ON GUARANTEED MINIMUM MATURITY BENEFITS AND FIRST-TO-DEFAULT TYPE PROBLEMS

Fenghao Yang

A DISSERTATION SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN Mathematics and Statistics
YORK UNIVERSITY
TORONTO, ONTARIO

November 2017

Abstract

A new class of exponential functionals arises when pricing certain equity-linked insurance products. We study the distribution of these exponential functionals using tools from Probability and Complex Analysis. In the case of the Kou process we obtain an explicit formula for the probability density function of the exponential functional and we apply this result to pricing equity-linked insurance products. As a by-product of this research we have also derived a new class of duality relations for hypergeometric functions.

In the second part of the thesis, we study correlation uncertainty in Credit Risk. The goal is to price analogues of first-to-default options under the assumption that the assets follow correlated stochastic processes with known marginal distributions and unknown dependence structure. We solve this problem using tools from Stochastic Analysis and Optimal Control Theory. We provide explicit solutions in some specific examples and numerical approximations in the more general case.

Acknowledgments

Before starting my PhD, I had already heard of Lévy processes, Martingale theorem, local time, those topics in Probability. They were hard for me at that time and I was somehow resistant to get involved with such topics. After finishing my PhD, I am happy to say that I have this dissertation and other three papers related to these topics. I attribute a large part of this success to both of my supervisors Tom Salisbury and Alexey Kuznetsov. They have spent a large amount of time discussing research problems with me, and without their support I could not have finished this dissertation and the papers. Beyond sharing their knowledge of probability, stochastic control, mathematical finance, complex analysis and special functions with me, they also taught me how to present the ideas in an easily understandable way. For all of these reasons, I am very grateful and thankful. Thank you, Tom. Thank you, Alexey.

My parents have been always supporting me without complaint even though I have not been going back home since pursuing my PhD. My girlfriend Gabriela has always been encouraging me, especially when I got desperate in the process of writing this dissertation. This dissertation is rightfully dedicated to them.

Contents

1 Overview

In this chapter, first we give a general outline of this dissertation; then we present a detailed overview for the purposes of literature review and new results summary; at the end, we provide the paper and authorship details.

1.1 General overview and introduction

In Mathematical Finance, stochastic processes are used to model the asset price (stock, fund, equity). The chosen process has to keep balance between reality and tractability: it needs to describe the phenomena in the real market as much as possible and it also has to give analytic solutions for derivative pricing.

Geometric Brownian motion is a well known process which has been widely used to model the underlying equity. It describes some important facts of the market, and also provides analytic solutions to option pricing. However, several empirical studies have shown that the geometric Brownian motion does not adequately explain some facts of empirical returns. One popular solution to this problem is to use Lévy processes to model log-returns of equities. Lévy processes, on one hand, have the jump term to capture more facts of the market and, on the other hand, they keep the tractability of the Brownian motion. For example, the Asian option pricing involves the average value of an exponentiated Lévy process, which is connected to the exponential functional. Many theoretical results have been obtained for the exponential functional of Lévy processes (see [9], [39], [40]), which makes an analytic solution for Asian option pricing approachable. Nowadays, equity-linked insurance products have been adopting Lévy processes to model the equity. However, the funding mechanism of such insurance product involves the average value and the final value of an exponentiated Lévy process. In order to model this product, a more general exponential functional is needed. In this dissertation we investigate that.

We employ analytical methods to derive the distribution of the general exponential functional. We work with integral transforms (Laplace, Mellin) to take probabilistic objects into the complex plane. This allows us to obtain an explicit expression directly, which is then inverted to get the distribution. With this result, any risk measure can be calculated.

The second part of this thesis is devoted to Credit Risk.

Credit risk has been essential for valuing assets. For a single asset, credit risk can be derived from its market information. However, it is not the case for portfolios consisting of multiple assets, such as CDOs (collateralized debt obligation). The main challenge lies in specifying the dependence among the assets. The most widely used mechanism for characterizing the dependence is the copula model initiated by Li [46]. Although such a model is easy for market calibration and credit risk valuation, the dependence among assets are introduced without regarding to their dynamics. This makes the model unreliable when the market becomes stressed, for example the crisis in 2008, where the mis-pricing of CDOs were a major contributor. In contrast, the structural model can incorporate the dependence quite naturally by assuming the assets follow correlated stochastic processes. Thus the correlation can directly reflect the dependence. In this dissertation we demonstrate such a way of valuing credit risk by pricing "first to default option". We investigate the extent of the price spread according to the dynamic of the correlation, since different settings of the correlation lead to different prices of the option.

We mainly use stochastic analysis and optimal control theory to investigate the pricing of the first to default option. If we take the correlation as the controlled object, then the problem is transformed into a stochastic control problem. By working with the theory of martingales, Itô's formula and Itô-Tanaka formula, we derive that our object is either a supermartingale or submartingale. This enables us to optimize the object to obtain the maximum and minimum price of the option, then their difference determines the price spread.

Generally speaking, the primary purpose of this dissertation is to demonstrate existing and new techniques for problems arising in mathematical finance by working with specific stochastic processes. These problems invariably involve two key theoretical objects, namely, the exponential functional and the verification theorem.

The two stochastic processes employed in this dissertation are precisely the Kou process and the two dimensional Brownian motion. The Kou process is an outstanding member in the family of hyper-exponential processes. It has been widely accepted to model the underlying asset, as it can explain some important empirical phenomena from the market and it leads to analytical solutions to a variety of option-pricing problems. The two dimensional Brownian motion is a suitable process to model two assets and the correlation is able to reflect the dynamic of the dependence.

A secondary purpose is to show two new research topics, which arise in the process of solving the insurance product problem and the option pricing problem, we take them as a bonus. The first one is a duality relation in special functions, which refers to an identity involving finite sums of products of two hypergeometric functions. We obtain the identity when studying the distribution of the general exponential functional. We find it may have independent interest in the special function area, so we generalize the identity to a large family of hypergeometric functions. In the literature, the first instances of such formulas have appeared in 1932 by Darling [16]. These results then have been expanded by Bailey [2] in 1933, and they have been greatly generalized recently by Beukers and Jouhet [6]. Here we present a very simple way to obtain such results. The second one is about the skew Brownian motion. Harrison and Shepp [31] have introduced a beautiful way to prove the existence and uniqueness of the solution for a stochastic differential equation (SDE), which involves the symmetric local time. They also have shown that the solution is exactly a skew Brownian motion. We modify the idea from Harrison and Shepp and then we apply it to a specific SDE involving the asymmetric local time. We get this SDE while dealing with the option pricing problem. Also in the process we provide a new construction of the skew Brownian motion, which may have independent interest in its area.

1.2 Detailed overview and summary of specific results

This dissertation is divided into two parts: Part I is intended as a literature review, and to introduce concepts, and analytical techniques that are used in Part II to derive new results. There are practically no new results in Part I (except for some discussion in Section 5.3), but the majority of definitions are made here. Part II consists of three chapters each corresponding to a new contribution to the literature, see the next section for paper and authorship details, and the summaries below for details of each contribution.

Part I: Literature review and overview of techniques

Chapter 2: Lévy processes

Here we briefly introduce Lévy processes, and we list the Lévy density and characteristic/Laplace exponent. Most importantly we introduce the Kou process which is used in Chapter 4 and Chapter 6. We explain why the Kou process is proposed and why it is important in option pricing. In the meanwhile, we show the analytically tractable property of the Kou process in Wiener-Hopf factorization and its application in pricing barrier options.

Chapter 3: Complex analysis and special functions

We mention the Identity theorem, Cauchy residue theorem, Liouville's theorem which are used in the dissertation. We also mention the Mellin transform and inverse Mellin transform, which are the essential tools in Chapter 4 and Chapter 6. Furthermore, we introduce hypergeometric functions, basic hypergeometic functions and Meijer G-functions which enable us to derive some explicit results in Chapter 4 and Chapter 6. Hypergeometric functions and basic hypergeometic functions are also the main objects in 8. As the reader may be unfamiliar with Meijer G-functions, we also show many properties of Meijer G-functions. Those properties will be mainly used in Chapter 6.

Chapter 4: Exponential functionals

The primary purpose of this chapter is to introduce the exponential functional

$$J_t = \int_0^t e^{X_s} \mathrm{d}s,$$

where X is a Lévy process. We give a literature review of the development of the exponential functionals in recent years. An easy way to deal with J_t is by replacing the time t with a random time

$e(q)$, which has an exponential distribution with parameter $q > 0$ and is independent of the process X. Here we use I_q to denote the exponential functional, which means $I_q = J_{e(q)}$. In the meanwhile, the verification result is introduced for determining the Mellin transform of I_q. We demonstrate how to apply such result to obtain the Mellin transform of I_q under the Kou process. Such transform has already been obtained by Cai and Kou [9] alternatively via the ordinary integro-differential equation approach. Furthermore, we show how to apply the theory of Meijer G-functions to obtain the probability density function of I_q, after its Mellin transform was obtained. The intention is to get the reader familiar with the Meijer G-function which appears frequently in Chapter 6.

Chapter 5: Brownian motion

We briefly state some basic concepts of continuous martingales and their connection with Brownian motions. Most of the results in Chapter 7 are under this framework. Furthermore we mention semimartingale and local time which are important for introducing the skew Brownian motion. We take the skew Brownian motion as a tool to solve one particular type of stochastic differential equation (SDE). In this process we follow the idea from Harrison and Shepp [31] but with some modification. Thus a new theorem regarding the existence and uniqueness of the solution to one particular SDE is presented. This theorem helps us to prove an important result in Chapter 7.

Part II: New results

Chapter 6: Guaranteed Minimum Death Benefit (GMDB)

GMDB is an equity-linked insurance product. The unique funding mechanism in this product gives rise to a generalized form of exponential functional as

$$J_{x,t} := xe^{X_t} + \int_0^t e^{X_s}\,\mathrm{d}s,$$

where x is non negative, X is a Lévy process and t is the expiry time of the contract for this product. It is easy to observe that the exponential functional J_t in Chapter 3 is just the special case of $J_{x,t}$ when $x = 0$. We still adopt the approach by replacing t with $e(q)$, namely $I_{x,q} := J_{x,e(q)}$. Therefore the object which will be studied in this chapter is $I_{x,q}$.

The investigation of the $I_{x,q}$ is a new topic. Feng and Volkmer [23, 24] dealt with the case where the process X is a Brownian motion. Despite that, we are not aware of any existing results for the case where the process X has jumps. In this Chapter, we investigate $I_{x,q}$ where X is a Lévy process. This is important, as we have already stated in Chapter 2 that the Lévy process can capture some important features in the real market comparing to the Brownian motion.

We derive the Mellin transform of $I_{x,q}$ explicitly for the case when X is a Lévy process and whose Lévy measure has exponentially decaying tails. The technique of measure change is an essential step to obtain this result. Here it is also important to note that in the process to obtain the Mellin transform of $I_{x,q}$, we have used some results which are already in the literature for I_q. Those results can be seen in Rivero [64], Kuznetsov [39], Patie and Savov [60].

In the remainder of this chapter we use the Kou process as an example to obtain the Mellin transform of $I_{x,q}$ and also get its probability density function $f_{x,q}(y)$. It can be easily checked that the Kou process satisfies the requirement of having exponentially decaying tails in its Lévy measure. We can observe that the formula of the Mellin transform of $I_{x,q}$ is complicated, which makes $f_{x,q}(y)$ hard to get. We derive a formula for $f_{x,q}(y)$ by using inverse Mellin transform, the theory of the Meijer G-function and Cauchy residue theory. The process to obtain its explicit formula is demonstrated in detail although it is not rigorous. However, after that a rigorous proof is presented to show that $f_{x,q}(y)$ is exactly the probability density function of $I_{x,q}$. The proof involves many computations related to special functions, and the essential step of the proof is one identity involving hypergeometric functions. This identity is a special case of Theorem 27 in Chapter 8. Finally, with the obtained $f_{x,q}(y)$, we can easily compute the tail distribution and tail expectation of $I_{x,q}$. These theoretical results can be directly applied to the computation of various risk measures in insurance companies' favor, such as the Value at risk (VaR), the Conditional tail expectation (CTE). We show that the analytical formulas are much more efficient and more accurate than the current approach used by the insurance industry—the Monte Carlo simulation.

Chapter 7: Optimal control in first to default problem

Suppose we have two assets A_1 and A_2 in a portfolio. We are interested in the probability that the first default happens before a fixed time T. The marginal distribution of the default time for each asset can be derived from its market information, but the dependence between these two assets is hard to specify. The copula model, initiated by Li [46], is widely used for characterizing the dependence. Although such model has the advantage in market calibration and credit risk valuation, the dependence between assets is introduced without regarding to their dynamics. This makes the model unreliable when the market become stressed. In order to avoid such situation, we assume the assets follow correlated stochastic processes, which can incorporate the dependence naturally. By setting the correlation dynamic, it can dynamically reflect the dependence. Since the correlation is not fixed, a different choice of correlation will lead to a different probability of first to default. We are interested in the highest and lowest probability and also the corresponding correlation. However, depending on the complexity of the stochastic processes, the computation can be very difficult, especially under dynamic correlation assumption. In order to make the computation feasible, we will use simple processes to model the assets. We let $S_t^1 = e^{\sigma_1 B_t^1}$ and $S_t^2 = e^{\sigma_2 B_t^2}$ represent the value of assets A_1 and A_2, where (B_t^1, B_t^2) is a two dimensional Brownian motion which starts from

(B_0^1, B_0^2) and satisfies

$$\mathrm{d}\langle B^1, B^2\rangle_t = \rho_t \mathrm{d}t. \tag{1}$$

Here ρ_t denotes the correlation and $\rho_t \in [-1, 1]$. The default happens when either asset reaches the value 1, which is equivalent to saying that B_t^1 or B_t^2 reaches 0. Of course, this is not a general geometric Brownian motion model, as it assumes that $\mu_1 = \sigma_1^2/2$ and $\mu_2 = \sigma_2^2/2$ in the geometric Brownian motion setting. However, in our case some closed form solutions do exist. We let $\tau = \inf\{0 \leq t \leq T : B_t^1 = 0 \text{ or } B_t^2 = 0\}$, thus our study is investigating

$$\max_{\rho_t} \mathbb{P}(\tau \leq T) \quad \text{and} \quad \min_{\rho_t} \mathbb{P}(\tau \leq T).$$

It is easy to obtain the answer for the minimum case, which will be shown in a simple proof. However, for the maximum case, it is a different story.

We generalize the problem by setting a payoff function f, where $f(B_\tau^1, B_\tau^2) = f(B_\tau^1 + B_\tau^2)$, which means if $B_\tau^1 = 0$, the payoff will be $f(B_\tau^2)$, and vice versa. Therefore this problem will be modeled as

$$\max_{\rho_t} \mathbb{E}(f(B_\tau^1, B_\tau^2)\mathbb{1}_{\{\tau<T\}}) \quad \text{and} \quad \min_{\rho_t} \mathbb{E}(f(B_\tau^1, B_\tau^2)\mathbb{1}_{\{\tau<T\}}).$$

We can observe that they are actually optimal control problems. Therefore by applying the optimality principle and Itô's formula, we can obtain a partial differential equation (PDE) for the optimal expected payoff. It also can be easily derived that the optimal choice of ρ_t only can switch between two values 1 and -1. Although it is hard to obtain the analytical solutions for the PDEs, some numerical results are demonstrated.

In order to decide the exact switch region as the correlation is changing from 1 to -1, we simplify the problem by getting rid of T. So the problem transforms to $\max \mathbb{E}(f(B_\tau^1, B_\tau^2))$ and $\min \mathbb{E}(f(B_\tau^1, B_\tau^2))$. We investigate two specific forms of the payoff function $f(x, y)$, the symmetric one $f(x, y) = (x+y)^\alpha$ and the asymmetric one $f(x, y) = (x+by)^\alpha$, the latter one implies the payoff will be more favorable if one particular asset defaults first.

For most of the cases in this Chapter, the essential approach to obtain the optimal expected value and the optimal choice of the correlation is by using the verification theorem. This theorem transforms the optimal control problem to either a supermartingale or a submartingale problem. By such theorem, letting the drift term of the supermartingale be 0 gives the optimal choice of ρ_t, and then applying Fatou's lemma gives the maximum expected value; letting the drift term of the submartingale be 0 gives the optimal choice of ρ_t, and then applying Doob's $\mathbb{L}^p-$inequality gives the minimum expected value. However, for $\alpha > 1$, it is proved that the maximum expected value is infinite, by applying theory of complex Brownian motion.

In the process of obtaining the optimal choice of ρ_t for the maximum case $\max \mathbb{E}(f(B_\tau^1, B_\tau^2))$ when $f(x, y) = (x + by)^\alpha$, we extract one topic which may have independent interest. The topic regards the solution of one particular SDE

$$\mathrm{d}B_t^1 = \mathrm{sgn}(\theta B_t^1 - B_t^2)\mathrm{d}B_t^2,$$

where B_t^2 is a Brownian Motion which starts from B_0^2 and $0 < \theta < 1$. We use the skew Brownian motion to prove the existence and uniqueness of the solution. Such idea has already been stated in Chapter 5 in detail. In the meanwhile, we also provide a new construction of the skew Brownian motion.

Chapter 8: Duality relations for hypergeometric functions

As we state in Chapter 6, one identity involving finite sums of products of two hypergeometric functions arises. Such an identity is called a duality relation in the special function area and in this chapter we expand this result for a more general family of hypergeometric functions. The first instances of such formulas have appeared in 1932 in the paper [16] by Darling. These results have been expanded by Bailey [2] in 1933, and recently they have been greatly generalized by Beukers and Jouhet [6], who have used the theory of D-modules of general linear differential (or difference) equations. In this Chapter we demonstrate our approach to derive such relations. The approach is elementary and is inspired by a simple fact that the sum of residues of a rational function is zero when the degree of the denominator is greater than one plus the degree of numerator. Such an approach is shown as a lemma in this chapter. Before presenting the main results, we demonstrate how to apply this approach to prove one simple identity in detail, such that readers can get familiar with the idea behind this approach. Furthermore, we show that the duality has an analogue in terms of basic hypergeomeric functions.

1.3 Published papers/Preprint

The contents of Chapter 6, 7 and 8 have either been published or in preparation. The results appearing in these chapters represent joint work with Runhuan Feng, Alexey Kuznetsov and Thomas Salisbury.

A modified version of: Chapter 6 has been submitted [22]; Chapter 7 has been in preparation; Chapter 8 has appeared in Journal of Mathematical Analysis and Applications [21].

2 Lévy processes

In this chapter, we first introduce Lévy processes and some of their properties, then we present the Kou process to which we will refer in Chapter 4 and Chapter 6. We show the tractable property of the Kou process in Wiener-Hopf factorization and demonstrate how to apply such result in pricing barrier options.

2.1 General Lévy processes

A random variable ξ is called infinitely divisible if for each $n \in \mathbb{N}$, there exist n i.i.d. random variables $\{\xi_i\}$ such that

$$\xi \stackrel{d}{=} \xi_1 + \cdots + \xi_n.$$

This is equivalent to saying that for any $n \in \mathbb{N}$, the distribution of ξ is the convolution of n identical distributions. By the famous Lévy-Khintchine Formula this is again equivalent to the statement that there exists a unique triple (μ, σ^2, Π) such that

$$\mathbb{E}[e^{\mathrm{i}z\xi}] = e^{-\Psi(z)}, z \in \mathbb{R},$$

where,

$$\Psi(z) = \frac{\sigma^2 z^2}{2} - \mathrm{i}\mu z - \int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}zx} - 1 - \mathrm{i}zx\mathbb{1}_{\{|x|<1\}})\Pi(\mathrm{d}x), \tag{2}$$

and where, $a \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\Pi(\mathrm{d}x)$ is a measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\Pi(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}\setminus\{0\}} \min(1, x^2)\Pi(dx) < \infty. \tag{3}$$

The function $\Psi(z)$ is called the characteristic exponent of ξ.

Definition 1. A Lévy process is an $\mathbb{R}$-valued stochastic process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ that has the following properties:

(i) The paths of X are right continuous with left limits $\mathbb{P}$-a.s.

(ii) $X_0 = 0$ $\mathbb{P}$-a.s.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

Given a Lévy process X we can use properties (iii) and (iv) from the Definition 1 to obtain that, for any $n \in \mathbb{N}$ we may write

$$X_t = \left(X_t - X_{\frac{n-1}{n}t}\right) + \left(X_{\frac{n-1}{n}t} - X_{\frac{n-2}{n}t}\right) + \cdots + \left(X_{\frac{2}{n}t} - X_{\frac{1}{n}t}\right) + X_{\frac{1}{n}t}. \tag{4}$$

Here the terms on the right-hand side are independent and identically distributed. This shows that X_t is an infinitely divisible random variable. Let $\Psi(z)$ be the characteristic exponent of X_1 as defined in (2). Then it follows from property (i) that

$$\mathbb{E}(e^{\mathrm{i}zX_t}) = e^{-t\Psi(z)}.$$

In other words, X is completely determined by the triple (μ, σ^2, Π) corresponding to the characteristic exponent of X_1. Accordingly, the function $\Psi(z)$ used in this context is also called the characteristic exponent of X. We have seen that every Lévy process is naturally associated with an infinitely divisible random variable. It is also true, although more difficult to show, that every infinitely divisible random variable ξ gives rise to a (unique up to equality in distribution) Lévy process X such that $\xi \overset{d}{=} X_1$ (see Theorem 2.1 in [44]).

Remark 1. The triple (μ, σ^2, Π) is called the the generating triple. The quantity σ^2 is known as the Gaussian component and the measure $\Pi(dx)$ is known as Lévy measure. The function in (2) is known as a cut-off function. Without additional restriction on $\Pi(dx)$, such a function is needed to ensure convergence of the integral. However, any function $h(x)$ that satisfies $h(x) = 1 + o(x)$ as $|x| \to 0$ and $h(x) = O(1/x)$ as $x \to \infty$ will suffice.

Typically we often classify Lévy processes by the characteristics of their sample paths. The first such classification deals with the amount of jump activity, which is measured by the number of discontinuities (jumps) of a sample path over any time interval. Each Lévy process has either almost surely finite jump activity or almost surely infinite activity. Therefore we can classify Lévy processes as either finite jump activity processes or infinite activity processes. We note that a Lévy process is a finite activity process if and only if the jumps follow a compound Poisson process. Namely

$$\Psi(z) = \frac{\sigma^2 z^2}{2} - \mathrm{i}\mu z - \lambda \int_{\mathbb{R}} (e^{\mathrm{i}zx} - 1) v(\mathrm{d}x),$$

where $\lambda \in \mathbb{R}^+$ and v is a probability measure (see Section 2 in [44]).

A process is called a finite variation process if its sample paths have almost surely finite total variation; it is called an infinite variation process if its sample paths have almost surely infinite total variation. As with jump activity, each Lévy process has either finite variation or infinite variation. Note that a process is a finite variation process if and only if

$$\int_{\mathbb{R}} \min(1,|x|)\Pi(\mathrm{d}x) < \infty \quad \text{and} \quad \sigma = 0,$$

see Section 2 in [44].

The term subordinator refers to a Lévy process whose paths are almost surely increasing. A Lévy process which is not a subordinator but has no negative jumps is called a spectrally positive process. Likewise, a Lévy process which is not the negative of a subordinator, but has no positive jumps is called a spectrally negative process. Spectrally positive and negative processes are called spectrally one-sided processes. Note that a scaled Brownian motion with drift is both a spectrally positive and a spectrally negative process. Lévy processes that have only positive or only negative jumps are called processes with one-sided jumps, or one-sided processes. A process with both positive and negative jumps is called a process with two-sided jumps or a two-sided process.

Often, we wish to work with the Laplace exponent of a Lévy process X, which we define as

$$\begin{aligned}
\psi(z) := \; &= \frac{1}{t}\log \mathbb{E}[e^{zX_t}] \\
&= -\Psi(-\mathrm{i}z) \\
&= \frac{\sigma^2 z^2}{2} + \mu z + \int_{\mathbb{R}\setminus 0} (e^{zx} - 1 - zx\mathbb{1}_{\{|x|<1\}})\Pi(\mathrm{d}x), \qquad z \in \mathrm{i}\mathbb{R}. \qquad (5)
\end{aligned}$$

Of course, this definition is rather pointless unless we can extend $\psi(z)$ beyond just the imaginary numbers. From [44] we have the following equivalent condition to the existence of $\psi(z)$ in terms of the Lévy measure $\Pi(\mathrm{d}x)$.

Theorem 1. *Let $\psi(z)$ be the Laplace exponent of a Lévy process with generating triple (μ, σ^2, Π). Then $\psi(z_0)$ is finite if and only if $\int_{|x|\geq 1} \mathrm{e}^{\mathrm{Re}(z_0)x}\Pi(\mathrm{d}x) < \infty$.*

See Theorem 3.6, in [44].

In this dissertation, we will focus on the case when the domain of $\psi(z)$ includes a vertical strip of $\mathbb{C}$ containing the origin. Therefore we will work with the Laplace exponent instead of the characteristic exponent.

2.2 Kou process

In option pricing, the standard geometric Brownian motion has been widely used under the Black-Scholes framework. Its analytical tractability not only gives explicit formula for the pricing of call and put options and also ensures explicitly pricing the path dependent options such as barrier options and lookback options. However, in several empirical studies (see Cont [15], Madan and Seneta [48], Carr et al. [11], Kou [36]), it was demonstrated that the geometric Brownian motion does not explain many stylized facts of empirical equity returns. Here we take two important facts as examples: the leptokurtic feature and the volatility smile. The leptokurtic feature means that the return distribution of assets may have a higher peak and two (asymmetric) heavier tails than those of the normal distribution. The volatility smile means that the implied volatility curve is a convex curve of the strike price. But under the Black-Scholes model framework, the volatility is assumed to be constant. Therefore many studies have been conducted to modify the Black-Scholes model to explain the two empirical phenomena. For example, Merton [53] has proposed the Normal jump-diffusion model, and Duffie et al. [17] have proposed Affine Jump-Diffusion model. However, those alternative models can only compute prices explicitly for call and put options, they can not get analytic solutions for those path-dependent options. Even the numerical methods for the path-dependent options are not easy, as the convergence rate of Monte Carlo simulation for path-dependent options are typically much slower than those for call and put options (see Boyle et al. [7]). This also makes it harder to persuade practitioners to switch from the Black-Scholes model to more realistic alternative models.

In 2002, Kou [36] proposed the double exponential jump-diffusion model (namely the Kou process), which can capture both facts: the leptokurtic feature and the volatility smile. Furthermore, it can give closed-form solutions for the pricing of the path-dependent options. The Kou process follows the idea of Merton's Normal jump-diffusion model, which incorporates jumps into the standard geometric Brownian motion by adding compound Poisson jumps. But instead of using normal jumps, Kou[36] used the asymmetric double exponential jumps. By combining the memoryless property of the exponential jumps with an approach based on differential equations and martingale, Kou and Wang [37] have computed the distribution of the first passage time and applied it in the path dependent options pricing in [38].

The Kou process is defined as follows:

$$X_t = \mu t + \sigma W_t + \sum_{j=1}^{N_t} \xi_i, \tag{6}$$

where $\sigma > 0$, $\mu \in \mathbb{R}$, N_t is a Poisson process with intensity λ, and $\{\xi_i\}_i$ are i.i.d. random variables with the probability density function

$$p_\xi(x) = p\rho e^{-\rho x} \mathbb{1}_{\{x>0\}} + (1-p)\hat{\rho} e^{\hat{\rho} x} \mathbb{1}_{\{x<0\}},$$

for some $p \in (0, 1)$ and $\rho, \hat{\rho} > 0$. Note that W_t, N_t and $\{\xi_i\}_i$ are independent in the definition.

The Laplace exponent is equal to

$$\psi(z) = \mu z + \frac{\sigma^2}{2} z^2 + \lambda p \frac{z}{\rho - z} - \lambda(1-p) \frac{z}{\hat{\rho} + z}. \tag{7}$$

For $q > 0$ the rational function $\psi(z) = q$ has four zeros $\{-\hat{\zeta}_2, -\hat{\zeta}_1, \zeta_1, \zeta_2\}$ and two poles $\{-\hat{\rho}, \rho\}$. This can be easily seen from (7), essentially by using the intermediate value theorem, that the four zeros are all real and satisfy the interlacing property

$$-\hat{\zeta}_2 < -\hat{\rho} < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho < \zeta_2.$$

Another attractive aspect of the Kou process is that it has explicit Wiener-Hopf factorization. Wiener-Hopf factorization is a powerful tool in investigating the first passage time or the extrema for Lévy processes (see Chapter 6 in [44]). As we have stated above, in order to price the path dependent options, one needs to investigate either the first passage time or the extrema processes. An alternative way, different from Kou and Wang [38], for the path dependent options pricing is by applying the Wiener-Hopf factorization. The essential idea is to get the Wiener-Hopf factors, which are related to the extrema processes. For most of the Lévy processes, they do not have explicit factorizations. However, the Kou process does have. Furthermore, it is also easy to obtain the probability density function of its extrema processes. In the following, we will show the approach to get the Wiener-Hopf factors for the Kou process and also demonstrate its application in pricing barrier options.

First we define extrema processes

$$S_t = \sup\{X_s : 0 \le s \le t\}, \qquad I_t = \inf\{X_s : 0 \le s \le t\}.$$

We introduce an exponential random variable $e(q)$ with parameter $q > 0$, which is independent of the process X_t. We use the following notation for the Laplace exponent of $S_{e(q)}$ and $I_{e(q)}$:

$$\phi_q^+(z) = \mathbb{E}[e^{zS_{e(q)}}], \qquad \phi_q^-(z) = \mathbb{E}[e^{zI_{e(q)}}].$$

The Wiener-Hopf factorization states that the random variables $S_{e(q)}$ and $X_{e(q)} - S_{e(q)}$ are independent; random variables $I_{e(q)}$ and $X_{e(q)} - S_{e(q)}$ have the same distribution. Thus we have

$$\frac{q}{q - \psi(z)} = \mathbb{E}[e^{zX_{e(q)}}] = \mathbb{E}[e^{zS_{e(q)}}]\mathbb{E}[e^{z(X_{e(q)} - S_{e(q)})}] = \phi_q^+(z)\phi_q^-(z).$$

Naturally one idea is to factor the function $q/(q - \psi(z))$ in such a way that we can identify the Laplace transforms of two infinitely divisible distributions with support on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively. But not all the Lévy processes can apply this approach, as it has a specific requirement for the suitability of the Laplace exponent $\psi(z)$. Luckily, the Kou process is suitable for this approach. With the Laplace exponent we have shown above, via simple algebra, we can get

$$\frac{q}{q - \psi(z)} = \frac{1 - \frac{z}{\rho}}{(1 - \frac{z}{\zeta_1})(1 - \frac{z}{\zeta_2})} \times \frac{1 + \frac{z}{\hat{\rho}}}{(1 + \frac{z}{\hat{\zeta}_1})(1 + \frac{z}{\hat{\zeta}_2})}. \tag{8}$$

From (8) we can obtain (the rigorous proof can be inferred from Chapter 6 in [44])

$$\phi_q^+(z) = \frac{1-\frac{z}{\rho}}{(1-\frac{z}{\zeta_1})(1-\frac{z}{\zeta_2})} = \frac{1-\frac{\zeta_1}{\rho}}{1-\frac{\zeta_1}{\zeta_2}} \times \frac{\zeta_1}{\zeta_1 - z} + \frac{1-\frac{\zeta_2}{\rho}}{1-\frac{\zeta_2}{\zeta_1}} \times \frac{\zeta_2}{\zeta_2 - z},$$

and

$$\phi_q^-(z) = \frac{1+\frac{z}{\hat{\rho}}}{(1+\frac{z}{\hat{\zeta}_1})(1+\frac{z}{\hat{\zeta}_2})} = \frac{1-\frac{\hat{\zeta}_1}{\hat{\rho}}}{1-\frac{\hat{\zeta}_1}{\hat{\zeta}_2}} \times \frac{\hat{\zeta}_1}{\hat{\zeta}_1 + z} + \frac{1-\frac{\hat{\zeta}_2}{\hat{\rho}}}{1-\frac{\hat{\zeta}_2}{\hat{\zeta}_1}} \times \frac{\hat{\zeta}_2}{\hat{\zeta}_2 + z}.$$

We can identify $\phi_q^+(z)$ as the Laplace transform of a random variable ξ^+ whose distribution is equal to a mixture of exponential distributions with parameters $\{\zeta_1, \zeta_2\}$. And likewise, $\phi_q^+(z)$ is the Laplace transform of a random variable ξ^-, where $-\xi^-$ is a mixture of exponential distributions with parameters $\{\hat{\zeta}_1, \hat{\zeta}_2\}$. Namely, we have $S_{e(q)} \stackrel{d}{=} \xi^+$ and $I_{e(q)} \stackrel{d}{=} \xi^-$.

