{v}_s)^2}{2c^3} + \frac{v_e^2(\hat{n} \cdot \vec{v}_e)}{2c^3} - \frac{v_s^2(\hat{n} \cdot \vec{v}_e)}{2c^3} + \frac{D(\vec{v}_e \cdot \vec{a}_s)}{2c^3} - \frac{(\hat{n} \cdot \vec{v}_e)(\vec{D} \cdot \vec{a}_s)}{2c^3} \\
&\quad - \frac{U_e(\hat{n} \cdot \vec{v}_e)}{c^3} + o(c^{-4})
\end{aligned} \tag{2.163}$$

Finally, after multiplying the terms from Eqs. (2.159), (2.161) and (2.163) while taking care to keep cross terms up to third order and with the following substitutions:

- $\frac{\hat{n} \cdot \vec{v}_e}{c} - \frac{\hat{n} \cdot \vec{v}_s}{c} = -\frac{\dot{D}}{c}$
- $\frac{v_e^2}{2c^2} - \frac{\vec{v}_e \cdot \vec{v}_s}{c^2} + \frac{v_s^2}{2c^2} = \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2}$
- $\frac{(\hat{n} \cdot \vec{v}_s)^3}{2c^3} - \frac{(\hat{n} \cdot \vec{v}_e)(\hat{n} \cdot \vec{v}_s)^2}{2c^3} = \frac{\dot{D}(\hat{n} \cdot \vec{v}_s)^2}{2c^3}$
- $\frac{(\hat{n} \cdot \vec{v}_e)(\vec{D} \cdot \vec{a}_s)}{2c^3} - \frac{3(\hat{n} \cdot \vec{v}_s)(\vec{D} \cdot \vec{a}_s)}{2c^3} = -\frac{(\hat{n} \cdot \vec{v}_s)(\vec{D} \cdot \vec{a}_s)}{c^3} - \frac{\dot{D}(\vec{D} \cdot \vec{a}_s)}{2c^3}$
- $\frac{U_s(\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{U_e(\hat{n} \cdot \vec{v}_e)}{c^3} + \frac{U_e(\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{U_s(\hat{n} \cdot \vec{v}_s)}{c^3} = -\frac{\Delta U \dot{D}}{c^3}$

we find:

$$\begin{aligned}
y_{1w} = & -\frac{\dot{D}}{c} + \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} + \frac{\Delta U}{c^2} + \Delta h \\
& + \frac{v_e^2 (\hat{n} \cdot \vec{v}_e)}{2c^3} - \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{2c^3} - \frac{|\vec{v}_e - \vec{v}_s|^2 (\hat{n} \cdot \vec{v}_s)}{2c^3} - \frac{D (\vec{D} \cdot \vec{j}_s)}{2c^3} \\
& + \frac{\dot{D} (\hat{n} \cdot \vec{v}_s)^2}{2c^3} - \frac{(\hat{n} \cdot \vec{v}_s) (\vec{D} \cdot \vec{a}_s)}{c^3} - \frac{\dot{D} (\vec{D} \cdot \vec{a}_s)}{2c^3} + \frac{D (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s}{2c^3} \\
& + 2 \frac{\mu_{\oplus}}{c^3} \left[\frac{\left(\frac{\vec{r}_e \cdot \vec{v}_e}{r_e} + \frac{\vec{r}_s \cdot \vec{v}_s}{r_s} \right) D - (r_e + r_s) \dot{D}}{\vec{r}_e \cdot \vec{r}_s + r_e r_s} \right] - \frac{\Delta U \dot{D}}{c^3} - \frac{D (\nabla U_s \cdot \vec{v}_s)}{c^3} + o(c^{-4})
\end{aligned} \tag{2.164}$$

where the first term is the non-relativistic Doppler shift, the second term is the transverse Doppler shift, the third term is due to our use of spacecraft state at t_3 , the fourth term is the Gravitational Redshift, the fifth term is due to the frequency biases, terms on the second and third lines are higher order kinematic terms while those on the last line are higher order gravitational terms including the Shapiro effect. Having included relative motion between spacecraft and station, this equation is a generalization of what we found in Eq. (2.64) for a geostationary satellite.

This final equation is equivalent to Eq. 61 in Sazhin et al. (2010) after correcting in their paper the term $-\frac{(\vec{n}\vec{a}_s)D^2}{2c^3}$ which should be $-\frac{(\vec{n}\vec{a}_s)D\dot{D}}{2c^3} = -\frac{\dot{D}(\vec{D}\cdot\vec{a}_s)}{2c^3}$. We have also reproduced the Doppler-related terms in the second order Eq. 3 in Litvinov et al. (2018) after noting that $\hat{n}$ and $\dot{D}$ in their paper are equivalent to $\hat{s}_{32}$ and $\hat{s}_{32} \cdot (\vec{v}_{s2} - \vec{v}_e)$ respectively using our notation and with the following substitution which can be derived by accounting for the

difference in the first order term using Eqs. (2.115) and (2.112):

$$-\frac{\dot{D}}{c} + \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} = -\frac{\hat{s}_{32} \cdot (\vec{v}_{s2} - \vec{v}_e)}{c} - \frac{v_{s2}^2 - v_e^2}{2c^2} + \frac{(\vec{v}_{s2} \cdot \hat{s}_{32})^2 - (\vec{v}_e \cdot \hat{s}_{32})(\vec{v}_{s2} \cdot \hat{s}_{32})}{c^2} + o(c^{-3}) \quad (2.165)$$

Note the times of each of the state vectors on the right match those used in their paper.

Finally, let us group the terms in Eq. (2.164) into a form that will be convenient later on:

$$y_{1w} = y_{1w}^{dop\ 1st} + y_{grav} + y_{1w}^{dop\ 2nd} + y_{1w}^{dop\ 3rd} + \Delta h + o(c^{-4}) \quad (2.166)$$

where:

$$y_{1w}^{dop\ 1st} = -\frac{\dot{D}}{c} \quad (2.167)$$

y_{grav} is as defined in Eq. (1.1), $y_{1w}^{dop\ 2nd}$ includes the remaining second order terms:

$$y_{1w}^{dop\ 2nd} = \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} \quad (2.168)$$

and $y_{1w}^{dop\ 3rd}$ includes the third order terms:

$$\begin{aligned} y_{1w}^{dop\ 3rd} = & \frac{v_e^2 (\hat{n} \cdot \vec{v}_e)}{2c^3} - \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{2c^3} - \frac{|\vec{v}_e - \vec{v}_s|^2 (\hat{n} \cdot \vec{v}_s)}{2c^3} - \frac{D (\vec{D} \cdot \vec{j}_s)}{2c^3} \\ & + \frac{\dot{D} (\hat{n} \cdot \vec{v}_s)^2}{2c^3} - \frac{(\hat{n} \cdot \vec{v}_s) (\vec{D} \cdot \vec{a}_s)}{c^3} - \frac{\dot{D} (\vec{D} \cdot \vec{a}_s)}{2c^3} + \frac{D (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s}{2c^3} \\ & + 2\frac{\mu_{\oplus}}{c^3} \left[\frac{\left(\frac{\vec{r}_e \cdot \vec{v}_e}{r_e} + \frac{\vec{r}_s \cdot \vec{v}_s}{r_s} \right) D - (r_e + r_s) \dot{D}}{\vec{r}_e \cdot \vec{r}_s + r_e r_s} \right] - \frac{\Delta U \dot{D}}{c^3} - \frac{D (\nabla U_s \cdot \vec{v}_s)}{c^3} \end{aligned} \quad (2.169)$$

2.10.3 Uplink Frequency Shift

Next, we consider the uplink signal sent by the ground station at t_1 and received by the spacecraft at t_2 . Similarly to the previous section, we can match the variables in Eq. (2.149) to those developed earlier in the chapter as follows:

$$\begin{aligned} t_E &= t_1, & d\tau_E &= d\tau_{e1}, & \vec{r}_{GE} &= \vec{r}_{e1}, & \vec{\beta}_{GE} &= \vec{\beta}_{e1} = \frac{\vec{v}_{e1}}{c} \\ t_O &= t_2, & d\tau_O &= d\tau_{s2}, & \vec{r}_{GO} &= \vec{r}_{s2}, & \vec{\beta}_{GO} &= \vec{\beta}_{s2} = \frac{\vec{v}_{s2}}{c} \end{aligned} \quad (2.170)$$

The direction opposite propagation will be given by $\hat{s}_G = \hat{s}_{21}$ and so the frequency shift during the uplink is given by:

$$1 + y_{up} = \frac{\nu_G(t_2)}{\nu_G(t_1)} \frac{d\tau_{e1}}{d\tau_{s2}} \frac{1 + \hat{s}_{21} \cdot \vec{\beta}_{s2}}{1 + \hat{s}_{21} \cdot \vec{\beta}_{e1}} + o(c^{-4}) \quad (2.171)$$

We will not develop an explicit expression for the uplink frequency shift in terms of state at t_3 as it is unnecessary in the next step of our derivation. Instead, we will use the above result directly when developing an expression for the 2way mode frequency shift.

2.10.4 Two Way Mode Frequency Shift

The overall frequency shift experienced in the 2way mode will consist of a shift on the uplink path between t_1 and t_2 followed by a shift on the downlink from t_2 to t_3 :

$$\frac{\nu_e}{\nu_{up}(t_1)} = (1 + y_{up}) F_{\nu_0} (1 + y_{dw}) \quad (2.172)$$

where ν_e is the observed frequency at the ground station at t_3 , ν_{up} is the frequency of the uplink tone emitted by the ground station at t_1 and F_{ν_0} is the multiplier between the nominal

uplink and downlink frequencies used in the phase-synchronization loop:

$$F_{8.4} = \frac{8.4}{7.2075} \quad F_{15} = \frac{15}{7.2075} \quad (2.173)$$

It is convenient to incorporate the factor F_{ν_0} into the definition of the 2w shift as follows:

$$1 + y_{2w} \equiv \frac{\nu_e}{F_{\nu_0} \nu_{up}(t_1)} \rightarrow y_{2w} = \frac{\nu_e - F_{\nu_0} \nu_{up}(t_1)}{F_{\nu_0} \nu_{up}(t_1)} \quad (2.174)$$

and so, from Eq. (2.172):

$$1 + y_{2w} = (1 + y_{up})(1 + y_{dw}) \quad (2.175)$$

Using Eq. (2.171) and Eq. (2.158) after noting that $y_{dw} = y_{1w}$, we arrive at:

$$1 + y_{2w} = \frac{\nu_G(t_2)}{\nu_G(t_1)} \frac{\nu_G(t_3)}{\nu_G(t_2)} \frac{d\tau_{e1}}{d\tau_{s2}} \frac{d\tau_{s2}}{d\tau_e} \frac{1 + \hat{s}_{21} \cdot \vec{\beta}_{s2}}{1 + \hat{s}_{21} \cdot \vec{\beta}_{e1}} \frac{1 + \hat{s}_{32} \cdot \vec{\beta}_e}{1 + \hat{s}_{32} \cdot \vec{\beta}_{s2}} + o(c^{-4}) \quad (2.176)$$

For convenience, we separate the overall shift into factors:

$$1 + y_{2w} = (1 + y_{2w}^{SH}) (1 + y_{2w}^T) (1 + y_{2w}^{UP}) (1 + y_{2w}^{DW}) + o(c^{-4}) \quad (2.177)$$

where:

$$\begin{aligned} 1 + y_{2w}^{SH} &= \frac{\nu_G(t_2)}{\nu_G(t_1)} \frac{\nu_G(t_3)}{\nu_G(t_2)} \\ 1 + y_{2w}^T &= \frac{d\tau_{e1}}{d\tau_{s2}} \frac{d\tau_{s2}}{d\tau_e} \\ 1 + y_{2w}^{UP} &= \frac{1 + \hat{s}_{21} \cdot \vec{\beta}_{s2}}{1 + \hat{s}_{21} \cdot \vec{\beta}_{e1}} \\ 1 + y_{2w}^{DW} &= \frac{1 + \hat{s}_{32} \cdot \vec{\beta}_e}{1 + \hat{s}_{32} \cdot \vec{\beta}_{s2}} \end{aligned} \quad (2.178)$$

Starting with the first factor related to the Shapiro effect, we recall that over the propagation time the difference between state at t_1 , t_2 and t_3 is not significant at third order. Upon inspection, we also note that Eq. (2.156) is insensitive to the direction of propagation. Therefore, we conclude that:

$$\frac{\nu_G(t_2)}{\nu_G(t_1)} \approx \frac{\nu_G(t_3)}{\nu_G(t_2)} = 1 + y_{1w}^{SH} \quad (2.179)$$

and so:

$$\begin{aligned} 1 + y_{2w}^{SH} &= (1 + y_{1w}^{SH})^2 \\ &= 1 + 2y_{1w}^{SH} + o(c^{-4}) \\ &= 1 + \frac{4\mu_{\oplus}}{c^3} \frac{\left(\frac{\vec{v}_e \cdot \vec{r}_e}{r_e} + \frac{\vec{v}_s \cdot \vec{r}_s}{r_s} \right) D - (r_e + r_s) \dot{D}}{\vec{r}_e \cdot \vec{r}_s + r_e r_s} + o(c^{-4}) \end{aligned} \quad (2.180)$$

The next factor in Eq. (2.177) is related to the rates of proper time:

$$1 + y_{2w}^T = \frac{d\tau_{e1}}{d\tau_{s2}} \frac{d\tau_{s2}}{d\tau_e} = \frac{d\tau_{e1}}{d\tau_e} = \frac{d\tau_e}{dt}(t_1) \frac{dt}{d\tau_e}(t_3) \quad (2.181)$$

Using Eq. (2.129) for the rate of proper time at t_1 , Eq. (2.39) for the rate at t_3 and the Taylor expansion from Eq. (2.63), we find:

$$\begin{aligned} 1 + y_{2w}^T &= \frac{1 + \frac{U_e}{c^2} - \frac{v_e^2}{2c^2} + 2\frac{D}{c^3} \vec{v}_e \cdot \vec{a}_e - 2\frac{D}{c^3} \nabla U_e \cdot \vec{v}_e + h_e}{1 + \frac{U_e}{c^2} - \frac{v_e^2}{2c^2} + h_e} + o(c^{-4}) \\ &= 1 + \frac{2\frac{D}{c^3} \vec{v}_e \cdot \vec{a}_e - 2\frac{D}{c^3} \nabla U_e \cdot \vec{v}_e}{1 + \frac{U_e}{c^2} - \frac{v_e^2}{2c^2} + h_e} + o(c^{-4}) \\ &= 1 + \left(2\frac{D}{c^3} \vec{v}_e \cdot \vec{a}_e - 2\frac{D}{c^3} \nabla U_e \cdot \vec{v}_e \right) \left(1 - \frac{U_e}{c^2} + \frac{v_e^2}{2c^2} - h_e \right) + o(c^{-4}) \\ &= 1 + 2\frac{D}{c^3} \vec{v}_e \cdot \vec{a}_e - 2\frac{D}{c^3} \nabla U_e \cdot \vec{v}_e + o(c^{-4}) \end{aligned} \quad (2.182)$$

where products from the cross-multiplication in the second-to-last step involving h_e were discarded as they will be below our threshold for significance. In fact, at third order, $\vec{v}_e \perp \vec{a}_e$ and $\vec{v}_e \perp \nabla U_e$, so $1 + y_{2w}^T \approx 1$. However, we will retain these additional terms in the final form of the expression as they lead to useful cancellations later on.

Now working on the factor $1 + y_{2w}^{UP}$, we start by substituting in Eqs. (2.118), (2.112) and (2.111):

$$\begin{aligned}
1 + y_{2w}^{UP} = & \left[1 + \left\{ \begin{aligned} & -\hat{n} - 2\frac{\vec{v}_e}{c} + \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_e}{c} \hat{n} - \frac{\dot{D}}{c} \hat{n} \\ & -\frac{2}{c^2} \left[(\hat{n} \cdot \vec{v}_e)^2 \hat{n} - v_e^2 \hat{n} - D\vec{a}_e + \left(\vec{D} \cdot \vec{a}_e \right) \hat{n} \right] \\ & -\frac{1}{2c^2} \left[(\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D\vec{a}_s - \left(\vec{D} \cdot \vec{a}_s \right) \hat{n} \right] \\ & -\frac{2(\hat{n} \cdot \vec{v}_e)}{c^2} \left[\dot{D} \hat{n} + (\hat{n} \cdot \vec{v}_s) \hat{n} + \vec{v}_e - \vec{v}_s \right] \\ & -2\frac{\vec{v}_e \cdot \vec{v}_s}{c^2} \hat{n} + o(c^{-3}) \end{aligned} \right\} \cdot \left[\frac{\vec{v}_s}{c} - \frac{D\vec{a}_s}{c^2} + \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{a}_s + \frac{D^2}{2c^3} \vec{j}_s + o(c^{-4}) \right] \right] \\
& \cdot \left[\frac{\vec{v}_e}{c} - 2\frac{D\vec{a}_e}{c^2} + 2\frac{\vec{D} \cdot (\vec{v}_s - \vec{v}_e)}{c^3} \vec{a}_e + 2\frac{D^2}{c^3} \vec{j}_e + o(c^{-4}) \right] \right]^{-1} \quad (2.183)
\end{aligned}$$

After some algebra and using Eq. (2.63) along with the following substitutions:

- $\frac{\vec{v}_s^2}{c^2} - 3\frac{\vec{v}_s \cdot \vec{v}_e}{c^2} + 2\frac{\vec{v}_e^2}{c^2} = 3\frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{v_e^2 - v_s^2}{2c^2}$
- $\frac{\dot{D}(\hat{n} \cdot \vec{v}_e)}{c^2} - \frac{\dot{D}(\hat{n} \cdot \vec{v}_s)}{c^2} = -\frac{\dot{D}^2}{c^2}$
- $5\frac{(\hat{n} \cdot \vec{v}_s)^2(\hat{n} \cdot \vec{v}_e)}{2c^3} - 3\frac{(\hat{n} \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e)^2}{c^3} + \frac{(\hat{n} \cdot \vec{v}_e)^3}{c^3} - \frac{(\hat{n} \cdot \vec{v}_s)^3}{2c^3} = -\frac{\dot{D}^3}{c^3} + \frac{\dot{D}(\hat{n} \cdot \vec{v}_s)^2}{2c^3}$

$$\bullet \quad 2 \frac{(\hat{n} \cdot \vec{v}_s)(\vec{D} \cdot \vec{a}_e)}{c^3} - 2 \frac{(\hat{n} \cdot \vec{v}_e)(\vec{D} \cdot \vec{a}_e)}{c^3} - 2 \frac{\dot{D}(\vec{D} \cdot \vec{a}_e)}{c^3} = 0$$

we arrive at:

$$\begin{aligned} 1 + y_{2w}^{UP} = & 1 - \frac{\dot{D}}{c} - \frac{\dot{D}^2}{c^2} + 3 \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{v_e^2 - v_s^2}{2c^2} - 2 \frac{\vec{D} \cdot \vec{a}_e}{c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} \\ & - \frac{\dot{D}^3}{c^3} - 3 \frac{v_s^2 (\hat{n} \cdot \vec{v}_e)}{2c^3} + \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{2c^3} + 2 \frac{(\vec{v}_e \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{(\vec{v}_e \cdot \vec{v}_s)(\hat{n} \cdot \vec{v}_s)}{c^3} \\ & + \frac{\dot{D} (\hat{n} \cdot \vec{v}_s)^2}{2c^3} + \frac{\dot{D} (\vec{D} \cdot \vec{a}_s)}{c^3} - \frac{(\hat{n} \cdot \vec{v}_e)(\vec{D} \cdot \vec{a}_s)}{2c^3} - \frac{(\hat{n} \cdot \vec{v}_s)(\vec{D} \cdot \vec{a}_s)}{2c^3} \\ & - 6 \frac{D(\vec{v}_e \cdot \vec{a}_e)}{c^3} + 5 \frac{D(\vec{v}_e \cdot \vec{a}_s)}{2c^3} + 4 \frac{D(\vec{v}_s \cdot \vec{a}_e)}{c^3} - 3 \frac{D(\vec{v}_s \cdot \vec{a}_s)}{2c^3} \\ & + 2 \frac{D(\vec{D} \cdot \vec{j}_e)}{c^3} - \frac{D(\vec{D} \cdot \vec{j}_s)}{2c^3} + o(c^{-4}) \end{aligned} \quad (2.184)$$

The final factor, $1 + y_{2w}^{DW}$, is found by substituting in Eqs. (2.115) and (2.112):

$$\begin{aligned} 1 + y_{2w}^{DW} = & \left[1 + \left\{ \begin{aligned} & \hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c} \hat{n} \\ & + \frac{1}{2c^2} \left[(\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D \vec{a}_s - (\vec{D} \cdot \vec{a}_s) \hat{n} \right] + o(c^{-3}) \end{aligned} \right\} \cdot \left[\frac{\vec{v}_e}{c} \right] \right] \\ & \left[1 + \left\{ \begin{aligned} & \hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c} \hat{n} \\ & + \frac{1}{2c^2} \left[(\hat{n} \cdot \vec{v}_s)^2 \hat{n} - v_s^2 \hat{n} + D \vec{a}_s - (\vec{D} \cdot \vec{a}_s) \hat{n} \right] + o(c^{-3}) \end{aligned} \right\} \cdot \left[\frac{\vec{v}_s}{c} - \frac{D \vec{a}_s}{c^2} + \frac{\vec{D} \cdot \vec{v}_s}{c^3} \vec{a}_s + \frac{D^2}{2c^3} \vec{j}_s + o(c^{-4}) \right] \right]^{-1} \end{aligned} \quad (2.185)$$

After some algebra and using Eq. (2.63) along with the following substitutions:

$$\begin{aligned} \bullet \quad & \frac{v_s^2}{c^2} - \frac{\vec{v}_e \cdot \vec{v}_s}{c^2} = \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} - \frac{v_e^2 - v_s^2}{2c^2} \\ \bullet \quad & \frac{(\hat{n} \cdot \vec{v}_s)^3}{2c^3} - \frac{(\hat{n} \cdot \vec{v}_s)^2 (\hat{n} \cdot \vec{v}_e)}{2c^3} = \frac{\dot{D} (\hat{n} \cdot \vec{v}_s)^2}{2c^3} \end{aligned}$$

we arrive at:

$$\begin{aligned}
1 + y_{2w}^{DW} = & 1 - \frac{\dot{D}}{c} + \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} - \frac{v_e^2 - v_s^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_s}{c^2} \\
& + \frac{v_s^2 (\hat{n} \cdot \vec{v}_e)}{2c^3} - 3 \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{2c^3} + \frac{(\vec{v}_e \cdot \vec{v}_s) (\hat{n} \cdot \vec{v}_s)}{c^3} \\
& + \frac{\dot{D} (\hat{n} \cdot \vec{v}_s)^2}{2c^3} + \frac{(\hat{n} \cdot \vec{v}_e) (\vec{D} \cdot \vec{a}_s)}{2c^3} - 3 \frac{(\hat{n} \cdot \vec{v}_s) (\vec{D} \cdot \vec{a}_s)}{2c^3} \\
& + \frac{D (\vec{v}_e \cdot \vec{a}_s)}{2c^3} - 3 \frac{D (\vec{v}_s \cdot \vec{a}_s)}{2c^3} - \frac{D (\vec{D} \cdot \vec{j}_s)}{2c^3} + o(c^{-4})
\end{aligned} \tag{2.186}$$

Finally, after multiplying the terms from Eqs. (2.180), (2.182), (2.184) and (2.186) while taking care to only keep cross terms up to third order and with the following substitution:

$$2 \frac{(\vec{v}_e \cdot \vec{v}_s) (\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{v_s^2 (\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{\dot{D} |\vec{v}_e - \vec{v}_s|^2}{c^3} = \frac{v_e^2 (\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{(\hat{n} \cdot \vec{v}_s) |\vec{v}_e - \vec{v}_s|^2}{c^3} \tag{2.187}$$

we find:

$$\begin{aligned}
y_{2w} = & -2 \frac{\dot{D}}{c} + 2 \frac{|\vec{v}_e - \vec{v}_s|^2}{c^2} + 2 \frac{\vec{D} \cdot \vec{a}_s}{c^2} - 2 \frac{\vec{D} \cdot \vec{a}_e}{c^2} \\
& + \frac{v_e^2 (\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{|\vec{v}_e - \vec{v}_s|^2 (\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{D (\vec{D} \cdot \vec{j}_s)}{c^3} + 2 \frac{D (\vec{D} \cdot \vec{j}_e)}{c^3} \\
& + \frac{\dot{D} (\hat{n} \cdot \vec{v}_s)^2}{c^3} - 2 \frac{(\hat{n} \cdot \vec{v}_s) (\vec{D} \cdot \vec{a}_s)}{c^3} - \frac{\dot{D} (\vec{D} \cdot \vec{a}_s)}{c^3} + 2 \frac{\dot{D} (\vec{D} \cdot \vec{a}_e)}{c^3} \\
& - \frac{\dot{D} |\vec{v}_e - \vec{v}_s|^2}{c^3} + 3 \frac{D (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s}{c^3} - 4 \frac{D (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_e}{c^3} \\
& + 4 \frac{\mu_{\oplus}}{c^3} \left[\frac{\left(\frac{\vec{r}_e \cdot \vec{v}_e}{r_e} + \frac{\vec{r}_s \cdot \vec{v}_s}{r_s} \right) D - (r_e + r_s) \dot{D}}{\vec{r}_e \cdot \vec{r}_s + r_e r_s} \right] - 2 \frac{D (\nabla U_e \cdot \vec{v}_e)}{c^3} + o(c^{-4})
\end{aligned} \tag{2.188}$$

where the first term is the non-relativistic Doppler shift, the second term is transverse Doppler shift, the third and fourth terms are due to our use of state at t_3 , terms on the second to

fourth lines are higher order kinematic terms and those on the last line are higher order gravitational terms including the Shapiro effect.

The Doppler-related second order terms in Eq. (2.188) are consistent with those in Eq. 4 of Litvinov et al. (2018) after taking into account the same notational differences mentioned at the end of the previous section and using the following substitution:

$$\frac{\hat{s}_{32}}{c} \Delta t = \frac{\vec{D}}{c^2} + o(c^{-3}) \quad (2.189)$$

along with the following which can again be derived by using Eqs. (2.115) and (2.112) to account for the differences in the first order term:

$$\begin{aligned} -2\frac{\dot{D}}{c} + 2\frac{|\vec{v}_e - \vec{v}_s|^2}{c^2} + 2\frac{\vec{D} \cdot \vec{a}_s}{c^2} = & -2\frac{\hat{s}_{32} \cdot (\vec{v}_{s2} - \vec{v}_e)}{c} - \frac{v_{s2}^2 - v_e^2}{c^2} + \frac{|\vec{v}_{s2} - \vec{v}_e|^2}{c^2} \\ & + 2\frac{(\vec{v}_{s2} \cdot \hat{s}_{32})^2 - (\vec{v}_e \cdot \hat{s}_{32})(\vec{v}_{s2} \cdot \hat{s}_{32})}{c^2} + o(c^{-3}) \end{aligned} \quad (2.190)$$

Finally, let us group the terms in Eq. (2.188) into a form that will be convenient later on:

$$y_{2w} = y_{2w}^{dop\ 1st} + y_{2w}^{dop\ 2nd} + y_{2w}^{dop\ 3rd} + o(c^{-4}) \quad (2.191)$$

where:

$$y_{2w}^{dop\ 1st} = -2\frac{\dot{D}}{c} \quad (2.192)$$

$y_{1w}^{dop\ 2nd}$ includes the remaining second order terms:

$$y_{2w}^{dop\ 2nd} = 2\frac{|\vec{v}_e - \vec{v}_s|^2}{c^2} + 2\frac{\vec{D} \cdot \vec{a}_s}{c^2} - 2\frac{\vec{D} \cdot \vec{a}_e}{c^2} \quad (2.193)$$

and $y_{1w}^{dop\ 3rd}$ includes all the third order terms:

$$\begin{aligned}
y_{2w}^{dop\ 3rd} = & \frac{v_e^2 (\hat{n} \cdot \vec{v}_e)}{c^3} - \frac{v_s^2 (\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{|\vec{v}_e - \vec{v}_s|^2 (\hat{n} \cdot \vec{v}_s)}{c^3} - \frac{D(\vec{D} \cdot \vec{j}_s)}{c^3} + 2 \frac{D(\vec{D} \cdot \vec{j}_e)}{c^3} \\
& + \frac{\dot{D}(\hat{n} \cdot \vec{v}_s)^2}{c^3} - 2 \frac{(\hat{n} \cdot \vec{v}_s)(\vec{D} \cdot \vec{a}_s)}{c^3} - \frac{\dot{D}(\vec{D} \cdot \vec{a}_s)}{c^3} + 2 \frac{\dot{D}(\vec{D} \cdot \vec{a}_e)}{c^3} \\
& - \frac{\dot{D}|\vec{v}_e - \vec{v}_s|^2}{c^3} + 3 \frac{D(\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s}{c^3} - 4 \frac{D(\vec{v}_e - \vec{v}_s) \cdot \vec{a}_e}{c^3} \\
& + 4 \frac{\mu_{\oplus}}{c^3} \left[\frac{\left(\frac{\vec{r}_e \cdot \vec{v}_e}{r_e} + \frac{\vec{r}_s \cdot \vec{v}_s}{r_s} \right) D - (r_e + r_s) \dot{D}}{\vec{r}_e \cdot \vec{r}_s + r_e r_s} \right] - 2 \frac{D(\nabla U_e \cdot \vec{v}_e)}{c^3}
\end{aligned} \tag{2.194}$$

2.10.5 Doppler-Compensated Frequency Shift

A significant challenge when performing a gravitational redshift experiment with a spacecraft is calculating the non-relativistic Doppler term in Eq. (2.164) with sufficient accuracy using reconstructed orbital state vectors. To work around this problem, Vessot & Levine implemented a Doppler compensation scheme on GP-A that combined 1way and 2way signals. A similar approach is used with RadioAstron by combining measurements from the two communication modes as follows:

$$\Delta y_{dc} \equiv y_{1w} - \frac{1}{2} y_{2w} \tag{2.195}$$

Using Eqs. (2.164) and (2.188), the suppression of the non-relativistic Doppler term is apparent with the remaining second and third order terms after some rearranging being given

by:

$$\begin{aligned} \Delta y_{dc} = & \left(1 - \frac{\dot{D}}{c}\right) \left[\frac{\Delta U}{c^2} - \frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_e}{c^2} \right] + \Delta h \\ & + \frac{D}{c^3} \left[2(\vec{v}_e - \vec{v}_s) \cdot \vec{a}_e - (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s - \vec{D} \cdot \vec{j}_e + \nabla U_e \cdot \vec{v}_e - \nabla U_s \cdot \vec{v}_s \right] + o(c^{-4}) \end{aligned} \quad (2.196)$$

what is of particular importance, is that the leading term is now the gravitational redshift and the kinematic terms are significantly smaller than those in y_{1w} and y_{2w} . This result is comparable to others in the literature, though the sole use of state vectors at t_3 , which is most convenient for data processing, is new. We can rewrite this equation in terms of the spacecraft state at t_2 and ground station state at t_3 by noting that over the short propagation time, and at third order, that $\vec{r}_{s2} \approx \vec{r}_s$, $\vec{v}_{s2} \approx \vec{v}_s$, $\hat{s}_{32} \approx \hat{n}$ and by using the following substitutions:

- $\frac{U_{s2}}{c^2} = \frac{U_s}{c^2} - \frac{D(\nabla U_s \cdot \vec{v}_s)}{c^3} + o(c^{-4})$
- $-\frac{|\vec{v}_{s2} - \vec{v}_e|^2}{2c^3} = -\frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} - \frac{D(\vec{v}_e \cdot \vec{a}_s)}{c^3} + \frac{D(\vec{v}_s \cdot \vec{a}_s)}{c^3} + o(c^{-4})$
- $\frac{\vec{D}_{32} \cdot \vec{a}_e}{c^2} = \frac{\vec{D} \cdot \vec{a}_e}{c^2} - \frac{D(\vec{v}_s \cdot \vec{a}_e)}{c^3} + o(c^{-4})$

where $\vec{D}_{32} \equiv |\vec{r}_{s2} - \vec{r}_e| \hat{s}_{32}$. And so we can write:

$$\begin{aligned} \Delta y_{dc}(t_2, t_3) = & \left(1 - \frac{\dot{D}}{c}\right) \left[\frac{U_{s2} - U_e}{c^2} - \frac{|\vec{v}_{s2} - \vec{v}_e|^2}{2c^2} + \frac{\vec{D}_{32} \cdot \vec{a}_e}{c^2} \right] + \Delta h \\ & + \frac{D}{c^3} \left[(2\vec{v}_e - \vec{v}_{s2}) \cdot \vec{a}_e - \vec{D} \cdot \vec{j}_e + \nabla U_e \cdot \vec{v}_e \right] + o(c^{-4}) \end{aligned} \quad (2.197)$$

where D and $\dot{D}$ have been calculated using $\vec{r}_{s2}$ instead of $\vec{r}_s$ and $\hat{s}_{32}$ instead of $\hat{n}$.

