BI-FREE EMBEDDABILITY OF CERTAIN MULTI-ALGEBRA INDEPENDENCES

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Abstract

Non-commutative probability is an abstraction of classical probability that arises from considering random variables that do not commute in multiplication. In this non-commutative setting, five different types of independence between variables arise that are unique but closely parallel the classical independence. For example, each type of independence has its own theory of probability associated including central limit theorems and stochastic processes.

More recently an independence between pairs of variables known as bi-free independence has received some attention. It is a remarkable fact that each of the five main types of independence can be embedded into this bi-free independence.

Shortly after the inception of bi-free independence, a generalization was introduced, spawning simultaneously several notions of independence between tuples of variables. This wave of new types of independence appears to have dramatically increased the potential avenues of research. One is left with the question of how these can be corralled, if possible.

Given the embeddability of the five main types of independence into bi-free probability, one naturally extends the question of embeddability to the more recent multi-algebra independences. Can these too be studied from the context of bi-free probability? The main results of this work will be an affirmative answer for a small collection of these, most notably the free-Boolean and free-free-Boolean independences for pairs and triples of algebras respectively. Some other simple embeddings are shown, as well as some partial negative results.

To Mom and Kristen

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Introduction

It is a well known fact in quantum physics that the act of making an observation can affect the observed. One consequence of this is that the order in which we perform observations matters. For example, observables like position and momentum of a state, classically considered as independent quantities, are naturally modelled by operators, say P and Q respectively, satisfying $PQ - QP = i\hbar$. That is, $PQ \neq QP$. Since the measurement of these quantities in a particular quantum state is fundamentally probabilistic, P and Q may then be seen as *non-commutative random variables*.

Classical probability is built to handle, first and foremost, independent random variables that commute. That is to say, if X and Y are independent random variables, then we must have $XY = YX$. To handle a non-commutative setting, a theory of *non-commutative probability* becomes natural.

Non-commutative probability could be reasonably called a generalization or abstraction of classical probability, but one which has found numerous applications. This type of probability can't necessarily be easily described in the classical way of a probability measure on a set of events, so an abstraction is necessary. Perhaps surprisingly, the natural abstraction allowed for several other types of probability to be studied.

The hallmark of a theory of probability is in its notion of independence. In non-commutative probability theory, independence provides a rule for the calculations of expectations of products of independent variables in terms of their individual moments. The commutative setting hides many such ways that this can occur, as we could always just commute all of our independent variables into groups before evaluating expectations. This provides a very clear benefit to studying in the non-commutative setting; patterns of independence themselves become clearer. This allows us to get a richer understanding of what independence is.

For example, in the 1980's, Voiculescu introduced the idea of *free probability*, a kind of non-commutative independence that shows up in a "maximally non-commutative" framework. Free probability turns out to be a very different notion of probability than the classical setting in some sense, but one which has a rich structure and many striking parallels with the classical setting. Among these parallels are a *free central limit theorem*, *free levy process*, and even some notions of *free stochastic integration*. And far from being an idle curiosity, this type of independence turns out to have applications ranging from random matrix theory to operator algebras. The reader is referred to [9] for some of these applications.

The success of free probability sparked much interest in non-commutative probability, leading to many other questions. One such question that naturally arose was '*what other kinds of non-commutative probability are there and how are they related to each other?*' Speicher showed in [21] that there are only three symmetric types of this independence namely classical (tensor) independence, free independence, and Boolean independence. In [10], Muraki found two antisymmetric independences called monotone and anti-monotone independence and was able to show that these five independences were a complete list, at least for independences between variables.

Voiculescu originally only considered independence between variables in his development of free probability, but later realized that one could study a notion of independence between *pairs* of variables. This gave birth to a new kind of independence (for pairs of variables) in a series of papers ([26], [27], [28], [29]), that was dubbed *bi-free independence*. His own observations and other's developments have shown bi-free probability to be a rich and interesting area with many extensions and parallels coming from free probability. One such observation was that both classical independence and free independence could be studied through the unified lens of bi-free probability. He then conjectured that Boolean independence could also be modeled therein. This conjecture was answered affirmatively by Skoufranis in [19]. Skoufranis also showed that Muraki's monotone independence between a pair of algebras could be modeled in a similar fashion.

The success of bi-free probability opened up many new questions about other types multi-algebra independences and sparked new research into types of mixtures of the five independences. For example, a notion of bi-Boolean independence for pairs of algebras was discussed in [3], while a notion that mixes Boolean, monotone, and tensor independences simultaneously was considered in [1]. Liu also defined a collection of multi-algebra independences, with emphasis on a free-Boolean independence for pairs of algebras in [5]. He later found a notion of free-free-Boolean independence for triples of algebras in [6]. The free-Boolean independence was upgraded to a more general setting in [7]. Voiculescu's question about whether bi-free probability can house the independences then extends to this setting. Can these multi-algebra independences be studied through bi-free probability as well?

This work investigates and answers in the affirmative some special cases of this question, as well as look at certain limitations that arise with others.

To answer this question satisfactorily, we will need to use various tools and faces of non-commutative probability in conjunction. For example, non-commutative probability turns out to wield powerful combinatorial tools in the form of *cumulants*, which are related to expectations (or moments). Each (symmetric) independence has its own version of cumulants. For example, to get a handle on its cumulants, free independence exploits the so-called non-crossing partitions, elucidated in [14] and [20]. Bi-free probability finds an analogous framework by exploiting the so-called bi-non-crossing partitions. This combinatorial framework is developed in [8], [12], and generalized in [11].

Aside from this combinatorial lens, we will need a robust understanding of certain types of operators acting on a *free product space*. It turns out that Boolean, monotone, and free independence are all naturally represented as special operators on the same type of space. Classical independence (often called *tensor independence*) acts instead on a tensor product space. This makes it difficult to discuss in the same context as the other independences, and thus it will largely not feature in this work.

Putting these pieces together, along with inspiration from Skoufranis' approach to embedding Boolean independence in bi-free probability, one arrives at an embedding of Liu's free-free-Boolean independence for triples of faces into bi-free probability as well. From this we also get for free an embedding of Liu's free-Boolean independence into bi-free probability.

The outline of this work is as follows.

In chapter 1, we will generalize classical probability to the non-commutative setting, and discuss the different notions of independence. Three different faces of independence play a central role and are compared and contrasted. Notably, independence is defined directly via a moment condition. An equivalent definition, of a combinatorial flavour, is that these independences are equivalent to a *vanishing of mixed cumulants*, defined on their own related combinatorial structures. Notably, this combinatorial lens will allow us to develop a calculus for evaluating moments via diagrams, rather than a tedious algebra. Lastly, we discuss independence as operators acting on special corresponding spaces.

The work in chapter 1 will pave the way for a clearer understanding of the multi-algebra independences of Voiculescu and Liu in chapter 2. In particular, we begin immediately with an operator action definition of these independences (rather than a moment condition). We look at the combinatorial nature of the bi-free independence, and at a similar, but related structure of LR-diagrams that will help us keep a careful accounting of related operator actions. Liu's multi-algebra independences are discussed with an emphasis on the free-free-Boolean independence for triples of algebras.

Lastly, we will discuss the issue of embedding independences into bi-freeness. The standard independences were shown by Voiculescu ([26]) and Skoufranis ([19]), but we discuss some limitations therein. The main result of this work follows, based on the work [15], where we show that free-free-Boolean independence can embed faithfully into a bi-freely independent family. An easy embedding for pairs of monotone-Boolean pairs is touched on afterwards. The chapter is finished off by a partial negative result about a failure of Liu's free-monotone independence to embed into bi-freeness.

Chapter 1

Background on Non-Commutative Probability

The non-commutative analog of classical probability allows for many patterns, hidden by commutativity, to become more obvious. For example, central to probability is its notion of independence. In the classical (commuting) setting, there is exactly one notion of independence. However, when commutativity is lost, a handful of other types of independence become visible. Optimistically, this deeper look into the notion of independence allows one to understand the meaning of independence in a richer way.

First we will take a closer look at the abstraction from classical probability to non-commutative probability. Then we will discuss four distinct notions of independence that arise in this framework. The remainder of the chapter will be devoted to studying similarities and differences between these notions of independence through three different (but equivalent) perspectives; a direct moment definition, a combinatorial theory of cumulants, and independence as an action of operators on corresponding special spaces.

1.1 Non-Commutative Probability Space and Representations

The usual setting of probability theory is to work in a probability triple $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is a σ-algebra of subsets of Ω, and P is a probability measure on $\mathcal{F}$. One then defines random variables as (Borel) measurable functions

$$X : \Omega \to \mathbb{C}$$

along with an expectation

$$\mathbb{E}(X) := \int_\Omega X(\omega) dP(\omega),$$

which is a unital linear functional on the space of random variables with finite moments.

One line of motivating non-commutative probability theory is then to look at the algebra, call it $\mathcal{A}$, of random variables with finite moments of all orders. It turns out that $\mathcal{A}$ encodes all the meaningful information about the original probability triple. The extent to which the original triple can be recovered will not be discussed much here, but we make an easy observation. To each element $E \in \mathcal{F}$, there is a characteristic function χ_E (which is a random variable) which has the property that $\chi_E^2 = \chi_E$ and $\mathbb{E}(\chi_E) = P(E)$. So we get the following associations

$$E \in \mathcal{F} \leftrightarrow X \in \mathcal{A} \text{ with } X^2 = X,$$

and

$$P(E) \leftrightarrow \mathbb{E}(X) \text{ where again } X^2 = X.$$

So we can recover at least the σ-algebra of events, and the probability measure just from looking at our projections and their expectations.

1.1.1 Non-Commutative Probability Spaces

Now we may forget about our probability triple and what we are left with is simply a unital algebra and a unital linear functional on that algebra. This abstract structure opens the door to studying non-commutative algebras with the same probabilistic ideas in mind. Hence the following definition.

Definition 1.1.1. A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional.

As the motivation might suggest, we often call elements of $\mathcal{A}$ *non-commutative random variables* and φ an *expectation*.

Example 1.1.2. A simple example of this is found in the algebra of $n \times n$ matrices over $\mathbb{C}$. In particular, if $\mathcal{A} = M_n(\mathbb{C})$ and $\varphi = \frac{1}{n} \operatorname{Tr}$ then $(\mathcal{A}, \varphi)$ is a non-commutative probability space.

Example 1.1.3. Another example that will pervade this work is that of a group non-commutative probability space. Let G be a discrete group, and form its *group algebra*

$$\mathbb{C}[G] = \operatorname{span}_{\mathbb{C}}\{\delta_g : g \in G\}$$

where

$$\delta_h \cdot \delta_g = \delta_{hg}.$$

Then $\mathbb{C}[G]$ has unit δ_e, where e is the identity of G. Note that $\mathbb{C}[G]$ is a commutative algebra exactly when G is abelian.

Next, let $\tau : \mathbb{C}[G] \to \mathbb{C}[G]$ be defined by

$$\tau(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise} \end{cases},$$

and extended linearly.

Then $(\mathbb{C}[G], \tau)$ is a non-commutative probability space.

For the purposes of this work, we also like to consider when our expectation takes values in a more general algebra. This is analogous to studying *conditional expectations* in the classical setting. So, throughout this work, we denote by B, a unital algebra over $\mathbb{C}$. We extend our definition of a non-commutative probability space to that of a B-valued probability space.

Definition 1.1.4. A *B-non-commutative probability space* is a pair $(\mathcal{A}, E)$ where $\mathcal{A}$ is a unital algebra containing B (with $1_{\mathcal{A}} = 1_B$) and $E : \mathcal{A} \to B$ is a linear map such that

$$E(b_1 a b_2) = b_1 E(a) b_2$$

for all $a \in \mathcal{A}$ and $b_1, b_2 \in B$.

A subalgebra A of $\mathcal{A}$ is called a *B-subalgebra* if $BAB \subseteq A$.

Let us take a moment to upgrade our examples from above to this more general setting.

Example 1.1.5. Consider $\mathcal{A} = M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ and $B = I_n \otimes M_m(\mathbb{C})$. Define $E : \mathcal{A} \to \mathcal{B}$ by

$$E(a_1 \otimes a_2) := \frac{1}{n} \operatorname{Tr}(a_1) I_n \otimes a_2$$

and extended linearly. Then $(\mathcal{A}, E)$ is a B-non-commutative probability space.

Example 1.1.6. If G is a discrete group with subgroup H, then let $\tau_H : \mathbb{C}[G] \to \mathbb{C}[H]$ be defined by

$$\tau_H(\delta_g) = \begin{cases} \delta_g & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases},$$

and extended linearly. Then $(\mathbb{C}[G], \tau_H)$ is a $\mathbb{C}[H]$-non-commutative probability space.

Distributions are key objects of study in probability, so it is sensible to ask how these are recovered in this abstract setting.

Definition 1.1.7. Let $(\mathcal{A}, E)$ be a B-non-commutative probability space. By *moment* we mean the expectation, $E(a)$ for some $a \in A$.

The *(joint) distribution* of a B-subalgebra A of $\mathcal{A}$ is the B-valued functional $E|_A$. Equivalently, it is the collection of all moments, $E(a)$ for $a \in A$.

The *joint distribution* of a family $\{A_k\}_{k\in K}$ of B-subalgebras of $\mathcal{A}$ is the functional $E|_{\bigvee_{k\in K} A_k}$, where $\bigvee_{k\in K} A_k$ denotes the B-subalgebra of $\mathcal{A}$ generated by the family $\{A_k\}_{k\in K}$. Equivalently, the joint distribution is the collection of all mixed moments

$$E(a_1 \cdots a_n)$$

for $n \in \mathbb{N}$, $i : \{1, \ldots, n\} \to K$, and $a_j \in A_{i(j)}$.

Remark 1.1.8. Throughout this work, we will concern ourselves with joint distributions of families of algebras, and whether or not they are preserved when represented.

One simplification we can make to the above definition, is that by combining adjacent elements of the same algebra in a moment, we may assume that $i(1) \neq i(2) \neq \cdots \neq i(n)$. One frequently observes this constraint, for example in the definitions of independence below.

Remark 1.1.9. A probabilist might more commonly associate the word *distribution* with a probability measure on $\mathbb{R}$ or $\mathbb{C}$, for example. A simple case of when these agree is for a compactly supported measure μ on $\mathbb{R}$. By knowing only the moments

$$\varphi_\mu(x^n) = \int_{\mathbb{R}} x^n d\mu(x)$$

one recovers (uniquely) the measure μ as a consequence of the Stone-Weierstrass theorem.

It is clear that not all distributions of random variables $a \in \mathcal{A}$ arise from probability measures on $\mathbb{R}$. A simple necessary condition is that $\varphi(a^n) \in \mathbb{R}$ for all $n \in \mathbb{N}$. One way to ameliorate this is to work in the setting of a $*$-probability space. That is, a non-commutative probability space $(\mathcal{A}, \varphi)$ that comes equipped with an *involution* $* : \mathcal{A} \to \mathcal{A}$ such that $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. A sufficient condition for the distribution of $a = a^* \in \mathcal{A}$ to coincide with the moments of a probability measure on $\mathbb{R}$ is then that this space can be faithfully represented on a Hilbert space (e.g. when $\mathcal{A}$ is a C^*-algebra and φ is faithful).

So one might naturally inquire as to when a collection of moments in our general algebraic setting is the collection of moments of a probability measure on $\mathbb{R}$. This is the content of moment problems like the *Hamburger moment problem*. While we won't concern ourselves with this distinction any further, we refer the curious reader to the monograph by Schmüdgen ([17]) for an extensive treatment of this problem.

1.1.2 Concrete Non-Commutative Probability Spaces and Representations

Every algebra acts on itself in a natural way. In this sense, elements of an algebra may be viewed as operators. Indeed, if $\mathcal{A}$ is an algebra over $\mathbb{C}$, then $\mathcal{A}$ is a vector space over $\mathbb{C}$, and every element $a \in \mathcal{A}$ is associated with an operator $L_a : \mathcal{A} \to \mathcal{A}$ defined by

$$L_a(x) = ax$$

for all $x \in \mathcal{A}$. Furthermore, if φ is a unital linear functional on $\mathcal{A}$, then we could write

$$L_a(x) = ax = \varphi(ax)1_{\mathcal{A}} + (ax - \varphi(ax)1_{\mathcal{A}}).$$

So that the projection p onto the span of $1_{\mathcal{A}}$ with kernel $\ker \varphi$ has the property that

$$pL_a 1_{\mathcal{A}} = \varphi(a)1_{\mathcal{A}}.$$

In the case where we are working over a more general algebra B, then we keep track of the possibility of left and right multiplication by B with the following structures.

Definition 1.1.10. A *B-B-bimodule with specified B-projection* is a triple $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ where $\mathcal{X} = B \oplus \mathring{\mathcal{X}}$ is a direct sum of B-B-bimodules and $p : \mathcal{X} \to B$ is the projection onto B with kernel $\mathring{\mathcal{X}}$.

Let $\mathcal{L}(\mathcal{X})$ denote the collection of $\mathbb{C}$-linear operators on $\mathcal{X}$, and define $E_{\mathcal{L}(\mathcal{X})} : \mathcal{L}(\mathcal{X}) \to B$ by

$$E_{\mathcal{L}(\mathcal{X})}(T) = pT1_B.$$

One readily observes that the pair $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$ is itself a B-non-commutative probability space. This allows us to represent elements of a B-non-commutative probability space as operators on B-B-bimodules with specified B-projections.

Definition 1.1.11. Let $(\mathcal{A}, E)$ be a B-non-commutative probability space, and let A be a B-subalgebra of $\mathcal{A}$. A *representation* of A is a B-B-bimodule with specified B-projection $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ and a B-homomorphism $\delta : A \to \mathcal{L}(\mathcal{X})$.

Say that δ is *faithful or distribution preserving* if it preserves expectations:

$$E_{\mathcal{L}(\mathcal{X})}(\delta(a)) = E(a).$$

In order to talk about independence, we will also need the notion of a representation for a family of B-subalgebras.

Definition 1.1.12. Let $\Gamma = \{A_k\}_{k \in K}$ be a family of B-subalgebras in a B-non-commutative probability space $(\mathcal{A}, E)$. A *representation* of Γ is a B-B-bimodule with specified B-projection along with a family $\delta = \{\delta_k\}_{k \in K}$ of homomorphisms $\delta_k : A_k \to \mathcal{L}(\mathcal{X})$.

We will say that δ is *expectation preserving* if for each $k \in K$ and $a \in A_k$, we have

$$E_{\mathcal{L}(\mathcal{X})}(\delta_k(a)) = E(a).$$

We will say that δ is *faithful or joint distribution preserving* if for every $n \in \mathbb{N}$ and $a_i \in A_{k_i}$ for $1 \leq i \leq n$, we have

$$E_{\mathcal{L}(\mathcal{X})}(\delta_{k_1}(a_1) \cdots \delta_{k_n}(a_{k_n})) = E(a_1 \cdots a_n).$$

Proposition 1.1.13. *Every B-non-commutative probability space $(\mathcal{A}, E)$, has a faithful representation.*

The verification for this follows the intuition from above. Note that if $\mathcal{A}$ is an algebra over B, then $\mathcal{A}$ itself is a B-B-bimodule which can be broken down as

$$\mathcal{A} = B \oplus \ker E_{\mathcal{A}}.$$

1.2 Independence in Non-Commutative Probability

The hallmark of probability theory is in its notion of *independence.* Independence between random variables (or more specifically their algebras) describes how their mixed moments are calculated solely from the knowledge of the individuals. This extends to tell us how the joint distributions of a family of independent algebras can be calculated in terms of the individual distributions.

The independence that one studies in classical probability has a direct translation to this non-commutative setting, which we will see now. Though, since we will work in the setting of B-valued probability, we pay some attention to detail. In particular, for the following generalization, we require that B is a commutative algebra.

For the remainder of this chapter, we fix our setting to be that of a B-valued probability space $(\mathcal{A}, E)$.

Notation 1.2.1. Let $\epsilon : \{1, \ldots, n\} \to K$ for some set K. Denote by $\ker \epsilon$ the set of equivalence classes of $\{1, \ldots, n\}$ by the relation $i \sim_\epsilon j$ if and only if $\epsilon(i) = \epsilon(j)$.

Definition 1.2.2. Let B be a unital commutative algebra. We say that a family $\{A_k\}_{k \in K}$ of unital B-subalgebras of $\mathcal{A}$ is *tensor (or classically, or commuting) independent over B* if $xy = yx$ whenever $x \in A_{k_1}$ and $y \in A_{k_2}$ and $k_1 \neq k_2$, $\epsilon : \{1, \ldots, n\} \to K$ and $x_i \in A_{\epsilon(i)}$, then

$$E(x_1 \cdots x_n) = \prod_{W \in \ker \epsilon} E(\overrightarrow{\prod_{i \in W}} x_i),$$

where $\overrightarrow{\prod}_{i \in W} x_i$ denotes the (index) ordered product of the x_i: if $W = \{i_1, i_2, \ldots, i_k\}$ where $i_1 < i_2 < \cdots < i_k$, then

$$\overrightarrow{\prod_{i \in W}} x_i = x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Remark 1.2.3. It might seem odd that in a setting of *non-commutative* probability, that we demand independent variables commute. However, it can be seen from demanding the type of moment formulas in the definition to occur, one needs commutativity at least *in distribution*. For example, it is mandatory that B is commutative in order for such moments to be well defined. Then whenever $b \in B$, since B is commutative,

$$E(xb) = E(x)b = bE(x) = E(bx).$$

Also if x and y are from independent algebras, then

$$E(xy) = E(x)E(y) = E(yx).$$

So since such expectations would need to factor through a quotient by the space of commutators between independent algebras anyways, the only faithful representations of this type of independence that can arise are from the setting in the definition above.

Example 1.2.4. Tensor independence shows up in the context of group algebras as follows. Let $\{G_k\}_{k\in K}$ be a family of subgroups of a group G having common (central) subgroup H and such that $G_i \cap G_j = H$ and $g_i g_j = g_j g_i$ whenever $g_i \in G_{k(i)}$, $g_j \in G_{k(j)}$ and $k(i) \neq k(j)$. Let $\bigoplus_{k\in K}^H G_k$ denote the (sub)group generated by the family $\{G_k\}_{k\in K}$ in G. Note that in the case when $H = \{e\}$, this group is the usual direct sum. Then $\{\mathbb{C}[G_k]\}_{k\in K}$ is tensor independent over $\mathbb{C}[H]$ in $(\mathbb{C}[\bigoplus_{k\in K}^H G_k], \tau_H)$.

Let us take a moment to show this fact. By linearity it suffices to consider monomials. Let $n \geq 1$ and $\epsilon : \{1, \ldots, n\} \to K$. Let $\text{ran}\epsilon = \{\epsilon_1, \ldots, \epsilon_m\}$. Then for $g_i \in G_{\epsilon(i)}$, we have, by the commutation relations between the G_k,

$$g_1 \cdots g_n = \left(\overrightarrow{\prod_{\epsilon(i)=\epsilon_1}} g_i \right) \cdots \left(\overrightarrow{\prod_{\epsilon(i)=\epsilon_m}} g_i \right).$$

For $1 \leq j \leq m$, denote by $\tilde{g_j} = \overrightarrow{\prod}_{\epsilon(i)=\epsilon_j} g_i$. Then if any $\tilde{g_j} \in G_{\epsilon_j} \setminus H$, then $\tilde{g_1} \cdots \tilde{g_m} \notin H$, implying

$$\tau_H(\delta_{g_1} \cdots \delta_{g_n}) = \tau_H(\delta_{\tilde{g_1}} \cdots \delta_{\tilde{g_m}}) = 0.$$

Otherwise, $\tilde{g_j} \in H$ for each $1 \leq j \leq m$, then $\tilde{g_1} \cdots \tilde{g_m} \in H$ so that

$$\begin{aligned}
\tau_H(\delta_{g_1} \cdots \delta_{g_n}) &= \tau_H(\delta_{\tilde{g_1}} \cdots \delta_{\tilde{g_m}}) \\
&= \delta_{\tilde{g_1}} \cdots \delta_{\tilde{g_m}} \\
&= \tau_H(\delta_{\tilde{g_1}}) \cdots \tau_H(\delta_{\tilde{g_m}}) \\
&= \tau_H \left(\overrightarrow{\prod_{\epsilon(i)=\epsilon_1}} \delta_{g_i} \right) \cdots \tau_H \left(\overrightarrow{\prod_{\epsilon(i)=\epsilon_m}} \delta_{g_i} \right).
\end{aligned}$$

Both cases affirm the definition of tensor independence.

While the above forms the more familiar notion of independence, the non-commutative setting spawns some other flavours of independence.

1.2.1 Free, Boolean, and Monotone Independence

Definition 1.2.5. A family $\{\mathcal{A}_k\}_{k\in K}$ of unital B-subalgebras of $\mathcal{A}$ is said to be *freely independent over B* if

$$E(a_1 \cdots a_n) = 0.$$

whenever $n \in \mathbb{N}$, $i : \{1, \ldots, n\} \to K$ with $i(j) \neq i(j+1)$ and $E(a_j) = 0$ for all $j \in \{1, \ldots, n\}$

Example 1.2.6. The conditions of this definition can seem unintuitive at first. But they arise from studying free products of groups as follows (c.f. [25]). Let G be a group, $H \leq G$, and $\{G_k\}_{k\in K}$ a family of subgroups with $H \subseteq G_k$. Say that the family $\{G_k\}_{k\in K}$ is *free with amalgamation over* H if

$$g_1 g_2 \cdots g_n \neq e \quad \text{whenever} \quad g_i \in G_{k_i} \setminus H \text{ and } k_1 \neq k_2 \neq \cdots \neq k_n.$$

Then the algebras $\{\mathbb{C}[G_k]\}_{k\in K}$ are freely independent over $\mathbb{C}[H]$ in $(\mathbb{C}[G], \tau_H)$.

If the family $\{G_k\}_{k\in K}$ is free over H, we will denote by $*_{k\in K}^H G_k$ the (sub)group generated by the family $\{G_k\}_{k\in K}$. We note that $*_{k\in K}^H G_k$ is generated by elements of the form

$$g_1 g_2 \cdots g_n$$

where $n \geq 0$ and $g_i \in G_{k(i)} \setminus H$ with $k(i) \neq k(i+1)$.

The family $\{\mathbb{C}[G_k]\}_{k\in K}$ is then freely independent over H in $(\mathbb{C}[*_{k\in K}^H G_k], \tau_H)$. Indeed, by linearity it suffices to consider $n \geq 1$, $g_i \in G_{k(i)} \setminus H$ with $k(i) \neq k(i+1)$. Then

$$\tau_H(\delta_{g_1} \cdots \delta_{g_n}) = \tau_H(\delta_{g_1 \cdots g_n}) = 0,$$

since $g_1 \cdots g_n \notin H$.

Definition 1.2.7. A family $\{\mathcal{A}_k\}_{k\in K}$ of B-subalgebras of $\mathcal{A}$ is said to be *Boolean independent over* B if whenever $n \in \mathbb{N}$, $i : \{1, \dots, n\} \to K$, and $i(j) \neq i(j+1)$ then

$$E(a_1 a_2 \cdots a_n) = E(a_1)E(a_2) \cdots E(a_n).$$

Remark 1.2.8. One observes that if we allow the A_k to be *unital* B-subalgebras of $\mathcal{A}$, then Boolean independence would imply that E is a homomorphism on each A_k. To avoid this, we typically consider the A_k's to be non-unital.

Remark 1.2.9. The naming convention for the tensor and free independences comes from the types of product spaces they arise from, which will be investigated in Section 1.4. The name *Boolean* might at first glance seem a little strange for this type of independence, but it was chosen ([22]) for its association with the *interval partitions*. See Remark 1.3.28 below.

Lastly, we have the so-called *monotone* independence.

Definition 1.2.10. A family $\{\mathcal{A}_k\}_{k\in K}$ of B-subalgebras of $\mathcal{A}$, where K is a totally ordered index set, is said to be *monotone independent over B* if

$$E(a_1 \cdots a_{i-1}a_i a_{i+1} \cdots a_n) = E(a_1 \cdots a_{i-1}E(a_i)a_{i+1} \cdots a_n)$$

whenever $a_j \in A_{k_j}$ and $k_i > k_{i-1}$ and $k_i > k_{i+1}$.

Remark 1.2.11. There are a couple of observations one might make about the above definition.

One might notice, for example, that there is an inherent asymmetry in this type of independence. That is, one cannot swap the roles of the variables to other independent algebras and have the moments factor out the same way. For example, considering x and y from separate algebras that are monotonically independent. If x comes from a lower-indexed algebra then

$$E(xyx) = E(xE(y)x).$$

But if instead x comes from a higher-indexed algebra, then

$$E(xyx) = E(x)E(y)E(x).$$

This is different from tensor, free, or Boolean independence, as the index of the algebras plays no role there.

Secondly, one might suppose we could relax the assumption that our index set is totally ordered to a partially ordered set and achieve a more general type of independence. However, this does not work as we may not be able to recover joint distributions of independent algebras from the individual joint distributions, as one would expect of a notion of *independence*. This is clarified slightly in Remark 1.2.13 below.

Remark 1.2.12. Initially, the independences don't appear to be different from each other. One finds for example, that when $x \in A_1$ and $y \in A_2$ and A_1 and A_2 are independent over B, we have

$$E(xy) = E(x)E(y)$$

for all three types of independence.

But suppose instead that we calculate the moment $E(xyxy)$. If the independence is tensor (and B is commutative),

$$E(xyxy) = E(x^2)E(y^2).$$

If the independence is Boolean

$$E(xyxy) = E(x)E(y)E(x)E(y).$$

If the independence is free, then

$$E(xyxy) = E(x)E(yE(x)y) + E(xE(y)x)E(y) - E(x)E(y)E(x)E(y).$$

Lastly if the independence is monotone, then

$$E(xyxy) = E(xE(y)x)E(y).$$

The tensor, Boolean, and monotone moments are simple calculations from the definition, but the free moment is not obvious. We take a moment to hint at how one can calculate the moment above.

The trick is known as *centering*. Let $\mathring{x} = x - E(x)$ and $\mathring{y} = y - E(y)$. Then $\mathring{x} \in A_1$ and $\mathring{y} \in A_2$ since A_1 and A_2 are unital B-algebras, and we have $E(\mathring{x}) = E(\mathring{y}) = 0$. This allows us, from the definition to make the claim that

$$E(\mathring{x}\mathring{y}\mathring{x}\mathring{y}) = 0.$$

Now to recover $E(xyxy)$, one simply replaces $\mathring{x}$ with $x - E(x)$ and $\mathring{y}$ with $y - E(y)$, and uses the linearity of E and some rearranging.

Remark 1.2.13. If $\{A_k\}_{k \in K}$ is a family of independent algebras, one would hope that the joint distribution of the family collectively is determined solely by the joint distributions of the individual A_k's.

We can check this by showing that expectations of arbitrary mixed moments in the family $\{A_k\}_{k \in K}$ may be computed solely through submoments on the individual A_k's. This is immediate from the definition of the tensor and Boolean independences. But it is less obvious for the free and monotone independences. We take a moment to elucidate the case of monotone independence now. The case of free independence is shown more easily from the cumulants in the following section.

For the monotone independence, let us fix an arbitrary mixed moment $E(a_1 \cdots a_n)$ where $a_i \in A_{k_i}$. Let us suppose that $k_i \neq k_{i+1}$ for $1 \leq i \leq n-1$ by possibly grouping together adjacent elements from the same algebras. An algorithm for computing such a moment is simply to find an index $j \in \{1, \ldots, n\}$ such that k_j is largest among $k_1, \ldots, k_n$. Then we have

$$E(a_1 \cdots a_{j-1} a_j a_{j+1} \cdots a_n) = E(a_1 \cdots a_{j-1} E(a_j) a_{j+1} \cdots a_n).$$

By possibly regrouping as mentioned above, one finds, inductively, that the original mixed moment may be computed in terms of the moments restricted to the A_k's.

Thus we have argued the following for the case of tensor, Boolean, and monotone independence.

Proposition 1.2.14. *Given B-valued probability spaces $(\mathcal{A}_1, E_1)$ and $(\mathcal{A}_2, E_2)$, a family $\{A_k\}_{k\in K}$ of tensor (respectively Boolean, monotone) independent B-subalgebras of $\mathcal{A}_1$, and faithful homomorphisms $\delta_k : A_k \to \mathcal{A}_2$, then the family $\delta = \{\delta_k\}$ is joint distribution preserving if and only if the family $\{\delta_k(A_k)\}_{k\in K}$ is tensor (respectively Boolean, monotone) independent in $(\mathcal{A}_2, E_2)$.*

Remark 1.2.15. Given the variety of different independences we get in this non-commutative setting, one might naturally inquire if there are any more, and if so, how many?

Perhaps surprisingly, in a strong sense, this is *the complete list*. That is, if we require, naturally, that joint distributions of families of independent algebras can be recovered from the joint distributions of their individual algebras (i.e. Proposition 1.2.14 holds), then the work of Speicher in [21] and Muraki in [10] show that these are the only types of independence that can exist.

This work, initiated by Schürmann in [18], uses the language of *universal* or *natural products* of algebras with expectations (non-commutative probability spaces). Speicher ([21]) showed that there are only two types of universal product (where moment formulas are not dependent on the indices of the algebras, e.g. as in Remark 1.2.11) constructions when the algebras are presumed to be unital, that of the tensor product and free product corresponding of course to tensor independence and free independence. In the non-unital case, the Boolean independence arises.

It was Muraki in [10] who relaxed the definition of *universal* to *natural* product (allowing for indices of algebras to matter), and showed that exactly two more exist: monotone and anti-monotone independence. Since anti-monotone is simply monotone with the reverse ordering on the index set, we restrict our investigations to the monotone case in this work.