We will take the up-and-in barrier call option as an example to demonstrate the application of the Wiener-Hopf factorization. This option gives buyers the right but not obligation to buy a stock at some expiry time $T > 0$ for strike price $K > 0$ on the condition that the option is valid if the stock price rises above some barrier $B > A_0$ prior to time T. In mathematical terms the quantity we are interested in is

$$U(A_0, K, B, T) := e^{-rT}\mathbb{E}\Big[(A_T - K)^+ \mathbb{1}\Big(\sup_{0\le t\le T} A_t > B\Big)\Big].$$

By factoring out the constant S_0 and dropping the discounting term, we can instead solve the equivalent problem by determining

$$f(T) := \mathbb{E}[(e^{X_T} - k)^+ \mathbb{1}(S_T > b)].$$

Here $k = K/A_0$ and $b = \log(B/A_0)$. Now if we take the Laplace transform of $f(t)$, we may replace the deterministic time T by the random time $e(q)$ which is independent of the process X_t. We define the function $F(q)$ as

$$F(q) = \int_0^\infty q e^{-qt} f(t) \mathrm{d}t = \mathbb{E}[(e^{X_{e(q)}} - k)^+ \mathbb{1}(S_{e(q)} > b)]. \tag{9}$$

Accordingly, we can solve our problem if we can determine $F(q)$ and then invert the Laplace transform to recover $f(t)$. By observing (9), we can write it as

$$F(q) = \mathbb{E}[(e^{X_{e(q)} - S_{e(q)}} e^{S_{e(q)}} - k)^+ \mathbb{1}(S_{e(q)} > b)].$$

Therefore we have rewritten the problems in terms of $S_{e(q)}$ and $X_{e(q)} - S_{e(q)}$, which are independent random variables as we mentioned above, also we have obtained probability density functions for $S_{e(q)}$ and $X_{e(q)} - S_{e(q)}$ (has the same distribution as $I_{e(q)}$). Therefore we are able to get an explicit expression for $F(q)$. More details can be seen in Jeannin and Pistorius [30].

3 Complex analysis and special functions

In this chapter, we mention some important theorems in complex analysis which are used in the dissertation. We also introduce the Mellin transform and inverse Mellin transform, which are the essential tools in Chapter 4 and Chapter 6. Furthermore, we include Meijer G-functions and many of their properties, and those properties will be mainly used in Chapter 6.

3.1 Some important theorems and Mellin transform

We introduce the identity theorem in the following, which will be used to prove Proposition 4 in Chapter 6.

Theorem 2. *Let D be a domain and suppose that f_1, f_2 are analytic functions defined on D. Then if $S = \{z \in D : f_1(z) = f_2(z)\}$ has a limit point in D, we must have $S = D$, which means $f_1(z) = f_2(z)$ for all $z \in D$.*

See Theorem 6.9, in [3].

There are some known facts coming from this theorem. For example, the exponential function e^z on the complex plane is the unique analytic function that agrees on the real line $\mathbb{R}$ with e^x.

Theorem 3. *If Γ is a simple closed curve, which traverses counterclockwise in the complex plane and f is analytic except for some points $z_1, z_2, \cdots, z_n$ inside the contour Γ, then*

$$\int_\Gamma f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n} \mathrm{Res}(f(z) : z = z_k).$$

See Theorem 10.5, in [3].

This theorem is called Cauchy residue theorem, which will be an essential tool for computing the probability density function of $I_{x,q}$ in Chapter 6.

The following is Liouville's theorem. we will use it to prove Lemma 5 in Chapter 8.

Theorem 4. *A bounded analytic function in the complex plane $\mathbb{C}$ is constant.*

See Theorem 5.10, in [3].

Definition 2. The Mellin transform of a function f is

$$M(s) = \int_0^\infty x^{s-1} f(x) \mathrm{d}x.$$

Note that, when we refer to the Mellin transform of a random variable X, it means $M(X, s) := E(X^{s-1})$.

Theorem 5. *If $M(s)$ is analytic in the strip $a < \mathrm{Re}(s) < b$, and if it tends to zero uniformly as $|Im(s)| \to \infty$ for any real value c between a and b, with its integral along such a line converging absolutely, then we have*

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s} M(s) \mathrm{d}s, \tag{10}$$

where the right hand side of (10) *is the inverse Mellin transform.*

See Theorem 4.3.3, in [69].

We will take the function $\Gamma(s)$ as an example to demonstrate how to apply the Cauchy residue theorem to obtain its Inverse Mellin transform. Letting $f(x) = e^{-x}$, we have the known result

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx.$$

Letting $g(x)$ denote the inverse Mellin transform

$$g(x) := \frac{1}{2\pi \mathrm{i}} \int_{1+\mathrm{i}\mathbb{R}} x^{-s} \Gamma(s) \mathrm{d}s. \tag{11}$$

Since we have the identity

$$\Gamma(s) = \frac{\Gamma(s+1)}{s},$$

so we can get for $\mathrm{Re}(s) < 0$

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}.$$

Here n is the smallest integer such that $\mathrm{Re}(s) + n + 1 > 0$. Therefore, by analytic continuation, $\Gamma(s)$ is analytic in the whole complex plane except at $0, -1, -2, \cdots, -n, \cdots$. We observe that those points are simple poles of $\Gamma(s)$.

Here we include Stirling's asymptotic formula for gamma functions, which will be used many times in the dissertation,

$$|\Gamma(a + \mathrm{i}b)| = \sqrt{2\pi} \exp(-\pi|b|/2 + (a - 1/2)\ln(|b|) + O(1)), \quad b \to \infty, \tag{12}$$

which holds uniformly in a on compact subsets of $\mathbb{R}$.

According to the asymptotic result (12), $\Gamma(s)$ decays exponentially as $\mathrm{Im}(s) \to \infty$. This fact enables us to use the infinite contour and the residue theorem to derive the following expression

$$\frac{1}{2\pi\mathrm{i}} \int_{1/2+\mathrm{i}\mathbb{R}} x^{-s}\Gamma(s)\mathrm{d}s = \mathrm{Res}(x^{-s}\Gamma(s) : s = 0) + \frac{1}{2\pi\mathrm{i}} \int_{-1/2+\mathrm{i}\mathbb{R}} x^{s}\Gamma(s)\mathrm{d}s. \tag{13}$$

With the same idea, we keep shifting the contour, we can get

$$\frac{1}{2\pi\mathrm{i}} \int_{1/2+\mathrm{i}\mathbb{R}} x^{-s}\Gamma(s)\mathrm{d}s = \sum_{j=0}^{m-1} \mathrm{Res}(x^{-s}\Gamma(s) : s = -j) + \frac{1}{2\pi\mathrm{i}} \int_{-m+1/2+\mathrm{i}\mathbb{R}} x^{s}\Gamma(s)\mathrm{d}s. \tag{14}$$

For any m, since we have

$$\Gamma(s) = \frac{1}{s+m} \frac{\Gamma(s+m+1)}{s(s+1)\cdots(s+m-1)}, \tag{15}$$

thus the residue of $\Gamma(s)$ at $-m$ is given by:

$$\mathrm{Res}(\Gamma(s) : s = -m) = \lim_{s \to -m} (s+m)\Gamma(s). \tag{16}$$

When $s = -m$,

$$\Gamma(s+m+1) = \Gamma(1) = 1,$$

and

$$s(s+1)\cdots(s+m-1) = (-1)^m m!.$$

So the residue at $-m$ is

$$\mathrm{Res}(\Gamma(s) : s = -m) = \frac{(-1)^m}{m!}. \tag{17}$$

Since x^s is an analytic function, thus $\mathrm{Res}(x^{-s}\Gamma(s) : s = -m) = x^m \mathrm{Res}(\Gamma(s) : s = -m) = \frac{(-1)^m x^m}{m!}$. By letting $m \to \infty$,

$$\frac{1}{2\pi\mathrm{i}} \int_{1/2+\mathrm{i}\mathbb{R}} x^{-s}\Gamma(s)\mathrm{d}s = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!} = e^{-x} \tag{18}$$

As $\Gamma(s)$ is analytic for $\mathrm{Re}(s) > 0$, by applying Cauchy residue theorem

$$\int_{1+\mathrm{i}\mathbb{R}} x^{-s}\Gamma(s)\mathrm{d}s = \int_{1/2+\mathrm{i}\mathbb{R}} x^{-s}\Gamma(s)\mathrm{d}s.$$

Therefore $g(x) = e^{-x} = f(x)$.

3.2 Hypergeometric function and Meijer G-function

Definition 3. We define the hypergeometric function

$$ {}_pF_r\left(\begin{matrix} b_1,\dots,b_p \\ a_1,\dots,a_r \end{matrix}\middle| z\right) := \sum_{k\geq 0} \frac{(b_1)_k\dots(b_p)_k}{(a_1)_k\dots(a_r)_k} \times \frac{z^k}{k!}, \tag{19}$$

where $(a)_k := \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. When $p < r+1$ it is an entire function of z and when $p = r+1$ the series in (19) converges only for $|z| < 1$ (though the function can be continued analytically in the cut complex plane).

We will also work with the regularized hypergeometric function

Definition 4.

$$ {}_p\Phi_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\middle| z\right) = \Gamma\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right] {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\middle| z\right), \tag{20}$$

where

$$ \Gamma\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right] := \frac{\prod_{i=1}^p \Gamma(a_i)}{\prod_{j=1}^q \Gamma(b_j)}. \tag{21}$$

This definition will be used widely in Chapter 6

Definition 5. We define *the q-Pochhammer symbol*

$$ (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}, \quad a \in \mathbb{C},\ |q| < 1,\ k \in \mathbb{Z}, \tag{22}$$

where $(w;q)_\infty := \prod_{j\geq 0}(1-wq^j)$. *The basic hypergeometric function* is defined as follows

$$ {}_{r+1}\phi_r\left(\begin{matrix} b_1,b_2,\dots,b_{r+1} \\ a_1,a_2,\dots,a_r \end{matrix}\middle| z\right) := \sum_{k\geq 0} \frac{(b_1;q)_k(b_2;q)_k\dots(b_{r+1};q)_k}{(a_1;q)_k(a_2;q)_k\dots(a_r;q)_k} \times \frac{z^k}{(q;q)_k}. \tag{23}$$

It is easy to see that the above series converges when $|q| < 1$ and $|z| < 1$.

In the following we define Meijer G-functions and discuss some of their properties. We begin with four non-negative integers m, n, p and q and two vectors $\mathbf{a} = (a_1,\dots,a_p) \in \mathbb{C}^p$ and $\mathbf{b} = (b_1,\dots,b_q) \in \mathbb{C}^q$ and define for $0 \leq m \leq q, 0 \leq n \leq p$,

$$ \mathcal{G}^{mn}_{pq}\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\middle| s\right) := \frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)}. \tag{24}$$

We denote

$$\underline{b}(m) := \min_{1\le j\le m} \mathrm{Re}(b_j), \quad \bar{a}(n) := \max_{1\le j\le n} \mathrm{Re}(a_j), \tag{25}$$

and we set $\underline{b}(0) = +\infty$ and $\bar{a}(0) = -\infty$. When the parameters m, n, $\mathbf{a}$ and $\mathbf{b}$ are fixed we will write simply $\underline{b} = \underline{b}(m)$ and $\bar{a} = \bar{a}(n)$.

Definition 6. Assume that parameters $m, n, p, q, \mathbf{a}$ and $\mathbf{b}$ satisfy the following two conditions

$$\text{Condition A:} \quad \bar{a} - 1 < \underline{b} \tag{26}$$

$$\text{Condition B:} \quad p + q < 2m + 2n. \tag{27}$$

We define the *Meijer G-function* as follows

$$G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) := \frac{1}{2\pi \mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} \mathcal{G}_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| s\right) x^{-s} \mathrm{d}s, \tag{28}$$

where $x > 0$ and $\lambda \in (-\underline{b}, 1 - \bar{a})$.

Let us explain why the Meijer G-function is well-defined. The condition (26) is needed because it separates the poles of $\Gamma(b_j + s)$ from the poles of $\Gamma(1 - a_j - s)$ in the numerator in (24), thus the function $s \mapsto \mathcal{G}_{pq}^{mn}(\mathbf{a}, \mathbf{b}|s)$ is analytic in the strip $-\underline{b} < \mathrm{Re}(s) < 1 - \bar{a}$. Condition (27) and the asymptotic result (12) for the gamma function ensure that the integrand in (28) converges to zero exponentially fast as $\mathrm{Im}(s) \to \infty$, and it is easy to check that (28) defines the Meijer G-function as an analytic function in the sector $|\arg(z)| < (m + n - (p + q)/2)\pi$.

Remark 2. Our definition of the Meijer G-function is sufficient for our purposes, but it is not the most general possible. One could relax conditions (26) and (27) by appropriately deforming the contour of integration in (28) . See Section 8.2 in Prudnikov et al. [62] for more details.

We record here some properties of the Meijer G-function, which are used elsewhere in this dissertation. These properties and many other results on Meijer G-functions can be found in Gradshteyn and Ryzhik [29]. In Section 8.4 in Prudnikov et al. [62] one can find an extensive collection of formulas expressing various special functions in terms of Meijer G-functions.

(i)

$$x^c G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}+c\\ \mathbf{b}+c\end{matrix}\middle| x\right). \tag{29}$$

(ii)

$$G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = G_{qp}^{nm}\left(\begin{matrix}1-\mathbf{b}\\ 1-\mathbf{a}\end{matrix}\middle| x^{-1}\right). \tag{30}$$

(iii) For any $\epsilon > 0$

$$G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = \begin{cases} O(x^{\underline{b}-\epsilon}), & \text{as } x \to 0^+, \\ O(x^{\bar{a}-1+\epsilon}), & \text{as } x \to +\infty. \end{cases} \tag{31}$$

(iv) Assume that $b_j - b_k \notin \mathbb{Z}$ for $1 \le j < k \le m$. If $p < q$ or $p = q$ and $|x| < 1$ we have

$$\begin{aligned} G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = \sum_{k=1}^{m} & \frac{\prod\limits_{\substack{1\le j\le m\\ j\neq k}} \Gamma(b_j - b_k) \prod\limits_{j=1}^{n} \Gamma(1 + b_k - a_j)}{\prod\limits_{j=m+1}^{q} \Gamma(1 + b_k - b_j) \prod\limits_{j=n+1}^{p} \Gamma(a_j - b_k)} \\ & \times x^{b_k} {}_pF_{q-1}\left(\begin{matrix} 1 + b_k - a_1, \dots, 1 + b_k - a_p \\ 1 + b_k - b_1, \dots, *, \dots, 1 + b_k - b_q \end{matrix}\middle| (-1)^{p-m-n} x\right), \end{aligned} \tag{32}$$

where the asterisk in the function ${}_pF_{q-1}$ denotes the omission of the k-th parameter. If $p > q$ or $p = q$ and $|x| > 1$, the corresponding representation of the Meijer G-function in terms of ${}_qF_{p-1}$ functions can be obtained using (30) and (32).

The following variant of (32) also will be used frequently:

$$\begin{aligned} G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = \pi^{m+n-p-1} \sum_{k=1}^{m} & \frac{\prod\limits_{j=n+1}^{p} \sin(\pi(a_j - b_k))}{\prod\limits_{\substack{1\le j\le m\\ j\neq k}} \sin(\pi(b_j - b_k))} \\ & \times x^{b_k} {}_p\Phi_{q-1}\left(\begin{matrix} 1 + b_k - a_1, \dots, 1 + b_k - a_p \\ 1 + b_k - b_1, \dots, *, \dots, 1 + b_k - b_q \end{matrix}\middle| (-1)^{p-m-n} x\right). \end{aligned} \tag{33}$$

Here ${}_p\Phi_{q-1}$ is the regularized hypergeometric function defined in (20). This formula can be easily derived from (32) by using *the reflection formula* for the Gamma function:

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \tag{34}$$

(v) If one of the parameter a_j (for $j = 1, 2, \cdots, n$) coincides with one of the parameters b_j (for $j = m + 1, m + 2, \cdots, q$), the order of the G-function decreases. For example

$$G_{pq}^{mn}\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_{q-1}, a_1 \end{matrix}\middle| x\right) = G_{p-1,q-1}^{m,n-1}\left(\begin{matrix} a_2, \cdots, a_p \\ b_1, \cdots, b_{q-1} \end{matrix}\middle| x\right). \tag{35}$$

An analogous relationship occurs when one of the parameters b_j (for $j = 1, 2, \cdots, m$) coincides with one of the parameters a_j (for $j = n + 1, \cdots, p$). In this case, it is m and not n that decreases by one unit.

$$G_{pq}^{mn}\left(\begin{matrix} a_1, \cdots, a_p \\ a_p, b_2 \cdots, b_q \end{matrix}\middle| x\right) = G_{p-1,q-1}^{m-1n}\left(\begin{matrix} a_1, \cdots, a_{p-1} \\ b_1, \cdots, b_{q-1} \end{matrix}\middle| x\right). \tag{36}$$

(vi)

$$\int_1^\infty x^{\alpha-1} G_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| zx\right)\mathrm{d}x = G_{p+1,q+1}^{m+1,n}\left(\begin{matrix}\mathbf{a}, 1-\alpha\\ -\alpha, \mathbf{b}\end{matrix}\middle| z\right). \tag{37}$$

(vii) For $p \leq q$ and $\mathrm{Re}(\alpha) > 0$,

$$\int_0^1 x^{\alpha-1} {}_pF_q\left(\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\middle| zx\right)\mathrm{d}x = \alpha^{-1} \times {}_{p+1}F_{q+1}\left(\begin{matrix}\alpha, a_1, \ldots, a_p\\ \alpha+1, b_1, \ldots, b_q\end{matrix}\middle| z\right). \tag{38}$$

4 Exponential functionals of Lévy processes

In this chapter we introduce exponential functionals of Lévy processes and their application in pricing the Asian option. We state a verification result for determining the Mellin transform of the exponential functional and we demonstrate how the verification result may be applied in the case of the Kou process. Furthermore, we show how to apply the theory of Meijer G-functions to obtain the probability density function of the exponential functional.

4.1 Introduction

The exponential functional of a Lévy process is a very interesting object, which has many applications in such areas as self-similar Markov processes and branching processes. An overview of this topic, covering both theory and applications, can be found in [5]. Recently it has been popularized in the finance literature by its applications to the pricing of Asian options in financial markets.

The exponential functional is the key to calculating Asian option price. Asian options are a special type of path dependent option contracts whose payoff is contingent upon the average price of underlying equity over the contract period. We denote the equity price: $A_t = A_0 e^{X_t}$. Here X_t is a stochastic process, A_0 is the initial equity value. In our assumption the measure $\mathbb{P}$ is risk neutral. We are interested in calculating the price of an arithmetic average, continuously monitored, fixed strike price Asian call option, which is given by

$$C(A_0, K, T) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T A_0 e^{X_u}\mathrm{d}u - K\right)^+\right].$$

Here T is the expiry time and K is the strike price. By factoring out the constants $\frac{1}{T}$ and A_0, we get

$$C(A_0, K, T) = e^{-rT} \times A_0/T \times f(TK/A_0, T),$$

where

$$f(k,t) = \mathbb{E}\left[\left(\int_0^t e^{X_u}\mathrm{d}u - k\right)^+\right]. \tag{39}$$

We see that determining $f(k,t)$ is equivalent to obtaining the option price, and $\int_0^t e^{X_u}\mathrm{d}u$ is the key to determining $f(k,t)$.

Let us define the exponential functional of a process X to be

$$J_t := \int_0^t e^{X_s}\mathrm{d}s.$$

There has been a vast amount of work in the literature devoted to the distribution of J_t. Yor [68] employs the Lamperti transformation relating the geometric Brownian motion and the exponential functional to a Bessel process. Linetsky [47] starts with an identity in distribution

$$J_t \overset{d}{=} U_t := e^{X_t} \int_0^t e^{-X_s}\, \mathrm{d}s,$$

and the fact that the latter is a diffusion process and then applies the eigenfunction expansion technique to determine the distribution of U_t. Vecer [67] applies the change of measure to produce a partial differential equation satisfied by the Asian option price. The above list is by no means comprehensive. More applications of exponential functionals of Brownian motion and references can be found in Carmona et al. [10] and Matsumoto and Yor [49, 50].

As we have mentioned in Chapter 2, several empirical studies have demonstrated that the geometric Brownian motion does not adequately explain many stylized facts of empirical equity returns, such as the asymmetric leptokurtic log-returns and the volatility smile. One popular solution to this problem is to use Lévy processes to model log-returns. When working with exponential functionals of Lévy processes, it is easier to study the distribution of the exponential functional of the form

$$I_q := J_{\mathrm{e}(q)} = \int_0^{\mathrm{e}(q)} e^{X_s}\, \mathrm{d}s, \tag{40}$$

where $\mathrm{e}(q)$ is an exponential random variable with mean $1/q$, independent of the process X.

Instead of investigating I_q directly, one good approach is to use the Mellin transform to determine the distribution of the exponential functional. The Mellin transform of I_q has some good properties which enable us to determine its expression. The Mellin transform of I_q is defined as

$$\mathcal{M}_q(s) := \mathbb{E}[I_q^{s-1}],$$

where $s \in 1 + \mathrm{i}\mathbb{R}$ (s can be extended to a strip later).

We denote the probability density function of I_q by $p(x)$. There has been an extensive literature covering the asymptotic behavior of $p(x)$ as $x \to \infty$ (see [51],[12],[13]) or as $x \to 0^+$ (see [8], [57]). At the same time, the distribution of I_q is known explicitly for some processes with one-sided jumps: standard Poisson process, Brownian motion with drift, one particular spectrally negative Lamperti-stable process (see for instance [12], [45], [58]), spectrally positive Lévy process satisfying the Cramér's condition (see for example [59]). In the last several years, the distribution of I_q have been obtained explicitly for processes with double-sided jumps. First of all, Cai and Kou [9] obtained the distribution of I_q for hyper-exponential Lévy processes. Additionally, they have shown I_q has the same distribution as a product of a sequence of independent gamma and beta random variables. These results were later extended to processes with jumps of rational transform in [39] and to meromorphic Lévy process in [40]. Furthermore, in [39], a verification technique based on a functional equation is proposed to identify the Mellin transform of I_q, which considerably simplifies the derivation of many results on exponential functionals. A rather easy way to compute the probability density function $p(x)$ explicitly has been presented in [39], which is based on the theory of Meijer G-functions. By now the analytical theory behind the exponental functionals I_q is rather well understood, see the papers by Patie and Savov [60, 61].

As stated above, the Mellin transform is the key to derive the distribution of I_q. Here we will introduce two important results regarding the expression of the Mellin transform. We will prove a lemma due to Maulik and Zwart [51], which shows that the Mellin transform satisfies a functional equation involving the Laplace exponent of the process. Then we will show a theorem due to Kuznetsov and Pardo [41], that any function satisfying this functional equation and some other technical condition must be the Mellin transform of I_q.

Lemma 1. *Let $q > 0$ and X be a Lévy process with Laplace exponent $\psi(s)$. If $s > 0$ and $q - \psi(s) > 0$, we have*

$$\mathcal{M}_q(s+1) = \frac{s}{q - \psi(s)} \mathcal{M}_q(s), \tag{41}$$

where the equality is interpreted to mean that both sides can be infinite.

Proof. We start by integrating the following identity

$$\frac{\mathrm{d}}{\mathrm{d}u}(J_t - J_u)^s = -s(J_t - J_u)^{s-1} e^{X_u},$$

over the interval $[0, t]$ to obtain

$$J_t^s = s \int_0^t (I_t - I_u)^{s-1} e^{X_u} \mathrm{d}u. \tag{42}$$

Now, we observe that

$$J_t - J_u = e^{X_u} \int_0^{t-u} e^{(X_{u+v} - X_u)} \mathrm{d}v. \tag{43}$$

Here we have the fact that the process $\tilde{X}$, defined by $\tilde{X}_v := X_{t+v} - X_t$, is independent of the process X up until time t and has the same distribution as X. From this fact we have $\int_0^{t-u} e^{(X_{u+v} - X_u)} \mathrm{d}v \overset{d}{=} J_{t-u}$. By plugging (43) into (42), taking expectation and applying Tonelli's theorem,

$$\mathbb{E}[J_t^s] = s \int_0^t e^{u\psi(s)} \mathbb{E}[J_{t-u}^{s-1}] \mathrm{d}u. \tag{44}$$

By plugging (44) into the equation below

$$\mathbb{E}[I_q^s] = q \int_0^\infty e^{-qt} \mathbb{E}[J_t^s] \mathrm{d}t,$$

and applying Tonelli's theorem again, with the fact that $q - \psi(s) > 0$, we yield the result

$$\mathbb{E}[I_q^s] = \frac{s}{q - \psi(s)} \mathbb{E}[I_q^{s-1}].$$

$\square$

Now we will state and prove the verification result. The statement of this theorem and the associated proof originally appeared in [41], but here we will use the statement and brief proof from [39].

Theorem 6. *Assume that Cramér's condition is satisfied: there exists $z_0 > 0$ such that the Laplace exponent $\psi(z)$ is finite for all $z \in (0, z_0)$ and $\psi(\theta) = q$ for some $\theta \in (0, z_0)$. If $f(s)$ satisfies the following three properties*

(i) $f(s)$ is analytic and zero-free in the strip $\mathrm{Re}(s) \in (0, 1+\theta)$,

(ii) $f(1) = 1$ and $f(s+1) = sf(s)/(q - \psi(s))$ for all $s \in (0, \theta)$,

(iii) $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ as $\mathrm{Im}(s) \to \infty$, $\mathrm{Re}(s) \in (0, 1+\theta)$,

then $\mathcal{M}_q(s) \equiv f(s)$ for $\mathrm{Re}(s) \in (0, 1+\theta)$.

Proof. We present the main steps of the proof here. First of all, the Cramér's condition and Lemma 2 in [64] imply that $\mathcal{M}_q(s)$ can be extended to an analytic function in the vertical strip $\mathrm{Re}(s) \in (0, 1+\theta)$. Since $|\mathcal{M}_q(s)| < \mathcal{M}_q(\mathrm{Re}(s))$, we see that $\mathcal{M}_q(s)$ is bounded in the strip $\mathrm{Re}(s) \in [\theta/2, 1+\theta/2]$. Also from the Cramér's condition, we see that the sufficient conditions of Lemma 1 are satisfied on the interval $(0, \theta)$, so we know $\mathcal{M}_q(s)$ satisfies the same functional equation as $f(s)$. Therefore the ratio $F(s) = \mathcal{M}_q(s)/f(s)$ is a periodic function: $F(s+1) = F(s)$; And due to condition (i) $F(s)$ can be extended to an analytic function in the entire complex plane. Finally, condition (iii) and boundedness of $\mathcal{M}_q(s)$ imply that $F(s) = o(\exp(2\pi|\mathrm{Im}(s)|))$ in the entire complex plane, and any function which is analytic, periodic with period equal to one, and which satisfies this upper bound must be identically equal to a constant. Since $F(1) = 1$, we conclude that $F(s) \equiv 1$, that is $\mathcal{M}_q(s) \equiv f(s)$. $\square$

This verification result is a convenient tool which allows us to explicitly identify the Mellin transform of I_q as a solution to the functional equation $f(s+1) = sf(s)/(q - \psi(s))$. What makes this equation analytically tractable is that if $s/(q - \psi(s))$ is a rational function, then the function $f(s)$ can be connected with Gamma functions.

In order to apply the verification result, we need a candidate function $f(s)$ that satisfies the three criteria. In the following section, we will show how to construct this function for the Kou process.

4.2 Application in the Kou process

We will take the Kou process as an example to show how to get the Mellin transform of I_q and then how to obtain the probability density function $p(x)$. First let us recall the Kou process.

The Kou process X is defined in Chapter 2, its Laplace exponent is equal to

$$\psi(z) = \mu z + \frac{\sigma^2}{2} z^2 + \lambda p \frac{z}{\rho - z} - \lambda(1-p)\frac{z}{\hat{\rho} + z}.$$

For $q > 0$ the rational function $\psi(z) = q$ has four zeros $\{-\hat{\zeta}_2, -\hat{\zeta}_1, \zeta_1, \zeta_2\}$ and two poles $\{-\hat{\rho}, \rho\}$ which satisfy the interlacing property

$$-\hat{\zeta}_2 < -\hat{\rho} < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho < \zeta_2.$$

Theorem 7. *Assume $q > 0$, for $\mathrm{Re}(s) \in (0, 1 + \zeta_1)$ we have*

$$\mathbb{E}[I_q^{s-1}] = A^{1-s}\Gamma(s)\frac{\mathcal{G}(s)}{\mathcal{G}(1)}, \tag{45}$$

where

$$\mathcal{G}(s) = \frac{\Gamma(1+\zeta_1 - s)\Gamma(1+\zeta_2 - s)\Gamma(\hat{\rho} + s)}{\Gamma(1+\rho - s)\Gamma(\hat{\zeta}_1 + s)\Gamma(\hat{\zeta}_2 + s)}, \tag{46}$$

and the constant $A = \frac{\sigma^2}{2}$.

Proof. Our approach is to solve the functional equation of Theorem 6 (ii)

$$f(s+1) = sf(s)/(q - \psi(s)), \tag{47}$$

and then verify that one of our solutions satisfies the remaining requirements of the verification result. In deriving a solution, we will take advantage of the fact that $\psi(s)$ is a rational function so that we may write

$$\frac{s}{q - \psi(s)} = \frac{s(\rho - s)(\hat{\rho} + s)}{A(\zeta_1 - s)(\zeta_2 - s)(s + \hat{\zeta}_1)(s + \hat{\zeta}_2)}, \tag{48}$$

where $A = \frac{\sigma^2}{2}$.

By observing equation (47) and each factor of (48), we find that it is very similar to the recursion formula for the gamma function

$$\Gamma(s+1) = s\Gamma(s).$$

This is precisely what we will use to find a solution. Let us consider each factor of (48) separately and solve simpler functional equation of the type

$$f_1(s+1) = (s+a)f_1(s), \qquad f_2(s+1) = (a-s)f_2(s),$$

and

$$f_3(s+1) = \frac{1}{s+a}f_3(s), \qquad f_4(s+1) = \frac{1}{a-s}f_4(s), \qquad f_5(s+1) = \frac{1}{A}f_5(s).$$

Here $a \in \{0, \zeta_1, \zeta_2, \hat{\zeta}_1, \hat{\zeta}_2, \rho, \hat{\rho}\}$. The first four equations can be solved by using the gamma function recursion formula. This approach yields solution of the form $f_1(s) = \Gamma(s+a)$, $f_2(s) = \frac{1}{\Gamma(a-s+1)}$, $f_3(s) = \frac{1}{\Gamma(s+a)}$ and $f_4(s) = \Gamma(a-s+1)$. The final equation can be solved by guessing. It has the solution $f_5(s) = (A)^{1-s}$. These facts together with our knowledge of the domain of the gamma function, demonstrate that the function (as a product of those functions we get)

$$f(s) = (A)^{1-s} \times \Gamma(s) \times \frac{\Gamma(1+\zeta_1-s)\Gamma(1+\zeta_2-s)}{\Gamma(1+\rho-s)} \times \frac{\Gamma(\hat{\rho}+s)}{\Gamma(\hat{\zeta}_1+s)\Gamma(\hat{\zeta}_2+s)} \tag{49}$$

solves equation (47) for $\mathrm{Re}(s) \in (0, \zeta_1)$ and is analytic and zero free for $\mathrm{Re}(s) \in (0, 1+\zeta_1)$. Further, the function $h(s) = \frac{f(s)}{f(1)}$ satisfies $h(1) = 1$. We observe that $h(s)$ can be a candidate of $\mathbb{E}[I_q^{s-1}]$.

Now we have some insight of how to get the candidate function, the remaining is to verify the criteria of Theorem 6. Namely, let us demonstrate that $h(s)$ is the Mellin transform of the exponential functional. We have already checked the condition in (i) and (ii). The definition of ρ and ζ_1 show that Cramér's condition is satisfied for $z_0 = \rho$ and $\theta = \zeta_1$. In order to check the asymptotic condition in (iii) holds, we use the asymptotic results (12), which ensures that we can write $|h(s)|^{-1}$ as

$$|h(s)|^{-1} = f(1) \times A^{\mathrm{Re}(s)-1} \times \exp(\pi|\mathrm{Im}(s)|/2 + c\ln(|\mathrm{Im}(s)|) + O(1)), \qquad |\mathrm{Im}(s)| \to \infty,$$

where c is a constant depends on $\mathrm{Re}(s)$. This shows that $|h(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ as $\mathrm{Im}(s) \to \infty$, namely the condition in (iii) holds. Therefore, $h(s) \equiv \mathcal{M}_q(s)$ for $\mathrm{Re}(s) \in (0, 1+\zeta_1)$. $\square$

Cai and Kou [9] have shown that I_q has the same distribution as a product of gamma and beta random variable for hyper-exponential Lévy processes. We will demonstrate it in the Kou process (the Kou process is a special case of hyper-exponential Lévy processes). In the following proposition, we let $G_{(\alpha,\beta)}$ stands for a gamma random variable with shape and scale parameters α and β respectively; we let $B_{(\alpha,\beta)}$ stands for a beta random variable with shape parameters α and β respectively.