The second order terms in Eq. (2.197) are equal to those in Eq. 1 of Vessot & Levine (1979). This equation is also equal to Eq. 17 in Blanchet et al. (2001) and Eqs. 5 and 6 in Litvinov et al. (2018) using $\Delta t \approx \frac{D}{c}$. Finally, Eq. (2.196) is similar to Eq. 90 in Turyshev et al. (2016) after effects not considered here are neglected.

As we will see in the following sections, the Doppler compensation scheme also reduces other systematics simplifying any such experiment. In preparation for adding other effects, we can group the terms in Eq. (2.196) in the following form:

$$\Delta y_{dc} = y_{grav} + \Delta y_{dop\ 2nd} + \Delta y_{dop\ 3rd} + \Delta y_{small} + \Delta h + o(c^{-4}) \quad (2.198)$$

where $\Delta y_{dop\ 2nd}$ includes the two second order kinematic terms:

$$\Delta y_{dop\ 2nd} = -\frac{|\vec{v}_e - \vec{v}_s|^2}{2c^2} + \frac{\vec{D} \cdot \vec{a}_e}{c^2} \quad (2.199)$$

$\Delta y_{dop\ 3rd}$ includes the most significant third order terms:

$$\Delta y_{dop\ 3rd} = -\frac{\dot{D}}{c^3} \left[\Delta U - \frac{1}{2} |\vec{v}_e - \vec{v}_s|^2 \right] - \frac{D}{c^3} \left[\vec{D} \cdot \vec{j}_e + (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s + \nabla U_s \cdot \vec{v}_s \right] \quad (2.200)$$

and finally, Δy_{small} includes the remaining third order terms which will be shown in Chapter 4 to be insignificant for the RadioAstron gravity experiment:

$$\Delta y_{small} = -\frac{\dot{D}}{c^3} \vec{D} \cdot \vec{a}_e + \frac{D}{c^3} [2(\vec{v}_e - \vec{v}_s) \cdot \vec{a}_e + \nabla U_e \cdot \vec{v}_e] \quad (2.201)$$

2.11 Antenna Phase-Centre Motion

An assumption that went unstated in the previous section was that the phase-centre of the ground station and spacecraft antennas is held at a fixed distance and orientation relative to the reference point for which state vectors are given. This is not always the case and so the motion of the phase-centre relative to the reference point, called antenna phase-centre motion (PCM), must be considered as it introduces a small additional Doppler shift. In this

section, expressions for the PCM effect will be developed based on our derivation in Litvinov et al. (2021). As the effect is small, we will only need to work to second order in $1/c$.

The phase-centre of RadioAstron’s HGCA is offset from the spacecraft’s centre-of-mass by the following vector given in a spacecraft-fixed frame (see Figure 2.4a):

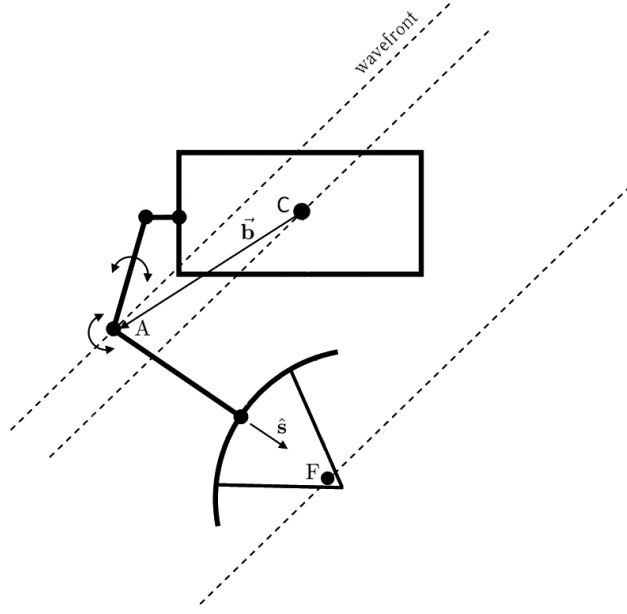
$$\vec{b}_s = [-2.299, 0, 2.546] \text{ m} \quad (2.202)$$

The orientation of the spacecraft-fixed frame relative to the ECI frame is given by a $\mathbb{R}^{3 \times 3}$ rotation matrix $R(t)$, so that in the inertial frame the offset of the phase-centre from the spacecraft center-of-mass is given by: $R(t) \vec{b}_s$.

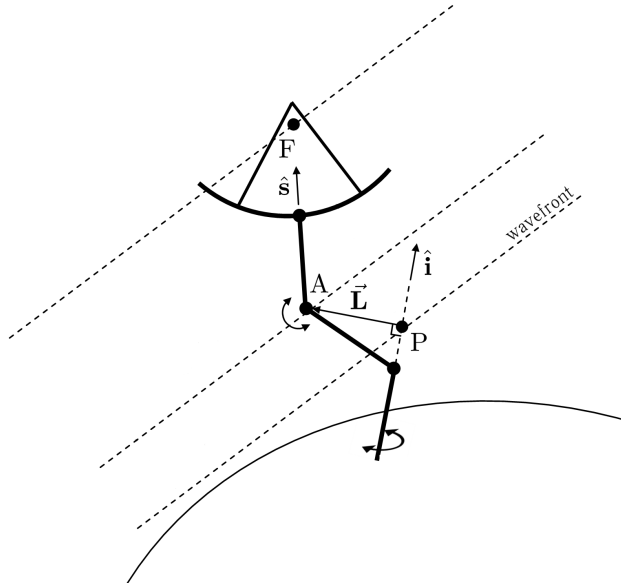
The antenna of the Pushchino ground station is of an Alt-Az design and does not have an offset between its axes and so does not manifest the PCM effect. However, the antenna of the Green Bank ground station utilizes an equatorial mount with an offset between its hour angle and declination axes of:

$$L_{GB} = +14.939,0 \text{ m} \quad (2.203)$$

where the $+$ indicates that the phase-centre is further from Earth’s center than the reference point (see Figure 2.4b). The pointing angle of the antenna in the ECI frame is required to determine the projection of this offset leading to the PCM effect. For a polar aligned telescope, this angle is given by the declination of the spacecraft $\theta(t)$ defined in Eq. (2.8). For simplicity, we will use the symbol L to denote the ground station antenna offset in the rest of this section where $L = L_{GB}$ for Green Bank and $L = 0$ for Pushchino.



(a) Spacecraft



(b) Green Bank

Figure 2.4: Schematic of the spacecraft HGCA (a) showing the vector $\vec{b}$ separating the center-of-mass C from the point A fixed to the antenna. As the antenna slews to follow the ground station by pointing towards $\hat{s}$, the distance travelled by the wavefront between points F and C will vary resulting in a frequency shift. The situation at Green Bank (b) is similar where the reference point of the telescope P fixed to the Earth is separated from the point A fixed to the antenna by the vector $\vec{L}$. A frequency shift arises as the distance travelled by the wavefront between points F and P varies with $\hat{s}$. Based on images from Litvinov et al. (2021).

2.11.1 One Way Mode Frequency Shift

Using Eq. 36 from Litvinov et al. (2021) but with our notation, the PCM effect in the 1way mode due to the ground station and spacecraft antennas will be:

$$y_{1w}^{pcm} = \frac{L}{c} \sin \theta_{32} \dot{\theta}_{32} - \frac{1}{c} R \vec{b}_s \cdot \dot{\hat{s}}_{23} + o(\delta\theta^2) \quad (2.204)$$

where θ_{32} and $\dot{\theta}_{32}$ refer to the pointing angle and its rate of change for when the spacecraft signal is emitted at t_2 and observed by the ground station at t_3 (see Figure 2.1). That is, we assume that the spacecraft is tracking where the ground station will be at the time of observation as opposed to where the ground station is at t_2 . The error in the pointing angle is given by $\delta\theta$. As the PCM effect is small and RadioAstron's pointing accuracy is sufficiently high, errors in pointing will be insignificant for our purposes so we will ignore them. We have also assumed that the spacecraft maintains a fixed orientation in the ECI frame over the propagation time, i.e. $\dot{R} = 0$. This was the case during the gravity experiments with RadioAstron and so is a reasonable assumption for our purposes.

Next, we develop an expression for $\sin \theta_{32} \dot{\theta}_{32}$ based solely on the state at t_3 . We start with Eq. (2.10) which allows us to write $\dot{\theta}$ in terms of θ and $\dot{\hat{s}}$:

$$\sin \theta_{32} \dot{\theta}_{32} = \sin \theta_{32} \sec \theta_{32} \hat{i} \cdot \dot{\hat{s}}_{32} \quad (2.205)$$

Using Eqs. (2.8) and (2.115), we can eliminate the dependence on the state at t_2 :

$$\begin{aligned} \sin \theta_{32} &= \hat{i} \cdot \left[\hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c} \hat{n} + o(c^{-2}) \right] \\ &= \hat{i} \cdot \hat{n} - \frac{\hat{i} \cdot \vec{v}_s}{c} + \frac{(\hat{n} \cdot \vec{v}_s)(\hat{i} \cdot \hat{n})}{c} + o(c^{-2}) \\ &= \sin \theta - \frac{\hat{i} \cdot \vec{v}_s}{c} + \frac{\sin \theta \hat{n} \cdot \vec{v}_s}{c} + o(c^{-2}) \end{aligned} \quad (2.206)$$

With this result we can derive $\sec \theta_{32}$:

$$\begin{aligned}\sec \theta_{32} &= \frac{1}{\sqrt{1 - \sin^2 \theta_{32}}} \\ &= \frac{1}{\sqrt{\cos^2 \theta + 2 \frac{\sin \theta \hat{i} \cdot \vec{v}_s}{c} - 2 \frac{\sin^2 \theta \hat{n} \cdot \vec{v}_s}{c} + o(c^{-2})}}\end{aligned}\tag{2.207}$$

Using the following Taylor expansion at $x = 0$:

$$\frac{1}{\sqrt{a+x}} = \frac{1}{\sqrt{a}} - \frac{x}{2a^{\frac{3}{2}}} + \dots\tag{2.208}$$

we find:

$$\sec \theta_{32} = \sec \theta - \frac{\sin \theta \hat{i} \cdot \vec{v}_s}{c \cos^3 \theta} + \frac{\sin^2 \theta \hat{n} \cdot \vec{v}_s}{c \cos^3 \theta} + o(c^{-2})\tag{2.209}$$

An expression for $\dot{\hat{s}}_{32}$ can be found by differentiating Eq. (2.115):

$$\begin{aligned}\dot{\hat{s}}_{32} &= \frac{d}{dt} \left[\hat{n} - \frac{\vec{v}_s}{c} + \frac{\hat{n} \cdot \vec{v}_s}{c} \hat{n} + o(c^{-2}) \right] \\ &= \dot{\hat{n}} - \frac{\vec{a}_s}{c} + \frac{\dot{\hat{n}} \cdot \vec{v}_s}{c} \hat{n} + \frac{\hat{n} \cdot \vec{a}_s}{c} \hat{n} + \frac{\hat{n} \cdot \vec{v}_s}{c} \dot{\hat{n}} + o(c^{-2})\end{aligned}\tag{2.210}$$

Substituting Eqs. (2.206), (2.209) and (2.210) into Eq. (2.205) we have:

$$\begin{aligned}\sin \theta_{32} \dot{\theta}_{32} &= \left[\sin \theta - \frac{\hat{i} \cdot \vec{v}_s}{c} + \frac{\sin \theta \hat{n} \cdot \vec{v}_s}{c} + o(c^{-2}) \right] \\ &\quad \left[\sec \theta - \frac{\sin \theta \hat{i} \cdot \vec{v}_s}{c \cos^3 \theta} + \frac{\sin^2 \theta \hat{n} \cdot \vec{v}_s}{c \cos^3 \theta} + o(c^{-2}) \right] \\ &\quad \hat{i} \cdot \left[\dot{\hat{n}} - \frac{\vec{a}_s}{c} + \frac{\dot{\hat{n}} \cdot \vec{v}_s}{c} \hat{n} + \frac{\hat{n} \cdot \vec{a}_s}{c} \hat{n} + \frac{\hat{n} \cdot \vec{v}_s}{c} \dot{\hat{n}} + o(c^{-2}) \right]\end{aligned}\tag{2.211}$$

After some algebra, this can be reduced to:

$$\begin{aligned} \sin \theta_{32} \dot{\theta}_{32} &= \sin \theta \dot{\theta} - \frac{\sec^2 \theta \dot{\theta}}{c} \left(\hat{i} \cdot \vec{v}_s - \sin \theta \hat{n} \cdot \vec{v}_s \right) \\ &\quad - \frac{\tan \theta}{c} \hat{i} \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] + o(c^{-2}) \end{aligned} \quad (2.212)$$

We also need an expression for $\dot{\hat{s}}_{23}$ which is simply:

$$\dot{\hat{s}}_{23} = \frac{d}{dt} \hat{s}_{23} = -\frac{d}{dt} \hat{s}_{32} = -\dot{\hat{s}}_{32} \quad (2.213)$$

Substituting Eqs. (2.212) and (2.210) into Eq. (2.204) we find:

$$\begin{aligned} y_{1w}^{pcm} &= \frac{L}{c} \left\{ \sin \theta \dot{\theta} - \frac{\sec^2 \theta \dot{\theta}}{c} \left(\hat{i} \cdot \vec{v}_s - \sin \theta \hat{n} \cdot \vec{v}_s \right) \right. \\ &\quad \left. - \frac{\tan \theta}{c} \hat{i} \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] + o(c^{-2}) \right\} \\ &\quad + \frac{1}{c} R \vec{b}_s \cdot \left[\dot{\hat{n}} - \frac{\vec{a}_s}{c} + \frac{\dot{\hat{n}} \cdot \vec{v}_s}{c} \hat{n} + \frac{\hat{n} \cdot \vec{a}_s}{c} \hat{n} + \frac{\hat{n} \cdot \vec{v}_s}{c} \dot{\hat{n}} + o(c^{-2}) \right] \end{aligned} \quad (2.214)$$

Finally, rearranging terms to separate orders we have:

$$\begin{aligned} y_{1w}^{pcm} &= \frac{L \sin \theta \dot{\theta}}{c} + \frac{R \vec{b}_s \cdot \dot{\hat{n}}}{c} \\ &\quad - \frac{L}{c^2} \left\{ \sec^2 \theta \dot{\theta} \left(\hat{i} \cdot \vec{v}_s - \sin \theta \hat{n} \cdot \vec{v}_s \right) \right. \\ &\quad \left. + \tan \theta \hat{i} \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] \right\} \\ &\quad - \frac{1}{c^2} R \vec{b}_s \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] + o(c^{-3}) \end{aligned} \quad (2.215)$$

2.11.2 Two Way Mode Frequency Shift

Next, we will derive an expression for the PCM effect in the 2way mode using Eq. 33 from Litvinov et al. (2021) modified to use our notation:

$$y_{2w}^{pcm} = \frac{L}{c} \left(\sin \theta_{12} \dot{\theta}_{12} + \sin \theta_{32} \dot{\theta}_{32} \right) - \frac{1}{c} R \vec{b}_s \cdot \left(\dot{\hat{s}}_{21} + \dot{\hat{s}}_{23} \right) + o(\delta\theta^2) \quad (2.216)$$

An expression for the signal propagation vector $\hat{s}_{12}$ can be found using Eqs. (2.118) and (2.6):

$$\begin{aligned} \hat{s}_{12} &= -\hat{s}_{21} \\ &= \hat{n} + 2 \frac{\vec{v}_e}{c} - \frac{\vec{v}_s}{c} - \frac{\hat{n} \cdot \vec{v}_e}{c} \hat{n} + (\hat{n} \cdot \vec{v}_s - \hat{n} \cdot \vec{v}_e) \frac{\hat{n}}{c} + o(c^{-2}) \\ &= \hat{n} + \frac{2}{c} [\vec{v}_e - (\hat{n} \cdot \vec{v}_e) \hat{n}] - \frac{1}{c} [\vec{v}_s - (\hat{n} \cdot \vec{v}_s) \hat{n}] + o(c^{-2}) \end{aligned} \quad (2.217)$$

Differentiating we find:

$$\begin{aligned} \dot{\hat{s}}_{12} &= \dot{\hat{n}} + \frac{2}{c} \left[\vec{a}_e - \left(\dot{\hat{n}} \cdot \vec{v}_e \right) \hat{n} - (\hat{n} \cdot \vec{a}_e) \hat{n} - (\hat{n} \cdot \vec{v}_e) \dot{\hat{n}} \right] \\ &\quad - \frac{1}{c} \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] + o(c^{-2}) \end{aligned} \quad (2.218)$$

As we found for the signal propagation vector in the previous section:

$$\dot{\hat{s}}_{21} = -\dot{\hat{s}}_{12} \quad (2.219)$$

Following procedure of the previous section using Eqs. (2.8) and (2.118), we arrive at:

$$\begin{aligned}
\sin \theta_{12} \dot{\theta}_{12} &= \sin \theta \dot{\theta} \\
&+ 2 \frac{\sec^2 \theta \dot{\theta}}{c} \left[\hat{i} \cdot \vec{v}_e - \sin \theta \hat{n} \cdot \vec{v}_e \right] \\
&+ 2 \frac{\tan \theta}{c} \hat{i} \cdot \left[\vec{a}_e - \left(\dot{\hat{n}} \cdot \vec{v}_e \right) \hat{n} - (\hat{n} \cdot \vec{a}_e) \hat{n} - (\hat{n} \cdot \vec{v}_e) \dot{\hat{n}} \right] \\
&- \frac{\sec^2 \theta \dot{\theta}}{c} \left[\hat{i} \cdot \vec{v}_s - \sin \theta \hat{n} \cdot \vec{v}_s \right] \\
&- \frac{\tan \theta}{c} \hat{i} \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] + o(c^{-2})
\end{aligned} \tag{2.220}$$

Substituting Eqs. (2.220), (2.212), (2.219) and (2.213) into Eq. (2.216) and rearranging to separate the orders we arrive at:

$$\begin{aligned}
y_{2w}^{pcm} &= 2 \frac{L \sin \theta \dot{\theta}}{c} + 2 \frac{R \vec{b}_s \cdot \dot{\hat{n}}}{c} \\
&+ \frac{2L}{c^2} \left\{ \begin{aligned} &\sec^2 \theta \dot{\theta} \left(\hat{i} \cdot \vec{v}_e - \sin \theta \hat{n} \cdot \vec{v}_e \right) \\ &+ \tan \theta \hat{i} \cdot \left[\vec{a}_e - \left(\dot{\hat{n}} \cdot \vec{v}_e \right) \hat{n} - (\hat{n} \cdot \vec{a}_e) \hat{n} - (\hat{n} \cdot \vec{v}_e) \dot{\hat{n}} \right] \end{aligned} \right\} \\
&- \frac{2L}{c^2} \left\{ \begin{aligned} &\sec^2 \theta \dot{\theta} \left(\hat{i} \cdot \vec{v}_s - \sin \theta \hat{n} \cdot \vec{v}_s \right) \\ &+ \tan \theta \hat{i} \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] \end{aligned} \right\} \\
&+ \frac{2}{c^2} R \vec{b}_s \cdot \left[\vec{a}_e - \left(\dot{\hat{n}} \cdot \vec{v}_e \right) \hat{n} - (\hat{n} \cdot \vec{a}_e) \hat{n} - (\hat{n} \cdot \vec{v}_e) \dot{\hat{n}} \right] \\
&- \frac{2}{c^2} R \vec{b}_s \cdot \left[\vec{a}_s - \left(\dot{\hat{n}} \cdot \vec{v}_s \right) \hat{n} - (\hat{n} \cdot \vec{a}_s) \hat{n} - (\hat{n} \cdot \vec{v}_s) \dot{\hat{n}} \right] \\
&+ o(c^{-3})
\end{aligned} \tag{2.221}$$

2.11.3 Doppler-Compensated Frequency Shift

The PCM effect after applying the Doppler compensation scheme from Eq. (2.195) will be given by:

$$\Delta y_{pcm} = y_{1w}^{pcm} - \frac{1}{2} y_{2w}^{pcm} \quad (2.222)$$

Using Eqs. (2.215) and (2.221) we see that the first order of the effect is suppressed with the remaining terms being:

$$\begin{aligned} \Delta y_{pcm} = & -\frac{L}{c^2} \left\{ \sec^2 \theta \dot{\theta} \left(\hat{i} \cdot \vec{v}_e - \sin \theta \hat{n} \cdot \vec{v}_e \right) \right. \\ & \left. + \tan \theta \hat{i} \cdot \left[\vec{a}_e - \left(\dot{\hat{n}} \cdot \vec{v}_e \right) \hat{n} - (\hat{n} \cdot \vec{a}_e) \hat{n} - (\hat{n} \cdot \vec{v}_e) \dot{\hat{n}} \right] \right\} \\ & - \frac{1}{c^2} R \vec{b}_s \cdot \left[\vec{a}_e - \left(\dot{\hat{n}} \cdot \vec{v}_e \right) \hat{n} - (\hat{n} \cdot \vec{a}_e) \hat{n} - (\hat{n} \cdot \vec{v}_e) \dot{\hat{n}} \right] + o(c^{-3}) \end{aligned} \quad (2.223)$$

This is equivalent to Eq. 42 in Litvinov et al. (2021) after allowing for notational differences.

2.12 Tropospheric Refraction

The speed at which an electromagnetic signal propagates through the atmosphere will vary depending on atmospheric composition and local conditions. The delay in receiving the signal can be thought of as a difference in path length, ΔL_{trop} , relative to transmission through vacuum and can be computed as follows:

$$\Delta L_{trop} = \int_P n ds - \int_G ds \quad (2.224)$$

where n is the index of refraction of the troposphere. The first integral computes the effective path length over the refracted path P while the second integral is the path length over the

purely geometric path G . In Hopfield (1963), the delay is approximated by recognizing that the difference between the refracted path and geometric path will be small. Doing both integrals over the geometric path we have:

$$\Delta L_{trop} \approx \int_G (n - 1) ds \quad (2.225)$$

Unfortunately, the index of refraction of the atmosphere varies from place-to-place and is a function of time. Therefore, performing this integral is very difficult in practice. It is common to simplify the problem by assuming that the variation in the delay with elevation angle can be described with a specific functional form as follows:

$$\Delta L_{trop}(E) = \Delta L_{trop}^z m(E) \quad (2.226)$$

where $m(E)$ is called a mapping function and ΔL_{trop}^z is the delay in the zenith direction. Niell (1996) proposed the following continued fraction form for the mapping function:

$$m(t, E) = \frac{1 + \frac{a(t)}{1 + \frac{b(t)}{1 + c(t)}}}{\sin E + \frac{a(t)}{\sin E + \frac{b(t)}{\sin E + c(t)}}} \quad (2.227)$$

where a , b and c are coefficients which depend on local conditions including temperature, height above the geoid and latitude but can also be estimated by integrating refractivity through atmospheric models. The delay is also often broken down into a component due to the less variable hydrostatic gases making up the bulk of the atmosphere and a significantly smaller wet component due to the variable water vapor content of the atmosphere driven by

local weather. This can be expressed as:

$$\Delta L_{trop}(t, E) = \Delta L_{trop}^{hz}(t) m_h(t, E) + \Delta L_{trop}^{wz}(t) m_w(t, E) \quad (2.228)$$

where h denotes the hydrostatic and w the wet components respectively. Note that separate mapping functions are needed to more accurately describe the relationship with elevation.

The zenith delay can be estimated from surface weather data. However, there are also models that can be used when weather data are not available. In particular, empirical models use past weather data to estimate the delay (for example, see Collins (1999)). These are appropriate when building a model to forecast a future outcome. Discrete models use ray-tracing through numerical weather models to estimate the actual delay over a given place and time in the past. We will use one such model, the Vienna Mapping Function 3 (VMF3) from Landskron & Böhm (2018a), to estimate tropospheric delays in this project. This model provides values for the delay and mapping function coefficients four times per day on a $1^\circ \times 1^\circ$ geographic grid. In addition, a complementary and consistent model describing the dependence of the delay on the azimuthal direction is also available (see Landskron & Böhm (2018b)). This model, called GRAD, accounts for anisotropy by introducing orthogonal north and east gradients as follows:

$$\begin{aligned} \Delta L_{trop}(t, A, E) = \Delta L_{trop}(t, E) &+ [G_N^h(t) \cos A + G_E^h(t) \sin A] m_{gh}(E) \\ &+ [G_N^w(t) \cos A + G_E^w(t) \sin A] m_{gw}(E) \end{aligned} \quad (2.229)$$

where $\Delta L_{trop}(t, E)$ is the delay from Eq. (2.228) and is considered to be the isotropic component of the delay while G_N and G_E are additional coefficients provided separately for the hydrostatic and wet components of the atmosphere. The gradient mapping function takes

on the much simpler form:

$$m_{gi}(E) = \frac{1}{\sin E \tan E + C_i} \quad (2.230)$$

where C_i is a coefficient dependent on the scale height of the atmosphere and takes on the values $C_h = 0.0031$ and $C_w = 0.0007$.

If the tropospheric delay were constant over time, then the frequency of signals passing through the atmosphere would be unaffected. However, as weather conditions change and as the line-of-sight to the spacecraft changes in elevation and azimuth, the delay will change giving rise to the following frequency shift:

$$y_{1w}^{trop} = -\frac{1}{c} \frac{d}{dt} \Delta L_{trop}(t_3, E(t_3), A(t_3)) \quad (2.231)$$

As the elevation and azimuth are functions of time themselves, we simplify this to:

$$y_{1w}^{trop} = -\frac{1}{c} \Delta \dot{L}_{trop}(t_3) \quad (2.232)$$

Note that we are using t_3 instead of t_2 as the propagation through the troposphere will occur near the ground station as opposed to near the spacecraft. The 2way mode frequency shift caused by passing through the atmosphere on the uplink and downlink paths can similarly be written as:

$$y_{2w}^{trop} = -\frac{1}{c} \left[\Delta \dot{L}_{trop}(t_1) + \Delta \dot{L}_{trop}(t_3) \right] \quad (2.233)$$

Assuming that the change in the troposphere over the propagation time can be adequately described by a linear function, we can use the following approximation:

$$\Delta \dot{L}_{trop}(t_1) = \Delta \dot{L}_{trop}(t_3) - \Delta \ddot{L}_{trop}(t_3) \Delta t_{2w}(t_3) \quad (2.234)$$

And so:

$$y_{2w}^{trop} = -\frac{1}{c} \left[2\Delta\dot{L}_{trop}(t_3) - \Delta\ddot{L}_{trop}(t_3) \Delta t_{2w}(t_3) \right] \quad (2.235)$$

Finally, after applying the Doppler compensation scheme from Eq. (2.195) we arrive at:

$$\Delta y_{trop} = -\frac{1}{2c} \Delta\ddot{L}_{trop} \Delta t_{2w} \quad (2.236)$$

This frequency shift is directly proportional to the second time derivative of the tropospheric delay which will be caused by changes in atmospheric conditions and by the change in elevation over time. As we are using a model which provides parameters every 6 hours and linearly interpolating in between, we are not readily able to estimate the change due to short-term atmospheric conditions. However, given that the dry component of the tropospheric delay, which accounts for the majority of the delay, does not change significantly over short timescales, we anticipate that the change due to elevation will dominate.

