LAPLACIANS ON NONISOTROPIC HEISENBERG GROUPS WITH MULTI-DIMENSIONAL CENTER

SHENGWEN YANG

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Abstract

We begin with the construction of the nonisotropic Heisenberg group with multi-dimensional center. The sub-Laplacian on it is introduced. By taking the inverse Fourier transform of the sub-Laplacian with respect to the center, we get a family of twisted Laplacians. They are proved to be globally hypoelliptic in the Schwartz space and in the Gelfand-Shilov spaces using the Green functions of the twisted Laplacians. Global regularity in a scale of Sobolev spaces of these twisted Laplacians is given.

Equally important are the heat semigroups generated by the twisted Laplacians in terms of Weyl transforms and the $L^p - L^q$ estimates of these heat semigroups are presented. Finally, the eigenvalues and eigenfunctions of the twisted bi-Laplacians on these Heisenberg groups are studied in the context of analytic number theory. Global hypoellipticty, essential self-adjointness, Sobolev spaces as well as the Sobolev estimates of the twisted bi-Laplacians are also introduced.

Dedication

TO MOM, DAD & MYSELF

Acknowledgements

Flight number CX826 is probably the most important and memorable flight in my life. It was the trip that spearheaded my journey into Mathematics as a student with engineering background. From the blissful tranquility of Rochester, NY to the rich diversity of Toronto, it has been truly a life changing experience.

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Notations

$\mathbb{R}^n$: Euclidean Space

$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$: Underlying Manifold of a Nonisotropic Heisenberg Group

$\mathbb{H}^n$: Heisenberg Group with One-Dimensional Center

$\mathbb{G}$: Nonisotropic Heisenberg Group with Multi-Dimensional Center

$[z, w]_j$: Symplectic Form

B_m: $n \times n$ Orthogonal Matrix

I_n: $n \times n$ Identity Matrix

$\mathfrak{g}$: Lie Algebra

e_j, e_k: Standard Unit Vector

X_j, Y_j, T_k: Left-invariant Vector Fields

$C^\infty(\mathbb{G})$: Smooth Function on $\mathbb{G}$

$[X_j, Y_l]$: Commutator

$\mathcal{L}$: Sub-Laplacian

$\mathbb{R}^{m*}$: Real Coordinate Space of Dimension m, without the Origin

λ: Vector

L^λ: λ-Twisted Laplacians

$e_k(x)$: Hermite Function of Order k

$H_k(x)$: Hermite Polynomial of Degree k

α, β, γ: Multi-Index

$e^\lambda_{\alpha,\beta}$: Special Hermite Function

B^t_λ: Transpose of B_λ

$V^\lambda(f,g)$: λ-Fourier-Wigner Transform

$W^\lambda(f,g)$: λ-Wigner Transform

W^λ_σ: λ-Weyl Transform

σ: Symbol

π_λ: Schrödinger Representation

$F *_\lambda G$: λ-Twisted Convolution of Functions F and G

κ^λ_τ: Heat Kernel of the λ-Twisted Laplacian

G^λ: Green Function of the λ-Twisted Laplacian

$K_{(n-1)/2}(x)$: Modified Bessel Function of Order $(n-1)/2$

$K_\nu(x)$: Modified Bessel Function of Order ν

$\Gamma(*)$: Gamma Function

$\mathcal{S}'$: Space of Tempered Distributions

$\mathcal{S}$: Schwartz Space

$\sup|*|$: Supremum

S_ν^μ: Gelfand–Shilov Space

L_0^λ: Closure of L^λ

$\{\varphi_l\}_{l=1}^\infty$, $\{\psi_l\}_{l=1}^\infty$: Sequence

$\mathcal{D}(*)$: Domain

$\Sigma(*)$: Spectrum

I: Identity Operator

$R(*)$: Range

$\rho(*)$: Resolvent Set

$\inf|*|$: Infimum

$H^{s,2,\lambda}$: L^2-Sobolev Space of Order s

$H^{-s,2,\lambda}$: Dual Space of $H^{s,2,\lambda}$

$(\ ,\ )_{s,2,\lambda}$: Inner Product on L^2-Sobolev Space

$\|\ \|_{s,2,\lambda}$: Norm on L^2-Sobolev Space

$e^{-\tau L^\lambda}$, $\tau > 0$: Heat Semigroup

$C^* - algebra$: Banach Algebra(Closed and Adjoint)

H: Hermite Operator

N: Angular Momentum

M: Twisted Bi-Laplacian

$d(k)$: Dirichlet Divisor of Eigenvalue k

$O(k^\varepsilon)$: Order of k^ε

K: Operator of Conjugation

$N(\mu)$: Counting Function

$\sum m_n(k)$: Sum of the Multiplicity of k

$J_{s,t}$: Bessel Potential

$H^{s,t,2}$: L^2-Sobolev Space of Order (s, t)

$i : H^{s_2,t_2,2} \hookrightarrow H^{s_1,t_1,2}$: Inclusion/Embedding

1 Introduction

The Euclidean space $\mathbb{R}^n$ has been one of the most important spaces on which mathematics is studied. It is a Riemannian manifold, and it is a group with respect to the usual addition of points in $\mathbb{R}^n$. The Heisenberg group, named after Werner Karl Heisenberg, German theoretical physicist in 20th century, is the simplest non-commutative nilpotent Lie group. Analysis on the Heisenberg group is a subject of continuing interest in various areas of mathematics from partial differential equations to geometry to number theory. More general Heisenberg groups are defined for higher dimensions in Euclidean space.

Previous research has been focused on the Heisenberg group $\mathbb{H}^n$ with one-dimensional center with emphases on the heat kernels and Green functions of the Laplacians. The main techniques used are the Wigner transforms and Weyl Transforms. Features like global hypoellipticity of the twisted Laplacians in the Schwartz space and the Gelfand–Shilov spaces, essential self-adjointness and estimates in Sobolev spaces are considered as well. These are well covered in the book [15] by Wong.

With the above background materials provided, extensions to Heisenberg groups with higher-dimensional center are introduced in [9] by Molahajloo and Wong. While twisted Laplacians related to Heisenberg groups with one-dimensional center are the Hamiltonian of a system of electrons moving in a $2n$-dimensional Euclidean space under the influence of a magnetic field, twisted Laplacians on Heisenberg groups with multi-dimensional center are analogs of the same system of electrons moving in the same space under the influence of a multi-dimensional analog of a magnetic field. Meanwhile, the twisted Laplacians on Heisenberg groups with multi-dimensional center are proved to be globally hypoelliptic in the Schwartz space and in the Gelfand–Shilov spaces using the Green functions of the twisted Laplacians.

The introduction of the nonisotropic Heisenberg groups with multi-dimensional center in [8] and the construction on these groups of the heat kernels and Green functions of twisted Laplacians in [9] are the foundations on which this dissertation is built. The contributions in this dissertation are the applications of the heat kernels and Green functions in the analysis of partial differential equations in general, and in the heat equations and the potential equations driven by the twisted Laplacians on these Heisenberg groups with higher complexities.

Furthermore, a formula for the heat semigroups generated by the twisted Laplacians in terms of Weyl transforms is shown in this thesis, and the $L^p - L^q$ estimates for the heat semigroups are also given. Global regularity in a scale of Sobolev spaces for these twisted Laplacians are presented in this thesis as well. Heat kernels and Green functions of twisted Laplacians related to Heisenberg groups with multi-dimensional center can be found in [9] by Molahajloo and Wong.

In addition to the aforementioned, the eigenvalues and eigenfunctions of the twisted bi-Laplacian on the Heisenberg group with multi-dimensional center are studied in the context of analytic number theory. These results are the analogs of the results obtained in Heisenberg groups with one-dimensional center in [25] by Gramchev, Pilipovic, Rodino and Wong. Properties such as global hypoellipticity, essential self-adjointness, Sobolev spaces, as well as Sobelov estimates of the twisted bi-Laplacian are also introduced in the last chapter.

The motivation for passing from the twisted Laplacians to the twisted bi-Laplacian is to change the infinite multiplicities of the eigenvalues of the twisted Laplacians to the finite multiplicities of the eigenvalues of the twisted bi-Laplacian, and after renormalizations, the eigenvalues are positive integers and we can count the number of eigenvalues, including multiplicities, up to any positive number. This entails the use of analytic number theory.

The following four sections in this chapter provide a recall of the nonisotropic Heisenberg group $\mathbb{G}$ with multi-dimensional center, a family of twisted Laplacians L^λ

on it parametrized by $\lambda \in \mathbb{R}^{m*}$ in the center, the heat kernels and the Green functions of these twisted Laplacians. To look at some aspects of analysis and partial differential equations on a class of nonisotropic Heisenberg groups with multi-dimensional center, we begin with a description of the Laplacians.

1.1 Nonisotropic Heisenberg Group with Multi-Dimensional Center

Let $B_1, B_2, \ldots, B_m$ be $n \times n$ orthogonal matrices with real entries such that

$$B_j^{-1} B_k = -B_k^{-1} B_j$$

for all $j, k = 1, 2, \ldots, m$ with $j \neq k$. Then we define the nonisotropic Heisenberg group $\mathbb{G}$ with multi-dimensional center to be the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ equipped with the binary operation given by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{2}[z, w] \right)$$

for all (z, t) and (w, s) in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, where $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $t, s \in \mathbb{R}^m$ and $[z, w] \in \mathbb{R}^m$ is given by

$$[z, w]_j = u \cdot B_j y - x \cdot B_j v, \quad j = 1, 2, \ldots, m.$$

The center Z of the nonisotropic Heisenberg group $\mathbb{G}$ with multi-dimensional center is m-dimensional and is given by

$$Z = \{(0, 0, t) : t \in \mathbb{R}^m\}.$$

The following proposition on the dimension of a nonisotropic Heisenberg group and the dimension of its center can be found in [8].

Proposition 1.1.1 *Let $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ be the underlying manifold of a nonisotropic Heisenberg group. Then*

$$m^2 \leq n.$$

Nonisotropic Heisenberg groups with multi-dimensional center are special cases of H-type groups in [5, 6, 7]. If $m = 1$ and $B_1 = -I_n$, where I_n is the $n \times n$ identity matrix, then we get back to the ordinary n-dimensional Heisenberg group $\mathbb{H}^n$. In the book [15], for the sake of simplifying the notation and making the presentation transparent, we have chosen to study the one-dimensional Heisenberg group $\mathbb{H}^1$ in detail.

Let $\mathfrak{g}$ be the Lie algebra of all left-invariant vector fields on $\mathbb{G}$. For $j = 1, 2, \ldots, n$, let $\gamma_{1,j} : \mathbb{R} \to \mathbb{G}$ and $\gamma_{2,j} : \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{1,j}(s) = (se_j, 0, 0)$$

and

$$\gamma_{2,j}(s) = (0, se_j, 0)$$

for all $s \in \mathbb{R}$, where e_j is the standard unit vector in $\mathbb{R}^n$ with 1 in the j^{th} position. For $k = 1, 2, \ldots, m$, let $\gamma_{3,k} : \mathbb{R} \to \mathbb{G}$ be the curve in $\mathbb{G}$ given by

$$\gamma_{3,k}(s) = (0, 0, se_k)$$

for all $s \in \mathbb{R}$, where e_k is the standard unit vector in $\mathbb{R}^m$ with 1 in the k^{th} position. For $j = 1, 2, \ldots, n$, we define the left-invariant vector fields X_j and Y_j by

$$\begin{aligned}
&(X_j f)(x, y, t) \\
=\;& \frac{d}{ds} f((x, y, t) \cdot \gamma_{1,j}(s))\Big|_{s=0} \\
=\;& \frac{d}{ds} f\left(x + se_j, y, \left(t_1 + \frac{1}{2}(B_1 y, se_j), \ldots, t_m + \frac{1}{2}(B_m y, se_j)\right)\right)\Big|_{s=0} \\
=\;& \frac{\partial f}{\partial x_j}(x, y, t) + \frac{1}{2}\sum_{k=1}^{m}(B_k y, e_j)\frac{\partial f}{\partial t_k}(x, y, t)
\end{aligned}$$

and

$$\begin{aligned}
&(Y_j f)(x, y, t) \\
=\;& \frac{d}{ds} f((x, y, t) \cdot \gamma_{2,j}(s))\Big|_{s=0} \\
=\;& \frac{d}{ds} f\left(x, y + se_j, \left(t_1 - \frac{1}{2}(x, sB_1 e_j), \ldots, t_m - \frac{1}{2}(x, sB_m e_j)\right)\right)\Big|_{s=0} \\
=\;& \frac{\partial f}{\partial y_j}(x, y, t) - \frac{1}{2}\sum_{k=1}^{m}(x, B_k e_j)\frac{\partial f}{\partial t_k}(x, y, t)
\end{aligned}$$

for all $(x, y, t) \in \mathbb{G}$ and all $f \in C^\infty(\mathbb{G})$. For $k = 1, 2, \ldots, m$, we define the vector field T_k by

$$\begin{aligned}
& (T_k f)(x, y, t) \\
= \; & \frac{d}{ds} f((x, y, t) \cdot \gamma_{3,k}(s)) \Big|_{s=0} \\
= \; & \frac{d}{ds} f(x, y, t + s e_k) \Big|_{s=0} \\
= \; & \frac{\partial f}{\partial t_k}(x, y, t)
\end{aligned}$$

for all $(x, y, t) \in \mathbb{G}$ and all $f \in C^\infty(\mathbb{G})$. We can easily check that

$$[X_j, Y_l] = -\sum_{k=1}^{m} (B_k)_{jl} T_k, \quad j, l = 1, 2, \ldots, n,$$

and the other commutators are zero.