1.3 The Combinatorics of Independence

One may observe from Remark 1.2.12, that the different independences yield different patterns in moment calculations. In fact, that patterns we saw are *universal* in the sense that if

$\{a_1, a_3\}$ is freely independent from $\{a_2, a_4\}$, regardless of the particular elements a_i, we have

$$E(a_1a_2a_3a_4) = E(a_1E(a_2)a_3)E(a_4) + E(a_1)E(a_2E(a_3)a_4) - E(a_1)E(a_2)E(a_3)E(a_4).$$

One attempt to describe the types of moments that appear on the right hand side of the above calculation by looking at the collection of indices that appear in the same parentheses at the same "level". For example, in the first piece, $E(a_1E(a_2)a_3)E(a_4)$, we might say that the variables with indices 1 and 3 appear at the same level, where as the variables with indices 2 and 4 are isolated. We could record this as three separate *blocks* $\{1, 3\}, \{2\}$, and $\{4\}$. For the second piece, we might describe the index blocks as $\{1\}, \{3\}$, and $\{2, 4\}$. The last piece has all separated pieces, so all of the singleton blocks.

We might take this a step further and use diagrams to depict this connection. To make such a diagram, one connects indices that are part of the same *block*. See Remark and Notation 1.3.6 for the diagrams that appear in our example.

In fact, each type of independence is associated with its own universal moment patterns similar to above. Exploiting these patterns yields one of non-commutative probability's most potent tools, known as the *cumulants*.

The moment calculations for the tensor and Boolean independence are more direct, so they are less obviously interesting. But we will develop this theory for these three independences in parallel to highlight the equivalence of this kind of thinking with the original definition.

Remark 1.3.1. Monotone independence moment calculations are not universal in the same way as the tensor, free, or Boolean independence. Indeed, their pattern depends on which algebra the variables come from. For example, if A_2 is monotonically independent from A_1 and $x, y \in A_1$ and $z \in A_2$, then

$$E(xzy) = E(xE(z)y).$$

But if instead $x, y \in A_2$ and $z \in A_1$, then

$$E(xzy) = E(x)E(z)E(y).$$

We end up with different universal moment patterns simply because of algebras that the variables come from. For this reason, we will omit discussion of the monotone independence in this section. Though it is interesting that Hasebe and Saigo have attempted to describe the monotone cumulants in an analogous way. The reader is referred to [24] and [23] for more on this topic.

1.3.1 Partition Lattices and Möbius Inversion

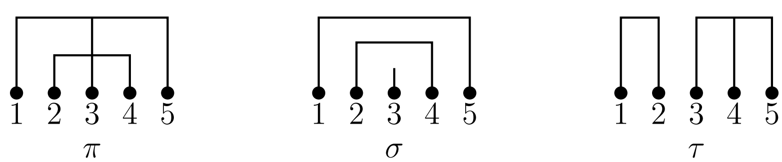
Definition 1.3.2. A *partition* of $S = \{1, \ldots, n\}$ is a collection $\pi = \{V_1, \ldots, V_k\}$ of non-empty subsets of S such that $V_i \cap V_j = \emptyset$ for all $i \neq j$, and $S = V_1 \cup \cdots \cup V_k$. Let $\mathcal{P}(n)$ denote the collection of partitions of $\{1, \ldots, n\}$. The elements V_i of π are called *blocks*. Let $\sim_\pi$ be the equivalence relation on S defined by π. That is, $i \sim_\pi j$ if i and j belong to the same block in π.

Say a partition π is *non-crossing* if whenever $1 \leq i_1 < j_1 < i_2 < j_2 \leq n$ with $i_1 \sim_\pi i_2$ and $j_1 \sim_\pi j_2$, then $i_1 \sim_\pi j_1$ (that is, they must all be part of the same block). Let $NC(n)$ denote the collection of non-crossing partitions of $\{1, \ldots, n\}$. π is an *interval partition* if every block of π is of the form $\{i, i+1, \ldots, i+j\}$ (that is, every block is an *interval*). Let $\mathcal{I}(n)$ denote the collection of interval partitions of $\{1, \ldots, n\}$.

Remark 1.3.3. It is easy to see straight from the definition that $\mathcal{I}(n) \subseteq NC(n) \subseteq \mathcal{P}(n)$.

Remark 1.3.4. To each partition there corresponds a diagram, that can be easier to work with visually. Some arguments and calculations that follow heavily draw from a more visual reasoning about partitions. The idea is simply to draw a node for every number (or index), and connect those numbers which are part of the same block with a line. Non-crossing partitions can be represented in a way that their lines do not cross.

For example, suppose $n = 5$, $\pi = \{\{1,3,5\},\{2,4\}\}$, $\sigma = \{\{1,5\},\{2,4\},\{3\}\}$, and $\tau = \{\{1,2\},\{3,4,5\}\}$. The corresponding partition diagrams may be depicted as



Then π is a partition which is *not* non-crossing, σ is a non-crossing partition but not an interval partition, and τ is an interval partition.

Definition 1.3.5. If $\pi, \sigma \in \mathcal{P}(n)$, say that $\pi \leq \sigma$ if every block of π is contained in a block of σ.

For each such $\sigma \leq \pi$ define

$$[\sigma, \pi] = \{\tau \in \mathcal{P}(n) : \sigma \leq \tau \leq \pi\}.$$

Remark and Notation 1.3.6. For $n \in \mathbb{N}$, the partition $\mathbf{1_n} = \{\{1, \ldots, n\}\}$, is the unique maximal element of $\mathcal{P}(n)$, and the element $\mathbf{0}_n = \{\{i\} : 1 \leq i \leq n\}$ is the unique minimal element of $\mathcal{P}(n)$.

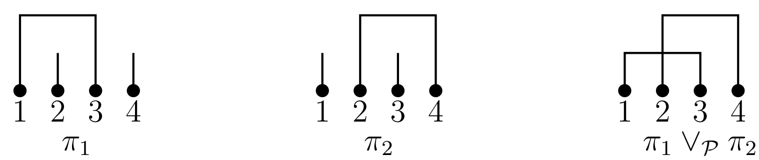
For any $\pi_1, \pi_2 \in \mathcal{P}(n)$, there is a unique largest partition $\pi_1 \wedge \pi_2 \in \mathcal{P}(n)$ with $\pi_1 \wedge \pi_2 \leq \pi_1, \pi_2$. If $\pi_1 = \{V_i\}_{i \in I}$ and $\pi_2 = \{W_j\}_{j \in J}$ then

$$\pi_1 \wedge \pi_2 = \{V_i \cap W_j : i \in I, j \in J, V_i \cap W_j \neq \emptyset\}.$$

It is also true that if $\pi_1, \pi_2 \in \mathcal{P}(n)$, there is a unique smallest partition $\pi_1 \vee_{\mathcal{P}} \pi_2$ with $\pi_1, \pi_2 \leq \pi_1 \vee_{\mathcal{P}} \pi_2$. This is defined by unioning blocks of π_1 and π_2 whenever they are not disjoint. For example, if $\pi_1 = \{\{1, 3\}, \{2\}, \{4\}\}$ and $\pi_2 = \{\{1\}, \{3\}, \{2, 4\}\}$, then

$$\pi_1 \vee_{\mathcal{P}} \pi_2 = \{\{1, 3\}, \{2, 4\}\}.$$

The corresponding diagrams are



One observes from the above example that $\pi_1, \pi_2 \in NC(4)$, but $\pi_1 \vee_{\mathcal{P}} \pi_2 \notin NC(4)$. It turns out that $NC(n)$ has its own join, $\vee_{NC}$ that remedies this defect, which can be seen as follows. If $\pi \in \mathcal{P}(n)$, define its *non-crossing closure*, $\overline{\pi}^{NC}$ to be that (non-crossing) partition which joins any blocks that cross in π (c.f. [4]). Then for $\pi_1, \pi_2 \in NC(n)$, define

$$\pi_1 \vee_{NC} \pi_2 := \overline{\pi_1 \vee_{\mathcal{P}} \pi_2}^{NC}.$$

For example, with π_1, π_2 as above, we have $\pi_1 \vee_{NC} \pi_2 = \mathbf{1}_4$.

These operations of join and meet, when restricted to $\mathcal{I}(n)$ are the same. So each of $\mathcal{P}(n)$, $NC(n)$, and $\mathcal{I}(n)$ are *lattices*.

The lattice of non-crossing partitions, introduced by Kreweras in [4], has a very rich combinatorial theory, which we take from only sparingly.

Inspired by Rota (c.f. [16]), one investigates certain functions on these partition lattices that respect the structure of the underlying partitions. This investigation yields a *Möbius inversion* which will allow us to formalize our moment-cumulant calculus.

In the following definition, let $L \in \{\mathcal{P}, NC, \mathcal{I}\}$.

Definition 1.3.7. Let $\{L(n)\}_{n\geq 1}$ be a family such that for each $n \geq 1$, $L(n)$ is a lattice of partitions of $\{1, \ldots, n\}$.

The L-Möbius function is the function

$$\mu_L : \bigcup_{n\geq 1} L(n) \times L(n) \to \mathbb{C}$$

defined by

$$\sum_{\substack{\tau \in L(n) \\ \pi \leq \tau \leq \sigma}} \mu_L(\tau, \sigma) = \sum_{\substack{\tau \in L(n) \\ \pi \leq \tau \leq \sigma}} \mu_L(\pi, \tau) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{otherwise} \end{cases}.$$

Example 1.3.8. Another familiar example of a lattice is the Boolean lattice of subsets of $\{1, \ldots, n\}$. Since the notation $\mathcal{P}(n)$ is already taken, we adopt the unconventional notation of $\mathcal{S}(n)$ to denote the set of all subsets of $\{1, \ldots, n\}$. Then for $S, T \in \mathcal{S}(n)$, we have

$$S \vee T = S \cup T \quad \text{and} \quad S \wedge T = S \cap T.$$

This implies that $\mathcal{S}$ is equipped with a Möbius function as above. In fact, it is relatively simple to calculate its values. If $S \subseteq T \subseteq \{1, \ldots, n\}$ then

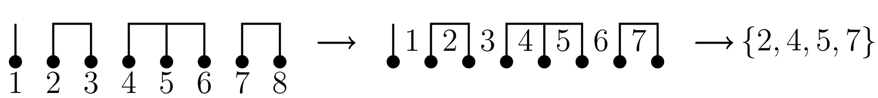
$$\mu_{\mathcal{S}}(S, T) = (-1)^{|T|-|S|}.$$

To see why, simply note that the interval $[S, T] = \{U : S \subseteq U \subseteq T\}$ is order isomorphic to the interval $[\emptyset, T \setminus S]$, and

$$0 = \sum_{U \subseteq T \setminus S} \mu_{\mathcal{S}}(U, T\setminus S) = \mu_{\mathcal{S}}(\emptyset, T\setminus S) + \sum_{i=1}^{|T\setminus S|} \binom{|T \setminus S|}{i} (-1)^{|T\setminus S| - i} = \mu_{\mathcal{S}}(\emptyset, T\setminus S) + (0 - (-1)^{|T\setminus S|}).$$

Inductively one verifies the expression for the Möbius function.

It turns out that the Boolean lattice of subsets is naturally order-isomorphic to the interval partitions on $\{1, \ldots, n+1\}$. Indeed, for an interval partition $\pi \in \mathcal{I}(n+1)$, let $S_\pi = \{i : i \sim_\pi i+1\}$. Then $\sigma \leq \pi$ implies $S_\sigma \subseteq S_\pi$. This is an isomorphism, which is depicted below, for example when $\pi = \{\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8\}\}$.



Under this isomorphism, we have that $i \in S_\pi^c$ if and only if $i \nsim_\pi i+1$, so that i is the largest element of a block of π. This implies that $|\pi| = |S_\pi^c| + 1$, which yields

$$|\pi| = |S_\pi^c| + 1 = (n+1) - |S_\pi|$$

and thus,

$$\mu_{\mathcal{I}}(\sigma, \pi) = \mu_{\mathcal{S}}(S_\sigma, S_\pi) = (-1)^{|S_\sigma| - |S_\pi|} = (-1)^{(n+1-|S_\pi|)-(n+1-|S_\sigma|)} = (-1)^{|\pi|-|\sigma|}.$$

We will use this observation to work with the Boolean cumulants shortly.

Notation 1.3.9. Note that we can define any of our partitions on any finite totally ordered set S. Indeed, one uses the order-isomorphism $\{1, \ldots, |S|\} \to S$ to import partitions onto S. Let $\mathcal{P}(S)$, $NC(S)$, $\mathcal{I}(S)$ be the images of the partitions under this isomorphism.

With this observation, if $\pi \in \mathcal{P}(n)$ and $V \in \pi$, then we will consider $V^c = \{1, \ldots, n\} \setminus V$, and

$$\pi|_{V^c} = \pi \setminus \{V\}.$$

It is a simple observation that if $\pi \in NC(n)$, then $\pi|_{V^c} \in NC(V^c)$.

For the remainder of this work, we will freely interchange between $\mathcal{P}(S)$ and $\mathcal{P}(|S|)$ (and similarly $NC(S)$ and $NC(|S|)$, etc.) where convenient.

One of the most useful structural properties of the non-crossing partitions is a recursive one, which will allow us to define our moment functions shortly.

Remark 1.3.10. If $\pi \in NC(n)$, then π must have an *interval block* V (i.e. $V = \{i, i+1, \ldots, i+\ell\}$). This can be seen by first selecting any block $W \in \pi$. If W is not an interval, then there exists $i, j \in W$ such that $i+1 < j$ and $k \notin W$ for any $i < k < j$. Then $\pi|_{\{i+1,\ldots,j-1\}}$ must be a non-crossing partition, and we can repeat the process to reach an interval block. This leads to an alternative, equivalent characterization of non-crossing partitions that π is non-crossing if and only if there is an interval block, $V \in \pi$, and $\pi|_{V^c}$ is non-crossing.

The discrepancy between the join operations on $\mathcal{P}(n)$ and $NC(n)$ leaves some questions about how these two lattices are related. The following shows that there is a certain *density* of $NC(n)$ in $\mathcal{P}(n)$. We will make use of this fact in a moment calculus later on.

Proposition 1.3.11. *Let $\pi \in \mathcal{P}(n)$, and let $\{\tau_1, \ldots, \tau_k\} \subseteq NC(n)$ be the collection of maximal elements of $[\mathbf{0}_n, \pi] \cap NC(n)$. Then*

$$\pi = \tau_1 \vee_{\mathcal{P}} \tau_2 \vee_{\mathcal{P}} \cdots \vee_{\mathcal{P}} \tau_k.$$

Proof. With the setup as above, let $\tau = \tau_1 \vee_{\mathcal{P}} \tau_2 \vee_{\mathcal{P}} \cdots \vee_{\mathcal{P}} \tau_k$ and suppose for contradiction that $\pi > \tau$.

Then by definition of the order, there must be some indices $i \neq j$ such that $i \sim_\pi j$ but $i \nsim_\tau j$. By definition of $\vee_{\mathcal{P}}$, it follows that $i \nsim_{\tau_m} j$ for any $1 \leq m \leq k$

But if $\sigma = \{\{k\} : k \neq i, j\} \cup \{\{i, j\}\}$, then $\sigma \in [\mathbf{0}_n, \pi] \cap NC(n)$, so there must be an element $\tau^* \geq \sigma$ which is maximal in $[\mathbf{0}_n, \pi] \cap NC(n)$. So there must be some $1 \leq m \leq k$ such that $\tau^* = \tau_m$. But this is a contradiction, as $i \sim_{\tau^*} j$ but $i \nsim_{\tau_m} j$. $\square$

It is not too hard to see that an analogous result about interval partitions is not possible. For example, consider $n = 3$, and $\pi = \{\{1, 3\}, \{2\}\}$. Then π cannot be seen as a join of interval partitions.

The interval partitions are somehow too sparse to retrieve the richer structure of $NC(n)$ or $\mathcal{P}(n)$.

1.3.2 Moment and Cumulant Functions

Now that we have the basic combinatorial framework set up, we look to connect this with our moment calculations.

Definition 1.3.12. For each $\pi \in NC(n)$, $a_1, \ldots, a_n \in \mathcal{A}$, define $E_{NC,\pi}(a_1, \ldots, a_n) \in B$ recursively as follows. Of the interval blocks of π, let V be the one with the highest indices (that is, if W is also an interval block of π then $i \in W, j \in V$ implies $i < j$). Then define

$$E_{NC,\pi}(a_1, \ldots, a_n) := E_{NC,\pi|_{V^c}}((a_1, \ldots, a_{i-1}E(a_i \cdots a_j), a_{j+1}, \ldots, a_n)|_{V^c}).$$

Definition 1.3.13. The *non-crossing moment function on* $\mathcal{A}$ is the map

$$\mathcal{E}_{NC} : \bigcup_{n \geq 1} (NC(n) \times \mathcal{A}^n) \to B$$

defined by

$$\mathcal{E}_{NC,\pi}(a_1, \ldots, a_n) = E_{NC,\pi}(a_1, \ldots, a_n).$$

Remark 1.3.14. Since $E_{\mathbf{1}_n}(a_1, \ldots, a_n) = E(a_1 \cdots a_n)$, which is n-linear (by linearity of E), it follows that for each $\pi \in NC(n)$ that $\mathcal{E}_{NC,\pi} : \mathcal{A}^n \to B$ is n-linear.

Notation 1.3.15. The recursive nature of the non-crossing partitions, as in Remark 1.3.10, allows the above definition. Computing moments in this way for general partitions $\pi \in \mathcal{P}(n)$ fails to be coherent. However, in the case when B is in the center of $\mathcal{A}$, we may simply define

$$E_{\mathcal{P},\pi}(a_1, \ldots, a_n) := \prod_{V \in \pi} E(\overrightarrow{\prod_{i \in V}} a_i).$$

In this setting, $E_{\mathcal{P}}|_{NC(n)} = E_{NC}$.

Thus we define the cumulants via Möbius inversion.

Definition 1.3.16. The *free cumulant function on $\mathcal{A}$* is the map

$$\kappa_* : \bigcup_{n \geq 1} (NC(n) \times \mathcal{A}^n) \to B$$

defined by

$$\kappa_{*,\pi}(a_1, \ldots, a_n) := \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi}} \mathcal{E}_{NC,\pi}(a_1, \ldots, a_n) \mu_{NC}(\sigma, \pi).$$

Definition 1.3.17. The *Boolean cumulant function on $\mathcal{A}$* is the map

$$\kappa_{\uplus} : \bigcup_{n \geq 1} (\mathcal{I}(n) \times \mathcal{A}^n) \to B$$

defined by

$$\kappa_{\uplus,\pi}(a_1, \ldots, a_n) := \sum_{\substack{\sigma \in \mathcal{I}(n) \\ \sigma \leq \pi}} \mathcal{E}_{NC,\pi}(a_1, \ldots, a_n) \mu_{\mathcal{I}}(\sigma, \pi).$$

Definition 1.3.18. When B is in the center of $\mathcal{A}$, the *tensor cumulant function on $\mathcal{A}$* is the map

$$\kappa_{\otimes} : \bigcup_{n \geq 1} (\mathcal{P}(n) \times \mathcal{A}^n) \to B$$

defined by

$$\kappa_{\otimes,\pi}(a_1, \ldots, a_n) := \sum_{\substack{\sigma \in \mathcal{P}(n) \\ \sigma \leq \pi}} E_{\mathcal{P},\pi}(a_1, \ldots, a_n) \mu_{\mathcal{P}}(\sigma, \pi).$$

Remark 1.3.19. For each of the above cumulant functions, it follows from the definition of the corresponding Möbius functions (and taking B to be in the center of $\mathcal{A}$ for the tensor cumulants) that

$$\mathcal{E}_\pi(a_1, \ldots, a_n) = \sum_{\substack{\sigma \in L(n) \\ \sigma \leq \pi}} \kappa_\sigma(a_1, \ldots, a_n).$$

Thus we arrive at the cumulants by taking a Möbius inversion of the moment functions (by the definition of the cumulants), and from the cumulants we arrive at the moments by summing the cumulants. This gives a kind of equivalence between these two functions.

Remark 1.3.20. We took effort to separate different types of cumulants, but from the remark above, it is not clear that they provide any different information. For example, if we want to evaluate the moment $E(a_1 \cdots a_n)$, in principle we could have used any of the cumulants to do so. However, as we will see next, each of the tensor, free, and Boolean independences are intimately linked with the tensor, free, and Boolean cumulants defined above.

1.3.3 Independence via Cumulants

In this subsection we state some results that independence is equivalent to vanishing of mixed cumulants of the associated type. This not only provides an equivalent definition of the three independences, but also sets us up, at least in the case of free independence, with a powerful calculus with which we can compute moments of independent variables.

We will take a moment to prove the case for the Boolean independence, as exemplar of the three. The first two lemmas demonstrate a certain *block-multiplicativity* that the cumulants have. This property is shared, in an appropriate form, by each of the cumulants with respect to their corresponding partition lattices.

Lemma 1.3.21. *Let $\sigma, \pi \in \mathcal{I}(n)$ with $\sigma \leq \pi$. Let $V_1, \ldots, V_k$ be the blocks of π in order (i.e. if $i < j$ then elements of V_i are less than those in V_j). Then*

$$\mu_{\mathcal{I}}(\sigma, \pi) = \prod_{i=1}^{k} \mu_{\mathcal{I}}(\sigma|_{V_i}, \pi|_{V_i}).$$

Proof. Since $\sigma \leq \pi$, we have

$$|\sigma| = \sum_{i=1}^{k} \left|\sigma|_{V_i}\right|.$$

Then

$$|\pi| - |\sigma| = \sum_{i=1}^{k} 1 - \left|\sigma|_{V_i}\right|,$$

so that, by the work done in Example 1.3.8, and since $\pi|_{V_i} = \mathbf{1}_{|V_i|}$,

$$\mu_{\mathcal{I}}(\sigma, \pi) = (-1)^{|\pi| - |\sigma|} = \prod_{i=1}^{k} (-1)^{1 - \left|\sigma|_{V_i}\right|} = \prod_{i=1}^{k} \mu_{\mathcal{I}}(\sigma|_{V_i}, \pi|_{V_i}).$$ □

Lemma 1.3.22. *If $\pi \in I(n)$ and $V_1, \cdots, V_k$ are the blocks of π in order, then*

$$\kappa_{\uplus,\pi}(a_1, \cdots, a_n) = \kappa_{\uplus,\pi|_{V_1}}((a_1, \cdots, a_n)|_{V_1}) \cdots \kappa_{\uplus,\pi|_{V_k}}((a_1, \cdots, a_n)|_{V_k})$$

Proof. First note that if $\sigma \in \mathcal{I}(n)$ with blocks $W_1, \ldots, W_m$ in order, then by repeatedly applying the definition of $E_{NC,\sigma}$,

$$E_{NC,\sigma}(a_1, \ldots, a_n) = E_{NC,\sigma|_{W_1}}((a_1, \ldots, a_n)|_{W_1}) \cdots E_{NC,\sigma|_{W_m}}((a_1, \ldots, a_n)|_{W_m}).$$

Then

$$\begin{aligned}
\kappa_{\uplus,\pi}(a_1, \ldots, a_n) &= \sum_{\sigma \leq \pi} E_{NC,\sigma}(a_1, \ldots, a_n) \\
&= \sum_{\sigma \leq \pi} E_{NC,\sigma|_{V_1}}((a_1, \ldots, a_n)|_{V_1}) \cdots E_{NC,\sigma|_{V_k}}((a_1, \ldots, a_n)|_{V_k}) \\
&= \sum_{\substack{\sigma_1 \leq V_1, \ldots, \sigma_k \leq V_k \\ \sigma_i \in \mathcal{I}(V_i)}} E_{NC,\sigma_1}((a_1, \ldots, a_n)|_{V_1}) \cdots E_{NC,\sigma_k}((a_1, \ldots, a_n)|_{V_k}) \\
&= \left(\sum_{\substack{\sigma_1 \leq \mathbf{1}_{V_1} \\ \sigma_1 \in \mathcal{I}(V_1)}} E_{NC,\sigma_1}((a_1, \ldots, a_n)|_{V_1}) \right) \cdots \left(\sum_{\substack{\sigma_k \leq \mathbf{1}_{V_k} \\ \sigma_k \in \mathcal{I}(V_k)}} E_{NC,\sigma_k}((a_1, \ldots, a_n)|_{V_k}) \right) \\
&= \kappa_{\uplus,\pi|_{V_1}}((a_1, \cdots, a_n)|_{V_1}) \cdots \kappa_{\uplus,\pi|_{V_k}}((a_1, \cdots, a_n)|_{V_k})
\end{aligned}$$

□

Proposition 1.3.23 (Vanishing Mixed Cumulants for Boolean Independence)**.** *A family $\{A_k\}_{k \in K}$ of unital B-subalgebras in a B-non-commutative probability space $(\mathcal{A}, E)$ is Boolean independent if and only if for all $n \in \mathbb{N}$, $\epsilon : \{1, \ldots, n\} \to K$ and $a_i \in A_{\epsilon(i)}$, we have*

$$\kappa_{\uplus,\mathbf{1}_n}(a_1, \ldots, a_n) = 0,$$

unless ϵ is constant.

Proof. First suppose that mixed cumulants vanish. Let $n \geq 1$ and $\epsilon : \{1, \ldots, n\} \to K$ such that $\epsilon(i) \neq \epsilon(i+1)$, and $a_i \in A_{\epsilon(i)}$. Then the only interval partition finer than $\ker \epsilon$ is $\mathbf{0}_n$.

This implies

$$\begin{aligned}
E(a_1 \cdots a_n) &= \sum_{\pi \in \mathcal{I}(n)} \kappa_{\uplus,\pi}(a_1, \dots, a_n) \\
&= \sum_{\substack{\pi \leq \ker \epsilon \\ \pi \in \mathcal{I}(n)}} \kappa_{\uplus,\pi}(a_1, \dots, a_n) \\
&= \kappa_{\uplus,\mathbf{0}_n}(a_1, \dots, a_n) \\
&= E_{NC,\mathbf{0}_n}(a_1, \dots, a_n) \\
&= E(a_1) \cdots E(a_n).
\end{aligned}$$

Conversely, suppose the family $\{A_k\}_{k \in K}$ is Boolean independent. We proceed by induction on n. In the case when $n = 1$, there is nothing to show. Now suppose that for all $1 \leq m \leq n-1$, non-constant $\epsilon : \{1, \dots, m\} \to K$, and $a_i \in A_{\epsilon(i)}$, we have

$$\kappa_{\uplus,\mathbf{1}_k}(a_1, \dots, a_k) = 0.$$

Now let $\epsilon : \{1, \dots, n\} \to K$ be non-constant and $a_i \in A_{\epsilon(i)}$. There is a unique largest interval partition $\tau \leq \ker \epsilon$, obtained by combining adjacent indices of the same $\ker \epsilon$-block.

Next, we see that by Boolean independence and the moment-cumulant formulas, we have

$$E(a_1 \cdots a_n) = E_{\tau,NC}(a_1, \dots, a_n) = \sum_{\substack{\sigma \in \mathcal{I}(n) \\ \sigma \leq \tau}} \kappa_{\uplus,\sigma}(a_1, \dots, a_n) \tag{1.1}$$

However, if we invoke the moment-cumulant formula directly, we get

$$E(a_1 \cdots a_n) = \sum_{\sigma \in \mathcal{I}(n)} \kappa_{\uplus,\sigma}(a_1, \dots, a_n). \tag{1.2}$$

If $\sigma \in \mathcal{I}(n)$ with $\sigma \neq \mathbf{1}_n$ and $\sigma \not\leq \ker \epsilon$, then there is some block $V \in \sigma$ such that $\epsilon|_V$ is non constant. Lemma 1.3.22 along with our induction hypothesis guarantee that therefore $\kappa_{\uplus,\sigma}(a_1, \dots, a_n) = 0$. Applying this to equation 1.2 yields

$$E(a_1 \cdots a_n) = \kappa_{\uplus,\mathbf{1}_n}(a_1, \dots, a_n) + \sum_{\substack{\sigma \in \mathcal{I}(n) \\ \sigma \leq \ker \epsilon}} \kappa_{\uplus,\sigma}(a_1, \dots, a_n)$$

Combining this with equation 1.1 yields

$$E(a_1 \cdots a_n) = \kappa_{\uplus,\mathbf{1}_n}(a_1, \dots, a_n) + E(a_1 \cdots a_n).$$

So $\kappa_{\uplus,\mathbf{1}_n}(a_1, \dots, a_n) = 0$, and by induction, we get our vanishing of mixed cumulants. $\square$

With similar arguments, making note that in the tensor case we have commutativity of B and between independent variables, we get the following.

Proposition 1.3.24 (Vanishing Mixed Cumulants for Tensor Independence)**.** *A family $\{A_k\}_{k\in K}$ of unital B-subalgebras in a B-non-commutative probability space $(\mathcal{A}, E)$ is tensor independent if and only if for all $n \in \mathbb{N}$, $\epsilon : \{1, \ldots, n\} \to K$ and $a_i \in A_{\epsilon(i)}$, we have*

$$\kappa_{\otimes,\mathbf{1}_n}(a_1, \ldots, a_n) = 0$$

unless ϵ is constant.

And lastly we have the analogous result for the free independence.

Proposition 1.3.25 (Vanishing Mixed Cumulants for Free Independence)**.** *A family $\{A_k\}_{k\in K}$ of unital B-subalgebras in a B-non-commutative probability space $(\mathcal{A}, E)$ is freely independent if and only if for all $n \in \mathbb{N}$, $\epsilon : \{1, \ldots, n\} \to K$ such that $a_i \in A_{\epsilon(i)}$, we have*

$$\kappa_{*,\mathbf{1}_n}(a_1, \ldots, a_n) = 0$$

unless ϵ is constant.

Remark 1.3.26. The moment conditions in the definitions of the tensor and Boolean independences are direct enough to show why the mixed cumulants vanish, as we saw in the Boolean case above. However, the moment conditions for free independence are less direct. While the proof still follows the same inductive idea on the moment condition, there are some more careful details that go into the argument.

For a nice treatment in the case where $B = \mathbb{C}$, the reader is referred to [14]. For the general case, see [20].

Corollary 1.3.27. *Proposition 1.2.14 holds for free independence as well.*

Remark 1.3.28. Example 1.3.8 showed that the collection of interval partitions on $\{1, \ldots, n+1\}$ is order-isomorphic to the Boolean algebra of subsets of $\{1, \ldots, n\}$. Thus Boolean independence was named for its characterization via this Boolean lattice of partitions.

The vanishing of mixed cumulants for independence results from above translate to convenient tools for calculating moments. A simple example of this is the following.

Corollary 1.3.29. *Let $\{A_k\}_{k\in K}$ be a family of freely independent (respectively Boolean independent, tensor independent) B-algebras in a B-non-commutative probability space $(\mathcal{A}, E)$. For any $n \in \mathbb{N}$, $\epsilon : \{1, \ldots, n\} \to K$, and $a_i \in A_{\epsilon(i)}$ for each $1 \leq i \leq n$, if $\ker \epsilon \in NC(n)$ (respectively $\ker \epsilon \in \mathcal{I}(n)$, $\ker \epsilon \in \mathcal{P}(n)$) then*

$$E(a_1 \cdots a_n) = E_{\ker \epsilon}(a_1, \cdots a_n).$$

Proof.

$$\begin{aligned} E(a_1 \cdots a_n) &= E_{\mathbf{1}_n}(a_1, \cdots, a_n) \\ &= \sum_{\pi \in L(n)} \kappa_\pi(a_1, \cdots, a_n) \\ &= \sum_{\substack{\pi \in L(n) \\ \pi \leq \ker \epsilon}} \kappa_\pi(a_1, \ldots, a_n) \qquad \text{(vanishing mixed cumulants)} \\ &= E_{\ker \epsilon}(a_1, \cdots, a_n). \end{aligned}$$

□

One finds in cumulants a very convenient tool for moment calculations. We saw above in Remark 1.2.12, that computing the moments of freely independent variables could be quite tedious if one goes directly for the definition. However, the use of cumulants can make many moment calculations much more tractable.

The following result demonstrates how the use of cumulants can give us a more direct understanding of the moment definition of free independence. We generalize this into our multi-algebra setting in subsequent chapters.