Proposition 1. *Let X be the Kou process, and assume $q > 0$, then*

$$I_q \stackrel{d}{=} \frac{1}{A} \frac{B_{(1,\hat{\zeta}_1)} B_{(\hat{\rho}+1,\hat{\zeta}_2-\hat{\rho})}}{G_{(\zeta_2,1)} B_{(\zeta_1,\rho-\zeta_1)}}. \tag{50}$$

Proof. First let us show that the Mellin transforms of a gamma random variable $G_{(\alpha,\beta)}$ and beta random variable $B_{(\alpha,\beta)}$ are given by

$$M(G,s) = \beta^{1-s} \frac{\Gamma(s+\alpha-1)}{\Gamma(\alpha)}, \quad \mathrm{Re}(s) + \alpha - 1 > 0,$$

$$M(B,s) = \frac{\Gamma(\alpha+s-1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+s-1)}, \quad \mathrm{Re}(s) + \alpha - 1 > 0,$$

respectively.

We can rearrange (45) in this way

$$\mathbb{E}[I_q^{s-1}] = \left(\frac{1}{A}\right)^{s-1} \frac{\Gamma(s)\Gamma(1+\hat{\zeta}_1)}{\Gamma(1)\Gamma(\hat{\zeta}_1+s)} \times \frac{\Gamma(\hat{\rho}+s)\Gamma(1+\hat{\zeta}_2)}{\Gamma(1+\hat{\rho})\Gamma(\hat{\zeta}_2+s)} \times \frac{\Gamma(1-s+\zeta_2)}{\Gamma(\zeta_2)} \times \frac{\Gamma(1+\zeta_1-s)\Gamma(\rho)}{\Gamma(\zeta_1)\Gamma(1-s+\rho)}, \tag{51}$$

since $\mathrm{Re}(s) \in (0, 1+\zeta_1)$, the condition $\mathrm{Re}(s) + \alpha - 1 > 0$ holds. Thus we know

$$\frac{\Gamma(s)\Gamma(1+\hat{\zeta}_1)}{\Gamma(1)\Gamma(\hat{\zeta}_1+s)} \quad \text{and} \quad \frac{\Gamma(\hat{\rho}+s)\Gamma(1+\hat{\zeta}_2)}{\Gamma(1+\hat{\rho})\Gamma(\hat{\zeta}_2+s)}$$

are the Mellin transforms of $B_{(1,\hat{\zeta}_1)}$ and $B_{(1+\hat{\rho},\hat{\zeta}_2-\hat{\rho})}$ respectively.

Since by the definition of Mellin transform, if the condition $\mathrm{Re}(2-s)+\alpha-1 > 0$ holds, we have

$$M(1/G, s) = M(G, 2-s) \quad \text{and} \quad M(1/B, s) = M(B, 2-s).$$

Therefore, we have

$$\frac{\Gamma(1-s+\zeta_2)}{\Gamma(\zeta_2)} \quad \text{and} \quad \frac{\Gamma(1+\zeta_1-s)\Gamma(\rho)}{\Gamma(\zeta_1)\Gamma(1-s+\rho)}$$

are the Mellin transform of $G_{(\zeta_2,1)}$ and $B_{(\zeta_1,\rho-\zeta_1)}$ respectively. It also can be checked that the condition $\mathrm{Re}(2-s) + \alpha - 1 > 0$ holds.

Therefore I_q has the same distribution as a product of gamma and beta random variables. □

Remark 3. We can observe from Theorem 7, the right-hand side of (45) has more gamma functions in the numerator than in the denominator. This fact and the asymptotic formula (12) imply that $\mathbb{E}[I_q^{s-1}]$ decreases to zero exponentially fast as $|\mathrm{Im}(s)| \to \infty$, which implies(via the inverse Mellin transform) that the probability density function of I_q is a smooth function on $\mathbb{R}^+$.

Formula (45) gives us the Mellin transform of I_q, which uniquely characterizes the distribution of I_q via the inverse Mellin transform

$$p(x) = \frac{1}{2\pi \mathrm{i}} \int_{1+\mathrm{i}\mathbb{R}} \mathbb{E}[I_q^{s-1}] x^{-s} \mathrm{d}s. \tag{52}$$

In fact, the function $p(x)$ can be computed explicitly, and this can be achieved in a variety of ways. One approach (which is quite general) is to use the fact that the integrand in the right-hand of (45) is an analytic function in $\mathbb{C}$ except those simple poles (it is easy to observe that since $\Gamma(s)\mathcal{G}(s)$ is the product of gamma functions), and whose residues can be computed explicitly, therefore by shifting the contour of integration in (52) we will obtain convergent series representations for $p(x)$. The second approach is to use the theory of Meijer G-functions, which will be shown in the following.

Proposition 2. *Assume that $q > 0$, then the density function $p(x)$ can be expressed in terms of the Meijer G-function as follows*

$$p(x) = \frac{A}{\mathcal{G}(1)} G^{2,2}_{3,4}\Big(\begin{array}{c} 1, 1-\hat{\rho}, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{array} \Big| \frac{1}{Ax} \Big). \tag{53}$$

Proof. By plugging formula (45) into (52) and applying the definition of the Meijer-G function (28),

$$\begin{aligned} p(x) &= \frac{1}{2\pi \mathrm{i}} \int_{1+\mathrm{i}\mathbb{R}} A^{1-s} \frac{\Gamma(s)\mathcal{G}(s)}{\mathcal{G}(1)} x^{-s} \mathrm{d}s \\ &= \frac{A}{\mathcal{G}(1)} \times \frac{1}{2\pi \mathrm{i}} \int_{1+\mathrm{i}\mathbb{R}} \frac{\Gamma(s)\Gamma(1+\zeta_1-s)\Gamma(1+\zeta_2-s)\Gamma(\hat{\rho}+s)}{\Gamma(1+\rho-s)\Gamma(\hat{\zeta}_1+s)\Gamma(\hat{\zeta}_2+s)} (Ax)^{-s} \mathrm{d}s \\ &= \frac{A}{\mathcal{G}(1)} G^{2,2}_{4,3}\Big(\begin{array}{c} -\zeta_1, -\zeta_2, \hat{\zeta}_1, \hat{\zeta}_2 \\ 0, \hat{\rho}, -\rho \end{array} \Big| Ax \Big). \end{aligned} \tag{54}$$

Note that both conditions 26 and 27 are satisfied, since in our case we have

$$a = \max(-\zeta_1, -\zeta_2) = -\zeta_1, \tag{55}$$

$$b = \min(0, \hat{\rho}) = 0. \tag{56}$$

Thus $1 \in (-b, 1-a)$. The desired result (53) is obtained from applying the property (30) to the Meijer-G function in (54). □

5 Brownian motion

In this chapter, we state the concepts of continuous martingales and stochastic integrals, most of the results in Chapter 7 are based on them. Furthermore, we mention semimartingale and local time, for the purpose of introducing the Skew Brownian motion. Finally, we demonstrate how to employ the Skew Brownian motion to solve one particular type of SDEs.

5.1 Martingale and Stochastic Integrals

Definition 7. A filtration on the measurable space $(\Omega, \mathcal{F})$ is an increasing family $(\mathcal{F}_t)_{t\geq 0}$, of sub-$\sigma$-algebras of $\mathcal{F}$. In other words, for each t we have a sub-σ-algebra $\mathcal{F}_t$ and $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$.

A process X on $(\Omega, \mathcal{F})$ is adapted to the filtration $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for each t. It is obvious to see X is adapted to its natural filtration $\mathcal{F}_t^0{=}\sigma(X_s, s \leq t)$. It is the introduction of a filtration which allows for the parameter t to be really thought of as *time*. Heuristically speaking, the σ-algebra $\mathcal{F}_t$ is the collection of events which may occur before or at time t or, in other words, the set of possible "past events" up to time t.

Definition 8. The process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is said to be a submartingale (respectively, a supermartingale) if, for every $0 \leq s < t < \infty$, we have $\mathbb{E}(X_t|\mathcal{F}_t) \geq X_s$ $\mathbb{P}$-a.s.(respectively, $\mathbb{E}(X_t|\mathcal{F}_t) \leq X_s$). If $\mathbb{E}(X_t|\mathcal{F}_t) = X_s$, then $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale.

A martingale is both a submartingale and supermartingale.

Definition 9. A stopping time T relative to the filtration $(\mathcal{F}_t)$ is a positive r.v. with values in $[0, \infty]$, such that for every t,

$$\{T \leq t\} \in \mathcal{F}_t.$$

The class of sets A in $\mathcal{F}_\infty$ such that $A \cap \{T \leq t\} \in \mathcal{F}_t$ for all t is a σ-algebra denoted by $\mathcal{F}_T$. The sets in $\mathcal{F}_t$ are thought of as events which may occur before stopping time T. A Stopping time is thought of as the first time some event happens, for example the first time the stochastic process hits some boundary.

We will introduce the Optional Sampling theorem in the following, this theorem will be used often in Chapter 7.

Theorem 8. *If X is a martingale and S, T are two bounded stopping times with $S \leq T$, then*

$$X_S = \mathbb{E}[X_T|\mathcal{F}_S] \quad a.s. \tag{57}$$

If X is uniformly integrable, the family $\{X_S\}$ where S runs through the set of all stopping times is uniformly integrable and if $S \leq T$, then

$$X_S = \mathbb{E}[X_T|\mathcal{F}_S] \quad a.s. \tag{58}$$

See proof in Theorem II 3.2 in [63]

It is obvious that the Optional Sampling theorem generalizes the properties of martingales from constant times to stopping times. For example, if we replace S with s and T with t, we can observe that it is exactly the martingale property in Definition 8. In Chapter 7, we will use the former argument in Theorem 8 with the bounded stopping times. Furthermore, there are similar results for supermartingale and submartingale with bounded stopping times.

Definition 10. Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous process with $X_0 = 0$ a.s. If there exists a nondecreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\{\mathcal{F}_t\}$, such that $\{X_t^{(n)} = X_{t\wedge T_n}, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale for each $n \geq 1$ and $\mathbb{P}[\lim_{n\to\infty} T_n = \infty] = 1$, then we say that X is a continuous local martingale.

Theorem 9. *If $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion and X is a measurable, adapted process with $\mathbb{P}[\int_0^t X_s^2 \mathrm{d}s < \infty] = 1$ for every $0 \leq t < \infty$, then the stochastic integral $\int_0^t X_s \mathrm{d}B_s$ is a continuous local martingale.*

An adapted process with continuous paths is progressively measurable, and a progressively measurable process must be measurable (For more details, see Proposition I 4.8 and Definition IV 1.14 of [63]). And for readers' convenience, all the processes appear in Chapter 7 are measurable and adapted.

We call $B = B^1 + \mathrm{i}B^2$ the planar Brownian motion, where (B^1, B^2) is a pair of independent Brownian motion. It is also called a complex Brownian motion. We will introduce an important theorem which is known as the conformal invariance of complex Brownian motion.

Theorem 10. *If F is an entire and non constant function, B_t is a complex Brownian motion, then $F(B_t)$ is a time-changed complex Brownian motion.*

See proof in Theorem V 2.5 of [63].

Time-changed Brownian motion has the same paths as Brownian motion but possibly runs at a different speed.

Theorem 11. *Suppose B is a complex Brownian motion that starts from $(a + ib)$ $(b \neq 0)$, and let τ denote the first time B hits the real axis, namely, $\tau = \min\{t \geq 0; B_t^2 = 0\}$. Then B_τ^1, namely the hitting position on the real axis, has a Cauchy distribution:*

$$\mathbb{P}(B_\tau^1 \in \mathrm{d}x) = \frac{1}{\pi} \frac{|b|}{(x-a)^2 + b^2} \mathrm{d}x \tag{59}$$

5.2 Semimartingale and Local time

Definition 11. A continuous$(\mathcal{F}_t, P)$-semimartingale is a continuous process X that can be written as $X = M + A$, where M is a continuous $(\mathcal{F}_t, P)$-local martingale and A is a continuous adapted process of finite variation.

For example submartingales and supermartingales are semimartingales.

We will introduce the Tanaka's formula given by Theorem VI 1.2 of [63].

First, let us define the function $\mathrm{sgn}(x)$

Definition 12. $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x \leq 0$.

The following theorem is called Tanaka's formula, which will be used often in Chapter 7.

Theorem 12. *Suppose X is a continuous semimartingale. For any real number a, there exists an increasing continuous process L^a called the local time of X at a such that,*

$$|X_t - a| = |X_0 - a| + \int_0^t \mathrm{sgn}(X_s - a)\mathrm{d}X_s + L_t^a,$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbb{1}_{\{X_s > a\}}\mathrm{d}X_s + 1/2L_t^a,$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbb{1}_{\{X_s \leq a\}}\mathrm{d}X_s + 1/2L_t^a.$$

In particular, $|X - a|$, $(X - a)^+$ and $(X - a)^-$ are semimartingales.

From Corollary VI 1.9 of [63], L_t^a has an equivalent definition.

Theorem 13. *If X is a continuous semimartingale, then, almost surely,*

$$L_t^a(X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{[a, a+\epsilon)}(X_s)\mathrm{d}\langle X, X\rangle_s. \tag{60}$$

for every a and t.

In this dissertation, we are interested in the local time of X at 0, and also we have the corresponding differential form of the equation for the local time. For convenience, we will write L_t^0 as L_t.

$$\begin{aligned}
\mathrm{d}|X_t| &= \operatorname{sgn}(X_t)dX_t + \mathrm{d}L_t(X), \\
\mathrm{d}(X_t)^+ &= \mathbb{1}_{\{X_t>0\}}dX_t + 1/2\mathrm{d}L_t(X), \\
\mathrm{d}(X_t)^- &= -\mathbb{1}_{\{X_t\le 0\}}dX_t + 1/2\mathrm{d}L_t(X).
\end{aligned}$$

A different definition of the sign function will give a different version of local time for the same semimartingale X. Another definition of the sign function $\widetilde{\operatorname{sgn}}(x)$ is

Definition 13. $\widetilde{\operatorname{sgn}}(x) = 1$ if $x > 0$, $\widetilde{\operatorname{sgn}}(x) = 0$ if $x = 0$ and $\widetilde{\operatorname{sgn}}(x) = -1$ if $x < 0$.

This sign function is symmetric and we will call the corresponding local time as symmetric local time (see Exercise VI 1.25 of [63]).

Theorem 14. *Suppose X is a semimartingale, for any real number a, there exists an increasing continuous process $\tilde{L}^a$ called the symmetric local time of X at a such that,*

$$|X_t - a| = |X_0 - a| + \int_0^t \widetilde{\operatorname{sgn}}(X_s - a)\mathrm{d}X_s + \tilde{L}_t^a.$$

$\tilde{L}_t^a$ also has an equivalent definition

$$\tilde{L}_t^a(X) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t \mathbb{1}_{(a-\epsilon,a+\epsilon)}(X_s)\mathrm{d}\langle X, X\rangle_s. \tag{61}$$

for every a and t.

We will let $\tilde{L}_t$ denote $\tilde{L}_t^0$ for the symmetric local time of X at 0. The differential form will be

$$\mathrm{d}|X_t| = \widetilde{\operatorname{sgn}}(X_t)\mathrm{d}X_t + \mathrm{d}\tilde{L}_t(X),$$

The lack of symmetry in the last two identities in the Theorem 12 is due to the choice below of using left derivatives and the choice of sign function. In this dissertation, we will use this notation to denote the asymmetric sign function. Also we will see the importance of the value of the sign function sgn(x) when $x = 0$ in one of our proofs in Chapter 7.

It is known that in Itô's formula, if X is a continuous semimartingale, and f is a C^2-function,

$$\mathrm{d}f(X_t) = f^{'}(X_t)\mathrm{d}X_t + \frac{1}{2}f^{''}(X_t)\mathrm{d}\langle X, X\rangle_t. \tag{62}$$

By generalizing the functions in C^2 to convex functions, this leads to the Itô-Tanaka formula

Theorem 15. *If f is the difference of two convex functions and if X is a continuous semimartingale*

$$f(X_t) = f(X_0) + \int_0^t f_-^{'}(X_t)\mathrm{d}X_t + \frac{1}{2}\int_{\mathbb{R}} L_t^a f^{''}(\mathrm{d}a). \tag{63}$$

In particular, $f(x)$ is a semimartingale.

Here if f is still C^2, $f_-^{'}(x) = f^{'}(x)$, and $f^{''}(x)$ will just be the second derivative. However, if f is only convex, $f_-^{'}(x)$ is the left-hand derivative of $f(x)$, and $f^{''}(x)$ will be a positive measure in the sense of distribution, which is associated with the increasing function $f_-^{'}(x)$ (see Appendix Sect.3 in [63]). Therefore, for f the difference of two convex functions, $f_-^{'}(x)$ is still the left-hand derivative of $f(x)$, but $f^{''}(x)$ will be a measure associated with $f_-^{'}(x)$. The measure is actually the difference of two positive measures (see Theorem VI 1.5 in [63]).

5.3 Skew Brownian motion

According to [31], the skew Brownian motion $X_\alpha = \{X_\alpha(t), t \geq 0\}$, indexed by $0 \leq \alpha \leq 1$, is a diffusion process that can be intuitively constructed by the following procedure. Let $Z = \{Z(t), t \geq 0\}$ be a reflecting Brownian motion on $[0, \infty)$ and consider the excursions of Z away from the origin. Change the sign of each excursion independently with probability $1 - \alpha$ so that a given excursion is positive with probability α and negative with probability $1 - \alpha$.

The following from [31] will explain how to construct the skew Brownian motion X_α. Let

$$\sigma_\alpha^2(x) = \begin{cases} (1-\alpha)^2, & \text{if } x \geq 0, \\ \alpha^2, & \text{if } x < 0. \end{cases} \tag{64}$$

Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion on some probability space and set

$$Y_\alpha(t) = B(T_\alpha(t)), \tag{65}$$

where the time change T_α is defined by

$$t = \int_0^{T_\alpha(t)} \mathrm{d}u/\sigma_\alpha^2(B(u)). \tag{66}$$

Thus Y_α is a diffusion in natural scale with state space $\mathbb{R}$, the speed measure of Y_α is

$$m_\alpha(\mathrm{d}x) = 2\mathrm{d}x/\sigma_\alpha^2(x). \tag{67}$$

Next we let

$$r_\alpha(x) = \begin{cases} x/(1-\alpha), & \text{if } x \geq 0, \\ x/\alpha, & \text{if } x < 0. \end{cases} \tag{68}$$

We define X_α by

$$X_\alpha(t) = r_\alpha(Y_\alpha(t)). \tag{69}$$

Thus the scale function of the diffusion X_α is

$$s_\alpha(x) = \begin{cases} (1-\alpha)x, & \text{if } x \geq 0, \\ \alpha x, & \text{if } x < 0. \end{cases} \tag{70}$$

We can see s_α is the inverse of r_α. The construction of X_α is given by (65) and (69).

Harrison and Shepp [31] have introduced a very nice way to prove the existence and uniqueness of solution for a specific stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + \beta \mathrm{d}\tilde{L}_t(X), \qquad -1 \leq \beta \leq 1. \tag{71}$$

Furthermore, it was shown that the solution is exactly a skew Brownian motion with parameter $\alpha = (1+\beta)/2$. As we mentioned above, $\tilde{L}_t(X)$ is the symmetric local time of X_t at 0, B_t is a standard Brownian motion.

A natural question will be why the traditional way of proving the existence and uniqueness of solution for SDE can not apply directly to this particular equation (71)?

The answer is very easy, the traditional argument works for the form of SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + a(X_t)\mathrm{d}t, \tag{72}$$

but such argument is very hard to directly apply on the $\mathrm{d}\tilde{L}_t(X)$ term.

In this dissertation, we are interested in the same type of SDE as (71), but the local time will be the asymmetric one L_t. However, we will use the same technique as in [31] and show that the solution in our case exists and it is also unique. Furthermore, it is a skew Brownian motion.

Theorem 16. *Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion on some probability space, then the equation*

$$\mathrm{d}X_t = \mathrm{d}B_t + \beta \mathrm{d}L_t(X) \qquad \beta < 1/2, \tag{73}$$

has a unique solution X_t, X_t is adapted to the filtration $\{\mathcal{F}_t^B\}$. Furthermore, X_t is a skew Brownian motion with parameter $\frac{1}{2(1-\beta)}$.

Proof. First we show uniqueness. Assume X_t solves (73) with $X_0 = x_0$. We let $Y_t = s_\alpha(X_t)$, where s_α is the function we defined in (70), and $\alpha = \frac{1}{2(1-\beta)}$ (it is easy to see $0 < \alpha < 1$). Let f be the left derivative of the function s_α.

$$f(x) = \begin{cases} (1-\alpha), & \text{if } x > 0, \\ \alpha, & \text{if } x \leq 0. \end{cases} \tag{74}$$

The second derivative of s_α is $(1-2\alpha)\delta_0$, where δ_0 is a point mass at 0.

By applying the Itô-Tanaka formula to $s_\alpha(X_t)$.

$$s_\alpha(X_t) = s_\alpha(X_0) + \int_0^t f(X_s)\mathrm{d}X_s + \frac{1}{2}L_t(X)(1-2\alpha), \tag{75}$$

therefore,

$$\begin{aligned} \mathrm{d}Y_t &= \mathrm{d}s_\alpha(X_t) \\ &= f(X_t)\mathrm{d}X_t + \frac{1}{2}\mathrm{d}L_t(X)(1-2\alpha) \\ &= f(X_t)(\mathrm{d}B_t + \beta\mathrm{d}L_t(X)) + \frac{1}{2}\mathrm{d}L_t(X)(1-2\alpha) \\ &= f(X_t)\mathrm{d}B_t + f(0)\beta\mathrm{d}L_t(X) + \frac{1}{2}\mathrm{d}L_t(X)(1-2\alpha) \\ &= f(X_t)\mathrm{d}B_t = f(Y_t)\mathrm{d}B_t. \end{aligned} \tag{76}$$

In the fourth step, we have used the fact that $L_t(X)$ increase only when $X_t = 0$. In the fifth step we have used $\alpha\beta + \frac{1}{2}(1-2\alpha) = 0$. In the last step we have used the fact X_t and Y_t have the same sign because of the definition of s_α.

Now we know Y_t is a diffusion process and

$$Y_t = \int_0^t f(Y_u)\mathrm{d}B_u. \tag{77}$$

The theorem of Nakao [55] says that this stochastic differential equation (77) has unique solution Y, and the solution is adapted to the filtration $\{\mathcal{F}_t^B\}$ of the Brownian motion. Since we have $Y(t) = s_\alpha(X_t)$, therefore X_t is also unique, and X_t is adapted to the filtration $\{\mathcal{F}_t^B\}$.

For the existence of solution to the SDE (73), start with Y_t which satisfies (77), then we recall the function r_α in (68) and define

$$X_t = r_\alpha(Y_t). \tag{78}$$

Again, here $\alpha = \frac{1}{2(1-\beta)}$. Let g be the left derivative of function r_α.

$$g(x) = \begin{cases} 1/(1-\alpha), & \text{if } x > 0, \\ 1/\alpha, & \text{if } x \leq 0. \end{cases} \tag{79}$$

The second derivative of r_α is $(1/(1-\alpha)-1/\alpha)\delta_0$.

By applying the Itô-Tanaka formula to $r_\alpha Y_t$,

$$r_\alpha(Y_t) = r_\alpha(Y_0) + \int_0^t g(Y_s)\mathrm{d}Y_s + \frac{1}{2}L_t(Y)(1/(1-\alpha)-1/\alpha), \tag{80}$$

therefore,

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}r_\alpha(Y_t) \\ &= g(Y_t)\mathrm{d}Y_t + \frac{1}{2}\mathrm{d}L_t(Y)(1/(1-\alpha)-1/\alpha) \\ &= g(Y_t)f(Y_t)\mathrm{d}B_t + \frac{1}{2}(1-\alpha)\mathrm{d}L_t(X)(1/(1-\alpha)-1/\alpha) \\ &= \mathrm{d}B_t + \beta \mathrm{d}L_t(X), \end{aligned} \tag{81}$$

in the last step we have used the fact $g(x)f(x) \equiv 1$ and $1 - 1/(2\alpha) = \beta$. In the third step we have used the fact $L_t(Y) = (1-\alpha)L_t(X)$, we will show how to get this in the following. By taking the definition of L_t

$$(Y_t)^+ = \int_0^t \mathbb{1}_{\{Y_s>0\}}\mathrm{d}Y_s + 1/2L_t(Y), \tag{82}$$

and $Y_t = s_\alpha(X_t)$, we have for $Y_t > 0$, $(Y_t)^+ = (1-\alpha)(X_t)^+$ and $\int_0^t \mathbb{1}_{\{Y_s>0\}}dY_s = \int_0^t \mathbb{1}_{\{X_s>0\}}\mathrm{d}(1-\alpha)X_s$. Thus the equation (82) can be written as

$$(1-\alpha)(X_t)^+ = \int_0^t \mathbb{1}_{\{X_s>0\}}\mathrm{d}(1-\alpha)X_s + 1/2L_t(Y). \tag{83}$$

Therefore we obtain $(1-\alpha)L_t(X) = L_t(Y)$.

Overall we have shown the existence and uniqueness of the solution to (73), now we are going to show that the solution X_t is a skew Brownian motion.

Corollary 4 in Chapter 3 Section 15 of [25] tells us that if $\tilde{B}_t$ is a standard Brownian motion and we define τ_t as

$$t = \int_0^{\tau_t} \frac{1}{\sigma^2(\tilde{B}_s)}\mathrm{d}s, \tag{84}$$

where $\sigma > 0$ is such that $\mathbb{P}(\int_0^\infty \frac{1}{\sigma^2(\tilde{B}_s)}\mathrm{d}s = +\infty) = 1$, then $Y_t = \tilde{B}_{\tau_t}$ will be a solution to the stochastic equation $\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}B_t$, for B_t another Brownian motion.

Let $Y_\alpha(t)$ be as in (65) so $r_\alpha(Y_\alpha(t))$ is a skew Brownian motion. By the above, we can obtain

$$\mathrm{d}Y_\alpha(t) = f(Y_\alpha(t))\mathrm{d}B_t,$$

for some Brownian motion B_t. In other words, $Y_\alpha(t)$ has the same law as a solution of (77), so X_t as in (78) has the law of a skew Brownian motion. $\square$

6 Guaranteed Minimum Death Benefit (GMDB)

In this chapter, we introduce the equity-linked insurance product—GMDB and its funding mechanism. We explain how such funding mechanism gives rise to a general exponential functional. We derive the Mellin transform of such general exponential functional for those Lévy processes whose Lévy measure has exponentially decaying tails. Furthermore, we use the Kou process as an example to get the probability density function of the exponential functional via inverting its Mellin transform. Additionally, with the obtained probability density function, we compute the tail distribution and the tail expectation, which can be directly applied to the computation of Var and CTE in insurance companies' favor. Finally, we demonstrate that the analytical formulas we obtained are much more efficient and more accurate than the Monte Carlo approach used by the insurance industry currently.

6.1 Introduction

Equity-linked insurance products allow policyholders to invest their premiums in equity market. In other words, the daily returns on the premium investments are directly linked to a particular equity index, such as S&P 500, or a particular equity fund of the policyholder's choosing. Upon selection, the premiums are transferred by the insurer to third-party fund managers. To illustrate the mathematical structure, we consider a simplified example. Let $\{F_t, t \geq 0\}$ denote the evolution of a policyholder's investment account and $\{S_t, t \geq 0\}$ denote that of an equity index. Then the equity-linking mechanism dictates that

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad t \geq 0, \tag{85}$$

where m is the rate of account-value-based management and expenses (M&E) fee per time unit. Among various products, variable annuities are of particular interest as they offer investors a selection of investments often with added guarantees which protect policyholders from severe losses on their investments. These added benefits can often be viewed as the insurance industry's counterparts of option contracts in financial markets. For example, a guaranteed minimum death benefit (GMDB) would guarantee that a policyholder's beneficiary receives the greater of the then-current account value and a guaranteed minimum amount upon the policyholder's death. For example, the guarantee, denoted by $\{G_t, t \geq 0\}$, is for the policyholder to recoup at least his/her initial investment with interest accrued at the risk-free rate, i.e. $G_t = F_0 e^{rt}$, where r is the yield rate per time unit on the insurer's assets backing up the GMDB liability. Denote by T_x the future lifetime of the policyholder, who is currently at age x. It is typically assumed in practice that the mortality model is independent of equity returns, i.e. T_x is independent of $\{S_t, t \geq 0\}$. Therefore, the payoff from the GMDB is given by

$$(G_{T_x} - F_{T_x})^+,$$

which resembles a put option in financial markets. Keep in mind, however, that without any guaranteed benefits the insurer would simply transfer the premiums to third party fund managers. Like other guaranteed benefits, the GMDB is technically an add-on provision to the base contract that provides additional benefits to the policyholder at an additional cost and from which the insurer assumes additional liability. Hence the GMDB is often referred to as a rider. Nonetheless, due to nonforfeiture regulations, the GMDB rider is typically offered on all variable annuity contracts.

While there are many common features of financial derivatives and embedded options in insurance products, a key difference is that financial derivatives are typically short-dated and insurance coverages last for decades. Due to the lack of long-dated options in the market, the risk management of equity-linked insurance is much more sophisticated than the trading of derivatives and plays a fundamental role to the success of insurance business. In this work, we consider a simplified model that captures the structure of the risk management problem for a variable annuity contract with a plain-vanilla GMDB.

Unlike many exchange-traded financial derivatives which require only an up-front fee, embedded options in equity-linked insurance products are often compensated by a stream of fee incomes. For example, fund managers typically charge a fixed percentage m per time unit per dollar of each policyholder's account and a portion of the fees, say m_d, is kicked back to the insurer to compensate for the GMDB rider. Here we consider the present value of the fee income collected continuously up until the time of the policyholder's death,

$$\int_0^{T \wedge T_x} e^{-rs} m_d F_s \, \mathrm{d}s,$$

where r is the yield rate on insurer's bonds backing up the GMDB liability. As in most cases fee incomes exceed the GMDB liability, insurers are interested in the present value of insurer's net liability (gross liability less fee income)

$$L := e^{-rT_x}(G_{T_x} - F_{T_x})_+ - \int_0^{T_x} e^{-rs} m_d F_s \, \mathrm{d}s.$$

A crucial task of risk management modeling is to quantify and assess the likelihood and severity of positive net liability, which leads to a loss to the insurer. Practitioners typically apply certain risk measures to empirical distributions of net liabilities developed from Monte Carlo simulations. The risk measures would then be used to form the basis of risk management decision making, such as setting up reserves and capitals, to provide a buffer against losses under adverse economic conditions. The most commonly used risk measures in the North American insurance industry is the conditional tail expectation,

$$\mathrm{CTE}_p(L) = \mathbb{E}[L | L > \mathrm{VaR}_p(L)],$$

where the Value-at-Risk is determined by

$$\mathrm{VaR}_p(L) := \inf\{y : \mathbb{P}[L \le y] \ge p\}.$$

Since the purpose of risk management is to analyze the severity of positive loss rather than negative loss (profit), we are interested in the risk measures CTE_p and VaR_p for $p > \xi := \mathbb{P}(L \le 0)$. In order to compute the above-mentioned risk measures, we need to compute for $V > \mathrm{VaR}_\xi$,

$$\mathbb{P}(L > V | T_x = t) = \mathbb{P}\left(e^{-rt}F_t + \int_0^t e^{-rs} m_d F_s \,\mathrm{d}s < F_0 - V\right).$$

It is clear that this rather unique funding mechanism in equity-linked insurance gives rise to a generalized form of exponential functional as defined:

$$J_{x,t} := xe^{X_t} + \int_0^t e^{X_s}\,\mathrm{d}s, \quad x \geq 0,$$

While any concern regarding fitting empirical data in the modeling of financial derivatives may carry over to that of equity-linked insurance, there is the additional question of the validity of such models for long-term projection. Nonetheless, the insurance industry has in the past two decades adopted many well-known equity return models from the financial industry, such as geometric Brownian motion, regime-switching geometric Brownian motion, etc. See American Academy of Actuaries publications [26], [43] and [27] for details on a selection of equity return models. Computations of risk measures for variable annuity guaranteed benefits based on exponential functionals of Brownian motion can be found in Feng and Volkmer [23, 24]. In this dissertation, we are interested in the exponential Lévy processes, primarily for two reasons: (i) such models have been shown to explain various stylized facts of empirical data and (ii) they often lead to analytical solutions, not only for pricing problems of exotic options, which are well-studied in finance literature, but also for risk measures of extreme liabilities in equity-linked insurance products, thereby providing fast algorithms for computation needed for capital requirement and other risk management purposes.