Theorem 1.1.2 *The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is generated by*

$$\{X_j, Y_l, [X_j, Y_l] : j, l = 1, 2, \ldots, n\}.$$

The sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ is defined by

$$\mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

Explicitly,

$$\begin{aligned}
\mathcal{L} \; = \; & -\Delta_x - \Delta_y - \frac{1}{4}(|x|^2 + |y|^2)\Delta_t \\
& + \sum_{j=1}^{n} \sum_{k=1}^{m} \left[-(B_k y, e_j) \frac{\partial}{\partial x_j} + (x, B_k e_j) \frac{\partial}{\partial y_j} \right] \frac{\partial}{\partial t_k},
\end{aligned}$$

where

$$\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},\ \Delta_y = \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2},\ \Delta_t = \sum_{k=1}^{m} \frac{\partial^2}{\partial t_k^2}.$$

Let $\mathbb{R}^{m*} = \mathbb{R}^m \setminus \{0\}$. Then by taking the inverse Fourier transform of the sub-Laplacian with respect to t, we get parametrized twisted Laplacians L^λ, $\lambda \in \mathbb{R}^{m*}$, given by

$$L^\lambda = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2+|y|^2)|\lambda|^2 - i\sum_{j=1}^{n}\left\{-(B_\lambda y, e_j)\frac{\partial}{\partial x_j} + (x, B_\lambda e_j)\frac{\partial}{\partial y_j}\right\}, \quad (1.1.1)$$

where

$$B_\lambda = \sum_{j=1}^{m} \lambda_j B_j.$$

1.2 Spectral Analysis of λ-Twisted Laplacians

For $k = 0, 1, 2, \ldots$, the Hermite function e_k of order k on $\mathbb{R}$ is defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define the function e_α on $\mathbb{R}^n$ by

$$e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n}.$$

For all $\lambda \in \mathbb{R}^{m*}$ and all multi-indices α and β in $(\mathbb{N} \cup \{0\})^n$, we define the special Hermite function $e^\lambda_{\alpha,\beta}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$e^\lambda_{\alpha,\beta}(q,p) = |\lambda|^{n/2} V^\lambda(e_\alpha, e_\beta)\left(\frac{q}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}p\right), \quad q, p \in \mathbb{R}^n,$$

where

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(B^t_\lambda q)\cdot y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy, \quad q, p \in \mathbb{R}^n, \quad (1.2.1)$$

for all f and g in $\mathcal{S}$. The matrix B^t_λ is the transpose of B_λ. In fact, $e^\lambda_{\alpha,\beta}$ is given by

$$e^\lambda_{\alpha,\beta}(q,p) = V^\lambda(e^\lambda_\alpha, e^\lambda_\beta)(q, \sqrt{|\lambda|}p), \quad q, p \in \mathbb{R}^n,$$

where

$$e^\lambda_\alpha(x) = |\lambda|^{n/4} e_\alpha(\sqrt{|\lambda|}x), \quad x \in \mathbb{R}^n.$$

Theorem 1.2.1 *$\{e^\lambda_{\alpha,\beta} : \alpha, \beta \in (\mathbb{N} \cup \{0\})^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$. For $l = 1, 2, \ldots, n$, we define the linear partial differential operators Z^λ_l and $\bar{Z}^\lambda_l$ by*

$$Z^\lambda_l = \frac{1}{|\lambda|} \sum_{j=1}^n (B_\lambda)_{jl} \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_l} + \frac{1}{2}\left(B^t_\lambda x\right)_l - i\frac{|\lambda|}{2} y_l,$$

$$\bar{Z}^\lambda_l = \frac{1}{|\lambda|} \sum_{j=1}^n (B_\lambda)_{jl} \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_l} - \frac{1}{2}\left(B^t_\lambda x\right)_l - i\frac{|\lambda|}{2} y_l,$$

where $-\bar{Z}^\lambda_l$ is a formal adjoint of Z^λ_l, i.e., $(Z^\lambda_l)^* = -\bar{Z}^\lambda_l$.

Then,

$$L^\lambda = -\frac{1}{2} \sum_{l=1}^n \left(Z^\lambda_l \bar{Z}^\lambda_l + \bar{Z}^\lambda_l Z^\lambda_l\right).$$

In the following lemma, e_l is the standard unit vector in $\mathbb{R}^n$.

Lemma 1.2.2 *For $l = 1, 2, \ldots, n$, and for all multi-indices α and β, we have both creation and annihilation operators respectively,*

$$\bar{Z}_l^\lambda e_{\alpha,\beta}^\lambda = i|\lambda|^{\frac{n}{2}} (2\beta_l + 2)^{\frac{1}{2}} e_{\alpha,\beta+e_l}^\lambda,$$

and

$$Z_l^\lambda e_{\alpha,\beta}^\lambda = i|\lambda|^{\frac{n}{2}} (2\beta_l)^{\frac{1}{2}} e_{\alpha,\beta-e_l}^\lambda, \quad \beta_l \neq 0.$$

The following theorem gives the spectral analysis of the λ-twisted Laplacian for all $\lambda \in \mathbb{R}^{m*}$.

Theorem 1.2.3 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all multi-indices α and β in $(\mathbb{N} \cup \{0\})^n$,*

$$L^\lambda e_{\alpha,\beta}^\lambda = |\lambda|^n (2|\beta| + n) e_{\alpha,\beta}^\lambda.$$

All definitions and results in this section can be found in [9].

1.3 λ-Weyl Transforms

We have defined the λ-Fourier-Wigner transform $V^\lambda(f,g)$ of f and g in $\mathcal{S}$ by (1.2.1). In fact,

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(B_\lambda^t q)\cdot x} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)}\, dx, \quad q, p \in \mathbb{R}^n.$$

It is easy to see that the λ-Fourier-Wigner transform is related to the ordinary Fourier-Wigner transform by

$$V^\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q, p), \quad q,p \in \mathbb{R}^n.$$

Note that

$$V^\lambda(f,g)(-q,-p) = \overline{V^\lambda(g,f)}(q,p), \quad q,p \in \mathbb{R}^n.$$

Now, we define the λ-Wigner transform $W^\lambda(f,g)$ of f and g in $L^2(\mathbb{R}^n)$ to be the Fourier transform of $V^\lambda(f,g)$. In fact, the λ-Wigner transform has the form

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p} f\left(\frac{B_\lambda^t x}{|\lambda|^2} + \frac{p}{2}\right)\overline{g\left(\frac{B_\lambda^t x}{|\lambda|^2} - \frac{p}{2}\right)}\, dp \quad (1.3.1)$$

for all x and ξ in $\mathbb{R}^n$, and and it is related to the ordinary Wigner transform by

$$W^\lambda(f,g)(x,\xi) = |\lambda|^{-n} W(f,g)\left(\frac{B_\lambda^t x}{|\lambda|^2}, \xi\right)$$

for all x, ξ in $\mathbb{R}^n$. Moreover,

$$W^\lambda(f,g) = \overline{W^\lambda(g,f)}, \quad f,g \in L^2(\mathbb{R}^n).$$

Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then we define the λ-Weyl transform $W_\sigma^\lambda f$ of f corresponding to the symbol σ by

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi) W^\lambda(f,g)(x,\xi)\, dx\, d\xi, \quad (1.3.2)$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Therefore using Parseval's identity, we have

$$\left(W_\sigma^\lambda f, g\right)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) V^\lambda(f,g)(q,p)\, dq\, dp.$$

Hence, formally, we can write

$$\left(W_\sigma^\lambda f\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\sigma}(q,p) \left(\pi_\lambda(q,p) f\right)(x)\, dq\, dp, \quad x \in \mathbb{R}^n,$$

where $\pi_\lambda(q,p)$ is the Schrödinger representation of $\mathbb{G}$ on $L^2(\mathbb{R}^n)$, i.e.,

$$\left(\pi_\lambda(q,p) f\right)(x) = e^{iq \cdot B_\lambda(x+p/2)} f(x+p), \quad x \in \mathbb{R}^n.$$

Proposition 1.3.1 *Let $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then the λ-Weyl transform W_σ^λ is given by*

$$W_\sigma^\lambda = W_{\sigma_\lambda},$$

where W_{σ_λ} is the ordinary Weyl transform corresponding to the symbol σ_λ given by

$$\sigma_\lambda(x,\xi) = \sigma(B_\lambda x, \xi), \quad x, \xi \in \mathbb{R}^n.$$

Let F and G be functions in $L^2(\mathbb{R}^{2n})$. Then the λ-twisted convolution $F *_\lambda G$ of F and G is the function on $\mathbb{R}^{2n}$ defined by

$$(F *_\lambda G)(z) = \int_{\mathbb{R}^{2n}} F(z-w) G(w) e^{\frac{i}{2}\lambda \cdot [z,w]}\, dw, \quad z \in \mathbb{R}^{2n}, \tag{1.3.3}$$

provided that the integral exists.

Theorem 1.3.2 *Let σ and τ be in $L^2(\mathbb{R}^{2n})$. Then*

$$W_\sigma^\lambda W_\tau^\lambda = W_\omega^\lambda,$$

where $\omega \in L^2(\mathbb{R}^{2n})$ and $\hat{\omega} = (2\pi)^{-n}(\hat{\sigma} *_\lambda \hat{\tau})$.

We have the following Moyal identity for the λ-Wigner transform and the λ-Fourier-Wigner transform.

Proposition 1.3.3 *For all f_1, f_2, g_1, g_2 in $L^2(\mathbb{R}^n)$,*

$$\big(W^\lambda(f_1, g_1), W^\lambda(f_2, g_2)\big) = |\lambda|^{-n}\,(f_1, f_2)\,\overline{(g_1, g_2)}$$

and

$$\big(V^\lambda(f_1, g_1), V^\lambda(f_2, g_2)\big) = |\lambda|^{-n}\,(f_1, f_2)\,\overline{(g_1, g_2)}.$$

1.4 Heat Kernels and Green Functions of λ-Twisted Laplacians

We recall in this section the heat kernel and the Green function of the λ-twisted Laplacian L^λ, $\lambda \in \mathbb{R}^{m*}$, given in [9]. We first give the heat kernel of the λ-twisted Laplacian L^λ, which is the kernel of the integral operator $e^{-\tau L^\lambda}$ for $\tau > 0$. The twisted convolution defined in (1.3.3) is the key ingredient in the following theorem.

Theorem 1.4.1 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and all $\tau > 0$,*

$$e^{-\tau L^\lambda} f = k_\tau^\lambda *_{-\lambda} f,$$

where

$$k_\tau^\lambda(z) = (2\pi)^{-n} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n} e^{-\frac{1}{4}|\lambda|\,|z|^2 \coth(|\lambda|^n\tau)}$$

for all $z \in \mathbb{R}^{2n}$.

As a corollary, the heat kernel κ_τ^λ of the λ-twisted Laplacian L^λ for $\lambda \in \mathbb{R}^{m*}$ is given by

$$\begin{aligned} \kappa_\tau^\lambda(z,w) &= k_\tau^\lambda(z-w) e^{-\frac{i}{2}\lambda\cdot[z,w]} \\ &= (2\pi)^{-n} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n} e^{-\frac{1}{4}|\lambda|\,|z-w|^2 \coth(\tau|\lambda|^n)} e^{-\frac{i}{2}\lambda\cdot[z,w]} \qquad (1.4.1) \end{aligned}$$

for all z and w in $\mathbb{R}^{2n}$.

The Green function G^λ of L^λ is the kernel of the inverse $(L^\lambda)^{-1}$. The Green function G^λ is related to the heat kernel κ_τ^λ of L^λ by

$$G^\lambda(z,w) = \int_0^\infty \kappa_\tau^\lambda(z,w)\, d\tau, \quad z, w \in \mathbb{R}^{2n}.$$

Let

$$g^\lambda(z) = \int_0^\infty k_\tau^\lambda(z)\, d\tau, \quad z \in \mathbb{R}^{2n}.$$

Then the Green function G^λ of L^λ is given by

$$G^\lambda(z,w) = e^{-\frac{i}{2}\lambda\cdot[z,w]} g^\lambda(z-w)$$

for all z and w in $\mathbb{R}^{2n}$. An explicit formula for g^λ is given in the following theorem.

Theorem 1.4.2 *For all* $z \in \mathbb{R}^{2n}$,

$$g^\lambda(z) = \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \frac{\Gamma(n/2)}{(\sqrt{|\lambda|}|z|)^{n-1}} K_{(n-1)/2}\left(\frac{1}{4}|\lambda|\,|z|^2\right),$$

where $K_{(n-1)/2}$ *is the modified Bessel function of order* $(n-1)/2$ *given by*

$$K_{(n-1)/2}(x) = \int_0^\infty e^{-x\cosh\delta}\cosh((n-1)\delta/2)\,d\delta, \quad x > 0.$$

2 Global Hypoellipticity of λ-Twisted Laplacians

2.1 Global Hypoellipticity in the Schwartz Space

The Green function is now used to prove that the λ-twisted Laplacian L^λ with $\lambda \in \mathbb{R}^{m*}$ is globally hypoelliptic. We start with an estimate of the modified Bessel function K_ν of order ν, where $\nu > 0$.

Lemma 2.1.1 *Let $\nu > 0$. Then for every positive number η with $\eta \geq \nu$, there exists a positive constant C_η such that*

$$|K_\nu(x)| \leq C_\eta x^{-\eta}, \quad x > 0.$$

Proof Let η be a positive number such that $\eta \geq \nu$. Using the asymptotic behavior of $K_\nu(x)$ for large x in [10], we know that

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$$

as $x \to \infty$. So, there exists a positive constant C'_η such that for sufficiently large x, say, $x \geq R'$,

$$K_\nu(x) \leq \sqrt{\frac{\pi}{2x}} e^{-x} \leq C'_\eta x^{-((1/2)+\eta)} < C'_\eta x^{-\eta}.$$

Using the asymptotic behavior of $K_\nu(x)$ for small x in [10], we have

$$K_\nu(x) \sim 2^{\nu-1}\Gamma(\nu)x^{-\nu}$$

as $x \to 0+$. Then there exists a positive constant C''_η such that

$$K_\nu(x) \leq C''_\eta x^{-\eta}$$

for sufficiently small and positive values of x, say, $x \leq R''$. Since $\frac{K_\nu(x)}{x^{-\eta}}$ is continuous on $(0, \infty)$, it follows that there exists a positive constant C'''_η for which

$$K_\nu(x) \leq C'''_\eta x^{-\eta}, \quad x \in [R'', R'],$$

and the lemma is proved with $C_\eta = \max(C'_\eta, C''_\eta, C'''_\eta)$. $\square$

We also need the following estimate.