2.13 Ionospheric Refraction

A radio frequency signal propagating through the ionosphere is also delayed due to the presence of free electrons. This can be treated in a similar way to Eqs (2.224) and (2.225) with the index of refraction of the plasma, from Vessot & Levine (1979), being given by:

$$n(\nu) = \sqrt{1 - \frac{\nu_p^2}{\nu^2 (1 \pm \frac{\nu_m}{\nu})}} \quad (2.237)$$

ν_p is the plasma frequency which is itself given by:

$$\nu_p = \sqrt{\frac{N_e e^2}{4\pi^2 \epsilon_o m_e}} \quad (2.238)$$

where N_e is the electron number density, ε_o is the permittivity of free space, e the charge of the electron and m_e its mass. ν_m is the electron gyromagnetic frequency and according to Petrie et al. (2011) is given by:

$$\nu_m = -\frac{e}{2\pi m_e} B \cos \theta \quad (2.239)$$

where B is the Earth's magnetic field strength and θ is the angle between the magnetic field vector and the direction of signal propagation. Using a typical value of $B = 5 \times 10^{-5}$ T and $\theta = 0$, we calculate $\nu_m = 1.4$ MHz so the ratio of ν_m/ν for the GHz signal frequencies of RadioAstron will be very small and can be neglected for our purposes. Using Eq. (2.83) to Taylor expand the remaining parts of the equation we arrive at the following:

$$n(\nu) = 1 - \frac{80.616 N_e}{2\nu^2} - \frac{1}{8} \left(\frac{80.616 N_e}{\nu^2} \right)^2 - o \left(\frac{N_e^3}{\nu^6} \right) \quad (2.240)$$

where we have used the values of the constants in ν_p to arrive at the factor of $80.616 \text{ m}^3 \text{ s}^{-2}$. The maximum electron number density in the ionosphere is on the order of $N_e \sim 1 \times 10^{12} \text{ e/m}^3$. At the uplink frequency of $\nu = 7.207,5$ GHz, the second order term in this expansion will be over six orders of magnitude smaller than the first. Given that the ionospheric shift is itself on the order of 10^{-13} , we can safely discard all but the first term in the expansion. Doing so and using this index of refraction from Eq. (2.225) we find:

$$\Delta L_{ion}(\nu) = -\frac{40.308}{\nu^2} \int_G N_e ds \quad (2.241)$$

The integral represents the total number of electrons along the path and is referred to as the slant total electron content (STEC) along the line-of-sight. The electron content of the

ionosphere will vary with time so it is more accurate to write:

$$\Delta L_{ion}(\nu, t) = -\frac{40.308}{\nu^2} \text{STEC}(t) \quad (2.242)$$

This variation leads to a corresponding frequency shift which over the downlink is given by:

$$y_{1w}^{ion} = -\frac{1}{c} \frac{d}{dt} \Delta L_{ion}(\nu_0, t_3) = \frac{40.308}{c \nu_0^2} \dot{\text{STEC}}(t_3) \quad (2.243)$$

where ν_0 is the nominal downlink frequency of the spacecraft (8.4 GHz or 15 GHz).

The rate of change of STEC can in principle be measured using two simultaneous and coherent signals transmitted at different frequencies. So long as the ionospheric effect is the only dispersive effect, all other frequency shifts will cancel when taking the difference of the frequency shift at the two frequencies. The difference due to the ionosphere will then be given by:

$$\Delta y(t) \equiv y_{\nu_1}(t) - y_{\nu_2}(t) = \frac{40.308}{c} \left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2} \right) \dot{\text{STEC}}(t) \quad (2.244)$$

and so:

$$\dot{\text{STEC}}(t) = \frac{c}{40.308} \left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2} \right)^{-1} \Delta y(t) \quad (2.245)$$

It was found in Belonenko et al. (2020) that this approach is not possible using the 8.4 GHz and 15 GHz downlink signals from RadioAstron measured by the Doppler tracking system. They suggest this is due to instrumental noise being larger than the ionospheric effect. This approach may still be possible with VLBI recordings of the spacecraft signal, though a study along the lines of Belonenko et al. (2020) has not yet been done. For this project, we simply wish to estimate the ionospheric effect, so we will use a different approach similar to what was done for the tropospheric delay.

STEC can be estimated from GNSS measurements of the electron content along the zenith direction by again using a mapping function:

$$\text{STEC}(z) = m_{ion}(z) \text{VTEC} \quad (2.246)$$

where VTEC is the vertical total electron content and z is the zenith distance ($z = \pi - E$). We will use the modified single-layer mapping function from Schaer (1999):

$$m_{ion}(z) = \left[1 - \left(\frac{R_{\oplus}}{R_{\oplus} + H} \right)^2 \sin^2(\alpha_m z) \right]^{-1/2} \quad (2.247)$$

where $R_{\oplus}$ is the mean radius of the Earth, H is the single layer height and α_m is an additional parameter that allows the mapping function to be tuned to better fit empirical measurements. We will use the following values for the constants:

$$R_{\oplus} = 6,371 \text{ km}, \quad H = 506.7 \text{ km}, \quad \alpha_m = 0.9782 \quad (2.248)$$

which are taken from Dach et al. (2007), who fitted JPL's more complex extended slab model mapping function. VTEC as a function of time will be read from NASA's Crustal Dynamics Data Information System (CDDIS) ionospheric data archive (Noll 2010) which provides measurements every 2 h on a global grid with $2.5^\circ \times 5^\circ$ resolution. Linear interpolation in time and space is then done to find VTEC over a particular point on Earth at a given time.

For the 2way mode frequency shift, we must take into account that the uplink and downlink signal frequencies are not the same:

$$y_{2w}^{ion} = \frac{40.308}{c} \left[\frac{\text{STEC}(t_1)}{\nu_{up}(t_1)^2} + \frac{\text{STEC}(t_3)}{\nu_{down}(t_2)^2} \right] \quad (2.249)$$

The uplink frequency for RadioAstron was not constant. Rather, it was shifted to account for the non-relativistic Doppler shift over the uplink path such that the signal received at

the spacecraft would be close to the nominal frequency of 7.207,5 GHz. In the 2way mode, the downlink frequency was phase-coherently generated from the uplink frequency received by the spacecraft which would have been shifted from that broadcast by the ground station. And so:

$$\nu_{down}(t_2) = F_{\nu_0}(1 + y_{up})\nu_{up}(t_1) = \nu_0 + \delta\nu_0(t_2) \quad (2.250)$$

where F_{ν_0} is given by Eq. (2.173) and $\delta\nu_0$ represents the error introduced by having computed the uplink frequency using the a priori predicted orbit rather than the true orbit of the spacecraft and only compensating for the non-relativistic Doppler shift. This error was necessarily small compared to the nominal frequency to simplify the spacecraft's electronics and in practice was $\delta\nu_0 < 100$ Hz. Using Eq. (2.250) we can write:

$$\frac{1}{\nu_{down}(t_2)^2} = \frac{1}{\nu_0^2} \left[1 + \frac{\delta\nu_0(t_2)}{\nu_0} \right]^{-2} \quad (2.251)$$

Using the following Taylor expansion at $x = 0$:

$$\frac{1}{(1+x)^2} = 1 - 2x + \dots \quad (2.252)$$

we find to first order that:

$$\frac{1}{\nu_{down}(t_2)^2} \approx \frac{1}{\nu_0^2} - 2\frac{\delta\nu_0(t_2)}{\nu_0^3} \quad (2.253)$$

At 8.4 GHz, the second term in this equation is 10^8 times smaller than the first. Given that the ionospheric shift is 10^{-13} or smaller, this second term will correspond to shifts smaller than 10^{-21} and can be neglected. And so for the purpose of calculating the ionospheric shift

we can use the following approximation:

$$\frac{1}{\nu_{down}(t_2)^2} \approx \frac{1}{\nu_0^2} \quad (2.254)$$

Similarly, by Eq. (2.250):

$$\begin{aligned} \frac{1}{F_{\nu_0}^2 (1 + y_{up})^2 \nu_{up}(t_1)^2} &\approx \frac{1}{\nu_0^2} \\ \frac{1}{\nu_{up}(t_1)^2} &\approx \frac{F_{\nu_0}^2}{\nu_0^2} + 2 \frac{F_{\nu_0}^2 y_{up}}{\nu_0^2} + \frac{F_{\nu_0}^2 y_{up}^2}{\nu_0^2} \end{aligned} \quad (2.255)$$

y_{up} will be dominated by $\frac{\dot{D}}{c}$ due to the non-relativistic Doppler shift which we will find out in the next chapter is of order 10^{-5} . Again, given that the ionospheric shift is 10^{-13} or smaller, the second and third terms will lead to shifts smaller than 10^{-18} and 10^{-23} respectively and can also be neglected. This leaves the following approximation which can be used when calculating the ionospheric shift:

$$\frac{1}{\nu_{up}(t_1)^2} \approx \frac{F_{\nu_0}^2}{\nu_0^2} \quad (2.256)$$

Assuming STEC does not change too rapidly over the propagation time, we can approximate the rate of change of STEC as follows:

$$\text{ST}\dot{\text{E}}\text{C}(t_1) = \text{ST}\dot{\text{E}}\text{C}(t_3) - \text{ST}\ddot{\text{E}}\text{C}(t_3) \Delta t_{2w} \quad (2.257)$$

Using the three preceding approximations, we can re-write Eq. (2.249) as follows:

$$y_{2w}^{ion} = \frac{40.308}{c \nu_0^2} \left[(F_{\nu_0}^2 + 1) \text{ST}\dot{\text{E}}\text{C}(t_3) - F_{\nu_0}^2 \text{ST}\ddot{\text{E}}\text{C}(t_3) \Delta t_{2w} \right] \quad (2.258)$$

Combining Eqs. (2.243) and (2.258) in the Doppler compensation scheme of Eq. (2.195) we

find the remaining shift to be:

$$\Delta y_{ion} = \frac{40.308}{2c \nu_0^2} \left[(1 - F_{\nu_0}^2) \text{STEC}(t_3) + F_{\nu_0}^2 \ddot{\text{STEC}}(t_3) \Delta t_{2w} \right] \quad (2.259)$$

where the first term corresponds to the difference in the ionospheric effect at the uplink and downlink frequencies while the second term is due to the change in the ionospheric shift due to changes in the electron number density and elevation over time. As we found with the tropospheric effect, we cannot readily estimate the change due to electron number density as we are using VTEC estimates from a model providing periodic estimates that are then linearly interpolated. However, it is reasonable to assume that the second derivative of STEC will usually be smaller than the first derivative and so the first term in the previous equation will dominate. Notice that the first term is proportional to the ionospheric shift in the 1way mode. Once again, the Doppler compensation scheme is found to be beneficial as it reduces the effect to less than 20% of what it is in the 1way mode at 8.4 GHz.

2.14 Overall Frequency Shift

Before we can incorporate PCM and atmospheric effects into Eq. (2.198), we note that these enter Eq. (2.150) as follows:

$$1 + y = (1 + y^E) (1 + y^{SH}) (1 + y^O) (1 + y^{trop}) (1 + y^{ion}) (1 + y^{pcm}) + o(c^{-4}) \quad (2.260)$$

Cross-terms with products of multiple effects (Doppler, PCM, troposphere and ionosphere) will be below the 10^{-15} threshold, and so our final equation for the overall frequency shift in

the 1way mode is:

$$y_{1w} = y_{1w}^{dop\ 1st} + y_{grav} + y_{1w}^{dop\ 2nd} + y_{1w}^{dop\ 3rd} + y_{1w}^{pcm} + y_{1w}^{trop} + y_{1w}^{ion} + \Delta h + o(c^{-4}) \quad (2.261)$$

Similarly, in the 2way mode it is:

$$y_{2w} = y_{2w}^{dop\ 1st} + y_{2w}^{dop\ 2nd} + y_{2w}^{dop\ 3rd} + y_{2w}^{pcm} + y_{2w}^{trop} + y_{2w}^{ion} + \Delta h + o(c^{-4}) \quad (2.262)$$

And finally, after applying the Doppler compensation scheme the overall frequency shift will be:

$$\Delta y_{dc} = y_{grav} + \Delta y_{dop\ 2nd} + \Delta y_{dop\ 3rd} + \Delta y_{pcm} + \Delta y_{trop} + \Delta y_{ion} + \Delta h + \Delta y_{small} + o(c^{-4}) \quad (2.263)$$

Chapter 3

Model Implementation

3.1 Overview

The theoretical model from the previous chapter has been implemented in software (ModelApp) using the C# programming language and Matlab. ModelApp has approximately 28,000 lines of code excluding those from external libraries. The components of ModelApp are illustrated in Figure 3.1. The main routine is written in C# and loads the spacecraft state vectors from file and computes ground station state vectors using library functions written in Matlab and C. Similarly, atmospheric and gravitational quantities are computed using scripts written in Matlab. The main routine then computes the various quantities described in the previous chapter and writes output to file. Finally, this output is read by routines written in Matlab to produce the figures in this chapter and the tables appearing in Chapter 4. For the purpose of illustrating the quantities computed using the dynamical model, a series of four orbits starting in December 2015 were selected as a sample interval for most of the figures appearing in this chapter.

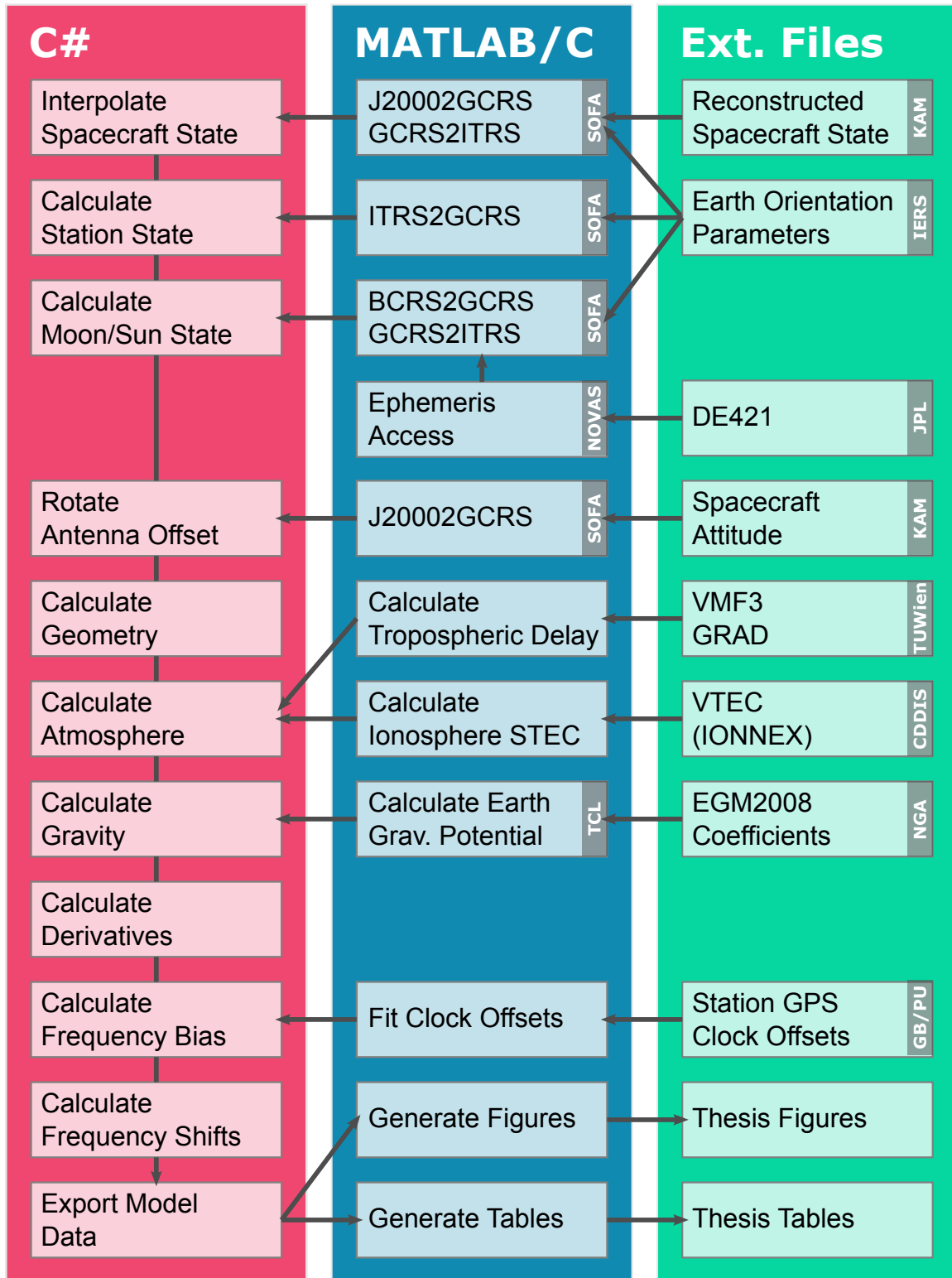


Figure 3.1: Software architecture implementing the dynamical model. Note that J20002GCRS, ITRS2GCRS and BCRS2GCRS are functions that transform vectors between reference frames.

3.2 Spacecraft State Vectors

Orbit determination for the RadioAstron spacecraft was performed by the Keldysh Institute of Applied Mathematics (KAM) of the Russian Academy of Sciences (see Zakhvatkin et al. 2014 and Zakhvatkin et al. 2020). KAM has conveniently provided files with the reconstructed spacecraft state vectors covering periods of time that include the scientific observing sessions. These files contain records with the Moscow time (UTC+3h) and the corresponding 3D position and velocity vectors in the J2000 reference frame. Records are provided in one second increments. Multiple versions of each file, corresponding to improvements in the orbit determination algorithms, also exist. ModelApp reads the most recent version of the files covering the period of time of interest and converts each record’s time to UTC and vectors from J2000 to GCRF. If the time needed lies between two records provided by KAM, state vectors are calculated by computing the Keplerian orbital elements of the records before and after, then interpolating each orbital element to the time needed before finally using the interpolated orbital elements to compute the state vectors at the time needed. The acceleration and jerk of the spacecraft are computed by taking derivatives of the velocity. This is done numerically when working with long intervals, such as that used for the figures plotted herein, or by taking derivatives of a polynomial fit of each component of the velocity vector in the case of individual sessions of ~ 1 hr duration. Spacecraft state vectors over the sample interval are plotted in Figure 3.2. The highly elliptical orbit of the spacecraft is obvious in Figures 3.3 and 3.4 in which the geocentric distance of the spacecraft is plotted. During the sample interval, perigee distances range from 39,000 km to 27,800 km and apogee distances from 319,600 km to 334,100 km. These ranges reflect the highly evolving nature of the spacecraft’s orbit.

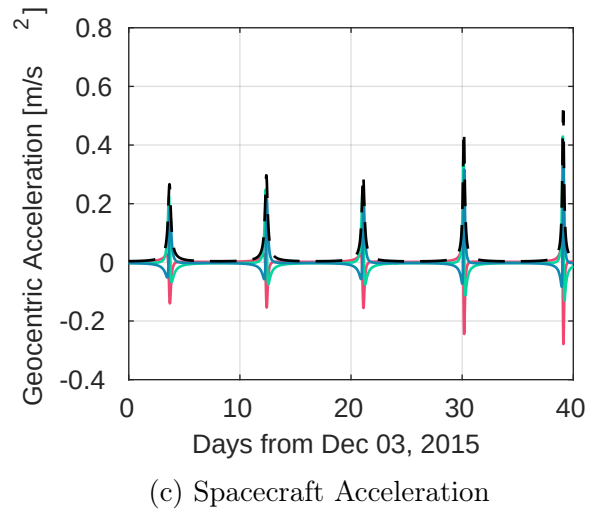
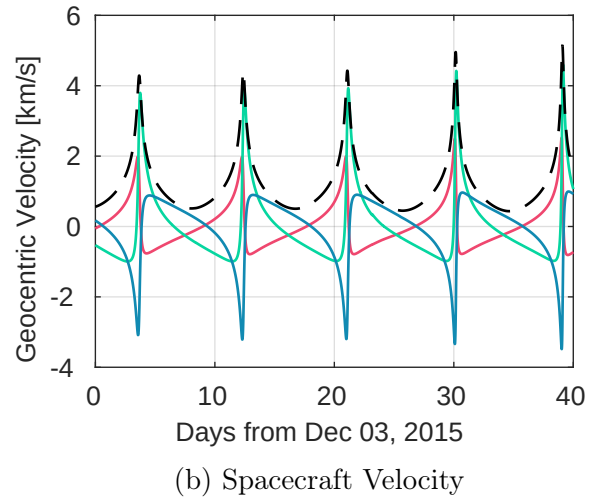
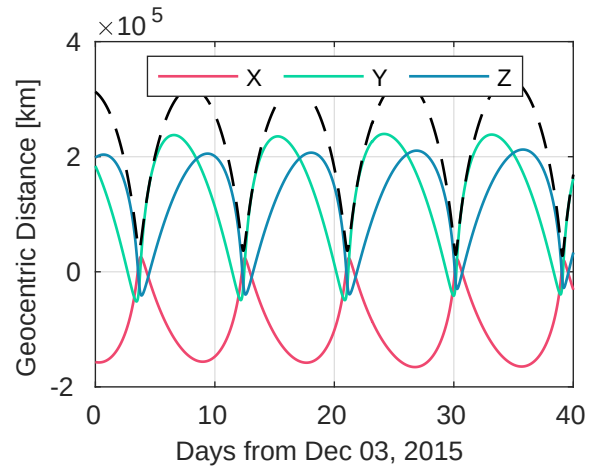


Figure 3.2: Spacecraft state vectors from orbit determination. The dashed lines represent the magnitude of each vector.

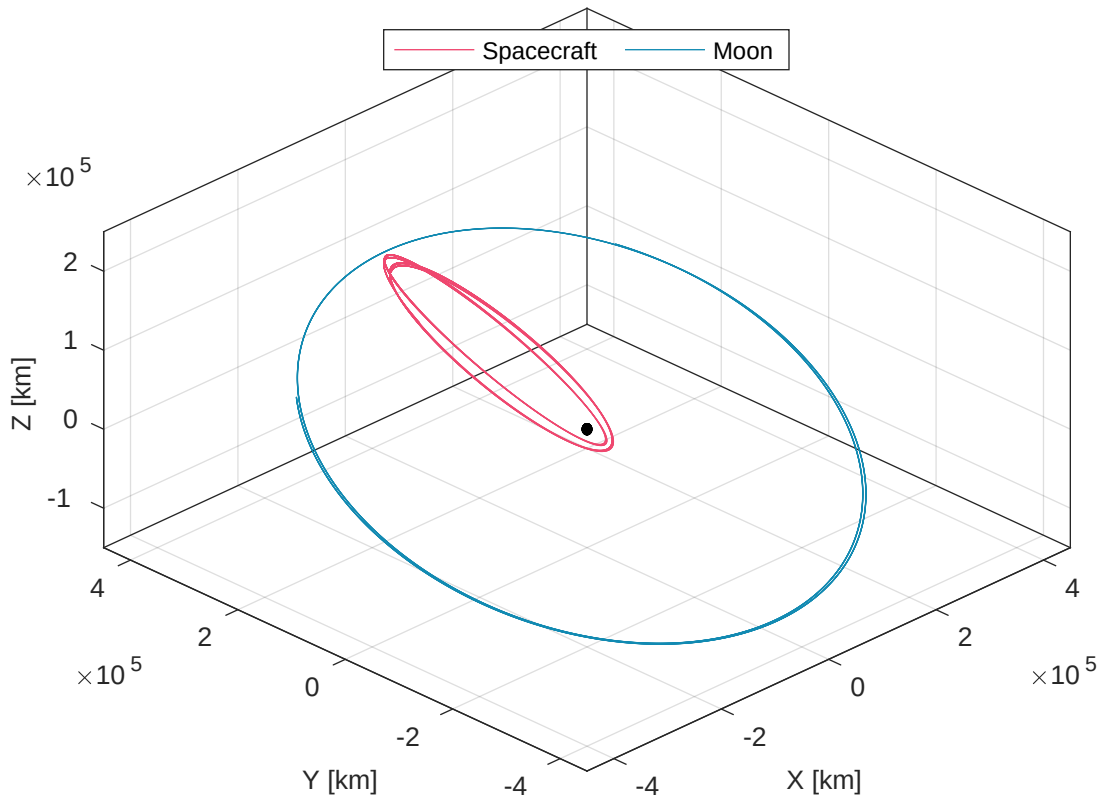


Figure 3.3: The positions of the spacecraft and Moon between December 3, 2015 and January 17, 2016 with the Earth represented to scale by the sphere at the origin. Note how the highly elliptical orbit of the spacecraft reaches distances that are on the order of the radius of the Moon's orbit. Also note the significant variation in the spacecraft's orbit, particular after close approaches to the Moon like the one that took place on December 27, 2015.

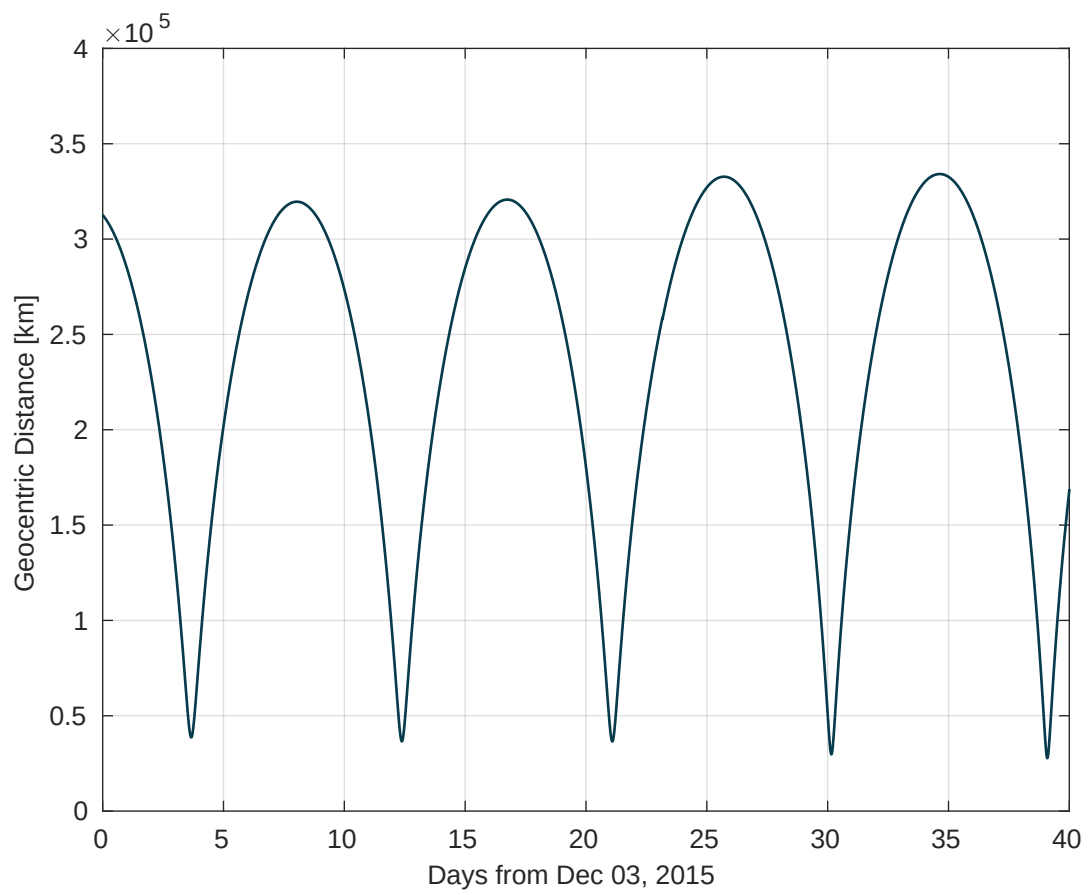


Figure 3.4: Spacecraft distance from Earth's centre-of-mass during selected orbits.