6.2 Exponential functional and its Mellin transform

We consider a Lévy process X, started from zero, and having the Laplace exponent $\psi(z) := \ln \mathbb{E}[\exp(zX_1)]$, $z \in \mathrm{i}\mathbb{R}$. The Lévy-Khintchine formula tells us that

$$\psi(z) = \sigma^2 z^2/2 + \mu z + \int_{\mathbb{R}} \left(e^{zx} - 1 - zx\mathbf{1}_{\{|x|<1\}}\right) \Pi(\mathrm{d}x), \quad z \in \mathrm{i}\mathbb{R},$$

where $\sigma \geq 0$, $\mu \in \mathbb{R}$ and the Lévy measure $\Pi(\mathrm{d}x)$ satisfies $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. We denote by $\mathrm{e}(q)$ an exponential random variable with mean $1/q$, which is independent of X, and we recall our definition of the exponential functional

$$I_{x,q} := xe^{X_{\mathrm{e}(q)}} + \int_0^{\mathrm{e}(q)} e^{X_s}\mathrm{d}s, \quad x \geq 0.$$

Remark 4. By using time-reversal it is easy to show that $I_{x,q} \stackrel{d}{=} U_{\mathrm{e}(q)}$, where U_t is the generalized Ornstein-Uhlenbeck process

$$U_t = xe^{X_t} + e^{X_t}\int_0^t e^{-X_s}\mathrm{d}s. \tag{86}$$

Note that U_t is a strong Markov process started from x with the generator

$$\mathcal{L}^{(U)}f(x) = \mathcal{L}^{(X)}\phi(\ln(x)) + f'(x),$$

where $\phi(x) := f(e^x)$ and $\mathcal{L}^{(X)}$ is the Markov generator of the Lévy process X. This result follows from [42, Proposition 2.3].

We define the Mellin transform of $I_{x,q}$

$$\mathcal{M}_{x,q}(s) = \mathbb{E}\left[(I_{x,q})^{s-1}\right]. \tag{87}$$

Initially $\mathcal{M}_{x,q}(s)$ is well defined on the vertical line $\mathrm{Re}(s) = 1$, later we will extend this function analytically into a certain vertical strip.

Everywhere in this section we will work under the following condition: the measure $\Pi(\mathrm{d}x)$ has exponentially decaying tails. In other words

$$\int_{\mathbb{R}\setminus(-1,1)} e^{\theta|x|}\Pi(\mathrm{d}x) < \infty, \quad \text{for some } \theta > 0. \tag{88}$$

The above condition implies that the Laplace exponent $\psi(z)$ is analytic in the strip $|\mathrm{Re}(z)| < \theta$ and it is convex on the real interval $z \in (-\theta, \theta)$.

Now we are going to extend the function $\mathcal{M}_{x,q}(s)$ analytically into a certain vertical strip.

Definition 14. For $q > 0$ we define

$$\Phi^+(q) = \sup\{z > 0 \,:\, \psi(z) < q\} \quad \text{and} \quad \Phi^-(q) = \inf\{z < 0 \,:\, \psi(z) < q\}.$$

Note that condition (88) implies that for every $q > 0$ we have $\Phi^+(q) > 0$ and $\Phi^-(q) < 0$. Furthermore, $\psi(z)$ is convex on the real interval $z \in (\Phi^-(q), \Phi^+(q))$, which ensures $q - \psi(z) > 0$ on this interval.

Proposition 3. *For all $q > 0$, $x \geq 0$ and $\mathrm{Re}(s) \in (0, 1 + \Phi^+(q))$ we have $|\mathcal{M}_{x,q}(s)| < \infty$.*

Proof. First, we need to mention that $|\mathcal{M}_{x,q}(s)| \leq \mathcal{M}_{x,q}(\mathrm{Re}(s))$. This is easy since

$$|\mathcal{M}_{x,q}(s)| = |\mathbb{E}(I_{x,q}^{s-1})| \leq \mathbb{E}|I_{x,q}^{s-1}| \leq \mathbb{E}(I_{x,q}^{\mathrm{Re}(s-1)}) = \mathcal{M}_{x,q}(\mathrm{Re}(s)). \tag{89}$$

Therefore, it is sufficient to prove $\mathcal{M}_{x,q}(\mathrm{Re}(s)) < \infty$.

Let us denote $\zeta = x\exp(X_{\mathrm{e}(q)})$ and $\eta = I_{0,q}$, so that $I_{x,q} = \zeta + \eta$. Note that

$$\begin{aligned}
\mathbb{E}[\zeta^w] &= x^w \mathbb{E}(\mathbb{E}[\exp(wX_t)|\mathrm{e}(q) = t]) \\
&= x^w \int_0^\infty e^{\psi(w)t} \times q e^{-qt} \mathrm{d}t \\
&= x^w \int_0^\infty q e^{-(q-\psi(w))t} \mathrm{d}t \\
&= x^w \frac{q}{q - \psi(w)} < \infty, \quad w \in (\Phi^-(q), \Phi^+(q))
\end{aligned}$$

and $\mathbb{E}[\eta^w] < \infty$ for all $w \in (-1, \Phi^+(q))$ (see Rivero [64, Lemma 2]).

When $0 < w < \min(\Phi^+(q), 1)$ we use Jensen's inequality and obtain

$$\mathbb{E}[(\zeta + \eta)^w] \leq \mathbb{E}[\zeta^w] + \mathbb{E}[\eta^w] < \infty.$$

If $\Phi^+(q) > 1$, then for $1 \leq w < \Phi^+(q)$ we use Minkowski's inequality to get

$$\mathbb{E}[(\zeta + \eta)^w]^{1/w} \leq \mathbb{E}[\zeta^w]^{1/w} + \mathbb{E}[\eta^w]^{1/w} < \infty.$$

Finally, when $-1 < w < 0$ we use the fact that the function $x \in (0, \infty) \mapsto x^w$ is decreasing and obtain

$$\mathbb{E}[(\zeta + \eta)^w] < \mathbb{E}[\eta^w] < \infty.$$

Thus we have proved that $\mathbb{E}[(I_{x,q})^w] = \mathbb{E}[(\zeta + \eta)^w] < \infty$ for all $w \in (-1, \Phi^+(q))$, which is equivalent to the statement of Proposition 3. $\square$

The following theorem is our main result in this section.

Theorem 17. *For $q > 0$ and $w \in (\max(-1, \Phi^-(q)), 0)$,*

$$\mathcal{M}_{x,q}(1 + w) = q\sin(\pi w)\mathcal{M}_{0,q}(1 + w) \times \left[-\frac{1}{2\mathrm{i}} \int_{c+\mathrm{i}\mathbb{R}} \frac{1}{z\sin(\pi z)\mathcal{M}_{0,q}(-z)} \times \frac{x^{-z}\mathrm{d}z}{\sin(\pi(w + z))} \right], \quad (90)$$

where $c \in (0, -w)$.

Before we prove Theorem 17, we need to establish several auxiliary results.

Lemma 2. *For $q > 0$ the function $F(s) = \mathcal{M}_{0,q}(s)/\Gamma(s)$ is analytic and zero-free in the vertical strip $\Phi^-(q) < \mathrm{Re}(s) < 1 + \Phi^+(q)$ and it satisfies*

$$F(s+1) = \frac{1}{q - \psi(s)} F(s), \quad \Phi^-(q) < \mathrm{Re}(s) < \Phi^+(q). \tag{91}$$

Proof. The functional equation follows from Maulik and Zwart [51, Lemma 2.1] (see also Carmona et al. [10, Proposition 3.1]). The fact that $F(s)$ is zero-free follows from the generalized Weierstrass product representation (see Patie and Savov [60, Theorem 2.1]). $\square$

Let us fix $q > 0$, $w \in (\Phi^-(q), 0)$ and define a new measure $\mathbb{Q}$

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{wX_t - t\psi(w)}. \tag{92}$$

Under the new measure $\mathbb{Q}$, the process X is a Lévy process with the Laplace exponent

$$\psi_{\mathbb{Q}}(z) = \psi(z+w) - \psi(w).$$

This is because

$$\begin{aligned} \psi_{\mathbb{Q}}(z) &= \ln \mathbb{E}_{\mathbb{Q}}[\exp(zX_1)] \\ &= \ln \mathbb{E}[\exp(wX_1 - \psi(w) + zX_1)] = \psi(w+z) - \psi(w). \end{aligned}$$

Let us define the exponential functional

$$\hat{J}_t = \int_0^t e^{-X_s} \mathrm{d}s. \tag{93}$$

Lemma 3. *For $w \in (\Phi^-(q), 0)$ we denote $\tilde{q} := q - \psi(w)$. Then for $0 < \mathrm{Re}(s) < 1 + w - \Phi^-(q)$*

$$\mathbb{E}_{\mathbb{Q}}\left[(\hat{J}_{e(\tilde{q})})^{s-1}\right] = \frac{\mathcal{M}_{0,q}(w)}{\Gamma(w)} \times \frac{\Gamma(s)\Gamma(1+w-s)}{\mathcal{M}_{0,q}(1+w-s)}. \tag{94}$$

Proof. Let us denote $Y_t = -X_t$: under the measure $\mathbb{Q}$ this is a Lévy process with the Laplace exponent $\psi_Y(z) = \psi(w-z) - \psi(w)$. Let us also denote $\theta := w - \Phi^-(q)$ and the function in the right-hand side of (94) by $f(s)$. According to Theorem 6, in order to establish Lemma 3 we need to check the following three conditions

(i) $f(s)$ is analytic and zero-free in the strip $\mathrm{Re}(s) \in (0, 1+\theta)$,

(ii) $f(1) = 1$ and $f(s+1) = sf(s)/(\tilde{q} - \psi_Y(s))$ for all $s \in (0, \theta)$,

(iii) $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ as $\mathrm{Im}(s) \to \infty$, $\mathrm{Re}(s) \in (0, 1+\theta)$.

Let us verify Condition (i). Since $w \in (\Phi^-(q), 0)$, we know $\mathcal{M}_{0,q}(w)/\Gamma(w)$ is analytic and zero-free from Lemma 2. With the condition $\Phi^-(q) < 1 + w - \mathrm{Re}(s) < 1$, we obtain that $\mathcal{M}_{0,q}(1 + w - s)/\Gamma(1 + w - s)$ is also analytic and zero-free from Lemma 2, therefore its inverse $\Gamma(1 + w - s)/\mathcal{M}_{0,q}(1 + w - s)$ is analytic and zero-free. Thus, Condition (i) is satisfied.

Let us check condition (ii): we use (91) and obtain that

$$\frac{\mathcal{M}_{0,q}(1 + w - s)}{\Gamma(1 + w - s)} = \frac{1}{q - \psi(w - s)} \frac{\mathcal{M}_{0,q}(w - s)}{\Gamma(w - s)},$$

and with the identity $\Gamma(s + 1) = s\Gamma(s)$, so we have

$$\begin{aligned} f(s + 1) &= \frac{\mathcal{M}_{0,q}(w)}{\Gamma(w)} \times \frac{\Gamma(s + 1)\Gamma(w - s)}{\mathcal{M}_{0,q}(w - s)} \\ &= \frac{\mathcal{M}_{0,q}(w)}{\Gamma(w)} \frac{1}{q - \psi(w - s)} \frac{s\Gamma(s)\Gamma(w - s + 1)}{\mathcal{M}_{0,q}(w - s + 1)} \\ &= \frac{s}{\tilde{q} - \psi_Y(s)} f(s). \end{aligned}$$

The last equality comes from the identity $q - \psi(w - s) = q - \psi(w) + \psi(w) - \psi(w - s) = \tilde{q} - \psi_Y(s)$.

To check condition (iii), we use the asymptotic result (12) to estimate the gamma functions, so we obtain that for any $\epsilon > 0$ small enough,

$$\Big|\frac{1}{\Gamma(1 + w - s)\Gamma(s)}\Big| = o(\exp((\epsilon + \pi)|\mathrm{Im}(s)|)),$$

as $|\mathrm{Im}(s)| \to \infty$; Also by the definition of the Mellin transform, similar to (89) , we have $|\mathcal{M}_{0,q}(1 + w - s)| \leq \mathcal{M}_{0,q}(1 + w - \mathrm{Re}(s))$; Thus, we have

$$\begin{aligned} \big|1/f(s)\big| &= \Big|\frac{\mathcal{M}_{0,q}(w)\mathcal{M}_{0,q}(1 + w - s)}{\Gamma(w)}\Big|\Big|\frac{1}{\Gamma(1 + w - s)\Gamma(s)}\Big| \\ &\leq \Big|\frac{\mathcal{M}_{0,q}(w)\mathcal{M}_{0,q}(1 + w - \mathrm{Re}(s))}{\Gamma(w)}\Big| \times o(\exp((\epsilon + \pi)|\mathrm{Im}(s)|)) = o(\exp(2\pi|\mathrm{Im}(s)|)). \end{aligned}$$

Thus all three conditions are satisfied and we have proved (94). □

Proof of Theorem 17: We recall that $I_{x,q}$ has the same distribution as $U_{\mathrm{e}(q)} = e^{X_{\mathrm{e}(q)}}(x + \hat{J}_{\mathrm{e}(q)})$, where $\hat{J}_t$ is defined by (93). Assume that $q > 0$ and $w \in (\max(-1, \Phi^-(q)), 0)$, so that $q - \psi(w) > 0$. According to Proposition 3, $\mathcal{M}_{x,q}(1 + w) < \infty$ and we can write

$$\mathcal{M}_{x,q}(1+w) = \mathbb{E}\left[I_{x,q}^w\right] = \mathbb{E}\left[U_{\mathrm{e}(q)}^w\right] = \mathbb{E}\left[e^{wX_{\mathrm{e}(q)}}(x + \hat{J}_{\mathrm{e}(q)})^w\right] = \int_0^\infty q e^{-qt} \mathbb{E}\left[e^{wX_t}(x + \hat{J}_t)^w\right] \mathrm{d}t. \quad (95)$$

The last equality comes from conditional expectation on $\mathrm{e}(q)$.

Next, we define the measure $\mathbb{Q}$ as in (92) and denote $\tilde{q} = q - \psi(w)$. From (95) we find

$$\begin{aligned}\mathcal{M}_{x,q}(1+w) &= \int_0^\infty q e^{-qt}\mathbb{E}\left[e^{wX_t}(x+\hat{J}_t)^w\right]\mathrm{d}t \\ &= \int_0^\infty q e^{-qt+\psi(w)t}\mathbb{E}\left[e^{wX_t-\psi(w)t}(x+\hat{J}_t)^w\right]\mathrm{d}t \\ &= \int_0^\infty q e^{-(q-\psi(w))t}\mathbb{E}_{\mathbb{Q}}\left[(x+\hat{J}_t)^w\right]\mathrm{d}t = \frac{q}{\tilde{q}}\mathbb{E}_{\mathbb{Q}}\left[(x+\hat{J}_{\mathrm{e}(\tilde{q})})^w\right].\end{aligned} \tag{96}$$

In the following, we take $\mathrm{Re}(z) \in (0, -w)$ and use (96) to compute $\int_0^\infty x^{z-1}\mathcal{M}_{x,q}(1+w)\mathrm{d}x$. First of all, we will show this integral is well defined.

As $x \to \infty$, $\mathcal{M}_{x,q}(1+w) = O(x^w)$, therefore $x^{\mathrm{Re}(z-1)}\mathcal{M}_{x,q}(1+w) = O(x^{\mathrm{Re}(z-1+w)})$ is integrable as $\mathrm{Re}(z+w-1) < -1$;

As $x \to 0$, $\mathcal{M}_{x,q}(1+w) = O(1)$, therefore $x^{\mathrm{Re}(z-1)}\mathcal{M}_{x,q}(1+w) = O(x^{\mathrm{Re}(z-1)})$ is integrable as $\mathrm{Re}(z-1) > -1$.

Thus we have

$$\begin{aligned}\int_0^\infty x^{z-1}\mathcal{M}_{x,q}(1+w)\mathrm{d}x &= \frac{q}{\tilde{q}}\int_0^\infty x^{z-1}\mathbb{E}_{\mathbb{Q}}\left[(x+\hat{J}_{\mathrm{e}(\tilde{q})})^w\right]\mathrm{d}x \\ &= \frac{q}{\tilde{q}}\mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty x^{z-1}(x+\hat{J}_{\mathrm{e}(\tilde{q})})^w\mathrm{d}x\right] \\ &= \frac{q}{\tilde{q}}\mathbb{E}_{\mathbb{Q}}\left[\left(\hat{J}_{\mathrm{e}(\tilde{q})}\right)^{z+w}\int_0^\infty y^{z-1}(y+1)^w\mathrm{d}y\right] \\ &= \frac{q}{\tilde{q}}\mathbb{E}_{\mathbb{Q}}\left[\left(\hat{J}_{\mathrm{e}(\tilde{q})}\right)^{z+w}\right]\times\frac{\Gamma(z)\Gamma(-w-z)}{\Gamma(-w)}. \\ &= \frac{q}{\tilde{q}}\frac{\mathcal{M}_{0,q}(w)}{\Gamma(w)}\times\frac{\Gamma(1+z+w)\Gamma(-z)}{\mathcal{M}_{0,q}(-z)}\times\frac{\Gamma(z)\Gamma(-w-z)}{\Gamma(-w)},\end{aligned} \tag{97}$$

where we used Fubini's theorem in the second step, change of variables $x = J_{\mathrm{e}(\tilde{q})}y$ in the third step, the well-known beta-function integral in the fourth step

$$\int_0^\infty y^{z-1}(y+1)^w\mathrm{d}y = \int_0^\infty \Big(\frac{y}{y+1}\Big)^{z-1}\Big(1-\frac{y}{y+1}\Big)^{-w-z-1}\mathrm{d}\Big(\frac{y}{y+1}\Big) = \frac{\Gamma(z)\Gamma(-w-z)}{\Gamma(-w)},$$

and Lemma 3 in the fifth step.

Note here, we almost obtained formula (90) by applying inverse Mellin transform to the right hand side of (97), we just need to verify that it is integrable in the imaginary direction for z. For arbitrary $c \in (0, -w)$, according to Theorems 2.7 and 3.3 in [61], we have an upper bound

$|1/\mathcal{M}_{0,q}(-z)| = O(\exp((\pi+\epsilon)|\mathrm{Im}(z)|)$ (for any $\epsilon > 0$) as $|z| \to \infty$ along the vertical line $c + \mathrm{i}\mathbb{R}$. By combining it with the asymptotic result (12), we obtain

$$\Big|\frac{\Gamma(1+z+w)\Gamma(-z)}{\mathcal{M}_{0,q}(-z)} \times \Gamma(z)\Gamma(-w-z)\Big| = O(\exp(-(\pi-\epsilon)|\mathrm{Im}(z)|)),$$

as $|\mathrm{Im}(z)| \to \infty$. Therefore the right hand side of (97) is integrable in the imaginary direction and we can use the inverse Mellin transform.

Finally, from (91) we find that

$$\frac{1}{\tilde{q}}\frac{\mathcal{M}_{0,q}(w)}{\Gamma(w)} = \frac{1}{q-\psi(w)}\frac{\mathcal{M}_{0,q}(w)}{\Gamma(w)} = \frac{\mathcal{M}_{0,q}(1+w)}{\Gamma(1+w)}.$$

We also use the reflection formula for the gamma function and rewrite (97) in the form

$$\int_0^\infty x^{z-1}\mathcal{M}_{x,q}(1+w)\mathrm{d}x = -\frac{\pi q \sin(\pi w)\mathcal{M}_{0,q}(1+w)}{z\sin(\pi z)\mathcal{M}_{0,q}(-z)\sin(\pi(w+z))},$$

from which formula (90) follows by the inverse Mellin transform. □

6.3 Case study: Kou process

6.3.1 Mellin transform of $I_{x,q}$ and its probability density function

In this section we demonstrate how Theorem 17 can be used to compute explicitly the density of the exponential functional $I_{x,q}$ for the Kou process. Let us recall the definition of the Kou process from Chapter 2. A Kou process X_t is defined as

$$X_t = \mu t + \sigma W_t + \sum_{j=1}^{N_t} \xi_i, \tag{98}$$

where $\sigma > 0$, $\mu \in \mathbb{R}$, N_t is a Poisson process with intensity λ and $\{\xi_i\}_i$ are i.i.d. random variables with the probability density function

$$p_\xi(x) = p\rho e^{-\rho x}\mathbb{1}_{\{x>0\}} + (1-p)\hat{\rho}e^{\hat{\rho}x}\mathbb{1}_{\{x<0\}},$$

for some $p \in (0,1)$ and $\rho, \hat{\rho} > 0$ (independent of N_t).

The Laplace exponent is equal to

$$\psi(z) = \mu z + \frac{\sigma^2}{2}z^2 + \lambda p\frac{z}{\rho - z} - \lambda(1-p)\frac{z}{\hat{\rho}+z}.$$

For $q > 0$ the rational function $\psi(z) = q$ has four zeros $\{-\hat{\zeta}_2, -\hat{\zeta}_1, \zeta_1, \zeta_2\}$ and two poles $\{-\hat{\rho}, \rho\}$ which satisfy the interlacing property

$$-\hat{\zeta}_2 < -\hat{\rho} < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho < \zeta_2.$$

The Mellin transform $\mathcal{M}_{0,q}(s)$ was computed in Cai and Kou [9] (see also [39]) and is also shown in Theorem 7 as follows,

$$\mathcal{M}_{0,q}(s) = A^{1-s}\Gamma(s)\frac{\mathcal{G}(s)}{\mathcal{G}(1)}. \tag{99}$$

Here $A = \sigma^2/2$ and

$$\mathcal{G}(s) := \Gamma\left[\begin{array}{c} 1+\zeta_1 - s,\ 1+\zeta_2 - s,\ \hat{\rho}+s \\ 1+\rho - s,\ \hat{\zeta}_1 + s,\ \hat{\zeta}_2 + s \end{array}\right].$$

In the above formula (and everywhere else in this dissertation) we use the notation

$$\Gamma\left[\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\right] := \frac{\prod_{i=1}^p \Gamma(a_i)}{\prod_{j=1}^q \Gamma(b_j)}. \tag{100}$$

Our first main result in this section is an explicit expression for the Mellin transform $\mathcal{M}_{x,q}(s)$.

Proposition 4. *For* $0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1$, *we have*

$$\mathcal{M}_{x,q}(s) = qA^{-s}\Gamma\left[\begin{array}{c} 1+\zeta_1 - s,\ 1+\zeta_2 - s,\ \hat{\rho}+s \\ 1-s,\ 1+\rho - s,\ \hat{\zeta}_1 + s,\ \hat{\zeta}_2 + s \end{array}\right] G^{3,3}_{4,5}\left(\begin{array}{c} 1-s, 1, -\rho, \hat{\rho} \\ 1-s, \hat{\zeta}_1, \hat{\zeta}_2, -\zeta_1, -\zeta_2 \end{array} \Bigg| \frac{1}{Ax}\right), \tag{101}$$

where G *is the Meijer G-function defined in* (28).

Proof. Formula (99) and Theorem 17 tell us that for $-(1 \wedge \hat{\zeta}_1) < w < -c < 0$ we have

$$\begin{aligned} \mathcal{M}_{x,q}(1+w) &= q\sin(\pi w)A^{-w}\Gamma\left[\begin{array}{c} 1+w, \zeta_1 - w,\ \zeta_2 - w,\ \hat{\rho}+1+w \\ \rho - w,\ \hat{\zeta}_1 + 1 + w,\ \hat{\zeta}_2 + 1 + w \end{array}\right] \\ &\quad \times \frac{-1}{2\mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma\left[\begin{array}{c} 1+\rho+z,\ \hat{\zeta}_1 - z,\ \hat{\zeta}_2 - z \\ -z, 1+\zeta_1 + z,\ 1+\zeta_2+z,\ \hat{\rho} - z \end{array}\right] \frac{A^{-1-z}x^{-z}\mathrm{d}z}{z\sin(\pi z)\sin(\pi(w+z))}. \end{aligned}$$

By using the reflection formula for the Gamma function we rewrite the above equation in the form

$$\begin{aligned} \mathcal{M}_{x,q}(1+w) &= qA^{-1-w}\Gamma\left[\begin{array}{c} \zeta_1 - w,\ \zeta_2 - w,\ \hat{\rho}+1+w \\ -w,\ \rho - w,\ \hat{\zeta}_1 + 1 + w,\ \hat{\zeta}_2 + 1 + w \end{array}\right] \\ &\quad \times \frac{1}{2\pi\mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma\left[\begin{array}{c} 1+w+z,\ z,\ 1+\rho+z,\ -w-z,\ \hat{\zeta}_1 - z,\ \hat{\zeta}_2 - z \\ \hat{\rho} - z,\ 1+\zeta_1+z,\ 1+\zeta_2+z \end{array}\right] (Ax)^{-z}\mathrm{d}z. \end{aligned}$$

By applying formula (28) we conclude that for all $-(1 \wedge \hat{\zeta}_1) < w < 0$

$$\begin{aligned} \mathcal{M}_{x,q}(1+w) = qA^{-1-w}\Gamma\left[\begin{matrix} \zeta_1 - w,\ \zeta_2 - w,\ \hat{\rho}+1+w \\ -w,\ \rho - w,\ \hat{\zeta}_1+1+w,\ \hat{\zeta}_2+1+w \end{matrix} \right] \\ \times G^{3,3}_{5,4}\left(\begin{matrix} 1+w, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2, 1+\zeta_1, 1+\zeta_2 \\ 1+w, 0, 1+\rho, 1-\hat{\rho} \end{matrix} \middle| Ax \right). \end{aligned} \tag{102}$$

Note that both conditions (26) and (27) are satisfied, since in our case we have

$$\begin{aligned} a &= \max(1+w, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2) = \max(1+w, 1-\hat{\zeta}_1) \in (0,1), \\ b &= \min(0, 1+w, 1+\rho) = 0, \end{aligned}$$

and $c \in (-b, 1-a)$.

By applying formula (30), we obtain

$$\mathcal{M}_{x,q}(1+w) = qA^{-1-w}\Gamma\left[\begin{matrix} \zeta_1 - w,\ \zeta_2 - w,\ \hat{\rho}+1+w \\ -w,\ \rho - w,\ \hat{\zeta}_1+1+w,\ \hat{\zeta}_2+1+w \end{matrix} \right] G^{3,3}_{4,5}\left(\begin{matrix} -w, 1, -\rho, \hat{\rho} \\ -w, \hat{\zeta}_1, \hat{\zeta}_2, -\zeta_1, -\zeta_2 \end{matrix} \middle| \frac{1}{Ax} \right). \tag{103}$$

By replacing the variable $w = s-1$, we let

$$\mathcal{N}_{x,q}(s) = qA^{-s}\Gamma\left[\begin{matrix} 1+\zeta_1 - s,\ 1+\zeta_2 - s,\ \hat{\rho}+s \\ 1-s,\ 1+\rho - s,\ \hat{\zeta}_1+s,\ \hat{\zeta}_2+s \end{matrix} \right] G^{3,3}_{4,5}\left(\begin{matrix} 1-s, 1, -\rho, \hat{\rho} \\ 1-s, \hat{\zeta}_1, \hat{\zeta}_2, -\zeta_1, -\zeta_2 \end{matrix} \middle| \frac{1}{Ax} \right). \tag{104}$$

Here $-(1\wedge\hat{\zeta}_1) < \mathrm{Re}(s-1) < 0$, namely $0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1$. We can observe that $\mathcal{N}_{x,q}(s)$ is analytic for $0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1$, and we know $\mathcal{M}_{x,q}(s)$ is also analytic for $0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1$. With the fact $\mathcal{N}_{x,q}(s)$ and $\mathcal{M}_{x,q}(s)$ agrees on the interval $(0 \vee (1-\hat{\zeta}_1), 1)$, Theorem 2 ensures $\mathcal{M}_{x,q}(s) = \mathcal{N}_{x,q}(s)$ for $0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1$. Therefore the desired result (101) is obtained. □

6.3.2 Informal derivation of the probability density function of $I_{x,q}$

For the rest of this section we will work under the following

Assumption 1: $\zeta_2 - \zeta_1 \notin \mathbb{N}$ and $\hat{\zeta}_2 - \hat{\zeta}_1 \notin \mathbb{N}$.

In the previous section we have obtained the Mellin transform of $I_{x,q}$ for the Kou process. Now a straightforward idea is to get the probability density function by using inverse Mellin transform

$$f_{x,q}(y) = \frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}\mathbb{R}} y^{-s} \mathcal{M}_{x,q}(s) \mathrm{d}s, \qquad 0 \vee (1-\hat{\zeta}_1) < c < 1. \tag{105}$$

By expressing the Meijer G-function in (101) via (32) and (20), we have

$$\mathcal{M}_{x,q}(s) = M_1 + M_2 + M_3, \tag{106}$$

where

$$\begin{aligned} M_1 &:= \frac{q}{A} \times \frac{(\hat{\rho}-1+s)(1+\rho-s)}{(1+\zeta_1-s)(1+\zeta_2-s)(\hat{\zeta}_1+s-1)(\hat{\zeta}_2+s-1)} (x)^{s-1} \\ &\times {}_4F_4 \left(\begin{matrix} 1-s, 1, 2-s+\rho, 2-s-\hat{\rho} \\ 2-s-\hat{\zeta}_1, 2-s-\hat{\zeta}_2, 2-s+\xi_1, 2-s+\xi_2 \end{matrix} \middle| \frac{1}{Ax} \right) \\ M_2 &:= qA^{-s}\Gamma\left[\begin{matrix} 1+\zeta_1-s, 1+\zeta_2-s, \hat{\rho}+s, 1-\hat{\zeta}_1-s \\ 1-s, 1+\rho-s, \hat{\zeta}_2+s \end{matrix} \right] \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))} (Ax)^{-\hat{\zeta}_1}, \\ &\times {}_3\Phi_3 \left(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \middle| \frac{1}{Ax} \right), \\ M_3 &:= qA^{-s}\Gamma\left[\begin{matrix} 1+\zeta_1-s, 1+\zeta_2-s, \hat{\rho}+s, 1-\hat{\zeta}_2-s \\ 1-s, 1+\rho-s, \hat{\zeta}_1+s \end{matrix} \right] \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_2))}{\sin(\pi(\hat{\zeta}_1-\hat{\zeta}_2))} (Ax)^{-\hat{\zeta}_2} \\ &\times {}_3\Phi_3 \left(\begin{matrix} \hat{\zeta}_2, 1+\hat{\zeta}_2+\rho, 1+\hat{\zeta}_2-\hat{\rho} \\ 1+\hat{\zeta}_2-\hat{\zeta}_1, 1+\hat{\zeta}_2+\zeta_1, 1+\hat{\zeta}_2+\zeta_2 \end{matrix} \middle| \frac{1}{Ax} \right). \end{aligned}$$

By observing M_2 and M_3, the idea is to apply the theory of Meijer G-functions on the gamma functions.