Lemma 2.1.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all multi-indices γ on $\mathbb{R}^{2n}$,*

$$\left|\partial_z^\gamma \left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| \leq \left(\frac{1}{2}m|\lambda|\right)^{|\gamma|} |w|^{|\gamma|}, \quad z, w \in \mathbb{R}^{2n}.$$

Proof Writing $z = x + iy$ and $w = u + iv$, where x, y, u and v are in $\mathbb{R}^n$, we have

$$[z, w]_j = u \cdot B_j y - x \cdot B_j v, \quad j = 1, 2, \ldots, m.$$

Then for $j = 1, 2, \ldots, m$,

$$[z, w]_j = \sum_{l=1}^{n} (B_j^t u)_l y_l - \sum_{l=1}^{n} (B_j v)_l x_l$$

and hence

$$e^{-\frac{i}{2}\lambda\cdot[z,w]} = e^{-\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j v)_l x_l}.$$

We also write

$$\partial_z^\gamma = \partial_x^\theta \partial_y^\phi,$$

where θ and ϕ are multi-indices on $\mathbb{R}^n$ with $|\theta + \phi| = |\gamma|$. For $k = 1, 2, \ldots, n$,

$$\begin{aligned} & \partial_{y_k}\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j v)_l x_l} \left(-\frac{i}{2}\sum_{j=1}^{m}\lambda_j (B_j^t u)_k\right) \end{aligned}$$

and hence

$$\begin{aligned} & \partial_{y_k}^{\phi_k}\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j v)_l x_l} \left(-\frac{i}{2}\sum_{j=1}^{m}\lambda_j (B_j^t u)_k\right)^{\phi_k}. \end{aligned}$$

So,

$$\begin{aligned} & \partial_{y}^{\phi}\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\ = \quad & e^{-\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{l}(B_j^t u)_l y_j} e^{\frac{i}{2}\sum_{j=1}^{m}\lambda_j \sum_{l=1}^{n}(B_j v)_l x_l} \prod_{k=1}^{n}\left(-\frac{i}{2}\sum_{j=1}^{m}\lambda_j (B_j^t u)_k\right)^{\phi_k}. \end{aligned}$$

Differentiating the preceding equation with respect to x to the order θ, we obtain

$$\begin{aligned}
& \partial_z^\gamma \left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\
= \;& \partial_x^\theta \partial_y^\phi \left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right) \\
= \;& e^{-\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j^t u)_l y_l} e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \sum_{l=1}^n (B_j v)_l x_l} \\
& \left[\prod_{k=1}^n \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right)^{\phi_k}\right] \left[\prod_{k=1}^n \left(\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j v)_k\right)^{\theta_k}\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& \left|\partial_z^\gamma \left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| \\
= \;& \left|\prod_{k=1}^n \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right)^{\phi_k}\right| \left|\prod_{k=1}^n \left(\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j v)_k\right)^{\theta_k}\right|.
\end{aligned}$$

If we let $\|B_j\|$ denote the operator norm of B_j for $j = 1, 2, \ldots, m$, then

$$\begin{aligned}
\left|\prod_{k=1}^n \left(-\frac{i}{2}\sum_{j=1}^m \lambda_j (B_j^t u)_k\right)^{\phi_k}\right| &\le \prod_{k=1}^n \left(\frac{1}{2}\sum_{j=1}^m |\lambda_j||B_j^t u|\right)^{\phi_k} \\
&\le \prod_{k=1}^n \left(\frac{1}{2}|\lambda| \left(\sum_{j=1}^m \|B_j\|\right) |u|\right)^{\phi_k} \\
&= \left(\frac{1}{2}|\lambda|\right)^{|\phi|} \left(\sum_{j=1}^m \|B_j\|\right)^{|\phi|} |u|^{|\phi|} \\
&\le \left(\frac{1}{2}|\lambda|\right)^{|\phi|} \left(\sum_{j=1}^m \|B_j\|\right)^{|\phi|} |w|^{|\phi|}.
\end{aligned}$$

Since B_j is an orthogonal matrix for $j = 1, 2, \ldots, m$, it follows that

$$\|B_j\| = 1, \quad j = 1, 2, \ldots, m,$$

and hence

$$\left|\prod_{k=1}^{n}\left(-\frac{i}{2}\sum_{j=1}^{m}\lambda_j(B_j^t u)_k\right)^{\phi_k}\right| \leq \left(\frac{1}{2}m|\lambda|\right)^{|\phi|}|w|^{|\phi|}.$$

Similarly,

$$\left|\prod_{k=1}^{n}\left(\frac{i}{2}\sum_{j=1}^{m}\lambda_j(B_j v)_k\right)^{\theta_k}\right| \leq \left(\frac{1}{2}m|\lambda|\right)^{|\theta|}|w|^{|\theta|}.$$

Thus,

$$\left|\partial_z^\gamma\left(e^{-\frac{i}{2}\lambda\cdot[z,w]}\right)\right| \leq \left(\frac{1}{2}m|\lambda|\right)^{|\gamma|}|w|^{|\theta|}$$

and the proof is complete. □

We can now give the global hypoellipticity of the λ-twisted Laplacian L^λ with $\lambda \in \mathbb{R}^{m*}$ in the Schwartz space.

Theorem 2.1.3 *Let $\lambda \in \mathbb{R}^{m*}$. Then the λ-twisted Laplacian L^λ is globally hypoelliptic in the sense that*

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), L^\lambda u \in \mathcal{S}(\mathbb{R}^{2n}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{2n}),$$

where $\mathcal{S}'(\mathbb{R}^{2n})$ is the space of all tempered distributions on $\mathbb{R}^{2n}$.

Proof Let $f = L^\lambda u$. Then for all $z \in \mathbb{R}^{2n}$,

$$\begin{aligned} u(z) &= ((L^\lambda)^{-1}f)(z) \\ &= \int_{\mathbb{R}^{2n}} g^\lambda(w)f(z-w)e^{-\frac{i}{2}\lambda\cdot[z,w]}dw, \end{aligned}$$

where

$$g^\lambda(z) = \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \frac{\Gamma(n/2)}{(\sqrt{|\lambda|}|z|)^{n-1}} K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right).$$

Let β be any multi-index. Then for all $z \in \mathbb{R}^{2n}$,

$$(\partial^\beta u)(z) = \int_{\mathbb{R}^{2n}} g^\lambda(w) \partial_z^\beta \left(f(z-w) e^{-\frac{i}{2}\lambda\cdot[z,w]} \right) dw.$$

To justify the interchange of differentiation and integration, we write for all $z \in \mathbb{R}^{2n}$,

$$\int_{\mathbb{R}^{2n}} |g^\lambda(w)| \left| \partial_z^\beta \left(f(z-w) e^{-\frac{i}{2}\lambda\cdot[z,w]} \right) \right| dw = I_1(z) + I_2(z),$$

where

$$I_1(z) = \int_{|w|\le 1} |g^\lambda(w)| \left| \partial_z^\beta \left(f(z-w) e^{-\frac{i}{2}\lambda\cdot[z,w]} \right) \right| dw$$

and

$$I_2(z) = \int_{|w|\ge 1} |g^\lambda(w)| \left| \partial_z^\beta \left(f(z-w) e^{-\frac{1}{2}i\lambda\cdot[z,w]} \right) \right| dw.$$

Using the hypothesis that $f \in \mathcal{S}(\mathbb{R}^n)$, the definition of $[z,w]$, the formula of Leibniz to the effect that

$$\partial_z^\beta \left(f(z-w) e^{-\frac{i}{2}\lambda\cdot[z,w]} \right) = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} (\partial^{\beta-\gamma} f)(z-w) \partial^\gamma \left(e^{-\frac{i}{2}\lambda\cdot[z,w]} \right)$$

and Lemma 2.1.2, we get

$$\sup_{z\in\mathbb{R}^{2n}} |I_1(z)| < \infty$$

and

$$\sup_{z\in\mathbb{R}^{2n}} |I_2(z)| < \infty.$$

Now, let α and β be arbitrary multi-indices with $\alpha \neq 0$. Then for all $z \in \mathbb{R}^{2n}$,

$$|z^\alpha (\partial^\beta u)(z)| \leq 2^{|\alpha|}(J_1(z) + J_2(z)),$$

where

$$J_1(z) = \int_{\mathbb{R}^{2n}} |w|^{|\alpha|} |g^\lambda(w)| \left| \partial_z^\beta \left(f(z-w) e^{-\frac{i}{2}\lambda \cdot [z,w]} \right) \right| dw$$

and

$$J_2(z) = \int_{\mathbb{R}^{2n}} |z-w|^{|\alpha|} |g^\lambda(w)| \left| \partial_z^\beta \left(f(z-w) e^{-\frac{i}{2}\lambda \cdot [z,w]} \right) \right| dw.$$

As in the case when $\alpha = 0$,

$$\sup_{z \in \mathbb{R}^{2n}} |J_1(z)| < \infty.$$

By breaking $\mathbb{R}^{2n}$ into $|w| \leq 1$ and $|w| \geq 1$, and using the same argument as in the case when $\alpha = 0$, we see that

$$\sup_{z \in \mathbb{R}^{2n}} |J_2(z)| < \infty,$$

and the proof is complete. $\square$

2.2 Global Hypoellipticity in Gelfand–Shilov Spaces

Let μ and ν be positive real numbers such that $\mu + \nu \geq 1$. Then the Gelfand–Shilov space $S_\nu^\mu(\mathbb{R}^n)$ is defined to be the set of all functions φ in $C^\infty(\mathbb{R}^n)$ for which there exists a positive constant C such that for all multi-indices α and β,

$$|x^\alpha (\partial^\beta \varphi)(x)| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^\nu (\beta!)^\mu, \quad x \in \mathbb{R}^n.$$

It can be shown that a function φ is in $S_\nu^\mu(\mathbb{R}^n)$ if and only if there exist positive constants C and ε such that for all muti-indices α,

$$|(\partial^\alpha\varphi)(x)| \leq C^{|\alpha|+1}(\alpha!)^\mu e^{-\varepsilon|x|^{1/\nu}}, \quad x \in \mathbb{R}^n.$$

This characterization tells us that a function in a Gelfand–Shilov space has exponential decay at infinity. Moreover, a function φ is in the Gelfand–Shilov space $S_\nu^\mu(\mathbb{R}^n)$ if and only if there exist positive constants C and ε such that

$$|\varphi(x)| \leq Ce^{-\varepsilon|x|^{1/\nu}}, \quad x \in \mathbb{R}^n,$$

and

$$|\hat{\varphi}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/\mu}}, \quad \xi \in \mathbb{R}^n.$$

It is worth pointing out that the Gelfand–Shilov space $S_1^1(\mathbb{R}^n)$ is the same as the test space F for Fourier hyperfunctions. In fact, F is the set of all functions φ in $C^\infty(\mathbb{R}^n)$ for which there exist positive constants C, ε and δ such that for all multi-indices α,

$$|(\partial^\alpha)(x)| \leq C\delta^{|\alpha|}\alpha! e^{-\varepsilon|x|}, \quad x \in \mathbb{R}^n.$$

We have the following theorem on the global hypoellipticity of the λ-twisted Laplacian L^λ in Gelfand–Shilov spaces.

Theorem 2.2.1 *Let μ and ν be positive real numbers with $\mu + \nu \geq 1$. Then*

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), L^\lambda u \in \mathcal{S}_\nu^\mu(\mathbb{R}^{2n}) \Rightarrow u \in S_\nu^\mu(\mathbb{R}^{2n}).$$

Proof Let $f \in S_\nu^\mu(\mathbb{R}^{2n})$. Then there exists a positive constant C such that for all multi-indices α and β,

$$|z^\alpha(\partial^\beta f)(z)| \le C^{|\alpha|+|\beta|+1}(\alpha!)^\nu(\beta!)^\mu, \quad z \in \mathbb{C}^n. \tag{2.2.1}$$

As in the proof of Theorem 2.1.3, we need to estimate $I_1(z)$. To do this, we use the inequality (2.2.1), the definition of $[z, w]$ and the Leibniz formula to obtain a positive constant C_1 such that

$$I_1(z) \le C_1^{|\beta|+1}(\beta!)^\mu \int_{|w|\le 1} |g^\lambda(w)|\, dw.$$

By Lemma 2.1.2, we see that there exists a positive constant C_2 such that

$$I_1(z) \le C_2^{|\beta|+1}(\beta!)^\mu, \quad z \in \mathbb{C}^n.$$

Similarly, there exists positive constants C_3 and C_4 such that

$$I_2(z) \le C_3^{|\beta|+1}(\beta!)^\mu \int_{|w|\ge 1} |g^\lambda(w)|\, dw$$

and

$$I_2(z) \le C_4^{|\beta|+1}(\beta!)^\mu, \quad z \in \mathbb{C}^n.$$

Then as in the proof of Theorem 2.1.3 again, we need to estimate $J_1(z)$ and $J_2(z)$. Using the same argument as in the case when $\alpha = 0$, we obtain a positive constants C_5 for which

$$J_1(z) \le C_5^{|\alpha|+|\beta|+1}(\alpha!)^\mu(\beta!)^\mu, \quad z \in \mathbb{C}^n.$$

Using the Leibniz formula and Lemma 2.1.2, we get a positive constant C_6 such that

$$J_2(z) \le C_6^{|\alpha|+|\beta|+1}(\alpha!)^{\nu}(\beta!)^{\mu}\int_{\mathbb{C}^n}|w|^{|\beta|}|g^{\lambda}(w)|\,dw, \quad z \in \mathbb{C}.$$

By breaking $\mathbb{C}^n$ into $|w| \le 1$ and $|w| \ge 1$ and using Lemma 2.1.2, the proof is complete. $\square$

2.3 Essential Self-Adjointness

Let $\lambda \in \mathbb{R}^{m*}$. Then using the explicit formula for the λ-twisted Laplacian L^{λ} given in (1.1.1), it can be checked easily that L^{λ} is a symmetric operator from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with dense domain $\mathcal{S}$. So, L^{λ} is closable and we denote the closure by L_0^{λ}.

Proposition 2.3.1 *Let $\lambda \in \mathbb{R}^{m*}$. Then L_0^{λ} is closed and symmetric.*

Proof We only need to prove that L_0^{λ} is symmetric. Let u and v be functions in the domain $\mathcal{D}(L_0^{\lambda})$ of L_0^{λ}. Then we can find sequences $\{\varphi_l\}_{l=1}^{\infty}$ and $\{\psi_l\}_{l=1}^{\infty}$ in $\mathcal{S}$ such that

$$\varphi_l \to u,$$

$$L^{\lambda}\varphi_l \to L_0^{\lambda}u,$$

$$\psi_l \to v$$

and

$$L^{\lambda}\psi_l \to L_0^{\lambda}v$$

in $L^2(\mathbb{R}^{2n})$ as $l \to \infty$. So, using the symmetry of L^λ as a linear operator from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with domain $\mathcal{S}$,

$$(L_0^\lambda u, v) = \lim_{l\to\infty}(L^\lambda \varphi_l, \psi_l) = \lim_{l\to\infty}(\varphi_l, L^\lambda \psi) = (u, L_0^\lambda v).$$

Therefore L_0^λ is symmetric. $\square$

For all $\lambda \in \mathbb{R}^{m*}$, let $\Sigma(L_0^\lambda)$ be the spectrum of L_0^λ. Then we have the following theorem.

Theorem 2.3.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then*

$$\Sigma(L_0^\lambda) = \{|\lambda|^n(2|\beta| + n) : \beta \in (\mathbb{N} \cup \{0\})^n\}.$$

Moreover, for every $\beta \in (\mathbb{N} \cup \{0\})^n$, the number $|\lambda|^n(2|\beta| + n)$ is an eigenvalue of L_0^λ with infinite multiplicity.