3.3 Ground Station State Vectors

The ITRF coordinates used for the Pushchino ground station are:

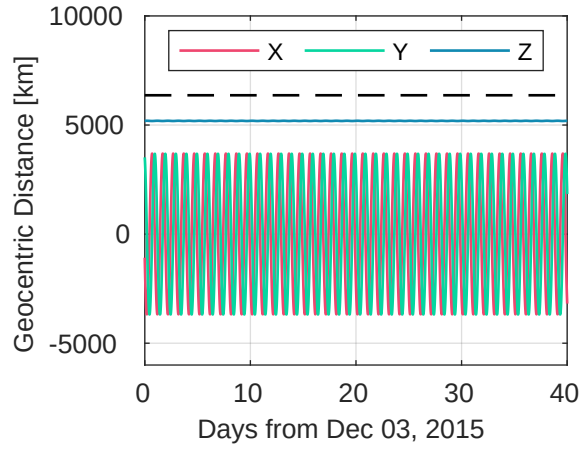
$$\begin{aligned}\vec{r}_{PU} &= [2916948.812, 2248648.203, 5190099.861] \text{ m} \\ \vec{v}_{PU} &= [-0.02161, 0.01269, 0.00665] \text{ m/yr} \\ N_{PU} &= 14.385 \text{ m}\end{aligned}\tag{3.1}$$

This position is from 1995 and was provided by KAM. Velocity is from the ITRF2008 reference system and the geoid height was looked up using EGM96. Coordinates for the Green Bank ground station are from Langston (2012) for April 26, 2011:

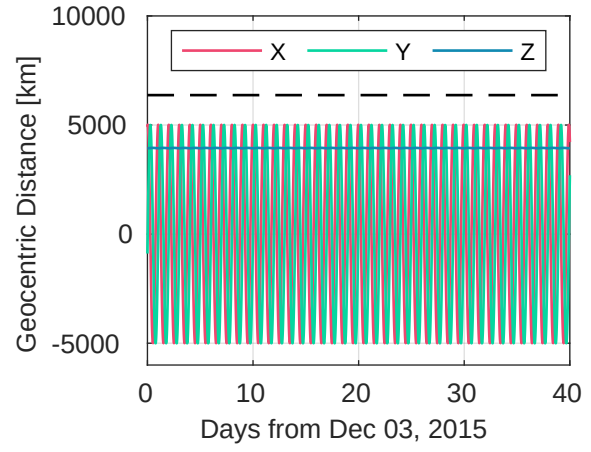
$$\begin{aligned}\vec{r}_{GB} &= [882879.7266, -4924482.2950, 3944130.6925] \text{ m} \\ \vec{v}_{GB} &= [-0.01487, -0.00057, 0.00144] \text{ m/yr} \\ N_{GB} &= -31.739 \text{ m}\end{aligned}\tag{3.2}$$

where the geoid height was also looked up in EGM96. ITRF position vectors are computed by ModelApp using the velocities of the ground station to translate the reference coordinates given at a particular epoch, as listed above, to the time of interest. Note that the relativistic Doppler shift caused by these velocities is exceedingly small and was ignored in the development of the theoretical model in the previous chapter.

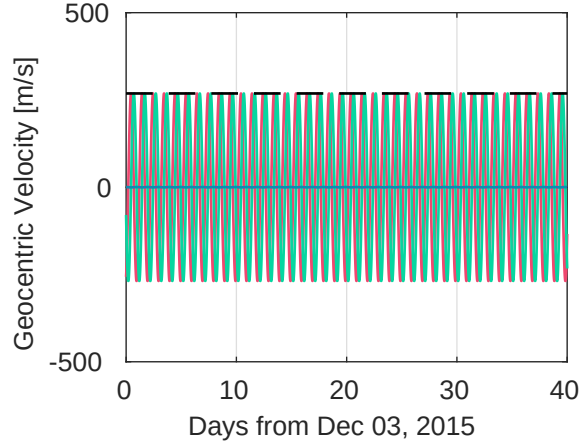
The rotation from ITRF to GCRF was done using an external library called the Tracker Component Library (TCL) from the Naval Research Laboratory (see Crouse 2017). TCL is written in Matlab and itself wraps a number of other libraries written in C including the Standard of Fundamental Astronomy (SOFA) library developed by the International Astronomical Union (IAU) which implements a highly accurate model of Earth’s rotation, precession, nutation and pole motion (see IAU SOFA Board 2019). SOFA utilizes EOP to achieve the highest accuracy possible including:



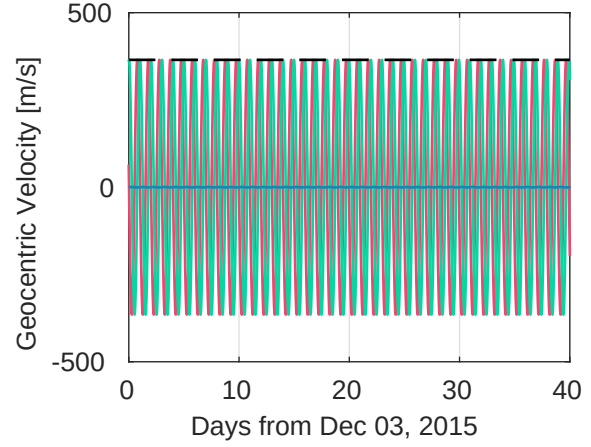
(a) Pushchino Position



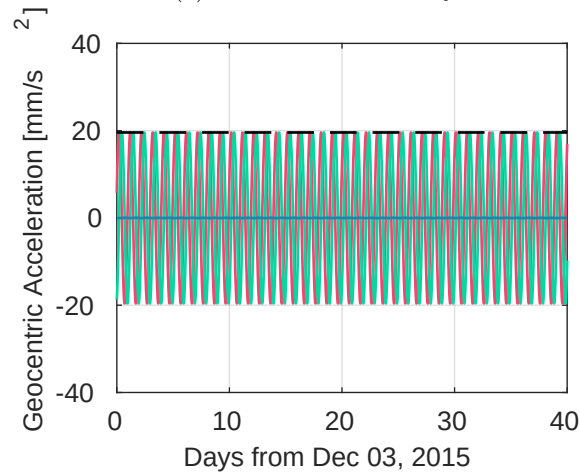
(b) Green Bank Position



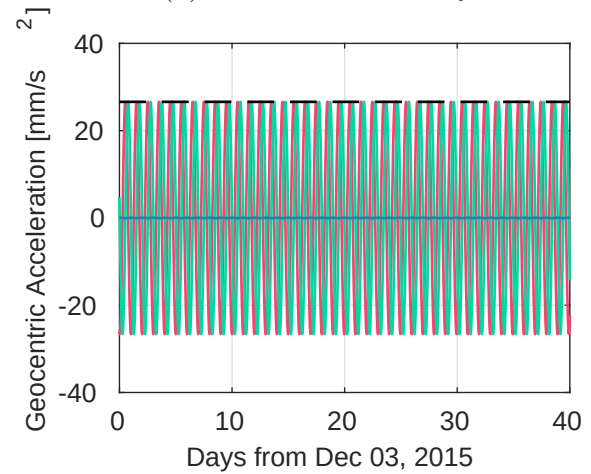
(c) Pushchino Velocity



(d) Green Bank Velocity



(e) Pushchino Acceleration



(f) Green Bank Acceleration

Figure 3.5: Ground station state vectors in the GCRF calculated using ITRF coordinates with the SOFA library. The dashed lines represent the magnitude of each vector.

- UT1-UTC Offset
- Length-of-Day (LOD) offset
- Coordinates of the pole (x_p, y_p)
- Celestial pole offsets (dx, dy)

These parameters are provided by IERS and contained within a file in TCL that covers the duration of the RadioAstron mission. The SOFA library provides both GCRF position and velocity vectors. The acceleration and jerk of the ground station, caused by Earth’s rotation, were computed in a similar fashion to those of the spacecraft by taking derivatives of the velocity. Ground station state vectors over the sample interval are plotted in Figure 3.5. Note that the GCRF z-axis is approximately aligned with the Earth’s rotation axis on January 1, 2000 and remains fairly closely aligned to it at the sample interval. Thus the ground station motion is essentially circular in the xy-plane.

3.4 Moon and Sun State Vectors

The position of the Moon and Sun are computed by ModelApp using JPL’s DE421 ephemeris (Folkner et al. 2009). Numerous interfaces exist to this ephemeris, including one in TCL. However, this was found to be too slow, so the interface written in C that is part of the NOVAS library from the US Naval Observatory (Kaplan et al. 2012) is used instead. Regardless of the interface chosen, the computed state vectors should be the same. The geocentric and spacecraft distances to the Moon and Sun are plotted in Figure 3.6. Interestingly, the Earth’s perihelion on January 2, 2016 happens to lie within the sample interval and is visible in Figure 3.6b as a minimum. A relatively close approach of the Moon is also visible in Figure 3.6c on day 24 when the spacecraft reached a lunar distance of $\sim 103,000$ km.

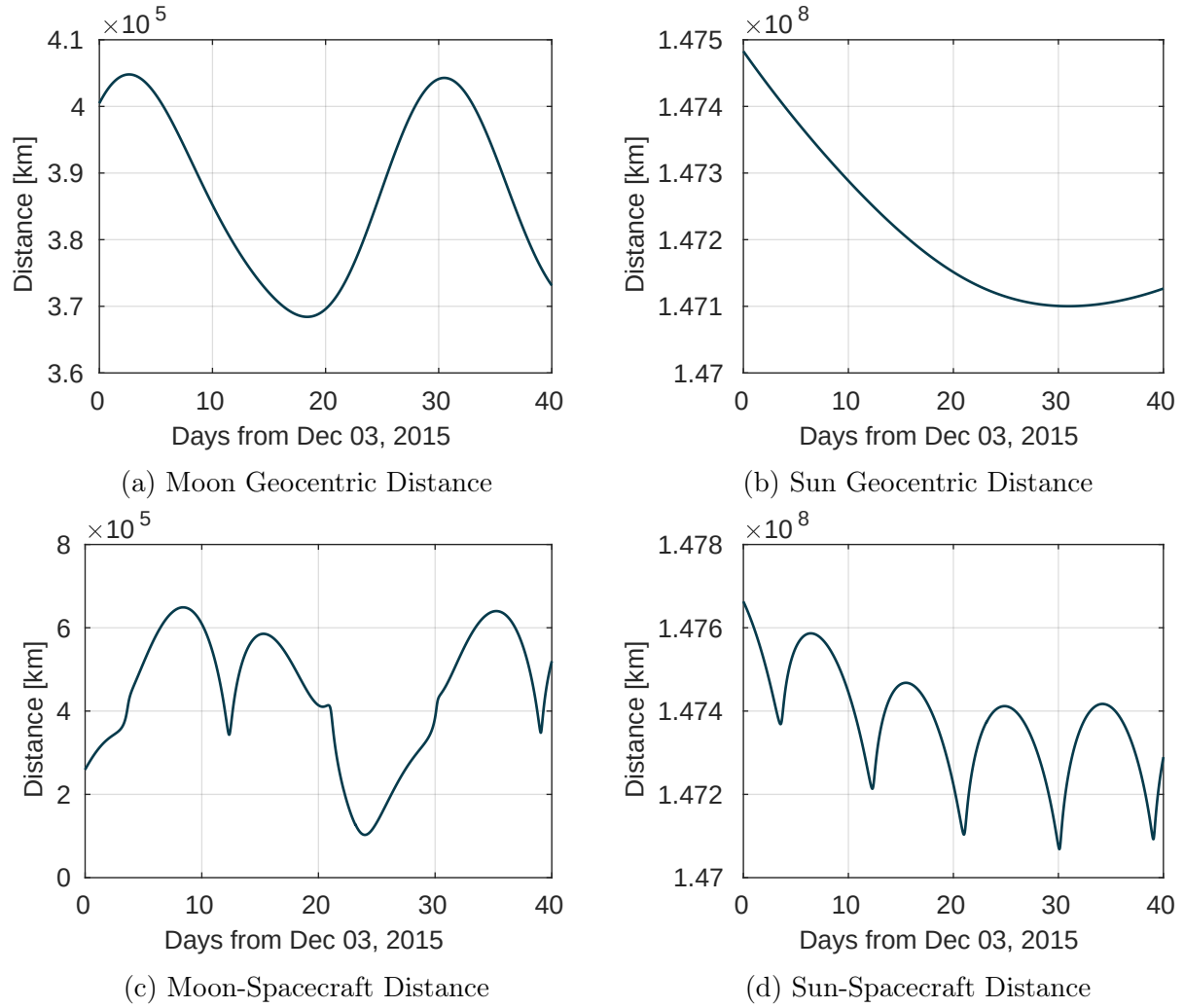


Figure 3.6: Moon and Sun distances from Earth's centre-of-mass (top) and from the spacecraft (bottom) during the selected interval with Earth's perihelion on January 2, 2016 appearing in the upper right plot and a relatively close approach to the Moon by the spacecraft in the lower left plot near day 24.

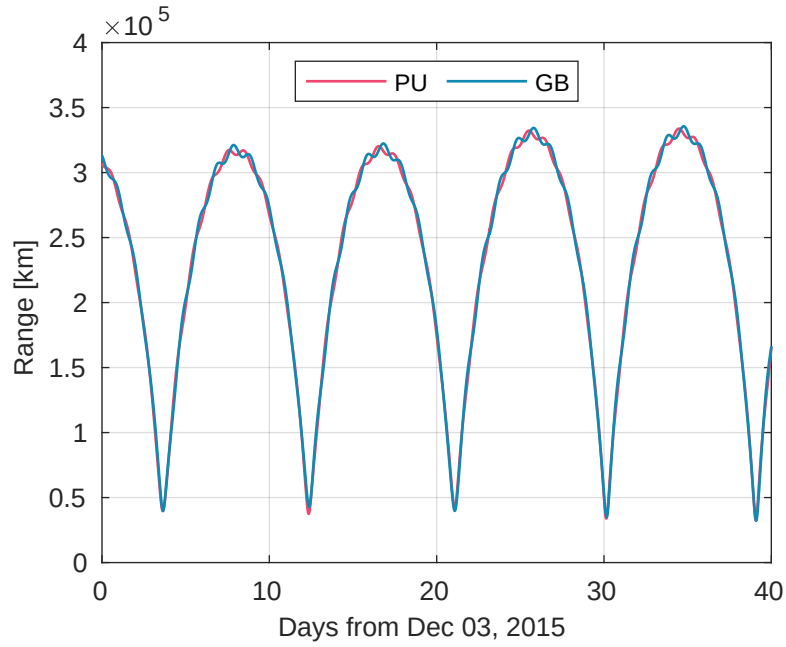
3.5 Geometric Calculations

Using the calculated state vectors, various geometric quantities are computed. Range or radial distance is calculated using Eq. (2.4) and is plotted in Figure 3.7a over the sample interval. Range rate or radial velocity is calculated using Eq. (2.6) and is plotted in Figure 3.7b. Diurnal variations due to Earth’s rotation superimposed on the overall variations due to the spacecraft’s orbit are visible in both plots. In particular, the change in radial velocity from positive to negative as the spacecraft passes apogee is apparent as is the rapid swing in the sign of the radial velocity when the spacecraft moves fastest as it rounds perigee.

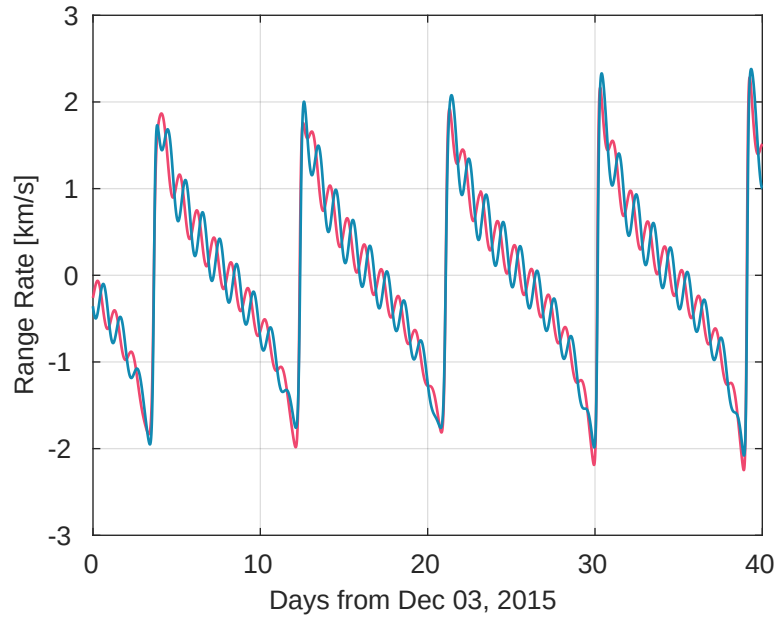
Declination and its rate of change are calculated using Eqs. (2.8) and (2.10) respectively. These are plotted in Figure 3.8. During the sample interval, the orientation of the orbit is such that maximum declination occurs when the spacecraft approaches perigee. Near perigee, the spacecraft goes through a rapid change in declination as it first plunges below the Earth’s equatorial plane before rapidly moving back towards positive declinations. This is characteristic of many Russian satellites whose elliptical orbits are optimized so that the spacecraft spends the majority of its time over the northern hemisphere.

Elevation and azimuth are calculated using the method described in Section 2.1.4 and are plotted in Figure 3.9. Again, variations due to the motion of the spacecraft and those due to Earth’s diurnal motion are evident. However, in this case the latter cause a much larger change as would be expected for angles measured relative to a local ENU frame.

The propagation time can also now be calculated using the state vectors and quantities derived from them. Both the one way and two way propagation times calculated using Eqs. (2.97) and (2.108) are plotted in Figure 3.10. These closely follow the radial distance in Figure 3.7a as to first order the propagation time is simply a multiple of D/c .

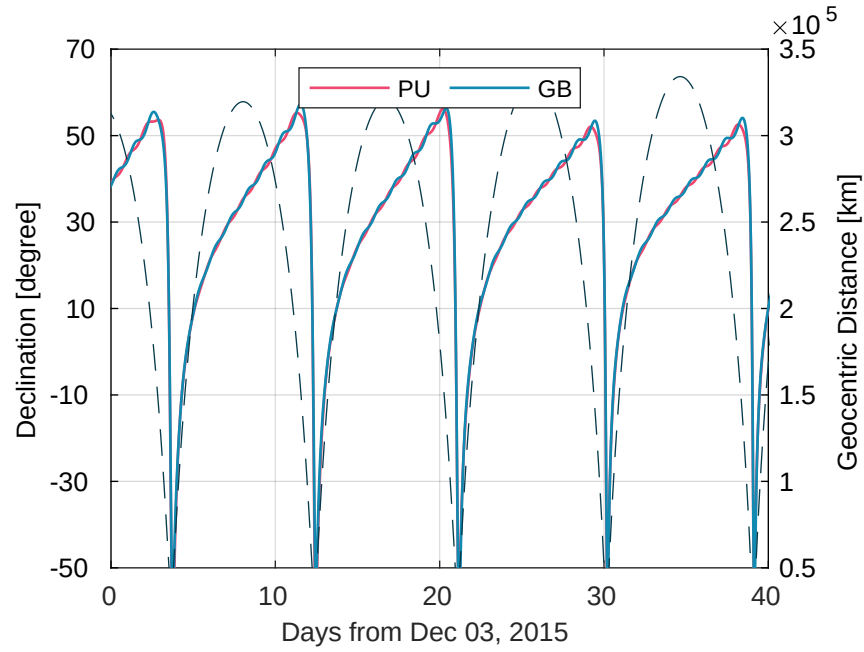


(a) Radial Distance

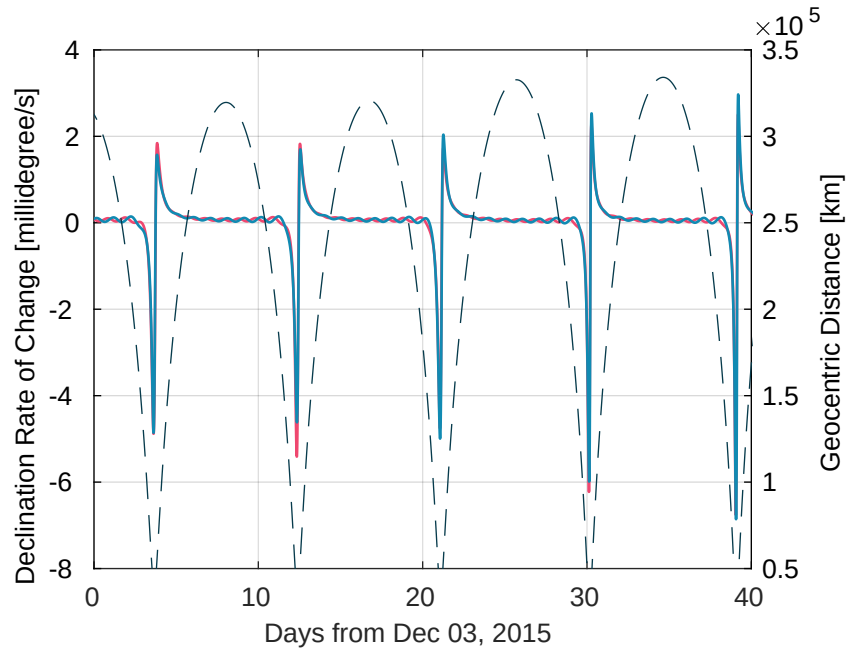


(b) Radial Velocity

Figure 3.7: Spacecraft radial distance and velocity relative to each ground station, also called range and range rate respectively, during selected orbits. Note the rapid change in radial velocity around perigee as the spacecraft transitions from moving towards Earth to moving away. Diurnal variations due to Earth's rotation are also clearly visible.

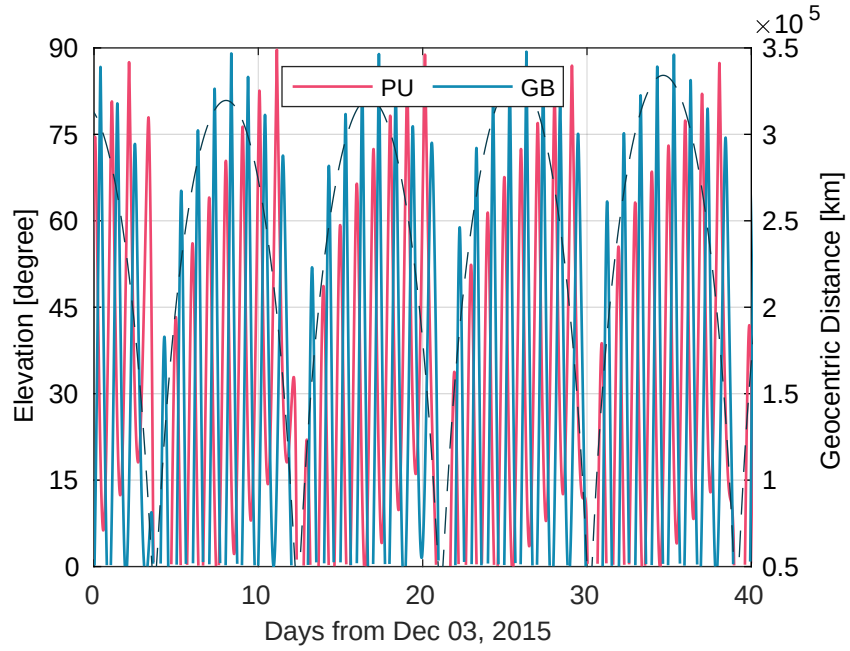


(a) Declination

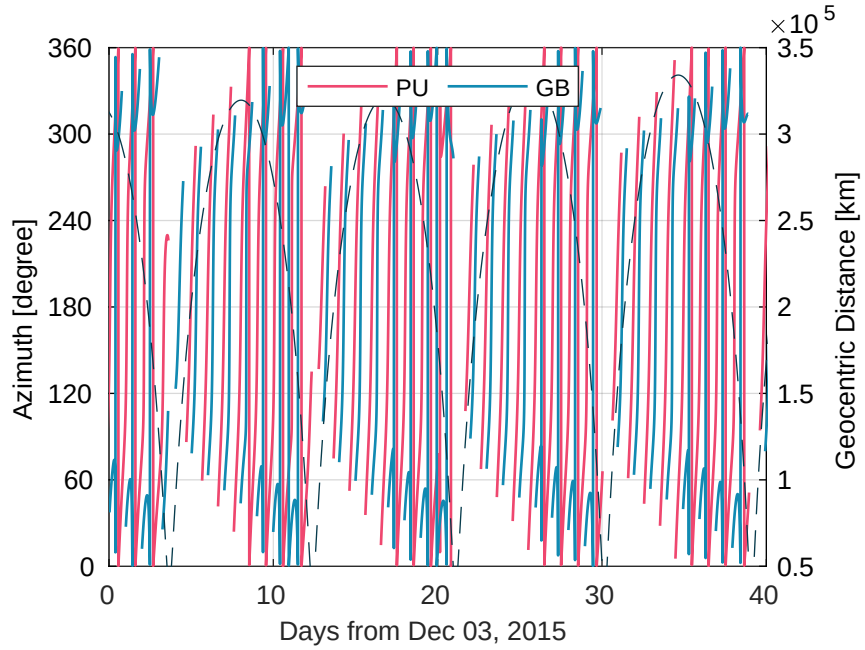


(b) Declination Rate of Change

Figure 3.8: Spacecraft declination within GCRF (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. Note the rapid change in declination near perigee when the spacecraft is moving the fastest.



(a) Elevation



(b) Azimuth

Figure 3.9: Spacecraft elevation and azimuth at each ground station (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. Angles are only plotted when the spacecraft is above the horizon. Trends due to the spacecraft's motion as well as Earth's rotation are clearly present.

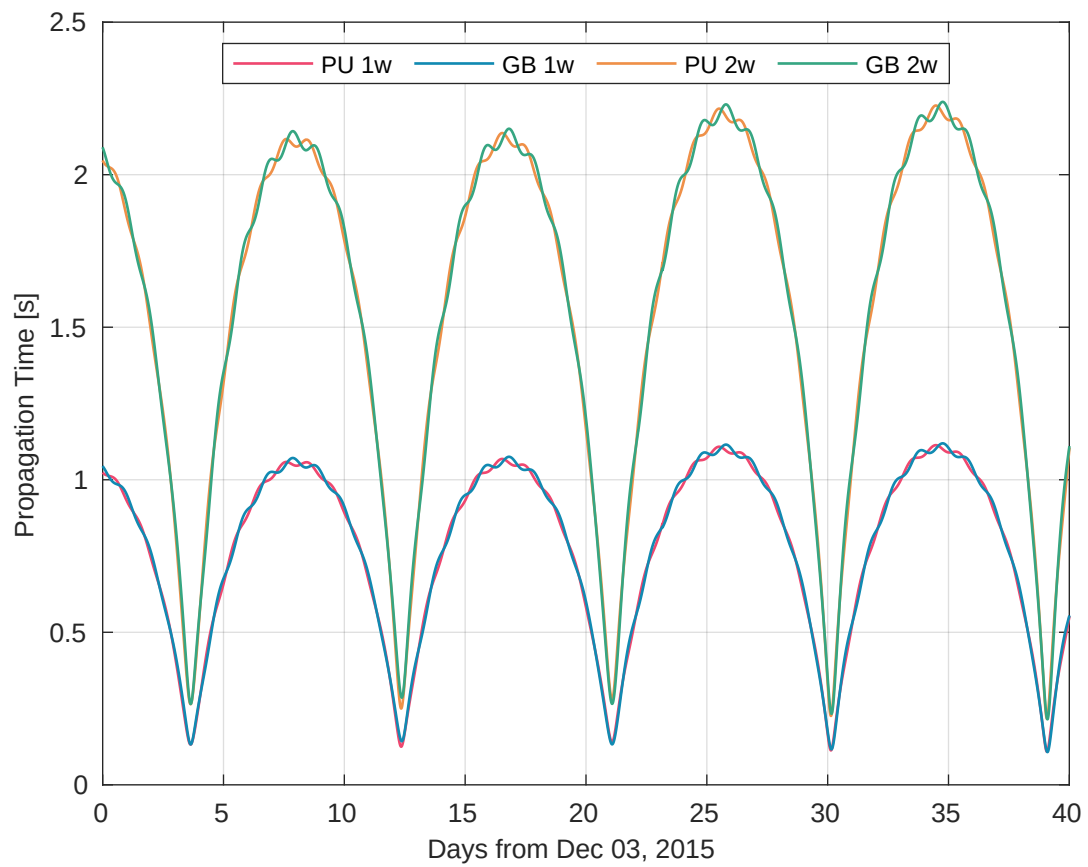


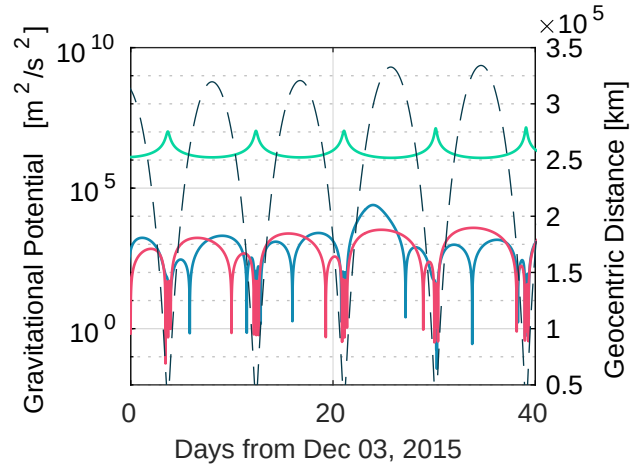
Figure 3.10: One way and two way propagation times which closely follow the trends in the radial distance seen in Figure 3.7a.

3.6 Gravitational Potential and Redshift

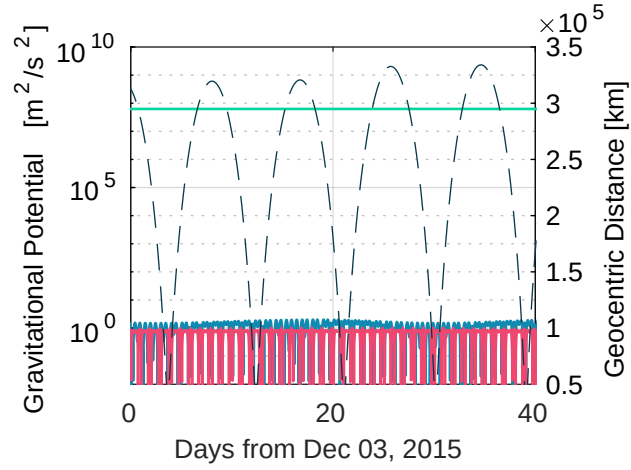
The Earth’s gravitational potential at the position of the spacecraft and each ground station was computed using TCL routines that implement Eq. (2.28) and make use of tide-free coefficients from NGA’s EGM2008 gravitational model up to spherical harmonic degree 20. The coefficients are adjusted for post-glacial rebound, polar motion, solid earth tides, oceanic tides and pole tides although these effects are very small and likely unnecessary for the RadioAstron gravity experiment. The calculated gravitational potentials are plotted in Figure 3.11 in green. For the ground stations, the gravitational potential is effectively constant during the sample interval. In contrast, for the spacecraft the potential varies along the orbit mostly due to the dominant monopole term proportional to $1/r$.