The formula (28) tells us for M_2 and M_3:

$$\frac{1}{2\pi \mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} A^{-s}\Gamma\left[\begin{matrix} 1+\zeta_1-s, 1+\zeta_2-s, \hat{\rho}+s, 1-\hat{\zeta}_1-s \\ 1-s, 1+\rho-s, \hat{\zeta}_2+s \end{matrix} \right] y^{-s} \mathrm{d}s = G^{1,3}_{4,3}\left(\begin{matrix} -\zeta_1, -\zeta_2, \hat{\zeta}_1, \hat{\zeta}_2 \\ \hat{\rho}, 0, -\rho \end{matrix} \middle| Ay \right),$$

and

$$\frac{1}{2\pi \mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} A^{-s}\Gamma\left[\begin{matrix} 1+\zeta_1-s, 1+\zeta_2-s, \hat{\rho}+s, 1-\hat{\zeta}_2-s \\ 1-s, 1+\rho-s, \hat{\zeta}_1+s \end{matrix} \right] y^{-s} \mathrm{d}s = G^{1,3}_{4,3}\left(\begin{matrix} -\zeta_1, -\zeta_2, \hat{\zeta}_2, \hat{\zeta}_1 \\ \hat{\rho}, 0, -\rho \end{matrix} \middle| Ay \right).$$

But for M_1, the theory of the Meijer G-function can not apply. The idea is to use an alternative method by applying residue theory on M_1. We can observe that M_1 is analytic in the half plane for $\mathrm{Re}(s) < 1-\hat{\zeta}_2$, the Cauchy residue theory implies that

$$\frac{1}{2\pi \mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} y^{-s} M_1 \mathrm{d}s = \lim_{c\to-\infty} \frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}\mathbb{R}} y^{-s} M_1 \mathrm{d}s.$$

Furthermore it can be proved that in the case for $0 < y \le x$, with the fact $(x/y)^s$ is bounded as $\mathrm{Re}(s) \to -\infty$, we have

$$\lim_{c\to-\infty} \frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}\mathbb{R}} y^{-s} M_1 ds = 0.$$

Therefore we obtain the probability density function for $y \le x$,

$$\begin{aligned} f_{x,q}(y) = &\Big\{ q(Ax)^{-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \\ &\times G^{1,3}_{4,3}\Big(\begin{matrix} -\zeta_1, -\zeta_2, \hat{\zeta}_1, \hat{\zeta}_2 \\ \hat{\rho}, 0, -\rho \end{matrix} \Big| Ay \Big) \Big\} \\ + &\Big\{ q(Ax)^{-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_2))}{\sin(\pi(\hat{\zeta}_1-\hat{\zeta}_2))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_2, 1+\hat{\zeta}_2+\rho, 1+\hat{\zeta}_2-\hat{\rho} \\ 1+\hat{\zeta}_2-\hat{\zeta}_1, 1+\hat{\zeta}_2+\zeta_1, 1+\hat{\zeta}_2+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \\ &\times G^{1,3}_{4,3}\Big(\begin{matrix} -\zeta_1, -\zeta_2, \hat{\zeta}_2, \hat{\zeta}_1 \\ \hat{\rho}, 0, -\rho \end{matrix} \Big| Ay \Big) \Big\}. \end{aligned}$$

By applying the formula (30), we have

$$G^{3,1}_{3,4}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay} \Big) = G^{1,3}_{4,3}\Big(\begin{matrix} -\zeta_1, -\zeta_2, \hat{\zeta}_1, \hat{\zeta}_2 \\ \hat{\rho}, 0, -\rho \end{matrix} \Big| Ay \Big).$$

Finally, we get for $y \le x$,

$$\begin{aligned} f_{x,q}(y) = &\Big\{ q(Ax)^{-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \\ &\times G^{3,1}_{3,4}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay} \Big) \Big\} \\ + &\Big\{ q(Ax)^{-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_2))}{\sin(\pi(\hat{\zeta}_1-\hat{\zeta}_2))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_2, 1+\hat{\zeta}_2+\rho, 1+\hat{\zeta}_2-\hat{\rho} \\ 1+\hat{\zeta}_2-\hat{\zeta}_1, 1+\hat{\zeta}_2+\zeta_1, 1+\hat{\zeta}_2+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \\ &\times G^{3,1}_{3,4}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay} \Big) \Big\}. \end{aligned}$$

For the case $y > x$, as $(x/y)^s$ is uniformly bounded as $\mathrm{Re}(s) > 0$, so we shift the contour to the right. The term M_1 has simple poles $\Big[\{j+\zeta_1\}_{j\ge1}, \{j+\zeta_2\}_{j\ge1}, \{j-\hat{\zeta}_1\}_{j\ge1}, \{j-\hat{\zeta}_2\}_{j\ge1}\Big]$. Thus computing the residue is easy but tedious. By doing some algebra on M_1,M_2 and M_3, we can derive that the $\mathcal{M}_{x,q}(s)$ satisfies

$$(q-\psi(s))\mathcal{M}_{x,q}(s+1) = qx^s + s\mathcal{M}_{x,q}(s). \tag{107}$$

Equation (107) shows

$$\mathcal{M}_{x,q}(s+1) = \frac{qx^s + s\mathcal{M}_{x,q}(s)}{q - \psi(s)}. \tag{108}$$

By recalling the Kou process, we know

$$q - \psi(s) = \frac{A(s-\zeta_1)(s-\zeta_2)(s+\hat{\zeta}_1)(s+\hat{\zeta}_2)}{(\rho - s)(\hat{\rho}+s)}. \tag{109}$$

Since $\mathcal{M}_{x,q}(s)$ is analytic in the vertical strip $0 < \mathrm{Re}(s) < 1 + \Phi^+(q)$, we can see $\mathcal{M}_{x,q}(s)$ have simple poles in $A = \{1+\zeta_1, 2+\zeta_1, \cdots, n+\zeta_1, \cdots\}$ and $B = \{1+\zeta_2, , 2+\zeta_2, \cdots, n+\zeta_2, \cdots\}$. As we shift the contour to the right half of the complex plane, the contour will pass those poles.

When we shift the contour integration to the right, we only need to compute the residues of the function $y^{-s}\mathcal{M}_{x,q}(s)$ on those simple poles.

By applying Cauchy residue theorem, we have

$$\begin{aligned} & \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} y^{-s}\mathcal{M}_{x,q}(s)\mathrm{d}s \\ = & \int_{c+1-\mathrm{i}\infty}^{c+1+\mathrm{i}\infty} y^{-s}\mathcal{M}_{x,q}(s)\mathrm{d}s - \sum_{\substack{c<a<c+1 \\ a\in A\cup B}} \mathrm{Res}(y^{-s}\mathcal{M}_{x,q}(s) : s = a). \end{aligned} \tag{110}$$

By repeating this contour shifting, we can get

$$\begin{aligned} & \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} y^{-s}\mathcal{M}_{x,q}(s)\mathrm{d}s \\ = & \int_{c+n-\mathrm{i}\infty}^{c+n+\mathrm{i}\infty} y^{-s}\mathcal{M}_{x,q}(s)\mathrm{d}s - \sum_{\substack{c<a<c+n \\ a\in A\cup B}} \mathrm{Res}(y^{-s}\mathcal{M}_{x,q}(s) : s = a). \end{aligned} \tag{111}$$

In employing this technique the conjecture is that as $n \to \infty$, the integral on the right-hand side of (111) is zero. Therefore, we guess the probability density function as

$$f_{x,q}(y) = -\sum_{n\geq 1} \mathrm{Res}(y^{-s}\mathcal{M}_{x,q}(s) : s = \zeta_1 + n) - \sum_{n\geq 1} \mathrm{Res}(y^{-s}\mathcal{M}_{x,q}(s) : s = \zeta_2 + n). \tag{112}$$

At the end we will prove this is true, but here we only show how to compute the density function explicitly.

In the next step, we compute the residues of $y^{-s}\mathcal{M}_{x,q}(s)$ by taking advantage of (108).

As $s \to \zeta_1$,

$$\begin{aligned}\frac{qx^s + s\mathcal{M}_{x,q}(s)}{q-\psi(s)} &= \frac{qx^s + s\mathcal{M}_{x,q}(s)}{(q-\psi(\zeta_1)) - \psi'(\zeta_1)(s-\zeta_1) + O((s-\zeta_1)^2)} \\ &= \frac{qx^s + s\mathcal{M}_{x,q}(s)}{-\psi'(\zeta_1)(s-\zeta_1)} + \frac{O((s-\zeta_1)^2)}{-\psi'(\zeta_1)(s-\zeta_1)\big(-\psi'(\zeta_1)(s-\zeta_1) + O((s-\zeta_1)^2)\big)} \\ &= \frac{qx^s + s\mathcal{M}_{x,q}(s)}{-\psi'(\zeta_1)(s-\zeta_1)} + O(1),\end{aligned}$$

where we used the Taylor series in the first step. Thus it is easy to observe as $s \to \zeta_1$, we obtain

$$\mathcal{M}_{x,q}(s+1) = \frac{qx^{\zeta_1} + \zeta_1\mathcal{M}_{x,q}(\zeta_1)}{-\psi'(\zeta_1)(s-\zeta_1)} + O(1) = \frac{qx^{\zeta_1} + \zeta_1\mathcal{M}_{x,q}(\zeta_1)}{-\psi'(\zeta_1)(s+1-(\zeta_1+1))} + O(1). \tag{113}$$

Therefore by the fact $\mathrm{Res}(1/(x-c) : x = c) = 1$, we can get

$$\mathrm{Res}(\mathcal{M}_{x,q}(s) : s = \zeta_1 + 1) = \frac{qx^{\zeta_1} + \xi_1\mathcal{M}_{x,q}(\zeta_1)}{-\psi'(\zeta_1)}.$$

We let

$$a := \frac{qx^{\zeta_1} + \zeta_1\mathcal{M}_{x,q}(\zeta_1)}{-\psi'(\zeta_1)},$$

for convenience. Therefore, as $s \to \zeta_1 + 1$, we have $\mathcal{M}_{x,q}(s) = a/(s-(\zeta_1+1)) + O(1)$.

By applying the recursion formula

$$\mathcal{M}_{x,q}(s+1) = \frac{qx^s + s\mathcal{M}_{x,q}(s)}{q-\psi(s)},$$

and letting $s \to \zeta_1 + 1$, we obtain

$$\begin{aligned}\mathcal{M}_{x,q}(s+1) &= \frac{qx^s}{q-\psi(s)} + \frac{s \times a}{(q-\psi(s))(s-(\zeta_1+1))} + O(1) \\ &= \frac{s \times a}{(q-\psi(s))(s+1-(\zeta_1+2))} + O(1).\end{aligned} \tag{114}$$

Here we use the conclusion $\mathcal{M}_{x,q}(s) = a/(s-(\zeta_1+1)) + O(1)$ as $s \to \zeta_1 + 1$, and also the fact that $qx^s/(q-\psi(s))$ is continuous at $\zeta_1 + 1$, so $qx^s/(q-\psi(s)) = O(1)$ as $s \to \zeta_1 + 1$.

Therefore, we can get

$$\mathrm{Res}(\mathcal{M}_{x,q}(s) : s = \zeta_1 + 2) = \frac{\zeta_1 + 1}{q - \psi(\zeta_1+1)}a.$$

By recursion, it is easy to compute

$$\mathrm{Res}(\mathcal{M}_{x,q}(s) : s = \zeta_1 + n) = a\prod_{j=1}^{n-1}\frac{\zeta_1 + j}{q-\psi(\zeta_1+j)} = \frac{qx^{\zeta_1} + \zeta_1\mathcal{M}_{x,q}(\zeta_1)}{-\psi'(\zeta_1)}\prod_{j=1}^{n-1}\frac{\zeta_1+j}{q-\psi(\zeta_1+j)}.$$

Similarly, we can also have

$$\mathrm{Res}(\mathcal{M}_{x,q}(s) : s = \zeta_2 + n) = \frac{qx^{\zeta_2} + \zeta_1 \mathcal{M}_{x,q}(\zeta_2)}{-\psi'(\zeta_2)} \prod_{j=1}^{n-1} \frac{\zeta_2 + j}{q - \psi(\zeta_2 + j)}.$$

Furthermore, $\mathrm{Res}(y^{-s}\mathcal{M}_{x,q}(s) : s = \zeta_1 + n) = y^{-(\zeta_1+n)} \times \mathrm{Res}(\mathcal{M}_{x,q}(s) : s = \zeta_1 + n)$ since y^{-s} is continuous at $\zeta_1 + n$.

Finally, we obtain

$$\begin{aligned}
&\sum_{n\geq 1} y^{-n-\zeta_1} \mathrm{Res}(\mathcal{M}_{x,q} : s = \zeta_1 + n) \\
=\ & \frac{qx^{\zeta_1} + \zeta_1 M(\zeta_1)}{-\psi'(\zeta_1)} y^{-1-\zeta_1} \sum_{n\geq 1} y^{-n+1} \prod_{j=1}^{n-1} \frac{(j+\zeta_1)(\rho - j - \zeta_1)(\hat{\rho} + j + \zeta_1)}{A(j)(j+\zeta_1-\zeta_2)(j+\zeta_1+\hat{\zeta}_1)(j+\zeta_1+\hat{\zeta}_2)} \\
=\ & \frac{qx^{\zeta_1} + \zeta_1 M(\zeta_1)}{-\psi'(\zeta_1)} y^{-1-\zeta_1} {}_3F_3 \left(\begin{array}{c} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{array} \middle| -\frac{1}{Ay} \right),
\end{aligned} \tag{115}$$

in the first step we replaced

$$\frac{\zeta_1 + j}{q - \psi(\zeta_1 + j)} = \frac{(j+\zeta_1)(\rho - j - \zeta_1)(\hat{\rho} + j + \zeta_1)}{A(j)(j+\zeta_1-\zeta_2)(j+\zeta_1+\hat{\zeta}_1)(j+\zeta_1+\hat{\zeta}_2)},$$

and in the second step we used the definition of hypergeometric function.

Similarly, we also get

$$\begin{aligned}
&\sum_{n\geq 1} y^{-n-\zeta_2} \mathrm{Res}(\mathcal{M}_{x,q} : s = \zeta_2 + n) \\
=\ & \frac{qx^{\zeta_2} + \zeta_2 M(\zeta_2)}{-\psi'(\zeta_2)} y^{-1-\zeta_2} \sum_{n\geq 1} y^{-n+1} \prod_{j=1}^{n-1} \frac{(j+\zeta_2)(\rho - j - \zeta_2)(\hat{\rho} + j + \zeta_2)}{A(j)(j+\zeta_2-\zeta_1)(j+\zeta_2+\hat{\zeta}_1)(j+\zeta_2+\hat{\zeta}_2)} \\
=\ & \frac{qx^{\zeta_2} + \zeta_2 M(\zeta_2)}{-\psi'(\zeta_2)} y^{-1-\zeta_2} {}_3F_3 \left(\begin{array}{c} 1+\zeta_2, 1+\zeta_2-\rho, 1+\zeta_2+\hat{\rho} \\ 1+\zeta_2-\zeta_1, 1+\zeta_2+\hat{\zeta}_1, 1+\zeta_2+\hat{\zeta}_2 \end{array} \middle| -\frac{1}{Ay} \right).
\end{aligned} \tag{116}$$

Therefore, when $y > x$, we obtain the probability density function

$$\begin{aligned}
f_{x,q}(y) =\ & \frac{qx^{\zeta_1} + \zeta_1 M(\zeta_1)}{\psi'(\zeta_1)} y^{-1-\zeta_1} {}_3F_3 \left(\begin{array}{c} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{array} \middle| -\frac{1}{Ay} \right) \\
& + \frac{qx^{\zeta_2} + \zeta_2 M(\zeta_2)}{\psi'(\zeta_2)} y^{-1-\zeta_2} {}_3F_3 \left(\begin{array}{c} 1+\zeta_2, 1+\zeta_2-\rho, 1+\zeta_2+\hat{\rho} \\ 1+\zeta_2-\zeta_1, 1+\zeta_2+\hat{\zeta}_1, 1+\zeta_2+\hat{\zeta}_2 \end{array} \middle| -\frac{1}{Ay} \right).
\end{aligned} \tag{117}$$

Let us make a summary about what we have done here, by applying the Cauchy residue theory and the recursion equation which $\mathcal{M}_{x,q}(s)$ satisfied, we compute the residue explicitly, and finally we get the probability density function when $y > x$ with the conjecture $\lim_{c\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}\mathbb{R}} y^{-s}\mathcal{M}_{x,q}(s)\mathrm{d}s = 0$.

We have obtained $f_{x,q}(y)$ explicitly. However, when $y > x$, we have mentioned that the conjecture is that the integral on the right-hand side of (111) is zero, thus we guessed the form of $f_{x,q}(y)$. However, we find it is very hard to prove that integral is zero. Therefore we try another way to verify that $f_{x,q}(y)$ is exactly the probability density function of $I_{x,q}$. The idea is to get the Mellin transform of $f_{x,q}(y)$ and to prove it is equal to the Mellin transform of $I_{x,q}$. The following is the details of the proof.

6.3.3 Probability density function of $I_{x,q}$

Theorem 18. *The probability density function of $I_{x,q}$ is as follows:*

for $y > x$,

$$f_{x,q}(y) := \Big\{ \frac{qx^{\zeta_1} + \zeta_1 \mathcal{M}_{x,q}(\zeta_1)}{\psi'(\zeta_1)} y^{-1-\zeta_1} {}_3F_3\Big(\begin{matrix} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ay} \Big) \Big\} \tag{118}$$
$$+ \Big\{ \text{the same expression with } \zeta_1 \text{ and } \zeta_2 \text{ interchanged} \Big\},$$

and for $0 < y \le x$,

$$f_{x,q}(y) := \Big\{ q(Ax)^{-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \tag{119}$$
$$\times G^{3,1}_{3,4}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay} \Big) \Big\}$$
$$+ \Big\{ \text{the same expression with } \hat{\zeta}_1 \text{ and } \hat{\zeta}_2 \text{ interchanged} \Big\}.$$

In the above formula, Φ denotes the regularized hypergeometric function, as defined in (20).

Proof. By applying formula (31), we check that for any $\epsilon > 0$ small enough,

$$f_{x,q}(y) = O(y^{\hat{\rho}-\epsilon}), \quad \text{as} \quad y \to 0, \tag{120}$$
$$f_{x,q}(y) = O(y^{-1-\zeta_1}), \quad \text{as} \quad y \to +\infty, \tag{121}$$

so the function $y^{s-1}f_{x,q}(y)$ is integrable for $0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1$.

For s in this strip we define

$$I_1(s) := \int_0^x f_{x,q}(y) y^{s-1} \mathrm{d}y, \qquad I_2(s) = \int_x^\infty f_{x,q}(y) y^{s-1} \mathrm{d}y,$$

and now our goal is to check that $I_1(s) + I_2(s) = \mathcal{M}_{x,q}(s)$ (where the right-hand side is given by (101)).

First we use formula (37) and obtain

$$\begin{aligned} I_1(s) = \Big\{ q A^{-\hat{\zeta}_1} x^{s-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho} - \hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2 - \hat{\zeta}_1))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \\ \times G^{4,1}_{4,5}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho, s+1 \\ s, 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \Big\} \\ + \Big\{ \text{the same expression with } \hat{\zeta}_1 \text{ and } \hat{\zeta}_2 \text{ interchanged} \Big\}. \end{aligned}$$

Similarly, by using formula (38) we find

$$\begin{aligned} I_2(s) = \Big\{ \frac{q x^{\zeta_1} + \zeta_1 \mathcal{M}_{x,q}(\zeta_1)}{\psi'(\zeta_1)} \frac{x^{s-1-\zeta_1}}{1+\zeta_1-s} {}_4F_4\Big(\begin{matrix} 1+\zeta_1-s, 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 2+\zeta_1-s, 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ax} \Big) \Big\} \\ + \Big\{ \text{the same expression with } \zeta_1 \text{ and } \zeta_2 \text{ interchanged} \Big\}. \end{aligned}$$

Let us outline the plan for proving the identity

$$I_1(s) + I_2(s) - \mathcal{M}_{x,q}(s) = 0, \quad \text{for all } s \text{ in the strip } 0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1. \tag{122}$$

First we use formula (32) and express all Meijer G-functions appearing in (122) in terms of hypergeometric functions. This would give us an expression involving products of two hypergeometric functions. After simplifying this expression we would obtain the following identity

$$\begin{aligned} \sum_{i=1}^{5} \frac{(a_i-\rho)(a_i+\hat{\rho})}{\prod\limits_{\substack{1\le j\le 5 \\ j\neq i}} (a_i-a_j)} \times {}_4F_4\Big(\begin{matrix} 1+a_i-\rho, 1+a_i+\hat{\rho}, 1+a_i, 1+a_i-s \\ 1+a_i-a_1, \dots, *, \dots, 1+a_i-a_5 \end{matrix} \Big| -\frac{1}{Ax} \Big) \\ \times {}_4F_4\Big(\begin{matrix} 1+\rho-a_i, 1-\hat{\rho}-a_i, -a_i, s-a_i \\ 1+a_1-a_i, \dots, *, \dots, 1+a_5-a_i \end{matrix} \Big| \frac{1}{Ax} \Big) = 0, \quad x \in \mathbb{R} \setminus \{0\}, \end{aligned} \tag{123}$$

where $[a_1, a_2, a_3, a_4, a_5] = [\zeta_1, \zeta_2, -\hat{\zeta}_1, -\hat{\zeta}_2, s-1]$ and the asterisk means that the term $1 + a_i - a_i$ is omitted. The identity (123) is known to be true: it is a special case of Theorem 27 in Chapter 8.

Remark 5. The above steps of the proof, while conceptually simple, require very long computations. At the same time, it is easy to confirm the validity of this identity by a numerical experiment: one simply needs to compute Meijer G-functions via (32) and the hypergeometric functions via series expansion (19), and check that (122) holds true with arbitrary choices of parameters.

We recall that $x > 0$, $q > 0$, the numbers $\{-\hat{\zeta}_2, -\hat{\zeta}_1, \zeta_1, \zeta_2\}$ and $\{-\hat{\rho}, \rho\}$ are the roots and the poles of the rational function $\psi(z) - q$ and they are known to satisfy the interlacing property

$$-\hat{\zeta}_2 < -\hat{\rho} < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho < \zeta_2.$$

Note that the function $\psi(z) - q$ can be factorized as follows

$$\psi(z) - q = A\frac{(z-\zeta_1)(z-\zeta_2)(z+\hat{\zeta}_1)(z+\hat{\zeta}_2)}{(z-\rho)(z+\hat{\rho})}, \tag{124}$$

where $A := \sigma^2/2$. This fact (and the result $\psi(0) = 0$) implies

$$q = A\frac{\zeta_1\zeta_2\hat{\zeta}_1\hat{\zeta}_2}{\rho\hat{\rho}}, \tag{125}$$

and

$$\psi'(\zeta_1) = A\frac{(\zeta_1-\zeta_2)(\zeta_1+\hat{\zeta}_1)(\zeta_1+\hat{\zeta}_2)}{(\zeta_1-\rho)(\zeta_1+\hat{\rho})}, \tag{126}$$

$$\psi'(\zeta_2) = A\frac{(\zeta_2-\zeta_1)(\zeta_2+\hat{\zeta}_1)(\zeta_2+\hat{\zeta}_2)}{(\zeta_2-\rho)(\zeta_2+\hat{\rho})}. \tag{127}$$

Finally, we recall that we work under the following assumptions

$$\zeta_2 - \zeta_1 \notin \mathbb{N}, \qquad \hat{\zeta}_2 - \hat{\zeta}_1 \notin \mathbb{N}, \qquad 0 \vee (1-\hat{\zeta}_1) < \mathrm{Re}(s) < 1.$$

Our goal is to verify the following identity

$$I_1(s) + I_2(s) - \mathcal{M}_{x,q}(s) = 0, \tag{128}$$

where we have computed earlier

$$\begin{aligned} I_1(s) = \Big\{ qA^{-\hat{\zeta}_1}x^{s-\hat{\zeta}_1}\frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))}\, {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax}\Big) \\ \times G^{4,1}_{4,5}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho, s+1 \\ s, 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ax}\Big)\Big\} \\ + \Big\{\text{the same expression with } \hat{\zeta}_1 \text{ and } \hat{\zeta}_2 \text{ interchanged}\Big\}, \end{aligned} \tag{129}$$

and

$$I_2(s) = \Big\{ \frac{qx^{\zeta_1} + \zeta_1 \mathcal{M}_{x,q}(\zeta_1)}{\psi'(\zeta_1)} \frac{x^{s-1-\zeta_1}}{1+\zeta_1-s} {}_4F_4\Big(\begin{matrix} 1+\zeta_1-s, 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 2+\zeta_1-s, 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ax} \Big) \Big\} \tag{130}$$

$$+ \Big\{ \text{the same expression with } \zeta_1 \text{ and } \zeta_2 \text{ interchanged} \Big\},$$

and

$$\mathcal{M}_{x,q}(s) = qA^{-s}\Gamma\Big[\begin{matrix} 1+\zeta_1-s, \ 1+\zeta_2-s, \ \hat{\rho}+s \\ 1-s, \ 1+\rho-s, \ \hat{\zeta}_1+s, \ \hat{\zeta}_2+s \end{matrix} \Big] \tag{131}$$

$$\times G^{3,3}_{4,5}\Big(\begin{matrix} 1-s, 1, -\rho, \hat{\rho} \\ 1-s, \hat{\zeta}_1, \hat{\zeta}_2, -\zeta_1, -\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big).$$

Our main tool will be the following result, which expresses the Meijer G-function as a sum of hypergeometric functions (see formula (32), (33)). Assume that $b_j - b_k \notin \mathbb{Z}$ for $1 \le j < k \le m$ and $p < q$. Then for $x > 0$, we have:

$$G^{mn}_{pq}\binom{\mathbf{a}}{\mathbf{b}}\Big| x\Big) = \pi^{m+n-p-1} \sum_{k=1}^{m} \frac{\prod\limits_{j=n+1}^{p} \sin(\pi(a_j - b_k))}{\prod\limits_{\substack{1\le j\le m \\ j\ne k}} \sin(\pi(b_j - b_k))} \tag{132}$$

$$\times x^{b_k} {}_p\Phi_{q-1}\Big(\begin{matrix} 1+b_k-a_1, \dots, 1+b_k-a_p \\ 1+b_k-b_1, \dots, *, \dots, 1+b_k-b_q \end{matrix} \Big| (-1)^{p-m-n} x \Big).$$

As the proof will be rather technical and will involve many tedious computations, let us explain the main steps and ideas behind the proof. The first step is to express all Meijer G-functions appearing in (129), (130) and (101) in terms of hypergeometric functions via (32) or (33). In the second step we will use the results of step one and we will rewrite the expression in (122) as a sum of products of two hypergeometric functions. In the third step our goal is to simplify the expression obtained in step two. In the fourth step we will show the (simplified) identity related to finite sums of products of hypergeometric functions.

Let us deal with the first step – expressing Meijer G-functions in terms of hypergeometric functions.

Step 1a. We define

$$f_1 := {}_4\Phi_4\Big(\begin{matrix} 1, 1-s, 2+\rho-s, 2-\hat{\rho}-s \\ 2-s-\hat{\zeta}_1, 2-s-\hat{\zeta}_2, 2-s+\zeta_1, 2-s+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big),$$

$$f_2 := {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big),$$

$$f_3 := {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_2, 1+\hat{\zeta}_2+\rho, 1+\hat{\zeta}_2-\hat{\rho} \\ 1+\hat{\zeta}_2-\hat{\zeta}_1, 1+\hat{\zeta}_2+\zeta_1, 1+\hat{\zeta}_2+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big),$$

and

$$a_1 := -\pi q A^{-s}\Gamma\left[\begin{array}{c} 1+\zeta_1-s, 1+\zeta_2-s, \hat\rho+s \\ 1-s, 1+\rho-s, \hat\zeta_1+s, \hat\zeta_2+s \end{array}\right]\frac{\sin(\pi(\hat\rho+s))}{\sin(\pi(\hat\zeta_1+s))\sin(\pi(\hat\zeta_2+s))}(Ax)^{s-1},$$
$$a_2 := \pi q A^{-s}\Gamma\left[\begin{array}{c} 1+\zeta_1-s, 1+\zeta_2-s, \hat\rho+s \\ 1-s, 1+\rho-s, \hat\zeta_1+s, \hat\zeta_2+s \end{array}\right]\frac{\sin(\pi(\hat\rho-\hat\zeta_1))}{\sin(\pi(s+\hat\zeta_1))\sin(\pi(\hat\zeta_2-\hat\zeta_1))}(Ax)^{-\hat\zeta_1},$$
$$a_3 := \pi q A^{-s}\Gamma\left[\begin{array}{c} 1+\zeta_1-s, 1+\zeta_2-s, \hat\rho+s \\ 1-s, 1+\rho-s, \hat\zeta_1+s, \hat\zeta_2+s \end{array}\right]\frac{\sin(\pi(\hat\rho-\hat\zeta_2))}{\sin(\pi(s+\hat\zeta_2))\sin(\pi(\hat\zeta_1-\hat\zeta_2))}(Ax)^{-\hat\zeta_2}.$$

Then formulas (101) and (33) give us

$$\mathcal{M}_{x,q}(s) = a_1 f_1 + a_2 f_2 + a_3 f_3. \tag{133}$$

Step 1b. We define $f_4 := f_1\big|_{s=\zeta_1}$ and $f_5 := f_1\big|_{s=\zeta_2}$, that is

$$f_4 := {}_4\Phi_4\left(\begin{array}{c} 1, 1-\zeta_1, 2+\rho-\zeta_1, 2-\hat\rho-\zeta_1 \\ 2, 2-\zeta_1-\hat\zeta_1, 2-\zeta_1-\hat\zeta_2, 2-\zeta_1+\zeta_2 \end{array}\Big| \frac{1}{Ax}\right)$$
$$f_5 := {}_4\Phi_4\left(\begin{array}{c} 1, 1-\zeta_2, 2+\rho-\zeta_2, 2-\hat\rho-\zeta_2 \\ 2, 2-\zeta_2-\hat\zeta_1, 2-\zeta_2-\hat\zeta_2, 2-\zeta_2+\zeta_1 \end{array}\Big| \frac{1}{Ax}\right).$$

In the same way we define

$$b_1 := a_1\big|_{s=\zeta_1}, \qquad b_2 := a_2\big|_{s=\zeta_1}, \qquad b_3 := a_3\big|_{s=\zeta_1},$$
$$c_1 := a_1\big|_{s=\zeta_2}, \qquad c_2 := a_2\big|_{s=\zeta_2}, \qquad c_3 := a_3\big|_{s=\zeta_2}.$$

Then (133) gives us

$$\begin{aligned} \mathcal{M}_{x,q}(\zeta_1) &= b_1 f_4 + b_2 f_2 + b_3 f_3, \\ \mathcal{M}_{x,q}(\zeta_2) &= c_1 f_5 + c_2 f_2 + c_3 f_3. \end{aligned} \tag{134}$$

Step 1c. We define

$$f_6 := {}_4\Phi_4\left(\begin{array}{c} 1+\zeta_1+\hat\rho, 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1-s, \\ 2+\zeta_1-s, 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat\zeta_1, 1+\zeta_1+\hat\zeta_2 \end{array}\Big| -\frac{1}{Ax}\right),$$
$$f_7 := {}_4\Phi_4\left(\begin{array}{c} 1+\zeta_2+\hat\rho, 1+\zeta_2, 1+\zeta_2-\rho, 1+\zeta_2-s \\ 2+\zeta_2-s, 1+\zeta_2-\zeta_1, 1+\zeta_2+\hat\zeta_1, 1+\zeta_2+\hat\zeta_2 \end{array}\Big| -\frac{1}{Ax}\right),$$
$$f_8 := {}_4\Phi_4\left(\begin{array}{c} 1+\hat\rho-\hat\zeta_1, 1-\hat\zeta_1, 1-\rho-\hat\zeta_1, 1-s-\hat\zeta_1 \\ 2-\hat\zeta_1-s, 1-\hat\zeta_1-\zeta_1, 1-\hat\zeta_1-\zeta_2, 1+\hat\zeta_2-\hat\zeta_1 \end{array}\Big| -\frac{1}{Ax}\right),$$

and

$$\begin{aligned}
d_1 &:= -\frac{\sin(\pi\zeta_1)\sin(\pi(\rho-\zeta_1))}{\sin(\pi(\zeta_2-\zeta_1))\sin(\pi(\hat{\zeta}_1+\zeta_1))}(Ax)^{-\zeta_1-1},\\
d_2 &:= -\frac{\sin(\pi\zeta_2)\sin(\pi(\rho-\zeta_2))}{\sin(\pi(\zeta_1-\zeta_2))\sin(\pi(\hat{\zeta}_1+\zeta_2))}(Ax)^{-\zeta_2-1},\\
d_3 &:= -\frac{\sin(\pi\hat{\zeta}_1)\sin(\pi(\rho+\hat{\zeta}_1))}{\sin(\pi(\zeta_1+\hat{\zeta}_1))\sin(\pi(\zeta_2+\hat{\zeta}_1))}(Ax)^{\hat{\zeta}_1-1},\\
d_4 &:= \Gamma\left[\begin{array}{c} 1+\zeta_1-s, 1+\zeta_2-s, 1-\hat{\zeta}_1-s, s+\hat{\rho} \\ s+\hat{\zeta}_2, 1-s, 1+\rho-s \end{array}\right](Ax)^{-s}.
\end{aligned}$$

Then formulas (32) and (33) give us

$$G^{4,1}_{4,5}\left(\begin{array}{c} 1-\hat{\rho}, 1, 1+\rho, s+1 \\ s, 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{array}\Big|\frac{1}{Ax}\right) = d_1 f_6 + d_2 f_7 + d_3 f_8 + d_4. \tag{135}$$

Step 1d. We define

$$f_9 := {}_4\Phi_4\left(\begin{array}{c} 1+\hat{\rho}-\hat{\zeta}_2, 1-\hat{\zeta}_2, 1-\rho-\hat{\zeta}_2, 1-s-\hat{\zeta}_2 \\ 2-\hat{\zeta}_2-s, 1-\hat{\zeta}_2-\zeta_1, 1-\hat{\zeta}_2-\zeta_2, 1+\hat{\zeta}_1-\hat{\zeta}_2 \end{array}\Big|-\frac{1}{Ax}\right),$$

and

$$\begin{aligned}
e_1 &:= -\frac{\sin(\pi\zeta_1)\sin(\pi(\rho-\zeta_1))}{\sin(\pi(\zeta_2-\zeta_1))\sin(\pi(\hat{\zeta}_2+\zeta_1))}(Ax)^{-\zeta_1-1},\\
e_2 &:= -\frac{\sin(\pi\zeta_2)\sin(\pi(\rho-\zeta_2))}{\sin(\pi(\zeta_1-\zeta_2))\sin(\pi(\hat{\zeta}_2+\zeta_2))}(Ax)^{-\zeta_2-1},\\
e_3 &:= -\frac{\sin(\pi\hat{\zeta}_2)\sin(\pi(\rho+\hat{\zeta}_2))}{\sin(\pi(\zeta_1+\hat{\zeta}_2))\sin(\pi(\zeta_2+\hat{\zeta}_2))}(Ax)^{\hat{\zeta}_2-1},\\
e_4 &:= \Gamma\left[\begin{array}{c} 1+\zeta_1-s, 1+\zeta_2-s, 1-\hat{\zeta}_2-s, s+\hat{\rho} \\ s+\hat{\zeta}_1, 1-s, 1+\rho-s \end{array}\right](Ax)^{-s}.
\end{aligned}$$

Then formulas (32) and (33) give us

$$G^{4,1}_{4,5}\left(\begin{array}{c} 1-\hat{\rho}, 1, 1+\rho, s+1 \\ s, 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_2, 1-\hat{\zeta}_1 \end{array}\Big|\frac{1}{Ax}\right) = e_1 f_6 + e_2 f_7 + e_3 f_9 + e_4. \tag{136}$$

Our next goal is to collect all these formulas and express the functions $I_1(s)$ and $I_2(s)$ as sums of products $f_i f_j$.