Proposition 1.3.30 (Cumulant Inclusion-Exclusion)**.** *Let $\{A_k\}_{k\in K}$ be a freely independent family of unital B-algebras in a B-non-commutative probability space $(\mathcal{A}, E)$. Let $n \in \mathbb{N}$, $\epsilon : \{1, \ldots, n\} \to K$ and $a_i \in A_{\epsilon(i)}$. Let $M \subseteq NC(n)$ be the set of distinct maximal elements of $[\mathbf{0}_n, \ker \epsilon] \cap NC(n)$. Then*

$$E(a_1 \cdots a_n) = \sum_{m=1}^{|M|} (-1)^{m+1} \sum_{\{\tau_1, \ldots, \tau_m\} \subseteq M} \mathcal{E}_{NC, \tau_1 \wedge \cdots \wedge \tau_m}(a_1, \ldots, a_n).$$

Proof. First since mixed cumulants vanish (Proposition 1.3.25),

$$E(a_1 \cdots a_n) = E_{\mathbf{1}_n}(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \kappa_{*,\pi}(a_1, \ldots, a_n) = \sum_{\substack{\pi \in NC(n) \\ \pi \leq \ker \epsilon}} \kappa_{*,\pi}(a_1, \ldots, a_n).$$

Next for each $\tau \in NC(n)$, let $S_\tau = [0, \tau] \cap NC(n)$. Note that $S_\tau \cap S_\sigma = S_{\tau\wedge\sigma}$. Let $U = \bigcup_{\tau\in M} S_\tau$. By Proposition 1.3.11, we have that $U = \{\pi \in NC(n) : \pi \leq \ker \epsilon\}$. Let $M = \{\tau_1, \ldots, \tau_{|M|}\}$. Define for each $I \subsetneq M$,

$$g(I) = \sum_{\pi\in\bigcap_{\tau\in I^c} S_\tau} \kappa_{*,\pi}(a_1, \ldots, a_n) = \sum_{\substack{\pi\in NC(n)\\ \pi\leq\bigwedge_{\tau\in I^c}\tau}} \kappa_{*,\pi}(a_1, \ldots, a_n) = E_{NC,\bigwedge_{\tau\in I^c}\tau}(a_1, \ldots, a_n),$$

and

$$f(I) = \sum_{\pi\in\left(\bigcap_{\tau\in I^c} S_\tau\right)\setminus\left(\bigcup_{\tau\in I} S_\tau\right)} \kappa_{*,\pi}(a_1, \ldots, a_n).$$

Note that $f(M) = 0$ and

$$g(M) = \sum_{\pi\in U} \kappa_{*,\pi}(a_1, \ldots, a_n) = \sum_{\substack{\pi\in NC(n)\\ \pi\leq\ker\epsilon}} \kappa_{*,\pi}(a_1, \ldots, a_n) = E(a_1 \cdots a_n).$$

Then for $I \subseteq M$, we have

$$\begin{aligned}\bigcup_{J\subseteq I}\left(\bigcap_{\tau\in J^c} S_\tau \setminus \bigcup_{\tau\in J} S_\tau\right) &= \bigcap_{\tau\in I^c} S_\tau \cap \left(\bigcup_{J\subseteq I}\left(\bigcap_{\tau\in I\setminus J} S_\tau \setminus \bigcup_{\tau\in J} S_\tau\right)\right)\\ &= \bigcap_{\tau\in I^c} S_\tau \cap U = \bigcap_{\tau\in I^c} S_\tau\end{aligned}$$

so that

$$g(I) = \sum_{J\subseteq I} f(J),$$

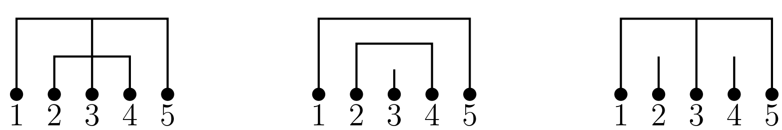
and thus by inclusion-exclusion (i.e. Möbius inversion on the Boolean lattice of subsets of

M), we have

$$\begin{aligned}
E(a_1 \cdots a_n) &= g(M) \\
&= \sum_{I \subseteq M} f(I) \\
&= \sum_{I \subsetneq M} f(I) && f(M) = 0 \\
&= \sum_{I \subsetneq M} \sum_{J \subseteq I} (-1)^{|I|-|J|} g(J) \\
&= \sum_{J \subsetneq M} \left(\sum_{J \subseteq I \subsetneq M} (-1)^{|I|-|J|} \right) g(J) \\
&= \sum_{J \subsetneq M} \left(\left(\sum_{J \subseteq I \subseteq M} (-1)^{|I|-|J|} \right) - (-1)^{|M|-|J|} \right) g(J) \\
&= \sum_{J \subsetneq M} (-1)^{|M|-|J|+1} g(J) && \sum_{J \subseteq I \subseteq M} (-1)^{|I|-|J|} = 0 \\
&= \sum_{m=1}^{|M|} (-1)^{m+1} \sum_{\{\tau_1, \ldots, \tau_m\} \subseteq M} E_{NC, \tau_1 \wedge \cdots \wedge \tau_m}(a_1, \ldots, a_n). && \square
\end{aligned}$$

Example 1.3.31. To demonstrate the power of the above proposition, we compute a couple of examples.

Let x and y be freely independent. Then the word $xyxyx$ has kernel $\pi = \{\{1,3,5\},\{2,4\}\}$. From inspecting the diagrams (below), one can see that the two maximal non-crossing partitions below π are $\sigma = \{\{1,5\},\{2,4\},\{3\}\}$ and $\tau = \{\{1,3,5\},\{2\},\{4\}\}$ which can be found visually by "cutting" the crossing in the two different ways.



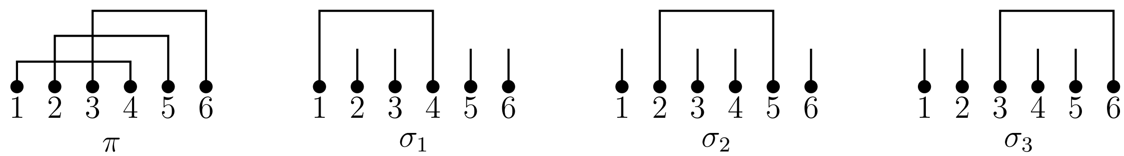
π σ τ

Then our calculus yields

$$\begin{aligned}
E(xyxyx) &= E_\sigma(x,y,x,y,x) + E_\tau(x,y,x,y,x) - E_{\sigma \wedge \tau}(x,y,x,y,x) \\
&= E(xE(yE(x)y)x) + E(xE(y)xE(y)x) - E(xE(y)E(x)E(y)x)
\end{aligned}$$

which would have been a pain to compute via centering.

Example 1.3.32. As a slightly more complicated example, suppose that x, y, z are mutually freely independent. The word $xyzxyz$ comes with kernel partition $\pi = \{\{1,4\},\{2,5\},\{3,6\}\}$. The diagrams show us that the maximal non-crossing partitions that lie below π are $\sigma_1 = \{\{1,4\},\{2\},\{3\},\{5\},\{6\}\}$, $\sigma_2 = \{\{2,5\},\{1\},\{3\},\{4\},\{6\}\}$ and $\sigma_3 = \{\{3,6\},\{1\},\{2\},\{4\},\{5\}\}$:



Note that $\sigma_i \wedge \sigma_j = \mathbf{0}_6$, so our calculus yields,

$$\begin{aligned}E(xyzxyz) &= E_{\sigma_1}(x,y,z,x,y,z) + E_{\sigma_2}(x,y,z,x,y,z) + E_{\sigma_3}(x,y,z,x,y,z) - 2E_{\mathbf{0}_6}(x,y,z,x,y,z)\\ &= E(xE(y)E(z)x)E(y)E(z) + E(x)E(yE(z)E(x)y)E(z) + E(x)E(y)E(zE(x)E(y)z)\\ &\quad - 2E(x)E(y)E(z)E(x)E(y)E(z).\end{aligned}$$

A calculation powered by diagrams, rather than by a tedious algebra of centering.

We will be able to make use of this diagram calculus in the next chapter.

1.4 Operator Actions on Special Product Spaces

Since subalgebras come with faithful representations as in Proposition 1.1.13, it is natural that the represented independent algebras themselves should exhibit the same notion of independence. But the issue is that for an independent family of algebras $\{A_k\}_{k\in K}$, each A_k may be represented on a different B-B-bimodule $\mathcal{X}_k$. This makes it difficult to discuss the notion of a joint distribution of the family $\{\delta_k(A_k)\}_{k\in K}$, as there is no way of multiplying elements together, and hence no way to make sense of mixed moments.

To be able to compare the representations then, we need a common space on which to represent the family.

Remark and Notation 1.4.1. Let $\Gamma = \{A_k\}_{k\in K}$ be a family of B-subalgebras in a B-non-commutative probability space $(\mathcal{A}, E)$, and $\delta = \{\delta_k\}_{k\in K}$ a corresponding family of representations with $\delta_k : A_k \to \mathcal{L}(\mathcal{X}_k)$. If $\gamma = \{\gamma_k\}_{k\in K}$ is a representation of the family $\{\mathcal{L}(\mathcal{X}_k)\}_{k\in K}$, then we can represent the family Γ by composition

$$\gamma \circ \delta := \{\gamma_k \circ \delta_k\}_{k\in K}.$$

Finding an appropriate common space to represent a family will thus be handled by taking an appropriate product of the representation spaces, which we investigate here. This view gives an equivalent definition of independence as actions of operators on a corresponding product space.

We begin with the most robust product construction, the *free product*, through which we will be able to define the Boolean and monotone products.

1.4.1 The Free Product

The free product of B-B-bimodules with specified B-projection will be built out of tensor products.

Definition 1.4.2. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be B-B-bimodules. Their *tensor product over B* is the B-B-bimodule

$$\mathcal{X}_1 \otimes_B \cdots \otimes_B \mathcal{X}_n$$

generated by the elements $\xi_1 \otimes \cdots \otimes \xi_n$ where $\xi_i \in \mathcal{X}_i$ and satisfying for all $b \in B$,

- $\xi_1 \otimes \cdots \otimes \xi_i \otimes b \cdot \xi_{i+1} \otimes \cdots \otimes \xi_n = \xi_1 \otimes \cdots \otimes \xi_i \cdot b \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n$,
- $b \cdot (\xi_1 \otimes \cdots \otimes \xi_n) = (b \cdot \xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$, and
- $(\xi_1 \otimes \cdots \otimes \xi_n) \cdot b = \xi_1 \otimes \cdots \otimes \xi_{n-1} \otimes (\xi_n \cdot b)$.

Definition 1.4.3. Given a family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ of B-B-bimodules with specified B-projection, we form their *reduced free product* which is a B-B-bimodule with specified B-projection $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ with

$$\mathcal{X} = B \oplus \bigoplus_{n\in\mathbb{N}} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{\mathcal{X}}_{i_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{i_n}.$$

For each $b \in B$, we will denote by L_b and R_b as respectively the left and right action by b on the free product. That is, if $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{X}$, then

$$L_b(\xi_1 \otimes \cdots \otimes \xi_n) = (b \cdot \xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n \quad \text{and} \quad R_b(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_{n-1} \otimes (\xi_n \cdot b).$$

For each $k \in K$, the algebra $\mathcal{L}(\mathcal{X}_k)$ acts on $\mathcal{X}$ on the left and right in a natural way, described as follows. For each $k \in K$, we collect the submodule generated by all those tensor words which do not start with elements of $\mathring{\mathcal{X}}_k$, and denote it by

$$\mathcal{X}_\ell(k) = B \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{k_1\neq k_2\neq\cdots\neq k_n \\ k_1\neq k}} \mathring{\mathcal{X}}_{k_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{k_n},$$

and let V_k be the natural isomorphism of B-B-bimodules

$$V_k : \mathcal{X} \to \mathcal{X}_k \otimes_B \mathcal{X}_\ell(k).$$

Similarly, we can denote the submodule generated by all those tensor words which do not end with the elements of $\mathring{\mathcal{X}}_k$. We denote this by

$$\mathcal{X}_r(k) = B \oplus \bigoplus_{n\geq 1} \bigoplus_{\substack{k_1\neq k_2\neq\cdots\neq k_n \\ k_n\neq k}} \mathring{\mathcal{X}}_{k_1} \otimes_B \cdots \otimes_B \mathring{\mathcal{X}}_{k_n}.$$

Let also W_k be the natural isomorphism of B-B-bimodules

$$W_k : \mathcal{X} \to \mathcal{X}_r(k) \otimes_B \mathcal{X}_k$$

For each $k \in K$, denote by $\lambda_k : \mathcal{L}(\mathcal{X}_k) \to \mathcal{L}(\mathcal{X})$ the unital homomorphism called the left regular representation of $\mathcal{L}(\mathcal{X}_k)$ on $\mathcal{X}$, defined by

$$\lambda_k(a) = V_k^{-1}(a \otimes \mathrm{Id})V_k,$$

and $\rho_k : \mathcal{L}(\mathcal{X}_k) \to \mathcal{L}(\mathcal{X})$ the unital homormorphism called the right regular representation of $\mathcal{L}(\mathcal{X}_k)$ on $\mathcal{X}$ defined by

$$\rho_k(a) = W_k^{-1}(\mathrm{Id} \otimes a)W_k.$$

Remark 1.4.4. It is straight forward to check that if $a \in \mathcal{L}(\mathcal{X}_k)$ then

$$\lambda_k(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \begin{cases} a\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n & \text{if } \xi_1 \in \mathring{\mathcal{X}}_k \\ a\mathbf{1}_B \otimes \xi_1 \otimes \cdots \otimes \xi_n & \text{otherwise} \end{cases}.$$

Similarly,

$$\rho_k(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \begin{cases} \xi_1 \otimes \xi_2 \otimes \cdots \otimes a\xi_n & \text{if } \xi_n \in \mathring{\mathcal{X}}_k \\ \xi_1 \otimes \cdots \otimes \xi_n \otimes a\mathbf{1}_B & \text{otherwise} \end{cases}.$$

It follows immediately that λ_k and ρ_k are distribution preserving representations of $\mathcal{L}(\mathcal{X}_k)$ on $\mathcal{X}$.

Proposition 1.4.5. *The families $\{\lambda_k(\mathcal{L}(\mathcal{X}_k))\}_{k\in K}$ and $\{\rho_k(\mathcal{L}(\mathcal{X}_k))\}_{k\in K}$ are freely independent over B.*

Proof. This is a simple observation that if $n \in \mathbb{N}$ and $\epsilon : \{1, \ldots, n\} \to K$ with $\epsilon(1) \neq \epsilon(2) \neq \cdots \neq \epsilon(n)$ and $a_i \in \mathcal{L}(\mathcal{X}_{\epsilon(i)})$, then

$$\lambda_{\epsilon(1)}(a_1) \cdots \lambda_{\epsilon(n)} a_n \mathbf{1}_B = a_1 \mathbf{1}_B \otimes \cdots \otimes a_n \mathbf{1}_B \in \mathcal{X}_{\epsilon(1)} \otimes_B \cdots \otimes_B \mathcal{X}_{\epsilon(n)},$$

so that

$$E_{\mathcal{L}(\mathcal{X})}(\lambda_{\epsilon(1)}(a_1) \cdots \lambda_{\epsilon(n)}(a_n)) = 0.$$

This affirms the definition of free independence over B. □

Corollary 1.4.6. *A family $\Gamma = \{A_k\}_{k\in K}$ of unital B-subalgebras of $\mathcal{A}$ is freely independent over B if and only if there is a family of representations $\delta = \{\delta_k\}_{k\in K}$ where $\delta_k : A_k \to \mathcal{L}(\mathcal{X}_k)$ such that $\lambda \circ \delta$ is a joint distribution preserving representation of Γ.*

Proof. Fix a family $\Gamma = \{A_k\}_{k\in K}$ of B-subalgebras of $\mathcal{A}$.

If there is a joint distribution preserving representation $\lambda \circ \delta$ for Γ, then from Proposition 1.4.5 above, it follows immediately that Γ is freely independent over B, making use of Corollary 1.3.27.

Now suppose that Γ is freely independent. Then for each $k \in K$, let $\mathcal{X}_k$ be as in Proposition 1.1.13, and $\delta_k : A_k \to \mathcal{L}(\mathcal{X}_k)$ be the corresponding distribution preserving representation. Then let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be the reduced free product of the family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$. Let λ_k be the left regular representation of $\mathcal{L}(\mathcal{X}_k)$ on $\mathcal{X}$.

Since Γ is freely independent, Corollary 1.3.27 says that computing a Γ-mixed moment reduces to knowing distributions of the A_k. Similarly, by Proposition 1.4.5, computing a mixed moment in the family $\{\lambda_k(\delta_k(A_k))\}_{k\in K}$ reduces to knowing the distributions of the individual $\lambda_k(\delta_k(A_k))$.

This combined with the fact that δ_k and λ_k are expectation preserving yields that $\lambda \circ \delta$ is joint distribution preserving, completing the proof. □

This corollary gives yet a third equivalent understanding of free independence, this time in terms of actions of operators on the free product space.

Example 1.4.7. Let $\{G_k\}_{k\in K}$ be a family of subgroups of a group G which is free over H. For each $k \in K$, as in Proposition 1.1.13 the space $(\mathbb{C}[G_k], \ker \tau_H, \tau_H)$ is a $\mathbb{C}[H]$-$\mathbb{C}[H]$-bimodule with specified $\mathbb{C}[H]$ projection. Next, denote by $*_{k\in K}^{\mathbb{C}[H]}\mathbb{C}[G_k]$ the corresponding free product $\mathbb{C}[H]$-$\mathbb{C}[H]$-bimodule as above.

Then we have

$$\mathbb{C}[*_{k\in K}^{H} G_k] \cong *_{k\in K}^{\mathbb{C}[H]}\mathbb{C}[G_k].$$

Briefly, this is the correspondence

$$\delta_{g_1}\cdots\delta_{g_n} \quad\longleftrightarrow\quad \delta_{g_1}\otimes\cdots\otimes\delta_{g_n}$$

when $g_i \in G_{k(i)} \setminus H$ and $k(i) \neq k(i+1)$.

It follows immediately that if $g \in G_k$ and L_g denotes the left multiplication by g operator on $\mathbb{C}[G_k]$ then the families $\{\mathrm{alg}(\{\lambda_k(L_g) : g \in G_k\})\}_{k\in K}$ and $\{\mathrm{alg}(\{\rho_k(L_g) : g \in G_k\})\}_{k\in K}$ are free over $\mathbb{C}[H]$.

1.4.2 The Boolean and Monotone Products

The Boolean and monotone product spaces, defined below, are subspaces of the free product space. This allows us to use the free product machinery to describe and work with them. As above, let $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ be a family of B-B-bimodules with specified B-projections, and let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be their reduced free product with amalgamation over B.

Definition 1.4.8. The *Boolean product* of the family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ is the B-B-bimodule with specified B-projection $(\mathcal{X}_{\uplus}, \mathring{\mathcal{X}}_{\uplus}, p_{\uplus})$ where

$$\mathring{\mathcal{X}}_{\uplus} = \bigoplus_{k\in K} \mathring{\mathcal{X}}_k.$$

For each $k \in K$ define the k^{th} *Boolean projection* $P_{\uplus,k} : \mathcal{X} \to \mathcal{X}$ to be the natural projection in the free product onto $B \oplus \mathring{\mathcal{X}}_k$.

Definition 1.4.9. The *monotone product* of the family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$, when K is totally ordered, is the B-B-bimodule with specified B-projection $(\mathcal{X}_{\rhd}, \mathring{\mathcal{X}}_{\rhd}, p_{\rhd})$ where

$$\mathring{\mathcal{X}}_{\rhd} = \bigoplus_{n\geq 1}\bigoplus_{k_1>\cdots>k_n} \mathring{\mathcal{X}}_{k_1}\otimes_B\cdots\otimes_B \mathring{\mathcal{X}}_{k_n}.$$

The *anti-monotone product* of the family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$, when K is totally ordered, is the B-B-bimodule with specified B-projection $(\mathcal{X}_{\lhd}, \mathring{\mathcal{X}}_{\lhd}, p_{\lhd})$ where

$$\mathring{\mathcal{X}}_{\lhd} = \bigoplus_{n\geq 1}\bigoplus_{k_1<\cdots<k_n} \mathring{\mathcal{X}}_{k_1}\otimes_B\cdots\otimes_B \mathring{\mathcal{X}}_{k_n}.$$

For each $k \in K$ we define

- the k^{th} *monotone projection* $P_{\triangleright,k}$, to be the natural projection onto

$$\mathcal{X}(\triangleright,k) = B \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{k\geq k_1>k_2>\cdots>k_n} \mathring{\mathcal{X}}_{k_1}\otimes_B\cdots\otimes_B\mathring{\mathcal{X}}_{k_n}\right),$$

- the k^{th} *anti-monotone projection* $P_{\triangleleft,k}$, to be the natural projection onto

$$\mathcal{X}(\triangleleft,k) = B \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{k_1<k_2<\cdots<k_n\leq k} \mathring{\mathcal{X}}_{k_1}\otimes_B\cdots\otimes_B\mathring{\mathcal{X}}_{k_n}\right).$$

Remark 1.4.10. While we won't explicitly talk about anti-monotone independence, the anti-monotone spaces will help us define monotone independent operators with the right regular action on the free product, as seen below.

We collect a couple of easy facts regarding these projections.

Lemma 1.4.11. *With notations as above, fix $k,k'\in K$ and $T\in\mathcal{L}(\mathcal{X}_k)$. We have*

a) $\lambda_k(T)P_{\uplus,k} = P_{\uplus,k}\lambda_k(T)$ *and* $\rho_k(T)P_{\uplus,k} = P_{\uplus,k}\rho_k(T)$,

b) $\lambda_k(T)P_{\uplus,k} = \rho_k(T)P_{\uplus,k}$,

c) $P_{\uplus,k}P_{\uplus,k'} = p$,

d) If K is totally ordered and k_0 is least in K, then

$$P_{\triangleright,k_0} = P_{\triangleleft,k_0} = P_{\uplus,k_0}.$$

e) If K is totally ordered and $k>k'$ then

$$P_{\triangleright,k}P_{\triangleright,k'} = P_{\triangleright,k'} = P_{\triangleright,k'}P_{\triangleright,k} \quad \textit{and} \quad P_{\triangleleft,k}P_{\triangleleft,k'} = P_{\triangleleft,k'} = P_{\triangleleft,k'}P_{\triangleleft,k}$$

f) If K is totally ordered, then

$$P_{\triangleright,k}P_{\triangleleft,k} = P_{\uplus,k'\leq k}.$$

where $P_{\uplus,k}$ is the projection onto the Boolean product of the family $\{\mathcal{X}_{k'},\mathring{\mathcal{X}}_{k'},p_{k'}\}_{k'\leq k}$, $\mathcal{X}(\uplus,k'\leq k)$.

Proof. These follow very quickly from the definitions. With a few easy observations.

For b), note that for any $\eta \in \mathcal{X}$, we have $P_{\uplus,k}\eta \in \mathcal{X}_k$ so that

$$\lambda_k(T)P_{\uplus,k}\eta = TP_{\uplus,k}\eta = \rho_k(T)P_{\uplus,k}\eta.$$

c) follows from the fact that $B \oplus \mathring{\mathcal{X}}_i \cap B \oplus \mathring{\mathcal{X}}_j = B$.

e) is the simple observation that if $k > k'$ then

$$\mathcal{X}(\rhd, k') \leq \mathcal{X}(\rhd, k) \quad \text{and} \quad \mathcal{X}(\lhd, k') \leq \mathcal{X}(\lhd, k). \qquad \square$$

f) is again a simple intersection of the subspaces $\mathcal{X}(\rhd, k)$ and $\mathcal{X}(\lhd, k)$. Indeed, let $k_1 \neq k_2 \in K$, then by the total ordering on K, either $k_1 < k_2$, in which case

$$\mathring{\mathcal{X}}_{k_1} \otimes_B \mathring{\mathcal{X}}_{k_2} \notin \mathcal{X}(\rhd, k)$$

or $k_1 > k_2$ in which case

$$\mathring{\mathcal{X}}_{k_1} \otimes_B \mathring{\mathcal{X}}_{k_2} \notin \mathcal{X}(\lhd, k).$$

Furthermore if $k' > k$ then $P_{\lhd,k}(\mathring{\mathcal{X}}_{k'}) = 0$. The elements that survive the product of projections are exactly those elements in $\mathcal{X}(\uplus, k' \leq k)$.

The two results that follow, shown by Liu in [5] in the case when $B = \mathbb{C}$, show that one could have equivalently defined Boolean and monotone independence as operators on different factors of the Boolean or monotone products, in the same spirit as we did with free independence above.

We look at these results in the amalgamated setting as a sanity check that nothing is lost in generalization (c.f. [5] Section 2).

Proposition 1.4.12. *Given a family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ and their reduced free product $(\mathcal{X}, \mathring{\mathcal{X}}, p)$, the family $\{P_{\uplus,k}\lambda_k(\mathcal{L}(\mathcal{X}_k))P_{\uplus,k}\}_{k\in K}$ is Boolean independent in $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$.*

Proof. We check this by looking at an arbitrary moment. The result is seen by a simple induction on the length of the moment word we are interested in computing. But we do the simple, illustrative case of length 2.

Let $a_1 \in \mathcal{L}(\mathcal{X}_{k_1})$ and $a_2 \in \mathcal{L}(\mathcal{X}_{k_2})$ with $k_1 \neq k_2$. Then

$$\begin{aligned} E_{\mathcal{L}(\mathcal{X})}(P_{\uplus,k_1}\lambda_{k_1}(a_1)P_{\uplus,k_1}P_{\uplus,k_2}\lambda_{k_2}(a_2)P_{\uplus,k_2}) &= pP_{\uplus,k_1}\lambda_{k_1}(a_1)P_{\uplus,k_1}P_{\uplus,k_2}\lambda_{k_2}(a_2)P_{\uplus,k_2}1_B \\ &= pP_{\uplus,k_1}\lambda_{k_1}(a_1)P_{\uplus,k_1}pP_{\uplus,k_2}\lambda_{k_2}(a_n)P_{\uplus,k_n}1_B \qquad (*) \\ &= pP_{\uplus,k_1}\lambda_{k_1}(a_1)P_{\uplus,k_1}E_{\mathcal{L}(\mathcal{X})}(P_{\uplus,k_n}\lambda_{k_n}(a_n)P_{\uplus,k_n}) \\ &= pP_{\uplus,k_1}\lambda_{k_1}(a_1)P_{\uplus,k_1}1_B E_{\mathcal{L}(\mathcal{X})}(P_{\uplus,k_n}\lambda_{k_n}(a_n)P_{\uplus,k_n}) \\ &= E_{\mathcal{L}(\mathcal{X})}(P_{\uplus,k_1}\lambda_{k_1}(a_1)P_{\uplus,k_1})E_{\mathcal{L}(\mathcal{X})}(P_{\uplus,k_n}\lambda_{k_n}(a_n)P_{\uplus,k_n}) \end{aligned}$$

where $(*)$ comes from the fact that $P_{\uplus,k_1}$ and $P_{\uplus,k_2}$ are projections, and Lemma 1.4.11.

Inductively, one arrives at the definition of Boolean independence for an arbitrary moment. $\square$

Remark and Notation 1.4.13. With notations as in Definition 1.4.8, for each $k \in K$, let $\gamma_{\uplus,k} : \mathcal{L}(\mathcal{X}_k) \to \mathcal{L}(\mathcal{X}_\uplus)$ be defined by

$$\gamma_{\uplus,k}(T) = P_{\uplus,k}\lambda_k(T)P_{\uplus,k}.$$

Then it is easy to see that $\gamma_\uplus = \{\gamma_{\uplus,k}\}_{k\in K}$ is an expectation preserving representation of the family $\{\mathcal{L}(\mathcal{X}_k)\}_{k\in K}$.

Corollary 1.4.14. *In a B-non-commutative probability space $(\mathcal{A}, E)$, a family $\Gamma = \{A_k\}_{k\in K}$ of B-subalgebras of $\mathcal{A}$ is Boolean independent if and only if there is a family of representations $\delta = \{\delta_k\}_{k\in K}$ such that if $\gamma_\uplus$ is defined as in Remark and Notation 1.4.13, then $\gamma_\uplus \circ \delta$ is a joint distribution preserving representation of Γ.*

The argument is the same as that of Corollary 1.4.6.

Lemma 1.4.15. *Fix a family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ with K totally ordered, and their reduced free product $(\mathcal{X}, \mathring{\mathcal{X}}, p)$. If $k_1 < k_2$ and $a \in \mathcal{L}(\mathcal{X}_{k_2})$, then for any $\eta \in \mathcal{X}(\rhd, k_1)$*

$$P_{\rhd,k_1}\lambda_{k_2}(a)\eta = E_{\mathcal{L}(\mathcal{X}_{k_2})}(a)\eta = P_{\lhd,k_1}\rho_{k_2}(a)\eta.$$

Consequently, the families $\{P_{\rhd,k}\lambda_k(\mathcal{L}(\mathcal{X}_k))P_{\rhd,k}\}_{k\in K}$ and $\{P_{\lhd,k}\rho_k(\mathcal{L}(\mathcal{X}_k))P_{\lhd,k}\}_{k\in K}$ are monotone independent in $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$.

Proof. Note that

$$\lambda_{k_2}(a)\eta = (a - pa)1_B \otimes \eta + pa\eta = (a1_B - E_{\mathcal{L}(\mathcal{X}_{k_2})}(a)1_B) \otimes \eta + E_{\mathcal{L}(\mathcal{X}_{k_2})}(a) \cdot \eta$$

But since $\eta \in \mathcal{X}(\rhd, k_1)$, we have $(a1_B - E_{\mathcal{L}(\mathcal{X}_{k_2})}(a)1_B) \otimes \eta \in \mathcal{X}(\rhd, k_2) \ominus \mathcal{X}(\rhd, k_1)$, so that

$$P_{\rhd,k_1}\lambda_{k_2}(a)\eta = E_{\mathcal{L}(\mathcal{X}_{k_2})}(a)\eta.$$

The other equality is similar, paying attention to the right action.

Now to see that the family is monotone independent, fix $n \in \mathbb{N}$, $a_i \in \mathcal{L}(\mathcal{X}_{k_i})$, and suppose $k_j > k_{j-1}$ and $k_j > k_{j+1}$. Then let

$$\hat{a} = P_{\rhd,k_1}\lambda_{k_1}(a_1)P_{\rhd,k_1}\cdots P_{\rhd,k_{j-1}}\lambda_{k_{j-1}}(a_{j-1})P_{\rhd,k_{j-1}}$$

and

$$\eta = P_{\rhd,k_{j+1}}\lambda_{k_{j+1}}(a_{j+1})P_{\rhd,k_{j+1}}\cdots P_{\rhd,k_n}\lambda_{k_n}(a_n)P_{\rhd,k_n}1_B$$

Then

$$\begin{aligned} p\hat{a}P_{\rhd,k_j}\lambda_{k_j}(a_j)P_{\rhd,k_j}\eta &= p\hat{a}P_{\rhd,k_{j-1}}P_{\rhd,k_j}\lambda_{k_j}(a_j)P_{\rhd,k_j}P_{\rhd,k_{j+1}}\eta \\ &= p\hat{a}P_{\rhd,k_{j-1}}\lambda_{k_j}(a_j)P_{\rhd,k_{j+1}}\eta \\ &= p\hat{a}E_{\mathcal{L}(\mathcal{X})}(\lambda_{k_j}(a_j))\eta \end{aligned}$$

verifying the definition of monotone independence. That the other family is monotone is a similar argument. □

Remark and Notation 1.4.16. With notations as in Definition 1.4.9, for each $k \in K$, let $\gamma_{\rhd,k,\ell} : \mathcal{L}(\mathcal{X}_k) \to \mathcal{L}(\mathcal{X}_{\rhd})$ be defined by

$$\gamma_{\rhd,k,\ell}(T) = P_{\rhd,k}\lambda_k(T)P_{\rhd,k}.$$

Also, let $\gamma_{\lhd,k,r} : \mathcal{L}(\mathcal{X}_k) \to \mathcal{L}(\mathcal{X}_{\rhd})$ be defined by

$$\gamma_{\lhd,k}(T) = P_{\lhd,k}\rho_k(T)P_{\lhd,k}.$$

Then it is easy to see that $\gamma_{\rhd} = \{\gamma_{\rhd,k}\}_{k\in K}$ and $\gamma_{\lhd} = \{\gamma_{\lhd,k,r}\}_{k\in K}$ are expectation preserving representations of the family $\{\mathcal{L}(\mathcal{X}_k)\}_{k\in K}$.

Corollary 1.4.17. *In a B-non-commutative probability space $(\mathcal{A}, E)$, a family $\{A_k\}_{k\in K}$, K a totally ordered index set of unital B-subalgebras of $\mathcal{A}$ is monotone independent if and only if there is a family of representations $\delta = \{\delta_k\}_{k\in K}$ such that if $\gamma_{\rhd}$ (equivalently $\gamma_{\lhd}$) is defined as in Remark and Notation 1.4.16, then $\gamma_{\rhd} \circ \delta$ (equivalently $\gamma_{\lhd} \circ \delta$) is a joint distribution preserving representation of Γ.*

The proof of this is similar to that of Corollary 1.4.6.

1.4.3 The Tensor Product

We finish off with the demonstration that the classical independence also arises from an action of operators. For this, we will require that B is in the center of $\mathcal{A}$. Furthermore, given our commutative setting, we will work simply with (left) B-modules rather than B-B-bimodules.

Remark 1.4.18. This is a fair restriction, given our motivation to study B-B-bimodules in the first place, as representations of B-non-commutative probability spaces. In this setting, our algebra $\mathcal{A}$ is naturally a B-B-bimodule. However, when B is in the center of $\mathcal{A}$, left and right multiplication by B are the same. Hence $\mathcal{A}$ can be more simply viewed as a B-module.

The type of independence that arises in classical probability corresponds to representing the algebras on a *tensor product*.

For simplicity and clarity, we will work only with a finite index set $K = \{1, \dots, n\}$. While infinite tensor products exist, they appear most naturally in more structurally rich settings, such as for von Neumann algebras. A purely algebraic approach to infinite tensor products of modules has been attempted in [13].

Definition 1.4.19. Given a family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{1\leq k\leq n}$ of B-modules with specified B-projections. Define the *tensor product* $\mathcal{X}_\otimes = \mathcal{X}_1 \otimes_B \cdots \otimes_B \mathcal{X}_n$ to be the B-module with specified B-projection defined by

$$p_\otimes = p_1 \otimes \cdots \otimes p_n.$$

Remark 1.4.20. Analogous with the B-B-bimodule tensor product, we have the following relations on the tensor product for the B-action:

$$b \cdot (\xi_1 \otimes \cdots \otimes \xi_k \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes b \cdot \xi_k \otimes \cdots \otimes \xi_n,$$

for any $1 \leq k \leq n$. In the above definition, we will identify freely the copies of B in each $\mathcal{X}_k$ and whenever convenient,

$$1_B \otimes \xi_k = \xi_k = \xi_k \otimes 1_B.$$

Definition 1.4.21. For each $a \in \mathcal{L}(\mathcal{X}_k)$, let

$$T_k(a) = I_{\mathcal{X}_1} \otimes \cdots \otimes I_{\mathcal{X}_{k-1}} \otimes a \otimes I_{\mathcal{X}_{k+1}} \otimes \cdots \otimes I_{\mathcal{X}_n}.$$

Proposition 1.4.22. *For each $1 \leq k \leq n$, the map $a \mapsto T_k(a)$ is an expectation preserving homomorphism, and the family $\{T_k(\mathcal{L}(\mathcal{X}_k))\}_{1\leq k\leq n}$ is tensor independent in $(\mathcal{L}(\mathcal{X}_\otimes), E_{\mathcal{L}(\mathcal{X}_\otimes)})$.*

Proof. First note that if $a \in \mathcal{L}(X_k)$, then

$$E_{L(\mathcal{X}_\otimes)}(T_k(a)) = p_1 \otimes \cdots \otimes p_n(T_k(a)1_B) = p_k a 1_B = E_{\mathcal{L}(\mathcal{X}_k)}(a).$$

So expectations are preserved.