Proof It follows from Theorem 1.2.3 that every number $|\lambda|^n(2|\beta| + n)$ with $\beta \in (\mathbb{N} \cup \{0\})^n$ is an eigenvalue of L_0^λ with infinite multiplicity and hence is an element of $\Sigma(L_0^\lambda)$. Now, let $\mu \in \mathbb{C}$. Suppose that

$$\mu \neq |\lambda|^n(2|\beta| + n)$$

for all $\beta \in (\mathbb{N} \cup \{0\})^n$. If we can prove that the range $R(L_0^\lambda - \mu I)$ of $L_0^\lambda - \mu I$ is dense in $L^2(\mathbb{R}^{2n})$, where I is the identity operator on $L^2(\mathbb{R}^{2n})$, and there exists a positive constant C such that

$$\|(L_0^\lambda - \mu I)u\|_{L^2(\mathbb{R}^{2n})} \geq C\|u\|_{L^2(\mathbb{R}^{2n})}, \quad u \in \mathcal{S}(L_0^\lambda),$$

then μ lies in the resolvent set $\rho(L_0^\lambda)$ and the proof is then complete. Let M be the subspace of $L^2(\mathbb{R}^{2n})$ consisting of all finite linear combinations of elements in $\{e_{\alpha,\beta}^\lambda : \alpha, \beta \in (\mathbb{N} \cup \{0\})^n\}$. Then by Theorem 1.2.1, M is dense in $L^2(\mathbb{R}^{2n})$. Let $f \in M$. Then we can write

$$f = \sum_{|\alpha| \leq N_1} \sum_{|\beta| \leq N_2} a_{\alpha,\beta} e_{\alpha,\beta}^\lambda,$$

where N_1 and N_2 are positive integers and

$$a_{\alpha,\beta} \in \mathbb{C}, \quad |\alpha| \leq N_1, |\beta| \leq N_2.$$

Let

$$u = \sum_{|\alpha| \leq N_1, |\beta| \leq N_2} \frac{a_{\alpha,\beta}}{|\lambda|^n(2|\beta| + n) - \mu} e_{\alpha,\beta}^\lambda.$$

Let $C_\mu = \inf_{\beta \in (\mathbb{N} \cup \{0\})^n} ||\lambda|^n(2|\beta| + n) - \mu|$. Since

$$\mu \neq |\lambda|^n(2|\beta| + n)$$

for all $\beta \in (\mathbb{N} \cup \{0\})^n$, it follows that $C_\mu > 0$. Therefore $u \in \mathcal{S}$. Furthermore,

$$\begin{aligned}
& (L_0^\lambda - \mu I)u = (L^\lambda - \mu I)u \\
= & \sum_{|\alpha| \le N_1} \sum_{|\beta| \le N_2} \frac{a_{\alpha,\beta}}{|\lambda|^n(2|\beta| + n) - \mu} L^\lambda e_{\alpha,\beta}^\lambda \\
= & \sum_{|\alpha| \le N_1} \sum_{|\beta| \le N_2} a_{\alpha,\beta} e_{\alpha,\beta}^\lambda \\
= & f.
\end{aligned} \tag{2.3.1}$$

Therefore $f \in R(L_0^\lambda - \mu I)$. So, $M \subseteq R(L_0^\lambda - \mu I)$. This proves that $R(L_0^\lambda - \mu I)$ is dense in $L^2(\mathbb{R}^{2n})$. Let $u \in \mathcal{D}(L_0^\lambda)$. Then using the symmetry of L^λ, Theorem 1.2.1 and Parseval's identity,

$$\begin{aligned}
\|(L_0^\lambda - \mu I)u)\|_{L^2(\mathbb{R}^{2n})}^2 &= \left\| \sum_\alpha \sum_\beta ((L_0^\lambda - \mu I)u, e_{\alpha,\beta}^\lambda) e_{\alpha,\beta}^\lambda \right\|_{L^2(\mathbb{R}^{2n})}^2 \\
&= \left\| \sum_\alpha \sum_\beta (u, ((L_0^\lambda)^* - \overline{\mu} I) e_{\alpha,\beta}^\lambda) e_{\alpha,\beta}^\lambda \right\|_{L^2(\mathbb{R}^{2n})}^2 \\
&= \left\| \sum_\alpha \sum_\beta (u, (L^\lambda - \overline{\mu} I) e_{\alpha,\beta}^\lambda) e_{\alpha,\beta}^\lambda \right\|_{L^2(\mathbb{R}^{2n})}^2 \\
&= \left\| \sum_\alpha \sum_\beta (u, (|\lambda|^n(2|\beta| + n) - \overline{\mu}) e_{\alpha,\beta}^\lambda) e_{\alpha,\beta}^\lambda \right\|_{L^2(\mathbb{R}^{2n})}^2 \\
&= \left\| \sum_\alpha \sum_\beta (|\lambda|^n(2|\beta| + n) - \mu)(u, e_{\alpha,\beta}^\lambda) e_{\alpha,\beta}^\lambda \right\|_{L^2(\mathbb{R}^{2n})}^2 \\
&= \sum_\alpha \sum_\beta \left| |\lambda|^n(2|\beta| + n) - \mu \right|^2 |(u, e_{\alpha,\beta}^\lambda)|^2.
\end{aligned}$$

Thus,

$$\|(L_0^\lambda - \mu I)u\|_{L^2(\mathbb{R}^{2n})} \geq C_\mu \|u\|_{L^2(\mathbb{R}^{2n})}, \quad u \in \mathcal{D}(L_0^\lambda).$$

□

By Theorem X.1 on page 136 of [11] and the preceding theorem, we see that for all $\lambda \in \mathbb{R}^{m*}$, L_0^λ is self-adjoint and hence the λ-twisted Laplacian L^λ given by (1.1.1) from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with dense domain $\mathcal{S}$ is essentially self-adjoint.

2.4 Sobolev Spaces

Let $s \in \mathbb{R}$. Then for all $\lambda \in \mathbb{R}^{m*}$, we define the L^2-Sobolev space $H^{s,2,\lambda}$ of order s by

$$H^{s,2,\lambda} = \left\{ u \in \mathcal{S}'(\mathbb{R}^{2n}) : \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s} |(u, e_{\alpha,\beta}^\lambda)|^2 < \infty \right\}.$$

It is easy to see that $H^{s,2,\lambda}$ is an inner product space with inner product $(\ ,\ )_{s,2,\lambda}$ and norm $\|\ \|_{s,2,\lambda}$ given by

$$(u, v)_{s,2,\lambda} = \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s}(u, e_{\alpha,\beta}^\lambda)(e_{\alpha,\beta}^\lambda, v)$$

and

$$\|u\|_{s,2,\lambda}^2 = \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s} |(u, e_{\alpha,\beta}^\lambda)|^2$$

for all u and v in $H^{s,2,\lambda}$.

Theorem 2.4.1 *$H^{s,2,\lambda}$ is a Hilbert space with respect to the inner product $(\,,\,)_{s,2,\lambda}$.*

Proof If $s \geq 0$, then the domain $\mathcal{S}((L_0^\lambda)^s)$ of the self-adjoint operator $(L_0^\lambda)^s$ from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ is a Banach space with respect to the norm $|\ |_s$ given by

$$|u|_s^2 = \|(L_0^\lambda)^s u\|_{L^2(\mathbb{R}^{2n})}^2 + \|u\|_{L^2(\mathbb{R}^{2n})}^2, \quad u \in \mathcal{D}((L_0^\lambda)^s).$$

Obviously,

$$\|(L_0^\lambda)^s u\|_{L^2(\mathbb{R}^{2n})}^2 = \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta| + n)^{2s} |(u, e_{\alpha,\beta}^\lambda)|^2 = \|u\|_{s,2,\lambda}^2.$$

So, $\|\ \|_{s,2,\lambda}$ is a norm in $H^{s,2,\lambda}$ and hence $H^{s,2,\lambda}$ is complete with respect to $\|\ \|_{s,2,\lambda}$. Let $s < 0$. Then $H^{s,2,\lambda}$ is the dual space of $H^{-s,2,\lambda}$ and is hence complete. $\square$

From the proof of the preceding theorem, we can have a characterization of the domain $\mathcal{D}(L_0^\lambda)$ of the closure of the λ-twisted Laplacian.

Theorem 2.4.2 *Let $\lambda \in \mathbb{R}^{m*}$. Then $\mathcal{D}(L_0^\lambda) = H^{1,2,\lambda}$.*

The following estimates can be considered to be the analogs for the λ-twisted Laplacian of the Agmon–Douglis–Nirenberg inequalities for elliptic boundary-value problems in [1] and globally elliptic pseudo-differential operators on $\mathbb{R}^n$ in [16].

Theorem 2.4.3 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all $s \in \mathbb{R}$,*

$$\|u\|_{s+1,2,\lambda} = \|L_0^\lambda u\|_{s,2,\lambda}, \quad u \in H^{s+1,2,\lambda}.$$

Proof Let $u \in H^{s+1,2,\lambda}$. Then

$$\begin{aligned}
\|L_0^\lambda u\|_{s,2,\lambda}^2 &= \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta|+n)^{2s}|(L_0^\lambda u, e_{\alpha,\beta}^\lambda)|^2 \\
&= \sum_\alpha \sum_\beta |\lambda|^{2ns}(2|\beta|+n)^{2s}|\lambda|^{2n}(2|\beta|+n)^2|(u, e_{\alpha,\beta}^\lambda)|^2 \\
&= \sum_\alpha \sum_\beta |\lambda|^{2n(s+1)}(2|\beta|+n)^{2(s+1)}|(u, e_{\alpha,\beta}^\lambda)|^2 \\
&= \|u\|_{s+1,2,\lambda}^2.
\end{aligned}$$

□

We give a corollary as a result on the global regularity of the λ-twisted Laplacian on Sobolev spaces.

Theorem 2.4.4 *Let $\lambda \in \mathbb{R}^{m*}$. Then for all $s \in \mathbb{R}$,*

$$u \in \mathcal{S}', L^\lambda u \in H^{s,2,\lambda} \Rightarrow u \in H^{s+1,2,\lambda}.$$

Remark 2.4.5 There is a loss of one derivative globally on $\mathbb{R}^{2n}$ because the operator L_0^λ with $\lambda \in \mathbb{R}^{m*}$ is not globally elliptic on $\mathbb{R}^{2n}$ as defined in [16], notwithstanding its ellipticity at every point in $\mathbb{R}^{2n}$.

3 Heat Semigroups and Inverses of λ-Twisted Laplacians

The aim of this chapter is to look at some aspects of analysis and partial differential equations on a class of nonisotropic Heisenberg groups with multi-dimensional center. We begin with a formula for the heat semigroup $e^{-\tau L^\lambda}$, $\tau > 0$, on $\mathbb{G}$ is given in the following theorem. It is an analog of the formula for the heat semigroup generated by the twisted Laplacian in the Heisenberg group with one-dimensional center given in [14].

3.1 Heat Semigroups Generated by λ-Twisted Laplacians on Heisenberg Groups with Multi-Dimensional Center

Theorem 3.1.1 *Let $f \in L^2(\mathbb{R}^{2n})$. Then for all $\lambda \in \mathbb{R}^{m*}$ with $|\lambda| = 1$ and $\tau > 0$,*

$$e^{-\tau L^\lambda} f = (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)} V^\lambda(W_{\hat{f}}^\lambda e_\beta, e_\beta).$$

Proof Let $f \in \mathcal{S}$. Then for $\tau > 0$, we have

$$e^{-\tau L^\lambda} f = \sum_\beta \sum_\alpha e^{-\tau|\lambda|^n(2|\beta|+n)} (f, e^\lambda_{\alpha,\beta}) e^\lambda_{\alpha,\beta}, \tag{3.1.1}$$

where the series is convergent in $L^2(\mathbb{R}^{2n})$. Now, using the λ-Wigner transform and the Plancherel theorem,

$$\begin{aligned}
(f, e^\lambda_{\alpha,\beta}) &= \int_{\mathbb{R}^{2n}} f(z)\overline{V^\lambda(e_\alpha, e_\beta)(z)}\, dz \\
&= \int_{\mathbb{R}^{2n}} \hat{f}(\zeta)\overline{V^\lambda(e_\alpha, e_\beta)^\wedge(\zeta)}\, d\zeta \\
&= \int_{\mathbb{R}^{2n}} \hat{f}(\zeta)\overline{W^\lambda(e_\alpha, e_\beta)(\zeta)}\, d\zeta \\
&= (2\pi)^{n/2}(W^\lambda_{\hat{f}} e_\beta, e_\alpha).
\end{aligned} \tag{3.1.2}$$

Similarly, for all $g \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$(e^\lambda_{\alpha,\beta}, g) = \overline{(g, e^\lambda_{\alpha,\beta})} = (2\pi)^{n/2}\overline{(W^\lambda_{\hat{g}} e_\beta, e_\alpha)} = (2\pi)^{n/2}(e_\alpha, W^\lambda_{\hat{g}} e_\beta). \tag{3.1.3}$$

So, by (3.1.1), (3.1.2) and (3.1.3), together with Fubini's theorem and Parseval's identity, we get

$$\begin{aligned}
(e^{-\tau L^\lambda} f, g) &= (2\pi)^n \sum_\beta \sum_\alpha e^{-\tau(2|\beta|+n)} (W^\lambda_{\hat{f}} e_\beta, e_\alpha)(e_\alpha, W^\lambda_{\hat{g}} e_\beta) \\
&= (2\pi)^n \sum_\beta e^{-\tau(2|\beta|+n)} \sum_\alpha (W^\lambda_{\hat{f}} e_\beta, e_\alpha)(e_\alpha, W^\lambda_{\hat{g}} e_\beta) \\
&= (2\pi)^n \sum_\beta e^{-\tau(2|\beta|+n)} (W^\lambda_{\hat{f}} e_\beta, W^\lambda_{\hat{g}} e_\beta)
\end{aligned} \tag{3.1.4}$$

for all $\tau > 0$. Using the property of the λ-Weyl transform and Plancherel's theorem,

$$\begin{aligned}
(W^\lambda_{\hat{f}} e_\beta, W^\lambda_{\hat{g}} e_\beta) &= (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} W^\lambda_{\hat{f}} e_\beta(z) \overline{W^\lambda_{\hat{g}} e_\beta(z)} dz \\
&= (2\pi)^{-n/2} \overline{\int_{\mathbb{R}^{2n}} \hat{g}(z) W^\lambda(e_\beta, W^\lambda_{\hat{f}} e_\beta)(z)\, dz} \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} W^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)(z) \overline{\hat{g}(z)}\, dz \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} W^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)^\vee(z) \overline{g(z)} dz \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^{2n}} V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)(z) \overline{g(z)}\, dz \qquad (3.1.5)
\end{aligned}$$

for all multi-indices β. By (3.1.4) and (3.1.5),

$$\begin{aligned}
(e^{-\tau L^\lambda} f, g) &= (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)} (V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta), g) \\
&= (2\pi)^{n/2} \left(\sum_\beta e^{-\tau(2|\beta|+n)} V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta), g \right)
\end{aligned}$$

for all f and g in $\mathcal{S}(\mathbb{R}^n)$ and all $\tau > 0$. Thus,

$$e^{-\tau L^\lambda} f = (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)} V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta)$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and all $\tau > 0$. $\square$

3.2 An L^p–L^2 Estimate

We begin with the following improvement of Theorem 11.1 in [13].