Step 2a. We define

$$h_1 = qA^{-\hat{\zeta}_1} x^{s-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho} - \hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2 - \hat{\zeta}_1))},$$
$$h_2 = qA^{-\hat{\zeta}_2} x^{s-\hat{\zeta}_2} \frac{\sin(\pi(\hat{\rho} - \hat{\zeta}_2))}{\sin(\pi(\hat{\zeta}_1 - \hat{\zeta}_2))},$$

and from formulas (129), (135) and (136) we obtain

$$\begin{aligned} I_1(s) = (h_1 d_4) f_2 + (h_2 e_4) f_3 + (h_1 d_1) f_2 f_6 + (h_1 d_2) f_2 f_7 \\ + (h_1 d_3) f_2 f_8 + (h_2 e_1) f_3 f_6 + (h_2 e_2) f_3 f_7 + (h_2 e_3) f_3 f_9. \end{aligned} \tag{137}$$

Step 2b. We define

$$g_1 := \frac{1}{\psi'(\zeta_1)} \frac{x^{s-1-\zeta_1}}{1+\zeta_1-s} \Gamma\left[\begin{matrix} 2+\zeta_1-s, 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \\ 1+\zeta_1+\hat{\rho}, 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1-s \end{matrix} \right],$$
$$g_2 := \frac{1}{\psi'(\zeta_2)} \frac{x^{s-1-\zeta_2}}{1+\zeta_2-s} \Gamma\left[\begin{matrix} 2+\zeta_2-s, 1+\zeta_2-\zeta_1, 1+\zeta_2+\hat{\zeta}_1, 1+\zeta_2+\hat{\zeta}_2 \\ 1+\zeta_2+\hat{\rho}, 1+\zeta_2, 1+\zeta_2-\rho, 1+\zeta_2-s \end{matrix} \right],$$

and from formulas (130) and (134) we obtain

$$\begin{aligned} I_2(s) = (qx^{\zeta_1} g_1) f_6 + (qx^{\zeta_2} g_2) f_7 + (\zeta_1 b_1 g_1) f_4 f_6 + (\zeta_1 b_2 g_1) f_2 f_6 \\ + (\zeta_1 b_3 g_1) f_3 f_6 + (\zeta_2 c_1 g_2) f_5 f_7 + (\zeta_2 c_2 g_2) f_2 f_7 + (\zeta_2 c_3 g_2) f_3 f_7. \end{aligned} \tag{138}$$

Step 2c. Using all the previous results (formulas (133), (137) and (138)) we rewrite the identity (122) in an equivalent form

$$\begin{aligned} I_1(s) + I_2(s) - \mathcal{M}_{x,q}(s) = {} & (h_1 d_4) f_2 + (h_2 e_4) f_3 + (h_1 d_1) f_2 f_6 + (h_1 d_2) f_2 f_7 \\ & + (h_1 d_3) f_2 f_8 + (h_2 e_1) f_3 f_6 + (h_2 e_2) f_3 f_7 + (h_2 e_3) f_3 f_9 \\ & + (qx^{\zeta_1} g_1) f_6 + (qx^{\zeta_2} g_2) f_7 + (\zeta_1 b_1 g_1) f_4 f_6 + (\zeta_1 b_2 g_1) f_2 f_6 \\ & + (\zeta_1 b_3 g_1) f_3 f_6 + (\zeta_2 c_1 g_2) f_5 f_7 + (\zeta_2 c_2 g_2) f_2 f_7 + (\zeta_2 c_3 g_2) f_3 f_7 \\ & - (a_1 f_1 + a_2 f_2 + a_3 f_3) = 0. \end{aligned} \tag{139}$$

Now our goal is to simplify the long sum in (139). First we will deal with cancellations and then we will use a certain transformation of hypergeometric functions.

Step 3a. Using the reflection formula for the Gamma function (34) we check that

$$\begin{aligned} \zeta_1 b_3 g_1 &= -h_2 e_1, \\ \zeta_1 b_2 g_1 &= -h_1 d_1, \\ \zeta_2 c_2 g_2 &= -h_1 d_2, \\ \zeta_2 c_3 g_2 &= -h_2 e_2, \\ a_2 &= h_1 d_4, \\ a_3 &= h_2 e_4. \end{aligned}$$

These identities allow us to simplify the expression in (139) as follows

$$\begin{aligned} I_1(s) + I_2(s) - \mathcal{M}_{x,q}(s) = (qx^{\zeta_1} + \zeta_1 b_1 f_4) g_1 f_6 + (qx^{\zeta_2} + \zeta_2 c_1 f_5) g_2 f_7 \\ + (h_1 d_3) f_2 f_8 + (h_2 e_3) f_3 f_9 - a_1 f_1. \end{aligned} \tag{140}$$

Step 3b. In this step we will simplify (140) via the following result

$$1 + z \Big(\prod_{i=1}^{3} \frac{\alpha_i - 1}{\beta_i - 1} \Big) \times {}_4F_4 \Big(\begin{matrix} 1, \alpha_1, \alpha_2, \alpha_3 \\ 2, \beta_1, \beta_2, \beta_3 \end{matrix} \Big| z \Big) = {}_3F_3 \Big(\begin{matrix} \alpha_1 - 1, \alpha_2 - 1, \alpha_3 - 1 \\ \beta_1 - 1, \beta_2 - 1, \beta_3 - 1 \end{matrix} \Big| z \Big). \tag{141}$$

The above identity can be easily established by comparing the coefficients of the Taylor series of both sides. Applying identity (141) we obtain

$$qx^{\zeta_1} + \zeta_1 b_1 f_4 = qx^{\zeta_1} f_{10}, \tag{142}$$

$$qx^{\zeta_2} + \zeta_2 c_1 f_5 = qx^{\zeta_2} f_{11}, \tag{143}$$

where

$$f_{10} := {}_3F_3 \Big(\begin{matrix} -\zeta_1, 1 + \rho - \zeta_1, 1 - \hat{\rho} - \zeta_1 \\ 1 - \zeta_1 - \hat{\zeta}_1, 1 - \zeta_1 - \hat{\zeta}_2, 1 - \zeta_1 + \zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big),$$

$$f_{11} := {}_3F_3 \Big(\begin{matrix} -\zeta_2, 1 + \rho - \zeta_2, 1 - \hat{\rho} - \zeta_2 \\ 1 - \zeta_2 - \hat{\zeta}_1, 1 - \zeta_2 - \hat{\zeta}_2, 1 - \zeta_2 + \zeta_1 \end{matrix} \Big| \frac{1}{Ax} \Big).$$

Formulas (140), (142) and (143) give us an equivalent form of the identity $I_1(s) + I_2(s) - \mathcal{M}_{x,q}(s) = 0$ as follows

$$(qx^{\zeta_1} g_1) f_6 f_{10} + (qx^{\zeta_2} g_2) f_7 f_{11} + (h_1 d_3) f_2 f_8 + (h_2 e_3) f_3 f_9 - a_1 f_1 = 0. \tag{144}$$

Step 4. By simplifying the coefficients (again, using the reflection formula for the Gamma function (34)) one can check that the left-hand side in (144) is a finite (that is, non-infinite) multiple of

$$\begin{aligned} H(x) := \sum_{i=1}^{5} \frac{(\alpha_i - \rho)(\alpha_i + \hat{\rho})}{\prod\limits_{\substack{1 \le j \le 5 \\ j \ne i}} (\alpha_i - \alpha_j)} \times {}_4F_4 \Big(\begin{matrix} 1 + \alpha_i - \rho, 1 + \alpha_i + \hat{\rho}, 1 + \alpha_i, 1 + \alpha_i - s \\ 1 + \alpha_i - \alpha_1, \dots, *, \dots, 1 + \alpha_i - \alpha_5 \end{matrix} \Big| - \frac{1}{Ax} \Big) \\ \times {}_4F_4 \Big(\begin{matrix} 1 + \rho - \alpha_i, 1 - \hat{\rho} - \alpha_i, -\alpha_i, s - \alpha_i \\ 1 + \alpha_1 - \alpha_i, \dots, *, \dots, 1 + \alpha_5 - \alpha_i \end{matrix} \Big| \frac{1}{Ax} \Big), \end{aligned} \tag{145}$$

where $[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] = [\zeta_1, \zeta_2, -\hat{\zeta}_1, -\hat{\zeta}_2, s - 1]$ and the asterisk means that the term $1 + \alpha_i - \alpha_i$ is omitted. The identity $H(x) \equiv 0$ is a special case of Theorem 27 in Chapter 8. To see this, we should set $p = r = 4$ and

$$\{a_i\}_{1 \le i \le 5} = \{\zeta_1, \zeta_2, -\hat{\zeta}_1, -\hat{\zeta}_2, s - 1\}, \qquad \{b_i\}_{1 \le i \le 4} = \{1 + \rho, 1 + \hat{\rho}, 0, s\}, \qquad \{m_i\}_{1 \le i \le 4} = \{1, 1, 0, 0\},$$

in the notion of Theorem 27. $\square$

Now we have proved that $f_{x,q}(y)$ is exactly the probability density function of $I_{x,q}$. At the beginning, we used the Mellin inverse transform to obtain the $f_{x,q}(y)$, but when $y > x$, our argument is not solid, we tried to compare the Mellin transform of $f_{x,q}(y)$ to $\mathcal{M}_{x,q}$, one can do a quick numerical check that theses two are equal. While we were doing the theoretical proof of the equality, we found the identity $H(x) \equiv 0$ in (145), which has triggered the topic in Chapter 8 as a generalization of this identity for hypergeometric functions.

At the end of this section, we are going to graph the probability density function $f_{x,q}(y)$ under the different choice of parameters.

$$\begin{aligned}
&\text{(Parameter set A)} && \mu = 0.119161, \sigma = 0.100499, \lambda = 1, p = 0.3, \rho = 20, \hat{\rho} = 10, \\
& && x = 1/0.0035, q = 0.785; \\
&\text{(Parameter set B)} && \mu = 0.064186, \sigma = 0.144395, \lambda = 0.00005, p = 0.3, \rho = 0.1, \hat{\rho} = 0.2, \\
& && x = 1/0.0035, q = 0.785; \\
&\text{(Parameter set C)} && \mu = -0.681815, \sigma = 1.399405, \lambda = 0.142833, p = 0.889337, \rho = 1.063917, \\
& && \hat{\rho} = 0.481293, x = 1.970593, q = 1.120256;
\end{aligned}$$

which gives the roots for $\phi(z) = q$ according to each set of parameters

$$\begin{aligned}
\text{set A}: \quad & \zeta_1 = 6.120373734933308, \zeta_2 = 21.755413426264614, \\
& \hat{\zeta}_1 = 6.7183299869653, \hat{\zeta}_2 = 34.75358031941775; \\
\text{set B}: \quad & \zeta_1 = 0.099998073227489, \zeta_2 = 6.129225774533752, \\
& \hat{\zeta}_1 = 0.199991222181719, \hat{\zeta}_2 = 12.286187878832155; \\
\text{set C}: \quad & \zeta_1 = 0.934010684229182, \zeta_2 = 1.632740553575107, \\
& \hat{\zeta}_1 = 0.469390641058205, \hat{\zeta}_2 = 0.818415354958329;
\end{aligned}$$

From Figure 1, Figure 2 and Figure 3, we can observe that the probability density function is continuous but is not differentiable at the point $y = x$.

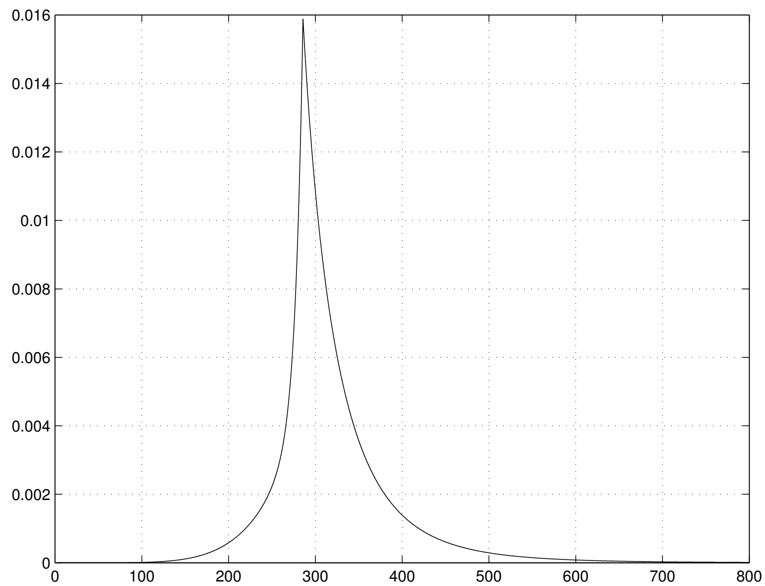


Figure 1: Probability density function $f_{x,q}(y)$ for set A

6.3.4 Distribution and expectation of $I_{x,q}$

In the following we compute the distribution function of $I_{x,q}$ and the tail expectation of $I_{x,q}$. These results will be used in Section 6.4.

Corollary 1. *For* $y \geq x$

$$\mathbb{P}(I_{x,q} > y) = \Big\{ \frac{qx^{\zeta_1} + \zeta_1 \mathcal{M}_{x,q}(\zeta_1)}{\zeta_1 \psi'(\zeta_1)} y^{-\zeta_1} {}_3F_3\Big(\begin{matrix} 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho}, \zeta_1 \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ay} \Big) \Big\} \tag{146}$$
$$+ \Big\{ \text{the same expression with } \zeta_1 \text{ and } \zeta_2 \text{ interchanged} \Big\}$$

and for $0 < y < x$

$$\mathbb{P}(I_{x,q} < y) = \Big\{ \frac{q}{A}(Ax)^{-\hat{\zeta}_1} \frac{\sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))} {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax} \Big) \tag{147}$$
$$\times G^{3,1}_{3,4}\Big(\begin{matrix} -\hat{\rho}, \rho, 1 \\ \zeta_1, \zeta_2, -\hat{\zeta}_1, -\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay} \Big) \Big\}$$
$$+ \Big\{ \text{the same expression with } \hat{\zeta}_1 \text{ and } \hat{\zeta}_2 \text{ interchanged} \Big\}.$$

Proof. Formula (146) can be easily obtained from (118) and (38). First we look at formula (118), for $y \geq x$,

$$\mathbb{P}(I_{x,q} > y) = \int_y^\infty f_{x,q}(s) \mathrm{d}s. \tag{148}$$

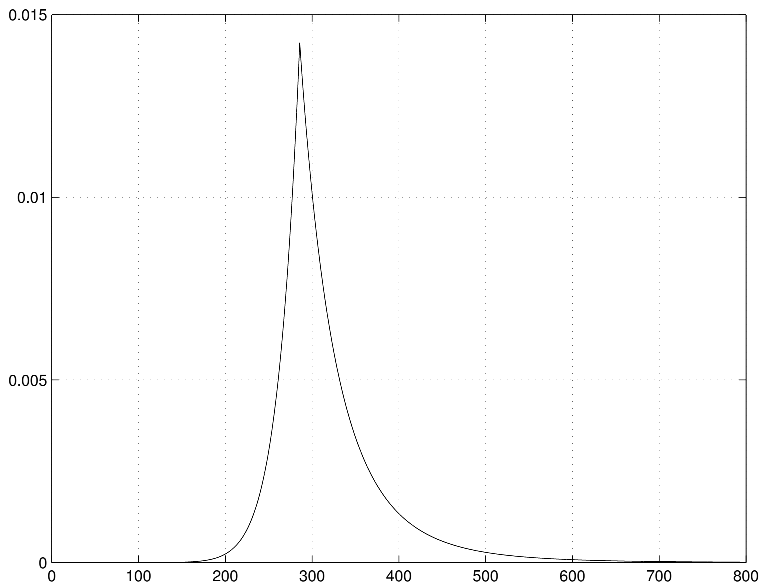


Figure 2: Probability density function $f_{x,q}(y)$ for set B

By observing (118), we can find it is essentially

$$\begin{aligned}
& \int_y^\infty s^{-1-\zeta_1} {}_3F_3\Big(\begin{matrix} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{As}\Big)\mathrm{d}s \\
= & \int_y^\infty -y^{-\zeta_1}(\frac{y}{s})^{\zeta_1-1} {}_3F_3\Big(\begin{matrix} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ay}\cdot\frac{y}{s}\Big)\mathrm{d}\frac{y}{s} \\
= & \; y^{-\zeta_1}\int_0^1 (t)^{\zeta_1-1} {}_3F_3\Big(\begin{matrix} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho} \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ay}\cdot t\Big)\mathrm{d}t \\
= & \; \frac{y^{-\zeta_1}}{\zeta_1} {}_3F_3\Big(\begin{matrix} 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho}, \zeta_1 \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ay}\Big).
\end{aligned} \tag{149}$$

Here in the first step and second step we use variable change: $t = y/s$, in the third step we apply (38).

Similarly, we find

$$\begin{aligned}
& \int_y^\infty s^{-1-\zeta_2} {}_3F_3\Big(\begin{matrix} 1+\zeta_2, 1+\zeta_2-\rho, 1+\zeta_2+\hat{\rho} \\ 1+\zeta_2-\zeta_1, 1+\zeta_2+\hat{\zeta}_1, 1+\zeta_2+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{As}\Big)\mathrm{d}s \\
= & \; \frac{y^{-\zeta_2}}{\zeta_2} {}_3F_3\Big(\begin{matrix} 1+\zeta_2-\rho, 1+\zeta_2+\hat{\rho}, \zeta_2 \\ 1+\zeta_2-\zeta_1, 1+\zeta_2+\hat{\zeta}_1, 1+\zeta_2+\hat{\zeta}_2 \end{matrix} \Big| -\frac{1}{Ay}\Big),
\end{aligned} \tag{150}$$

thus we get (146).

For $0 < y < x$,

$$\mathbb{P}(I_{x,q} < y) = \int_0^y f_{x,q}(s)\mathrm{d}s. \tag{151}$$

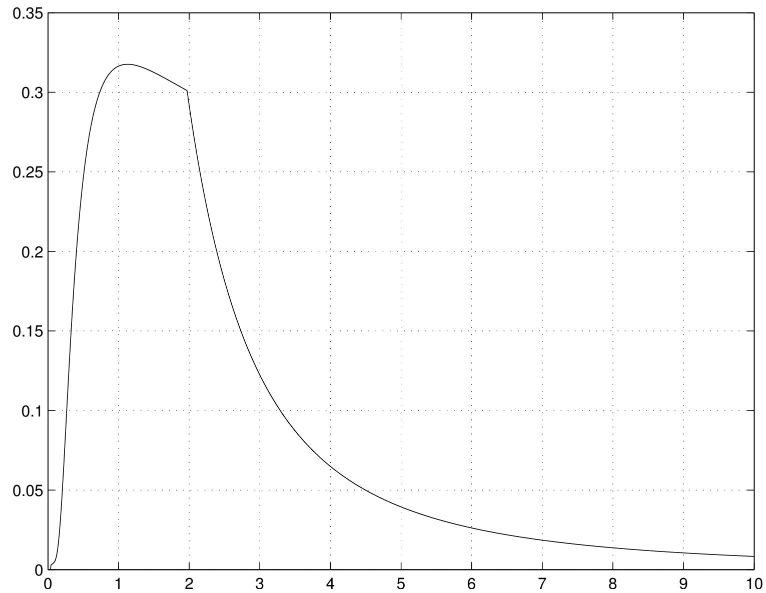


Figure 3: Probability density function $f_{x,q}(y)$ for set C

By observing (119), we can find it is essentially

$$
\begin{aligned}
& \int_0^y G_{3,4}^{3,1}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{As}\Big) \mathrm{d}s \\
= \; & y \int_1^{\infty} (t)^{-2} G_{3,4}^{3,1}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay} \cdot t\Big) \mathrm{d}t \\
= \; & y G_{4,5}^{4,1}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho, 2 \\ 1, 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay}\Big) \\
= \; & y G_{3,4}^{3,1}\Big(\begin{matrix} 1-\hat{\rho}, 1+\rho, 2 \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay}\Big) \\
= \; & y \cdot \frac{1}{Ay} G_{3,4}^{3,1}\Big(\begin{matrix} -\hat{\rho}, \rho, 1 \\ \zeta_1, \zeta_2, -\hat{\zeta}_1, -\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay}\Big).
\end{aligned}
$$

Here in the first step we change variable $t = y/s$, in the second step we apply (37), in the third step we apply (36), in the fourth step we apply (29).

Similarly, we find

$$
\begin{aligned}
& \int_0^y G_{3,4}^{3,1}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho \\ 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_2, 1-\hat{\zeta}_1 \end{matrix} \Big| \frac{1}{As}\Big) \mathrm{d}s \\
= \; & y \cdot \frac{1}{Ay} G_{3,4}^{3,1}\Big(\begin{matrix} -\hat{\rho}, \rho, 1 \\ \zeta_1, \zeta_2, -\hat{\zeta}_2, -\hat{\zeta}_1 \end{matrix} \Big| \frac{1}{Ay}\Big),
\end{aligned}
$$

thus we can get (147). □

Corollary 2. *Assume that $\zeta_1 > 1$, then for $y \geq x$*

$$\mathbb{E}[I_{x,q}\mathbf{1}_{\{I_{x,q}>y\}}] = \Big\{\frac{qx^{\zeta_1} + \zeta_1\mathcal{M}_{x,q}(\zeta_1)}{\psi'(\zeta_1)(\zeta_1-1)} y^{1-\zeta_1}\, {}_4F_4\Big(\begin{matrix} 1+\zeta_1, 1+\zeta_1-\rho, 1+\zeta_1+\hat{\rho}, \zeta_1-1 \\ 1+\zeta_1-\zeta_2, 1+\zeta_1+\hat{\zeta}_1, 1+\zeta_1+\hat{\zeta}_2, \zeta_1 \end{matrix} \Big| -\frac{1}{Ay}\Big)\Big\} \tag{152}$$

$$+ \Big\{\text{the same expression with } \zeta_1 \text{ and } \zeta_2 \text{ interchanged}\Big\},$$

and for $0 < y < x$

$$\mathbb{E}[I_{x,q}\mathbf{1}_{\{I_{x,q}<y\}}] = \Big\{\frac{qy^2 \sin(\pi(\hat{\rho}-\hat{\zeta}_1))}{(Ax)^{\hat{\zeta}_1}\sin(\pi(\hat{\zeta}_2-\hat{\zeta}_1))}\, {}_3\Phi_3\Big(\begin{matrix} \hat{\zeta}_1, 1+\hat{\zeta}_1+\rho, 1+\hat{\zeta}_1-\hat{\rho} \\ 1+\hat{\zeta}_1-\hat{\zeta}_2, 1+\hat{\zeta}_1+\zeta_1, 1+\hat{\zeta}_1+\zeta_2 \end{matrix} \Big| \frac{1}{Ax}\Big) \tag{153}$$
$$\times G^{4,1}_{4,5}\Big(\begin{matrix} 1-\hat{\rho}, 1, 1+\rho, 3 \\ 2, 1+\zeta_1, 1+\zeta_2, 1-\hat{\zeta}_1, 1-\hat{\zeta}_2 \end{matrix} \Big| \frac{1}{Ay}\Big)\Big\}$$
$$+ \Big\{\text{the same expression with } \hat{\zeta}_1 \text{ and } \hat{\zeta}_2 \text{ interchanged}\Big\}.$$

Proof. Same steps as in the proof of Corollary 1. □

6.4 Application to GMDB

As we have discussed in the introduction, due to the continual collection of management fees as a fixed percentage of policyholders' account value, exponential functionals arise naturally in the analysis of insurer's liabilities to variable annuity guaranteed benefits. In this section we use the obtained theoretical results to compute various risk measures for the guaranteed minimum death benefit (GMDB), which is one of the most common types of investment guarantees in the market.

Assume that the equity index $\{S_t, t \geq 0\}$ is modeled by an exponential Lévy process

$$S_t := S_0 e^{X_t}, \qquad t \geq 0,$$

where X is the Kou process, as defined in (98). Assume, also, that the policyholder's investment account is driven by the equity-linking mechanism as in (85). Recall that the GMDB net liability from an insurer's viewpoint is given by

$$L := e^{-rT_x}(F_0 e^{rT_x} - F_{T_x})_+ - \int_0^{T_x} e^{-rs} m_d F_s \, \mathrm{d}s. \tag{154}$$

Due to the independence of mortality risk and equity risk, we obtain an expression of $\mathbb{P}(L > V)$ for $V \geq \mathrm{VaR}_\xi > 0$,

$$\mathbb{P}(L > V) = \int_0^\infty P(t, K) f(t) \, \mathrm{d}t, \tag{155}$$

where f is the probability density function of T_x, $K := (F_0 - V)/(m_d F_0)$ and

$$P(t, K) := \mathbb{P}\left(x e^{X_t^*} + \int_0^t e^{X_s^*} \mathrm{d}s < K\right)$$

with $x = 1/m_d$. The underlying Lévy process X^* is the same as the process X in (98), but with μ replaced by

$$\mu^* := \mu - r - m.$$

The Laplace transform of P with respect to t is given by

$$\tilde{P}(q, K) := \int_0^\infty e^{-qt} P(t, K) \mathrm{d}t = \frac{1}{q} \mathbb{P}(I_{x,q} < K).$$

Similarly, we can show that

$$\mathrm{CTE}_p(L) = F_0 - \frac{m_d F_0}{1-p} \int_0^\infty Z(t, K) f(t) \mathrm{d}t, \tag{156}$$

where

$$Z(t, K) := \mathbb{E}\left[\left(x e^{X_t^*} + \int_0^t e^{X_s^*} \mathrm{d}s\right) \mathbb{1}_{\left\{x e^{X_t^*} + \int_0^t e^{X_s^*} \mathrm{d}s < K\right\}}\right].$$

Its Laplace transform with respect to t is given by

$$\tilde{Z}(q, K) := \int_0^\infty e^{-qt} Z(t, K) \mathrm{d}t = \frac{1}{q} \mathbb{E}\left[I_{x,q} \mathbb{1}_{\{I_{x,q} < K\}}\right].$$

A common model for human mortality in the literature is the so-called Gompertz-Makeham law of mortality, which assumes that the death rate μ_x is the sum of a constant A (to account for death due to accidents) and a component Bc^x (to account for aging):

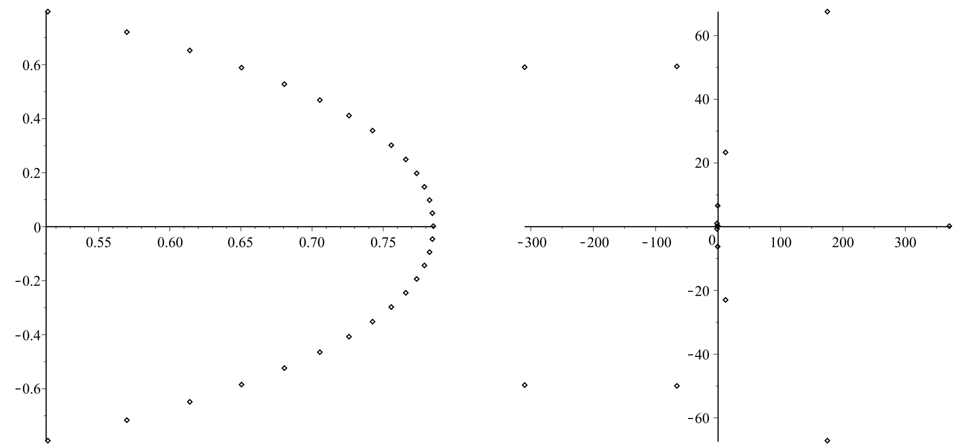
$$\mu_x = A + Bc^x, \qquad A > 0, B > 0, c > 1.$$

Its probability density function f is given by

$$f(t) = (A + Bc^{x+t}) \exp\left\{-At - \frac{Bc^x(c^t - 1)}{\ln c}\right\}. \tag{157}$$

As shown in Feng and Jing [20], we can always use a decomposition of a Hankel matrix to approximate f by a combination of exponential functions with complex components and complex weights,

$$f(t) \approx \sum_{i=1}^M w_i e^{-s_i t}, \qquad \Re(s_i) > 0.$$



(a) Bases

(b) Weights

Figure 4: Approximating exponential sum

There are many known methods in the literature for such approximations, most of which utilizes only real components and real weights. However, the Hankel matrix method has the advantage of using relatively small number of terms. Then, for large enough M,

$$\mathbb{P}(L > V) \approx \sum_{i=1}^{M} w_i \tilde{P}(s_i, K). \tag{158}$$

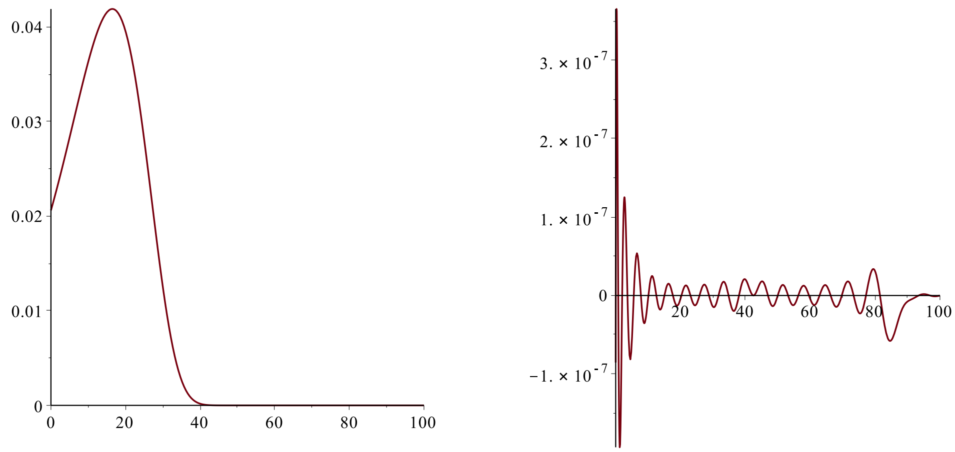
Similarly, we can approximate the CTE risk measure by

$$\mathrm{CTE}_p(L) \approx F_0 - \frac{m_d F_0}{1-p} \sum_{i=1}^{M} w_i \tilde{Z}(s_i, K). \tag{159}$$

Let us illustrate the application to GMDB with a numerical example.

(i) Survival model. Suppose that the variable annuity contract under consideration is issued to a 65-year-old, whose survival model is determined by the Gompertz-Makeham law of mortality with the probability density given in (157) where $x = 65, A = 0.0007, B = 0.00005, c = 10^{0.04}$. Using the Hankel matrix method, we approximate the mortality density by a combination of $M = 15$ terms of exponential functions. The bases and weights of the 15-term exponential sum are shown in Figure 4. In Figure 5, we show the plot of the original density function as well as the error from the 15-term approximating exponential sum. It is clear from the plots that the maximum error is controlled,

$$\sup_{t \in [0,100]} \left| f(t) - \sum_{i=1}^{M} w_i e^{-s_i t} \right| < 10^{-6}.$$



(a) Mortality density (b) Approximation error

Figure 5: Approximation of mortality density

(ii) Equity model. Suppose that the variable annuity contract is invested in a single equity fund which is driven by either of the following two models

1. Geometric Brownian motion (GBM): Here we use a standard model from the insurance industry calibrated to monthly S&P 500 total return data from December 1955 to December 2003 inclusive. The model is also known to pass the calibration criteria for equity return models set by the AAA (c.f. AAA report [26, p.35]).

$$\mu_1 = 0.064161, \sigma_1 = 0.16.$$

2. Exponential Lévy process with bilateral exponential jumps (Kou): we employ two sets of parameters for comparison with the GBM model.