It is immediate from the definition to note that if $i \neq j$ and $a_1 \in \mathcal{L}(\mathcal{X}_i)$ and $a_2 \in \mathcal{L}(X_j)$ then

$$T_i(a_1)T_j(a_2)\eta = T_j(a_2)T_i(a_1)\eta$$

for any $\eta \in \mathcal{X}_\otimes$.

Lastly, we evaluate mixed moments. Namely if $m \in \mathbb{N}$, $\epsilon : \{1, \ldots, m\} \to \{1, \ldots, n\}$ and $a_i \in \mathcal{L}(\mathcal{X}_{\epsilon(i)})$ for each $1 \leq i \leq m$, then

$$\begin{aligned}
E_{\mathcal{L}(\mathcal{X}_\otimes)}(T_{\epsilon(1)}(a_1) \cdots T_{\epsilon(m)}(a_m)) &= (p_1 \otimes \cdots \otimes p_n) T_{\epsilon(1)}(a_1) \cdots T_{\epsilon(m)}(a_m) 1_B \\
&= p_1 \prod_{k \in \epsilon^{-1}(1)}^{\rightarrow} a_k 1_B \otimes \cdots \otimes p_n \prod_{k \in \epsilon^{-1}(n)}^{\rightarrow} a_k 1_B \\
&= E_1\left(\prod_{k \in \epsilon^{-1}(1)}^{\rightarrow} a_k\right) \cdots E_n\left(\prod_{k \in \epsilon^{-1}(n)}^{\rightarrow} a_k\right) \\
&= E_{\mathcal{L}(\mathcal{X}_\otimes)}\left(T_1\left(\prod_{k \in \epsilon^{-1}(1)}^{\rightarrow} a_k\right)\right) \cdots E_{\mathcal{L}(\mathcal{X}_\otimes)}\left(T_n\left(\prod_{k \in \epsilon^{-1}(n)}^{\rightarrow} a_k\right)\right) \\
&= E_{\mathcal{L}(\mathcal{X}_\otimes)}\left(\prod_{k \in \epsilon^{-1}(1)}^{\rightarrow} (T_1(a_k))\right) \cdots E_{\mathcal{L}(\mathcal{X}_\otimes)}\left(\prod_{k \in \epsilon^{-1}(n)}^{\rightarrow} (T_n(a_k))\right)
\end{aligned}$$

where the final two equalities come from the fact that the maps $a \mapsto T_k(a)$ are expectation preserving homomorphisms. This verifies the moment condition for tensor independence. $\square$

Corollary 1.4.23. *A family $\Gamma = \{A_k\}_{k \in K}$ of unital B-subalgebras in a B-non-commutative probability space $(\mathcal{A}, E)$ is tensor independent if and only if there is a family of representations $\ell = \{\ell_k\}_{k \in K}$ where $\ell_k : A_k \to \mathcal{L}(\mathcal{X}_k)$ such that $T \circ \ell = \{T_k \circ \ell_k\}_{k \in K}$ is a distribution preserving representation of Γ.*

This follows from Proposition 1.4.22 and the same logic as in Corollary 1.4.6.

Chapter 2

Bi-Free Probability and Multi-Algebra Independence

In Chapter 1, we looked at the notions of independence in non-commutative probability and looked at three different, but equivalent ways of viewing them; a direct moment condition, a vanishing of mixed cumulants, and operators acting on special spaces. In this chapter, we look at notions of independence between tuples of algebras, which are extended naturally from the operator perspective of independence. In particular, we saw that free, Boolean, and monotone independent operators all conceivably act on the same space (the reduced free product). This allows us to consider pairs of algebras, where one algebra may be Boolean acting, while another is freely acting.

We begin with discussion of the original and most successful multi-algebra independence in non-commutative probability; Voiculescu's *bi-freeness*. This is defined through operator actions, but reinforced by a theory of vanishing mixed cumulants. Afterwards, we discuss Liu's generalization of bi-freeness with a closer look at *free-Boolean* and *free-free-Boolean* independences. To aid us in our investigations, we also look at a combinatorial accounting of operators on the free product known as LR-diagrams

2.1 Bi-Free Independence for Families of Pairs of Faces

Voiculescu's original formlation of free independence through operator actions was through the left-regular representation on the free product as in Section 1.4. But he later noticed that by studying simultaneously left and right regular acting operators, one could define a new type of independence for families of pairs of algebras (one algebra containing left-acting operators, one containing right-acting operators) that was similar in many ways to the types of independence we saw earlier. This was dubbed *bi-free independence for pairs of faces.* In the case when $B = \mathbb{C}$, this was defined in [26] as follows.

Definition 2.1.1. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. A family $\{(A_k^\ell, A_k^r)\}_{k\in K}$ of pairs of unital sub-algebras of $\mathcal{A}$ is said to be *bi-free* if there exist vector spaces with specified projection $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ and unital homomorphisms $\ell_k : A_k^\ell \to \mathcal{L}(\mathcal{X}_k)$, $r_k : A_k^r \to \mathcal{L}(\mathcal{X}_k)$ such that the joint distribution of $\{(A_k^\ell, A_k^r)\}_{k\in K}$ with respect to φ is equal to the joint distribution of the family

$$\{((\lambda_k \circ \ell_k)(A_k^\ell), (\rho_k \circ r_k)(A_k^r))\}_{k\in K}$$

in $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$ where $\mathcal{X}$ is the reduced free product of the $\mathcal{X}_k$, and λ_k and ρ_k are respectively the left and right regular representations.

We will look at the more general setting when we condition on an algebra B momentarily, but for now, this simpler case will serve as a stepping stone.

Remark and Notation 2.1.2. Distributions for families of pairs of faces have essentially the same meaning as for families of algebras, but in order to calculate an arbitrary moment, we need to decide if a particular element is coming from the left or the right face. Thus to describe an arbitrary moment for a family $\Gamma = \{(A_{k,\ell}, A_{k,r})\}_{k\in K}$ of pairs of faces will require an $n \in \mathbb{N}$, a colouring $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, an index function $\epsilon : \{1, \ldots, n\} \to K$, and elements $a_i \in A_{\epsilon(i)}^{\chi(i)}$ for $1 \leq i \leq n$.

The joint distribution of the family Γ is then known by computing all possible moments

$$E(a_1 \cdots a_n)$$

over all such n, χ, ϵ, a_i as chosen above.

For the remainder of the work, a colouring indicating which face an element comes from, will be denoted by χ, as above.

Remark and Notation 2.1.3. To each $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ we define a permutation s_χ on $\{1, \ldots, n\}$ as follows. Let

$$\chi^{-1}(\ell) = \{i_1 < \cdots < i_p\} \quad \text{and } \chi^{-1}(r) = \{i_{p+1} > \cdots > i_n\},$$

then define $s_\chi(j) = i_j$. We will also say that $i \prec_\chi j$ when $s_\chi^{-1}(i) < s_\chi^{-1}(j)$.

The motivation for this permutation comes from viewing an action of a product of left and rights on a specified vector on, in our case, a free-product B-B-bimodule. This action is naturally imagined in the form of a *double-ended queue*. Right acting operators have their multiplication reversed, hence the opposite ordering of $\chi^{-1}(r)$. For more on this, the reader is referred to the work in which this idea originates, [8].

Definition 2.1.4. Let $n \in \mathbb{N}$, and $\chi : \{1, \ldots, n\} \to \{\ell, r\}$. A χ-*interval* is a subset $S \subseteq \{1, \ldots, n\}$ such that $s_\chi^{-1} \cdot S$ is an interval, where s_χ is the permutation as above.

Remark 2.1.5. Our original definitions of independence came from rules for evaluating mixed moments. However, this definition is prescribed via operator actions. This definition keeps clear the inspiration behind the idea, that we want to look simultaneously at left and right operators on the free product.

Given our success in proving the equivalences of our independences with representations as operators in the last chapter, one might inquire if there are moment conditions that characterize bi-free independence. It was I. Charlesworth in [2] came up with the following equivalent characterization.

Proposition 2.1.6 (Theorem 7, [2])**.** *A family $\{(A_k^\ell, A_k^r)\}_{k\in K}$ of pairs of unital subalgebras in a non-commutative probability space $(\mathcal{A}, \varphi)$ is bi-free if and only if for every*

- $n \in \mathbb{N}$,
- $\chi : \{1, \ldots, n\} \to \{\ell, r\}$,
- $\epsilon : \{1, \ldots, n\} \to K$,
- $a_i \in A_{\epsilon(i)}^{\chi(i)}$, *and*
- *Whenever $S \subseteq \{1, \ldots, n\}$ is a maximal χ-interval such that $i, j \in S$ implies $\epsilon(i) = \epsilon(j)$, and*

$$\varphi(\overrightarrow{\prod_{i\in S}} a_i) = 0,$$

we have

$$\varphi(a_1 \cdots a_n) = 0.$$

Note that this moment condition is precisely the condition for freeness, when we forget the left and right colourings (e.g. when $\chi \equiv \ell$). In this case a maximal χ-interval is simply a maximal interval. An extension of this to the amalgamated setting was only achieved in [2] under the assumption that B is a *Banach algebra*.

There is also an equivalent combinatorial theory of bi-free independence that we will investigate shortly.

Remark 2.1.7. In Remark 1.2.15, we pointed out there are at most five unique types of *natural* independence between algebras, yet here is yet a new one. However our remark is exculpated by the fact that bi-free independence is between *pairs of algebras* rather than single algebras.

As we will see below, this idea of looking for independence between tuples of algebras extends beyond bi-freeness. However, by looking at the individual algebras in the pair, if there is to be an independence between them, then it must be one of the types presented in the previous chapter.

Indeed, one can note easily that if a family $\{(A_k^\ell, A_k^r)\}_{k\in K}$ is bi-free, then the family $\{A_k^\ell\}_{k\in K}$ is free and so is the family $\{A_k^r\}_{k\in K}$ (by Corollary 1.4.6). It also turns out that A_k^ℓ and $A_{k'}^r$ are classically independent whenever $k \neq k'$.

So this type of independence for pairs of algebras seems to be a direct upgrade of the notion of independence between individual algebras. The extent to which it is an upgrade will be a main theme of Chapter 3.

2.1.1 B-B-non-commutative probability space and amalgamated bi-free independence

The definition of bi-free independence above works for non-commutative probability spaces, but we are interested in the general setting of B-non-commutative probability. But we run into a technicality in such a generalization, that we need to respect the relationship between left and right B-action on the free product and left and right regular representations of operators in our algebras. In particular it should be the case that if $T \in \mathcal{L}(\mathcal{X}_k)$, $\xi \in \mathcal{X}_k$, and

$b \in B$, then

$$\begin{aligned}
\lambda_k(T)R_b\xi &= \lambda_k(T)(\xi \cdot b) \\
&= V(T \otimes I)(\xi \cdot b \otimes 1_B) \\
&= V(T \otimes I)(\xi \otimes b \cdot 1_B) \\
&= VT\xi \otimes b \cdot 1_B \\
&= VT\xi \cdot b \otimes 1_B \\
&= T\xi \cdot b \\
&= R_b T\xi \\
&= R_b\lambda_k(T)\xi.
\end{aligned}$$

The cases when $\xi \in \mathcal{X} \ominus \mathcal{X}_k$ are easier to see, making use of Remark 1.4.4. So in particular, we must have

$$\lambda_k(T)R_b = R_b\lambda_k(T) \quad \text{and by similar arguments} \quad \rho_k(T)L_b = L_b\rho_k(T).$$

Thus, we need a finer abstract structure in which to house these amalgamated bi-free families.

Definition 2.1.8. A *B-B-non-commutative probability space* is a triple $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ where $\mathcal{A}$ is a unital algebra, $\varepsilon : B \otimes B^{\mathrm{op}} \to \mathcal{A}$ is a unital homomorphism such that $\varepsilon|_{B\otimes 1_B}$ and $\varepsilon|_{1_B \otimes B}$ are injective, and $E_{\mathcal{A}} : \mathcal{A} \to B$ is a linear map such that

$$E_{\mathcal{A}}(\varepsilon(b_1 \otimes b_2)T) = b_1 E_{\mathcal{A}}(T) b_2$$

for all $b_1, b_2 \in B$ and $T \in \mathcal{A}$, and

$$E_{\mathcal{A}}(T\varepsilon(b \otimes 1_B)) = E_{\mathcal{A}}(T\varepsilon(1_B \otimes b))$$

for all $b \in B$ and $T \in \mathcal{A}$.

We define the *left* and *right algebras* of $\mathcal{A}$ respectively by

$$\mathcal{A}_\ell := \{T \in \mathcal{A} \mid T\varepsilon(1_B \otimes b) = \varepsilon(1_B \otimes b)T \text{ for all } b \in B\}$$

and

$$\mathcal{A}_r := \{T \in \mathcal{A} \mid T\varepsilon(b \otimes 1_B) = \varepsilon(b \otimes 1_B)T \text{ for all } b \in B\}.$$

Often, we use L_b to denote $\varepsilon(b \otimes 1_B)$ and R_b to denote $\varepsilon(1_B \otimes b)$ for simplicity.

Remark 2.1.9. The main point of a B-B-non-commutative probability space is motivated by highlighting the difference between left and right actions of B, as operators on an underlying B-B-bimodule.

One sees that, for example (c.f. Remark 2.3.5 [19]), if $(\mathcal{A}, \varepsilon, E)$ is a B-B-noncommutative probability space, then $(\mathcal{A}_\ell, E)$ is a B-non-commutative probability space with $B \cong \varepsilon(B \otimes 1_B)$. Similarly, if (A, E) is a B-non-commutative probability space, then $\mathcal{A}$ is a natural B-B-bimodule with specified B-projection (c.f. Proposition 1.1.13). Then let $\varepsilon : B \otimes B^{\mathrm{op}} \to \mathcal{L}(\mathcal{A})$ be defined by extension of

$$\varepsilon(b_1 \otimes b_2) := L_{b_1} R_{b_2}.$$

Then $(\mathcal{L}(\mathcal{A}), \varepsilon, E_{\mathcal{L}(\mathcal{A})})$ is a B-B-non-commutative probability space.

Proposition 1.1.13 shows a simple but useful faithful representation of B-non-commutative probability spaces into spaces of oeprators. This representation can be refined in the case of B-B-non-commutative probability spaces, with a bit more attention to detail. In particular, we want to make sure that the left and right algebras of $\mathcal{A}$ map to left and right operators on our representation space respectively.

Theorem 2.1.10 ([11], Theorem 3.2.4)**.** *Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-non-commutative probability space. Then there is a B-B-bimodule with a specified B-projection $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ and a unital homomorphism $\theta : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ such that $\theta(L_{b_1} R_{b_2}) = L_{b_1} R_{b_2}$,*

$$\theta(\mathcal{A}_\ell) \subseteq \mathcal{L}_\ell(\mathcal{X}), \quad \theta(\mathcal{A}_r) \subseteq L_r(\mathcal{X}), \quad \text{and} \quad E_{\mathcal{L}(\mathcal{X})}(\theta(T)) = E_{\mathcal{A}}(T)$$

for all $b_1, b_2 \in B$ and $T \in \mathcal{A}$.

Because the details are necessary for later use, we mention the basic construction that verifies the above result. One may observe that it is of a very similar structure as Proposition 1.1.13, obfuscated only slightly by the fact that left or right action by B, when performed first, should not make any difference.

Given $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$, let

$$\mathcal{X} = B \oplus \left(\ker(E_{\mathcal{A}})/\mathrm{span}\{TL_b - TR_b | T \in \mathcal{A}, b \in B\}\right).$$

From here, a unital homomorphism $\theta : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ is defined with the required properties by way of

$$\theta(T)(b) = E_{\mathcal{A}}(TL_b) \oplus q(TL_b - L_{E_{\mathcal{A}}(TL_b)})$$

and

$$\theta(T)(q(A)) = E_{\mathcal{A}}(TA) \oplus q(TA - L_{E_{\mathcal{A}}(TA)})$$

for each $A \in \mathcal{A}$ with $E_{\mathcal{A}}(A) = 0$, $T \in \mathcal{A}$, and $b \in B$. Here q is the quotient map

$$q : \ker E_{\mathcal{A}} \to \ker(E_{\mathcal{A}})/\operatorname{span}\{TL_b - TR_b | T \in \mathcal{A}, b \in B\}.$$

The B-B-bimodule structure on $\mathcal{X}$ is then defined by

$$b \cdot \xi = \theta(L_b)(\xi) \quad \text{and} \quad \xi \cdot b = \theta(R_b)(\xi)$$

for all $b \in B$ and $\xi \in \mathcal{X}$.

Definition 2.1.11. Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-non-commutative probability space. A *pair of B-faces of $\mathcal{A}$* is a pair (A^ℓ, A^r) of subalgebras of $\mathcal{A}$ such that

$$\varepsilon(B \otimes 1_B)A^\ell\varepsilon(B \otimes 1_B) \subseteq \mathcal{A}_\ell \quad \text{and} \quad \varepsilon(1_B \otimes B)A^r\varepsilon(1_B \otimes B) \subseteq \mathcal{A}_r.$$

We will say that A^ℓ (respectively A^r) is *unital* if $\varepsilon(B \otimes 1_B) \subseteq A^\ell$ (respectively $\varepsilon(1_B \otimes B) \subseteq A^r$).

Definition 2.1.12. A *representation* of a family $\Gamma = \{(A_k^\ell, A_k^r)\}_{k\in K}$ of pairs of B-faces in a B-B-non-commutative probability space $(\mathcal{A}, \varepsilon, E)$, is a B-B-bimodule with specified B-projection $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ and a family of homomorphisms $\ell_k : \mathcal{A}_k^\ell \to \mathcal{L}_\ell(\mathcal{X})$ and $r_k : \mathcal{A}_k^r \to \mathcal{L}_r(\mathcal{X})$ which are unital when their domains are.

We will say that the representation is joint-distribution preserving if for every $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$, and $a_i \in \mathcal{A}_{\epsilon(i)}^{\chi(i)}$,

$$E(a_1 \cdots a_n) = E_{\mathcal{L}(\mathcal{X})}(\chi(1)_{\epsilon(1)}(a_1) \cdots \chi(n)_{\epsilon(i)}(a_n)).$$

Then, bi-free independence for pairs of B-faces is defined as follows.

Definition 2.1.13. A family $\Gamma = \{(A_k^\ell, A_k^r)\}_{k\in K}$ of pairs of unital B-faces of $\mathcal{A}$ is said to be *bi-free over B* if there exist homomorphisms $\ell_k : A_k^\ell \to \mathcal{L}_\ell(\mathcal{X}_k)$ and $r_k : A_k^r \to \mathcal{L}_r(\mathcal{X}_k)$ and if $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ is the reduced free product of the family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$, then $\{(\lambda_k \circ \ell_k, \rho_k \circ r_k)\}_{k\in K}$ is a joint distribution preserving representation of Γ.

We record an immediate simple but useful observation about the definition.

Proposition 2.1.14. *If $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ is a family of B-B-bimodules with specified B projections, and $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ is their free product, then the family $\{(\lambda_k(\mathcal{L}(\mathcal{X}_k)), \rho_k(\mathcal{L}(\mathcal{X}_k)))\}_{k\in K}$ is bi-free over B.*

Proof. This is the simple observation that $\lambda_k(\mathcal{L}(\mathcal{X}_k))$ and $\rho_k(\mathcal{L}(\mathcal{X}_k))$ act naturally as lefts and rights on $\mathcal{X}_k$ which when viewed as a subspace of $\mathcal{X}$ is invariant for these actions. $\square$

Example 2.1.15. Our recurring example of group algebras extends to this bi-free setting as well. Let $\{G_k\}_{k\in K}$ be a family of groups which is free over H in G (c.f. Example 1.4.7).

Now consider simultaneously the left and right acting operators on the free product. In particular, the proposition above implies the family $\{(\mathrm{alg}(\{\lambda_k(L_g) : g \in G_k\}), \mathrm{alg}(\{\rho_k(L_g) : g \in G_k\}))\}_{k\in K}$ is bi-free over $\mathbb{C}[H]$.

2.1.2 Bi-Non-Crossing Partitions

In line with the other independences, bi-free independence comes with its own cumulants. These cumulants make use of a similar non-crossing structure as the free cumulants.

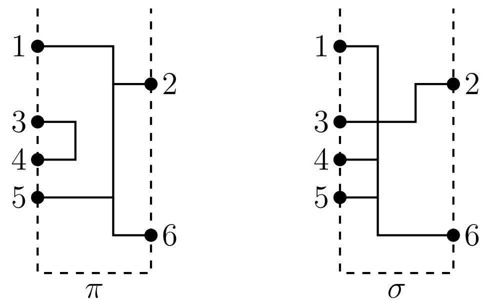
In particular, for any $\chi : \{1, \dots, n\} \to \{\ell, r\}$, s_χ, the permutation from Remark and Notation 2.1.3 extends is an order-isomorphism from $(\{1, \dots, n\}, <)$ to $(\{1, \dots, n\}, \prec_\chi)$. So we can use this to transform the non-crossing partitions into the so-called *bi-non-crossing* partitions.

Definition 2.1.16. A partition $\pi \in \mathcal{P}(n)$ is said to be *bi-non-crossing* with respect to χ if the permutation $s_\chi^{-1} \cdot \pi$ is non-crossing. Denote by $BNC(\chi)$ the set of bi-non-crossing partitions with respect to χ.

It is a simple observation that s_χ upgrades to a natural order-isomorphism of $\mathcal{P}(n)$, so that $BNC(\chi)$ is isomorphic to $NC(n)$, granting us that $BNC(\chi)$ is itself a lattice.

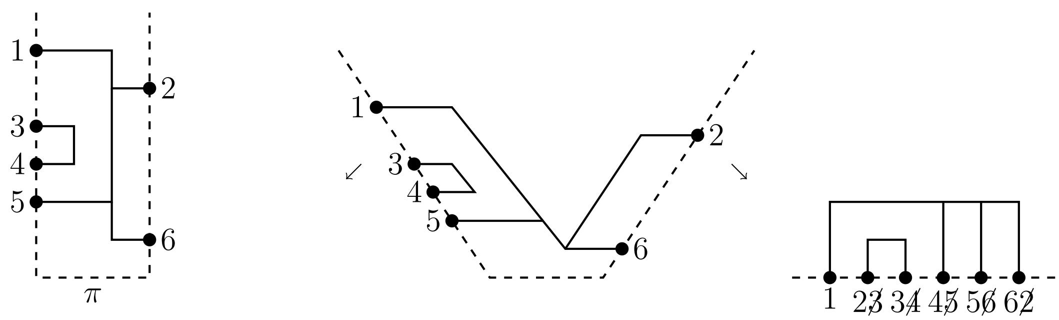
There is a visual representation associated to each of these partitions, a *bi-non-crossing diagram*. Let $\pi \in BNC(\chi)$, and create its bi-non-crossing diagram as follows. Along two parallel vertical dashed lines, place nodes labeled 1 to n from top to bottom, such that the nodes on the left line correspond to those values k such that $\chi(k) = \ell$ and the nodes on the right correspond to those values of k such that $\chi(k) = r$. Connect nodes together whose labels share a π block with lines. This can be done in a way so that lines from different blocks do not cross.

Example 2.1.17. Let $\chi : \{1, 2, 3, 4, 5, 6\} \to \{\ell, r\}$ such that $\chi^{-1}(\{\ell\}) = \{1, 3, 4, 5\}$, $\chi^{-1}(\{r\}) = \{2, 6\}$, let $\pi = \{\{1, 2, 5, 6\}, \{3, 4\}\}$ and $\sigma = \{\{1, 4, 5, 6\}, \{2, 3\}\}$. Then π is bi-non-crossing with respect to χ, while σ is not. This is visually apparent from their diagrams below.



In bi-non-crossing diagrams, we call the vertical lines in each block *spines*.

Remark 2.1.18. There is a simple visual way of relating a partition $\pi \in BNC(\chi)$ with a partition $\sigma_\pi \in NC(n)$. One simply takes the diagram for π, flattens it by pulling the left and right sides apart, and then reindexes the nodes to appear in order. Take for example π as in the example above. The process of finding σ_π is illustrated below



The partition σ_π is then read from the diagram on the right by replacing each node with $\{1, \ldots, 6\}$ from left to right in order. That is, $\sigma_\pi = \{\{1, 4, 5, 6\}, \{2, 3\}\}$.

Since $BNC(\chi)$ is a lattice that is isomorphic to $NC(n)$, it comes with its own Möbius function.

Definition 2.1.19. The *bi-non-crossing Möbius function* is the function

$$\mu_{BNC} : \bigcup_{n \geq 1} \bigcup_{\chi:\{1,\ldots,n\}\to\{\ell,r\}} BNC(\chi) \times BNC(\chi) \to \mathbb{C}$$

defined by

$$\sum_{\substack{\tau \in BNC(\chi) \\ \pi \leq \tau \leq \sigma}} \mu_{BNC}(\tau, \sigma) = \sum_{\substack{\tau \in BNC(\chi) \\ \pi \leq \tau \leq \sigma}} \mu_{BNC}(\pi, \tau) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{otherwise} \end{cases}.$$

2.1.3 Bi-Multiplicative Functions and Bi-Free Cumulants

We need the correct functions which exploit the combinatorial structure of these partitions, in the same spirit as our moment functions from Chapter 1.

Definition 2.1.20. Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-non-commutative probability space. Fix $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\pi \in BNC(\chi)$, and $a_1, \ldots, a_n \in \mathcal{A}$. Define

$$E_\pi(a_1, \ldots, a_n) \in B$$

recursively as follows. Let V be the block of π with the largest minimum element. Then,

- If $\pi = 1_\chi$ then
$$E_\pi(a_1, \ldots, a_n) = E_{\mathcal{A}}(a_1 \cdots a_n).$$
- If there is some $k \in \{1, \ldots n-1\}$ such that $V = \{k+1, \ldots, n\}$, then
$$E_\pi(a_1, \ldots, a_n) = E_{\pi|_{V^c}}(a_1, \ldots, a_k L_{E_{\mathcal{A}}(a_{k+1} \cdots a_n)}).$$
- Otherwise, $\min(V)$ is adjacent to a spine. Let W be the block of π corresponding to the spine adjacent to $\min(V)$. Let k be the smallest element of W such that $k > \min(V)$. Then define
$$E_\pi(a_1, \ldots, a_n) = \begin{cases} E_{\pi|_{V^c}}((a_1, \ldots, a_{k-1}, L_{E_{\pi|_V}((a_1, \ldots, a_n)|_V)} a_k, \ldots, a_n)|_{V^c}) & \text{if } \chi(\min(V)) = \ell \\ E_{\pi|_{V^c}}((a_1, \ldots, a_{k-1}, R_{E\pi|_V((a_1, \ldots, a_n)|_V)} a_k, \ldots, a_n)|_{V^c}) & \text{if } \chi(\min(V)) = r \end{cases}.$$

Remark 2.1.21. The function above that evaluates bi-non-crossing partition indexed moments is nearly identical to the non-crossing case from Definition 1.3.12, except that we pay attention to left and right acting operators.

Definition 2.1.22. Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-noncommutative probability space. The *bi-free operator-valued moment function*

$$\mathcal{E} : \bigcup_{n \geq 1} \bigcup_{\chi : \{1, \ldots, n\} \to \{\ell, r\}} BNC(\chi) \times \mathcal{A}_{\chi(1)} \times \cdots \times \mathcal{A}_{\chi(n)} \to B$$

is defined, for each $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\pi \in BNC(\chi)$, and $a_k \in \mathcal{A}_{\chi(k)}$, by

$$\mathcal{E}_\pi(a_1, \ldots, a_n) = E_\pi(a_1, \ldots, a_n).$$

Similarly, the *bi-free operator-valued cumulant function*

$$\kappa_{**} : \bigcup_{n\geq 1} \bigcup_{\chi:\{1,\ldots,n\}\to\{\ell,r\}} BNC(\chi) \times \mathcal{A}_{\chi(1)} \times \cdots \times \mathcal{A}_{\chi(n)} \to B$$

is defined, for each $\chi : \{1,\ldots,n\} \to \{\ell, r\}$, $\pi \in BNC(\chi)$, and $a_k \in \mathcal{A}_{\chi(k)}$, by

$$\kappa_{**,\pi}(a_1,\ldots,a_n) = \sum_{\substack{\sigma\in BNC(\chi)\\ \sigma\leq\pi}} \mathcal{E}_\sigma(a_1,\ldots,a_n)\mu_{BNC}(\sigma,\pi).$$

The following result establishes the equivalence of the combinatorial approach with the operator approach from the definition.

Theorem 2.1.23 (Theorem 8.1.1, [11])**.** *Let $\Gamma = \{(A_k^\ell, A_k^r)\}_{k\in K}$ be a family of pairs of B-faces in a B-B-non-commutative probability space $(\mathcal{A}, \varepsilon, E)$. Then Γ is bi-free over B if and only if for all $n \geq 0$, $\chi : \{1,\ldots,n\} \to \{\ell, r\}$, $\epsilon : \{1,\ldots,n\} \to K$, and $a_i \in A_{\epsilon(i)}^{\chi(i)}$,*

$$\kappa_{**,\mathbf{1}_n}(a_1,\ldots,a_n) = 0$$

unless ϵ is constant.

Remark 2.1.24. That $BNC(\chi)$ is isomorphic to $NC(n)$, and given our diagram equivalence as in Remark 2.1.18, one finds an immediate translation of the cumulant inclusion-exclusion (Proposition 1.3.30) to the bi-free setting.

Proposition 2.1.25 (Bi-Free Cumulant Inclusion-Exclusion)**.** *Let $\{(A_k^\ell, A_k^r)\}_{k\in K}$ be a bi-freely independent family of pairs of unital B-faces in a B-B-non-commutative probability space $(\mathcal{A}, \varepsilon, E)$. Let $n \in \mathbb{N}$, and $\epsilon : \{1,\ldots,n\} \to K$, $\chi : \{1,\ldots,n\} \to \{\ell, r\}$ and $a_i \in A_{\epsilon(i)}^{\chi(i)}$. Let $M \subseteq BNC(\chi)$ be the set of distinct maximal elements of $[\mathbf{0}_n, \ker\epsilon] \cap BNC(\chi)$. Then*

$$E(a_1\cdots a_n) = \sum_{m=1}^{|M|} (-1)^{m+1} \sum_{\{\tau_1,\ldots,\tau_m\}\subseteq M} \mathcal{E}_{BNC,\tau_1\wedge\cdots\wedge\tau_m}(a_1,\ldots,a_n).$$

2.2 Liu's Multi-Algebra Independences

Liu noticed in [5], that Voiculescu's idea of simultaneously considering left and right acting operators could be generalized. As we saw in section 1.4, both the Boolean and monotone independences also arise naturally through certain operator actions on the free product.

So for example, by simultaneously considering left-regular acting operators on a free product with Boolean operators, one could define a notion of *free-Boolean* independence for pairs of algebras, or a *Boolean-monotone* independence. Liu efficiently captured this generalization in the case when $B = \mathbb{C}$, which we generalize to the amalgamated setting as follows.

2.2.1 Liu's Pair-Algebra Independences

Definition 2.2.1. Let $(\mathcal{A}, E, \varepsilon)$ be a B-B-non-commutative probability space. Let $\Gamma = \{(A_k^\ell, A_k^r)\}_{k\in K}$ be a family of pairs of B-faces of $\mathcal{A}$. If there exist representations $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ with corresponding homomorphisms $\ell_k : A_k^\ell \to \mathcal{L}_\ell(\mathcal{X}_k)$ and $r_k : A_k^r \to \mathcal{L}_r(\mathcal{X}_k)$ and $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ is the corresponding reduced free product, then say Γ is

- *free-Boolean independent* if each A_k^ℓ is unital and $\{(\lambda_k \circ \ell_k, \gamma_{\uplus,k} \circ r_k)\}_{k\in K}$ is joint-distribution preserving,
- *free-monotone independent* if each A_k^ℓ is unital and $\{(\lambda_k \circ \ell_k, \gamma_{\lhd,k} \circ r_k)\}_{k\in K}$ is joint-distribution preserving,
- *Boolean-Boolean independent* if $\{(\gamma_{\uplus,k} \circ \ell_k, \gamma_{\uplus,k} \circ r_k)\}_{k\in K}$ is joint-distribution preserving,
- *monotone-monotone independent* if $\{(\gamma_{\rhd,k} \circ \ell_k, \gamma_{\lhd,k} \circ r_k)\}_{k\in K}$ is joint-distribution preserving,
- *monotone-Boolean independent* if $\{(\gamma_{\rhd,k} \circ \ell_k, \gamma_{\uplus,k} \circ r_k)\}_{k\in K}$ is joint-distribution preserving,

where $\gamma_{\uplus,k}$, $\gamma_{\rhd,k}$ and $\gamma_{\lhd,k}$ are as in Remark and Notations 1.4.13 and 1.4.16.

Remark 2.2.2. Given the equivalences of independences with operator actions, the naming convention of the independences above is natural. For example, if $\{(A_k^\ell, A_k^r)\}_{k\in K}$ is free-Boolean independent, then $\{A_k^\ell\}_{k\in K}$ is a freely independent family and $\{A_k^r\}_{k\in K}$ is a Boolean independent family (c.f. Section 1.4). Likewise with the monotone mixtures.