Theorem 3.2.1 *Let $\sigma \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq 2$. Then for $\lambda \in \mathbb{R}^{m*}$, the λ-Weyl transform $W^{\lambda}_{\hat{\sigma}}$, is originally defined on $\mathcal{S}(\mathbb{R}^n)$. Moreover,*

$$\|W^{\lambda}_{\hat{\sigma}}\|_* \leq (2\pi)^{-n/2}(\frac{2}{|\lambda|})^{n(1-(2/p'))}\|\sigma\|_{L^p(\mathbb{R}^{2n})},$$

where $\| \; \|_*$ is the norm in the Banach space of all bounded linear operators from $L^p(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

Proof Let $\sigma \in L^2(\mathbb{R}^{2n})$. Then for all f and g in $L^2(\mathbb{R}^n)$, we get by (1.3.2), the Schwarz inequality and the Plancherel theorem,

$$\begin{aligned} |(W^{\lambda}_{\hat{\sigma}}f, g)_{L^2(\mathbb{R}^n)}| &\leq (2\pi)^{-n/2}\|\hat{\sigma}\|_{L^2(\mathbb{R}^{2n})}\|W^{\lambda}(f,g)\|_{L^2(\mathbb{R}^{2n})} \\ &= (2\pi)^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^{2n})}\|W^{\lambda}(f,g)\|_{L^2(\mathbb{R}^{2n})}. \end{aligned}$$

By Moyal's identity in Proposition 1.3.3,

$$\|W^{\lambda}(f,g)\|_{L^2(\mathbb{R}^{2n})} = \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$

Therefore

$$|(W^{\lambda}_{\hat{\sigma}}f, g)_{L^2(\mathbb{R}^n)}| \leq (2\pi)^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}, \quad f, g \in L^2(\mathbb{R}^n).$$

So,

$$\|W^{\lambda}_{\hat{\sigma}}f\|_{L^2(\mathbb{R}^n)} \leq (2\pi)^{-n/2}\|\sigma\|_{L^2(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n).$$

Now, let $\sigma \in L^1(\mathbb{R}^{2n})$. Then for all f and g in $L^2(\mathbb{R}^n)$, we get by (1.3.2) and Hölder's inequality,

$$|(W_{\hat{\sigma}}^\lambda f, g)_{L^2(\mathbb{R}^{2n})}| \leq (2\pi)^{-n/2} \|\hat{\sigma}\|_{L^1(\mathbb{R}^{2n})} \|W^\lambda(f,g)\|_{L^\infty(\mathbb{R}^{2n})}.$$

By (1.3.1) and the Minkowski's inequality,

$$\begin{aligned} & \|W^\lambda(f,g)\|_{L^\infty(\mathbb{R}^{2n})} \\ \leq\ & (2\pi)^{-n/2}|\lambda|^{-n} \left[\int_{\mathbb{R}^n} \left| f\left(\frac{B_\lambda^t x}{|\lambda|^2} + \frac{p}{2}\right)\right|^2 dp\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} \left| g\left(\frac{B_\lambda^t x}{|\lambda|^2} - \frac{p}{2}\right)\right|^2 dp\right]^{\frac{1}{2}} \\ =\ & (2\pi)^{-n/2}|\lambda|^{-n} 2^n \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Let $\sigma \in L^p(\mathbb{R}^{2n})$. Then by the Riesz-Thorin theorem, we get for all f in $L^2(\mathbb{R}^n)$,

$$\begin{aligned} & \|W_{\hat{\sigma}}^\lambda f\|_{L^2(\mathbb{R}^n)} \\ \leq\ & [(2\pi)^{-n/2}]^{2/p'} [(2\pi)^{-n/2} 2^n |\lambda|^{-n}]^{1-(2/p')} \|\sigma\|_{L^p(\mathbb{R}^{2n})} \|f\|_{L^2(\mathbb{R}^n)} \\ =\ & (2\pi)^{-n/2} (\frac{2}{|\lambda|})^{n(1-(2/p'))} \|\sigma\|_{L^p(\mathbb{R}^{2n})} \|f\|_{L^2(\mathbb{R}^n)}. \end{aligned} \qquad (3.2.1)$$

By (3.2.1), the proof is complete. □

Theorem 3.2.2 *For $\tau > 0$, the heat semigroup $e^{-\tau L^\lambda}$ with $|\lambda| = 1$, initially defined on $\mathcal{S}(\mathbb{R}^{2n})$, can be extended to a unique bounded linear operator from $L^p(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$, which we again denote by $e^{-\tau L^\lambda}$, and*

$$\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^{2n})} \leq 2^{n(1-(2/p'))} \frac{1}{[2\sinh\,\tau]^n} \|f\|_{L^p(\mathbb{R}^{2n})}$$

for all $f \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq 2$.

Proof By Theorem 3.1.1, Minkowski's inequality and the Moyal identity for the λ-Fourier-Wigner transform, we get for all $f \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\begin{aligned}
\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^{2n})} &\leq (2\pi)^{n/2} \sum_\beta e^{-\tau(2|\beta|+n)} \|V^\lambda(W^\lambda_{\hat{f}} e_\beta, e_\beta))\|_{L^2(\mathbb{R}^{2n})} \\
&= (2\pi)^{n/2} |\lambda|^{-n} \sum_\beta e^{-\tau(2|\beta|+n)} \|W^\lambda_{\hat{f}} e_\beta\|_{L^2(\mathbb{R}^n)} \|e_\beta\|_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{n/2} |\lambda|^{-n} \sum_\beta e^{-\tau|\lambda|^n(2|\beta|+n)} \|W^\lambda_{\hat{f}} e_\beta\|_{L^2(\mathbb{R}^n)} \qquad (3.2.2)
\end{aligned}$$

for $\tau > 0$. So, by (3.2.2) and Theorem 3.2.1, we get for $\tau > 0$,

$$\begin{aligned}
\|e^{-\tau L^\lambda} f\|_{L^2(\mathbb{R}^{2n})} &\leq (\frac{2}{|\lambda|})^{n(1-(2/p'))} \sum_\beta e^{-\tau(2|\beta|+n)} \|f\|_{L^p(\mathbb{R}^{2n})} \\
&= (2)^{n(1-(2/p'))} \frac{1}{[2\sinh\,\tau]^n} \|f\|_{L^p(\mathbb{R}^{2n})}.
\end{aligned}$$

□

3.3 L^p–L^∞ Estimates, $1 \leq p \leq \infty$

We begin with the following theorem.

Theorem 3.3.1 *Let $\lambda \in \mathbb{R}^{m*}$, Then for all $\tau > 0$ and $1 \leq p \leq \infty$,with $|\lambda| = 1$, $e^{-\tau L^\lambda} : L^p(\mathbb{R}^{2n}) \to L^\infty(\mathbb{R}^{2n})$ is a bounded linear operator. More precisely,*

$$\|e^{-\tau L^\lambda} f\|_{L^\infty(\mathbb{R}^{2n})} \leq (2\pi)^{-n/p} \frac{1}{[\sinh\tau]^{n/p}} \frac{1}{[\cosh\tau]^{n/p'}} \|f\|_{L^p(\mathbb{R}^{2n})}$$

for all $f \in L^p(\mathbb{R}^{2n})$.

Proof Using the formula (1.4.1) for the heat kernel κ_τ^λ of L^λ,

$$|\kappa_\tau^\lambda(z,w)| \le a_\tau$$

for all z and w in $\mathbb{C}^n$ and $\tau > 0$, where

$$a_\tau = (2\pi)^{-n}\frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n}.$$

So, for all $f \in L^1(\mathbb{R}^{2n})$,

$$|(e^{-\tau L^\lambda}f)(z)| \le \int_{\mathbb{C}^n} |\kappa_\tau^\lambda(z,w)|\,|f(w)|\,dw = a_\tau \|f\|_{L^1(\mathbb{R}^{2n})}$$

for all $z \in \mathbb{C}^n$. Therefore

$$\|e^{-\tau L^\lambda}f\|_{L^\infty(\mathbb{R}^{2n})} \le a_\tau \|f\|_{L^1(\mathbb{R}^{2n})}.$$

Now, for all $f \in L^\infty(\mathbb{R}^{2n})$,

$$\begin{aligned} |(e^{-\tau L^\lambda}f)(z)| &\le \|f\|_{L^\infty(\mathbb{R}^{2n})}\int_{\mathbb{C}^n}|\kappa_\tau^\lambda(z,w)|\,dw \\ &\le a_\tau \int_{\mathbb{C}^n} e^{-\frac{1}{4}|\lambda|\,|w|^2\coth(\tau|\lambda|^n)}dw\,\|f\|_{L^\infty(\mathbb{R}^{2n})} \end{aligned}$$

for all $z \in \mathbb{C}^n$. Thus,

$$\begin{aligned} \|e^{-\tau L^\lambda}f\|_{L^\infty(\mathbb{R}^{2n})} &\le a_\tau \frac{(4\pi)^n}{|\lambda|^n[\coth(|\lambda|^n\tau)]^n}\|f\|_{L^\infty(\mathbb{R}^{2n})} \\ &= \frac{1}{[\cosh(|\lambda|^n\tau)]^n}\|f\|_{L^\infty(\mathbb{R}^{2n})}. \end{aligned}$$

Using the Riesz-Thorin Theorem, we have

$$\|e^{-\tau L^\lambda}f\|_{L^\infty(\mathbb{R}^{2n})} \le (2\pi)^{-n/p}\frac{1}{[\sinh\tau]^{n/p}}\frac{1}{[\cosh\tau]^{n/p'}}\|f\|_{L^p(\mathbb{R}^{2n})}.$$

$\square$

3.4 L^p–L^q Estimates, $1 \le p \le 2$, $2 \le q \le \infty$

Using Theorem 3.2.2 and Theorem 3.3.1, we have the following theorem.

Theorem 3.4.1 *For $\tau > 0$, the heat semigroup $e^{-\tau L^\lambda}$ with $|\lambda| = 1$, initially defined on $\mathcal{S}(\mathbb{R}^n)$, can be extended to a bounded linear operator from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$, which we again denote by $e^{-\tau L^\lambda}$ and*

$$\|e^{-\tau L^\lambda} f\|_{L^q(\mathbb{R}^{2n})} \le \frac{2^{n(1-2/p')(2/q)}}{[\sinh \tau]^{(2n/q)+(n/p)(1-(2/q))}[\cosh \tau]^{(n/p')(1-(2/q))}} \|f\|_{L^p(\mathbb{R}^{2n})}$$

for all $f \in L^p(\mathbb{R}^{2n})$.

Proof By Riesz-Thorin Theorem, let $\alpha_1, \alpha_2, \beta_1$, and β_2 be real numbers in $[0,1]$. To interpolate, let α be such that $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and

$$\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$$

$$\beta = \theta\beta_1 + (1-\theta)\beta_2,$$

where $\theta \in [0,1]$. Then, $A : L^{1/\alpha}(\mathbb{R}^{2n}) \to L^{1/\beta}(\mathbb{R}^{2n})$ is a bounded linear operator. Moreover, when $\alpha_1 = 1$, $\beta_1 = 0$, and $M_1 = a_\tau = (2\pi)^{-n}\frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n}$,

$$\|Af\|_{1/\beta_1} \le M_1 \|f\|_{1/\alpha_1},$$

i.e., $A : L^{1/\alpha_1}\left(\mathbb{R}^{2n}\right) \to L^{1/\beta_1}\left(\mathbb{R}^{2n}\right)$, is a bounded linear operator.

When $\alpha_2 = 0$, $\beta_2 = 0$, and $M_2 = a_\tau \frac{4\pi}{|\lambda|^n \coth(|\lambda|^n\tau)} = (2\pi)^{-n} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n} \frac{4\pi}{|\lambda|^n \coth(|\lambda|^n\tau)}$,

$$\|Af\|_{1/\beta_1} \leq M_1 \|f\|_{1/\alpha_1},$$

i.e., $A : L^{1/\alpha_1}\left(\mathbb{R}^{2n}\right) \to L^{1/\beta_1}\left(\mathbb{R}^{2n}\right)$, is a bounded linear operator.

When $\alpha = \theta \cdot 1 + (1-\theta) \cdot 0 = 1/2$, $\beta = \theta \cdot 0 + (1-\theta) \cdot 0 = 0$, then $\theta = 1/2$,

$$\|Af\|_{1/\beta} \leq M_1^\theta M_2^{1-\theta} \|f\|_{1/\alpha},$$

i.e., $A : L^{1/\alpha}\left(\mathbb{R}^{2n}\right) \to L^{1/\beta}\left(\mathbb{R}^{2n}\right)$, is a bounded linear operator.

Now, let $1/\beta = q$, $1/\alpha = p$, and $\theta = 1/p$, then the heat semigroup $e^{-\tau L^\lambda}$ with $|\lambda| = 1$, initially defined on $\mathcal{S}(\mathbb{R}^n)$, is proved to be a bounded linear operator from $L^p(\mathbb{R}^{2n})$ into $L^q(\mathbb{R}^{2n})$.

$\square$

4 Heat Kernels and Green Functions of Sub-Laplacians

In this chapter, we use the heat kernel κ_τ^λ of the λ-twisted Laplacian to find the heat kernel of the sub-Laplacian by taking the Fourier transform with respect to the parameter λ. Similarly, the same technique is also applied in obtaining the Green function $\mathcal{G}$ of the sub-Laplacian using the Green function G^λ of the λ-twisted Laplacian.

4.1 Heat Kernels of Sub-Laplacians

To obtain the heat kernel of the sub-Laplacian, we need some preparation first. The group convolution of two measurable functions f and g on $\mathbb{G}$ is defined by

$$(f *_{\mathbb{G}} g)(z,t) = \int_{\mathbb{G}} f\left((z,t) \cdot_{\mathbb{G}} (w,s)^{-1}\right) g(w,s)\mathrm{d}w\mathrm{d}s, \quad z \in \mathbb{R}^{2n}, t \in \mathbb{R}^m,$$

if the integral exists. Moreover, we denote by f_λ the ordinary Fourier transform of f with respect to the t variable evaluated at the point $\lambda \in \mathbb{R}^m$. More precisely,

$$f_\lambda(z) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} f(z,t)\mathrm{d}t, \quad z \in \mathbb{R}^{2n}.$$

We need the following theorem.