The tidal gravitational potentials of the Moon and Sun are also plotted in Figure 3.11 in blue and red respectively. These are computed using Eq. (2.33) with the position vectors of the spacecraft or ground station and those of the Moon and Sun. The tidal potentials for the ground stations are very small but for the spacecraft achieve more significant values, especially when the spacecraft is far from Earth. Note the maximum in the spacecraft tidal potential due to the Moon in Figure 3.11a when the spacecraft makes its closest lunar approach on day 24.

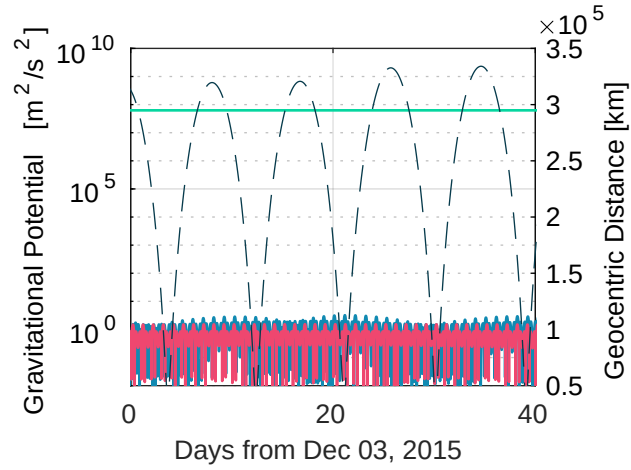
The gravitational redshift caused by the different gravitational potentials at the spacecraft and ground stations is calculated using Eq. (1.1). The redshift due to the Earth is plotted in Figure 3.12 and is the expected value for the gravitational redshift assuming no violation. The tidal redshifts due to the Moon and Sun are plotted in Figure 3.13. As the tidal potentials at the ground stations are small, the tidal redshifts very closely follow the change in the tidal potentials at the spacecraft.



(a) Spacecraft



(b) Pushchino



(c) Green Bank

Figure 3.11: Gravitational potentials (solid lines) at the spacecraft and ground station locations due to the Earth (green), Moon (blue) and Sun (red). Spacecraft geocentric distance (dashed lines) is plotted against right axes.

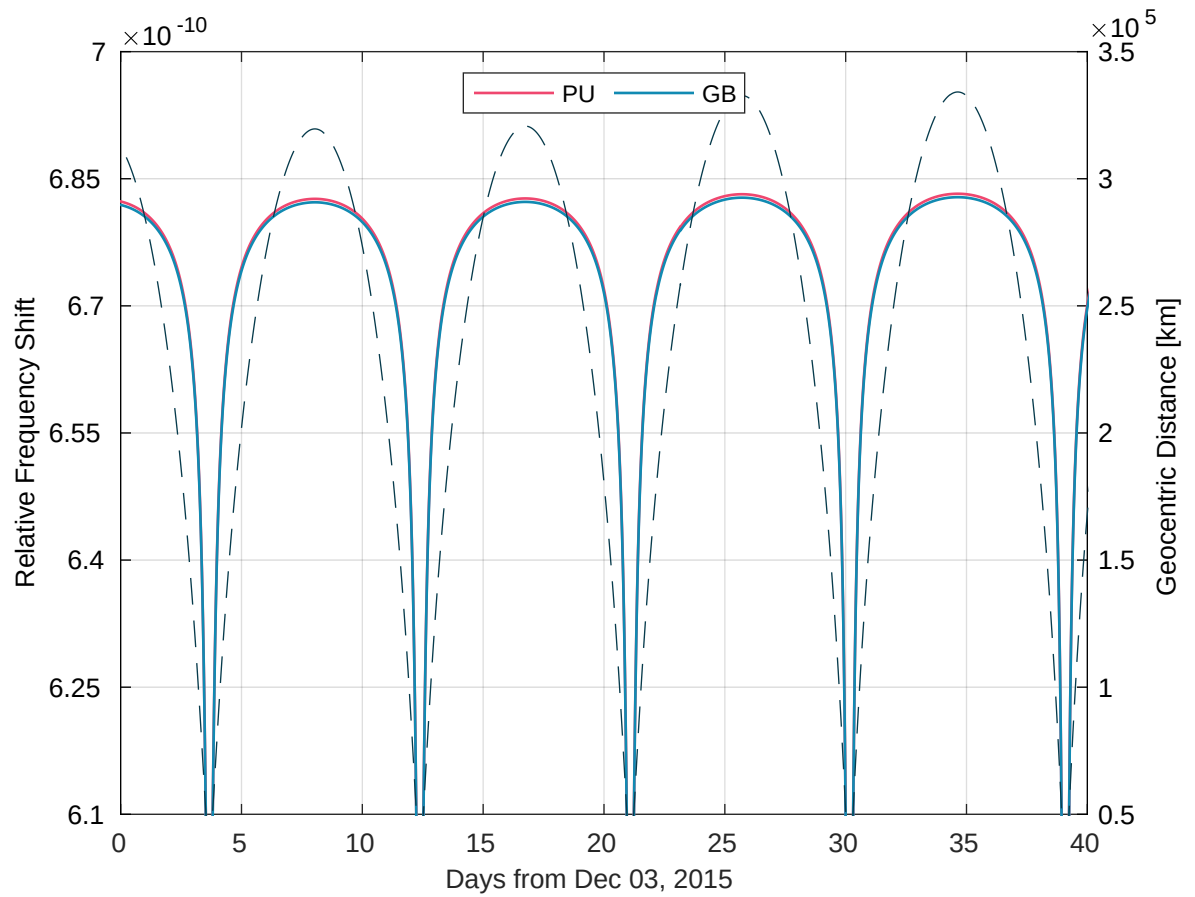


Figure 3.12: Frequency shift caused by the Gravitational Redshift (solid line) that reaches a maximum near apogee when the difference between Earth's gravitational potential at the locations of the spacecraft and ground station is greatest. Spacecraft geocentric distance (dashed line) is plotted against right axis.

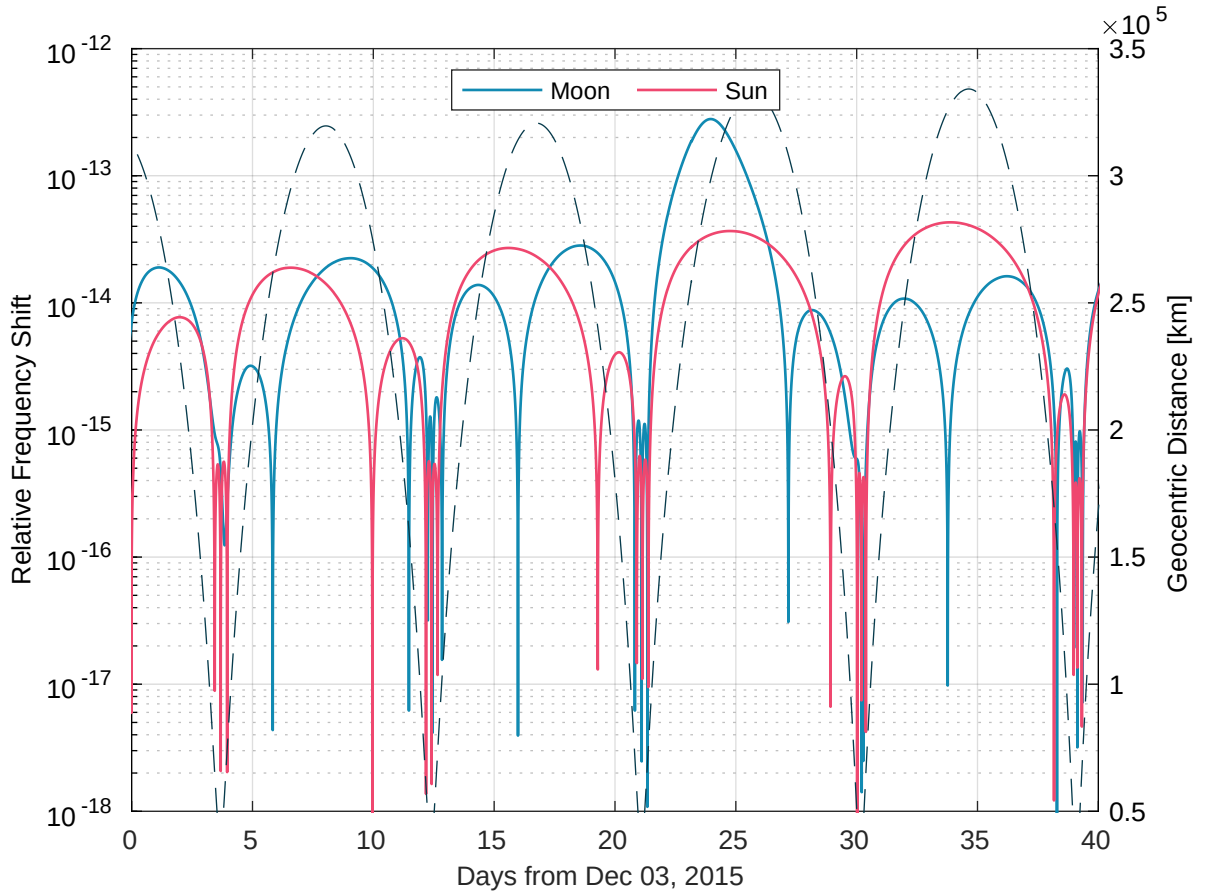


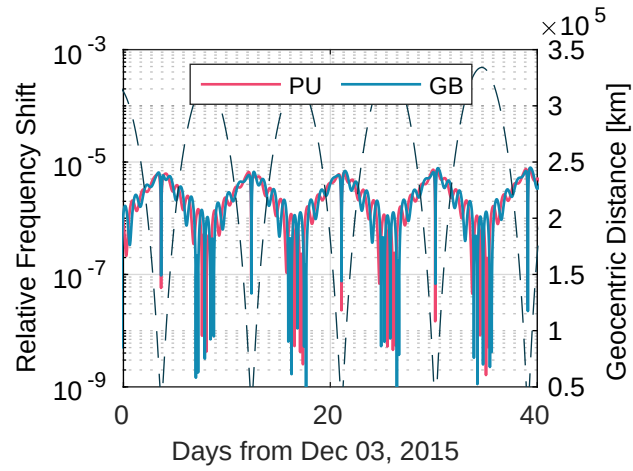
Figure 3.13: Frequency shift caused by the Gravitational Redshift due to the Moon and Sun (solid lines). Spacecraft geocentric distance (dashed line) is plotted against right axis. Note the increase in the Moon's tidal effect near day 24 when the spacecraft makes its closest approach to the Moon within the interval as seen in Figure 3.6c.

3.7 Relativistic Doppler Shift

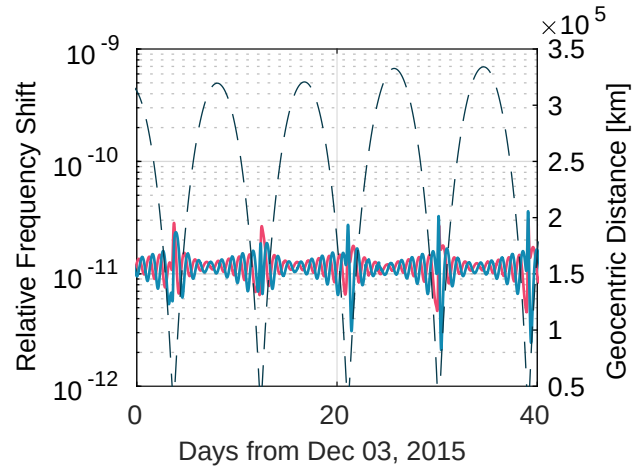
The relativistic Doppler shift is the dominant effect in the 1way and 2way mode frequency shifts. The effect in 1way mode is computed using Eq. (2.164) and plotted in Figure 3.14. The effect has been divided between three plots in order to show how the first order effect is typically 10^5 times larger than the rest of the effect. This highlights the importance of cancelling the first order effect when attempting to measure the gravitational redshift that is three orders of magnitude smaller. The relative frequency shift in the 2way mode from Eq. (2.188) is similarly plotted in Figure 3.15. Notice that the first order effect is simply a factor of two larger as compared to 1way while second order effects are often a factor of 10 larger.

As shown earlier, the non-relativistic Doppler effect is absent in the relative frequency shift after applying the Doppler compensation scheme from Eq. (2.195). The remaining frequency shifts from Eq. (2.196) are plotted in Figure 3.16. Notice that the remaining shift is dominated by the second order frequency shift from the 2way mode. However, the effect is now on the same order of magnitude as the gravitational redshift which greatly increases the accuracy of the experiment. If only the 1way mode were used, accuracy would be limited by the uncertainty in orbit determination limiting how accurately the first order term can be calculated.

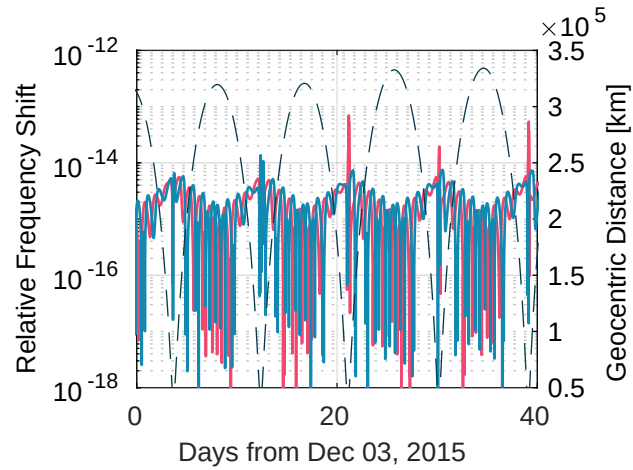
In Section 2.10.5, the third order terms from Eq. (2.196) were separated into significant terms appearing in $\Delta y_{dop\ 3rd}$ from Eq. (2.200) and insignificant terms in Δy_{small} from Eq. (2.201). Figure 3.17 plots each term over a single orbit between December 16 and 24, 2015. During this orbit, we can see the terms in Δy_{small} do not exceed 5×10^{-16} which provides an initial justification for the distinction between significant and insignificant terms. This will be studied further in the next chapter.



(a) First Order

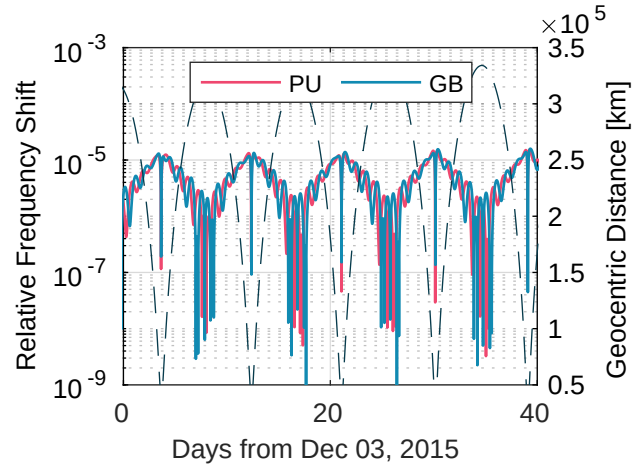


(b) Second Order

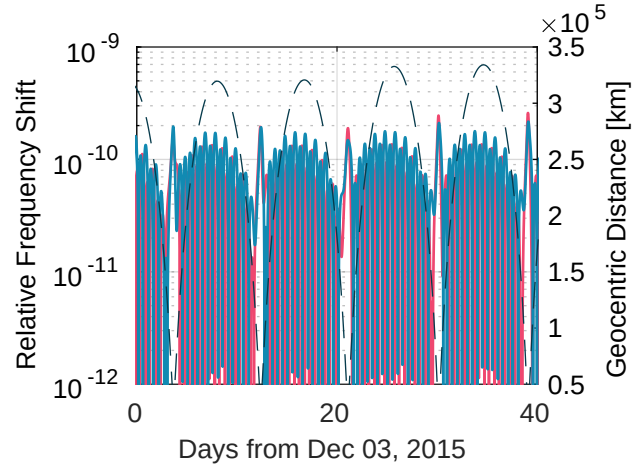


(c) Third Order

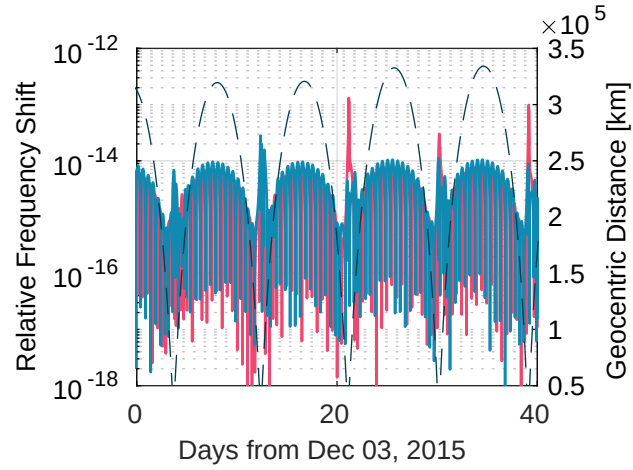
Figure 3.14: Frequency shifts due to the Doppler effect in 1way mode (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. Note the middle plot excludes shifts from Figures 3.12 and 3.13.



(a) First Order

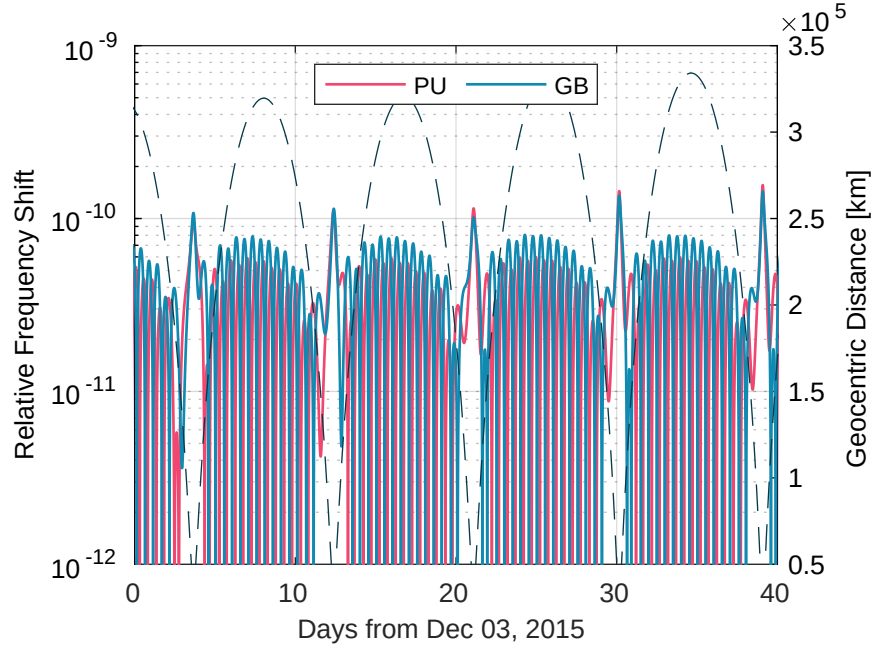


(b) Second Order

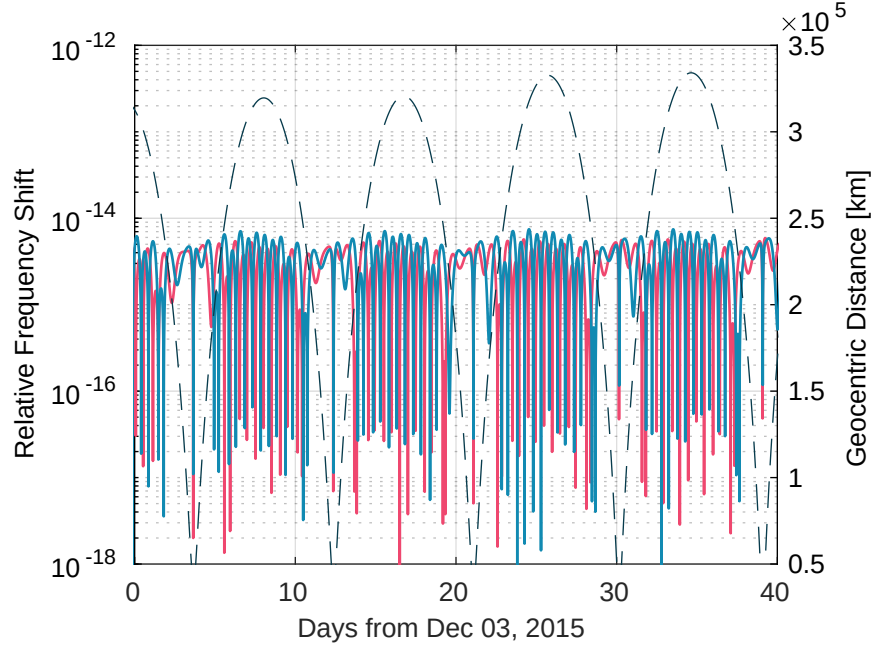


(c) Third Order

Figure 3.15: Frequency shifts due to the Doppler effect in 2way mode (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes.

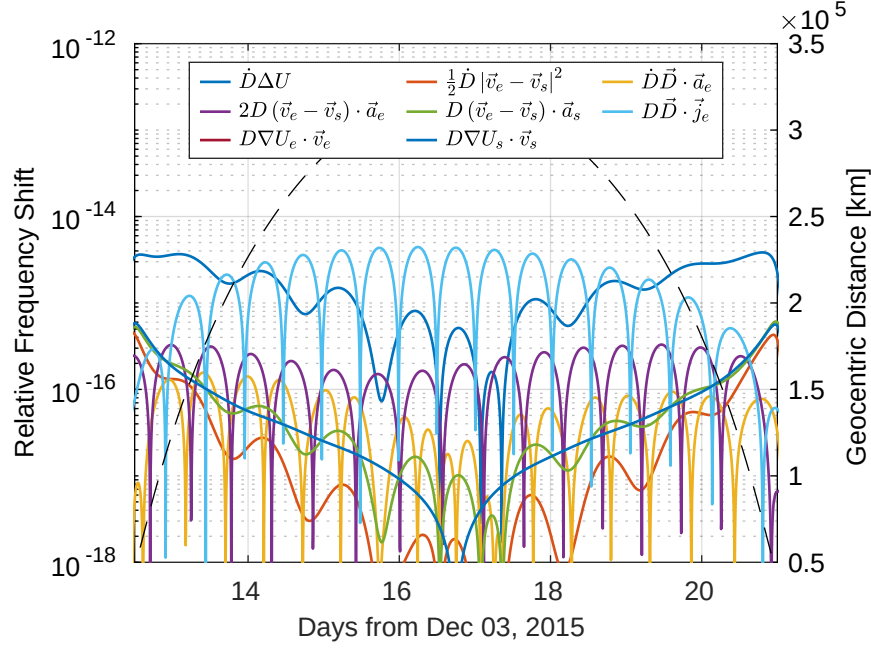


(a) Second Order

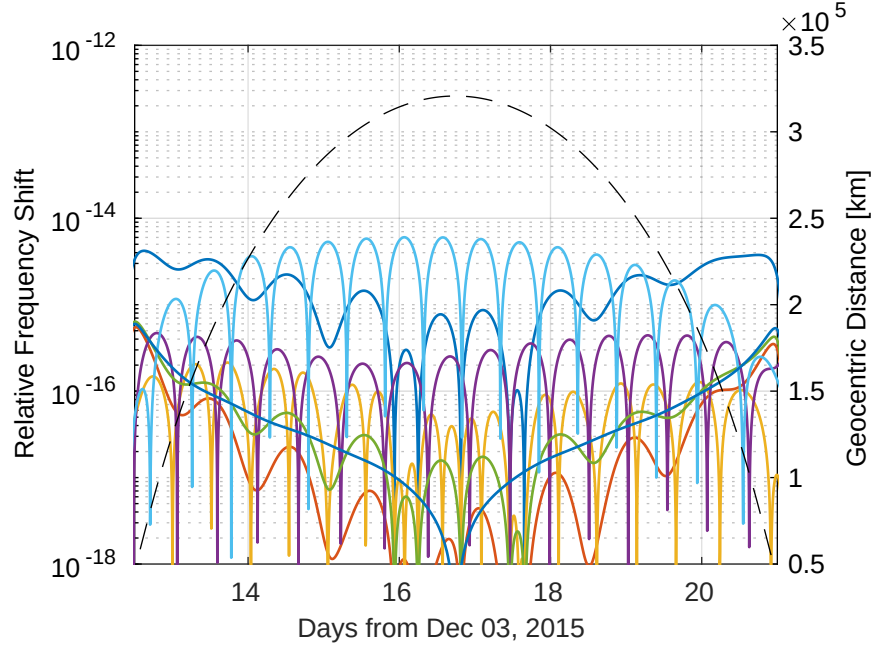


(b) Third Order

Figure 3.16: Frequency shifts due to the Doppler effect after Doppler compensation is applied (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. At first order, the Doppler effect is completely cancelled leaving only second order or smaller effects. Note the top plot excludes shifts from Figures 3.12 and 3.13 to reveal the smaller kinematic effect.



(a) Pushchino



(b) Green Bank

Figure 3.17: Third order terms in the frequency shift due to the Doppler effect after Doppler compensation is applied (solid lines). The sum of these terms is equivalent to Figure 3.16b. Spacecraft geocentric distance (dashed lines) is plotted against right axes. Only a single orbit between December 16 to 24, 2015 is shown to highlight the various effects. Note each term has a factor of c^{-3} not included in the legend in (a).

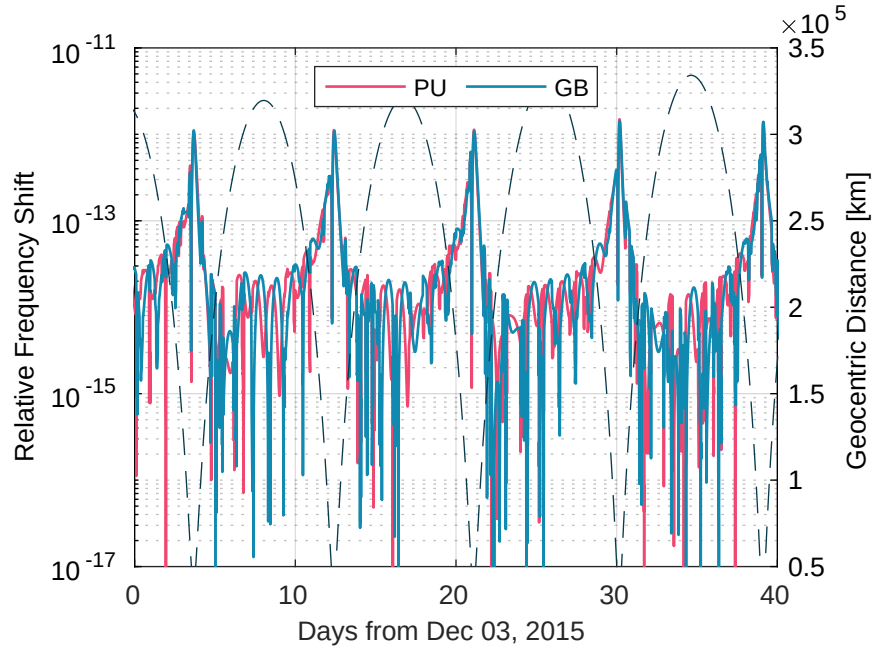
3.8 Antenna Phase-Centre Motion

The PCM effect described in Section 2.11 arose from two separate sources: the spacecraft’s antenna and the ground station’s antenna. The latter effect is not present at the Pushchino ground station as its Alt-Az design does not have an offset between its axes like at the Green Bank ground station. In contrast, the spacecraft effect is present when observing with either of the ground stations.

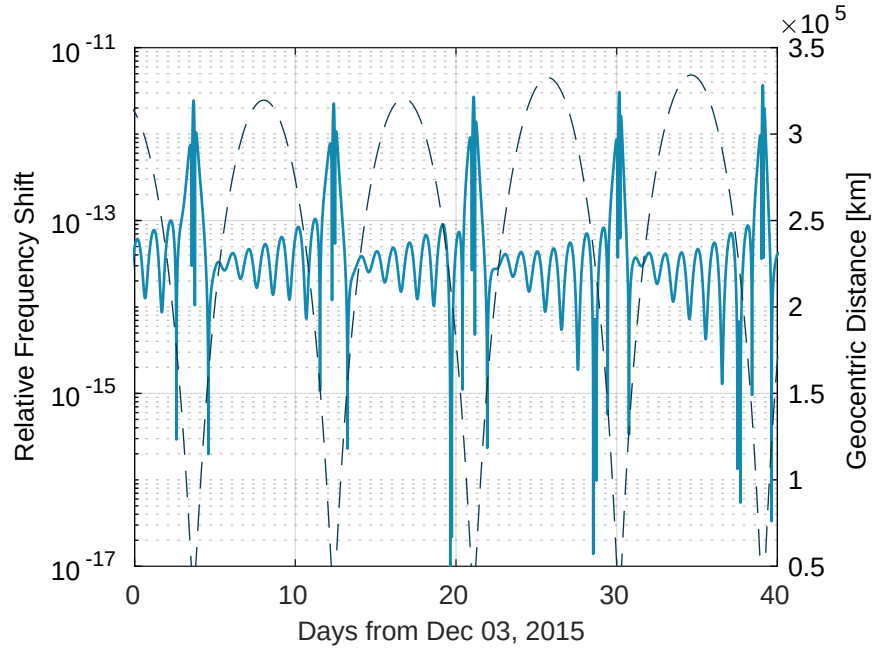
To compute the spacecraft PCM effect, the orientation of the spacecraft within the GCRF is needed. This was recorded by onboard star trackers on RadioAstron and included in downlinked telemetry. Attitude as a function of time is provided by KAM in 1 m increments in the form of four component quaternions representing a 3D rotation. Linear interpolation is used to calculate the components of the quaternion representing the appropriate rotation at any given point in time. This rotation is then applied to the spacecraft antenna offset from Eq. (2.202) to rotate it from the spacecraft fixed-frame in which it’s components are defined into the J2000 frame. The rotated vector is then rotated again from the J2000 frame to the GCRF and used to compute the spacecraft’s PCM effect.