However, one might notice that Liu's definition does not include mixtures with tensor independent operators, as they don't naturally act on the same space. Other attempts at combining tensor independence with the Boolean and monotone independences have been introduced in [1], though they do not follow the same idea as Liu.

Remark 2.2.3. The above list is not complete, for we could have also considered notions of *monotone-Boolean*, *monotone-free*, or *Boolean-free* independences, which can be defined analogously. However, at least in the case of $B = \mathbb{C}$ the difference between independences such as *free-monotone* and *monotone-free* are largely cosmetic. It is only in the general amalgamated case that we may need to concern ourselves with left and right operators appearing in the algebras.

Henceforth we will be careful about the orderings of the independences when appropriate, but make no comment about choosing, for example *free-Boolean* independence to study over *Boolean-free* independence.

Remark 2.2.4. One property we might naturally demand for a notion of independence to satisfy is that the joint distribution of a family of pairs of faces should depend only on individual (joint) distributions of the pairs. This is indeed satisfied by Liu's independences, however we will delay justification to Section 2.3 when we have a more potent tool to use.

Furthermore, it is easily verified that if $\{(A_k^\ell, A_k^r)\}_{k\in K}$ is an independent family of pairs of B-faces in any of the senses above, then the representations (ℓ_k, r_k) are necessarily expectation preserving. Indeed since λ_k, ρ_k, $\gamma_{\uplus,k}$, $\gamma_{\lhd,k}$, and $\gamma_{\rhd,k}$ are expectation preserving from $\mathcal{L}(\mathcal{X}_k)$, and their compositions with each ℓ_k and r_k preserve joint distributions, by looking at expectations of word-length $n = 1$, that the maps ℓ_k and r_k must be expectation preserving.

One consequence of this is that the individual families $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in k}$ and $\{(\ell_k, r_k)\}_{k\in K}$ matter only insofar as they faithfully represent each pair (A_k^ℓ, A_k^r). So for example, we may choose a suitable space, $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ on which to represent each pair (A_k^ℓ, A_k^r) without fear of losing information about the joint-distribution. We will take advantage of this fact.

Remark 2.2.5. It turns out from the definition of the representations, Boolean-Boolean independence turns out not to yield much more of interest than simply Boolean independence. Indeed, if $\{(A_k^\ell, A_k^r)\}_{k\in K}$ is a Boolean-Boolean independent family, then for $a \in A_k^\ell$, we have

$$\begin{aligned}
\gamma_{\uplus,k}(\ell_k(a))L_b &= P_{\uplus,k}\lambda_k(\ell_k(a))P_{\uplus,k}L_b \\
&= P_{\uplus,k}\rho_k(\ell_k(a))L_b && \text{(Lemma 1.4.11)} \\
&= L_b P_{\uplus,k}\rho_k(\ell_k(a)) \\
&= L_b\gamma_{\uplus,k}(\ell_k(a))
\end{aligned}$$

This shows that we have $\gamma_{\uplus,k}(\ell_k(A_k^\ell)) \subseteq \mathcal{L}_r(\mathcal{X})$. By symmetry, if $a \in A_k^r$, then $\gamma_{\uplus,k}(r_k(a)) \in \mathcal{L}_\ell(\mathcal{X})$. So the supposed left acting Boolean operators, act also like rights in distribution, while the supposed right acting Boolean operators act like lefts in distribution. The only thing separating the definition of a Boolean-Boolean independent family from being a Boolean independent family, therefore, is simply the constituents of each of the faces. In particular, we may embed Boolean-Boolean independence into Boolean independence as follows.

Let $\{(A_k^\ell, A_k^r)\}_{k\in K}$ be a Boolean-Boolean independent family of pairs of B-faces in a B-B-non-commutative probability space $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$. From Remark 2.2.4 above, we may let, for each $k \in K$, $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ be a copy of the B-B-bimodule with specified B-projection from Theorem 2.1.10, with $\ell_k = \theta|_{A_k^\ell}$ and $r_k = \theta|_{A_k^r}$. Now consider $A_k = \varepsilon(B \otimes 1_B)\text{-alg}(A_k^\ell, A_k^r)$, the smallest $\varepsilon(B \otimes 1_B)$-algebra generated by the pair A_k^ℓ and A_k^r in $\mathcal{A}$. Then A_k^ℓ and A_k^r embed naturally in A_k, and $\theta|_{\mathcal{A}_k}$ extends ℓ_k and r_k. It follows that $\gamma_{\uplus} \circ \theta$ preserves joint distributions for the family $\{(A_k^\ell, A_k^r)\}_{k\in K}$ by checking moments.

Note that the above shows that given any Boolean-Boolean independent family $\{(A_k^\ell, A_k^r)\}_{k\in K}$, we can find a Boolean independent family $\{\tilde{A}_k\}_{k\in K}$ (above $\tilde{A}_k = \gamma_{\uplus,k}(\theta(A_k))$) that houses it faithfully. When $B = \mathbb{C}$, the above construction is not even required, as Boolean-Boolean independence is simply Boolean independence.

A non-trivial notion of *bi-Boolean independence* was studied in [3], but requires simultaneous consideration of two expectations, hence arises in a different context to that we are considering.

The monotone-Boolean and monotone-monotone independences are more interesting than the Boolean-Boolean case. We will have more to say about these independences in Chapter 3.

2.2.2 Liu's Triple-Algebra Independences

In fact, Liu extended this idea to include triples of algebras. We again generalize this definition to the amalgamated setting.

Definition 2.2.6. In a B-B-non-commutative probability space $(\mathcal{A}, E, \varepsilon)$, a *triple of B-faces* is a triple (A^ℓ, A^r, A^m) of subalgebras of $\mathcal{A}$ such that (A^ℓ, A^r) is a pair of B-faces of $\mathcal{A}$, and

$$\varepsilon(B \otimes 1_B)A^m\varepsilon(B \otimes 1_B) \subseteq A^m \subseteq A_\ell,$$

and we will say that A^m is *unital* if $\varepsilon(B \otimes 1_B) \subseteq A^m$.

Remark 2.2.7. We've had to make one seemingly unnatural concession in the definition above, that the third algebra in such a triple of B-faces is also comprised of left operators.

However, since all of the representations we discussed in Section 1.4 are built around left and right regular representations on the free product, we are forced to make such a choice.

As with pairs of B-faces, we also have representations for triples of B-faces, defined in the expected way.

Definition 2.2.8. A *representation* of a family $\Gamma = \{(A_k^\ell, A_k^r, A_k^m)\}_{k\in k}$ of triples of B-faces is simply a representation of the family $\{(A_k^\ell, A_k^r)\}_{k\in K}$ together with homomorphisms $m_k : A_k^m \to \mathcal{L}(\mathcal{X}_k)$ which are unital when their domains are. The representation is joint-distribution preserving if for every $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell, r, m\}$, $\epsilon : \{1, \ldots, n\} \to K$, and $a_i \in A_{\epsilon(i)}^{\chi(i)}$,

$$E(a_1 \cdots a_n) = E_{\mathcal{L}(\mathcal{X})}(\chi(1)_{\epsilon(1)}(a_1) \cdots \chi(n)_{\epsilon(n)}(a_n)).$$

Definition 2.2.9. Let $(\mathcal{A}, E, \varepsilon)$ be a B-B-noncommutative probability space. Given a family $\Gamma = \{(A_k^\ell, A_k^r, A_k^m)\}_{k\in K}$ of triples of B-faces in $\mathcal{A}$. Suppose there exists a family representations $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ with corresponding homomorphisms $\ell_k : A_k^\ell \to \mathcal{L}_\ell(\mathcal{X}_k)$ and $r_k : A_k^r \to \mathcal{L}_r(\mathcal{X}_k)$, and $m_k : A_k^m \to \mathcal{L}_\ell(\mathcal{X}_k)$. Let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be the corresponding reduced free product. Say the family Γ is

- *free-free-Boolean independent* if A_k^ℓ and A_k^r are unital and $\{(\lambda_k \circ \ell_k, \rho_k \circ r_k, \gamma_{\uplus,k} \circ \ell_k)\}_{k\in K}$ is joint-distribution preserving,
- *free-free-monotone independent* if A_k^ℓ and A_k^r are unital and $\{(\lambda_k \circ \ell_k, \rho_k \circ r_k, \gamma_{\rhd,k} \circ \ell_k)\}_{k\in K}$ is joint-distribution preserving,
- *free-Boolean-monotone independent* if A_k^ℓ is unital and $\{(\lambda_k \circ \ell_k, \gamma_{\uplus,k} \circ r_k, \gamma_{\rhd,k} \circ \ell_k)\}_{k\in K}$ is joint-distribution preserving,
- *free-monotone-monotone* if A_k^ℓ is unital and $\{(\lambda_k \circ \ell_k, \gamma_{\lhd,k} \circ r_k, \gamma_{\rhd,k} \circ \ell_k)\}_{k\in K}$ is joint-distribution preserving.

Remark 2.2.10. One might hope for a notion of *tri-freeness* by following the extension from pairs to triples of B-faces. However, given the set up and Remark 2.2.7, there is no clear idea of how one could define such an independence. For the same reason, notions of *tri-Boolean* and *tri-monotone* independence are superfluous.

Remark 2.2.11. One can see from the definition above that

- both free-monotone and monotone-monotone independences embed naturally into free-monotone-monotone families,
- bi-free independence embeds into both the free-free-Boolean and free-free-monotone independences, and
- slightly less obviously (perhaps using anti-homomorphisms if necessary), free-monotone independence embeds into the free-free-monotone independence.

We have omitted a definition of a Boolean-Boolean-monotone independence or a free-Boolean-Boolean independence due to Remark 2.2.5, where we argued that Boolean-Boolean independence is nothing more interesting than simply Boolean independence.

We will spend more time looking at the free-Boolean and free-free-Boolean independences in the following subsection.

2.2.3 Free-Free-Boolean Independence

The main result of this work has to do with the free-free-Boolean independence as studied by Liu in [6]. This will have immediate consequences for the free-Boolean independence studied in [5] and [7].

Remark 2.2.12. Notice that from the definition above, if $\{(A_k^\ell, A_k^r, A_k^b)\}_{k\in K}$ is a free-free-Boolean independent family, then $\{(A_k^\ell, A_k^r)\}_{k\in K}$ is a bi-free family. It is also immediate that the family $\{A_k^b\}_{k\in K}$ is a Boolean independent family.

Furthermore, it is readily verified that in fact, the family of pairs of algebras $\{(A_k^\ell, A_k^b)\}_{k\in K}$ are free-Boolean independent with amalgamation over B in the sense of [7], noting that A_k^b is a B-B-bimodule by left and right multiplication by $\epsilon(B \otimes 1_B)$.

Lastly, in the case where $B = \mathbb{C}$, this definition readily conforms with a free-free-Boolean independent family as defined in [6].

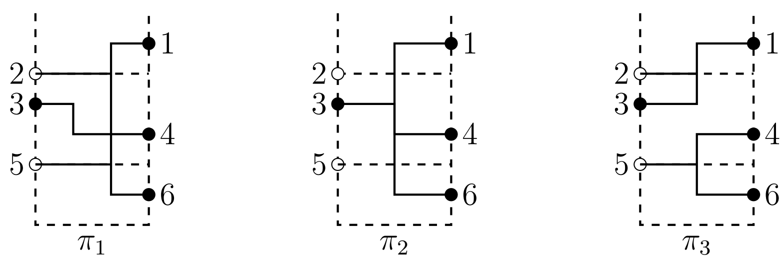
The lack of monotone components in this type of independence allows for a more direct combinatorial theory of cumulants. To record these, we not only need to keep track of left and right acting freely independent operators, but also the Boolean independent ones. For this, we will consider functions $\chi : \{1, \dots, n\} \to \{\ell, r, b\}$. Liu defined with these the following partitions.

Definition 2.2.13. Let $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell, r, b\}$, and $\omega : \{\ell, r, b\} \to \{\ell, r\}$ defined by $\omega(\ell) = \omega(b) = \ell$ and $\omega(r) = r$. Then say $\pi \in \mathcal{P}(n)$ is *interval bi-non-crossing* with respect to χ if $\pi \in BNC(\omega \circ \chi)$, and whenever $i < j < k$, $i \sim_\pi k$, and $\chi(j) = b$, then $i \sim_\pi j$.

Example 2.2.14. Let us see a simple example, and take a look at their diagrams. Let $n = 6$ and $\chi^{-1}(\ell) = \{3\}$, $\chi^{-1}(b) = \{2, 5\}$, and $\chi^{-1}(r) = \{1, 4, 6\}$.

Let $\pi_1 = \{\{1, 2, 5, 6\}, \{3, 4\}\}$, $\pi_2 = \{\{1, 3, 4, 6\}, \{2\}, \{5\}\}$, and $\pi_3 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$. Then π_1 and π_2 are *not* interval bi-non-crossing, since $\pi_1 \notin BNC(\omega \circ \chi)$ and $1 \sim_{\pi_2} 3$ but $1 \nsim_{\pi_2} 2$. However, π_3 is interval bi-non-crossing with respect to χ.

These partitions are easier understood visually. We use a similar style of diagram as in the bi-non-crossing partitions, adopting the convention that b-labelled numbers appear on the left since in the definition, the Boolean operators act on the left. We distinguish them by colouring their nodes differently. We also add a dashed line horizontally from the b-labelled numbers To highlight the fact that a block should not be allowed to cross over them without collecting them in the block. The diagrams corresponding to the partitions above are given below.

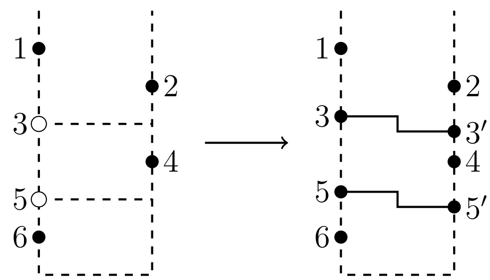


Visually, π_1 has a crossing and π_2 contains a block which crosses the dashed lines without including the corresponding b-labelled numbers.

Remark 2.2.15. One might note the striking similarity of the interval bi-non-crossing partition diagrams in connection with the bi-non-crossing diagrams. The exception is the colourings and the dashed lines. However the dashed lines act as a certain barrier that blocks may not cross without joining with it. However, with some thought, the bi-non-crossing diagrams can handle this exception too, by inflating the domain a bit.

In particular, by replacing each b-labelled number with a left-right pair that we demand stay connected, the lack of crossings for our diagrams force any block that passes over this pair to include it.

Take for example $\chi^{-1}(\ell) = \{1, 6\}$, $\chi^{-1}(b) = \{3, 5\}$ and $\chi^{-1}(r) = \{2, 4\}$. Then any interval bi-non-crossing partition corresponds to a block diagram on the frame below (left). To such a diagram, we associate with a bi-non-crossing diagram on $\{1, 2, 3, 3', 4, 5, 5', 6\}$ demanding that $3 \sim 3'$ and $5 \sim 5'$ as seen below (right). Such diagrams have all the same combinatorial properties as the interval bi-non-crossing partitions.



This process of converting interval bi-non-crossing partitions faithfully into bi-non-crossing partitions will be formalized in Chapter 3, which will help us understand how the free-free-Boolean independence can be embedded into bi-free independence.

Remark and Notation 2.2.16. For each $n \in \mathbb{N}$ and $\chi : \{1, \ldots, n\} \to \{\ell, r, b\}$, we have that $IBNC(\chi)$ is a lattice. This means that it comes with its own Möbius function

$$\mu_{IBNC} : \bigcup_{n\geq 1} \bigcup_{\chi:\{1,\ldots,n\}\to\{\ell,r,b\}} IBNC(\chi) \times IBNC(\chi) \to \mathbb{C}$$

defined by

$$\sum_{\substack{\tau\in IBNC(\chi)\\ \pi\leq\tau\leq\sigma}} \mu_{IBNC}(\tau, \sigma) = \sum_{\substack{\tau\in IBNC(\chi)\\ \pi\leq\tau\leq\sigma}} \mu_{IBNC}(\pi, \tau) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{otherwise} \end{cases}.$$

This allows us to define the free-free-Boolean cumulants as follows.

Definition 2.2.17. The *free-free-Boolean operator-valued cumulant function*

$$\kappa_{**\uplus} : \bigcup_{n\geq 1} \bigcup_{\chi:\{1,\ldots,n\}\to\{\ell,r,b\}} IBNC(\chi) \times \mathcal{A}_{\omega(\chi(1))} \times \cdots \times \mathcal{A}_{\omega(\chi(n))} \to B$$

is defined, for each $\chi : \{1, \ldots, n\} \to \{\ell, r, b\}$, $\pi \in IBNC(\chi)$, and $a_k \in \mathcal{A}_{\omega(\chi(k))}$, by

$$\kappa_{**\uplus,\pi}(a_1, \ldots, a_n) = \sum_{\substack{\sigma\in IBNC(\chi)\\ \sigma\leq\pi}} \mathcal{E}_\sigma(a_1, \ldots, a_n)\mu_{IBNC}(\sigma, \pi).$$

Liu showed the following result, establishing the equivalence of the unamalgamated ($B = \mathbb{C}$) free-free-Boolean independence, and vanishing of mixed cumulants (c.f. Propositions 1.3.23, 1.3.24, 1.3.25, and 2.1.23)

Theorem 2.2.18 (Theorem 5.5, [6])**.** *A family $\{(A_k^\ell, A_k^r, A_k^b)\}_{k\in K}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$ is free-free-Boolean independent if and only if for $n \in \mathbb{N}$, $\chi : \{1, \dots, n\} \to \{\ell, r, b\}$, $\epsilon : \{1, \dots, n\} \to K$, and $a_i \in A_{\epsilon(i)}^{\chi(i)}$ we have*

$$\kappa_{**\uplus,\pi}(a_1, \dots, a_n) = 0$$

whenever ϵ is not constant.

Remark 2.2.19. Now we have equivalent characterizations of the free-free-Boolean independence with operators and cumulants, one might hope for some moment conditions to add on to this. Doing this for the general (amalgamated) case seems to be a dead end at the moment, as these would imply moment conditions for amalgamated bi-free independence.

In [7], Liu and Zhong showed some moment conditions for the amalgamated free-Boolean independence. The idea revolved around the fact that if a string of operators from the same pair of B-faces contained within it a Boolean acting operator, then the whole block acted like a Boolean operator. This combined with the moment conditions for free independence yielded their moment conditions (with several details to check).

It is likely that when $B = \mathbb{C}$, that one could adapt the ideas of Liu and Zhong together with Charlesworth's moment conditions for bi-freeness to achieve moment conditions for free-free-Boolean independence over $\mathbb{C}$.

2.3 The LR-Diagram Calculus on the Free Product

So far, we have focused solely on the expectations of strings of operators, but we will make use of an accounting of their vector components as well. Since we are working with bi-freely independent operators, it is useful to know how a vector like $Z_1 \cdots Z_n 1_B$ decomposes as a sum of pure tensors in the free product. This seems like a daunting task, but luckily there is a combinatorial tool, similar to the bi-non-crossing partitions that help us accomplish exactly that when our operators come from bi-free pairs.

Throughout this section, in addition to a function $\chi : \{1, \dots, n\} \to \{\ell, r\}$, let $\epsilon : \{1, \dots, n\} \to K$ for some fixed index set K.

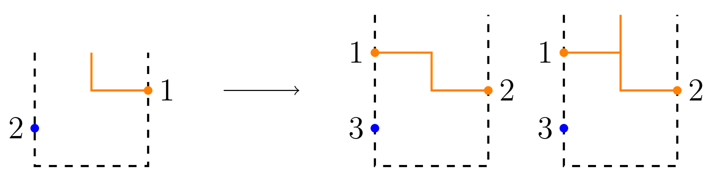
Definition 2.3.1. The set $LR(\chi, \epsilon)$ of *shaded LR diagrams* is defined recursively. When $n = 0$, $LR(\chi, \epsilon)$ contains only an empty diagram. If $n > 0$, let $\overline{\chi}(k) = \chi(k-1)$ and $\overline{\epsilon}(k) = \epsilon(k-1)$ for $k \in \{2, \dots, n\}$. Then, each $D \in LR(\overline{\chi}, \overline{\epsilon})$ corresponds to two unique elements of $LR(\chi, \epsilon)$ by the following processes:

- First, add to the top of D a node on the side corresponding to $\chi(1)$, shaded by $\epsilon(1)$.
- If a string of shade $\epsilon(1)$ extends from the top of D, connect it to the added node.
- Then, choose to either extend a string from the added node to the top of the new diagram or not, and extend any other strings from D to the top of the new diagram.

We denote by $LR_k(\chi, \epsilon)$, the subset of $LR(\chi(\epsilon))$ consisting of diagrams with exactly k strings that reach the top gap.

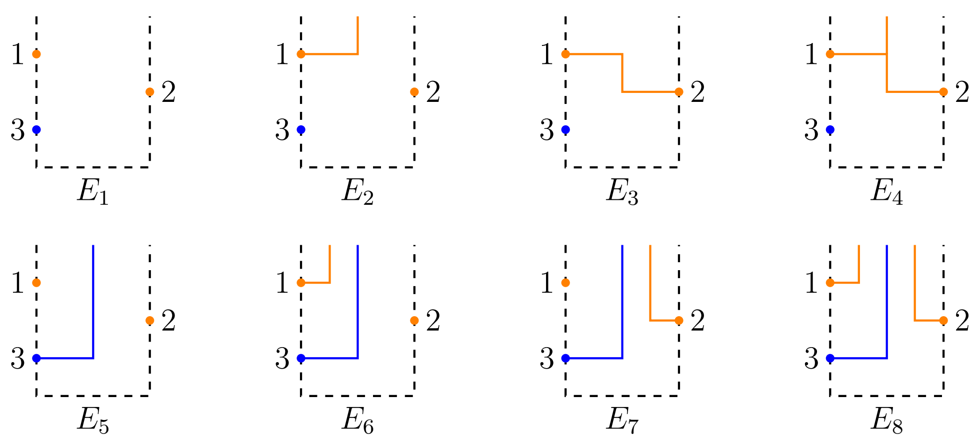
We refer to the vertical portions of strings in the diagrams as *spines* and horizontal segments connecting nodes to spines as *ribs*.

Example 2.3.2. As an example, consider $\chi = (\ell, r, \ell)$ and $\epsilon = ({}', {}', {}'')$, then $\overline{\chi} = (r, \ell)$ and $\overline{\epsilon} = ({}', {}'')$. Consider the diagram $D \in LR(\overline{\chi}, \overline{\epsilon})$ formed by extending a spine from the node 1 to the top gap, and leaving the node 2 isolated. Then the two diagrams we get from extending D are $D_1, D_2 \in LR(\chi, \epsilon)$, pictured below.



D D_1 D_2

Pictured below is the complete collection $LR(\chi, \epsilon)$ with χ and ϵ as above.



Remark 2.3.3. Note that any diagram in $LR_0(\chi, \epsilon)$ corresponds to a partition $\pi \in BNC(\chi)$ by defining the blocks in π to be those collections of numbers connected by strings in the diagram.

In the above example $\{E_1, E_3\}$ are the two diagrams in $LR_0(\chi, \epsilon)$ corresponding to the bi-non-crossing partitions $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$ and $\pi_2 = \{\{1, 2\}, \{3\}\}$.

Note that not all bi-non-crossing partitions appear in $LR_0(\chi, \epsilon)$. Indeed, one might first observe that the $\pi \in BNC(\chi)$ which appear in $LR_0(\chi, \epsilon)$ have the property that $\pi \leq \ker \epsilon$. But even then it is not true that we necessarily get *all* such partitions.

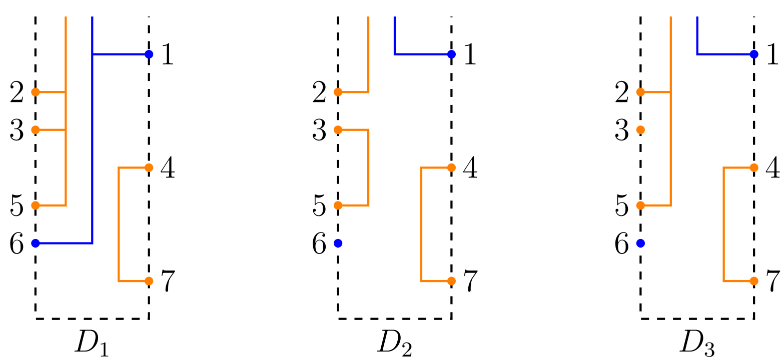
Consider for example $\chi \equiv \ell$ on $\{1, 2, 3\}$, and ϵ constant as well. One sees that the partition $\pi = \{\{1, 3\}, \{2\}\} \in BNC(\chi)$ but it is not an element of $LR_0(\chi, \epsilon)$.

The LR diagrams themselves are not quite sufficient, since in order to perform an accurate accounting, we need to be able to cut spines between ribs. The reason for this appears in the proof of Theorem 2.3.8 below.

Definition 2.3.4. Let $D_1, D_2 \in LR(\chi, \epsilon)$, we say that D_1 is a *lateral refinement* of D_2, and write $D_1 \leq_{\text{lat}} D_2$ if D_2 can be made from D_1 by a sequence of cuts to D_1's spines between ribs (not cutting along the top-gap).

Let $LR^{\text{lat}}(\chi, \epsilon)$ denote the closure of $LR(\chi, \epsilon)$ under lateral refinement.

Example 2.3.5. Consider for example the following diagrams. Below we have $D_2 \leq_{\text{lat}} D_1$ but D_2 is not itself an LR-diagram, as it cannot be made through the recursive definition. We also have $D_3 \leq D_1$ but $D_3 \not\leq_{\text{lat}} D_1$ as there is no lateral refinement that can isolate the node 3.



Definition 2.3.6. Let $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k \in K}$ be B-B-bimodules with specified projections. Let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be their reduced free product with amalgamation over B and for each $k \in K$ let λ_k

and ρ_k be respectively the left and right regular representations of $\mathcal{L}(\mathcal{X}_k)$ on $\mathcal{X}$. For each $i \in \{1, \dots, n\}$, let $T_i \in \mathcal{L}_{\chi(i)}(\mathcal{X}_{\epsilon(i)})$. Define $\mu_i(T_i) = \lambda_{(\epsilon(i))}(T_i)$ if $\chi(i) = \ell$ and $\mu_i(T_i) = \rho_{\epsilon(i)}(T_i)$ if $\chi(i) = r$. For each $D \in LR^{\text{lat}}(\chi, \epsilon)$ we define $E_D(\mu_1(T_1), \dots, \mu_n(T_n))$ recursively as follows. Apply the same recursive process as in Definition 2.1.20 until every block of D has a spine reaching the top. If every block of D has a spine reaching the top gap, enumerate them from left to right according to their spines as $V_1, \dots, V_m$ with $V_j = \{i_{j,1} < \cdots < i_{j,q_j}\}$, and set

$$E_D(\mu_1(T_1), \dots, \mu_n(T_n)) = [(1 - p_{\epsilon(i_{1,1})})T_{i_{1,1}} \cdots T_{i_{1,q_1}} 1_B] \otimes \cdots \otimes [(1 - p_{\epsilon(i_{m,1})})T_{i_{m,1}} \cdots T_{i_{m,q_m}} 1_B].$$

Remark 2.3.7. Given the notations as above, a simple but useful remark is that we have

$$E_D(\mu_1(T_1), \dots, \mu_n(T_n)) \in \mathring{\mathcal{X}}_{\epsilon(i_{1,1})} \otimes \cdots \otimes \mathring{\mathcal{X}}_{\epsilon(i_{m,1})}.$$

We have the following result.

Theorem 2.3.8 ([11], Lemma 7.1.3)**.** *With notation as in Definition 2.3.6,*

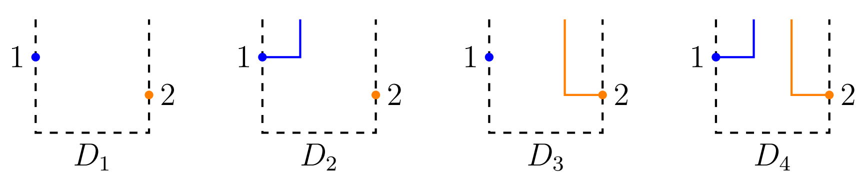
$$\mu_1(T_1) \cdots \mu_n(T_n) 1_B = \sum_{D \in LR^{lat}(\chi,\epsilon)} c_D E_D(\mu_1(T_1), \dots, \mu_n(T_n)),$$

where c_D are constants depending only on the diagram D.

Example 2.3.9. As a simple example, consider $S \in \mathcal{L}_\ell(\mathcal{X}_1)$ and $T \in \mathcal{L}_r(\mathcal{X}_2)$ and $\mathcal{X} = \mathcal{X}_1 * \mathcal{X}_2$. Then since $\mathcal{L}_{\mathcal{X}}(a) = pa1_B$, we have

$$\begin{aligned}
\lambda_1(S)\rho_2(T)1_B &= (pS1_B + (1-p)S1_B) \otimes (pT1_B + (1-p)T1_B) \\
&= E_{\mathcal{L}(\mathcal{X})}(\lambda_1(S))E_{\mathcal{L}(\mathcal{X})}(\rho_2(T)) + R_{E_{\mathcal{L}(\mathcal{X})}(\rho_2(T))}(1-p)S1_B \\
&\quad + L_{E_{\mathcal{L}(\mathcal{X})}(\lambda_1(S))}(1-p)T1_B + (1-p)S1_B \otimes (1-p)T1_B \\
&= E_{D_1}(\lambda_1(S), \rho_2(T)) + E_{D_2}(\lambda_1(S), \rho_2(T)) + E_{D_3}(\lambda_1(S), \rho_2(T)) + E_{D_4}(\lambda_1(S), \rho_2(T)),
\end{aligned}$$

where the D_i are



In Lemma 7.1.3 of [11], the constants c_D are made explicit, but we have no need for such precision. Keeping them as abstract constants will suffice and help to keep the notational noise to a minimum.

In fact, the proof of the above result can be extended more generally, as it details a recursive way to add on to the diagram-indexed vectors, E_D, with more operators. We start with some notation to help us keep track.

Definition 2.3.10. Let $i \leq n \in \mathbb{N}$. Let $\tilde{\chi} = \chi|_{\{i,\ldots,n\}}$, $\tilde{\epsilon} = \epsilon|_{\{i,\ldots,n\}}$, and $D_0 \in LR^{\text{lat}}(\tilde{\chi}, \tilde{\epsilon})$. Say a diagram $D \in LR^{\text{lat}}(\chi, \epsilon)$ is a χ-*extension* of D_0 if there is some diagram $D' \in LR^{\text{lat}}(\chi, \epsilon)$ with $D'|_{\{i,\ldots,n\}} = D_0$ and D can be laterally refined from D' by only making spine-cuts (between ribs) above i in the diagram D'.

If $S \subseteq LR^{\text{lat}}(\tilde{\chi}, \tilde{\epsilon})$, denote by S^{χ}, the set of diagrams in $LR^{\text{lat}}(\chi, \epsilon)$ which are χ-extensions of diagrams in S.

Remark 2.3.11. It is immediate from the definition above that if $i_1 \leq i_2$, $\chi_0 = \chi|_{\{i_1,\ldots,n\}}$ and $\tilde{\chi} = \chi_0|_{\{i_2,\ldots,n\}}$, then

$$(S^{\chi_0})^{\chi} = S^{\chi}$$

for any $S \subseteq LR^{\text{lat}}(\tilde{\chi}, \tilde{\epsilon})$

The purpose of this definition is that it contains the process described in the proof of Theorem 2.3.8 for applying new (left and right) operators to diagram-indexed vectors. With this notation in hand, we extend Theorem 2.3.8 to the following result.

Lemma 2.3.12. *Let $i \leq n$ and other notations be as above. Then for any $S \subseteq LR^{lat}(\tilde{\chi}, \tilde{\epsilon})$ and constants $c_{D,S}$ for each $D \in S$, there are constants $c'_{D,S}$ such that*

$$\mu_1(T_1) \cdots \mu_{k-1}(T_{i-1}) \left(\sum_{D \in S} c_{D,S} E_D(\mu_k(T_i), \ldots, \mu_n(T_n)) \right) = \sum_{D \in S^{\chi}} c'_{D,S} E_D(\mu_1(T_1), \ldots, \mu_n(T_n))$$

Proof. The proof is the same as for [11, Lemma 7.1.3], as the process inductively added on to existing diagrams. The only part that differs now is that we may end up with different constants scaling the vectors E_D, since we are starting with a potentially smaller collection of diagrams than in Lemma 7.1.3. of [11]. In the same way however, these constants depend only on the diagram D and the collection of diagrams S that we use. □

Notation 2.3.13. We want to combine this result with our Boolean and monotone projections. To this end, for each $k \in K$, we define the sets

$$LR^{\text{lat}}_{\uplus}(\chi, \epsilon, k) := \{D \in LR^{\text{lat}}(\chi, \epsilon) \mid \text{if a spine of } D \text{ reaches the top gap then it has colour } k\}$$

and (when K is totally ordered)

$$LR^{\mathrm{lat}}_{\triangleright}(\chi,\epsilon,k) := \{D \in LR^{\mathrm{lat}}(\chi,\epsilon) \mid {}^{\text{spine colours of } D \text{ reaching the top gap decrease}}_{\text{from left to right and are bounded above by } k}\}, \text{ and}$$
$$LR^{\mathrm{lat}}_{\triangleleft}(\chi,\epsilon,k) := \{D \in LR^{\mathrm{lat}}(\chi,\epsilon) \mid {}^{\text{spine colours of } D \text{ reaching the top gap increase}}_{\text{from left to right and are bounded above by } k}\}$$

By Remark 2.3.7, the set $LR^{\mathrm{lat}}_{\star}(\chi,\epsilon,k)$ is precisely the collection of LR-diagrams which do not get annihilated by the projection $P_{\star,k}$, where $\star \in \{\uplus, \triangleright, \triangleleft\}$.