Theorem 4.1.1 *Let f and g be functions in $L^1(\mathbb{G})$. Then for all nonzero $\lambda \in \mathbb{R}^m$,*

$$(f *_{\mathbb{G}} g)_\lambda = (2\pi)^{m/2} f_\lambda *_{-\lambda} g_\lambda.$$

Proof For all $z \in \mathbb{R}^{2n}$,

$$\begin{aligned}(f *_{\mathbb{G}} g)_\lambda &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} \left(f *_\lambda g\right)(z,t)\mathrm{d}t \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} \left(\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} f\left(z-w, t-s-\frac{1}{2}[z,w]\right) g(w,s)\mathrm{d}w\mathrm{d}s \right) \mathrm{d}t.\end{aligned}$$

Let $t' = t - \frac{1}{2}[z,w]$. Then

$$(f *_{\mathbb{G}} g)_\lambda = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} e^{-it'\cdot\lambda} f\left(z-w, t'-s\right) g(w,s) e^{-\frac{i}{2}\lambda\cdot[z,w]}\mathrm{d}w\mathrm{d}s\mathrm{d}t'.$$

On the other hand, for all $z \in \mathbb{R}^{2n}$, we get

$$\begin{aligned}\left(f_\lambda *_{-\lambda} g_\lambda\right)(z) &= \int_{\mathbb{R}^{2n}} f_\lambda(z-w) g_\lambda(w) e^{-\frac{i}{2}\lambda\cdot[z,w]}\mathrm{d}w \\ &= (2\pi)^{-m/2} \int_{\mathbb{R}^{2n}} \left\{ \int_{\mathbb{R}^m} f(z-w, \cdot - s) g(w,s)\mathrm{d}s \right\}^\wedge (\lambda) e^{-\frac{i}{2}\lambda\cdot[z,w]}\mathrm{d}w \\ &= (2\pi)^{-m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} e^{-it\cdot\lambda} f(z-w, t-s) g(w,s) e^{-\frac{i}{2}\lambda\cdot[z,w]}\mathrm{d}w\mathrm{d}s\mathrm{d}t.\end{aligned}$$

and the proof is complete. $\square$

Now, we consider the initial-value problem given by

$$\begin{cases} \frac{\partial u}{\partial \tau}(z,t,\tau) = -(\mathcal{L}u)(z,t,\tau), & z \in \mathbb{R}^{2n}, t \in \mathbb{R}^m, \tau > 0, \\ u(z,t,0) = f(z,t), & z \in \mathbb{R}^{2n}, t \in \mathbb{R}^m. \end{cases}$$

By taking the inverse Fourier transform with respect to t and evaluated at λ, we get an initial-value problem for the heat equation governed by the λ-twisted Laplacian L^λ, i.e.,

$$\begin{cases} \frac{\partial u_\lambda}{\partial \tau}(z,\tau) = -\left(L^\lambda u_\lambda\right)(z,\tau), \\ u_\lambda(z,0) = f_\lambda(z), \end{cases}$$

for all $z \in \mathbb{R}^{2n}, \tau > 0$ and $\lambda \in \mathbb{R}^{m*}$. By Theorem 4.1.1,

$$u_\lambda(z,\tau) = \left(k_\tau^\lambda *_{-\lambda} f_\lambda\right)(z), \quad z \in \mathbb{R}^{2n}, \tau > 0,$$

for all $\lambda \in \mathbb{R}^{m*}$. Therefore by taking the Fourier transform with respect to λ and evaluated at t, and using Theorem 4.1.1, we get the solution of the initial-value problem governed by the sub-Laplacian given by

$$u(z,t,\tau) = (2\pi)^{-m/2} \left(f *_{\mathbb{G}} K_\tau\right)(z,t), \quad z \in \mathbb{R}^{2n}, t \in \mathbb{R}^m, \tau > 0,$$

where K_τ is the Fourier transform of the heat kernel of k_τ^λ with respect to λ and evaluated at t. So, the heat kernel of $\mathcal{L}$ is given in the following theorem.

Theorem 4.1.2 *For all* f *in* $L^2(\mathbb{G}), e^{-\tau\mathcal{L}}f = f *_{\mathbb{G}} K_\tau$, *where*

$$K_\tau(z,t) = (2\pi)^{-(n+m)} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} \frac{|\lambda|^n}{[2\sinh(|\lambda|^n\tau)]^n} e^{-\frac{1}{4}|\lambda||z|^2\coth(|\lambda|^n\tau)} \mathrm{d}\lambda$$

for all $(z,t) \in \mathbb{G}$.

Hence the heat kernel κ_τ of $\mathcal{L}$ is given by

$$\kappa_\tau((z,t),(w,s)) = K_\tau\left(z-w, t-s+\frac{1}{2}[z,w]\right)$$

for all (z,t) and (w,s) in $\mathbb{G}$.

4.2 Green Functions of Sub-Laplacians

The Green function $\mathcal{G}$ of the sub-Laplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{G}$ with multi-dimensional center is the kernel of $\mathcal{L}^{-1}$. More precisely, the Green function $\mathcal{G}$ is given by

$$\mathcal{L}^{-1}f = f *_{\mathbb{G}} \mathcal{G}$$

for all suitable functions f on $\mathbb{G}$.

As in the case of the heat kernel of the sub-Laplacian $\mathcal{L}$, the Green function $\mathcal{G}$ is obtained by taking the Fourier transform of G^λ with respect to λ and evaluated at t. Therefore by Theorem 1.4.2, we have the following theorem.

Theorem 4.2.1 *The Green function $\mathcal{G}$ of $\mathcal{L}$ is given by*

$$\mathcal{G}(z,t) = \frac{c_n}{|z|^{n-1}} \int_{\mathbb{R}^m} e^{-i\lambda\cdot t} \frac{1}{|\lambda|^{(n-1)/2}} K_{(n-1)/2}\left(\frac{1}{4}|\lambda||z|^2\right) \mathrm{d}\lambda,$$

where

$$c_n = (2\pi)^{-m} \frac{(\sqrt{2}\pi)^{-n}}{2\sqrt{2\pi}} \Gamma(n/2).$$

5 Spectral Properties of the Twisted Bi-Laplacian

The eigenvalues and eigenfunctions of the twisted bi-Laplacian on the Heisenberg group with multi-dimensional center are studied in the context of analytic number theory. Properties such as global hypoellipticity, essential self-adjointness, Sobolev spaces, as well as Sobelov estimates of the twisted bi-Laplacian are also introduced in this chapter.

5.1 Twisted Bi-Laplacian

The λ-twisted Laplacian L^λ on the Heisenberg group with multi-dimensional center on $\mathbb{R}^{2n}$ is the second-order partial differential operator given by

$$L^\lambda = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2+|y|^2)|\lambda|^2 - i\sum_{j=1}^{n}\left\{-(B_\lambda y, e_j)\frac{\partial}{\partial x_j} + (x, B_\lambda e_j)\frac{\partial}{\partial y_j}\right\}, \quad (5.1.1)$$

where

$$B_\lambda = \sum_{j=1}^{m} \lambda_j B_j$$

and

$$\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},\ \Delta_y = \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}.$$

Thus, the λ-twisted Laplacian L^λ on the Heisenberg group with multi-dimensional center is the Hermite operator

$$H = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2)|\lambda|^2$$

perturbed by the partial differential operator $-iN$, where

$$N = \sum_{j=1}^{n} \left\{ -(B_\lambda y, e_j)\frac{\partial}{\partial x_j} + (x, B_\lambda e_j)\frac{\partial}{\partial y_j} \right\}.$$

The twisted Laplacian on the Heisenberg group with multi-dimensional center appears in harmonic analysis naturally in the context of heat kernels and green functions of sub-Laplacians on Heisenberg groups with multi-dimensional center [18, 24, 12], and the essential self-adjointness and global hypoellipticity of the twisted Laplacian on Heisenberg groups with multi-dimensional center can be found in [17, 25, 26]. In the paper [4], it is shown that L^λ is globally hypoelliptic in the Schwartz space $S(\mathbb{R}^{2n})$ in the sense that

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), L^\lambda u \in \mathcal{S}(\mathbb{R}^{2n}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{2n}),$$

where $\mathcal{S}'(\mathbb{R}^{2n})$ is the space of all tempered distributions on $\mathbb{R}^{2n}$. Moreover, L^λ is essentially self-adjoint, and the spectrum $\Sigma(L_0^\lambda)$ of the closure L_0^λ is given by a sequence of eigenvalues, where $|\lambda| = 1$, i.e. the unit vector on the surface of a unit ball. In fact,

$$\Sigma(L_0^\lambda) = \{(2|\beta| + n) : \beta \in (\mathbb{N} \cup \{0\})^n\}.$$

By Theorem 2.2.1, each eigenvalue has infinite multiplicity.

Renormalizing the λ-twisted Laplacian L^λ to the partial differential operator P given by

$$P = \frac{1}{2}(L^\lambda - n + 2). \tag{5.1.2}$$

Now that λ is a unit vector, we see that the eigenvalues of L^λ are the natural numbers $1, 2, \ldots,$ and each eigenvalue, as in the case of L^λ, has infinite multiplicity. Now, the transpose $(L^\lambda)^t$ of the twisted Laplacian L^λ is given by

$$(L^\lambda)^t = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2)|\lambda|^2 + i \sum_{j=1}^{n} \left\{ -(B_\lambda y, e_j)\frac{\partial}{\partial x_j} + (x, B_\lambda e_j)\frac{\partial}{\partial y_j} \right\} \tag{5.1.3}$$

and after renormalization, we get the transpose Q of P given by

$$Q = \frac{1}{2}((L^\lambda)^t - n + 2). \tag{5.1.4}$$

The aim of this chapter is to analyze the twisted bi-Laplacian M on the Heisenberg group with multi-dimensional center defined by

$$M = QP = PQ = \frac{1}{4}(L^\lambda - n + 2)((L^\lambda)^t - n + 2), \tag{5.1.5}$$

where P and Q commute because it can be shown by easy computations that H and N commute.

The impetus for our study of the twisted bi-Laplacian M on the Heisenberg group with multi-dimensional center comes from its non-standard spectral properties connected with analytic number theory. We observe that the twisted bi-Laplacian M is globally hypoelliptic in the Schwartz space $S(\mathbb{R}^{2n})$, i.e.,

$$u \in \mathcal{S}'(\mathbb{R}^{2n}), Mu \in \mathcal{S}(\mathbb{R}^{2n}) \Rightarrow u \in \mathcal{S}(\mathbb{R}^{2n}).$$

Moreover, we prove that M is essentially self-adjoint on $L^2(\mathbb{R}^{2n})$. If we denote the unique closed extension of M on $L^2(\mathbb{R}^{2n})$ again by M, then the resolvent $R(\lambda)$ of M given by

$$R(\lambda) = (M - \lambda I)^{-1},$$

where I is the identity operator on $L^2(\mathbb{R}^{2n})$, is a compact operator for all λ in the resolvent set of M. The novelty of the twisted bi-Laplacian M with respect to P and Q considered separately is the compactness of the resolvent. This is due to the favorable combination of the properties of the two factors. The spectrum of M is then given by a sequence of isolated eigenvalues and each eigenvalue has finite multiplicity. In order to describe the spectral properties of M precisely, let us first recall that the λ-Fourier-Wigner transform $V^\lambda(f, g)$ of two functions f and g in $S(\mathbb{R}^n)$

is the function in $S(\mathbb{R}^{2n})$ given by

$$V^\lambda(f,g)(q,p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(B_\lambda^t q)\cdot y} f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)} dy, \quad q,p \in \mathbb{R}^n. \quad (5.1.6)$$

It is easy to see that the λ-Fourier-Wigner transform is related to the ordinary Fourier-Wigner transform by

$$V^\lambda(f,g)(q,p) = V(f,g)(B_\lambda^t q, p), \quad q,p \in \mathbb{R}^n,$$

$$V^\lambda(f,g)(-q,-p) = \overline{V^\lambda(g,f)}(q,p), \quad q,p \in \mathbb{R}^n.$$

For $k = 1, 2, \ldots$, the Hermite function e_k of order k on $\mathbb{R}$ is defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R}, \quad (5.1.7)$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}. \quad (5.1.8)$$

For any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define the function e_α on $\mathbb{R}^n$ by

$$e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_n}.$$

We fix a nonzero vector $\lambda \in \mathbb{R}^{m*}$. Let α and β be multi-indices in $(\mathbb{N} \cup \{0\})^n$. Then we define the special Hermite function $e_{\alpha,\beta}^\lambda$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$e_{\alpha,\beta}^\lambda(q,p) = |\lambda|^{n/2} V^\lambda(e_\alpha, e_\beta)\left(\frac{q}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}p\right), \quad q,p \in \mathbb{R}^n. \quad (5.1.9)$$

In fact, $e^\lambda_{\alpha,\beta}$ is given by

$$e^\lambda_{\alpha,\beta}(q,p) = |\lambda|^{n/2} V(e^\lambda_\alpha, e^\lambda_\beta)(q, \sqrt{|\lambda|}p), \quad q,p \in \mathbb{R}^n,$$

where

$$e^\lambda_\alpha(x) = |\lambda|^{n/4} e_\alpha(\sqrt{|\lambda|}x), \quad x \in \mathbb{R}^n.$$

It can be shown that $\{e^\lambda_{\alpha,\beta} : \alpha, \beta \in (\mathbb{N} \cup \{0\})^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2n})$. See, for example, Proposition 4.1 in [19].

Theorem 5.1.1 *Let λ be a unit vector. The eigenvalues and eigenfunctions of the twisted bi-Laplacian on the Heisenberg group with multi-dimensional center are, respectively, the natural numbers $1, 2, 3, \ldots,$ and the functions $e^\lambda_{\alpha,\beta}$, where α and β are multi-indices in $(\mathbb{N} \cup \{0\})^n$. More precisely, for $k = 1, 2, 3, \ldots$, the eigenfunctions corresponding to the eigenvalue k are all the eigenfunctions $e^\lambda_{\alpha,\beta}$, such that*

$$(|\alpha| + 1) \cdot (|\beta| + 1) = k.$$

By means of Theorem 5.1.1 and some classical results in analytic number theory [21, 22], we can now give an estimate on the counting function $N(\mu)$ defined as the number of eigenvalues of M less than or equal to μ. In other words, we can figure out how many ways the eigenvalue k can be multiplied.