The PCM effect in the 1way mode from Eq. (2.215) is plotted in Figure 3.18. Notice that at times the effect exceeds 10^{-12} making this a significant effect. This is the case for both the spacecraft and Green Bank. The effect in the 2way mode from Eq. (2.221) is similar and is plotted in Figure 3.19.

Strikingly, the effect is reduced by over four orders of magnitude after applying the Doppler compensation scheme. This can be seen in Figure 3.20 which plots the computed values of Eq. (2.223). Thus, PCM is not significant in the sample interval when measuring the gravitational redshift using the Doppler compensation scheme. In the next chapter, we’ll revisit this in the context of the specific sessions in which the gravity experiment will be performed with RadioAstron.

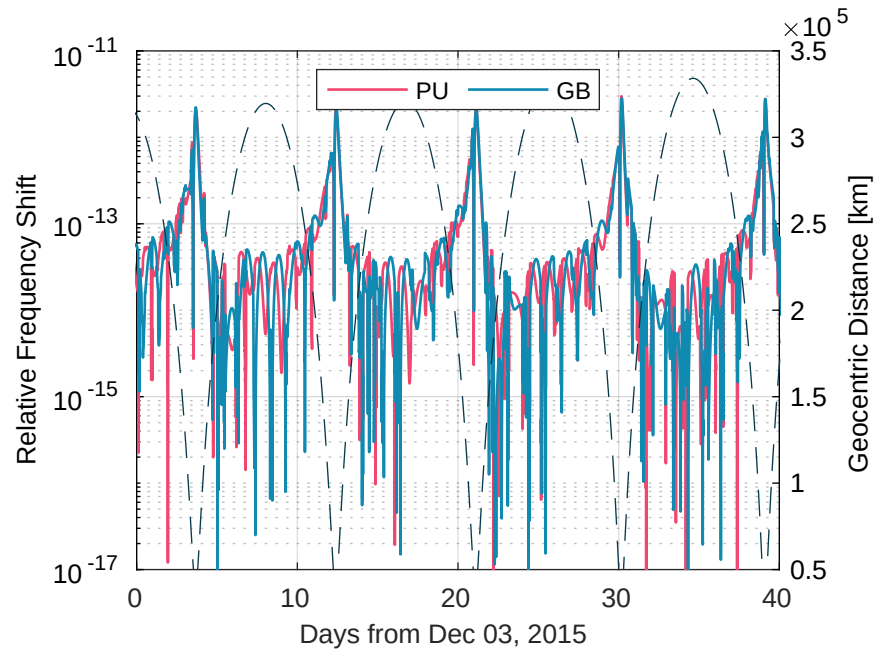


(a) Spacecraft

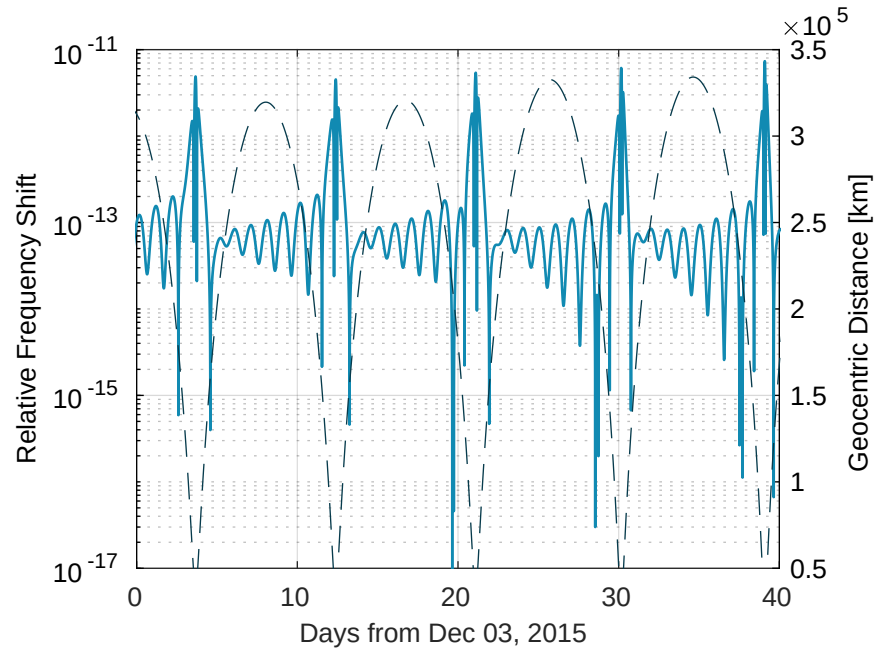


(b) Green Bank

Figure 3.18: Frequency shifts due to PCM in 1way mode (solid lines) arising from the spacecraft and ground station antennas. Spacecraft geocentric distance (dashed lines) is plotted against right axes. The Pushchino antenna does not have an antenna offset to give rise to a PCM effect and is thus absent from the lower plot.

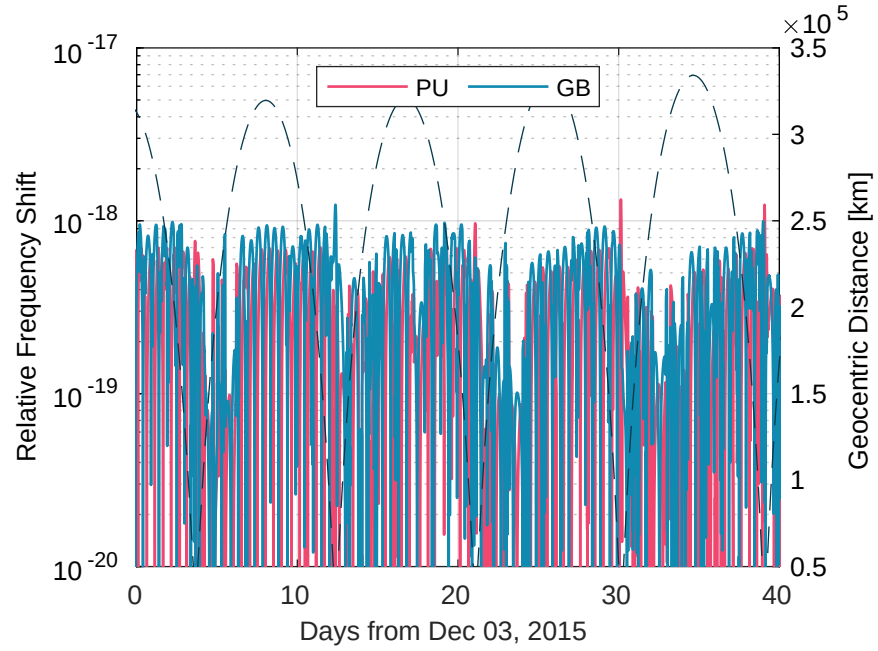


(a) Spacecraft

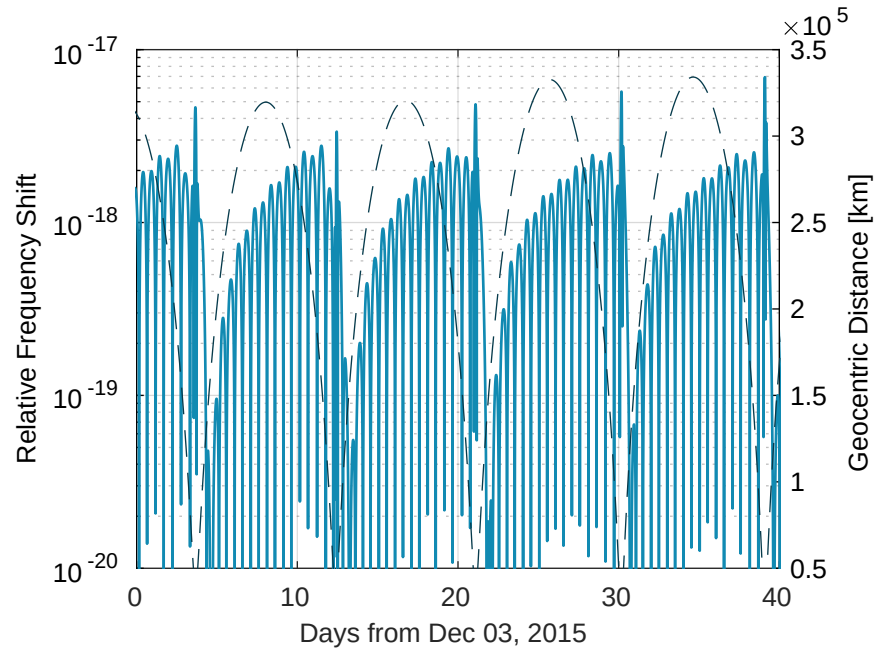


(b) Green Bank

Figure 3.19: Frequency shift due to PCM in 2way mode (solid lines) arising from the spacecraft and ground station antennas. Spacecraft geocentric distance (dashed lines) is plotted against right axes. The Pushchino antenna does not have an antenna offset to give rise to a PCM effect and is thus absent from the lower plot.



(a) Spacecraft



(b) Green Bank

Figure 3.20: Frequency shift due to PCM after Doppler compensation is applied (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. The Pushchino antenna does not have an antenna offset to give rise to a PCM effect and is thus absent from the lower plot.

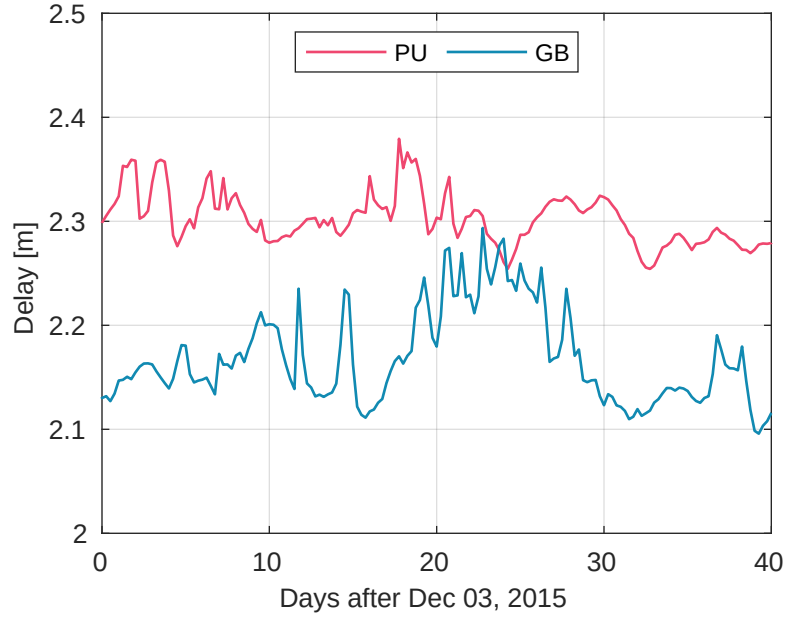
3.9 Tropospheric Refraction

The tropospheric delay is calculated using the VMF3 and GRAD models (Landskron & Böhm 2018a,b) based on grid data from the Technical University of Wien in Austria (TUWien). A Matlab script provided by Landskron & Böhm is used to read the files and perform the necessary spatial and temporal interpolations. This script also implements the mapping function from Eqs. (2.227) and (2.230). The zenith tropospheric delay calculated in this way is plotted in Figure 3.21a. As this does not include the effect of elevation, we clearly see the effect of intra-day variations in the troposphere. In particular, variations of up to 10% can occur over the span of a few days due to the volatile nature of the wet component of the atmosphere. The overall delay is greater at Pushchino because of differences in temperature and pressure due to its higher latitude.

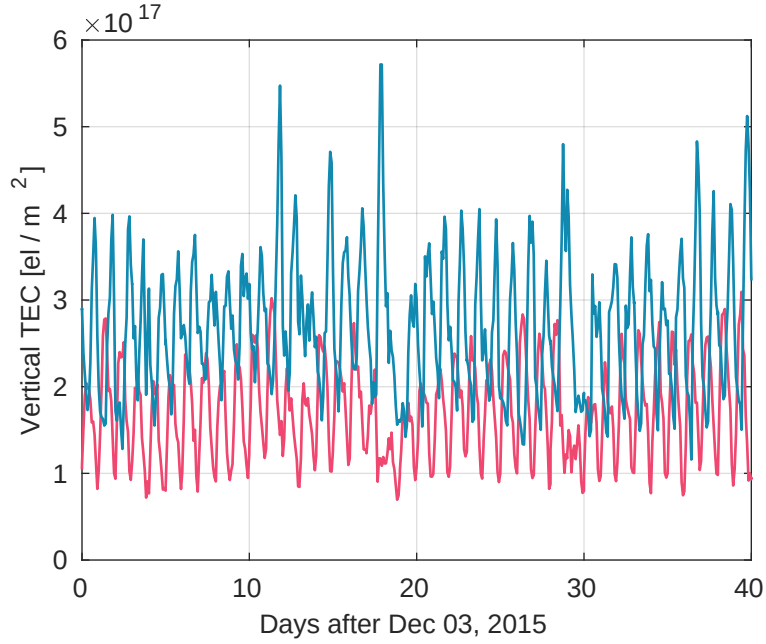
The tropospheric frequency shift in the 1way mode is plotted in Figure 3.22a using the calculated tropospheric delays and Eq. (2.232). Similarly, the 2way mode frequency shift is plotted in Figure 3.23a using Eq. (2.235). The strong diurnal effect is due to the change in elevation seen in Figure 3.9a. After applying the Doppler compensation scheme from Eq. (2.236), the residual tropospheric shift becomes insignificant more than half the time as seen in Figure 3.24a. Thus, the effect cannot be neglected without first confirming that the change of elevation in a given session is sufficiently small to render the residual effect after applying the Doppler compensation scheme insignificant.

3.10 Ionospheric Refraction

The ionospheric delay depends on the electron number density in the ionosphere. This is calculated using VTEC data from CDDIS (Noll 2010) measured using GNSS measurements fit to a global model. The gridded data are downloaded in IONEX file format (Schaer et al.

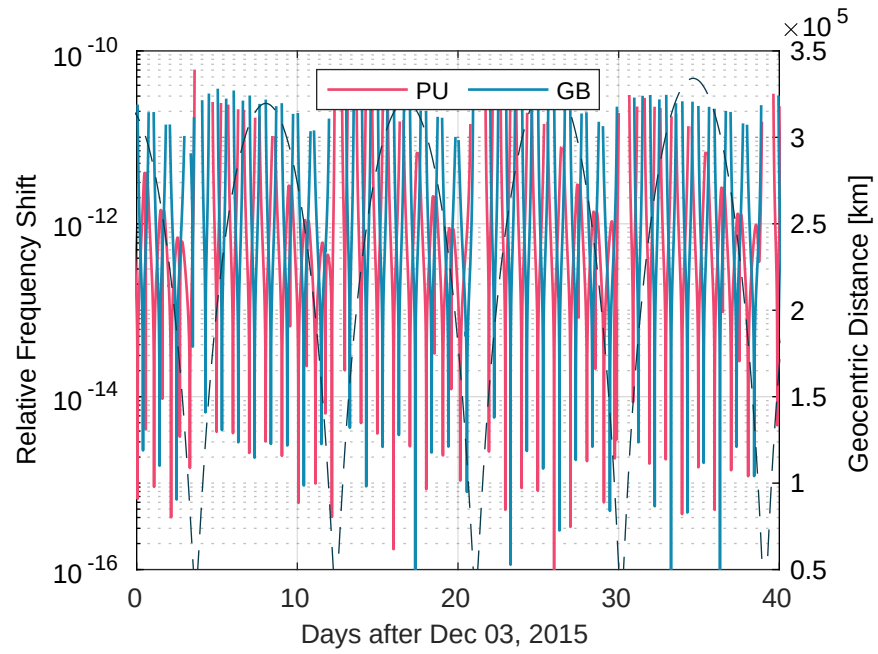


(a) Tropospheric Zenith Delay

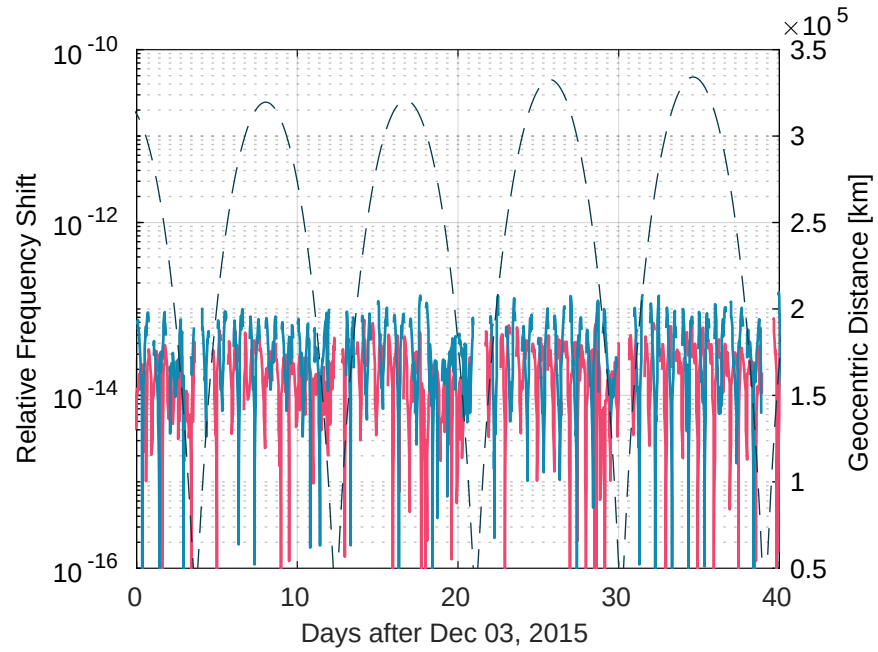


(b) Ionospheric VTEC

Figure 3.21: Key atmospheric parameters estimated during selected orbits. Tropospheric zenith delay, equivalent to an additional path length through the troposphere, is estimated using the VMF3 discrete atmospheric model which incorporates the effect of daily weather. Ionospheric VTEC estimated from GNSS with diurnal variations in the ionosphere accounting for the periodicity.

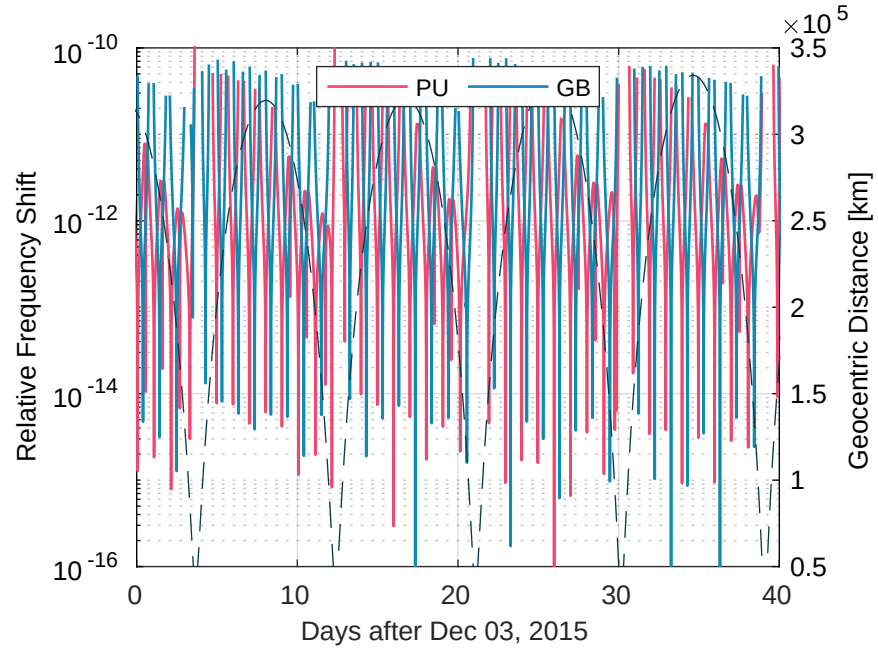


(a) Tropospheric Shift

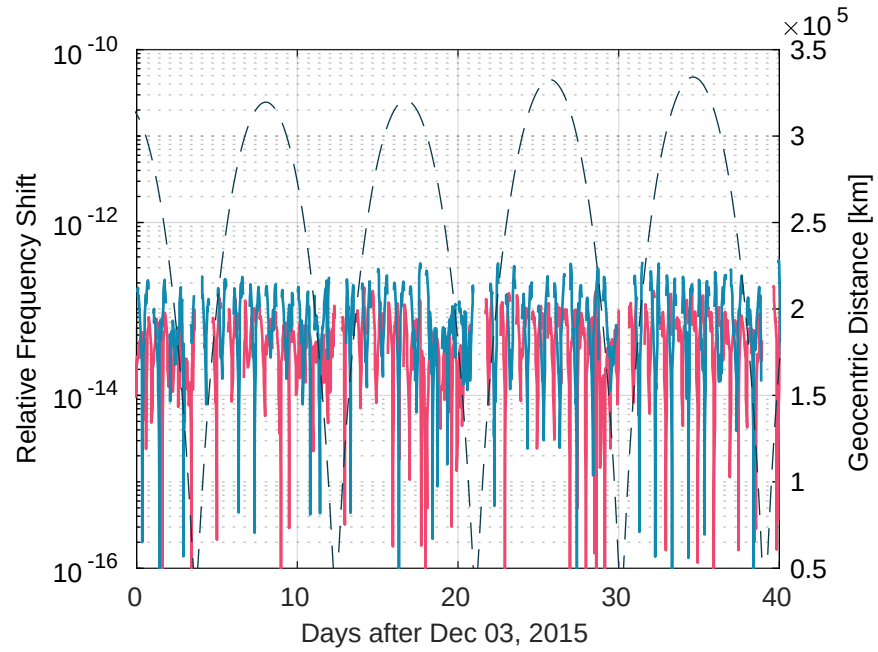


(b) Ionospheric Shift

Figure 3.22: Frequency shifts due to the troposphere and ionosphere when the spacecraft is above the local horizon in 1way mode (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. Note that (b) is given for the 8.4 GHz band. The ionospheric shift at 15 GHz is of a similar form though 69% smaller.

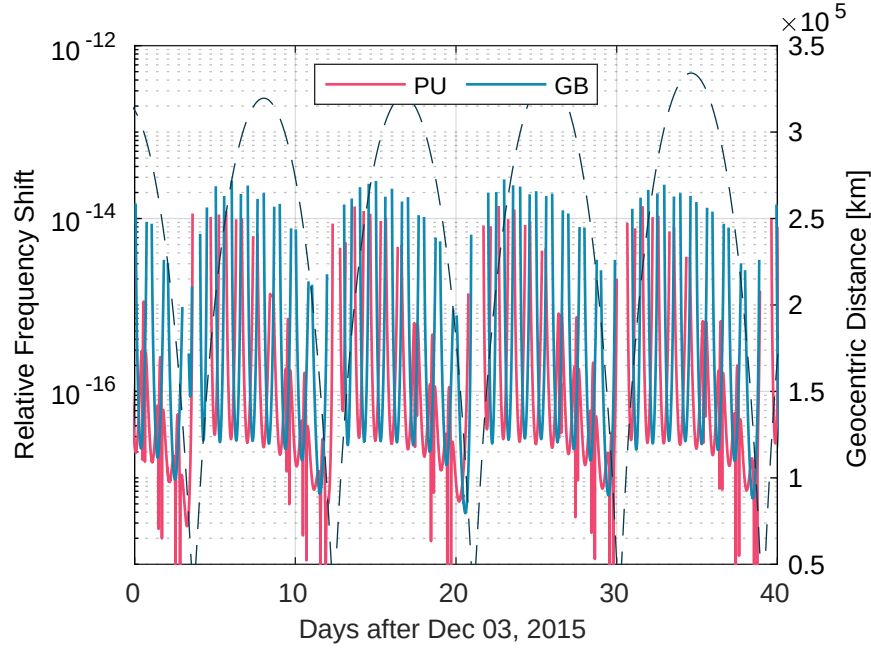


(a) Tropospheric Shift

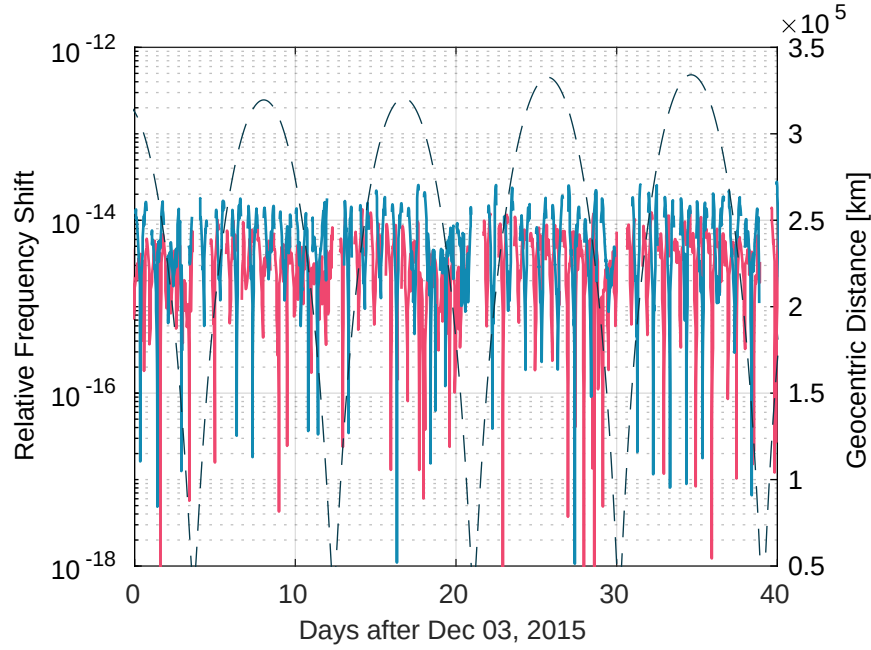


(b) Ionospheric Shift

Figure 3.23: Frequency shifts due to the troposphere and ionosphere when the spacecraft is above the local horizon in 2way mode (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. Note that (b) is given for the 8.4 GHz band. The ionospheric shift at 15 GHz is of a similar form though 29% smaller.



(a) Tropospheric Shift



(b) Ionospheric Shift

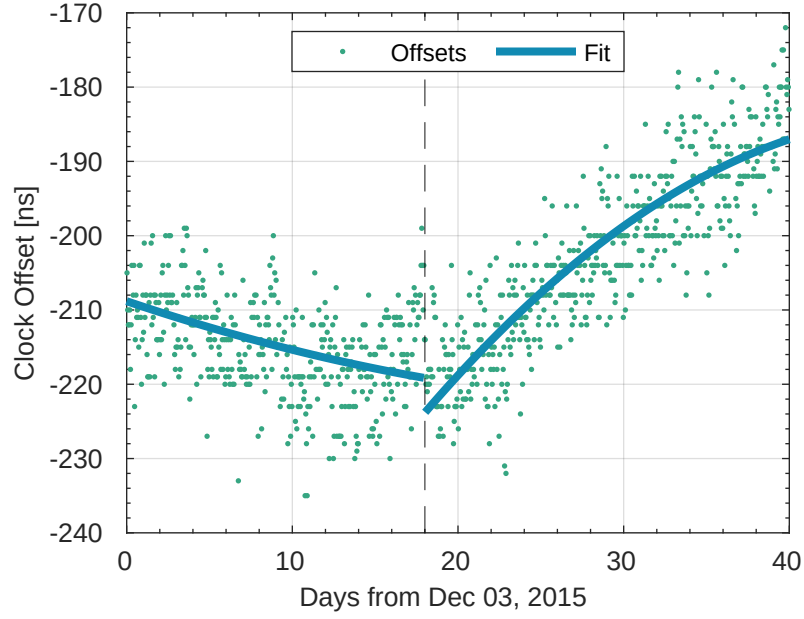
Figure 3.24: Frequency shifts due to the troposphere and ionosphere when the spacecraft is above the local horizon after Doppler compensation is applied (solid lines). Spacecraft geocentric distance (dashed lines) is plotted against right axes. Note that (b) is given for the 8.4 GHz band. The ionospheric shift at 15 GHz is of a similar form but larger by a factor of ~ 3 due to the larger difference between uplink and downlink frequencies.

1998) and interpolated spatially and temporally in a manner similar to the tropospheric delay data. This is implemented in a Matlab script that also calculates STEC using the mapping function from Eq. (2.247). VTEC data are plotted in Figure 3.21b showing the dominant diurnal variations in the electron content of the ionosphere caused by dissociation during daylight hours. During the northern hemisphere’s winter, the angle at which the Sun’s rays penetrate the ionosphere increases with latitude. Therefore, solar flux will be higher at lower latitudes and correspondingly the electron content will be higher. This explains why VTEC is larger over the Green Bank ground station ($\phi = 38.4^\circ$) than over Pushchino ($\phi = 54.8^\circ$).