   (Parameter set A) $\mu_2 = 0.119161, \sigma_2 = 0.100499, \lambda = 1, p = 0.3, \rho = 20, \hat{\rho} = 10;$

   (Parameter set B) $\mu_2 = 0.064186, \sigma_2 = 0.144395, \lambda = 0.00005, p = 0.3, \rho = 0.1, \hat{\rho} = 0.2.$

   The parameters are chosen so that the first two moments of X_1 are kept the same for both the GBM model and the Kou model, i.e.

$$\mu_1 = \mu_2 + \frac{\lambda p}{\rho} - \frac{\lambda(1-p)}{\hat{\rho}},$$
$$\sigma_1^2 = \sigma_2^2 + \frac{2\lambda p}{\rho^2} + \frac{2\lambda(1-p)}{\hat{\rho}^2}.$$

   The first set of parameters leads to relatively frequent occurrence of small jumps, whereas the second set of parameters is chosen to exhibit relatively rare occurrence of large jumps.

	$\lambda = 1$	$\lambda = 0.01$	$\lambda = 0.0001$	$\lambda = 0.000001$	GBM ($\lambda = 0$)
$\mathbb{P}(L > 0.2)$	0.4794368114	0.0954727742	0.0927572184	0.0927302874	0.0927300396
$\mathbb{P}(L > 0.4)$	0.3313624187	0.03327852158	0.03185715421	0.03184312600	0.03184298681
$\mathbb{P}(L > 0.6)$	0.1787553560	0.06201911742	0.005797295345	0.005793340382	0.005793300500

Table 1: Tail probabilities for the GMDB net liability

	Analytic	Monte Carlo ($N = 1,000$)	Monte Carlo ($N = 100,000$)
$\mathbb{P}(L > 0.2)$	0.4794368114	0.4787000000 (0.0154956700)	0.4796620000 (0.0015078343)
Time	11.097	68.422203	7107.196853
$\mathbb{P}(L > 0.4)$	0.3313624187	0.3342000000 (0.0143218640)	0.3321305000 (0.0013534030)
Time	10.912	–	–
$\mathbb{P}(L > 0.6)$	0.1787553560	0.1780 (0.0105481353)	0.1794875000 (0.0011432358)
Time	10.463	–	–

Table 2: Tail probabilities for the GMDB net liability with $\lambda = 1$

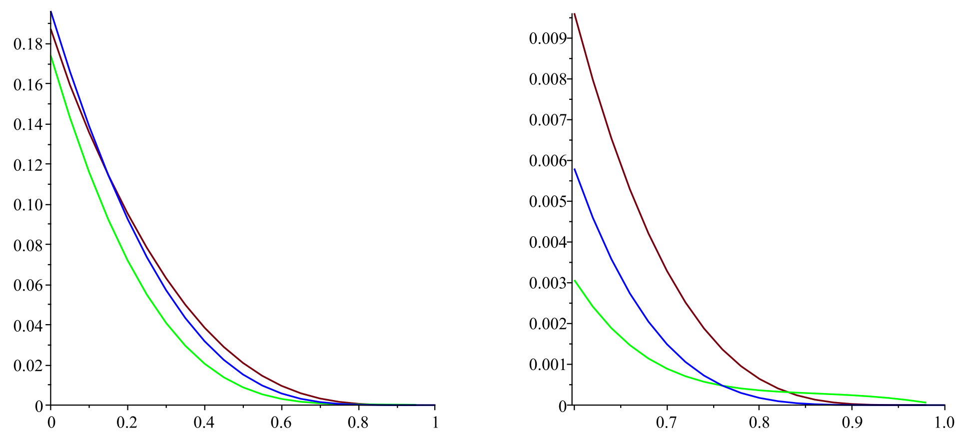
(iii) Fee schedule. The initial purchase payment is assumed to be $F_0 = 1$. The guarantee level starts off at $G_0 = 1$ and the yield rate on the insurer's assets backing up the GMDB liability is given by $r = 0.02$. The mortality and expenses (M&E) fee is charged at the rate of $m = 0.01$ per dollar of the policyholder's investment account per time unit. The GMDB rider charge rate is assumed to be 35% of the M&E fee rate, i.e. $m_d = 0.0035$.

Recall that the GBM model is in fact a special case of the Kou model. Hence we shall first use tail probabilities of the GMDB net liability under the GBM model as benchmarks against which the accuracy of corresponding results under the Kou model can be tested. In Table 1, the last row of tail probabilities are computed by formula (158) where $\tilde{\mathbb{P}}(s, K)$ is determined by formulas in Feng and Volkmer [24, Proposition 3.4]. The rest of the table are by formula (158) where $\tilde{\mathbb{P}}(s, K)$ is determined by formulas in Corollary 1. For the ease of direct comparison with the GBM model, we set for the Kou model

$$\mu_2 = 0.064161, \sigma_2 = 0.16, p = 0.3, \rho = 20, \hat{\rho} = 10.$$

As expected, Table 1 indicates that the tail probability of the GMDB net liability under the Kou model converges point-wise to the corresponding result under the GBM model, as the intensity rate λ of jumps declines to zero.

We can also test the accuracy of results on tail probabilites of GMDB net liability against those



(a) All positive liabilities

(b) Extremely large liabilities

Figure 6: Tail probability of GMDB net liability

resulting from a Monte Carlo method. Take the case of $\lambda = 1$ for example in Table 2. For the Monte Carlo method, we first employ an acceptance-rejection method to generate policyholders' remaining lifetimes from the Gompertiz-Makeham law of mortality in (157). In each experiment, we simulate N sample paths of the equity index based on the exponential Levy model from the beginning to policyholders' times of death. Under each sample path, we determine the GMDB net liability by the Riemman sum corresponding to (154) with a step size of 0.01. The GMDB payment is assumed to be payable at the end of the time step upon death. The tail probabilities $\mathbb{P}(L > 0.2), \mathbb{P}(L > 0.4), \mathbb{P}(L > 0.6)$ are estimated respectively by the number of sample paths under which the GMDB net liability surpasses the thresholds $0.2, 0.4, 0.6$, respectively, divided by the total number of sample paths N. In Table 2, we report tail probability results from both analytic formulas and estimates from Monte Carlo simulations. Computing time is reported in seconds. All algorithms based on the Monte Carlo method are implemented in Matlab (version 2016a) whereas results from analytic formulas are obtained in Maple (version 2016.1). In addition, each Monte Carlo result is the mean of estimates from 20 independent experiments and the corresponding sample standard deviation is quoted in brackets. Observe that Monte Carlo simulations are very time consuming to reach accuracy up to three decimal places. Therefore, it is worthwhile performing the above analysis to develop analytic formulas, as they are in general much more efficient and more accurate than Monte Carlo simulations.

Owing to the analytic formulas developed in this dissertation, the computational algorithm for tail probability is very efficient, enabling us to plot the tail probability function. The visualization of tail probabilities allows us to develop an understanding of the impact of jumps to the overall riskiness of insurer's liability. For example, we plot tail probability functions of the GMDB rider under the GBM model and the Kou models. In Figure 6, the blue line represents the tail probability function under the GBM model whereas the red line and green line represent the tail probability

	$\text{VaR}_{0.85}$	$\text{VaR}_{0.9}$	$\text{VaR}_{0.95}$	$\text{VaR}_{0.9999}$
Parameter set A	0.069344	0.187615	0.349984	0.868025
Parameter set B	0.038537	0.132969	0.266704	0.967712
	$\text{CTE}_{0.85}$	$\text{CTE}_{0.9}$	$\text{CTE}_{0.95}$	$\text{CTE}_{0.9999}$
Parameter set A	0.295863	0.380809	0.498331	0.890319
Parameter set B	0.226736	0.298245	0.401757	0.983389

Table 3: Risk measures for the GMDB net liability

function under the Kou models with parameter sets A and B respectively. The horizontal axis shows the level of net liability as a percentage of initial purchase payment and the vertical axis measures the corresponding tail probability. Figure 6(a) appears to indicate that the models with jumps tend to result in smaller probability of losses (positive net liability), which may be counterintuitive. This is likely caused by the fact that parameter sets A and B for the Kou models introduce smaller volatilities of white noise than that in the GBM model, which implies that larger probability masses are concentrated around negative net liabilities (profits for the insurer). The presence of jumps appears to play a role for generating extremely large liabilities, as shown in Figure 6(b). The tail probability in the Kou model with large jumps, represented by the green line, has a fatter tail than that in the GBM model, represented by blue line. The tail probability in the Kou model with smaller jumps, represented by the red line, also has a fatter tail, although to a less extent than the Kou model with large jumps. This is not surprising, as the equity index in Kou models with jumps can drop faster than the GBM can, thereby leading to severe losses for the insurer in extreme cases. This experiment shows that Kou models tend to produce more conservative estimates of insurer's net liabilities at the far right tail than the standard GBM model used in practice.

Next we illustrate the computation of risk measures for the GMDB net liability. The $\text{CTE}_{0.9}$ risk measure is commonly used to determine risk-based capitals for variable annuity guarantee products in the US. First we use the expression in (158) to determine tail probability of GMDB net liability for various levels and then employ a bisection root search algorithm to determine the exact quantiles. The algorithm terminates when the search interval narrows down to a width less than 10^{-7}. Then all results in Table 3 are rounded to nearest sixth decimal place. Then the VaR results are fed into the algorithm for determining the CTE based on the expression (159). Note that in Table 3 both quantile and CTE risk measures at confidence levels $p = 0.85, 0.9, 0.95$ for the model with parameter set A are larger than those in the model with parameter set B, which is consistent with the observation in Figure 6(a) that tail probability for the model with parameter set A (red line) tends to dominate that for the model with parameter set B (green line). However, if we move to the far right tail, the quantile and CTE risk measures at $p = 0.9999$ for the model with parameter set A become less than those for the model with parameter set B, confirmed by the reversed dominance in Figure 6(b). Again the comparison of risk measures show that infrequent occurrence of large jumps only increases the tail probability at extremely high levels of liabilities whereas frequent occurrence of small jumps may significantly increase the tail probability at more modest levels of liabilities, which are often of interest to insurance applications.

7 Optimal control in first to default problem

In this chapter, we let a two dimensional Brownian motion with dynamic correlation represent the value processes of two assets. We demonstrate the way of valuing credit risk by pricing "first to default option". Since different choice of the correlation leads to different price for the option, we investigate the highest price and the lowest price according to the dynamic of the correlation. We show how we transform this first-to-default problem to an optimal control problem and how we solve it analytically. Additionally, we extract one topic related to the skew Brownian motion in the process of obtaining the optimal results, such topic has independent interest.

7.1 Introduction

Credit risk is essential for valuing assets. For a single asset, the credit risk can be derived from its market spread information, such as spreads on corporate bonds or on single-asset credit default swaps (see Duffie and Singleton [18]). However, the same is not true for portfolios consisting of assets, such as CDOs (collateralized debt obligation). The main challenge lies in specifying the dependence among the assets, given their marginal distributions. The most widely used mechanism for characterizing this dependence is the copula model initiated by Li [46]. This constructs the joint probability distribution for the time to default from its marginal distribution. Based on this model, Monte Carlo simulation has been employed to evaluate the product, see Joshi and Kainth [35], Chen and Glasserman [14]. A fast procedure is also proposed by Hull and White [33] by using fast Fourier transforms. The advantage of the copula model is its flexibility, computational speed and ease of calibration. However, the correlations in this model are introduced without regard to the dynamics of the underlying assets. This means that such a model provides no means to include the dynamics of correlation and asset value changes, which makes it unreliable when the credit market becomes stressed. In contrast, there are models dealing with credit dynamics, such as multifirm structrual credit model, which is based on Merton [52] by connecting the default with the asset value process. This model can incorporate the dependence quite naturally by assuming that the assets follow correlated stochastic processes. However, the computation can be very difficult, depending on the complexity of the stochastic processes. Especially, the Monte Carlo simulation is also very hard to implement under dynamic correlation assumptions. Hurd [34] has modeled the credit risk for multi assets by using time-changed Brownian motions to model the assets value processes, but with a strong condition that the time-changed Brownian motions are conditionally independent. There is always a trade-off between the choice of assets value processes and the setting of their correlation.

In this chapter, suppose we have two assets A_1 and A_2. We are interested in the probability that the first default happens before a fixed time T. In order to dynamically reflect the dependence, we assume the assets follow correlated stochastic processes and set the correlation between them dynamic. Since a different choice of correlation will lead to a different first-to-default probability, we are interested in the highest and lowest probability and also the corresponding correlation. However, under the dynamic correlation assumption, the computation can be very difficult, depending on the complexity of the stochastic processes. In order to make the computation feasible, we will use simple processes to model the assets by trade off. We let $S_t^1 = e^{\sigma_1 B_t^1}$ and $S_t^2 = e^{\sigma_2 B_t^2}$ represent the value of assets A_1 and A_2, where (B_t^1, B_t^2) is a two dimensional Brownian motion which starts from (B_0^1, B_0^2) and satisfies

$$\mathrm{d}\langle B^1, B^2\rangle_t = \rho_t \mathrm{d}t. \tag{160}$$

Here ρ_t denotes the correlation and $\rho_t \in [-1, 1]$. The default happens when either asset reaches the value 1, which is equivalent to that B_t^1 or B_t^2 reaches 0. Of course, this is not a general geometric Brownian motion model, as it assumes that $\mu_1 = \sigma_1^2/2$ and $\mu_2 = \sigma_2^2/2$ in the geometric Brownian motion setting. However, in our case some closed form solutions do exist.

We let $\tau = \inf\{0 \le t \le T : B_t^1 = 0 \text{ or } B_t^2 = 0\}$ be the time the first default happens, thus our problem transforms to investigating

$$\max_{\rho_t} \mathbb{P}(\tau \le T) \quad \text{and} \quad \min_{\rho_t} \mathbb{P}(\tau \le T).$$

We will give a quick answer to the minimum case. If we let $\tau_1 = \inf\{0 \le t \le T : B_t^1 = 0\}$ and $\tau_2 = \inf\{0 \le t \le T : B_t^2 = 0\}$, so $\tau = \tau_1 \wedge \tau_2$. For any choice of ρ_t, we have

$$\mathbb{P}(\tau \le T) = \mathbb{P}(\tau_1 \le T \text{ or } \tau_2 \le T). \tag{161}$$

Since we know the facts $\mathbb{P}(A \cup B) \ge \mathbb{P}(A)$ and $\mathbb{P}(A \cup B) \ge \mathbb{P}(B)$, therefore we have

$$\mathbb{P}(\tau \le T) \ge \max\{\mathbb{P}(\tau_1 \le T), \mathbb{P}(\tau_2 \le T)\}.$$

Also we know if we choose $\rho_t \equiv 1$, the first time the two dimensional Brownian motion hits the boundary is the same as the time one dimensional Brownian motion hits zero by starting from $B_0^1 \wedge B_0^2$. This fact tells us that $\mathbb{P}(\tau \le T) = \max\{\mathbb{P}(\tau_1 \le T), \mathbb{P}(\tau_2 \le T)\}$, and this probability is very easy to compute (the first hitting time of one dimensional Brownian motion).

Therefore we have

$$\min_{\rho_t} \mathbb{P}(\tau \le T) \ge \max\{\mathbb{P}(\tau_1 \le T), \mathbb{P}(\tau_2 \le T)\},$$

and the equality holds when $\rho_t \equiv 1$. Furthermore, we find

$$\min_{\rho_t} \mathbb{P}(\tau \le T) = \frac{2}{\sqrt{2\pi T}} \int_m^\infty e^{-\frac{x^2}{2T}} \mathrm{d}x,$$

where $m = B_0^1 \wedge B_0^2$.

However, the answer to the maximum case is not that easy, and also for the minimum case, we want to generalize the problem and see if it can lead to some non-trivial correlation choice. We will let f be a payoff function, $f(B_\tau^1, B_\tau^2) = f(B_\tau^1 + B_\tau^2)$, which means if $B_\tau^1 = 0$, the payoff will be $f(B_\tau^2)$, and vice versa. It is obvious that this payoff function treats B_t^1 and B_t^2 equally. If we want to give more preference to the asset B_t^2, we can set $f(B_\tau^1, B_\tau^2) = f(B_\tau^1 + bB_\tau^2)$.

This problem will be modeled as

$$\max_{\rho_t} \mathbb{E}(f(B^1_\tau, B^2_\tau)\mathbb{1}_{\{\tau<T\}}) \quad \text{and} \quad \min_{\rho_t} \mathbb{E}(f(B^1_\tau, B^2_\tau)\mathbb{1}_{\{\tau<T\}}).$$

Now we can observe that they are actually optimal control problems, and ρ_t is the controlled object. Therefore we will apply the optimal control approach to solve such problems. In the literature, Merton [54] has applied the optimal control approach to solve the optimal portfolio selection problem. In Merton's problem, the objects that need to be controlled are the the consumption and the allocation to the risky asset. Huang, Milevsky and Salisbury [32] have applied the approach to find the optimal initiation strategy, in their problem, the object to be controlled is the optimal initiation time. Rogers [65] has given a very detailed explanation of the theory behind the stochastic optimal control problems.

If we let

$$v(B^1_t, B^2_t, T-t) := \max_{\{\rho_s, t\le s\le T\}} \mathbb{E}(f(B^1_\tau, B^2_\tau)\mathbb{1}_{\{\tau<T\}})|B^1_t, B^2_t),$$

then for $t_1 < t_2$, by applying the optimality principle, we have

$$\mathbb{E}[v(B^1_{t_2}, B^2_{t_2}, T-t_2)|\mathcal{F}_{t_1}] \le v(B^1_{t_1}, B^2_{t_1}, T-t_1), \tag{162}$$

where $\mathcal{F}_{t_1} = \sigma(B^1_t, B^2_t, t \le t_1)$. The underlying meaning of equation (162) is very straightforward, for the maximum case, if we take the optimal choice of ρ_t earlier, we are supposed to get the larger expected value. If we take the optimal choice since the beginning, the expected value will be the largest.

From (162), we can observe that $v(B^1_t, B^2_t, T-t)$ is a supermartingale and if we take the optimal choice, it will be a martingale. Therefore if the function v is C^2, we can apply the Itô's formula

$$\mathrm{d}v(B^1_t, B^2_t, T-t) = -v_t\mathrm{d}t + v_x\mathrm{d}B^1_t + v_y\mathrm{d}B^2_t + \frac{1}{2}v_{xx}\mathrm{d}t + \frac{1}{2}v_{yy}\mathrm{d}t + \rho_t v_{xy}\mathrm{d}t. \tag{163}$$

In order for $v(B^1_t, B^2_t, T-t)$ to be a supermartingale for any choice of ρ_t and to be martingale for the optimal choice of ρ_t, then the value function must satisfy

$$\max_{\rho_t\in[-1,1]}\{-v_t + \frac{1}{2}v_{xx} + \frac{1}{2}v_{yy} + \rho_t v_{xy}\} = 0. \tag{164}$$

From the equation, one can easily observe that: In order to obtain the maximum, we obtain

$$\rho_t = \begin{cases} 1 & \text{if } \ v_{xy} > 0, \\ -1 & \text{if } \ v_{xy} \le 0, \end{cases} \tag{165}$$

which gives us the following partial differential equation. The value function $v(x, y, T-t)$ needs to satisfy

$$-v_t + \frac{1}{2}v_{xx} + \frac{1}{2}v_{yy} + |v_{xy}| = 0, \tag{166}$$

with the terminal condition

$$v(x,y,0)=\begin{cases}0 & \text{if } \ x>0 \text{ and } y>0,\\ f(x) & \text{if } \ y=0,\\ f(y) & \text{if } \ x=0,\end{cases} \tag{167}$$

and the boundary condition

$$v(x,y,T-t)=\begin{cases}f(x) & \text{if } \ y=0 \text{ and } 0\le t\le T,\\ f(y) & \text{if } \ x=0 \text{ and } 0\le t\le T.\end{cases} \tag{168}$$

We can get a similar result for the minimum case: If $v_{xy}<0$, $\rho_t=1$; otherwise $\rho_t=-1$. The corresponding partial differential equation will be

$$-v_t+\frac{1}{2}v_{xx}+\frac{1}{2}v_{yy}-|v_{xy}|=0, \tag{169}$$

with the terminal condition

$$v(x,y,0)=\begin{cases}0 & \text{if } \ x>0 \text{ and } y>0,\\ f(x) & \text{if } \ y=0,\\ f(y) & \text{if } \ x=0,\end{cases} \tag{170}$$

and the boundary condition

$$v(x,y,T-t)=\begin{cases}f(x) & \text{if } \ y=0 \text{ and } 0\le t\le T,\\ f(y) & \text{if } \ x=0 \text{ and } 0\le t\le T.\end{cases} \tag{171}$$

It is obvious that in both cases we only know the boundary condition partially.

We have found that, if we want to find the maximum or minimum expected payoff, the correlation ρ_t can only be 1 or -1 for any given t. But we still have no idea when it will be 1 and when it will be -1.

From the viewpoint of PDE, by observing the equations in (166) and (169), we can find that they are nonlinear equations, which makes it hard to obtain their analytic solutions. One approach is using numerical methods to solve these equations to get the value function, and then obtain the sign of the mixed second order derivative to decide the behavior of the correlation ρ_t. The hard part of this method is that we only know the boundary condition partially. Thus in order to solve the equation numerically on a bounded region, we need to assume the unknown boundary condition. We will show some numerical examples for the maximum case in Section 7.4, the goal is to demonstrate the different switch region of the correlation according to different payoff function.

In order to get analytic results from this problem, we need to reduce the complexity. Our idea is to get rid of T, which means there is no longer an expiration date. Therefore it is reasonable to expect that the optimal choice of ρ_t is not impacted by the time remaining; it is only impacted by the location. The problem will transform to

$$\max_{\rho_t} \mathbb{E}(f(B^1_\tau, B^2_\tau)) \quad \text{and} \quad \min_{\rho_t} \mathbb{E}(f(B^1_\tau, B^2_\tau)).$$

We will explicitly find the optimal choice of correlation with respect to certain payoff functions. We will investigate two forms of the payoff function, the symmetric one $f(x, y) = (x + y)^\alpha$ and the asymmetric one $f(x, y) = (x + by)^\alpha$, the latter one implies the payoff will be more favorable if one particular asset defaults first.

Remark 6. Suppose that at time 0, we buy a portfolio consisting of 20 shares of asset B_1 with initial value 5 each, and 10 shares of asset B_2 with initial value 10 each, so totally we spend 200 to purchase the portfolio. We also buy 20 shares of our option, which has the payoff $f(x, y) = x + y/2$. As soon as the default happens: if asset B_1 defaults, the portfolio and the options we have purchased will be equal to 20 shares of B_2 with current value (10 shares from the option and 10 shares from the the portfolio), which is the same as if we had invested 200 only on B_2 at the beginning; if asset B_2 defaults, the portfolio and the options we have purchased will be 40 shares of B_1 with current value (20 shares from the option and 20 shares from the the portfolio), which is the same as if we had invested 200 only on B_1 at the beginning. Above all, upon the default of either asset, the option plus the portfolio results in what it would have been if the buyer had only bought the surviving asset with the amount of money that was spent on the portfolio at the beginning. In that sense, this option is a kind of a lookback option.

Before we present our results, we will introduce the essential idea we have used across the whole chapter to obtain the minimum or the maximum expected value and the optimal choice of correlation respectively.

If a function $v(x, y) \geq 0$ satisfies the following conditions:

(i) $v(x, y) = f(x, y)$ on the boundary,

(ii) $v(B^1_{t\wedge\tau}, B^2_{t\wedge\tau})$ is a supermartingale for every choice of ρ_t,

(iii) $v(B^1_{t\wedge\tau}, B^2_{t\wedge\tau})$ is a martingale for some choice of ρ_t,

(iv) for this choice of ρ_t, the optional sampling theorem applies,

then we have

$$v(B^1_0, B^2_0) = \max_{\rho_t} \mathbb{E}(f(B^1_\tau, B^2_\tau)|B^1_0, B^2_0).$$

For the minimum case we have a similar idea. If a function $v(x,y) \geq 0$ satisfies the following conditions:

(i) $v(x,y) = f(x,y)$ on the boundary,

(ii) $v(B^1_{t\wedge\tau}, B^2_{t\wedge\tau})$ is a submartingale for every choice of ρ_t,

(iii) $v(B^1_{t\wedge\tau}, B^2_{t\wedge\tau})$ is a martingale for some choice of ρ_t,

(iv) for this choice of ρ_t, the optional sampling theorem applies,

then we have

$$v(B^1_0, B^2_0) = \min_{\rho_t} \mathbb{E}(f(B^1_\tau, B^2_\tau)|B^1_0, B^2_0).$$

The maximum and minimum in the above statement mean the following. We take the maximum or minimum over all probability measures $\mathbb{P}$ for which there is a filtration $\mathcal{F}_t$ such that B^1_t and B^2_t are Brownian motions and semimartingales with respect to $\mathcal{F}_t$, with $\mathrm{d}\langle B^1, B^2\rangle_t = \rho_t \mathrm{d}t$.

Remark 7. For the maximum case, from (ii), we have $v(B^1_0, B^2_0) \geq \mathbb{E}[v(B^1_{t\wedge\tau}, B^2_{t\wedge\tau})]$ for every choice of ρ_t; by applying Fatou's lemma, we find $v(B^1_0, B^2_0) \geq \mathbb{E}[v(B^1_\tau, B^2_\tau)]$, which means $v(B^1_0, B^2_0) \geq \max_{\rho_t} \mathbb{E}[v(B^1_\tau, B^2_\tau)]$; then from (iii), for the particular choice of ρ_t, we have $v(B^1_0, B^2_0) = \mathbb{E}[v(B^1_{t\wedge\tau}, B^2_{t\wedge\tau})]$; the remaining issue is to show $v(B^1_0, B^2_0) = \mathbb{E}[v(B^1_\tau, B^2_\tau)]$ holds for this particular ρ_t, in other words, this issue arises because our problem has an infinite horizon.

Remark 8. For the minimum case, the approach is similar. The difference is we can not apply the Fatou's lemma here to extend from $t \wedge \tau$ to τ. Therefore we come up with another way for this step, which will be presented in the proof.

7.2 Symmetric payoff function

As we mentioned above, our payoff function here is $f(x,y) = (x+y)^\alpha$. We will discuss the maximum and minimum cases in this section separately.

7.2.1 Maximum case

For $\alpha > 0$, the maximum case will be different as $\alpha \leq 1$ and $\alpha > 1$.

Theorem 19. *For $0 < \alpha \leq 1$, $\max_{\rho_t} \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = (B_0^1 + B_0^2)^\alpha$ and $\rho_t \equiv -1$ is the optimal choice to get the maximum.*

Proof. By applying Itô's formula, we obtain

$$\mathrm{d}(B_t^1 + B_t^2)^\alpha = \alpha(B_t^1 + B_t^2)^{\alpha-1}\mathrm{d}(B_t^1 + B_t^2) + \alpha(\alpha-1)(B_t^1 + B_t^2)^{\alpha-2}(1+\rho_t)\mathrm{d}t. \tag{172}$$

We have $1 + \rho_t \geq 0$, $(B_t^1 + B_t^2)^{\alpha-2} > 0$ and $\alpha(\alpha - 1) \leq 0$, so $(B_t^1 + B_t^2)^\alpha$ is a local supermartingale for any choice of ρ_t, and it is a local martingale when $\rho_t \equiv -1$. We define the sequence of increasing stopping times $T_n = \inf\{t : B_t^1 \leq 1/n \text{ or } B_t^2 \leq 1/n\} \wedge n$, it is easy to observe that $T_n < \tau$, and $T_n \to \tau$ as $n \to \infty$.

$$\begin{aligned}(B_{t\wedge T_n}^1 + B_{t\wedge T_n}^2)^\alpha = (B_0^1 + B_0^2)^\alpha \quad &+ \int_0^{t\wedge T_n} \alpha(B_s^1 + B_s^2)^{\alpha-1}\mathrm{d}(B_s^1 + B_s^2) \\ &+ \int_0^{t\wedge T_n} \alpha(\alpha-1)(B_s^1 + B_s^2)^{\alpha-2}(1+\rho_s)\mathrm{d}s.\end{aligned} \tag{173}$$

As $(B_s^1 + B_s^2)^{\alpha-1} < n^{1-\alpha}$ when $0 \leq s \leq (t \wedge T_n)$, namely $(B_s^1 + B_s^2)^{\alpha-1}$ is bounded. By applying Theorem 9, we know $(B_{t\wedge T_n}^1 + B_{t\wedge T_n}^2)^\alpha$ is a supermartingale. Since T_n is bounded, the optional sampling theorem implies that

$$\mathbb{E}(B_{T_n}^1 + B_{T_n}^2)^\alpha \leq (B_0^1 + B_0^2)^\alpha. \tag{174}$$

By applying Fatou's lemma

$$\mathbb{E}(B_\tau^1 + B_\tau^2)^\alpha \leq \liminf_n \mathbb{E}(B_{T_n}^1 + B_{T_n}^2)^\alpha \leq (B_0^1 + B_0^2)^\alpha. \tag{175}$$

Therefore

$$\max_{\rho_t} \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] \leq (B_0^1 + B_0^2)^\alpha. \tag{176}$$

If we take $\rho_t \equiv -1$, then equality holds in (174), and $\mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = (B_0^1 + B_0^2)^\alpha$ by Dominated Convergence Theorem. $\square$

Let us make a summary here: In the first step, we find the value function $v(B_\tau^1, B_\tau^2) = (B_\tau^1 + B_\tau^2)^\alpha$ which satisfies $v(x, y) = f(x, y)$ on the boundary; In the second step, we show $v(B_{t\wedge T_n}^1, B_{t\wedge T_n}^2)$ is a supermartingale for every choice of ρ_t, and $v(B_{t\wedge T_n}^1, B_{t\wedge T_n}^2)$ is a martingale for $\rho_t \equiv -1$; In the third step, we apply optional sampling theorem as T_n is bounded; In the last step, we use Fatou's lemma to extend T_n to τ and apply Dominated Convergence Theorem to ensure the equality holds.

Theorem 20. *For $\alpha > 1$, $\max_{\rho_t} \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = \infty$.*

Before presenting the proof, we need to establish one result related to the exit point of Brownian motion in the complex plane.

Definition 15. We define $Z_t := W_t^1 + \mathrm{i}W_t^2$, where i is the imaginary unit, W_t^1 and W_t^2 are independent Brownian motions. For $\beta \in (0, 2\pi)$, we denote a cone by $C := \{re^{\mathrm{i}\theta} | 0 \le r, 0 \le \theta \le \beta\}$. The first exit time of Z_t from C is defined as $\tau_1 := \inf\{t | Z_t \in \mathbb{C} \setminus C\}$.

Lemma 4. *For $\beta \in (0, 2\pi)$, we let Z_t start at $ye^{\mathrm{i}\theta}(0 < \theta < \beta)$. If $\frac{\alpha\beta}{\pi} \ge 1$, we have $\mathbb{E}(|Z_{\tau_1}|^\alpha) = \infty$.*

Proof. We let $Y_t := g(Z_t)$, where $g(z) = z^{\pi/\beta}$. By applying the conformal transformation, we know Y_t is a time changed Brownian motion starting from $y^{\pi/\beta} e^{\mathrm{i}\pi\theta/\beta}$ and Y_{τ_1} is the hitting position on the real axis. By Theorem 10, Y_{τ_1} has Cauchy distribution, which means

$$\mathbb{E}[|Y_{\tau_1}|^c] = \infty, \text{ for } c \ge 1. \tag{177}$$

By the fact $Y_{\tau_1} = (Z_{\tau_1})^{\pi/\beta}$, we obtain

$$\mathbb{E}[|Z_{\tau_1}|^\alpha] = \mathbb{E}[|Y_{\tau_1}|^{\frac{\alpha\beta}{\pi}}].$$

If $\frac{\alpha\beta}{\pi} \ge 1$, we have $\mathbb{E}[|Z_{\tau_1}|^\alpha] = \infty$.

□

Proof of Theorem 20: Given $\alpha > 1$, we let $\beta = \frac{\pi}{\alpha}$, so $\beta \in (0, \pi)$. By letting $\rho = -\cos(\beta)$, so $|\rho| < 1$. We can find particular B_t^1, B_t^2 with constant correlation ρ,

$$\begin{aligned} B_t^1 &= \rho W_t^2 + \sqrt{1-\rho^2} W_t^1 \\ B_t^2 &= W_t^2. \end{aligned} \tag{178}$$

Here $\mathrm{d}\langle W^2, W^1 \rangle_t = 0$. By writing

$$\begin{pmatrix} B_t^1 \\ B_t^2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}, \tag{179}$$

we have

$$\begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}} & \frac{-\rho}{\sqrt{1-\rho^2}} \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} B_t^1 \\ B_t^2 \end{pmatrix} = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}. \tag{180}$$

When the Brownian motion (B_t^1, B_t^2) starts from (x, y) and hits the boundary of the first quadrant, this is equivalent to (W_t^1, W_t^2) starting from $(\frac{1}{\sqrt{1-\rho^2}}x - \frac{\rho}{\sqrt{1-\rho^2}}y, y)$ and hitting the boundary of $C = \{re^{i\theta}|0 \leq r, 0 \leq \theta \leq \beta\}$. When $B_\tau^2 = 0$, from (180), we have $W_\tau^2 = 0$ and $W_\tau^1 = \frac{1}{\sqrt{1-\rho^2}}B_\tau^1$; when $B_\tau^1 = 0$, from (180), we have $W_\tau^2 = B_\tau^2$ and $W_\tau^1 = \frac{-\rho}{\sqrt{1-\rho^2}}B_\tau^2$; above all, we can write

$$B_\tau^1 + B_\tau^2 = \sqrt{1-\rho^2} \times \sqrt{(W_\tau^2)^2 + (W_\tau^1)^2} = \sqrt{1-\rho^2} \times |Z_{\tau_1}|. \tag{181}$$

Since here $\frac{\alpha\beta}{\pi} = 1$, we can apply Lemma 4, so $\mathbb{E}(B_\tau^1 + B_\tau^2)^\alpha = (\sqrt{1-\rho^2})^\alpha \mathbb{E}|Z_{\tau_1}|^\alpha = \infty$ for $\rho = -\cos(\beta)$. Therefore we have $\max_{\rho_t} \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = \infty$. $\square$

We found that when $\alpha > 1$, the maximum expected value will be infinity, so it is impossible to find the optimal choice of correlation. Now we are going to study the minimum case. As we will show, the value in the minimum case is finite for all $\alpha > 0$.