Corollary 5.1.2 *Let λ be a unit vector. For all μ in $[0, \infty)$,*

$$N(\mu) = \sum_{k \le \mu} m_n(k) = \frac{1}{((n+1)!)^2} \sum_{k \le \mu} k^{n-1} d(k).$$

For each eigenvalue k of M, $m_n(k)$ stands for the multiplicity of the eigenvalue k. For all $\varepsilon > 0$, the number $d(k)$ of Dirichlet divisors of the eigenvalue k is given by

$$d(k) = O(k^{\varepsilon}).$$

In Section 5.2, we prove Theorem 5.1.1 and Corollary 5.1.2 on the spectral properties and the related number-theoretical properties of the twisted bi-Laplacian M on the Heisenberg group with multi-dimensional center.

5.2 The Spectrum and Analytic Number Theory

Let us first prove Theorem 5.1.1. Using the fact that M is self-adjoint with compact resolvent, which will be proved in subsequent sections, we see that M has isolated eigenvalues with finite multiplicity, and we can have an orthonormal basis for $L^2(\mathbb{R}^{2n})$ consisting of eigenfunctions of M. Then Theorem 5.1.1 follows from the following proposition.

Proposition 5.2.1 *Let the operator M be as in* (5.1.5) *and the functions $e^{\lambda}_{\alpha,\beta}$, $\alpha, \beta \in (\mathbb{N} \cup \{0\})^n$, be as in* (5.1.8). *Then*

$$Me^{\lambda}_{\alpha,\beta} = (|\alpha| + 1)(|\beta| + 1)e^{\lambda}_{\alpha,\beta}. \tag{5.2.1}$$

Proof Let us start with the operators L^{λ} and $(L^{\lambda})^t$. Then we have

$$(L^{\lambda})^t e^{\lambda}_{\alpha,\beta} = (2|\alpha| + n)e^{\lambda}_{\alpha,\beta} \tag{5.2.2}$$

and

$$L^\lambda e^\lambda_{\alpha,\beta} = (2|\beta| + n)e^\lambda_{\alpha,\beta}. \tag{5.2.3}$$

(5.2.2) is in fact Theorem 22.2 in [12]. The proof is similar to that of the spectral analysis of the simple harmonic oscillator in quantum mechanics by looking at the corresponding creation operator and annihilation operator. For a proof of (5.2.3), observe that

$$(L^\lambda)^t = KL^\lambda K, \tag{5.2.4}$$

where K is the operator of conjugation given by

$$(Kf)(x, y) = \overline{f(x, y)},\ x, y \in \mathbb{R}^{2n}$$

for all measurable functions f on $\mathbb{R}^{2n}$. On the other hand, we see from (5.1.6) and (5.1.7) that

$$e_\alpha(-x) = (-1)^{|\alpha|} e_\alpha(x), x \in \mathbb{R}^n$$

for α and β be multi-indices in $(\mathbb{N} \cup \{0\})^n$,

$$\begin{aligned}
&\overline{e^\lambda_{\alpha,\beta}(x,y)} \\
=\ & \overline{V^\lambda(e_\alpha, e_\beta)\left(\frac{x}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}y\right)} \\
=\ & V^\lambda(e_\beta, e_\alpha)\left(-\frac{x}{\sqrt{|\lambda|}}, -\sqrt{|\lambda|}y\right) \\
=\ & (-1)^{|\alpha|+|\beta|} V^\lambda(e_\beta, e_\alpha)\left(\frac{x}{\sqrt{|\lambda|}}, \sqrt{|\lambda|}y\right) \\
=\ & (-1)^{|\alpha|+|\beta|} e^\lambda_{\beta,\alpha}(x,y)
\end{aligned} \tag{5.2.5}$$

for all x and y in $\mathbb{R}^n$. Hence, by (5.2.2), (5.2.4) and (5.2.5)

$$\begin{aligned}
& (L^\lambda)^t e^\lambda_{\alpha,\beta} \\
=\ & \overline{L^\lambda \overline{e^\lambda_{\alpha,\beta}}} \\
=\ & (-1)^{|\alpha|+|\beta|} \overline{L^\lambda e^\lambda_{\beta,\alpha}} \\
=\ & (-1)^{|\alpha|+|\beta|} | \overline{(2|\alpha|+n) e^\lambda_{\beta,\alpha}} \\
=\ & (-1)^{|\alpha|+|\beta|} (2|\alpha|+n) \overline{e^\lambda_{\beta,\alpha}} \\
=\ & (-1)^{|\alpha|+|\beta|} (2|\alpha|+n) (-1)^{|\alpha|+|\beta|} e^\lambda_{\alpha,\beta} \\
=\ & (2|\alpha|+n) e^\lambda_{\alpha,\beta}.
\end{aligned} \tag{5.2.6}$$

Therefore (5.2.3) is proved. For the renormalized operators P in (5.1.2) and Q in (5.1.4), we deduce from (5.2.2) and (5.2.3)

$$Pe^\lambda_{\alpha,\beta} = \frac{1}{2}(L^\lambda - n + 2) e^\lambda_{\alpha,\beta}$$

and

$$Qe^{\lambda}_{\alpha,\beta} = \frac{1}{2}((L^{\lambda})^t - n + 2)e^{\lambda}_{\alpha,\beta}$$

for α and β in $(\mathbb{N} \cup \{0\})^n$. Since $M = PQ$, it follows that (5.2.1) is proved. $\square$

Proof of Corollary 5.1.2. For $k = 1, 2, \ldots,$ the multiplicity of the eigenvalue k is given by the number $d(k)$ of the Dirichlet divisors of the eigenvalue k. Then the counting function $N(\mu)$ coincides with the sum of the multiplicities of k with $k \leq \mu$, i.e.,

$$N(\mu) = \sum_{k \leq \mu} m_n(k),$$

where $m_n(k) = P_1 P_2 d(k)$, and n is the dimension. P_1 and P_2 are the numbers of partitions of $|\alpha|$ and $|\beta|$ into n nonnegative integers respectively. In other words, P_1 is the number of solutions of $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha_1 \geq \ldots \geq \alpha_n \geq 1$, then the $n!$ permutation of $\alpha = (\alpha_1, \ldots, \alpha_n)$ yield solutions of positive integer solutions of $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Thus,

$$n! P_1 \geq \begin{pmatrix} |\alpha| - 1 \\ n - 1 \end{pmatrix}.$$

P_1 can also be the number of solutions of $|\alpha| - n = \alpha_1 + \ldots + \alpha_n$, $\alpha_1 \geq \ldots \geq \alpha_n \geq 1$, and $\alpha_i = \alpha_j + (n - j)$, $1 \leq i, j \leq n$, then α_j's are distinct and $\alpha_1 + \ldots + \alpha_n = |\alpha| + n(n-1)/2$. Therefore,

$$n! P_1 \leq \begin{pmatrix} |\alpha| + \frac{n(n-1)}{2} - 1 \\ n - 1 \end{pmatrix}.$$

Now we have

$$P_1 = \frac{|\alpha|^{n-1}}{(n+1)!}\left(1 + \frac{n(n-1)}{2|\alpha|} + O(|\alpha|^{-2})\right).$$

Similarly,

$$P_2 = \frac{|\beta|^{n-1}}{(n+1)!}(1 + \frac{n(n-1)}{2|\beta|} + O(|\beta|^{-2})).$$

So,

$$\begin{aligned}
& P_1P_2 \\
= \; & \frac{(|\alpha||\beta|)^{n-1}}{((n+1)!)^2}\left(1 + \frac{n(n-1)}{2|\alpha|} + O(|\alpha|^{-2})\right)\left(1 + \frac{n(n-1)}{2|\beta|} + O(|\beta|^{-2})\right) \\
= \; & \frac{(k)^{n-1}}{((n+1)!)^2}\left(1 + \frac{n(n-1)}{2|\alpha|} + O(|\alpha|^{-2}\right) + \frac{n(n-1)}{2|\beta|} + \frac{n^2(n-1)^2}{4k} \\
+ \; & O(|\alpha|^{-2}|\beta|^{-1}) + O(|\beta|^{-2}) + O(|\alpha|^{-1}|\beta|^{-2}) + O(k^{-2})) \\
= \; & \frac{(k)^{n-1}}{((n+1)!)^2}(1 + O(1) + O(k^{-2})) \textit{ as } k \to \infty. \qquad (5.2.7)
\end{aligned}$$

Subsequently,

$$m_n(k) = P_1P_2d(k) = \frac{(k)^{n-1}}{((n+1)!)^2}(1 + O(1) + O(k^{-2}))d(k).$$

For all $\varepsilon > 0$, $d(k) = O(k^\varepsilon)$, thus,

$$\begin{aligned}
& m_n(k) \\
= \; & \frac{(k)^{n-1}}{((n+1)!)^2}d(k) + \frac{(k)^{n-1}}{((n+1)!)^2}(O(k^\varepsilon) + O(k^{\varepsilon-2})) \\
= \; & \frac{(k)^{n-1}}{((n+1)!)^2}d(k) + O(k^{\varepsilon+n-1}) + O(k^{\varepsilon+n-3}). \qquad (5.2.8)
\end{aligned}$$

Then the counting function $N(\mu)$ becomes

$$N(\mu) = \sum_{k \leq \mu} m_n(k) = \frac{1}{((n+1)!)^2} \sum_{k \leq \mu} (k)^{n-1} d(k) + O(k^{\varepsilon+n-1}) + O(k^{\varepsilon+n-3}).$$

We complete the proof of Corollary 6.1.2.

5.3 Global Hypoellipticity and Essential Self-Adjointness

Since L^λ and $(L^\lambda)^t$, as well as P and Q, are globally hypoelliptic in the Schwartz space $S(\mathbb{R}^{2n})$, it follows that the same is true for $M = PQ$. Since M is a symmetric operator from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ with dense domain $S(\mathbb{R}^{2n})$, it is closable and we denote the closure again by M, which can easily be shown to be symmetric.

Theorem 5.3.1 *M is self-adjoint.*

Proof Since M is closed and symmetric, it follows that M is self-adjoint if we can prove that the resolvent set $\rho(M)$ of M contains a real number. (See, for instance, page 137 of [11].) It is sufficient to prove that the range $R(M)$ of M is dense in $L^2(\mathbb{R}^{2n})$ and there exists a positive constant C such that

$$\|Mu\|_2 \geq C\|u\|_2, u \in D(M).$$

The density follows from the global hypoellipticity of M in $S(\mathbb{R}^{2n})$, and the fact that $0 \in \rho(M)$ then follows from (3.3.1). To prove the inequality (5.3.1), let $\varphi \in S(\mathbb{R}^{2n})$.

Then, by Parseval's identity and Proposition 6.2.1,

$$\begin{aligned}
&\|M\varphi\|_2^2 \\
= &\left\|\sum_\alpha\sum_\beta |\lambda|^{2n}(|\alpha|+1)(|\beta|+1)(\varphi, e^\lambda_{\alpha,\beta})e^\lambda_{\alpha,\beta}\right\|^2 \\
= &\sum_\alpha\sum_\beta |\lambda|^{4n}(|\alpha|+1)^2(|\beta|+1)^2|(\varphi, e^\lambda_{\alpha,\beta})|^2 \\
\geq &|\lambda|^{4n}\sum_\alpha\sum_\beta |(\varphi, e^\lambda_{\alpha,\beta})|^2 = |\lambda|^{4n}\|\varphi\|_2^2. \qquad (5.3.1)
\end{aligned}$$

□

Remark 5.3.2 *More generally, we may consider for all real numbers s and t, the self-adjoint operator P^sQ^t for which*

$$P^sQ^te^\lambda_{\alpha,\beta} = |\lambda|^{n(s+t)}(|\beta|+1)^s(|\alpha|+1)^te^\lambda_{\alpha,\beta}$$

and

$$\|P^sQ^t\varphi\|_2 \geq \|\varphi\|_2, \varphi \in S(\mathbb{R}^{2n}),$$

where α and β are multi-indices in $(\mathbb{N}\cup\{0\})^n$.

5.4 Sobolev Spaces

For all real numbers s and t, and for all tempered distributions u on $\mathbb{R}^{2n}$, we define the Bessel potential $J_{s,t}$ formally by the Fourier series

$$J_{s,t}u = \sum_\alpha\sum_\beta |\lambda|^{-n(s+t)}(|\alpha|^2+1)^{-s/2}(|\beta|^2+1)^{-t/2}u(\overline{e^\lambda_{\alpha,\beta}})e^\lambda_{\alpha,\beta},$$

where $u(\overline{e^\lambda_{\alpha,\beta}})$ is the action of the tempered distribution u on the Schwartz function $\overline{e^\lambda_{\alpha,\beta}}$. Then we define the L^2-Sobolev space of order (s,t) by

$$H^{s,t,2} = \left\{ u \in S'(\mathbb{R}^{2n}) : J_{s,t}u \in L^2(\mathbb{R}^{2n}) \right\}.$$

By the Plancherel formula,

$$H^{s,t,2} = \{u \in S'(\mathbb{R}^{2n}) : \sum_\alpha \sum_\beta |\lambda|^{2n(s+t)}(|\alpha|^2+1)^s(|\beta|^2+1)^t |u(\overline{e^\lambda_{\alpha,\beta}})|^2 < \infty\}.$$

It is easy to see that $H^{s,t,2}$ is an inner product space with inner product $(\,,\,)_{s,t,2}$ and norm $\|\ \ \|_{s,t,2}$ given by

$$(u,v)_{s,t,2} = \sum_\alpha \sum_\beta |\lambda|^{2n(s+t)}(|\alpha|^2+1)^s(|\beta|^2+1)^t (u, e^\lambda_{\alpha,\beta})(e^\lambda_{\alpha,\beta}, v)$$

and

$$\|u\|^2_{s,t,2} = \sum_\alpha \sum_\beta |\lambda|^{2n(s+t)}(|\alpha|^2+1)^s(|\beta|^2+1)^t |(u, e^\lambda_{\alpha,\beta})|^2,$$

for all u and v in $H^{s,t,2}$.