It is worth noting that since lateral refinements may only cut spines between ribs (not at the top gap), it follows that $D \in LR^{\mathrm{lat}}_{\uplus}(\chi,\epsilon,k)$ if D has either no spines reaching the top gap, or exactly one spine reaching the top gap of colour k.

We denote the complement of $LR^{\mathrm{lat}}_{\star}(\chi,\epsilon,k)$ in $LR^{\mathrm{lat}}_{\star}(\chi,\epsilon)$ simply by $LR^{\mathrm{lat}}_{\star}(\chi,\epsilon,k)^c$. When $\star = \uplus$, this is the collection of diagrams with at least one spine reaching the top gap with a different colour than k.

Combining Theorem 2.3.8 with Remark 2.3.7 yields the following.

Lemma 2.3.14. *With notations as in Definition 2.3.6 and Notation 2.3.13, let $k \in K$. Then we have for $\star \in \{\uplus, \triangleright, \triangleleft\}$ (and K totally ordered if $\star \neq \uplus$),*

$$P_{\star,k} \sum_{D \in S} c_{D,S} E_D(\mu_1(T_1), \ldots, \mu_n(T_n)) = \sum_{D \in LR^{lat}_{\star}(\chi,\epsilon,k) \cap S} c_{D,S} E_D(\mu_1(T_1), \ldots, \mu_n(T_n)),$$

and consequently

$$(1 - P_{\star,k}) \sum_{D \in S} c_{D,S} E_D(\mu_1(T_1), \ldots, \mu_n(T_n)) = \sum_{D \in LR^{lat}_{\star}(\chi,\epsilon,k)^c \cap S} c_{D,S} E_D(\mu_1(T_1), \ldots, \mu_n(T_n)).$$

We can combine our observations to get the following technical lemma which serves as an extension of the LR-diagram calculus, but now including the Boolean projections.

Lemma 2.3.15. *With notations as above, for $m \geq 0$, let $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and define*

$$\mu'_j(T_j) := \begin{cases} \mu_j(T_j) & \text{if } j \notin \{i_1, \ldots, i_m\} \\ P_{\star_{i_j},\epsilon(j)} \mu_j(T_j) & \text{otherwise} \end{cases},$$

where $\star_{i_j} \in \{\uplus, \triangleright, \triangleleft\}$. Then there is some $S \subseteq LR^{lat}(\chi,\epsilon)$ and constants $c_{D,S}$ such that

$$\mu_1(T_1) \cdots \mu_n(T_n) 1_B = \mu'_1(T_1) \cdots \mu'_n(T_n) 1_B + \sum_{D \in S} c_{D,S} E_D(\mu_1(T_1), \ldots, \mu_n(T_n)),$$

where $S \subseteq \bigcup_{j=1}^m S^{\chi}_{i_j}$, and $S_{i_j} = LR^{lat}_{\star_{i_j}}(\chi|_{\{i_j,\ldots,n\}}, \epsilon|_{\{i_j,\ldots,n\}}, \epsilon(i_j))^c$.

Corollary 2.3.16. *If $\Gamma = \{(A_k^\ell, A_k^r)\}_{k\in K}$ is a family of pairs of B-faces that are either bi-free, or independent in any sense as in Definition 2.2.1 then the joint distribution of Γ depends only on the joint distributions of the individual pairs (A_k^ℓ, A_k^r).*

Similarly, for $\Gamma = \{(A_k^\ell, A_k^r, A_k^m)\}_{k\in K}$ a family of triples of B-faces independent in the sense of Definition 2.2.9, then the joint distribution of Γ depends only on the joint distributions of the individual triples (A_k^ℓ, A_k^r, A_k^m).

This is a direct consequence of Lemma 2.3.15, as if Γ is an independent family, then joint Γ moments are computed via left and right regular actions on a free product with a mixture of the Boolean and monotone projections as necessary. Lemma 2.3.15 guarantees, via the LR-diagrams that this splits into submoments built from the individual pairs or triples.

Corollary 2.3.17. *For $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ and $(\mathcal{X}, \mathring{\mathcal{X}}, p)$, their reduced free product, the family $\Gamma = \{(\gamma_{k,\ell}(\mathcal{L}(\mathcal{X}_k)), \gamma_{k,r}(\mathcal{L}(\mathcal{X}_k)))\}_{k\in K}$ is*

- *bi-free if $\gamma_{k,\ell} = \lambda_k$ and $\gamma_{k,r} = \rho_k$,*
- *free-Boolean if $\gamma_{k,\ell} = \lambda_k$ and $\gamma_{k,r} = \gamma_{\uplus,k}$,*
- *free-monotone if $\gamma_{k,\ell} = \lambda_k$ and $\gamma_{k,r} = \gamma_{\lhd,k}$,*
- *monotone-Boolean if $\gamma_{k,\ell} = \gamma_{\rhd,k}$ and $\gamma_{k,r} = \gamma_{\uplus,k}$, and*
- *monotone-monotone if $\gamma_{k,\ell} = \gamma_{\rhd,k}$ and $\gamma_{k,r} = \gamma_{\lhd,k}$.*

Proof. This follows from the corollary above and the oberservation that each representation above is injective and expectation preserving. $\square$

A similar result can be stated regarding Liu's triple algebra independences.

Chapter 3

Bi-Free Embeddability

Bi-free probability has seen remarkable success in its ability to extend the ideas of free probability. Not only do we have a robust cumulant theory, as we saw in the last chapter, but it turns out that we can describe several types of independence from the single context of bi-freeness. In fact, all of the single-algebra independences embed into bi-free probability, which allows each to be studied with the bi-free cumulants. The main theme of this chapter will be to explore some extensions of these embeddings to Liu's independences, and also tease out some limitations.

In the first section, we will see how the single-algebra independences arise in the context of bi-free independence. Afterwards we show how even free-free-Boolean independence for *triples* of B-faces still embeds into a bi-free framework, allowing for the bi-free combinatorics to be used. We even get an embedding of monotone-Boolean and Boolean-Free-Monotone independences into bi-free probability. We round off this chapter with a negative result; it appears that Liu's free-monotone independence lacks the same ability to be described by bi-freeness.

3.1 The Main Four Independences in Bi-Free Probability

In the original paper introducing bi-freeness, [26], Voiculescu showed that both free independence and classical independence arose naturally in the framework of bi-free independence. In particular, he noted that if (A_1, B_1) and (A_2, B_2) were two bi-freely independent pairs of faces in a non-commutative probability space $(\mathcal{A}, \varphi)$, then it is easily verified that

- A_1 and B_2 (symmetrically B_1 and A_2) are classically independent, and
- A_1 and A_2 (symmetrically B_1 and B_2) are freely independent.

Voiculescu raised the question afterwards if bi-free independence also housed Boolean independence in an analogous way.

Skoufranis in [19] was the one to answer this in the affirmative. In addition, he was able to show that a pair of monotonically independent algebras could also be housed in a bi-free structure.

We briefly consider the ideas here now, as they elucidate some considerations going forward.

Definition 3.1.1. Given a family (of B-algebras, pairs of B-faces, triples of B-faces), $\Gamma = \{\Gamma_k\}_{k\in k}$, a *bi-free embedding* will be a family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$ and a family of homomorphisms $\delta_k : \Gamma_k \to \mathrm{alg}(\lambda_k(\mathcal{L}(\mathcal{X}_k)), \rho_k(\mathcal{L}(\mathcal{X}_k)))$ such that the representation $\delta = \{\delta_k\}_{k\in K}$ is joint distribution preserving.

3.1.1 Embedding of Tensor Independence

One might observe from Voiculescu's observations above that tensor independence was only for pairs of algebras. A natural question one might raise is if this extends any further. For example, can we faithfully embed (in a universal way) three tensor independent algebras as bi-free operators? We take a moment to give a (partial) negative answer to this here. It suffices to consider the case when $B = \mathbb{C}$.

Suppose that A_1, A_2, A_3 are three tensor independent algebras. We suppose for sake of contradiction that there are vector spaces with specified projection $(\mathcal{X}_i, \mathring{\mathcal{X}}_i, p_i)$, $a_i \in A_i$, and operators $T_i, S_i \in \mathcal{L}(\mathcal{X}_i)$ for $i = 1, 2, 3$ so that if $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ is the reduced free product of our three vector spaces, then the maps $\delta_i(a_i) = \lambda_i(T_i)\rho_i(S_i)$ are joint distribution preserving.

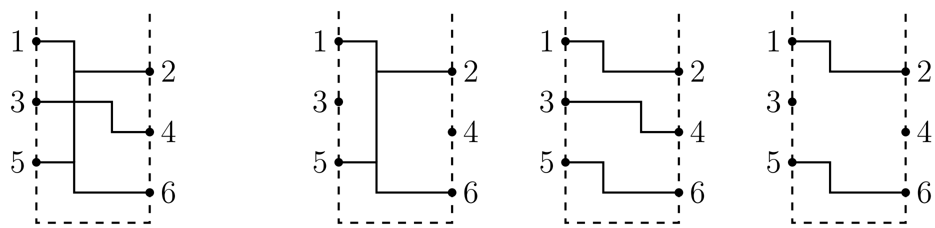
While our argument only represents each element $a_i \in A_i$ as a left operator multiplied by a right operator on the free product, this argument easily extends to any product of lefts and rights.

To reason about the failures of such an embedding, we will make good use of our Bi-Free Cumulant Inclusion-Exclusion result (Proposition 2.1.25).

Given the setup, we can evaluate the moment

$$\varphi(a_1a_2a_1) = E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\delta_2(a_2)\delta_1(a_1)) = E_{\mathcal{L}(\mathcal{X})}(\lambda_1(T_1)\rho_1(S_1)\lambda_2(T_2)\rho_2(S_2)\lambda_1(T_1)\rho_1(S_1))$$

by considering corresponding partitions. In particular, for the above moment, we have $n = 6$, $\chi^{-1}(\ell) = \{1, 3, 5\}$, $\chi^{-1}(r) = \{2, 4, 6\}$, and $\ker\epsilon = \{\{1, 2, 5, 6\}, \{3, 4\}\}$. Then there are two maximal elements of $BNC(\chi)$ which sit below $\ker\epsilon$, namely $\pi_1 = \{\{1, 2, 5, 6\}, \{3\}, \{4\}\}$ and $\pi_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, and $\pi_1 \wedge \pi_2 = \{\{1, 2\}, \{3\}, \{4\}, \{5, 6\}\}$. This is easily seen from their diagrams:



$\ker\epsilon$ $\qquad$ π_1 $\qquad$ π_2 $\qquad$ $\pi_1 \wedge \pi_2$

So by Proposition 2.1.25 and that since the δ_i are joint distribution preserving,

$$\begin{aligned}
E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\delta_2(a_2)\delta_1(a_1)) &= E_{\pi_1} + E_{\pi_2} - E_{\pi_1\wedge\pi_2} \\
&= E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) \\
&\quad + E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1))E_{\mathcal{L}(\mathcal{X})}(\delta_2(a_2))E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)) \\
&\quad + E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1))E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2))E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)) \\
&= \varphi(a_1^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) + \varphi(a_1)\varphi(a_2)\varphi(a_1) \\
&\quad + \varphi(a_1)^2E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2))
\end{aligned}$$

However, we expected

$$E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\delta_2(a_2)\delta_1(a_1)) = \varphi(a_1a_2a_1) = \varphi(a_1^2)\varphi(a_2)$$

It is readily seen that, for example by taking $\varphi(a_1) = 0$ and $\varphi(a_1^2) = 1$, that two of the terms in the sum above get annihilated, leaving only

$$\varphi(a_2) = \varphi(a_1a_2a_1) = \varphi(a_1^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) = E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)),$$

we must have $\varphi(a_2) = E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2))$. By symmetry this must hold for each of a_1, a_2, a_3.

So in particular, if $\varphi(a_i) = 0$, it must be the case that either $E_{\mathcal{L}(\mathcal{X})}(\lambda_i(T_i)) = 0$ or $E_{\mathcal{L}(\mathcal{X})}(\rho_i(S_i)) = 0$. This gives the following lemma immediately, which helps to reduce our calculations.

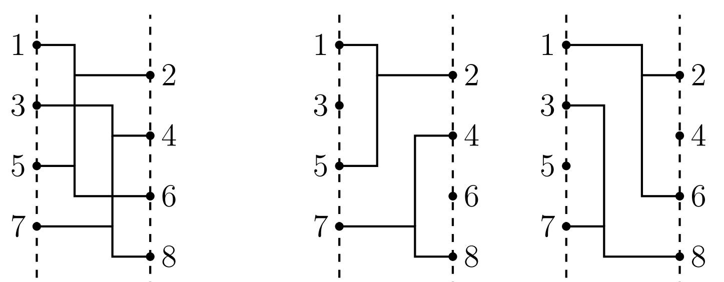
Lemma 3.1.2. *For vector spaces with specified projection $(\mathcal{X}_i, \mathring{\mathcal{X}}_i, p_i)$, $T_i, S_i \in \mathcal{L}(\mathcal{X}_i)$ for $i = 1, 2, 3$ and reduced free product $(\mathcal{X}, \mathring{\mathcal{X}}, p)$. Let $n \in \mathbb{N}$, and $\epsilon : \{1, \ldots, n\} \to \{1, 2, 3\}$. Let χ : $\{1, \ldots, 2n\}$ such that $\chi^{-1}(\ell) = \{1, 3, \ldots, 2n-1\}$ and $\pi \in BNC(\chi)$. If $E_{\mathcal{L}(\mathcal{X})}(\lambda_i(T_i))E_{\mathcal{L}(\mathcal{X})}(\rho_i(S_i)) = 0$ and π contains a block of the form $\{2k, 2k+1\}$ or both single blocks $\{2k\}, \{2k+1\}$ for some $1 \leq k \leq n$ and $\varphi(a_{\epsilon(k)}) = 0$, then*

$$E_\pi(\lambda_{\epsilon(1)}(T_{\epsilon(1)}), \rho_{\epsilon(1)}(S_{\epsilon(1)}), \ldots, \lambda_{\epsilon(n)}(T_{\epsilon(n)}), \rho_{\epsilon(n)}(S_{\epsilon(n)})) = 0.$$

Next consider the moment

$$E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\delta_2(a_2)\delta_1(a_1)\delta_2(a_2)) = \varphi(a_1a_2a_1a_2) = \varphi(a_1^2)\varphi(a_2^2).$$

If we demand $\varphi(a_1^2) = 1 = \varphi(a_2^2)$ and $\varphi(a_1) = \varphi(a_2) = 0$, then one finds, using the bi-free cumulant inclusion-exclusion, that all partition-indexed moments vanish by the above lemma, except for those corresponding to $\sigma_1 = \{\{1, 2, 5\}, \{3\}, \{4, 7, 8\}, \{6\}\}$ and $\sigma_2 = \{\{1, 2, 6\}, \{3, 7, 8\}, \{4\}, \{5\}\}$, both of which are maximal below $\ker \epsilon$ and which have diagrams below:



$\ker \epsilon$ σ_1 σ_2

Then,

$$\begin{aligned}E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\delta_2(a_2)\delta_1(a_1)\delta_2(a_2)) &= E_{\sigma_1} + E_{\sigma_2}\\ &= E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\lambda_1(T_1))E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)\delta_2(a_2))E_{\mathcal{L}(\mathcal{X})}(\rho_1(S_1))\\ &\quad + E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\rho_1(S_1))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2))E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2)\delta_2(a_2))E_{\mathcal{L}(\mathcal{X})}(\lambda_1(T_1))\end{aligned}$$

By tensor independence, we should have that

$$\varphi(a_1a_2a_1a_2) = \varphi(a_1^2)\varphi(a_2^2) = 1.$$

However we know that

$$\varphi(a_i) = 0 = E_{\mathcal{L}(\mathcal{X})}(\lambda_i(T_i))E_{\mathcal{L}(\mathcal{X})}(\rho_i(S_i)),$$

implying either $E_{\mathcal{L}(\mathcal{X})}(\lambda_i(T_i)) = 0$ or $E_{\mathcal{L}(\mathcal{X})}(\rho_i(S_i)) = 0$. This necessarily annihilates one of E_{σ_1} or E_{σ_2} above, but not both, as otherwise our moment ends up as 0 instead of 1. Without loss of generality $E_{\sigma_1} \neq 0$. Then we must have

$$E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2)) \neq 0 \quad \text{and} \quad E_{\mathcal{L}(\mathcal{X})}(\rho_1(S_1)) \neq 0$$

implying

$$E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) = 0 \quad \text{and} \quad E_{\mathcal{L}(\mathcal{X})}(\lambda_1(T_1)) = 0.$$

By repeating this observation with the moment $\varphi(a_1a_3a_1a_3)$, we are forced to conclude that

$$E_{\mathcal{L}(\mathcal{X})}(\lambda_3(T_3)) \neq 0 \quad \text{so that} \quad E_{\mathcal{L}(\mathcal{X})}(\rho_3(S_3)) = 0.$$

But now since $E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) = E_{\mathcal{L}(\mathcal{X})}(\rho_3(S_3)) = 0$, our diagram calculus forces

$$E_{\mathcal{L}(\mathcal{X})}(\delta_2(a_2)\delta_3(a_3)\delta_2(a_2)\delta_3(a_3)) = 0 \neq 1 = \varphi(a_2a_3a_2a_3),$$

a contradiction.

The argument above can be extended analogously to whenever $\delta_i(a_i)$ is a product of elements in
$\lambda_i(\mathcal{L}(\mathcal{X}_i)) \cup \rho_i(\mathcal{L}(\mathcal{X}_i))$. This yields the following.

Proposition 3.1.3. *There is no universal bi-free monomial embedding of three tensor independent algebras.*

A universal bi-free monomial representation will mean a monomial p in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ such that given an independent family $\{A_k\}_{k\in K}$ and expectation preserving representations $\ell_k : \mathcal{A}_k \to \mathcal{L}(\mathcal{X}_k)$ there are operators $(T_{1,\ell_k(a)}, \ldots, T_{n,\ell_k(a)}) \in \mathcal{L}(\mathcal{X}_k)^n$ and $(S_{1,\ell_k(a)}, \ldots, S_{m,\ell_k(a)}) \in \mathcal{L}(\mathcal{X}_k)^m$ such that if

$$\delta_k(a) = p(\lambda_k(T_{1,\ell_k(a)}), \ldots, \lambda_k(T_{n,\ell_k(a)}), \rho_k(S_{1,\ell_k(a)}), \ldots, \rho_k(S_{m,\ell_k(a)})),$$

we have that $\delta = \{\delta_k\}_{k\in K}$ is a joint distribution preserving B-linear representation of the family $\{A_k\}_{k\in K}$.

3.1.2 Embedding of Boolean Independence

Boolean independence has no such limitation. In [19], Skoufranis showed that any Boolean independent family can be faithfully embedded in bi-free probability. We draw heavy inspiration from Skoufranis' approach in the next section, so we take a moment to highlight some of the details here.

The structures that house Boolean independence are more complicated than the classical and free settings described above. Skoufranis came up with the following abstract structures:

Definition 3.1.4 (Skoufranis, [19]). Let $\{(C_k, D_k)\}_{k\in K}$ be a family of pairs of B-faces in B-B-non-commutative probability space $(\mathcal{A}, E, \varepsilon)$. For each $k \in K$ let

$$C'_k \subseteq C_k \quad \text{and} \quad D'_k \subseteq D_k \cap \mathcal{A}_\ell$$

be subsets such that for all $b \in B$, $L_b C'_k, C'_k L_b \subseteq C'_k$. Say that $\{(C'_k, D'_k)\}_{k\in K}$ is a *bi-free Boolean B-system* if for all $k \in K$ and $n \geq 0$

1. $(C'_k)^2 = \{0\} = (D'_k)^2$
2. $E(C'_k(D'_k C'_k)^n) = 0$, and
3. $E(D'_k(C'_k D'_k)^n) = 0$

The purpose of this definition is to capture Boolean independence in the following sense.

Theorem 3.1.5 (Corollary 4.2.2., [19]). *Let $\{(C'_k, D'_k)\}_{k\in K}$ be a bi-free Boolean B-system. Then $\{alg(C'_k, D'_k)\}_{k\in K}$ is a Boolean independent family over B.*

Not only can Boolean independence arise in bi-free probability, but any Boolean independent family can embed into one of these bi-free structures. The idea for this embedding is as follows.

Fix a Boolean independent family $\{A_k\}_{k\in K}$ in a B-non-commutative probability space $(\mathcal{A}, E)$. For each $k \in K$ let $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ be a copy of $(\mathcal{A} \oplus \mathcal{A}, \ker E \oplus \mathcal{A}, E \oplus 0)$. Then for $a, \xi_1, \xi_2 \in \mathcal{A}$, consider the operators T_a and S_{1_B} in $\mathcal{L}(\mathcal{X}_k)$ defined by

$$T_a(\xi_1 \oplus \xi_2) = a\xi_2 \oplus 0 \quad \text{and} \quad S_{1_B}(\xi_1 \oplus \xi_2) = 0 \oplus \xi_1.$$

Then the mappings $a \mapsto \lambda_k(T_a)\rho_k(S_{1_B})$ map the family $\{A_k\}_{k\in K}$ into a bi-free Boolean B-system in a joint distribution preserving way.

Thus Skoufranis showed the following.

Theorem 3.1.6 (Theorem 4.2.4., [19])**.** *For any Boolean indendent family* $\Gamma = \{A_k\}_{k\in K}$ *there exists a bi-free, joint distribution preserving, linear embedding* $\delta = \{\delta_k\}_{k\in K}$ *into a bi-free Boolean* B*-system.*

The advantage of this linear embedding is that we may use appropriate bi-free cumulants in place of the Boolean cumulants. The bi-free cumulants are constructed as in Remark 2.2.15, taking $\chi \equiv b$. We will be able to extend these ideas to the case of embedding free-free-Boolean independent families into bi-free probability in the next section.

3.1.3 Embedding of Monotone Independence

Skoufranis also showed the following result for a pair of monotone independent algebras.

Proposition 3.1.7 (Construction 5.2.1., [19])**.** *Any pair,* A_1, A_2 *of monotonically independent algebras has a faithful bi-free linear embedding.*

The embedding Skoufranis found was to consider, as in the setting of Boolean embedding as above the maps for each $a \in A_1$, $\delta_1(a) = \lambda_1(T_a)\rho_1(S_{1_B})$. While for each $a \in A_2$, $\delta_2(a) = \lambda_2(D_a)$, where

$$D_a(\xi_1 \oplus \xi_2) = a\xi_1 \oplus a\xi_2$$

for $\xi_1, \xi_2 \in \mathcal{A}$. Then $\{\delta_1, \delta_2\}$ turns out to be a joint distribution preserving representation of $\{A_1, A_2\}$.

Remark 3.1.8. It is worth noting in passing that under this representation, the elements of A_1 are acting the same as the Boolean operators as above, while the elements of A_2 are acting like left regular operators.

Given the asymmetric nature of monotone independence (i.e. if A_2 is monotone independent from A_1 then A_1 is *not* monotone independent from A_2) it is immediately dubious to hope for an embedding of more than two monotone independent algebras. Let us lend more specific justification to this idea now, in similar style to how we showed a limitation for the tensor independence above.

In particular, we will show that there is no bi-free embedding of three monotone independent algebras. It suffices to consider $B = \mathbb{C}$ again.

Suppose we have three monotone independent algebras A_1, A_2, and A_3. We consider elements $a_i \in A_i$. As with the tensor independence, for clarity, we make the assumption that for each $i = 1, 2, 3$, there are vector spaces with specified projection $(\mathcal{X}_i, \mathring{\mathcal{X}}_i, p_i)$, $(\mathcal{X}, \mathring{\mathcal{X}}, p)$, their reduced free product, and operators $T_i, S_i \in \mathcal{L}(\mathcal{X}_i)$ such that

$$\delta_i(a_i) = \lambda_i(T_i)\rho_i(S_i)$$

where $\{\delta_1, \delta_2, \delta_3\}$ is joint distribution preserving.

By monotone independence, we have

$$= E_{\mathcal{L}(\mathcal{X})}(\lambda_1(T_1)\rho_1(S_1)\lambda_2(T_2)\rho_2(S_2)\lambda_1(T_1)\rho_1(S_1)) = \varphi(a_1a_2a_1) = \varphi(a_2)\varphi(a_1)\varphi(a_2).$$

Computing the right hand side follows the same calculus we performed in subsection 3.1.1. Then the partitions that appear are the same, π_1, π_2, and $\pi_1 \wedge \pi_2$, so that

$$E_{\mathcal{L}(\mathcal{X})}(\delta_2(a_2)\delta_1(a_1)\delta_2(a_2)) = E_{\pi_1} + E_{\pi_2} - E_{\pi_1\wedge\pi_2}.$$

Setting $\varphi(a_2) = 0$, $\varphi(a_2^2) = 1$, and $\varphi(a_1) \neq 0$ and applying Lemma 3.1.2 annihilates E_{π_2} and $E_{\pi_1\wedge\pi_2}$.

Then for δ to be joint distribution preserving, we must have

$$0 = \varphi(a_2a_1a_2) = E_{\pi_1} = E_{\mathcal{L}(\mathcal{X})}(\lambda_1(T_1))E_{\mathcal{L}(\mathcal{X})}(\rho_1(S_1))$$

implying either $E_{\mathcal{L}(\mathcal{X})}(\lambda_1(T_1)) = 0$ or $E_{\mathcal{L}(\mathcal{X})}(\rho_1(S_1)) = 0$.

An identical argument with $\varphi(a_3) = 0$ and $\varphi(a_3^2) = 1$ shows that if $\varphi(a_2) \neq 0$ then

$$E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2)) = 0 \quad \text{or} \quad E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) = 0.$$

So now if $\varphi(a_1^2) = 1$ with $\varphi(a_1) = 0$ and $\varphi(a_2) = 1$, then

$$\varphi(a_1a_2a_1) = \varphi(a_1^2)\varphi(a_2) = 1$$

But

$$\begin{aligned} E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1)\delta_2(a_2)\delta_1(a_1)) &= E_{\pi_1} \\ &= \varphi(a_1^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(T_2))E_{\mathcal{L}(\mathcal{X})}(\rho_2(S_2)) \\ &= 0 \end{aligned}$$

a contradiction.

As with the tensor case, this argument is easily extended to when $\delta_i(a_i)$ is a product of elements in $\lambda_i(\mathcal{L}(\mathcal{X}_i)) \cup \rho_i(\mathcal{L}(\mathcal{X}_i))$. This yields the following.

Proposition 3.1.9. *There is no universal bi-free monomial embedding of three monotone independent algebras.*

3.2 Free-Boolean and Free-Free-Boolean Independence Embeddability

Inspired by Skoufranis' approach to embedding Boolean independent families into bi-free families of operators, we set out to show that we can do something analogous to Liu's free-Boolean independence and free-free-Boolean independence. In fact, by Remark 2.2.12 it suffices to handle the case of embedding free-free-Boolean independence with amalgamation over B. So it is this case we focus on for now. We begin with defining an abstract bi-free structure which will house the operators that will combine to give the free-free-Boolean independence. Afterwards, we show how any free-free-Boolean independent family can be embedded in this way. Then we show that such abstract structures preserve the free-free-Boolean independence, guaranteeing a distribution preserving embedding. To do so, we will show how the free-free-Boolean cumulants can naturally be described by the bi-free cumulants. This section closely follows the work [15].

3.2.1 Abstract Bi-Free Free-Free-Boolean Structures

We begin by defining a structure with bi-free independence that produces free-free-Boolean independence. We proceed with inspiration from [19]. Since we are trying to model a free-free-Boolean independent system, we upgrade Skoufranis' bi-free Boolean B-systems (Definition 3.1.4) with additional bi-free pairs of B-faces.

Definition 3.2.1. Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-non-commutative probability space and fix a family $\{(A_k^\ell, A_k^r, C_k, D_k)\}_{k\in K}$ of quadruples of subalgebras of $\mathcal{A}$ such that each of (A_k^ℓ, A_k^r) are pairs of B-faces of $\mathcal{A}$ and the family of pairs $\{(\mathrm{alg}(A_k^\ell, C_k), \mathrm{alg}(A_k^r, D_k))\}_{k\in K}$ are bi-free over B. For each $k \in K$, let $C_k' \subseteq C_k$ and $D_k' \subseteq D_k \cap \mathcal{A}_\ell$ be subsets such that $L_b C_k' \subseteq C_k'$ and $C_k' L_b \subseteq C_k'$ for all $b \in B$. We say that $\{(A_k^\ell, A_k^r, C_k', D_k')\}_{k\in K}$ is a *bi-free ffb B-system* if for all $n \geq 0$ and $k \in K$,

1. $C_k' \mathbb{A}_k C_k' = \{0\} = D_k' \mathbb{A}_k D_k'$,
2. $E_{\mathcal{A}}(\mathbb{A}_k C_k' \mathbb{A}_k (D_k' \mathbb{A}_k C_k')^n \mathbb{A}_k) = 0$, and
3. $E_{\mathcal{A}}(\mathbb{A}_k D_k' \mathbb{A}_k (C_k' \mathbb{A}_k D_k')^n \mathbb{A}_k) = 0$

where $\mathbb{A}_k := \mathrm{alg}(A_k^\ell, A_k^r)$.

Remark 3.2.2. This extends the notion of a bi-free Boolean B-system from [19]. In particular, since the algebras $\mathcal{A}_k^\ell$ and $\mathcal{A}_k^r$ are unital, it follows that the family of pairs $\{(C_k', D_k')\}_{k\in K}$ is a bi-free boolean B-system. It was shown in [19] that therefore the algebras $\{\mathrm{alg}(C_k' D_k')\}_{k\in K}$ Boolean independent with amalgamation over B.

Such a bi-free ffb B-system as defined in Definition 3.2.1 forms the triple of B-faces

$$\{(A_k^\ell, A_k^r, \mathrm{alg}(C_k' D_k'))\}_{k\in K}.$$

Indeed, since $D_k' \in \mathcal{A}_r \cap \mathcal{A}_\ell$, and $L_b C_k', C_k' L_b \subseteq C_k'$, we have that

$$\varepsilon(B \otimes 1_B)\mathrm{alg}(C_k' D_k')\varepsilon(B \otimes 1_B) \subseteq \mathrm{alg}(C_k' D_k').$$

We will show that such a triple of faces is free-free-Boolean independent with amalgamation over B. But first, let us see how such a collection can house any free-free-Boolean independent family.

3.2.2 Bi-Free Operators for free-free-Boolean families

The following construction motivates our definition of bi-free ffb B-system. This will allow us to embed free-free-Boolean families into bi-free families. Our strategy follows the idea of Construction 4.2.3 of [19], but tweaked to use an appropriate B-B-bimodule with specified B-projection.

Let us fix a B-B-noncommutative probability space $(\mathcal{A}, E, \varepsilon)$. Fix a free-free-Boolean independent family $\{(A_k^\ell, A_k^r, A_k^b)\}_{k\in K}$ in $\mathcal{A}$. Let

$$\mathcal{A}' = B \oplus \ker(E)/\mathrm{span}\{TL_b - TR_b : T \in \mathcal{A}, b \in B\}$$

as in Theorem 2.1.10, and let $\theta : \mathcal{A} \to \mathcal{A}'$ be the corresponding embedding.

For each $k \in K$ let

$$\mathcal{Y}_k = \mathcal{A}' \oplus \mathcal{A}' \quad \text{ and } \mathring{\mathcal{Y}}_k = \ker(E_{\mathcal{A}})/\mathrm{span}\{TL_b - TR_b : T \in \mathcal{A}, b \in B\} \oplus \mathcal{A}'.$$

That is, each $(\mathcal{Y}_k, \mathring{\mathcal{Y}}_k, q_k)$ is a copy of the same B-B-bimodule with specified B-projection ($q_k = E \oplus 0$).

Now, for each $k \in K$ and any $Z \in A_k^b$, consider the operators $T_Z, S_{1_B} \in \mathcal{L}(\mathcal{Y}_k)$ defined by

$$T_Z(\xi_1 \oplus \xi_2) = \theta(Z)\xi_2 \oplus 0 \quad \text{ and } \quad S_{1_B}(\xi_1 \oplus \xi_2) = 0 \oplus \xi_1,$$

for all $\xi_1, \xi_2 \in \mathcal{A}'$. It is simple to see that $S_{1_B} \in \mathcal{L}_\ell(\mathcal{Y}_k) \cap \mathcal{L}_r(\mathcal{Y}_k)$ and that $T_Z \in \mathcal{L}_\ell(\mathcal{Y}_k)$. Indeed, let $\xi_1, \xi_2 \in \mathcal{A}'$ and $b \in B$, then

$$S_{1_B}((\xi_1 \oplus \xi_2) \cdot b) = S_{1_B}(\theta(R_b)\xi_1 \oplus \theta(R_b)\xi_2) = 0 \oplus \theta(R_b)\xi_1 = (0 \oplus \xi_1) \cdot b = (S_{1_B}(\xi_1 \oplus \xi_2)) \cdot b,$$

showing that $S_{1_B} \in \mathcal{L}_\ell(\mathcal{Y}_k)$. One can show similarly that $S_{1_B} \in \mathcal{L}_r(\mathcal{Y}_k)$. Showing that $T_Z \in \mathcal{L}_\ell(\mathcal{Y}_k)$ is similar, but makes use of the fact that $Z \in A_k^b \subseteq \mathcal{A}_\ell$ so that $ZR_b = R_bZ$.

Also, for any $Z \in \mathcal{A}$ and any $k \in K$, define $D_Z \in \mathcal{L}(\mathcal{Y}_k)$ by

$$D_Z(\xi_1 \oplus \xi_2) = \theta(Z)\xi_1 \oplus \theta(Z)\xi_2.$$

If $Z \in \mathcal{A}_\ell$, then $D_Z \in \mathcal{L}_\ell(\mathcal{Y}_k)$. Similarly if $Z \in \mathcal{A}_r$ then $D_Z \in \mathcal{L}_r(\mathcal{Y}_k)$.