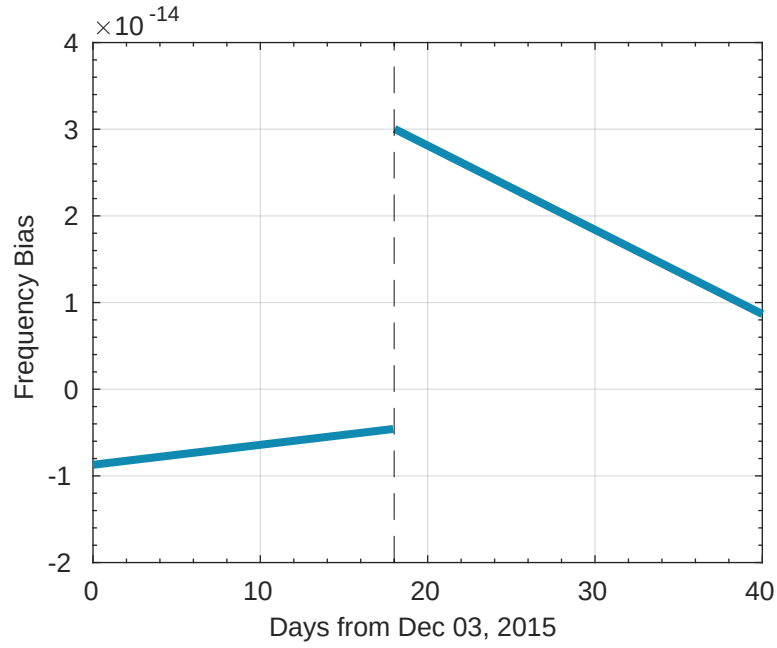
Unlike with the troposphere, the ionospheric frequency shift is nearly always significant even after applying the Doppler compensation scheme using Eq. (2.259). The 1way mode frequency shift using Eq. (2.243) is plotted in Figure 3.22b while the 2way mode using Eq. (2.258) is plotted in Figure 3.23b. Both reach the 10^{-13} level though most of the time are on the order of 10^{-14} . The frequency shift after apply the Doppler compensation scheme, plotted in Figure 3.24b, is reduced but remains on the order of 10^{-14} and thus the effect of the ionosphere cannot be neglected.

3.11 Frequency Bias

Each ground station provides a time series of clock offsets (ΔT) between the local H-maser and GPS time. These are plotted in Figure 3.25a for the Green Bank ground station and in Figure 3.26a for Pushchino. As the change in the offset is smaller for Green Bank, it is easier to see the scatter in the measurements caused by the lower short-term stability of the clocks onboard the GPS satellites. However, in the long-term, the GPS clocks are steered to follow TT and thus have very good long-term stability. Thus, the overall trend in the offset is due to the drift of the local H-maser. To first order, an H-maser that is not periodically adjusted will drift linearly in frequency which corresponds to a quadratic trend in offset. Typically this

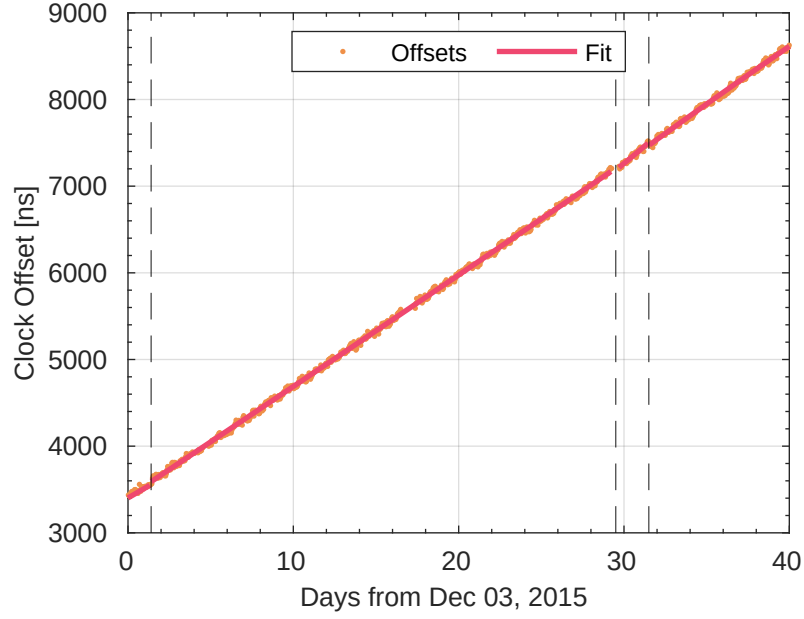


(a) Clock Offset

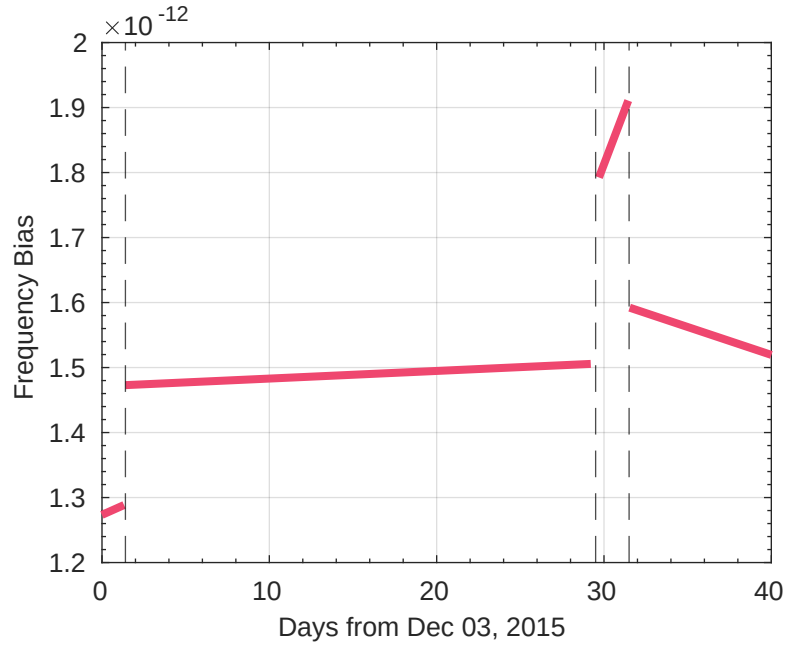


(b) Frequency Bias

Figure 3.25: Green Bank H-maser offset (a) measured relative to TT as realized by GPS time. The frequency bias relative to TT (b) is the rate of change of the clock offset. Dashed lines indicate where jumps in the offset and frequency bias occurred.



(a) Clock Offset



(b) Frequency Bias

Figure 3.26: Pushchino H-maser offset (a) measured relative to TT as realized by GPS time. The frequency bias relative to TT (b) is the rate of change of the clock offset. Dashed lines indicate where jumps in the offset and frequency bias occurred.

takes place over only limited intervals as an H-maser on the ground can easily be adjusted to keep its drifts within a reasonable range. For example, in Figure 3.25a, we can see that the H-maser at Green Bank was adjusted to bring the offset back closer to zero.

To measure the station frequency bias relative to TT ($d\Delta T/dTT$), we first divide the offset time series from each station into a series of intervals in which a quadratic fit is done to the clock offset data. The derivative of these fits are a measure of the frequency bias in each interval. This is plotted in Figure 3.25b for Green Bank and in Figure 3.26b for Pushchino. The vertical dashed lines delineate the separate intervals in which the fits were done. We can see that during the sample interval, the H-maser at Pushchino had a very significant frequency bias relative to TT.

The frequency bias relative to TCG (h_e), which we need when measuring the gravitational redshift, is related to the frequency bias relative to TT by Eq. (2.72). This equation uses the station's height above or below the geoid to account for the difference in proper time at the station versus on the geoid. The frequency biases of the two stations relative to TCG are plotted in Figure 3.27. Notice that the difference between the bias relative to TT and TCG is very small but still significant as it is on the order of 10^{-15} .

Unfortunately, clock offset data are not available for RadioAstron so a different method must be used to measure the SHM's clock offset (h_s). The approach proposed in Litvinov et al. (2018) is to use the Doppler compensation scheme to measure the residual frequency shift after all other effects have been accounted for in sessions at approximately the same distance from the Earth. The gravitational redshift in these sessions will be approximately the same and so a violation of the form of Eq. (1.2) will have the same magnitude. Thus, any variation in the residual frequency shift from session to session will be due to a change in the frequency bias of the SHM. This measurement is beyond the scope of this project, but it represents a critical step in performing a gravity experiment with RadioAstron.

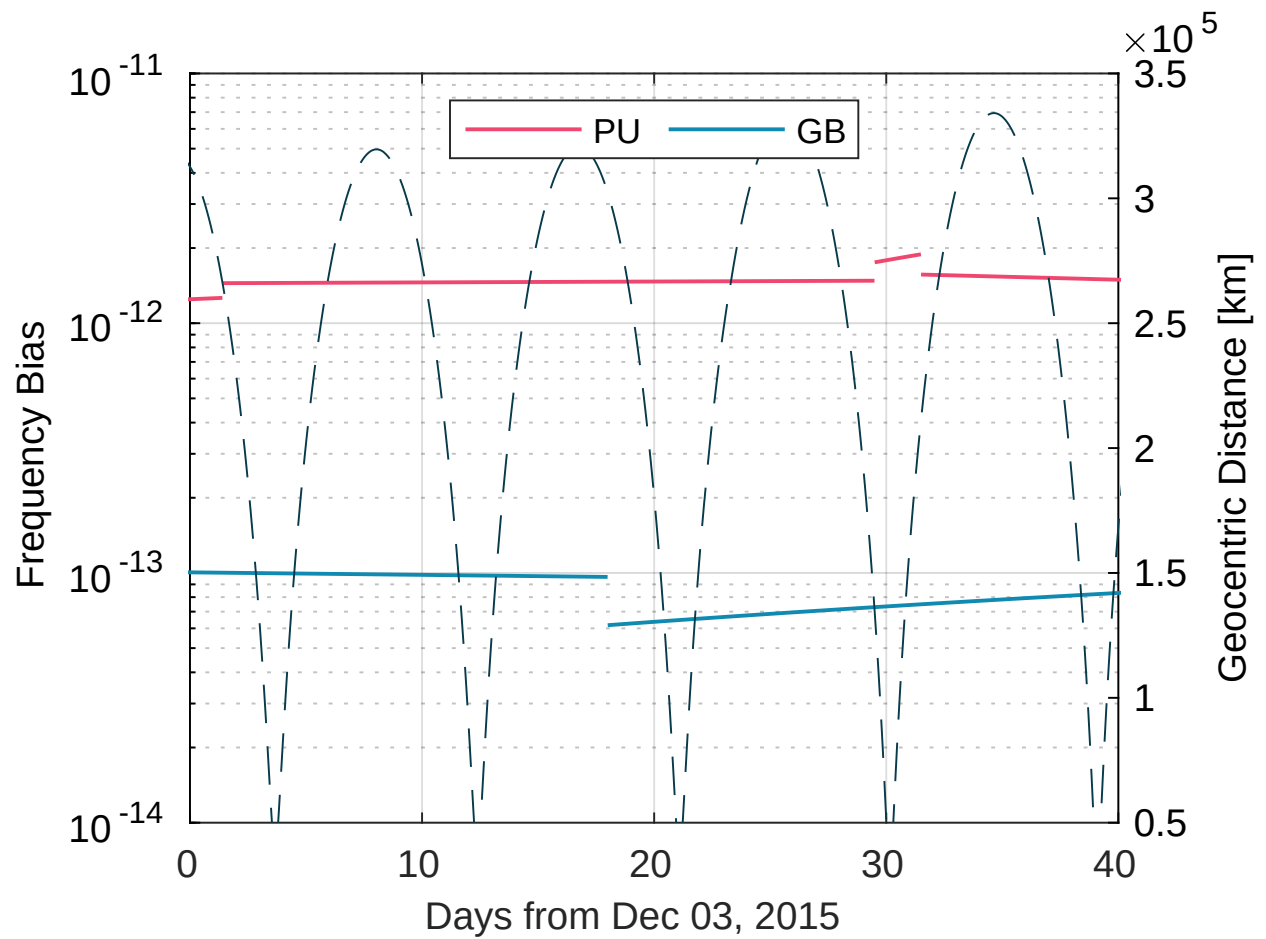
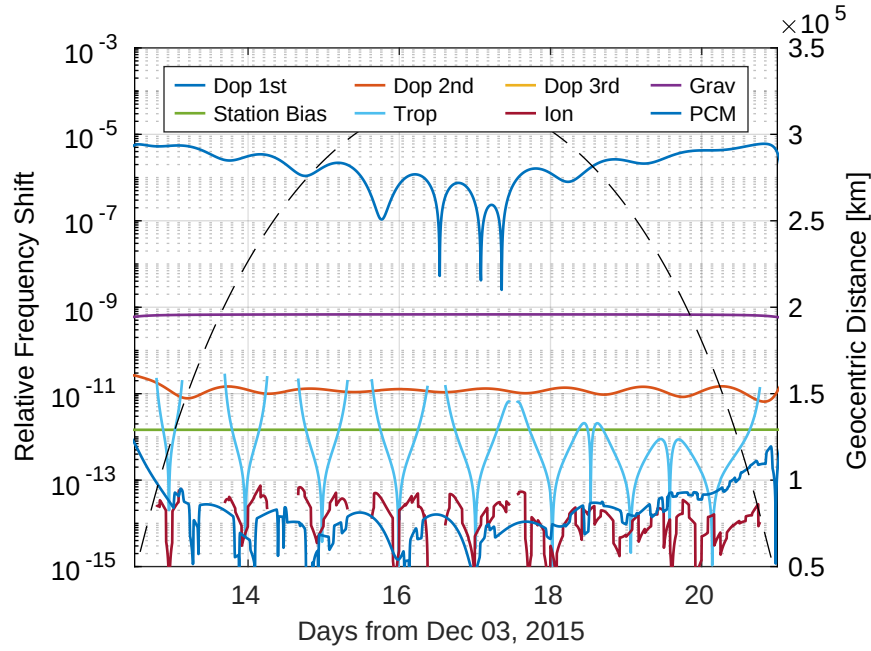


Figure 3.27: Frequency bias of the tracking station H-masers relative to TCG (solid lines) calculated from their clock offsets relative to GPS time and their station heights relative to the geoid. Spacecraft geocentric distance (dashed line) is plotted against right axis.

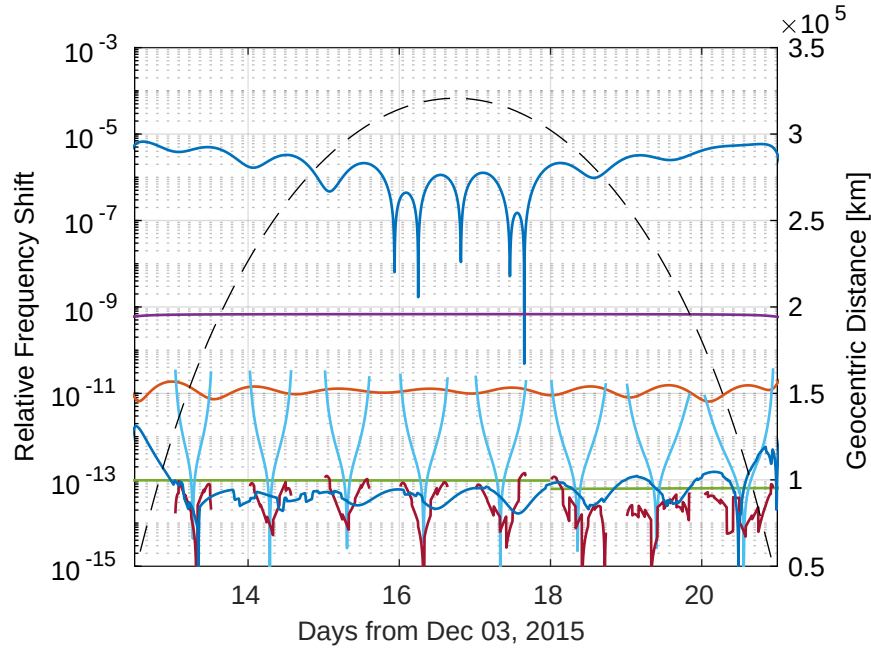
3.12 Overall Frequency Shift

Having now computed all the frequency shifts in the dynamical model, we can now summarize the components of the overall frequency shift observed in each mode. The terms in the 1way mode total shift from Eq. (2.261) is plotted in Figure 3.28 for a single orbit between December 16 to 24, 2015. As expected, most of the time, the non-relativistic Doppler shift is many orders of magnitude larger than the gravitational redshift. Also note that all of the effects, except for the third order Doppler shift, are above the 10^{-15} threshold and therefore significant. The various terms making up the overall frequency shift in the 2way mode from Eq. (2.262) over the same orbit are plotted in Figure 3.29. Note how the third order Doppler shift now becomes significant.

The impact of applying the Doppler compensation scheme from Eq. (2.195) is very clear in Figure 3.30 in which the terms from Eq. (2.263) are plotted. As the non-relativistic Doppler shift has been cancelled, the largest remaining effect is the gravitational redshift. Note how the effects due to PCM and the troposphere have also been reduced and are no longer significant. We will find in the next chapter that the tropospheric shift is not always insignificant. However, it is always significantly reduced. This clearly demonstrates the importance of the Doppler compensation scheme when performing the gravity experiment with RadioAstron.

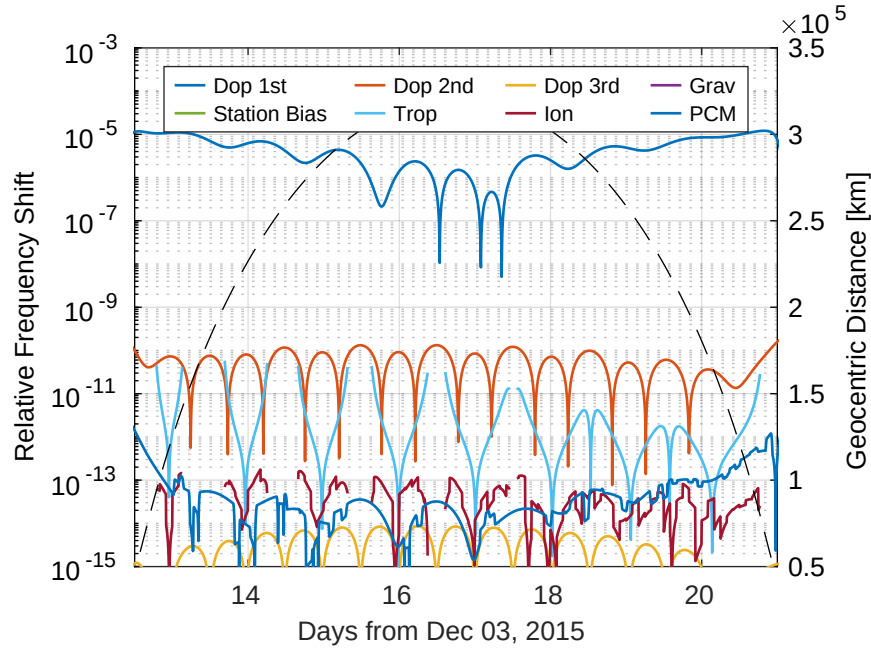


(a) Pushchino

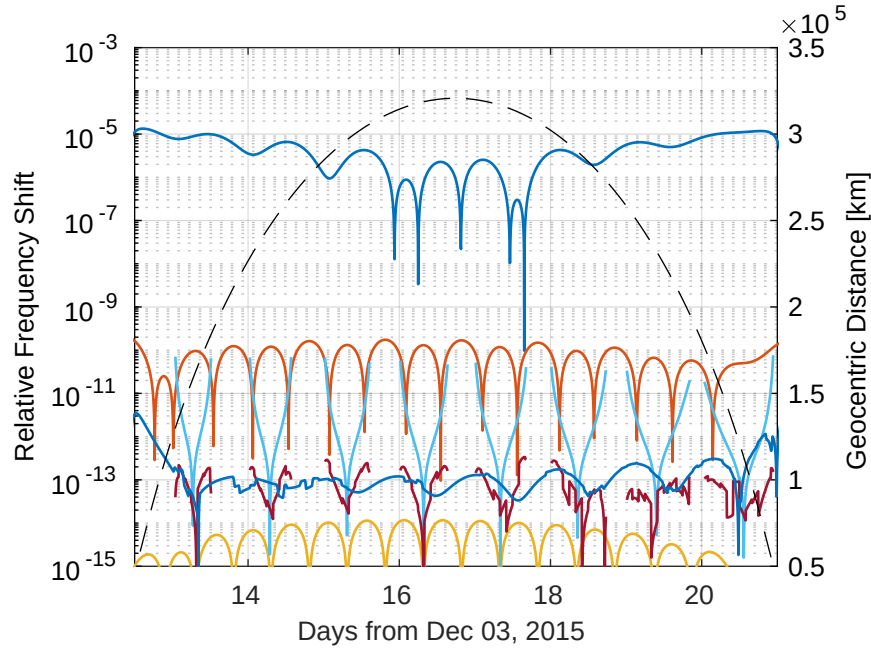


(b) Green Bank

Figure 3.28: Summary of frequency shifts in 1way mode (solid lines) for each tracking station over a single orbit between December 16 to 24, 2015. Spacecraft geocentric distance (dashed lines) is plotted against right axes. Legend in the upper figure also applies to the lower figure. Note the third order Doppler shift is below 10^{-15} and so it falls outside the plotted range.

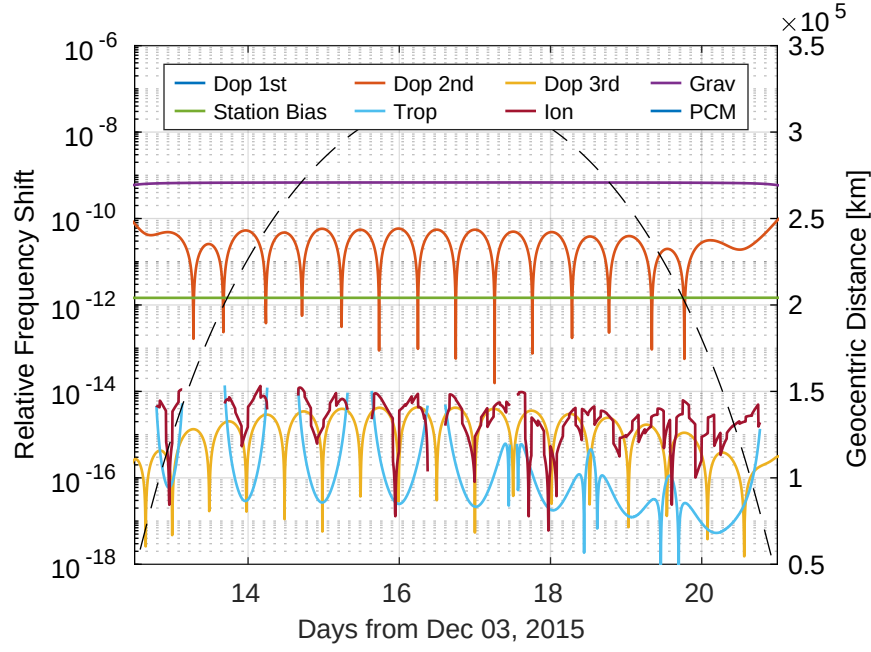


(a) Pushchino

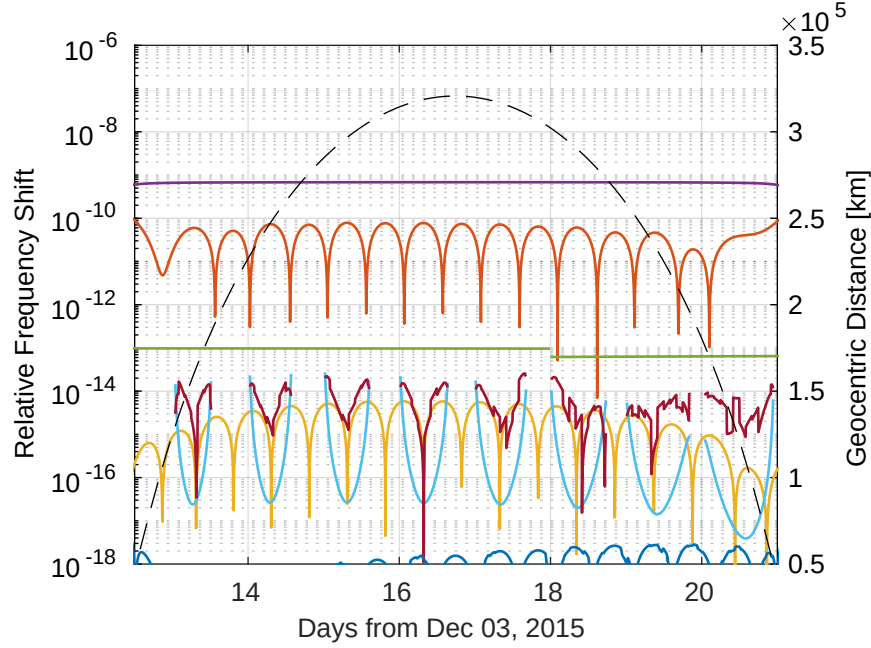


(b) Green Bank

Figure 3.29: Summary of frequency shifts in 2way mode (solid lines) for each tracking station over a single orbit between December 16 to 24, 2015. Spacecraft geocentric distance (dashed lines) is plotted against right axes. Legend in the upper figure also applies to the lower figure. Note the gravitational redshift and station frequency bias are not present in this mode.



(a) Pushchino



(b) Green Bank

Figure 3.30: Summary of frequency shifts after Doppler compensation is applied (solid lines) for each tracking station over a single orbit between December 16 to 24, 2015. Spacecraft geocentric distance (dashed lines) is plotted against right axes. Legend in the upper figure also applies to the lower figure. Note the non-relativistic Doppler shift is completely suppressed and the spacecraft PCM effect is reduced below 10^{-18} falling outside the plotted range.

Chapter 4

Interleaved Sessions

In the previous chapter we looked at output from the dynamical model for a specific sample interval. Now, we consider the interleaved sessions in which a Doppler compensation scheme is possible with RadioAstron. For each available session, we estimate the maximum absolute value of each of the major terms making up the overall frequency shift after the Doppler compensation scheme has been applied. This analysis provides an indication of the importance of the various phenomena from the previous chapter and which of these must be subtracted from the overall frequency shift to arrive at the gravitational redshift in a pure form and at the required level of accuracy.

The interleaved sessions were performed between 2015 and 2017 until the SHM became unreliable in July 2017. Each of the sessions were about 1 hr in length and included numerous switches, one every few minutes, between 1way and 2way communication modes. These switches allow the frequency shift from each mode to be fit over time so that at any given point during the session, a frequency shift from one mode and an interpolated shift from the other can be combined to implement the Doppler compensation scheme.

Interleaved sessions using the Green Bank tracking station are listed in Table 4.1 and with the Pushchino tracking station in Table 4.2. These include the maximum of the absolute

value of each of the terms in Eq. (2.263) with the exception of Δh . A significant challenge in performing a gravity experiment with RadioAstron is determining the frequency bias of the SHM. As previously noted, RadioAstron lacked a method to measure this bias directly and so it must be determined indirectly by some other means. This challenge is beyond the scope of this project, so only the station frequency bias is listed in the tables.

During the interleaved sessions, the gravitational redshift varied from as low as 5.3×10^{-10} to a maximum near apogee of 6.8×10^{-10} . Rather than measuring the absolute value of the gravitational redshift, it is convenient to measure the variation of the redshift over the orbit. So with these interleaved sessions, a maximum variation of $\Delta Z = 1.5 \times 10^{-10}$ is available for conducting the experiment.

In Chapter 2, we split the relativistic Doppler terms in the Doppler compensated shift into $\Delta y_{dop\ 2nd}$, $\Delta y_{dop\ 3rd}$ and Δy_{small} suggesting that the latter was insignificant. We can now see in Tables 4.1 and 4.2 that indeed Δy_{small} never rises above 6×10^{-16} and thus is not relevant to the gravity experiment. In contrast, $\Delta y_{dop\ 2nd}$ exceeds 30% of the gravitational redshift in some sessions. So it is essential that this term be computed with sufficient accuracy to exceed the 10^{-15} threshold. Finally, $\Delta y_{dop\ 3rd}$, although four orders of magnitude smaller than $\Delta y_{dop\ 2nd}$, is still above the threshold and must be included.

As was the case in the sample interval, the remaining PCM effect after applying the Doppler compensation scheme, Δy_{pcm} , is well below our threshold and can be neglected when using the interleaved sessions. Atmospheric effects, however, cannot be ignored. Δy_{ion} averages 3.6×10^{-15} and exceeds the threshold in all but six of the sessions. Δy_{trop} also exceeds the threshold in seven sessions and must be considered during those sessions.

The frequency bias turns out to be one of the most significant systematic effects that must be considered when performing the gravity experiment with RadioAstron. The bias of the spacecraft is known to exceed 10^{-11} at times, see Figure 3 in Litvinov et al. (2018). However, the station frequency biases, h_e , are also significant as they exceed 7×10^{-14} during

all interleaved sessions for both ground stations.

Let us finally consider the impact of incorrectly using state at t_3 in Eq. (2.197) instead of using Eq. (2.196) which is the correct formula when doing so. The difference will be given by subtracting the two equations and ignoring the time-related subscripts:

$$\Delta_{23} \equiv \Delta y_{dc} - \Delta y_{dc}(t_2, t_3) = \frac{D}{c^3} [\vec{v}_s \cdot \vec{a}_e + (\vec{v}_e - \vec{v}_s) \cdot \vec{a}_s + \nabla U_s \cdot \vec{v}_s] + o(c^{-4}) \quad (4.1)$$

The maximum absolute value of Δ_{23} is also included in Tables 4.1 and 4.2. We see that during sessions closer to Earth, the difference becomes larger and for some it is significant as it rises above the 10^{-15} threshold. Therefore, care must be taken, particularly with sessions near perigee, to use Eq. (2.196) if working with only t_3 state. Alternatively, the correct t_2 state for the spacecraft must be matched to each frequency observation at t_3 by using Eq. (2.97) to calculate the propagation time. As the later introduces an additional step in data processing, we prefer the former approach.