Theorem 5.4.1 *$H^{s,t,2}$ is a Hilbert space with respect to the inner product $(\,,\,)_{s,t,2}$.*

Proof We note that the domain $D(P^sQ^t)$ of the self-adjoint operator P^sQ^t from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ is a Banach space with respect to the graph norm $|\ \ |_{s,t}$ given by

$$|u|^2_{s,t} = \|P^sQ^tu\|^2_2 + \|u\|^2_2, u \in D(P^sQ^t).$$

But for all u in $D(P^sQ^t)$,

$$\|P^sQ^tu\|_2^2 = \sum_\alpha \sum_\beta |\lambda|^{2n(s+t)}(|\alpha|+1)^{2s}(|\beta|+1)^{2t}|(u, e^\lambda_{\alpha,\beta})|^2 \sim \|u\|^2_{s,t,2}.$$

Since

$$\begin{aligned}
& \|P^sQ^tu\|_2^2 \\
= & \sum_\alpha \sum_\beta |\lambda|^{2ns}(|\alpha|+1)^{2s}|\lambda|^{2nt}(|\beta|+1)^{2t}|(u, e^\lambda_{\alpha,\beta})|^2 \\
\geq & \sum_\alpha \sum_\beta |\lambda|^{2ns}(|\alpha|^2+1)^{s}|\lambda|^{2nt}(|\beta|^2+1)^{t}|(u, e^\lambda_{\alpha,\beta})|^2 \\
= & \|u\|^2_{s,t,2}
\end{aligned} \tag{5.4.1}$$

and

$$\begin{aligned}
& \|P^sQ^tu\|_2^2 \\
= & \sum_\alpha \sum_\beta |\lambda|^{2ns}(|\alpha|+1)^{2s}|\lambda|^{2nt}(|\beta|+1)^{2t}|(u, e^\lambda_{\alpha,\beta})|^2 \\
\leq & C\sum_\alpha \sum_\beta |\lambda|^{2ns}(|\alpha|^2+1)^{s}|\lambda|^{2nt}(|\beta|^2+1)^{t}|(u, e^\lambda_{\alpha,\beta})|^2 \\
= & C\|u\|^2_{s,t,2},
\end{aligned} \tag{5.4.2}$$

where C is a positive constant. Thus, the equivalent norm makes sense, and this completes the proof. □

From the proof of Theorem 5.4.1, we get the following result.

Corollary 5.4.2 $D(M) = H^{1,1,2}$.

We have the following Sobolev embedding theorem for these new Sobolev spaces.

Theorem 5.4.3 *For all real numbers s_1, t_1, s_2 and t_2 such that $s_1 \le s_2$ and $t_1 \le t_2$,*

$$H^{s_2,t_2,2} \subseteq H^{s_1,t_1,2},$$

and the inclusion $i : H^{s_2,t_2,2} \hookrightarrow H^{s_1,t_1,2}$ is a bounded linear operator. Moreover, if $s_1 < s_2$ and $t_1 < t_2$ then the inclusion $i : H^{s_2,t_2,2} \hookrightarrow H^{s_1,t_1,2}$ is a compact operator.

That the inclusion is a bounded linear operator is easy to see. To prove the part on compact embedding, we need some preparation.

Proposition 5.4.4 *For real numbers s, t, u and v, the linear operator $J_{u,v} : H^{s,t,2} \hookrightarrow H^{s+u,t+v,2}$ is a surjective isometry.*

Proposition 5.4.5 *Let δ_1 and δ_2 be positive numbers. Then the embedding $i : H^{0,0,2} \hookrightarrow H^{-\delta_1,-\delta_2,2}$ is a compact operator.*

Proof Let $\{u_l\}_{l=1}^{\infty}$ be a bounded sequence in $H^{0,0,2}$, i.e.,

$$\|u_l\|_{0,0,2} \le C,$$

where C is a positive constant. So, there existed a subsequence $\{u_{l_m}\}_{m=1}^{\infty}$ of $\{u_l\}_{l=1}^{\infty}$ such that $u_{l_m} \to u$ weakly for some u in $H^{0,0,2}$ as $m \to \infty$. Thus, for $m = 1, 2, \ldots,$

$$\|u_{l_m} - u\|^2_{-\delta_1,-\delta_2,\,2} = \sum_{\alpha} \sum_{\beta} |\lambda|^{-2n(\delta_1+\delta_2)}(|\alpha|^2+1)^{-\delta_1}(|\beta|^2+1)^{-\delta_2}|(u_{l_m} - u, e^{\lambda}_{\alpha,\beta})|^2.$$

Now, for every positive number ε, we can find a positive integer N such that

$$|\lambda|^{-2n\delta_1}(|\alpha|^2+1)^{-\delta_1} < \varepsilon$$

and

$$|\lambda|^{-2n\delta_2}(|\beta|^2+1)^{-\delta_2} < \varepsilon,$$

whenever $|\alpha|, |\beta| > N$. So, for $m = 1, 2, \ldots,$

$$\begin{aligned} & \sum_{|\alpha|=N+1}^{\infty} \sum_{|\beta|=N+1}^{\infty} |\lambda|^{-2n(\delta_1+\delta_2)}(|\alpha|^2+1)^{-\delta_1}(|\beta|^2+1)^{-\delta_2}|(u_{l_m}-u, e^{\lambda}_{\alpha,\beta})|^2 \\ \leq \ & \varepsilon^2 \sum_{|\alpha|=N+1}^{\infty} \sum_{|\beta|=N+1}^{\infty} \|u_{l_m}-u\|^2_{0,0,2} \leq \varepsilon^2 \|u_{l_m}-u\|^2 \leq \varepsilon^2 (C+\|u\|_{0,0,2})^2 \end{aligned} \quad (5.4.3)$$

and

$$\begin{aligned} & \sum_{|\alpha|=N+1}^{\infty} \sum_{|\beta|=0}^{\infty} |\lambda|^{-2n(\delta_1+\delta_2)}(|\beta|^2+1)^{-\delta_1}(|\alpha|^2+1)^{-\delta_2}|(u_{l_m}-u, e^{\lambda}_{\alpha,\beta})|^2 \\ + \ & \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=N+1}^{\infty} |\lambda|^{-2n(\delta_1+\delta_2)}(|\beta|^2+1)^{-\delta_1}(|\alpha|^2+1)^{-\delta_2}|(u_{l_m}-u, e^{\lambda}_{\alpha,\beta})|^2 \\ \leq \ & 2\varepsilon(C+\|u\|_{0,0,2})^2. \end{aligned} \quad (5.4.4)$$

Since $u_{l_m} \to u$ weakly in $H^{0,0,2}$ as $m \to \infty$, we can pick a positive integer m_0 such that

$$\sum_{|\alpha|=0}^{N} \sum_{|\beta|=0}^{N} |(u_{l_m}-u, e^{\lambda}_{\alpha,\beta})|^2 < \varepsilon,$$

whenever $m_0 \leq m$. Thus,

$$\|u_{l_m}-u\|_{-\delta_1,-\delta_2,2} \leq (\varepsilon^2+2\varepsilon)(C+\|u\|_{0,0,2})^2 + \varepsilon.$$

This proves that the embedding $i : H^{0,0,2} \hookrightarrow H^{-\delta_1,-\delta_2,2}$ is compact. $\square$

Proof of the Compact Embedding in Theorem 5.4.3

The embedding $i : H^{s_2,t_2,2} \hookrightarrow H^{s_1,t_1,2}$ can be decomposed into the composition $J_{s_2,t_2} \circ i \circ J_{-s_2,-t_2}$ of the linear operators

$$J_{-s_2,-t_2} : H^{s_2,t_2,2} \hookrightarrow H^{0,0,2},$$

$$i : H^{0,0,2} \hookrightarrow H^{s_1-s_2,t_1-t_2,2}$$

and

$$J_{s_2\ ,t_2} : H^{s_1-s_2,t_1-t_2,2} \hookrightarrow H^{s_1,t_1,2}.$$

Since $s_1 < s_2$ and $t_1 < t_2$, it follows that $i : H^{0,0,2} \hookrightarrow H^{s_1-s_2,t_1-t_2,2}$ is compact. Therefore the compactness of $i : H^{s_2,t_2,2} \hookrightarrow H^{s_1,t_1,2}$ is proved.

As an application of the compact embedding, we give the following result.

Corollary 5.4.6 *The resolvent $R(\mu)$ of the twisted bi-Laplacian M on the Heisenberg Group with multi-dimensional center given by*

$$R(\mu) = (M - \mu I)^{-1}$$

is a compact operator from $L^2(\mathbb{R}^{2n})$ into $L^2(\mathbb{R}^{2n})$ for all μ in the resolvent set $\rho(M)$ of M.

Proof Since $D(M) = H^{1,1,2}$, it follows that for all μ in $\rho(M)$, $R(\mu) : H^{1,1,2} \to H^{0,0,2}$ is a bounded linear operator. Using the compact embedding in Theorem 5.4.3, $R(\mu) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ is compact, and the embedding $i : H^{1,1,2} \to L^2(\mathbb{R}^{2n})$ is compact. This completes the proof. $\square$

5.5 Sobelov Estimates

The following theorem gives another measure of the global hypoellipticity of the twisted bi-Laplacian M on the Heisenberg group with multi-dimensional center using the Sobolev spaces introduced in the previous section.

Theorem 5.5.1 *For all real numbers s and t, we get*

$$u \in S'(\mathbb{R}^{2n}), Mu \in H^{s,t,2} \Rightarrow u \in H^{s+1,t+1,2}.$$

Moreover,

$$\|u\|_{s+1,t+1,2} \leq \|Mu\|_{s,t,2} \leq 2\|u\|_{s+1,t+1,2},\ u \in H^{s+1,t+1,2}.$$

Proof Let $u \in H^{s+1,t+1,2}$. Then from Proposition 5.2.1 and the fact that

$$(|\alpha| + 1)^2 \geq |\alpha|^2 + 1,$$

$$(|\beta| + 1)^2 \geq |\beta|^2 + 1$$

as well as

$$(|\alpha|+1)^2(|\beta|+1)^2 \leq 4(|\alpha|^2+1)(|\beta|^2+1),$$

we get

$$\begin{aligned}
& \|Mu\|^2_{s,t,2} \\
= \ & \sum_\alpha \sum_\beta |\lambda|^{2n(s+1)}(|\alpha|+1)^{2(s+1)}|\lambda|^{2n(t+1)}(|\beta|+1)^{2(t+1)}|(u, e^\lambda_{\alpha,\beta})|^2 \\
\geq \ & \sum_\alpha \sum_\beta |\lambda|^{2n(s+1)}(|\alpha|^2+1)^{(s+1)}|\lambda|^{2n(t+1)}(|\beta|^2+1)^{(t+1)}|(u, e^\lambda_{\alpha,\beta})|^2 \\
= \ & \|u\|^2_{s+1,t+1,2} \qquad (5.5.1)
\end{aligned}$$

and

$$\begin{aligned}
& \|Mu\|^2_{s,t,2} \\
= \ & \sum_\alpha \sum_\beta |\lambda|^{2n(s+1)}(|\alpha|+1)^{2(s+1)}|\lambda|^{2n(t+1)}(|\beta|+1)^{2(t+1)}|(u, e^\lambda_{\alpha,\beta})|^2 \\
\leq \ & 4\sum_\alpha \sum_\beta |\lambda|^{2n(s+1)}(|\alpha|^2+1)^{2(s+1)}|\lambda|^{2n(t+1)}(|\beta|^2+1)^{2(t+1)}|(u, e^\lambda_{\alpha,\beta})|^2 \\
= \ & 4\|u\|^2_{s+1,t+1,2}. \qquad (5.5.2)
\end{aligned}$$

Thus, the inequalities are proved. □

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Appendix

Plancherel Theorem/Formula

Let $f \in L^2(\mathbb{R}^n)$, $\widehat{f}$ is the Fourier transform of f in the sense of

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

for all x and ξ in $\mathbb{R}^n$. Then,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi$$

and

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)} dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\hat{g}(\xi)} d\xi.$$

Such theorem is also applicable to norm, as in

$$\|\hat{f}\| = \|f\|.$$

Parseval's Identity

Let B be an orthonormal basis of the Hilbert space H, i.e., an orthonormal set which is total in the sense that the linear span of B is dense in H.

Then for every $x \in H$,

$$\sum_n |(x, e_n)|^2 = (x, x) = \|x\|^2$$

where e_n are unit vectors in B.

Schwarz Inequality

For all vectors u and v in Hilbert space, an inner product of u and v exists if and only if

$$|(u, v)|^2 \leq (u, u) \cdot (v, v),$$

or written in the following form

$$|(u, v)| \leq \|u\| \|v\|.$$

Moreover, the two sides are equal if and only if u and v are linearly dependent.

Minkowski's Inequality

Let f be a Lebesgue measurable in $L^p(\mathbb{R}^n)$, and $1 \leq p < \infty$. Then,

$$\left(\int \left| \int f(x, y) dy \right|^p dx \right)^{1/p} \leq \int \left(\int |f(x, y)|^p dx \right)^{1/p} dy, \quad x, y \in \mathbb{R}^n.$$

Riesz-Thorin Theorem

Let $\alpha_1, \alpha_2, \beta_1$, and β_2 be real numbers in $[0, 1]$. Suppose that

$$A : L^{1/\alpha_1}(\mathbb{R}^n) \to L^{1/\beta_1}(\mathbb{R}^n)$$

and

$$A : L^{1/\alpha_2}(\mathbb{R}^n) \to L^{1/\beta_2}(\mathbb{R}^n)$$

are bounded linear operators such that there exist positive constants M_1 and M_2 for which

$$\|Af\|_{1/\beta_1} \leq M_1 \|f\|_{1/\alpha_1}, \quad f \in L^{1/\alpha_1}(\mathbb{R}^n)$$

and

$$\|Af\|_{1/\beta_2} \leq M_2 \|f\|_{1/\alpha_2}, \quad f \in L^{1/\alpha_2}(\mathbb{R}^n).$$

If, for $0 \leq \theta \leq 1$, we let

$$\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$$

and

$$\beta = \theta\beta_1 + (1-\theta)\beta_2,$$

then $A : L^{1/\alpha_1}(\mathbb{R}^n) \to L^{1/\beta_1}(\mathbb{R}^n)$ is a bounded linear operator and

$$\|Af\|_{1/\beta} \leq M_1^{\theta} M_2^{1-\theta} \|f\|_{1/\alpha}, \quad f \in L^{1/\alpha}(\mathbb{R}^n).$$

Leibniz Formula

Let f and g be n-times differentiable functions, then the product fg is also n-times differentiable and its nth derivative is given by

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient and $f^{(j)}$ denotes the jth derivative of f.

Fubini Theorem

Let X and Y be two finite measure spaces, and $X \times Y$ is given the product measure. If $X \times Y$ is integrable, meaning that f is a measurable function and

$$\int_{X\times Y} |f(x,y)|\mathrm{d}(x,y) < \infty$$

then

$$\int_X \left(\int_Y f(x,y)\mathrm{d}y\right) \mathrm{d}x = \int_Y \left(\int_X f(x,y)\mathrm{d}x\right) \mathrm{d}y = \int_{X\times Y} f(x,y)\mathrm{d}(x,y).$$

A special case of Fubini's theorem is in the interchange of the summations, as in

$$\sum_x \sum_y a_{xy} = \sum_y \sum_x a_{xy}$$

where a_{xy} are non-negative for all x and y.

Hölder's Inequality

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $g \in L^p(\mathbb{R}^n)$, and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq \|f\|_p \|g\|_{p'}.$$