Let $(\mathcal{Y}, \mathring{\mathcal{Y}}, q)$ be the reduced free product of the family $\{(\mathcal{Y}_k, \mathring{\mathcal{Y}}_k, q_k)\}_{k \in K}$. For each $k \in K$ let λ_k and ρ_k be respectively the left and right regular representations of $\mathcal{L}(\mathcal{Y}_k)$ on $\mathcal{Y}$. For $Z \in \mathcal{A}$, define

$$T_{k,Z} := \lambda_k(T_Z) \quad \text{ and } S_{k,1_B} := \rho_k(S_{1_B}).$$

For each $k \in K$, define $C_k' = \{T_{k,Z} \mid Z \in A_k^b\}$ and $D_k' = \{S_{k,1_B}\}$. Let

$$\tilde{A}_k^\ell = \{\lambda_k(D_Z) \mid Z \in A_k^\ell\} \quad \text{ and } \quad \tilde{A}_k^r = \{\rho_k(D_Z) \mid Z \in A_k^r\}.$$

Notice that for each $k \in K$, $(\tilde{A}_k^\ell, \tilde{A}_k^r)$ is a pair of B-faces of $\mathcal{L}(\mathcal{Y})$.

Then we check that the family $\{(\tilde{A}_k^\ell, \tilde{A}_k^r, C_k', D_k')\}_{k \in K}$ is a bi-free ffb B-system in $\mathcal{L}(\mathcal{Y})$. For property (1), fix $k \in K$ and $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ and let $Z_1, \ldots, Z_n \in \mathcal{A}$ such that $Z_i \in A_k^{\chi(i)}$. For convenience, let $Z = Z_1 \cdots Z_n$ so that $D_Z = D_{Z_1} \cdots D_{Z_n}$. Then for any $\xi_1 \oplus \xi_2 \in \mathcal{Y}_k$,

$$S_{1_B} D_Z S_{1_B}(\xi_1 \oplus \xi_2) = S_{1_B} D_Z(0 \oplus \xi_1) = S_{1_B}(0 \oplus \theta(Z)\xi_1) = 0.$$

Similarly, for any $A_1, A_2 \in A_k^b$,

$$T_{A_1} D_Z T_{A_2}(\xi_1 \oplus \xi_2) = T_{A_1} D_Z(\theta(A_2)\xi_2 \oplus 0) = T_{A_1}(\theta(Z)\theta(A_2)\xi_2 \oplus 0) = 0.$$

Now stepping up to the free product, for each $1 \le i \le n$ let D_{Z_i}' denote $\lambda_k(D_{Z_i})$ if $\chi(i) = \ell$ and $\rho_k(D_{Z_i})$ if $\chi(i) = r$. Since $Z_i \in A_k^{\chi(i)}$ for each $1 \le i \le n$, if $\eta \in \mathcal{Y}_k \subseteq \mathcal{Y}$ then

$$\rho_k(S_{1_B}) D_{Z_1}' \cdots D_{Z_n}' \rho_k(S_{1_B})\eta = S_{1_B} D_{Z_1} \cdots D_{Z_n} S_{1_B}\eta = 0,$$

and

$$\lambda_k(T_{A_1})D'_{Z_1}\cdots D'_{Z_n}\lambda_k(T_{A_2})\eta = T_{A_1}D_{Z_1}\cdots D_{Z_n}T_{A_2}\eta = 0.$$

If $\eta \in \mathcal{Y}\ominus\mathcal{Y}_k$, then we have that for any $X_1, X_2 \in \mathcal{L}(\mathcal{Y}_k)$

$$\lambda_k(X_1)\rho_k(X_2)\eta = \rho_k(X_2)\lambda_k(X_1)\eta.$$

With this in mind, form the (ordered) products

$$Z^\ell = \prod_{\substack{1\le i\le n\\ \chi(i)=\ell}} Z_i \quad \text{and} \quad Z^r = \prod_{\substack{1\le i\le n\\ \chi(i)=r}} Z_i.$$

Then, by commuting lefts with rights in their action on $\eta \in \mathcal{Y}\ominus\mathcal{Y}_k$ and combining with our understanding of the annihilating action from above,

$$\rho_k(S_{1_B})D'_{Z_1}\cdots D'_{Z_n}\rho_k(S_{1_B})\eta = \lambda_k(D_{Z^\ell})(\rho_k(S_{1_B}D_{Z^r}S_{1_B})\eta) = \lambda_k(D_{Z^\ell})0 = 0.$$

Similarly,

$$\lambda_k(T_{A_1})D'_{Z_1}\cdots D'_{Z_n}\lambda_k(T_{A_2})\eta = \rho_k(D_{Z^r})(\lambda_k(T_{A_1}D_{Z^\ell}T_{A_2})\eta) = \rho_k(D_{Z^r})0 = 0.$$

Combining these observations together, we get that property (1) holds.

Property (2) of Definition 3.2.1 is easy to see, as if Z is as above and $A \in A_k^b$ then

$$T_A D_Z(1_B \oplus 0) = T_A(\theta(Z)1_B \oplus 0) = 0.$$

Passing to the free product by left and right regular representations does not make a difference here as we are only acting on the element $1_B \in \mathcal{Y}_k$.

Lastly, let $n \in \mathbb{N}$, $A_1, \ldots, A_n \in A_k^b$, and $Z_1, Z_2, \ldots, Z_{n+1} \in \mathrm{alg}(A_k^\ell, A_k^r)$. Note that

$$\begin{aligned}
T_{A_n}D_{Z_n}S_{1_B}D_{Z_{n+1}}1_B &= T_{A_n}D_{Z_n}S_{1_B}(\theta(Z_{n+1})1_B \oplus 0)\\
&= T_{A_n}D_{Z_n}(0 \oplus \theta(Z_{n+1})1_B)\\
&= T_{A_n}(0 \oplus \theta(Z_n)\theta(Z_{n+1})1_B)\\
&= \theta(A_n)\theta(Z_n)\theta(Z_{n+1})1_B \oplus 0\\
&= \theta(A_nZ_nZ_{n+1})1_B.
\end{aligned}$$

So inductively,

$$\begin{aligned}
D_{Z_1}S_{1_B}D_{Z_2}T_{A_1}D_{Z_3}S_{1_B}\cdots T_{A_n}D_{Z_{n+1}}S_{1_B}D_{Z_{n+1}}1_B &= D_{Z_1}S_{1_B}(\theta(Z_2A_1Z_3\cdots A_nZ_{n+2})1_B \oplus 0)\\
&= D_{Z_1}(0 \oplus \theta(Z_2A_1Z_3\cdots A_nZ_{n+2})1_B)\\
&= 0 \oplus \theta(Z_1Z_2A_1\cdots A_nZ_{n+2})1_B,
\end{aligned}$$

which has expectation 0. So property (3) follows as well.

So our family $\{(\tilde{A}_k^\ell, \tilde{A}_k^r, C_k', D_k')\}_{k\in K}$ is a bi-free ffb B system. We get the following.

Proposition 3.2.3. *Given a free-free-Boolean independent family $\Gamma = \{(A_k^\ell, A_k^r, A_k^b)\}_{k\in K}$ in a B-B-non-commutative probability space $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$, there exists a bi-free ffb B-system $\{(\tilde{A}_k^\ell, \tilde{A}_k^r, C_k', D_k')\}_{k\in K}$ inside a B-B-non-commutative probability space $(\mathcal{A}', E_{\mathcal{A}'}, \varepsilon')$ and for each $k \in K$ there are injective B-linear maps $\alpha_{\ell,k} : A_k^\ell \to \tilde{A}_k^\ell$, $\alpha_{r,k} : A_k^r \to \tilde{A}_k^r$ and $\alpha_{b,k} : A_k^b \to alg(C_k' D_k')$ such that whenever $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell, r, b\}$, and $Z_i \in A_k^{\chi(i)}$ for all $1 \le i \le n$, we have*

$$E_{\mathcal{A}'}(\alpha_{\chi(1),k}(Z_1) \cdots \alpha_{\chi(n),k}(Z_n)) = E_{\mathcal{A}}(Z_1 \cdots Z_n).$$

Proof. Using the notation as in the construction above, we let $\mathcal{A}' = \mathcal{L}(\mathcal{Y})$, $E_{\mathcal{A}'} = E_{\mathcal{L}(\mathcal{Y})}$, and $\varepsilon'(b_1 \otimes b_2) = L_{b_1} R_{b_2} \in \mathcal{L}(\mathcal{Y})$. Then define for each $k \in K$

$$\alpha_{\ell,k}(Z) = \lambda_k(D_Z), \quad \alpha_{r,k}(Z) = \rho_k(D_Z), \quad \text{and} \quad \alpha_{b,k}(Z) = T_{k,Z} S_{k,1_B}$$

for each $Z \in A_k^\ell, A_k^r, A_k^b$ respectively. Then each of these maps are B-linear and the family $\{(\alpha_{\ell,k}(A_k^\ell), \alpha_{r,k}(A_k^r), \{T_{k,Z} : Z \in A_k^b\}, \{S_{k,1_B}\}\}_{k\in K}$ is a bi-free ffb B-system, as we showed above. That these maps are jointly expectation preserving can be seen to also come from our calculation for property (3) above. □

We have not yet shown that the free-free-Boolean independence is preserved, so we may not yet extend to the joint distribution preservation of Γ. In order to demonstrate independence, we make use of the bi-free cumulants. We successfully show independence below with Theorem 3.2.11.

3.2.3 Bi-Free Cumulants for Free-Free-Boolean Independence

Next, we make use of the bi-free cumulants to form free-free-Boolean cumulants. To do so, with inspiration coming from our construction above and from Remark 2.2.15, we will split the Boolean-acting operators into a left and a right piece. So we need a way of connecting any colouring $\hat{\chi} : \{1, \ldots, n\} \to \{\ell, r, b\}$ with an appropriate ℓ-r colouring.

Definition 3.2.4. Given a map $\hat{\chi} : \{1, \ldots, n\} \to \{\ell, r, b\}$, define the *$\ell$-$r$-replacement* of $\hat{\chi}$, to be the map $\chi : \{1, \ldots, n + |\hat{\chi}^{-1}(\{b\})|\} \to \{\ell, r\}$ defined as follows. Let $f_{\hat{\chi}}(i) =$

$i + |\hat{\chi}^{-1}(\{b\}) \cap [1, i-1]|$. Then,

$$\chi(i) := \begin{cases} \hat{\chi}(f_{\hat{\chi}}^{-1}(i)) & \text{if } i \in f_{\hat{\chi}}(\{1, \ldots, n\} \setminus \hat{\chi}^{-1}(\{b\})) \\ \ell & \text{if } i \in f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\})) \\ r & \text{otherwise} \end{cases}.$$

So in other words, χ is ℓ-r-replaced, if it replaces all instances of b with ℓ, r in order. To do so, we must inflate the domain of $\hat{\chi}$ appropriately.

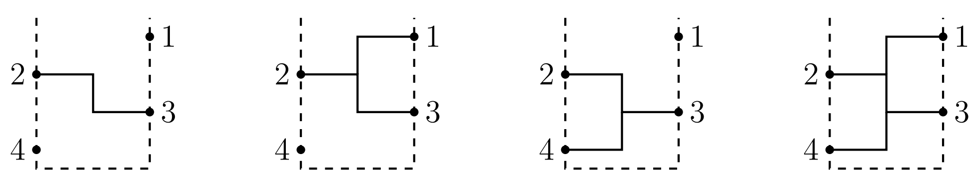
Example 3.2.5. As a simple example, consider $\hat{\chi} = \ell brb$ then the ℓ-r-replacement of $\hat{\chi}$ is the map

$$\chi = \ell(\ell r)r(\ell r) = \ell\ell rr\ell r.$$

Now we collect all of those partitions which respect this replacement. That is, we want to keep those elements together in our partitions who came from a 'b' colouring. We will show shortly that partition-moments which separate these elements will vanish.

Definition 3.2.6. Given a map $\hat{\chi} : \{1, \ldots, n'\} \to \{\ell, r, b\}$, form its ℓ-r-replacement $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ where $n = n' + |\hat{\chi}^{-1}(\{b\})|$. Let $f_{\hat{\chi}}(i) = i + |\hat{\chi}^{-1}(\{b\}) \cap [1, i-1]|$ for each $1 \leq i \leq n'$ and define $BNC_{\text{ffb}}(\hat{\chi})$ to be the subset of $BNC(\chi)$ such that $\pi \in BNC_{\text{ffb}}(\hat{\chi})$ if and only if whenever $i \in V \in \pi$ and $i \in f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\}))$, it follows that $i + 1 \in V$.

Example 3.2.7. As an example, consider the map $\hat{\chi} = rb\ell$. Then it's ℓ-r-replacement is the map $\chi = r\ell r\ell$. So $BNC_{\text{ffb}}(\hat{\chi})$ is the set of bi-non-crossing partitions π that lie above $0_{BNC_{\text{ffb}}(\hat{\chi})} = \{\{1\}, \{2, 3\}, \{4\}\}$. There are four of these, each is pictured below.



These boolean pairs (i.e. pairs of numbers that replaced a 'b' colouring) act as "walls" so that we cannot connect nodes on opposite sides of a boolean pair without including the boolean pair in the block.

Remark 3.2.8. Let us notice a couple easy facts about our set of lattices. Firstly note that if $\hat{\chi}^{-1}(\{b\}) = \emptyset$, then its ℓ-r-replacement $\chi = \hat{\chi}$, and in particular,

$$BNC_{\text{ffb}}(\hat{\chi}) = BNC(\chi).$$

Furthermore, for any $\hat{\chi} : \{1, \ldots, n'\} \to \{\ell, r, b\}$, if $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ is its ℓ-r-replacement where $n = n' + |\hat{\chi}^{-1}(\{b\})|$ and $f_{\hat{\chi}}(i) = i + |\hat{\chi}^{-1}(\{b\}) \cap [1, i-1]|$ then $BNC_{\text{ffb}}(\hat{\chi})$ is an interval of the lattice $BNC(\chi)$ with maximum element $1_{BNC(\chi)}$ and minimum element $0_{BNC_{\text{ffb}}(\hat{\chi})}$ defined by

$$0_{BNC_{\text{ffb}}(\hat{\chi})} = \{\{i\}, \{j, j+1\} : i \in f_{\hat{\chi}}(\{1, \ldots, n'\} \setminus \hat{\chi}^{-1}(\{b\})), j \in f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\})\}$$

This means that μ_{BNC} can be restricted to BNC_{ffb}. That is, if $\pi, \sigma \in BNC_{\text{ffb}}(\hat{\chi})$ then

$$\sum_{\substack{\tau \in BNC_{\text{ffb}}(\hat{\chi}) \\ \pi \le \tau \le \sigma}} \mu_{BNC}(\tau, \sigma) = \sum_{\substack{\tau \in BNC_{\text{ffb}}(\hat{\chi}) \\ \pi \le \tau \le \sigma}} \mu_{BNC}(\pi, \tau) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{otherwise} \end{cases}.$$

So the Möbius function on $BNC(\chi)$ acts as a Möbius function on $BNC_{\text{ffb}}(\hat{\chi})$ when restricted, giving us a Möbius inversion therein.

Now we aim to show that we can indeed use the bi-free cumulants as free-free-Boolean cumulants. To do so, we look at the moments if a bi-free ffb B-system, with our bi-non-crossing partitions above in mind.

Theorem 3.2.9. *Let $\{(A_k^\ell, A_k^r, C_k', D_k')\}$ be a bi-free ffb B-system in a B-B-non-commutative probability space $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$. Fix $\hat{\chi} : \{1, \ldots, n'\} \to \{\ell, r, b\}$. Let $\chi = \{1, \ldots, n\} \to \{\ell, r\}$ be the ℓ-r-replacement of $\hat{\chi}$, where $n = n' + |\hat{\chi}^{-1}(\{b\})|$. For each $i \in \{1, \ldots, n'\}$ define $f_{\hat{\chi}}(i) = i + |\hat{\chi}^{-1}(b) \cap [1, i-1]|$. Let $\epsilon : \{1, \ldots, n\} \to K$ be such that $\epsilon(i) = \epsilon(i+1)$ whenever $i \in f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\}))$. Now let*

$$Z_i \in \begin{cases} \mathcal{A}_{\epsilon(i)}^{\chi(i)} & \textit{if } i \in f_{\hat{\chi}}(\{1, \ldots, n\} \setminus \hat{\chi}^{-1}(\{b\})) \\ C'_{\epsilon(i)} & \textit{if } i \in f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\}) \\ D'_{\epsilon(i)} & \textit{otherwise} \end{cases}.$$

If $\pi \in BNC(\chi)$ and $\pi \le \epsilon$, then

$$\mathcal{E}_\pi(Z_1, \ldots, Z_n) = 0$$

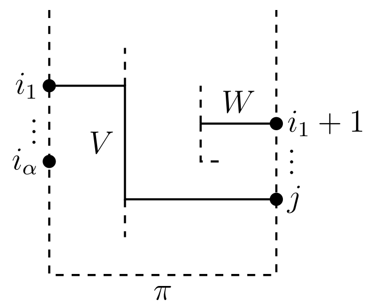
unless $\pi \in BNC_{\text{ffb}}(\hat{\chi})$.

Proof. Our goal amounts to showing that if π is a partition which splits up a boolean pair, then the corresponding π-moment must be zero.

Fix $\pi \in BNC(\chi)$. Consider the set $f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\})) = \{i_1 < i_2 < \cdots < i_m\}$. We argue that whenever $i_j \in V \in \pi$, then either $i_j + 1 \in V$, or the corresponding π-moment is zero. We proceed inductively.

For the base case we show that if $V \in \pi$ with $i_1 \in V$, then $i_1 + 1 \in V$, otherwise $\mathcal{E}_\pi(Z_1, \ldots, Z_n) = 0$. Indeed, suppose that there is some block $W \in \pi$ with $W \neq V$ such that $i_1 + 1 \in W$. We consider some cases.

Case 1: There is some $j > i_1$ such that $\chi(j) = r$ and $j \in V$. If this is the case $\chi^{-1}(\{\ell\}) \cap [i_1, n] \cap W = \emptyset$. So since $\pi \in BNC(\chi)$, W does not contain any i_α with $1 \leq \alpha \leq m$.



So the ordered product $\prod_{i \in W} Z_i$ is an alternating product of elements from $\mathbb{A}_{\epsilon(W)} = \mathrm{alg}(A^\ell_{\epsilon(W)}, A^r_{\epsilon(W)})$ and elements of $D'_{\epsilon(W)}$. In fact, since $A^\ell_{\epsilon(W)}$ and $A^r_{\epsilon(W)}$ are B-faces, we can add left and right multiplications by B without changing this fact. In particular, for any $b_{1,1}, b_{1,2}, b_{2,1}, \ldots, b_{n,1}, b_{n,2}, b_{n+1} \in B$, we have that the ordered product

$$\left(\prod_{i \in W} L_{b_{i,1}} R_{b_{i,2}} Z_i \right) \cdot L_{b_{n+1}}$$

is still an alternating product of elements from $\mathbb{A}_{\epsilon(W)}$ and $D'_{\epsilon(W)}$. Thus by property (3) in the definition of bi-free ffb B-systems, we have

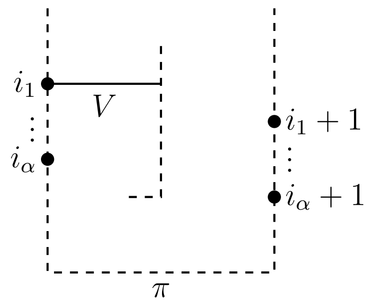
$$\mathcal{E}_{\pi|_W}((L_{b_{1,1}} R_{b_{1,2}} Z_1, L_{b_{2,1}} R_{b_{2,2}} Z_2, \ldots, L_{b_{n,1}} R_{b_{n,2}} Z_n L_{b_{n+1}})|_W) = 0.$$

Now by the reduction process in Definition 2.1.20 (of bi-multiplicative functions), there will be a stage where we must evaluate an expectation of the form above, where the elements $b_{1,1}, \ldots, b_{2,n}, b_{n+1}$ act as potential placeholders for other expectations of blocks. So from above, it follows from bi-multiplicativity, that

$$\mathcal{E}_\pi(Z_1, \ldots, Z_n) = 0.$$

Case 2: There are no elements $j > i_1$ with $\chi(j) = r$ and $j \in V$.

If this is the case, since $\pi \in BNC(\chi)$, V will not contain any elements $i_\alpha + 1$ for any $1 \leq \alpha \leq m$.



So similar to above, the ordered product of elements indexed by V is an alternating product of elements from $\mathbb{A}_{\epsilon(V)}$ and elements of $C'_{\epsilon(V)}$. The same is true if we include left and right multiplications by elements of B. That is, for any $b_{1,1}, b_{1,2}, b_{2,1}, \ldots, b_{n,1}, b_{n,2}, b_{n+1} \in B$, we have that the ordered product

$$\left(\prod_{i \in V} L_{b_{i,1}} R_{b_{i,2}} Z_i\right) \cdot L_{b_{n+1}}$$

is an alternating product of elements from $\mathbb{A}_{\epsilon(V)}$ and elements of $C'_{\epsilon(V)}$. Thus by property (2) of the definition of bi-free ffb B-systems, we have

$$\mathcal{E}_{\pi|_V}((L_{b_{1,1}} R_{b_{1,2}} Z_1, L_{b_{2,1}} R_{b_{2,2}} Z_2, \ldots, L_{b_{n,1}} R_{b_{n,2}} Z_n L_{b_{n+1}})|_V) = 0.$$

Bi-multiplicativity then implies

$$\mathcal{E}_\pi(Z_1, \ldots, Z_n) = 0.$$

Now suppose that for $k \geq 1$, we have that whenever $i_j \in V \in \pi$, it follows that $i_j + 1 \in V$, for all $1 \leq j \leq k$. Let $V \in \pi$ such that $i_{k+1} \in V$. Suppose that $i_{k+1} + 1 \in W \neq V$. We consider cases similarly as above.

Firstly, if there is some $\alpha > i_{k+1}$ such that $\chi(\alpha) = r$ and $\alpha \in V$, then $W \cap \chi^{-1}(\{\ell\}) \cap [i_{k+1}+1, n] = \emptyset$, since $\pi \in BNC(\chi)$. By the hypothesis above we have that for any $1 \leq j \leq k$, $i_j \in W$ if and only if $i_j + 1 \in W$. This implies, since π is bi-non-crossing, that if such a j exists, then in fact $i_k, i_k + 1 \in W$. If this is the case, then letting

$$A_1 = \prod_{i \in W \cap [i_k+2, i_{k+1}]} Z_i \in \mathbb{A}_{\epsilon(W)} \quad \text{and} \quad A_2 = \prod_{i \in W \cap [i_{k+1}+2, n]} Z_i$$

where the products are ordered. So property (1) of the definition of bi-free ffb B-system says that $Z_{i_k+1}A_1Z_{i_{k+1}+1} = 0$, yielding

$$\prod_{i\in W} Z_i = \left(\prod_{i\in W\cap[1,i_k]} Z_i\right) \cdot Z_{i_k+1}A_1Z_{i_{k+1}+1}A_2 = 0,$$

hence the corresponding $\pi|_W$ moment is zero. If instead $i_k \notin W$, then property (3) guarantees that

$$\mathcal{E}_{\pi|_W}((Z_1, \ldots, Z_n)|_W) = 0.$$

As above, we can upgrade this with left and right multiplications by elements of B to make use of bi-multiplicativity granting

$$\mathcal{E}_\pi(Z_1, \ldots, Z_n) = 0.$$

Otherwise, if there is no such $\alpha > i_{k+1}$ such that $\chi(\alpha) = r$ and $\alpha \in V$, then $V \cap \chi^{-1}(\{r\}) \cap [i_{k+1}, n] = \emptyset$. Let $j > k$ be the smallest natural number such that $i_j \in V$, if it exists. Define the ordered products

$$A_1 = \prod_{i\in V\cap[i_{k+1}+1,i_j-1]} Z_i \in \mathbb{A}_{\epsilon(V)} \quad \text{and} \quad A_2 = \prod_{i\in V\cap[i_j+1,n]} Z_i,$$

then property (1) again says that $Z_{i_{k+1}}A_1Z_{i_j} = 0$, so that the ordered product

$$\prod_{i\in V} Z_i = \left(\prod_{i\in V\cap[1,i_{k+1}-1]} Z_i\right) Z_{i_{k+1}}A_1Z_{i_j}A_2 = 0,$$

so the corresponding $\pi|_V$ moment is zero. If no such j exists, then by hypothesis, for any $1 \le j \le k$, $i_j \in V$ if and only if $i_j + 1 \in V$. This implies that

$$\mathcal{E}_{\pi|_V}((Z_1, \ldots, Z_n)|_V) = 0$$

as this is a moment of the form of property (2) in the definition of bi-free ffb B-system. We may upgrade this observation with left and right multiplications by elements of B and use bi-multiplicativity to get that

$$\mathcal{E}_\pi(Z_1, \ldots, Z_n) = 0. \qquad \square$$

This result grants us use of the bi-free cumulants to characterize the free-free-Boolean independence as follows. Note the analog between this result and Theorem 2.1.23.

Theorem 3.2.10. *Let $\{(A_k^\ell, A_k^r, C_k', D_k')\}_{k\in K}$ be a bi-free ffb B-system in a B-B-non-commutative probability space $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$. Let $\hat{\chi}:\{1,\ldots,n'\}\to\{\ell,r,b\}$, and define $f_{\hat{\chi}}(i) = i + |\hat{\chi}^{-1}(\{b\})\cap[0,i-1]|$. Let $\chi:\{1,\ldots,n\},\to\{\ell,r\}$ be the ℓ-r-replacement of $\hat{\chi}$, where $n = n' + |\hat{\chi}^{-1}(\{b\})|$. Let $\epsilon:\{1,\ldots,n\}\to K$ be such that $\epsilon(i)=\epsilon(i+1)$ for all $i\in f(\hat{\chi}^{-1}(\{b\}))$. For each $i\in\{1,\ldots,n\}$ if*

$$Z_i \in \begin{cases} A_{\epsilon(i)}^{\chi(i)} & \text{if } i\in f_{\hat{\chi}}(\{1,\ldots,n'\}\cap\hat{\chi}^{-1}(\{b\})) \\ C_k' & \text{if } i\in f_{\hat{\chi}}(\hat{\chi}^{-1}(\{b\})) \\ D_k' & \text{otherwise} \end{cases},$$

then

$$E(Z_1Z_2\cdots Z_n) = \sum_{\pi\in BNC_{ffb}(\hat{\chi})} \kappa_\pi(Z_1,\ldots,Z_n).$$

Equivalently,

$$\kappa_{1_\chi}(Z_1,\ldots,Z_n) = \sum_{\pi\in BNC_{ffb}(\hat{\chi})} \mathcal{E}_\pi(Z_1,\ldots,Z_n)\mu_{BNC}(\pi,1_\chi).$$

Lastly, $\kappa_{1_\chi}(Z_1,\ldots,Z_n)=0$ unless ϵ is constant.

Proof. Using the fact that the pairs $\{(\mathrm{alg}(A_k^\ell, C_k'), \mathrm{alg}(A_k^r, D_k'))\}_{k\in K}$ are bi-free and Theorem 2.1.23,

$$\begin{aligned}
E_{\mathcal{A}}(Z_1Z_2\cdots Z_n) &= \sum_{\pi\in BNC(\chi)} \sum_{\substack{\sigma\in BNC(\chi)\\ \pi\le\sigma\le\epsilon}} \mu_{BNC}(\pi,\sigma)\mathcal{E}_\pi(Z_1,Z_2,\ldots,Z_n) \\
&= \sum_{\pi\in BNC_{\mathrm{ffb}}(\chi)} \sum_{\substack{\sigma\in BNC(\chi)\\ \pi\le\sigma\le\epsilon}} \mu_{BNC}(\pi,\sigma)\mathcal{E}_\pi(Z_1,Z_2,\ldots,Z_n) \qquad \text{(Theorem 3.2.9)} \\
&= \sum_{\pi\in BNC_{\mathrm{ffb}}(\chi)} \sum_{\substack{\sigma\in BNC_{\mathrm{ffb}}(\chi)\\ \pi\le\sigma\le\epsilon}} \mu_{BNC}(\pi,\sigma)\mathcal{E}_\pi(Z_1,Z_2,\ldots,Z_n) \qquad \text{(Remark 3.2.8)} \\
&= \sum_{\substack{\sigma\in BNC_{\mathrm{ffb}}(\chi)\\ \sigma\le\epsilon}} \sum_{\substack{\pi\in BNC_{\mathrm{ffb}}(\chi)\\ \pi\le\sigma}} \mu_{BNC}(\pi,\sigma)\mathcal{E}_\pi(Z_1,Z_2,\ldots,Z_n) \\
&= \sum_{\substack{\sigma\in BNC_{\mathrm{ffb}}(\chi)\\ \sigma\le\epsilon}} \sum_{\substack{\pi\in BNC(\chi)\\ \pi\le\sigma}} \mu_{BNC}(\pi,\sigma)\mathcal{E}_\pi(Z_1,Z_2,\ldots,Z_n) \qquad \text{(Theorem 3.2.9)} \\
&= \sum_{\sigma\in BNC_{\mathrm{ffb}}(\chi)} \kappa_\sigma(Z_1,Z_2,\ldots,Z_n).
\end{aligned}$$

The last claim follows again from the fact that the pairs $\{(\mathrm{alg}(A_k^\ell, C_k'), \mathrm{alg}(A_k^r, D_k'))\}_{k\in K}$ are bi-free. □

So these special bi-free cumulants will act as free-free-Boolean cumulants once we can show that our bi-free ffb B-systems give rise to free-free-Boolean independence, which we will do next.

3.2.4 Independence of our abstract structures

We saw in section 5.2 a construction that embeds any free-free-Boolean family into a bi-free ffb B-system, but there was a lingering question about whether the joint distribution was preserved. With the combinatorial tools that we have just developed, we can now show that bi-free ffb B-systems do in fact build free-free-Boolean independent families with amalgamation over B, thus ensuring that our embedding preserves joint distribution.

Theorem 3.2.11. *If $\{(A_k^\ell, A_k^r, C_k', D_k')\}_{k\in K}$ is a bi-free ffb B-system in a B-B-non-commutative probability space $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ then the family $\Gamma = \{(A_k^\ell, A_k^r, alg(C_k'D_k'))\}_{k\in K}$ is free-free-Boolean independent with amalgamation over B.*

Proof. This argument compares an arbitrary moment in the family Γ with a corresponding free-free-Boolean represented moment as in the definition of free-free-Boolean independence. To do so, we set up several notations.

For each $k \in K$ let $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$ be a copy of the B-B-bimodule with specified B-projection that appears in Theorem 2.1.10 and let $\ell_k : A_k^\ell \to \mathcal{L}_\ell(\mathcal{X}_k)$, $r_k : A_k^r \to \mathcal{L}_r(\mathcal{X}_k)$, and $m_k : A_k^b := \mathrm{alg}(C_k'D_k') \to \mathcal{L}(\mathcal{X}_k)$ all be defined as their corresponding restrictions of the map θ as defined in Theorem 2.1.10. Let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be the reduced free product of the family $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k\in K}$, let λ_k and ρ_k denote the corresponding left and right regular representations respectively.

Henceforth, let $n \in \mathbb{N}$, $\hat{\chi} : \{1, \ldots, n\} \to \{\ell, r, b\}$, $\hat{\epsilon} : \{1, \ldots, n\} \to K$, and $Z_1, \ldots, Z_n \in \mathcal{A}$ such that $Z_i \in A_{\hat{\epsilon}(i)}^{\hat{\chi}(i)}$. For those i such that $\hat{\chi}(i) = b$, we restrict, without loss of generality, the Z_i to be of the form $Z_i = S_{C,i}S_{D,i}$ where $S_{C,i} \in C_{\hat{\epsilon}(i)}'$ and $S_{D,i} \in D_{\hat{\epsilon}(i)}'$.

Now let $m = n + |\hat{\chi}^{-1}(\{b\})|$ and $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ be defined by

$$f(i) = i + |\hat{\chi}^{-1}(\{b\}) \cap [1, i-1]|.$$

We want to split up the Boolean factors so that we can make use of bi-freeness and our

LR-diagram calculus. So with this idea, define for each $j \in \{1, \ldots, m\}$

$$T_j := \begin{cases} Z_{f^{-1}(j)} & \text{if } j \in f(\hat{\chi}^{-1}\{\ell, r\}) \\ S_{C,i} & \text{if } i \in \hat{\chi}^{-1}(\{b\}) \text{ and } f(i) = j \\ S_{D,i} & \text{if } i \in \hat{\chi}^{-1}(\{b\}) \text{ and } f(i) + 1 = j \end{cases}.$$

For convenience, also let $\chi : \{1, \ldots, m\} \to \{\ell, r\}$ be the ℓ-r-replacement of $\hat{\chi}$, and let $\epsilon : \{1, \ldots, m\} \to K$ be defined from $\hat{\epsilon}$ by $\epsilon(f(i)) = \hat{\epsilon}(i)$, and if $j \in \{1, \ldots, m\} \setminus f(\{1, \ldots, n\})$, let $\epsilon(j) = \epsilon(j-1)$ so that $\epsilon(f(i)) = \epsilon(f(i)+1)$ whenever $i \in \hat{\chi}^{-1}(\{b\})$.