Table 4.1: Green Bank Interleaved Sessions

Session	Date	Distance [km]	Maximum Absolute Value								
			y_{grav}	$\Delta y_{dop\ 2nd}$	$\Delta y_{dop\ 3rd}$	Δy_{pcm}	Δy_{trop}	Δy_{ion}	Δy_{small}	h_e	Δ_{23}
RAKS13AH	2015-11-27	60,205	6.3e-10	7.0e-11	4.3e-15	3.6e-19	6.0e-18	4.4e-15	1.3e-16	1.0e-13	8.0e-16
RAKS13AI	2015-11-29	205,129	6.7e-10	6.6e-11	3.5e-15	5.8e-20	3.4e-17	3.3e-15	5.7e-16	1.0e-13	4.4e-17
RAKS13AL	2015-12-22	246,727	6.8e-10	2.7e-11	5.0e-15	1.8e-18	1.1e-16	4.4e-15	4.6e-16	6.3e-14	9.7e-17
RAKS13AM	2015-12-23	58,350	6.3e-10	6.5e-11	3.8e-15	1.8e-18	5.8e-16	8.4e-15	1.8e-16	6.5e-14	9.9e-16

Table 4.2: Pushchino Interleaved Sessions

Session	Date	Distance [km]	Maximum Absolute Value								
			y_{grav}	$\Delta y_{dop\ 2nd}$	$\Delta y_{dop\ 3rd}$	Δy_{pcm}	Δy_{trop}	Δy_{ion}	Δy_{small}	h_e	Δ_{23}
RAKS13AB	2015-10-24	176,218	6.7e-10	4.6e-12	1.7e-15	2.8e-19	1.2e-17	2.8e-15	3.4e-16	7.5e-13	3.3e-16
RAKS13AC	2015-10-25	46,350	6.1e-10	8.8e-11	2.9e-15	3.0e-19	3.1e-16	8.8e-15	8.6e-17	7.9e-13	7.7e-16
RAKS13AD	2015-10-31	247,644	6.8e-10	1.1e-11	4.1e-15	7.0e-19	5.4e-17	4.6e-15	2.4e-16	8.2e-13	2.3e-17
RAKS13AE	2015-11-02	58,499	6.3e-10	6.4e-11	3.5e-15	6.7e-19	8.6e-15	4.1e-15	1.6e-16	8.4e-13	9.2e-16
RAKS13AF	2015-11-18	191,913	6.7e-10	1.7e-11	1.2e-15	6.9e-19	1.2e-17	5.6e-15	1.7e-16	1.1e-12	2.4e-16
RAKS13AG	2015-11-19	66,849	6.4e-10	6.3e-11	4.4e-15	4.8e-19	5.8e-18	1.8e-15	1.4e-16	1.1e-12	7.6e-16
RAKS13AJ	2015-12-06	65,091	6.3e-10	6.4e-11	4.2e-15	3.2e-19	1.1e-17	1.6e-15	7.6e-17	1.4e-12	7.9e-16

Table 4.2 – continued from previous page

Session	Date	Distance [km]	Maximum Absolute Value								
			y_{grav}	$\Delta y_{dop\ 2nd}$	$\Delta y_{dop\ 3rd}$	Δy_{pcm}	Δy_{trop}	Δy_{ion}	Δy_{small}	h_e	Δ_{23}
RAKS13AK	2015-12-08	262,632	6.8e-10	4.4e-11	1.8e-15	9.4e-20	1.3e-16	4.5e-15	9.2e-17	1.5e-12	7.9e-17
RAKS13AN	2016-05-28	245,144	6.8e-10	4.2e-11	4.5e-15	5.3e-19	3.7e-17	5.5e-15	3.8e-16	8.7e-14	1.1e-17
RAKS13AO	2016-05-29	127,266	6.6e-10	4.4e-11	5.2e-15	4.3e-19	2.0e-17	5.3e-15	4.0e-16	8.7e-14	3.2e-16
RAKS13AP	2016-05-29	48,103	6.2e-10	8.6e-11	8.0e-15	3.5e-19	1.5e-15	2.7e-15	6.3e-17	8.7e-14	2.5e-15
RAKS13AQ	2016-06-14	181,892	6.7e-10	1.3e-11	1.4e-15	7.3e-19	1.9e-16	1.8e-15	1.2e-16	9.9e-14	2.6e-16
RAKS13AR	2016-06-15	65,895	6.4e-10	7.2e-11	7.2e-15	3.5e-19	1.4e-17	3.5e-15	3.2e-16	9.6e-14	1.3e-15
RAKS13AS	2016-06-15	37,445	5.9e-10	1.8e-10	9.2e-15	8.2e-19	8.8e-16	2.7e-15	2.8e-16	9.6e-14	4.9e-15
RAKS17AC	2016-07-11	22,946	5.3e-10	2.4e-10	8.5e-15	2.0e-18	4.2e-14	2.0e-14	1.3e-16	9.2e-14	6.4e-15
RAKS17AD	2016-07-11	56,385	6.2e-10	7.6e-11	7.5e-15	5.5e-19	8.1e-18	6.7e-16	1.9e-16	9.2e-14	1.5e-15
RAKS17AF	2016-07-20	28,490	5.5e-10	1.5e-10	8.8e-15	1.5e-19	2.2e-17	4.0e-16	8.0e-17	9.1e-14	3.7e-15
RAKS17AG	2016-07-20	62,242	6.3e-10	6.3e-11	6.9e-15	1.0e-19	1.3e-18	1.7e-15	1.9e-16	9.1e-14	1.3e-15
RAKS17AI	2016-07-28	30,075	5.7e-10	1.6e-10	8.8e-15	9.1e-19	4.3e-15	2.3e-15	1.6e-16	9.1e-14	4.4e-15
RAKS17AJ	2016-07-28	75,835	6.4e-10	4.4e-11	5.9e-15	6.3e-19	1.2e-17	2.3e-15	1.3e-16	9.1e-14	1.1e-15
RAKS17AW	2016-09-29	51,125	6.1e-10	7.8e-11	3.5e-15	4.5e-19	2.0e-16	5.0e-15	5.0e-17	8.7e-14	9.7e-16
RAKS17AY	2016-09-29	80,535	6.4e-10	5.1e-11	4.3e-15	4.3e-19	4.6e-18	8.8e-16	2.4e-17	8.7e-14	6.9e-16
RAKS17AZ	2016-09-30	128,093	6.6e-10	3.0e-11	4.2e-15	1.2e-19	3.4e-18	1.3e-15	2.7e-16	8.7e-14	2.6e-16

Table 4.2 – continued from previous page											
Session	Date	Distance [km]	Maximum Absolute Value								Δ_{23}
			y_{grav}	$\Delta y_{dop\ 2nd}$	$\Delta y_{dop\ 3rd}$	Δy_{pcn}	Δy_{trop}	Δy_{ion}	Δy_{small}	h_e	
RAKS17BA	2016-09-30	154,835	6.7e-10	1.6e-11	4.0e-15	5.5e-19	5.6e-18	2.6e-15	1.7e-16	8.7e-14	1.9e-16
RAKS17BB	2016-11-14	51,898	6.1e-10	8.2e-11	1.6e-15	2.7e-19	5.0e-15	5.0e-15	1.6e-16	8.4e-14	4.3e-16
RAKS17BC	2016-11-14	61,691	6.3e-10	6.9e-11	3.5e-15	4.1e-19	4.3e-17	2.2e-15	1.5e-16	8.4e-14	6.1e-16
RAKS17BD	2016-11-15	168,852	6.7e-10	1.8e-11	2.1e-15	2.2e-19	4.4e-18	3.2e-15	2.3e-16	8.4e-14	3.2e-16
RAKS17BE	2016-11-16	215,750	6.8e-10	1.5e-11	3.5e-15	3.6e-19	1.2e-17	3.1e-15	3.0e-16	8.4e-14	2.6e-17
RAKS17BF	2016-11-24	69,420	6.3e-10	5.6e-11	1.4e-15	2.6e-19	6.2e-15	8.4e-15	5.4e-17	8.3e-14	3.7e-16
RAKS17BG	2016-11-24	83,526	6.4e-10	5.5e-11	2.7e-15	1.7e-19	2.3e-17	6.8e-15	5.9e-17	8.3e-14	4.0e-16
RAKS17BH	2016-11-24	127,983	6.6e-10	2.3e-11	3.7e-15	5.5e-19	2.3e-17	4.0e-15	1.9e-16	8.3e-14	2.2e-16
RAKS17BI	2016-11-26	225,518	6.8e-10	2.2e-11	3.2e-15	2.8e-19	9.7e-18	2.1e-15	3.8e-16	8.3e-14	6.0e-17
RAKS17BJ	2016-12-04	81,392	6.4e-10	5.2e-11	2.4e-15	3.3e-19	5.6e-17	4.8e-15	4.6e-17	8.3e-14	4.4e-16
RAKS17BK	2016-12-05	146,364	6.7e-10	1.4e-11	3.5e-15	4.3e-19	2.6e-18	3.7e-15	2.0e-16	8.2e-14	1.4e-16
RAKS17BL	2016-12-06	236,152	6.8e-10	1.9e-11	3.4e-15	1.7e-19	1.3e-17	3.2e-15	3.6e-16	8.2e-14	6.4e-17
RAKS17BM	2016-12-06	277,843	6.8e-10	3.5e-11	5.6e-16	5.0e-19	1.4e-17	1.8e-15	2.6e-16	8.2e-14	5.3e-17
RAKS17BN	2016-12-14	94,489	6.5e-10	4.3e-11	2.5e-15	4.4e-19	4.0e-17	2.6e-15	1.0e-16	8.2e-14	4.4e-16
RAKS17BO	2016-12-15	237,649	6.8e-10	2.7e-11	3.0e-15	4.7e-19	1.1e-17	2.8e-15	4.0e-16	8.2e-14	8.1e-17
RAKS17BP	2016-12-16	283,410	6.8e-10	3.4e-11	7.9e-16	4.8e-19	1.6e-17	1.6e-15	2.2e-16	8.2e-14	4.3e-17

Table 4.2 – continued from previous page											
Session	Date	Distance [km]	Maximum Absolute Value								Δ_{23}
			y_{grav}	$\Delta y_{dop\ 2nd}$	$\Delta y_{dop\ 3rd}$	Δy_{pcn}	Δy_{trop}	Δy_{ion}	Δy_{small}	h_e	
RAKS17BQ	2017-03-12	82,330	6.4e-10	4.6e-11	2.4e-15	5.1e-19	4.9e-17	6.1e-16	1.2e-16	7.6e-14	4.8e-16
RAKS17BR	2017-03-12	149,692	6.7e-10	2.2e-11	3.1e-15	4.7e-19	3.3e-18	1.7e-15	3.3e-16	7.6e-14	8.1e-17
RAKS17BS	2017-03-13	218,235	6.8e-10	2.4e-11	6.0e-16	3.5e-19	1.1e-17	2.7e-15	1.9e-16	7.6e-14	6.7e-17
RAKS17BT	2017-03-26	248,258	6.8e-10	5.8e-11	1.1e-15	1.3e-19	9.4e-17	7.8e-16	3.0e-16	7.6e-14	3.5e-17
RAKS17BU	2017-03-29	62,181	6.3e-10	6.8e-11	9.3e-16	3.1e-19	3.2e-16	2.2e-15	2.1e-17	7.5e-14	2.8e-16
RAKS17BV	2017-03-29	96,078	6.5e-10	3.5e-11	3.5e-15	6.3e-19	1.4e-18	3.2e-15	1.4e-16	7.5e-14	3.7e-16
RAKS17BW	2017-04-12	242,680	6.8e-10	5.7e-11	1.9e-15	8.9e-20	6.9e-17	1.7e-15	3.2e-16	7.4e-14	3.6e-17
RAKS17BX	2017-04-14	59,888	6.2e-10	6.3e-11	8.2e-16	5.2e-19	6.8e-15	4.2e-15	5.9e-17	7.4e-14	3.6e-16
RAKS17BY	2017-04-15	67,705	6.3e-10	6.2e-11	2.2e-15	4.9e-19	5.7e-17	1.7e-15	5.6e-17	7.4e-14	4.9e-16
RAKS17BZ	2017-04-20	260,936	6.8e-10	6.0e-11	2.1e-15	1.3e-19	5.7e-17	4.4e-15	2.5e-16	7.4e-14	2.6e-17
RAKS17CA	2017-04-20	253,923	6.8e-10	4.2e-11	2.3e-15	4.7e-19	8.9e-16	6.0e-16	1.1e-16	7.4e-14	3.3e-17
RAKS17CB	2017-04-23	57,900	6.2e-10	7.3e-11	1.7e-15	3.0e-19	4.5e-16	3.2e-15	1.0e-16	7.4e-14	3.5e-16
RAKS17CC	2017-04-23	67,396	6.3e-10	5.5e-11	3.0e-15	5.9e-19	1.3e-17	5.6e-15	7.9e-17	7.4e-14	4.9e-16

Chapter 5

Discussion

The need for an accurate dynamical model can be put into perspective by considering preliminary results from the gravity experiment with RadioAstron. The frequency of the downlink signals from RadioAstron were measured directly using the Doppler tracking system in 40 ms time steps during scientific observations. In addition, during the interleaved sessions, high-bandwidth recordings of the spacecraft signals were made at the tracking stations and, during a subset of the sessions, at additional GRTs using a VLBI backend and data recorder. The latter allows the frequency of the spacecraft signal to be reconstructed with post-processing and potentially at higher accuracy.

An important limit on the level of accuracy achievable is the frequency stability of the equipment used. This is driven by both the frequency standards employed as well as any additional equipment involved in distributing timing references throughout the apparatus used to transmit, receive and record the spacecraft signal. In the case of the Doppler tracking data, frequency stability is limited by the latter as compared to a VLBI backend at a GRT. We can see this by comparing ADEV measurements of the spacecraft signal as shown in Figure 5.1 from Litvinov et al. (2017). When tested prior to launch in the laboratory, the SHM demonstrated a stability of $\sigma_\tau = 2 \times 10^{-15}$ over an averaging interval of $\sim 3,000$ s, the

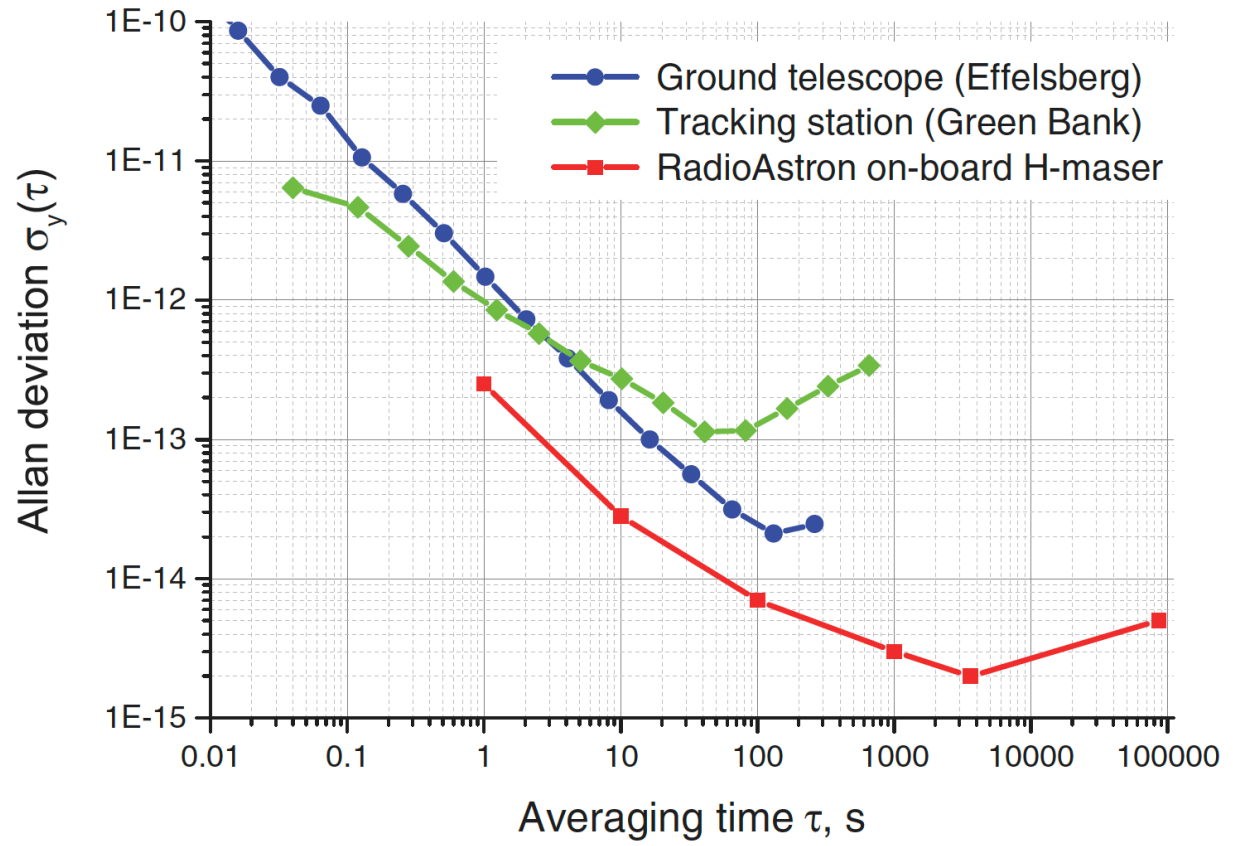


Figure 5.1: The frequency stability of the SHM signal recorded by the Green Bank tracking station (green), a GRT (blue) and in the laboratory prior to launch (red). From Litvinov et al. (2017).

length of a typical scientific session (see red curve in Figure 5.1). However, when measured with the highly stable VLBI backend at Effelsberg, a large GRT in Germany, stability on the order of 10^{-14} was observed (see blue curve in Figure 5.1). According to the Lavochkin Association, the state-owned company that built the Spektr-R spacecraft, this difference is due to destabilizing factors that are present in the space environment. When the spacecraft signal was measured with the Doppler tracking equipment at the tracking stations, frequency stability was $> 3 \times 10^{-13}$, more than two orders of magnitude worse than in the laboratory (see green curve in Figure 5.1).

While the frequency stability of Doppler tracking data is limited, the data set is plentiful and relatively simple to analyse. Between February 2012 and the end of 2017, tracking data are available from over 4,000 sessions. However, in $> 90\%$ of these the spacecraft utilized the 1way communication mode which does not allow the Doppler compensation scheme to be applied. Thus, measuring the gravitational redshift using these sessions is not just limited by the frequency stability of tracking data, but also by the accuracy with which the non-relativistic Doppler shift from Eq. (2.164) can be calculated. For example, the estimated uncertainty of the spacecraft velocity of $\delta v_s \sim 1.4 \text{ mm/s}$ from Zakhvatkin et al. (2020) corresponds to an error in $\dot{D}/c$ of approximately 5×10^{-12} . Errors due to the instability of frequency measurement and the velocity from the reconstructed orbit are random in nature and should fall by the square root of the number of sessions when combined. Thus, we would expect the uncertainty of the gravitational redshift measurement using the Doppler tracking data after 4,000 sessions to be on the order of 8×10^{-14} .

An analysis of the 1way Doppler tracking data was reported in Nunes et al. (2020a) where the gravitational redshift was not measured in absolute terms, but instead by looking for its variation, ΔZ , in pairs of sessions separated by up to 4.5 d. This technique of explicitly differencing sessions was used to cancel the unknown value of h_s . The average value of ΔZ over these sessions was 2.5×10^{-11} . Simplistically, we would expect that with this variation

in the redshift and the previously estimated measurement uncertainty, an uncertainty in the violation parameter of $\delta\varepsilon \sim 8 \times 10^{-14}/2.5 \times 10^{-11} \sim 3 \times 10^{-3}$ should be possible. Indeed, this is what was found for the statistical uncertainty in Nunes et al. (2020a). However, a systematic trend an order of magnitude larger was also found in the data limiting the accuracy of the measurement. An earlier version of the dynamical model was used in the analysis which lacked an implementation of PCM and used an empirical tropospheric model. It is possible that these shortcomings may have contributed to the trend. In addition, the coordinates of the Pushchino tracking station have an uncertainty of ~ 10 m which corresponds to a further uncertainty of $\sim 10^{-12}$ that is not entirely random and thus does not fall with the number of sessions in the same way as the other sources of uncertainty mentioned. Finally, the technique of using explicit differences itself may be particularly susceptible to systematics and so a different technique to compensate for h_s is used in subsequent analyses.

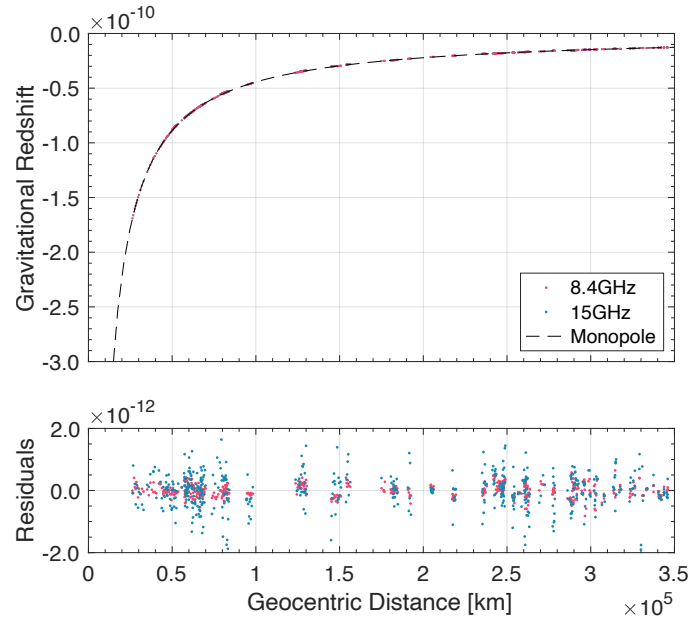
Interleaved sessions are also included in the Doppler tracking data in which the Doppler compensation scheme can be applied. The analysis of these is nearing completion with a publication of results forthcoming. Rather than using explicit differences, the evolution of h_s over time is fit simultaneously with the violation parameter measured as a trend against gravitational redshift. This approach should be less sensitive to systematics that do not correlate directly with the gravitational redshift. Although the number of interleaved sessions is significantly less, the uncertainty due to the non-relativistic Doppler shift is absent. Of the 164 interleaved sessions, the majority are near the spacecraft’s apogee and allow the evolution of h_s to be monitored over time. The remaining 60 sessions occurred at a variety of gravitational redshifts allowing the violation parameter to be measured. These are grouped into 20 experiments over individual orbits with an average of $\Delta Z \sim 5 \times 10^{-11}$ between sessions. Assuming that the frequency stability of the Doppler tracking data is the main source of error, rather than the dynamical model or other systematics, we would expect an uncertainty in the violation parameter of $\delta\varepsilon \sim 3 \times 10^{-13}/5 \times 10^{-11}/\sqrt{20} \sim 1 \times 10^{-3}$. Preliminary results

presented in Nunes et al. (2019) and Nunes et al. (2020b) as well as the most recent result of $\varepsilon = (0.2 \pm 2.0) \times 10^{-3}$ (Nunes et al. 2021), obtained using the latest version of ModelApp, do not appear to be affected by systematic errors unlike in the 1way analysis. As an example of the use of ModelApp, a number of plots related to this result are included in Figure 5.2. This result also indicates that the dynamical model is at least as accurate as what is implied by the stability of Doppler tracking measurements given that the result of $\varepsilon = 0$ within the uncertainty is consistent with the most accurate measurements of the gravitational redshift to date (see Herrmann et al. (2018) and Delva et al. (2018)).

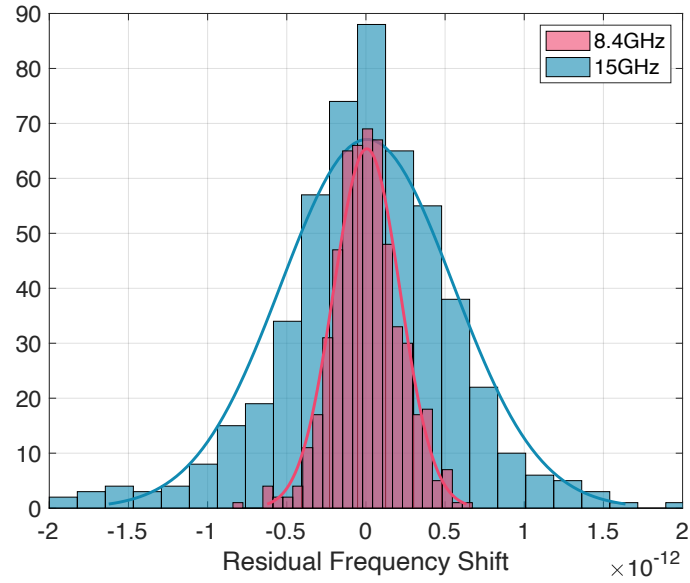
Ultimately, the highest accuracy will be achieved using the VLBI recordings of the spacecraft signal. The error budget from Table 1 of Litvinov et al. (2017) suggests that $\delta\varepsilon \sim 2.5 \times 10^{-5}$ may be possible based on repeating the experiment 30 times and assuming:

- Random sources of error: $\sigma_{stat} \sim 4 \times 10^{-14}$
- Systematic sources of error: $\sigma_{sys} \sim 2 \times 10^{-15}$
- Average variation in $\Delta U/c^2$: $\Delta Z \sim 3.0 \times 10^{-10}$

Unfortunately, in reality the number of experiments was fewer (20) and the average variation in the gravitational redshift over these experiments significantly less ($\Delta Z \sim 5 \times 10^{-11}$). Based on preliminary analysis, this implies that the achievable accuracy using the VLBI data from one station is only $\delta\varepsilon \sim 2 \times 10^{-4}$. However, simultaneous VLBI recordings were made at multiple GRTs during many sessions which, when combined, would reduce the random error. On average, 3.3 telescopes were involved in each session and so an uncertainty of $\delta\varepsilon \sim 1 \times 10^{-4}$ may be possible. Errors in the dynamical model can thus not exceed $\Delta Z \delta\varepsilon / 2 \sim 2.5 \times 10^{-15}$ so as not to become a limiting factor. As shown in this report, the dynamical model described herein is compatible with this requirement as it includes all known physical effects down to the 10^{-15} level.



(a) Observed Gravitational Redshift



(b) Residual Frequency Shift

Figure 5.2: An example use of ModelApp in the RadioAstron gravity experiment. Upper plot in (a) is the gravitational redshift measured using Doppler tracking data for each of the downlink signal frequencies relative to an observer at infinity. The dominant monopole component of the Earth's gravitational field is also shown for comparison (dashed line). The residuals are the difference between the observed and modeled gravitational redshift. The histogram (b) is of the residual frequency shift after fitting ε and h_s . The solid line represents a Gaussian fit of the residuals. From Nunes et al. (2021).

Chapter 6

Conclusions

Tests of EEP including measurements of the gravitational redshift continue to be important in finding a way towards reconciling our understandings of the universe at both subatomic and cosmological scales. The Russian space-VLBI mission RadioAstron, although never designed for a gravity experiment, provides a platform on which to look for violations of GR's predicted gravitational redshift in the Earth's gravitational field. Dedicated interleaved sessions are being used to implement a Doppler compensation scheme that may reach an accuracy slightly exceeding that of GP-A, and probes much larger distances from Earth.

In this work we have developed, in detail, the mathematical foundation for a dynamical model including contributions from the relativistic Doppler shift to third order in v/c , PCM, the non-dispersive effect of the troposphere, the dispersive effect of the ionosphere and the frequency bias between spacecraft and tracking station H-maser frequency standards. We have also shown, as have others, that the Doppler compensation scheme of GP-A significantly reduces systematics, particular the entire contribution from the non-relativistic Doppler shift. However, unlike in previously published work, we have done so solely in terms of the time of observation which simplifies model implementation.

The formulae from Chapter 2 have been implemented in C# and Matlab in a program

called ModelApp that itself relies on standard software libraries that implement the most accurate available models for Earth’s orientation, rotation and gravitational field. Spacecraft state vectors are interpolated from the reconstructed orbit of the spacecraft. ModelApp also uses data from GNSS-based models of the troposphere and ionosphere to calculate atmospheric effects. Output from ModelApp was presented in Chapter 3 showing the highly dynamic nature of most of the effects that make up the overall frequency shift the spacecraft’s signals. The relative importance of these effects becomes apparent when comparing their separate contributions.

In Chapter 4, we showed that for the gravity experiment with RadioAstron, only the PCM effect can be neglected after applying the Doppler compensation scheme. Other effects, while reduced, still exceed the 10^{-15} level in at least a subset of sessions. In the case of the troposphere, in particular, this is not what is usually assumed as the effect of the troposphere is thought to cancel almost completely. We have also shown that the residual effect of the ionosphere, even at the GHz frequencies of the spacecraft’s communication signal, almost always exceeded 1×10^{-15} , sometimes very significantly so, and must be taken into account. Finally, we found that the difference between the spacecraft state vectors at the time of emission as compared to the time of observation by the tracking station is significant in a subset of sessions. This underscores the importance of using the correct mathematical formulation, as presented herein when using state solely at observation time, or alternately, correctly associating spacecraft state at emission time with ground station state at observation time.

Finally, in the discussion of Chapter 5, we reviewed preliminary results from the RadioAstron gravity experiment that utilized ModelApp. We concluded that ModelApp correctly predicts effects on the order of 10^{-13} as no violation of the gravitational redshift predicted by GR has been found when analysing the Doppler tracking data from the interleaved sessions. This being consistent with other independent measurements of the gravitational redshift.

VLBI recordings of the same sessions promise to reduce this limit by two orders of magnitude. As the mathematical model developed herein is sufficient to support this more accurate analysis, ModelApp promises to make an important contribution to the final measurement of the gravitational redshift with RadioAstron. Our revised estimate for the accuracy that might be achieved by this measurement is $\delta\varepsilon \sim 1 \times 10^{-4}$, which is 4 times less accurate than previously estimated. However, while this only marginally exceeds the accuracy of GP-A, these measurements would still represent the first and only direct measurements of the gravitational redshift in the Earth's gravitational field over distances reaching the orbit of the Moon.

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