Furthermore, for each $1 \leq i \leq m$ let

$$\mu_i(T_i) = \begin{cases} \lambda_{\epsilon(i)}(\ell_{\epsilon(i)}(T_i)) & \text{if } \chi(i) = \ell \\ \rho_{\epsilon(i)}(r_{\epsilon(i)}(T_i)) & \text{if } \chi(i) = r \end{cases}, \quad \text{and} \quad \mu_i'(T_i) = \begin{cases} P_{\uplus,\epsilon(i)}\mu_i(T_i) & \text{if } i \in f(\hat{\chi}^{-1}(\{b\}) \\ \mu_i(T_i) & \text{otherwise} \end{cases},$$

and

$$\tilde{\mu}_i(Z_i) = \begin{cases} \lambda_{\epsilon(i)}(\ell_{\epsilon(i)}(Z_i)) & \text{if } \hat{\chi}(i) = \ell \\ \rho_{\epsilon(i)}(r_{\epsilon(i)}(Z_i)) & \text{if } \hat{\chi}(i) = r \\ P_{\uplus,\epsilon(i)}\lambda_{\epsilon(i)}(m_{\epsilon(i)}(Z_i))P_{\uplus,\epsilon(i)} & \text{if } \hat{\chi}(i) = b \end{cases}.$$

With all of our setup in hand, we note that the Z_i are chosen to correspond to an arbitrary moment of Γ for which we would like to compare with the corresponding free-free-Boolean moment. That is, our proof will be complete if we can demonstrate that

$$E_{\mathcal{A}}(Z_1 \cdots Z_n) = E_{\mathcal{L}(\mathcal{X})}(\tilde{\mu}_1(Z_1) \cdots \tilde{\mu}_n(Z_n)).$$

The elements T_i will help us form a bridge between these two moments. To see how, first note that by definition of the T_i, we have $Z_1 \cdots Z_n = T_1 \cdots T_m$, so that

$$E_{\mathcal{A}}(Z_1 \cdots Z_n) = E_{\mathcal{A}}(T_1 \cdots T_m) \tag{3.1}$$

Since the family $\{(\mathrm{alg}(A_k^\ell, C_k'), \mathrm{alg}(A_k^r, D_k'))\}_{k \in K}$ is bi-free over B, we have

$$E_{\mathcal{A}}(T_1 \cdots T_m) = E_{\mathcal{L}(\mathcal{X})}(\mu_1(T_1) \cdots \mu_m(T_m)) = p\mu_1(T_1) \cdots \mu_m(T_m)1_B. \tag{3.2}$$

Next, for each $i \in \hat{\chi}^{-1}(\{b\})$, we have

$$\begin{aligned}
\mu'_{f(i)}(T_{f(i)})\mu'_{f(i)+1}(T_{f(i)+1}) &= P_{\epsilon(f(i))}\mu_{f(i)}(T_{f(i)})\mu_{f(i)+1}(T_{f(i)+1}) \\
&= P_{\uplus,\epsilon(f(i))}\lambda_{\epsilon(f(i))}(\theta(S_{C,i}))\rho_{\epsilon(f(i)+1)}(\theta(S_{D,i})) \\
&= P_{\uplus,\epsilon(f(i))}\lambda_{\epsilon(f(i))}(\theta(S_{C,i}))\rho_{\epsilon(f(i))}(\theta(S_{D,i})) \\
&= P_{\uplus,\epsilon(f(i))}\lambda_{\epsilon(f(i))}(\theta(S_{C,i}))\lambda_{\epsilon(f(i))}(\theta(S_{D,i})) \qquad \text{(Lemma 1.4.11)} \\
&= P_{\uplus,\epsilon(f(i))}\lambda_{\epsilon(f(i))}(\theta(S_{C,i}S_{D,i})) \\
&= P_{\uplus,\epsilon(f(i))}\lambda_{\epsilon(f(i))}(m_i(Z_i))P_{\uplus,\epsilon(f(i))} \\
&= P_{\uplus,\hat{\epsilon}(i)}\lambda_{\hat{\epsilon}(i)}(m_i(Z_i))P_{\uplus,\hat{\epsilon}(i)} \\
&= \tilde{\mu}_i(Z_i),
\end{aligned}$$

where we made use of Lemma 1.4.11 to replace the $\rho_{\epsilon(f(i)+1)}$ with $\lambda_{\epsilon(f(i)+1)}$ (which is the same as $\lambda_{\epsilon(f(i))}$), and to commute those operators with the projections in the antepenultimate equality. From this we can see that, in fact,

$$\mu'_1(T_1)\cdots\mu'_m(T_m) = \tilde{\mu}_1(Z_1)\cdots\tilde{\mu}_n(Z_n). \tag{3.3}$$

For each $i \in \hat{\chi}^{-1}(\{b\})$, form the sets

$$S_{f(i)} = LR^{\text{lat}}(\chi|_{\{f(i),\ldots,m\}}, \epsilon|_{\{f(i),\ldots,m\}}, \epsilon(f(i)))^c$$

Then by Lemma 2.3.15, there is some $S \subseteq \bigcup_{i\in\hat{\chi}^{-1}(\{b\})} S^{\chi}_{f(i)}$ and constants $c_{D,S}$ such that

$$\mu_1(T_1)\cdots\mu_n(T_n)1_B = \mu'_1(T_1)\cdots\mu'_n(T_n)1_B + \sum_{D\in S} c_{D,S}E_D(\mu_1(T_1),\ldots,\mu_n(T_n)). \tag{3.4}$$

Now from equation (3.3), we only need to show that

$$E_{\mathcal{L}(\mathcal{X})}\left(\sum_{D\in S} c_{D,S}E_D(\mu_1(T_1),\ldots,\mu_m(T_m))\right) = 0. \tag{3.5}$$

It is immediate from Remark 2.3.7 that if $D \in LR^{\text{lat}}(\chi,\epsilon) \setminus LR_0^{\text{lat}}(\chi,\epsilon)$, then

$$E(c_{D,S}E_D(\mu_1(T_1),\ldots,\mu_m(T_m))) = 0,$$

so we only need to consider those diagrams D which correspond to partitions $\pi \in BNC(\chi)$ with $\pi \le \epsilon$. To this end, let $D \in S = \bigcup_{j=1}^{\ell} S^{\chi}_{i_j} \cap LR_0^{\text{lat}}(\chi,\epsilon)$ and let π be its corresponding bi-non-crossing partition. This means that there is some $i \in \hat{\chi}^{-1}(\{b\})$ such that D is a χ-extension of a diagram in $S_{f(i)}$. But by definition of $S_{f(i)}$, $T_{f(i)}$ is not in the same block as $T_{f(i)+1}$. So it remains true that any χ-extension of such a diagram also leaves $T_{f(i)}$ and $T_{f(i)+1}$ disconnected. Thus, by Theorem 3.2.9 and bi-free independence, it follows that

$$E_D(\mu_1(T_1), \ldots, \mu_m(T_m)) = \mathcal{E}_\pi(T_1, \ldots, T_m) = 0.$$

Combining these observations grants us equation (3.5).

We conclude from equations (3.1), (3.2), (3.4), (3.3), and (3.5), that

$$E_{\mathcal{A}}(Z_1 \cdots Z_n) = E_{\mathcal{L}(\mathcal{X})}(\tilde{\mu}_1(Z_1) \cdots \tilde{\mu}_n(Z_n)),$$

granting us, by the arbitrary choice of moment, that the family Γ is free-free-Boolean independent over B. $\square$

So after connecting free-free-Boolean independent families with bi-free structures, in the form of our ffb B-systems, we see that the free-free-Boolean cumulants can be defined in terms of the bi-free cumulants. Thus, any appropriate combinatorial tools from bi-free probability helps elucidate the combinatorics of free-free-Boolean independence. This diminishes the need to develop the tools of another lattice, such as the so-called *interval-bi-non-crossing partitions* of Liu in [6].

3.3 Monotone-Boolean Embeddability

We turn our sights next on a couple of comparably easy embeddings. Namely, Skoufranis' approach to embedding a pair of monotone independent algebras involved Boolean-like operators and free operators (c.f. Remark 3.1.8). This suggests that we should be able to find an embedding of a pair of monotone-Boolean independent pairs of B-faces into free-Boolean operators quite simply.

Lemma 3.3.1. *If (A_1^ℓ, A_1^b) and (A_2^ℓ, A_2^b) are a free-Boolean independent pair of pairs of B-faces in a B-B-non-commutative probability space $(\mathcal{A}, E, \varepsilon)$ and $\tilde{A}_1^b$ is the $\varepsilon(B \otimes 1_B)$-algebra generated by A_1^b and 1_B, then the pairs $\{(\tilde{A}_1^b, A_1^b), (A_2^\ell, A_2^b)\}$ are monotone-Boolean independent.*

Proof. By Corollary 2.3.16 we may choose our representation spaces as we like. So let $(\mathcal{X}_1, \mathring{\mathcal{X}}_1, p_1)$ and $(\mathcal{X}_2, \mathring{\mathcal{X}}_2, p_2)$ be copies of the B-B-bimodule with specified B-projection as in Theorem 2.1.10, with corresponding expectation preserving homomorphisms $\theta_i : \mathcal{A} \to \mathcal{L}(\mathcal{X}_i)$. Let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ their reduced free product. For convenience, set $\ell_i = \theta_i|_{A_i^\ell}$ and $r_i = \theta_i|_{A_i^b}$. Then by independence, we have that $\{(\lambda_1 \circ \ell_1, \gamma_{\uplus,1} \circ r_1), (\lambda_2 \circ \ell_2, \gamma_{\uplus,2} \circ r_2)\}$ is joint-distribution preserving. Let $\tilde{A}_1^\ell$ be the $\varepsilon(B \otimes 1_B)$-algebra generated by A_1^b and 1_B, and let $\tilde{\ell}_1 = \theta_1|_{\tilde{A}_1^\ell}$.

Now, let $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to \{1, 2\}$ and $a_i \in C_{\epsilon(i)}^{\chi(i)}$ where $C_1^\ell = \tilde{A}_1^\ell$, $C_2^\ell = A_2^\ell$, and $C_i^r = A_i^b$ for $i = 1, 2$. Let

$$\delta_i(a_i) = \begin{cases} \gamma_{\uplus,1}(\ell_1(a_i)) & \text{if } \chi(i) = \ell, \epsilon(i) = 1 \\ \lambda_2(\ell_2(a_i)) & \text{if } \chi(i) = \ell, \epsilon(i) = 2 \\ \gamma_{\uplus,2}(r_{\epsilon(i)}(a_i)) & \text{otherwise} \end{cases}$$

and

$$\delta_i'(a_i) = \begin{cases} \gamma_{\rhd,1}(\ell_1(a_i)) & \text{if } \chi(i) = \ell, \epsilon(i) = 1 \\ \gamma_{\rhd,2}(\ell_2(a_i)) & \text{if } \chi(i) = \ell, \epsilon(i) = 2 \\ \gamma_{\uplus,2}(r_{\epsilon(i)}(a_i)) & \text{otherwise} \end{cases}$$

From Lemma 1.4.11 d) and some considerations in Remark 2.2.5, it follows that $\gamma_{\rhd,1} = \gamma_{\uplus,1}$ as representations of $\mathcal{L}(\mathcal{X}_1)$, so in fact we have $\delta_i(a_i) = \delta_i'(a_i)$ whenever $\epsilon(i) = 1$ and $\chi(i) = \ell$.

Furthermore, it is an easy observation that both $\mathcal{X}(\rhd, 2)$ and $\mathcal{X} \ominus \mathcal{X}(\rhd, 2)$ are invariant for the algebras $\gamma_{\uplus,i}(r_i(A_i^b))$ for $i = 1, 2$, $\gamma_{\uplus,1}(\tilde{\ell}_1(\tilde{A}_1^\ell))$, and $\lambda_2(\ell_2(A_2^\ell))$. So in particular, the projections $P_{\rhd,2}$ commute with all these operators. This fact yields,

$$\delta_1'(a_1) \cdots \delta_n'(a_n) 1_B = \delta_1(a_1) \cdots \delta_n(a_n) P_{\rhd,2} 1_B = \delta_1(a_1) \cdots \delta_n(a_n) 1_B$$

so that

$$E_{\mathcal{L}(\mathcal{X})}(\delta_1'(a_1) \cdots \delta_n'(a_n)) = E_{\mathcal{L}(\mathcal{X})}(\delta_1(a_1) \cdots \delta_n(a_n))$$

verifying that the pair of pairs $\{(\tilde{A}_1^\ell, A_1^b), (A_2^\ell, A_2^b)\}$ is monotone-Boolean independent. $\square$

This gives us a way to embed any monotone-Boolean independent pair into a free-Boolean independent one.

Proposition 3.3.2. *Let $\{(A_1^m, A_1^b), (A_2^m, A_2^b)\}$ be a monotone-Boolean independent pair of pairs of B-faces in a B-B-non-commutative probability space $(\mathcal{A}, E, \varepsilon)$. Then there is a free-Boolean independent pair $\{(\tilde{A}_1^\ell, \tilde{A}_1^b), (\tilde{A}_2^\ell, \tilde{A}_2^b)\}$ of pairs of B-faces in a B-B-non-commutative* *probability space* $(\mathcal{A}', E', \varepsilon')$ *and injective* B*-linear maps* $\alpha_{1,m} : A_1^m \to \tilde{A}_1^b$*,* $\alpha_{2,m} : A_2^m \to \tilde{A}_2^\ell$*, and* $\alpha_{i,b} : A_i^b \to \tilde{A}_i^b$ *for* $i = 1, 2$ *such that for all* $n \in \mathbb{N}$*,* $\chi : \{1, \ldots, n\} \to \{m, b\}$*,* $\epsilon : \{1, \ldots, n\} \to \{1, 2\}$*, and* $a_i \in A_{\epsilon(i)}^{\chi(i)}$ *for* $1 \leq i \leq n$*, we have*

$$E'(\alpha_{\epsilon(1),\chi(1)}(a_1) \cdots \alpha_{\epsilon(n),\chi(n)}(a_n)) = E(a_1 \cdots a_n).$$

Proof. Let $(\mathcal{X}_1, \mathring{\mathcal{X}}_1, p_1)$ and $(\mathcal{X}_2, \mathring{\mathcal{X}}_2, p_2)$ be copies of the B-B-bimodule with specified B-projection as in Theorem 2.1.10 with corresponding homomorphisms $\theta_i : \mathcal{A} \to \mathcal{L}(\mathcal{X}_i)$, and let $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ be their reduced free product. Let C_1 be the smallest $\varepsilon_{\mathcal{L}(\mathcal{X})}(1_B \otimes B)$-algebra generated by $\gamma_{\triangleright,1}(\theta_1(A_1^m)) = \gamma_{\uplus,1}(\theta_1(A_1^m))$ and $\gamma_{\uplus,1}(\theta_1(A_1^b))$. Then the pair $\{(\varepsilon(B \otimes 1_B), C_1), (\lambda_2(\theta_2(A_2^m), \gamma_{\uplus,2}(\theta_2(A_2^b))))\}$ of pairs of B-faces in the B-B-non-commutative probability space $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})}, \varepsilon_{\mathcal{L}(\mathcal{X})})$ is free-Boolean independent by Corollary 2.3.17.

Let $\alpha_{1,\chi}(a) = \gamma_{\uplus,1}(a)$ for all $a \in A_1^\chi$ and $\chi \in \{m, b\}$, $\alpha_{2,m}(a) = \lambda_2(\theta_2(a))$ for $a \in A_2^m$, and $\alpha_{2,b}(a) = \gamma_{\uplus,2}(\theta_2(a))$ for all $a \in A_2^b$, It is clear that the $\alpha_{i,\chi}$'s are injective and expectation preserving.

Lastly, since the pair $\{(\alpha_{1,m}(\theta_1(A_1^m)), C_1), (\lambda_2(\theta_2(A_2^m)), \gamma_{\uplus,2}(\theta_2(A_2^b)))\}$ are monotone-Boolean independent from Lemma 3.3.1 above, it follows that the B-linear representation $\alpha = \{(\alpha_{1,m}, \alpha_{1,b}), (\alpha_{2,m}, \alpha_{2,b})\}$ joint-distribution preserving, as required. □

Remark 3.3.3. Because of these embeddings, we can use our embedding of free-free-Boolean independent families into bi-free families to conclude that pairs of monotone-Boolean independent pairs embed faithfully into bi-free independence as well.

3.4 A Partial Negative Result

Though it would have been nice, it appears that not all of Liu's notions of independence embed faithfully, even for pairs, into bi-free probability. It appears that the monotone independence does not behave well with Voiculescu's left and right freely acting operators.

Somewhat strangely, though at first glance it seems like we should simply be able to use Skoufranis' embedding of monotone probability and equip it with additional freely acting operators, Liu's free-monotone independence seems to put up some resistance.

In the following, we make several assumptions which may reduce the generality of the investigation. However, the assumptions are motivated by an attempt to extend Skoufranis' ideas to the free-monotone setting.

In what follows, we fix for simplicity $B = \mathbb{C}$, and two vector spaces with specified projection $(\mathcal{X}_i, \mathring{\mathcal{X}}_i, \xi_i)$ for $i = 1, 2$, letting $(\mathcal{X}, \mathring{\mathcal{X}}, \xi)$ be their reduced free product. Then the pair of pairs

$$\{(\lambda_1(\mathcal{L}(\mathcal{X}_1)), \gamma_{\lhd,1}(\mathcal{L}(\mathcal{X}_1))), (\lambda_2(\mathcal{L}(\mathcal{X}_2)), \gamma_{\lhd,2}(\mathcal{L}(\mathcal{X}_2)))\}$$

are free-monotone independent in $(\mathcal{L}(\mathcal{X}), E_{\mathcal{L}(\mathcal{X})})$. Note that because for $i = 1, 2$, λ_i and $\gamma_{\lhd,i}$ are injective and expectation preserving from $\mathcal{L}(\mathcal{X}_i)$, it follows that we may represent, in an expectation preserving way, each of the pairs on their underlying spaces $\mathcal{X}_i$ by simply inverting the maps λ_i and $\gamma_{\lhd,i}$. That is, let $\ell_i : \lambda_i(\mathcal{L}(\mathcal{X}_i)) \to \mathcal{L}(\mathcal{X}_i)$ and $r_i : \gamma_{\lhd,i} \to \mathcal{L}(\mathcal{X}_i)$ be the inverse maps, then (ℓ_i, r_i) is an expectation preserving representation of the pair $(\lambda_i(\mathcal{L}(\mathcal{X}_i)), \rho_i(\mathcal{L}(\mathcal{X}_i)))$.

We claim that Skoufranis's approach does not work to represent this pair in bi-free operators.

Because the representations (ℓ_i, r_i) are expectation preserving, it is sufficient to consider bi-free operators on $\mathcal{X}$, that is, left and right regular representations of $\mathcal{L}(\mathcal{X}_i)$. One simplifying, but reasonable assumption we will make immediately is that $\delta_{i,\ell} : \lambda_i(\mathcal{L}(\mathcal{X}_i)) \to \lambda_i(\mathcal{L}(\mathcal{X}_i))$ is simply the identity. Alternatively, we could have sent the free, left-acting operators to the right-acting operators, however this would lead to symmetric considerations in what follows.

Remark 3.4.1. A direct application of the definition shows that $P_{\lhd,1}$ projects onto the space

$$\mathcal{X}(\lhd, 1) = \mathbb{C} \oplus \mathring{\mathcal{X}}_1,$$

and $P_{\lhd,2}$ projects onto the space

$$\mathcal{X}(\lhd, 2) = \mathbb{C} \oplus \mathring{\mathcal{X}}_1 \oplus \mathring{\mathcal{X}}_2 \oplus \mathring{\mathcal{X}}_1 \otimes \mathring{\mathcal{X}}_2.$$

Next, we consider the possible extensions of Skoufranis' ideas for the monotone operators. In particular, we consider a few different options for $\delta_{i,r} : \gamma_{\lhd,i} \to \mathcal{L}(\mathcal{X})$, and show how each of them fail.

First, choose $m \in \mathcal{L}(\mathcal{X}_2)$ with $E_{\mathcal{L}(\mathcal{X}_2)}(m) = 1$, $a_1 \in \mathcal{L}(\mathcal{X}_1)$ such that $E_{\mathcal{L}(\mathcal{X}_1)}(a_1) = 0$ and $E_{\mathcal{L}(\mathcal{X}_1)}(a_1^2) = 1$, $a_2 \in \mathcal{L}(\mathcal{X}_2)$ such that $E_{\mathcal{L}(\mathcal{X}_2)}(a_2) = 0$ and $E_{\mathcal{L}(\mathcal{X})}(a_2^2) = 1$ and $E_{\mathcal{L}(\mathcal{X}_2)}(ma) = 1$.

Skoufranis' construction suggests we should consider our second monotone algebra to be a simple (left or right) regular action on the free product. These are shown to be problematic in cases 1 and 2 below. Alternatively, the case where the second monotone algebra acts like a product of a left action and a right action (as in the case of the first monotone algebra of

Skoufranis' construction), each of which independently have null expectation, is handled in case 3.

Case 1: Suppose that $\delta_{2,r}(\gamma_{\triangleleft,2}(m)) = \rho_2(m)$.

Under the definition, we have

$$\begin{aligned}
E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)\lambda_2(a_2)\gamma_{\triangleleft,2}(m)\lambda_2(a_2)\lambda_1(a_1)) &= p\lambda_1(a_1)\lambda_2(a_2)P_{\triangleleft,2}\rho_2(m)P_{\triangleleft,2}\lambda_2(a_2)\lambda_1(a_1)\mathbf{1} \\
&= p\lambda_1(a_1)\lambda_2(a_2)P_{\triangleleft,2}\rho_2(m)P_{\triangleleft,2}a_2\mathbf{1}\otimes a_1\mathbf{1} \\
&= p\lambda_1(a_1)\lambda_2(a_2)P_{\triangleleft,2}\rho_2(m)0 \\
&= 0.
\end{aligned}$$

With case assumption, and assuming that the family $\delta = \{(\delta_{i,\ell}, \delta_{i,r})\}_{i=1,2}$ is joint-distribution preserving, we can compute with the bi-free cumulant inclusion-exclusion, for example that

$$\begin{aligned}
E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)\lambda_2(a_2)\gamma_{\triangleleft,2}(m)\lambda_2(a_2)\lambda_1(a_1)) &= p\lambda_1(a_1)\lambda_2(a_2)\rho_2(m)\lambda_2(a_2)\lambda_1(a_1)\mathbf{1} \\
&= E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(a_2)^2)E_{\mathcal{L}(\mathcal{X})}(\rho_2(m)) \\
&\quad + E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1))^2E_{\mathcal{L}(\mathcal{X})}(\lambda_2(a_2)\rho_2(m)\lambda_2(a_2)) \\
&\quad + E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1))^2E_{\mathcal{L}(\mathcal{X})}(\rho_2(m))E_{\mathcal{L}(\mathcal{X})}(\lambda_2(a_2)^2) \\
&= 1+0+0 \\
&\neq 0
\end{aligned}$$

a contradiction.

Case 2: $\delta_{2,r}(\gamma_{\triangleleft,2}(m)) = \lambda_2(m)$.

Under the definition we have

$$\begin{aligned}
E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)\gamma_{\triangleleft,2}(m)\lambda_2(a_2)\lambda_1(a_1)) &= p\lambda_1(a_1)P_{\triangleleft,2}\rho_2(m)P_{\triangleleft,2}\lambda_2(a_2)\lambda_1(a_1)\mathbf{1} \\
&= p\lambda_1(a_1)P_{\triangleleft,2}\rho_2(m)P_{\triangleleft,2}a_2\mathbf{1}\otimes a_1\mathbf{1} \\
&= p\lambda_1(a_1)\lambda_2(a_2)P_{\triangleleft,2}\rho_2(m)0 \\
&= 0.
\end{aligned}$$

But if we calculate under the case assumption and that the family δ is joint distribution preserving, we get

$$\begin{aligned}E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)\gamma_{\lhd,2}(m)\lambda_2(a_2)\lambda_1(a_1)) &= p\lambda_1(a_1)\lambda_2(m)\lambda_2(a_2)\lambda_1(a_1)\mathbf{1}\\ &= E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(m)\lambda_2(a_2))\\ &= E_{\mathcal{L}(\mathcal{X})}(\lambda_1(a_1)^2)E_{\mathcal{L}(\mathcal{X})}(\lambda_2(ma))\\ &= 1.\end{aligned}$$

Another contradiction.

Case 3: There exist $m_1, m_2 \in \mathcal{L}(\mathcal{X}_2)$ such that $E_{\mathcal{L}(\mathcal{X}_2)}(m_i) = 0$ for $i = 1, 2$ and $E_{\mathcal{L}(\mathcal{X}_2)}(m_1m_2) = E_{\mathcal{L}(\mathcal{X}_2)}(m)$, then suppose $\delta_{2,r}(m) = \lambda_2(m_1)\rho_2(m_2)$.

Suppose further that $m' \in \mathcal{L}(\mathcal{X}_1)$ such that $E_{\mathcal{L}(\mathcal{X}_1)}(m') = 0$ and $E_{\mathcal{L}(\mathcal{X}_1)}(m'^2) = 1$.

Since m is monotone independent to m', we should get

$$E_{\mathcal{L}(\mathcal{X})}(\gamma_{\lhd,1}(m')\gamma_{\lhd,2}(m)\gamma_{\lhd,1}(m')) = E_{\mathcal{L}(\mathcal{X})}(\gamma_{\lhd,1}(m')^2)E_{\mathcal{L}(\mathcal{X})}(\gamma_{\lhd,2}(m)).$$

However, under the case assumption, and assuming that the family δ is a joint distribution preserving representation,

$$\begin{aligned}E_{\mathcal{L}(\mathcal{X})}(\gamma_{\lhd,1}(m')\gamma_{\lhd,2}(m)\gamma_{\lhd,1}(m')) &= p\delta_{1,r}(\gamma_{\lhd,1}(m'))\lambda_2(m_1)\rho_2(m_2)\delta_{1,r}(\gamma_{\lhd,1}(m'))\mathbf{1}\\ &= p\delta_{1,r}(\gamma_{\lhd,1}(m'))\lambda_2(m_1)\rho_2(m_2)(\delta_{1,r}(\gamma_{\lhd,1}(m'))\mathbf{1})\\ &= p\delta_{1,r}(\gamma_{\lhd,1}(m'))(m_1\mathbf{1} \otimes \delta_{1,r}(\gamma_{\lhd,1}(m'))\mathbf{1} \otimes m_2\mathbf{1})\\ &= 0.\end{aligned}$$

A contradiction.

So in these three cases of reasonable assumptions, it seems that we cannot faithfully embed arbitrary free-monotone families into bi-free operators, at least by extending Skoufranis' ideas.

Chapter 4

Summary and Questions

Throughout this work, we have encountered several notions of independence. A main theme has been in investigating how these notions relate to each other. In particular, the multi-algebra independences demonstrate an ability to house, via an embedding, other types of independence, including other multi-algebra independences.

There are several simple such embeddings essentially by inclusion, such as

- Boolean independence sits trivially inside the free-Boolean, Boolean-monotone, and Boolean-Boolean independences. Similar claims are immediate with monotone and free independences.

- Free-Boolean sits inside free-free-Boolean independence and free-Boolean-monotone independence. Similar claims can be made about free-monotone independence and bi-free independence.

- Boolean-Boolean independence acts redundantly as Boolean operators, so it can be embedded into Boolean independence (c.f. Remark 2.2.5).

We have also shown that

- free-free-Boolean independence may be embedded into bi-free probability (section 3.2),

- monotone-Boolean may embedded into free-Boolean in pairs (Proposition 3.3.2),

- free-monotone however, seems to fail (section 3.4).

These observations suggest a certain "universal" independence among this collection of multi-algebra independences, formed by quadruples of faces. Quadruples and their representation can be defined in the expected way.

Definition 4.0.1. A quadruple of B-faces in a B-B-non-commutative probability space $(\mathcal{A}, E, \varepsilon)$ is a quadruple $(A^{\ell_1}, A^{r_1}, A^{\ell_2}, A^{r_2})$ of B-subalgebras of $\mathcal{A}$ where each of (A^{ℓ_1}, A^{r_1}) and (A^{ℓ_2}, A^{r_2}) are pairs of B-faces.

Then for a family $\Gamma = \{(A_k^{\ell_1}, A_k^{r_1}, A_k^{\ell_2}, A_k^{r_2})\}_{k\in K}$ of quadruples of B-faces, a *representation* is a B-B-bimodule with specified B-projection $(\mathcal{X}, \mathring{\mathcal{X}}, p)$ and a family of quadruples of homomorphisms (unital when their domains are) $\delta = \{(\delta_{\ell_1,k}, \delta_{r_1,k}, \delta_{\ell_2,k}, \delta_{r_2,k})\}_{k\in K}$ into $\mathcal{L}(\mathcal{X})$ such that $\{(\delta_{\ell_1,k}, \delta_{r_1,k})\}_{k\in K}$ is a representation of the family $\{(A_k^{\ell_1}, A_k^{r_1})\}_{k\in K}$ and $\{(\delta_{\ell_2,k}, \delta_{r_2,k})\}_{k\in K}$ is a representation of the family $\{(A_k^{\ell_2}, A_k^{r_2})\}_{k\in K}$.

We say that δ is joint-distribution preserving if for all $n \in \mathbb{N}$, $\chi : \{1, \ldots, n\} \to \{\ell_1, r_1, \ell_2, r_2\}$, $\epsilon : \{1, \ldots, n\} \to K$, and $a_i \in A_{\epsilon(i)}^{\chi(i)}$,

$$E(a_1 \cdots a_n) = E_{\mathcal{L}(\mathcal{X})}(\delta_{\chi(1),\epsilon(1)}(a_1) \cdots \delta_{\chi(n),\epsilon(n)}(a_n)).$$

Definition 4.0.2. Given a family $\Gamma = \{(A_k^{\ell_1}, A_k^{r_1}, A_k^{\ell_2}, A_k^{r_2})\}_{k\in K}$ of quadruples of B-faces, B-B-bimodules with specified B-projections $(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)$, and homomorphisms $\ell_{i,k} : A_k^{\ell_i} \to \mathcal{L}_\ell(\mathcal{X}_k)$ and $r_{i,k} : A_k^{r_i} \to \mathcal{L}_r(\mathcal{X}_k)$, say that Γ is *free-free-monotone-monotone* independent if for each $k \in K$ and B-face in the corresponding quadruple is unital, and

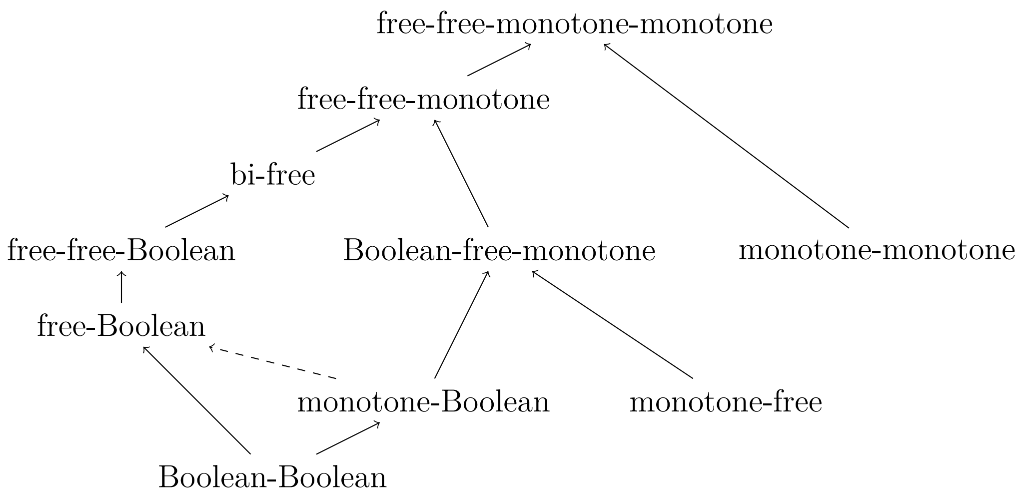
$$\{(\lambda_k \circ \ell_{1,k}, \rho_k \circ r_{1,k}, \gamma_{\rhd,k} \circ \ell_{2,k}, \gamma_{\lhd,k} \circ r_{2,k})\}_{k\in K}$$

is a distribution preserving representation of Γ.

Remark 4.0.3. It might seem that this process will never end, as we could define larger and larger tuples to make more complicated multi-algebra independences. However, given the relationships we have shown, the free-free-monotone-monotone is practically the "largest".

For example, we already saw how notions of *tri-free*, *tri-monotone*, or *tri-Boolean* indepenendences are redundant. Furthermore, by Remark 2.2.5 any Boolean-Boolean sub-independence can be simplified to a single Boolean, and if there is a free component, then by Section 3.2 free-Boolean independence sits inside of bi-free independence.

With this in mind, we can visualize the relationships between the multi-algebra independences. The following is a diagram of the embedding relationships we have encountered in this work. Arrows represent embeddability, while dashed arrows represent an embeddability reduced to pairs, due to the limitation on monotone embeddability.



There are several questions that arise from our investigations of these relationships, which are made visible in the diagram above. A few of these are mentioned below.

Question 1. *Is there any relationship, via universal embeddings, between free-monotone and monotone-monotone independences?*

Question 2. *Is there a universal embedding of free-Boolean independence into free-monotone independence?*

Question 3. *We saw a failure to embed free-monotone independence into bi-free independence, but can we find a universal embedding of bi-free independence into free-monotone independence?*

Question 4. *One of the most salient features of working in bi-free probability, compared to the monotone multi-algebra independences is in its cumulant theory. Can we exploit the bi-free cumulants to get insight about the types of independence that embed therein?*

Question 5. *How much power do we give up by working outside of bi-free probability, such as in the "universal" free-free-monotone-monotone independence, if we lack a robust cumulant theory therein?*

Question 6. *Can we relax any of our assumptions for any of the limitation arguments? For example, can we say that no universal bi-free polynomial embeddings of three tensor or monotone independent families exist?*

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