7.2.2 Minimum case

First we will show the result for minimum case when $0 < \alpha < 1$, then based on this result, we will prove the case for $\alpha \geq 1$.

Theorem 21. *For $0 < \alpha \leq 1$, $\min_{\rho_t} \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = (B_0^1 + B_0^2)^{\alpha-1}|B_0^1 - B_0^2|$ and the optimal choice to get the minimum is $\rho_t = -1$ when $B_t^1 \neq B_t^2$, $\rho_t \equiv 1$ once $B_t^1 = B_t^2$.*

Proof. For simplicity, we let $M_t := B_t^1 + B_t^2$, $N_t := B_t^1 - B_t^2$. By applying Tanaka's formula

$$\begin{aligned} \mathrm{d}M_t^{\alpha-1}|N_t| &= M_t^{\alpha-1}\mathrm{sgn}(N_t)\mathrm{d}N_t + |N_t|(\alpha-1)M_t^{\alpha-2}\mathrm{d}M_t \\ &\quad +|N_t|(\alpha-1)(\alpha-2)M_t^{\alpha-3}(1+\rho_t)\mathrm{d}t + M_t^{\alpha-1}\mathrm{d}L_t(N), \end{aligned} \tag{182}$$

where $L_t(N)$ is the local time of N_t at 0, it is an increasing continuous process and satisfies

$$\mathrm{d}|N_t| = \mathrm{sgn}(N_t)\mathrm{d}N_t + \mathrm{d}L_t(N).$$

We have $1 + \rho_t \geq 0$, $M_t^{\alpha-3} > 0$, $(\alpha-1)(\alpha-2) > 0$, $M_t^{\alpha-1} > 0$, and $dL_t(N) \geq 0$, so $M_t^{\alpha-1}|N_t|$ is a local submartingale for any choice of ρ_t. Under the optimal choice of ρ_t, by denoting $\tau^* = \inf\{t : B_t^1 = B_t^2\}$, we can observe that $N_t \equiv 0$ for $t \geq \tau^*$, thus we have

$$\mathrm{d}M_t^{\alpha-1}|N_t| = \begin{cases} M_t^{\alpha-1}\mathrm{sgn}(N_t)\mathrm{d}N_t, & t < \tau^*, \\ 0, & t \geq \tau^*, \end{cases} \tag{183}$$

which can be written as $\mathrm{d}M_t^{\alpha-1}|N_t| = M_t^{\alpha-1}\mathrm{sgn}(N_t)\mathbb{1}_{\{\tau^* > t\}}\mathrm{d}N_t$. By applying Theorem 9, we find $M_t^{\alpha-1}|N_t|$ is a local martingale provided $\rho_t = -1$ when $B_t^1 \neq B_t^2$ and $\rho_t \equiv 1$ once $B_t^1 = B_t^2$. We define the sequence of increasing stopping times $T_n = \inf\{t : B_t^1 \leq 1/n \text{ or } B_t^2 \leq 1/n\} \wedge n$. It is easy to see that $T_n < \tau$, and $T_n \to \tau$ as $n \to \infty$.

$$\begin{aligned} M_{t\wedge T_n}^{\alpha-1}|N_{t\wedge T_n}| &= M_0^{\alpha-1}|N_0| + \int_0^{t\wedge T_n} M_s^{\alpha-1}\mathrm{sgn}(N_s)\mathrm{d}N_s + \int_0^{t\wedge T_n} |N_s|(\alpha-1)M_s^{\alpha-2}\mathrm{d}M_s \\ &+ \int_0^{t\wedge T_n} |N_s|(\alpha-1)(\alpha-2)M_s^{\alpha-3}(1+\rho_s)\mathrm{d}s + \int_0^{t\wedge T_n} M_s^{\alpha-1}\mathrm{d}L_s(N). \quad (184) \end{aligned}$$

Since in formula (184), we have $(B_s^1+B_s^2)^{\alpha-1} \leq n^{1-\alpha}$ and $(B_s^1+B_s^2)^{\alpha-2} \leq n^{2-\alpha}$, when $0 \leq s \leq (t\wedge T_n)$, thus $M_{t\wedge T_n}^{\alpha-1}|N_{t\wedge T_n}|$ is a submartingale. Since T_n is bounded, optional sampling implies that

$$\mathbb{E}\big(M_{T_n}^{\alpha-1}|N_{T_n}|\big) \geq M_0^{\alpha-1}|N_0|. \quad (185)$$

Note that, we want to prove $\mathbb{E}\big(M_\tau^{\alpha-1}|N_\tau|\big) \geq M_0^{\alpha-1}|N_0|$, but the Fatou's lemma can not make the inequality in this direction, so will try another approach.

Since $0 < \alpha < 1$, we have

$$\begin{aligned} \mathbb{E}\big[\sup_n \big(M_{T_n}^{\alpha-1}|N_{T_n}|\big)^{1/\alpha}\big] &\leq (\frac{1}{1-\alpha})^{1/\alpha} \sup_n \mathbb{E}\big[\big(M_{T_n}^{\alpha-1}|N_{T_n}|\big)^{1/\alpha}\big] \\ &< (\frac{1}{1-\alpha})^{1/\alpha} \sup_n \mathbb{E}\big[\big(M_{T_n}^{\alpha}\big)^{1/\alpha}\big] \\ &= (\frac{1}{1-\alpha})^{1/\alpha}(M_0)\ . \quad (186) \end{aligned}$$

The first inequality comes from Doob's $\mathbb{L}^p-$inequality, the second one comes from $|N_{T_n}| < M_{T_n}$, and the final equality comes from $\mathbb{E}(M_{T_n}) = M_0$ by applying the optional sampling theorem.

By using the fact that $s \leq 1 + s^{1/\alpha}$ (for $s \geq 0$ and $1/\alpha \geq 1$), we have

$$\mathbb{E}\big[\sup_n M_{T_n}^{\alpha-1}|N_{T_n}|\big] < 1 + \mathbb{E}\big[\sup_n \big(M_{T_n}^{\alpha-1}|N_{T_n}|\big)^{1/\alpha}\big], \quad (187)$$

then the Dominated Convergence theorem implies

$$\mathbb{E}\big(M_\tau^{\alpha-1}|N_\tau|\big) = \lim_n \mathbb{E}\big(M_{T_n}^{\alpha-1}|N_{T_n}|\big) \geq M_0^{\alpha-1}|N_0|\ . \quad (188)$$

Since $(B_\tau^1 + B_\tau^2)^\alpha = (B_\tau^1 + B_\tau^2)^{\alpha-1}|B_\tau^1 - B_\tau^2|$, we find

$$\min_{\rho_t} \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] \geq (B_0^1 + B_0^2)^{\alpha-1}|B_0^1 - B_0^2|\ . \quad (189)$$

Now we take $\rho_t = -1$ when $B_t^1 \neq B_t^2$, $\rho_t \equiv 1$ once $B_t^1 = B_t^2$. Without loss of generality, we assume $B_0^1 \leq B_0^2$. If τ_0 is the first time B_t^1 exits $\big[0, \frac{B_0^1+B_0^2}{2}\big]$, then $\mathbb{P}(B_{\tau_0}^1 = 0) = \frac{B_0^2-B_0^1}{B_0^2+B_0^1}$. With the given choice, if $B_{\tau_0}^1 = 0$ then $\tau = \tau_0$ and $B_\tau^2 = B_0^1 + B_0^2$. If $B_{\tau_0}^1 \neq 0$ then $B_\tau^1 = B_\tau^2 = 0$ since Brownian motion hits 0. Therefore $\mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = (B_0^1 + B_0^2)^{\alpha-1}|B_0^1 - B_0^2|$. By symmetry, the same is true if $B_0^1 \geq B_0^2$.

□

Let us make a summary of the proof here: In the first step, we find the value function $v(B^1_\tau, B^2_\tau) = (B^1_\tau + B^2_\tau)^{\alpha-1}|B^1_\tau - B^2_\tau| = (B^1_\tau + B^2_\tau)^\alpha$, which satisfies $v(x,y) = f(x,y)$ on the boundary; In the second step, we show $v(B^1_{t\wedge T_n}, B^2_{t\wedge T_n})$ is a submartingale for every choice of ρ_t, and $v(B^1_{t\wedge T_n}, B^2_{t\wedge T_n})$ is a martingale for the choice of ρ_t: $\rho_t = -1$ when $B^1_t \neq B^2_t$ and $\rho_t = 1$ when $B^1_t = B^2_t$; In the third step, we apply optional sampling theorem as T_n is bounded; In the last step, we use Doob's $\mathbb{L}^p-$inequality to extend T_n to τ.

Theorem 22. *For $\alpha \geq 1$, $\min_{\rho_t} \mathbb{E}[(B^1_\tau + B^2_\tau)^\alpha] = |B^1_0 - B^2_0|^\alpha$ and the optimal choice to achieve the minimum is $\rho_t \equiv 1$.*

Proof. We will use two steps to prove this theorem. First, we prove that the equality holds for $\alpha = 1$; then we prove that the equality holds for $\alpha > 1$ based on the result of $\alpha = 1$.

From Theorem 21, when $0 < p < 1$ we have $\mathbb{E}[M^p_\tau] \geq M^{p-1}_0 |N_0|$ for any choice of ρ_t. Since M^p_τ is a continuous function of p, by applying Monotone Convergence Theorem, we have

$$\mathbb{E}[\lim_{p\to 1} M^p_\tau \mathbb{1}_{(M_\tau > 1)}] = \lim_{p\to 1} \mathbb{E}[M^p_\tau \mathbb{1}_{(M_\tau > 1)}],$$

by applying Dominated Convergence Theorem, we have

$$\mathbb{E}[\lim_{p\to 1} M^p_\tau \mathbb{1}_{(M_\tau \leq 1)}] = \lim_{p\to 1} \mathbb{E}[M^p_\tau \mathbb{1}_{(M_\tau \leq 1)}].$$

Therefore we get

$$\mathbb{E}[\lim_{p\to 1} M^p_\tau] = \lim_{p\to 1} \mathbb{E}[M^p_\tau],$$

and

$$\mathbb{E}[M_\tau] = \mathbb{E}[\lim_{p\to 1} M^p_\tau] = \lim_{p\to 1} \mathbb{E}[M^p_\tau] \geq \lim_{p\to 1} M^{p-1}_0 |N_0| = |N_0|. \tag{190}$$

Thus

$$\min_{\rho_t} \mathbb{E}[M_\tau] \geq |N_0|. \tag{191}$$

Now if we take $\rho_t \equiv 1$, then $\mathbb{E}[M_\tau] = |N_0|$.

For $\alpha > 1$, for any choice of ρ_t,

$$\mathbb{E}[M^\alpha_\tau] \geq (\mathbb{E}[M_\tau])^\alpha \geq |N_0|^\alpha,$$

the first inequality comes from Holder's inequality and the second comes from the previous result $E[M_\tau] \geq |N_0|$. Therefore

$$\min_{\rho_t} \mathbb{E}[M_\tau]^\alpha \geq |N_0|^\alpha. \tag{192}$$

Now if we take $\rho_t \equiv 1$, then $\mathbb{E}[M_\tau]^\alpha = |N_0|^\alpha$.

Overall, for $\alpha \geq 1$, $\min_{\rho_t} \mathbb{E}[(B^1_\tau + B^2_\tau)^\alpha] = |B^1_0 - B^2_0|^\alpha$. $\square$

α	B_0^1	B_0^2	max	min	Variation((max − min)/ min)
1	13	7	20	6	233%
0.5	13	7	4.4721	1.3416	233%
0.5	15	12	5.1962	0.5774	800%

Table 4: Range of option prices for symmetric payoff function

7.2.3 Conclusions

We have constructed closed form solutions for special boundary conditions, we are going to demonstrate the spread of the maximum and minimum in Table 4. For example, by setting the parameters $\alpha = 1, B_0^1 = 13, B_0^2 = 7$, we obtain that the maximum price $\max = 20$ and the minimum price $\min = 6$, which have a variation of 233%. This is large, so we conclude that uncertainty in the correlation ρ_t yields a wide range of option prices.

7.3 Asymmetric payoff function

In the above section, we got the behaviors of the bivariate Brownian motion according to optimal choices to obtain either the maximum or the minimum expected payoff. In this section we want to study more complicated behavior of the Brownian motion, so we will investigate a more general payoff function $(x+by)^\alpha$, here $0 \le b < 1$. Note that, when $b > 1$, $(B_\tau^1 + bB_\tau^2)^\alpha = b^\alpha(\frac{1}{b}B_\tau^1 + B_\tau^2)^\alpha$. Therefore without loss of generality, we only need to discuss $\max_{\rho_t} \mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha]$ and $\min_{\rho_t} \mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha]$ for $0 \le b < 1$. Furthermore, we provide a new way to construct skew Brownian motion in the proof of Theorem 24.

We will discuss the maximum and minimum cases separately.

7.3.1 Maximum case

For $\alpha > 1$, by applying the previous results in section 7.2 and the fact $0 \le b < 1$, we have

$$\max_{\rho_t} \mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha] \ge \max_{\rho_t} b^\alpha \mathbb{E}[(B_\tau^1 + B_\tau^2)^\alpha] = \infty. \tag{193}$$

For $0 < \alpha \le 1$, the maximum case will have some interesting results. Before showing the results, we need to introduce some definitions.

Definition 16. We define the functions $v_1(x, y)$ and $v_2(x, y)$ as follows:

$$v_1(x, y) = x\eta \left(\frac{x+y}{1+\theta}\right)^{\alpha-1} - (\theta x - y) b^\alpha (x+y)^{\alpha-1}, \tag{194}$$

$$v_2(x,y)=\begin{cases} b^\alpha(x+y)^{\alpha-1}+\frac{1}{\theta}(x-y)^{\alpha-1}+\frac{1}{\theta x-y}\left[\frac{y\eta}{\theta}\left(\frac{x-y}{1-\theta}\right)^{\alpha-1}-x\eta\left(\frac{x+y}{1+\theta}\right)^{\alpha-1}\right], & y\neq\theta x, \\ b^\alpha(x+y)^{\alpha-1}+\frac{1}{\theta}(x-y)^{\alpha-1}+x^{\alpha-1}\left[-\frac{\eta}{\theta}+\frac{\eta(\alpha-1)}{1-\theta}+\frac{\eta(\alpha-1)}{1+\theta}\right], & y=\theta x. \end{cases} \tag{195}$$

Here θ satisfies the equation

$$b^\alpha=\left(\frac{1+\theta}{1-\theta}\right)^{1-\alpha}\frac{1-\theta-\alpha}{1+\theta-\theta\alpha}, \tag{196}$$

and

$$\eta=\frac{b^\alpha\theta(1-\theta)(1+\theta)^\alpha+(1+\theta)(1-\theta)^\alpha}{(1-\theta)(1+\theta)+2(1-\alpha)\theta}. \tag{197}$$

Remark 9. $v_2(x,y)$ is a continuous function. This can be shown by applying L'Hospital's rule when $y=\theta x$. Furthermore, we let

$$z:=\frac{y}{x},\qquad m(z):=\frac{z}{\theta}\eta\left(\frac{1-z}{1-\theta}\right)^{\alpha-1}-\eta\left(\frac{1+z}{1+\theta}\right)^{\alpha-1},$$

it is obvious that $\frac{m(z)}{z-\theta}$ is $\mathbb{C}^2$ for $z\neq\theta$. By applying Taylor series, we have

$$\frac{m(z)}{z-\theta}=m'(\theta)+\frac{z-\theta}{2}m''(\theta)+\frac{(z-\theta)^2}{6}m'''(\theta)+o((z-\theta)^2),$$

thus in fact $\frac{m(z)}{z-\theta}$ is also $\mathbb{C}^2$ at $z=\theta$. Therefore $v_2(x,y)$ is a $\mathbb{C}^2$ function as is $v_1(x,y)$. It can be shown that this choice of η makes $v_2(x,\theta x)=0$.

Remark 10. Although we can not get θ explicitly via equation (196), it can be shown that the function

$$f(\theta):=\left(\frac{1+\theta}{1-\theta}\right)^{1-\alpha}\frac{1-\theta-\alpha}{1+\theta-\theta\alpha}$$

is decreasing and ranging over $(0,1-\alpha]$ as θ ranges over $[0,1-\alpha)$, which guarantees equation (196) has unique solution θ for $\beta^\alpha<1-\alpha$.

Theorem 23. *For $0<\alpha\leq 1$,*

(i) If $1-\alpha\leq b^\alpha\leq 1$, $\max_{\rho_t}\mathbb{E}[(B^1_\tau+bB^2_\tau)^\alpha]=(B^1_0+B^2_0)^{\alpha-1}(B^1_0+b^\alpha B^2_0)$ and $\rho_t\equiv-1$ is the optimal choice to achieve the maximum;

(ii) If $b^\alpha < 1-\alpha$, $\max_{\rho_t} \mathbb{E}[(B^1_\tau + bB^2_\tau)^\alpha] = v_1(B^1_0, B^2_0) + v_2(B^1_0, B^2_0)(\theta B^1_0 - B^2_0)^+$ and the optimal choice to get the maximum is $\rho_t = -1$ when $B^2_t \geq \theta B^1_t$ and $\rho_t = 1$ when $B^2_t < \theta B^1_t$.

Proof. (i) Let $M_t := B^1_t + B^2_t$, $C_t := B^1_t + b^\alpha B^2_t$ for simplicity. By applying Itô's formula, we find

$$\begin{aligned} \mathrm{d}M_t^{\alpha-1}C_t &= (\alpha-1)M_t^{\alpha-2}C_t \mathrm{d}M_t + M_t^{\alpha-1}\mathrm{d}C_t \\ &\quad +(\alpha-1)M_t^{\alpha-3}[B^1_t(\alpha+b^\alpha-1) + B^2_t(b^\alpha(\alpha-1)+1)](1+\rho_t)\mathrm{d}t \end{aligned} \tag{198}$$

We have $1+\rho_t \geq 0$, $M_t^{\alpha-3} > 0$, $\alpha - 1 \leq 0$, $b^\alpha(\alpha-1)+1 \geq 0$ and $\alpha + b^\alpha - 1 \geq 0$, so $M_t^{\alpha-1}C_t$ is a local supermartingale for any choice of ρ_t, and it is a local martingale when $\rho_t \equiv -1$. We define the sequence of increasing stopping times $T_n = \inf\{t : B^1_t \leq 1/n \text{ or } B^2_t \leq 1/n\} \wedge n$, it is easy to see that $T_n < \tau$, and $T_n \to \tau$ as $n \to \infty$. By applying the same technique used in the proof of Theorem 19, $M^{\alpha-1}_{t\wedge T_n} C_{t\wedge T_n}$ is a supermartingale. Since T_n is bounded, optional sampling implies that

$$\mathbb{E}(M^{\alpha-1}_{T_n} C_{T_n}) \leq M_0^{\alpha-1}C_0. \tag{199}$$

By applying Fatou's lemma, $\mathbb{E}(M^{\alpha-1}_\tau C_\tau) \leq M_0^{\alpha-1}C_0$. As $(B^1_\tau + bB^2_\tau)^\alpha = (B^1_\tau + B^2_\tau)^{\alpha-1}(B^1_\tau + b^\alpha B^2_\tau)$, we get

$$\max_{\rho_t} \mathbb{E}[(B^1_\tau + bB^2_\tau)^\alpha] \leq (B^1_0 + B^2_0)^{\alpha-1}(B^1_0 + b^\alpha B^2_0). \tag{200}$$

Now if we take $\rho_t \equiv -1$, $\mathbb{E}[(B^1_\tau + bB^2_\tau)^\alpha] = (B^1_0 + B^2_0)^{\alpha-1}(B^1_0 + b^\alpha B^2_0)$.

(ii) For simplicity, we let $v(B^1_t, B^2_t) := v_1(B^1_t, B^2_t) + v_2(B^1_t, B^2_t)(\theta B^1_t - B^2_t)^+$. By applying Tanaka's formula, we have

$$\begin{aligned} \mathrm{d}v(B^1_t, B^2_t) &= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} \times (\theta B^1_t - B^2_t)^+ + v_2\theta \mathbb{1}_{\{\theta B^1_t - B^2_t > 0\}}\right)\mathrm{d}B^1_t \\ &\quad + \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y} \times (\theta B^1_t - B^2_t)^+ - v_2\theta \mathbb{1}_{\{\theta B^1_t - B^2_t > 0\}}\right)\mathrm{d}B^2_t \\ &\quad + f(B^1_t, B^2_t)dt + \frac{v_2}{2}\mathrm{d}L_t(\theta B^1 - B^2). \end{aligned} \tag{201}$$

Here

$$\begin{aligned} f(x,y) :&= \frac{1}{2}\frac{\partial^2 v_1}{\partial x^2} + \rho \frac{\partial^2 v_1}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 v_1}{\partial y^2} \\ &\quad + (\theta x - y)^+\left[\frac{1}{2}\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial x \partial y}\rho + \frac{1}{2}\frac{\partial^2 v_2}{\partial y^2}\right] \\ &\quad - \left[(\rho - \theta)\frac{\partial v_2}{\partial x} + (1-\theta\rho)\frac{\partial v_2}{\partial y}\right]\mathbb{1}_{\{\theta x - y > 0\}}, \end{aligned} \tag{202}$$

where $L_t(\theta B^1 - B^2)$ is the local time of $\theta B^1_t - B^2_t$ at 0. It is an increasing continuous process and satisfies

$$\mathrm{d}(\theta B^1_t - B^2_t)^+ = \mathbb{1}_{\{\theta B^1_t - B^2_t > 0\}}\mathrm{d}(\theta B^1_t - B^2_t) + \frac{1}{2}\mathrm{d}L_t(\theta B^1 - B^2),$$

by the definition of local time, we know $\mathrm{d}L_t(\theta B^1 - B^2) = 0$ when $\theta B_t^1 - B_t^2 \neq 0$. Since $v_2(x, \theta x) = 0$, we have $v_2(B_t^1, B_t^2) = 0$ when $\theta B_t^1 = B_t^2$. Thus

$$\frac{v_2}{2}\mathrm{d}L_t(\theta B^1 - B^2) \equiv 0.$$

We write $f(x,y)$ as $g(x,y)I_{\{\theta x - y \leq 0\}} + h(x,y)I_{\{\theta x - y > 0\}}$, here g and h are both C^2.

$$\begin{aligned} g(B_t^1, B_t^2) &= (\alpha-1)(B_t^1 + B_t^2)^{\alpha-3}(1+\rho_t) \times \left(\frac{\eta}{(1+\theta)^{\alpha-1}}(\alpha-1) + b^\alpha(1+\theta-\theta\alpha)\right) B_t^1 \\ &\quad + (\alpha-1)(B_t^1 + B_t^2)^{\alpha-3}(1+\rho_t) \times \left(\frac{\eta}{(1+\theta)^{\alpha-1}} + b^\alpha(\alpha-1-\theta)\right) B_t^2, \end{aligned} \tag{203}$$

$$\begin{aligned} h(B_t^1, B_t^2) &= (\alpha-1)(B_t^1 - B_t^2)^{\alpha-3}(1-\rho_t) \times \left(-\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}} + (\alpha - 1 + \frac{1}{\theta})\right) B_t^1 \\ &\quad + (\alpha-1)(B_t^1 - B_t^2)^{\alpha-3}(1-\rho_t) \times \left(\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}}(\alpha-1) - \frac{\alpha-1+\theta}{\theta}\right) B_t^2. \end{aligned} \tag{204}$$

It can be shown that $g(B_t^1, B_t^2) \leq 0$ when $B_t^2 \geq \theta B_t^1$; similarly, it also can be shown $h(B_t^1, B_t^2) \leq 0$ when $B_t^2 < \theta B_t^1$. We will present it below.

From (203), we find that $g(B_t^1, B_t^2)$ has the opposite sign to

$$g_1(B_t^1, B_t^2) := \left(\frac{\eta}{(1+\theta)^{\alpha-1}}(\alpha-1) + b^\alpha(1+\theta-\theta\alpha)\right) + \left(\frac{\eta}{(1+\theta)^{\alpha-1}} + b^\alpha(\alpha-1-\theta)\right)\frac{B_t^2}{B_t^1}.$$

Clearly $g_1(B_t^1, B_t^2)$ is a linear function of $\frac{B_t^2}{B_t^1}$. By looking at the coefficient

$$\begin{aligned} & \frac{\eta}{(1+\theta)^{\alpha-1}} + b^\alpha(\alpha-1-\theta) \\ =\; & \frac{b^\alpha(\alpha-1)((1+\theta)^2 - 2\theta\alpha) + (1+\theta)^{2-\alpha}(1-\theta)^\alpha}{(1-\theta)(1+\theta) + 2(1-\alpha)\theta} \\ =\; & \left(\frac{1+\theta}{1-\theta}\right)^{1-\alpha} \frac{(1-\theta-\alpha)(\alpha-1)((1+\theta)^2 - 2\theta\alpha) + (1+\theta)(1-\theta)(1+\theta-\theta\alpha)}{(1+\theta-\theta\alpha)((1-\theta)(1+\theta) + 2(1-\alpha)\theta)} \\ =\; & \left(\frac{1+\theta}{1-\theta}\right)^{1-\alpha} \frac{(1-(1-\alpha)^2)(1-\theta^2) + 2\theta((1-\alpha) - (1-\alpha)^3)}{(1+\theta-\theta\alpha)((1-\theta)(1+\theta) + 2(1-\alpha)\theta)} > 0. \end{aligned} \tag{205}$$

The first and the second equality come by plugging (196) and (197) into the formula (205), the third equality comes by simple algebra. Thus $g_1(B_t^1, B_t^2)$ is an increasing function of $\frac{B_t^2}{B_t^1}$, and when $\frac{B_t^2}{B_t^1} = \theta$, we have

$$\begin{aligned} & \left(\frac{\eta}{(1+\theta)^{\alpha-1}}(\alpha-1) + b^\alpha(1+\theta-\theta\alpha)\right) + \left(\frac{\eta}{(1+\theta)^{\alpha-1}} + b^\alpha(\alpha-1-\theta)\right)\theta \\ =\; & \frac{1}{(1-\theta)(1+\theta) + 2(1-\alpha)\theta}\left(\frac{1+\theta}{1-\theta}\right)^{1-\alpha} \\ & \times \left(\beta^\alpha(1-\theta)^{2-\alpha}(1+\theta)^\alpha(1+\theta-\alpha\theta) - (1+\theta)(1-\theta)(1-\theta-\alpha)\right) = 0. \end{aligned} \tag{206}$$

The equality comes by plugging (196) and (197) into the formula (206) and doing simple algebra. Therefore, as $\frac{B_t^2}{B_t^1} \geq \theta$, we obtain $g_1(B_t^1, B_t^2) \geq 0$, which means $g(B_t^1, B_t^2) \leq 0$.

Similarly, From (204), we get that $h(B_t^1, B_t^2)$ has the opposite sign to

$$h_1(B_t^1, B_t^2) := \left(-\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}} + (\alpha - 1 + \frac{1}{\theta})\right)\frac{B_t^1}{B_t^2} + \left(\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}}(\alpha-1) - \frac{\alpha-1+\theta}{\theta}\right).$$

Here $h_1(B_t^1, B_t^2)$ is a linear function of $\frac{B_t^1}{B_t^2}$. By looking at the coefficient

$$\begin{aligned} & -\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}} + (\alpha - 1 + \frac{1}{\theta}) \\ = \quad & \frac{(1-\alpha)(1-\theta)(1-\theta) + 2\alpha(1-\alpha)\theta - \beta^\alpha(1-\theta)^{2-\alpha}(1+\theta)^\alpha}{(1-\theta)(1+\theta) + 2(1-\alpha)\theta} \\ = \quad & \frac{2\alpha(1-\alpha)\theta + \frac{\alpha\theta(1-\theta)(2\theta+\alpha-\theta\alpha)}{1+\theta-\theta\alpha}}{(1-\theta)(1+\theta) + 2(1-\alpha)\theta} > 0. \end{aligned} \tag{207}$$

The first and the second equality come by plugging (196) and (197) into the formula (207) and doing simple algebra. Therefore $h_1(B_t^1, B_t^2)$ is an increasing function of $\frac{B_t^1}{B_t^2}$, and when $\frac{B_t^1}{B_t^2} = \frac{1}{\theta}$, we have

$$\begin{aligned} & \left(-\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}} + (\alpha - 1 + \frac{1}{\theta})\right)\frac{1}{\theta} + \left(\frac{\eta}{\theta}\frac{1}{(1-\theta)^{\alpha-1}}(\alpha-1) - \frac{\alpha-1+\theta}{\theta}\right) \\ = \quad & \frac{1}{\theta((1-\theta)(1+\theta) + 2(1-\alpha)\theta)} \\ & \times \left(-\beta^\alpha(1+\theta)^\alpha(1-\theta)^{2-\alpha}(1+\theta-\theta\alpha) + (1-\theta)(1+\theta)(1-\theta-\alpha)\right) = 0. \end{aligned} \tag{208}$$

The equality comes by plugging (196) and (197) into the formula (208) and doing simple algebra. Therefore, as $\frac{B_t^1}{B_t^2} > \theta$, we get $h_1(B_t^1, B_t^2) > 0$, which means $h(B_t^1, B_t^2) \leq 0$.

We define the sequence of increasing stopping times $T_n = \inf\{t : B_t^1 \leq 1/n \text{ or } B_t^2 \leq 1/n\} \wedge n$. It is easy to see that $T_n < \tau$, and $T_n \to \tau$ as $n \to \infty$. By applying the same technique used in the proof of Theorem 19, $v(B_{t\wedge T_n}^1, B_{t\wedge T_n}^2)$ is a local supermartingale for any choice of ρ_t, and it is a local martingale for $\rho_t = -1$ when $B_t^2 \geq \theta B_t^1$ and $\rho_t = 1$ when $B_t^2 < \theta B_t^1$. Since T_n is bounded, optional sampling theorem implies that

$$\mathbb{E}(v(B_{t\wedge T_n}^1, B_{t\wedge T_n}^2)) \leq v(B_0^1, B_0^2) \ . \tag{209}$$

By applying Fatou's lemma, we get

$$\mathbb{E}(v(B_\tau^1, B_\tau^2)) \leq v(B_0^1, B_0^2).$$

As $(B^1_\tau + bB^2_\tau)^\alpha = v_1(B^1_\tau, B^2_\tau) + v_2(B^1_\tau, B^2_\tau)(\theta B^1_\tau - B^2_\tau)^+$, we get

$$\max_{\rho_t} \mathbb{E}[(B^1_\tau + bB^2_\tau)^\alpha] \le v_1(B^1_0, B^2_0) + v_2(B^1_0, B^2_0)(\theta B^1_0 - B^2_0)^+. \tag{210}$$

Now if we take $\rho_t = -1$ when $B^2_t \ge \theta B^1_t$ and $\rho_t = 1$ when $B^2_t < \theta B^1_t$, we will show in Theorem 24, that (B^1_t, B^2_t) with such correlation does exist. Therefore, we obtain $\mathbb{E}[(B^1_\tau + bB^2_\tau)^\alpha] = v_1(B^1_0, B^2_0) + v_2(B^1_0, B^2_0)(\theta B^1_0 - B^2_0)^+$ by applying Dominated Convergence Theorem.

□

As we mentioned above, we choose $\rho_t = -1$ when $B^2_t \ge \theta B^1_t$ and $\rho_t = 1$ when $B^2_t < \theta B^1_t$. One question arises naturally: does the Brownian motion (B^1_t, B^2_t) with such correlation exist? We will introduce one important result in the following, which is related to the skew Brownian motion (see Chapter 5) and has independent interest.

Theorem 24. *Assume that B^2_t is a Brownian Motion which starts from B^2_0 and $0 < \theta < 1$. Then the SDE $\mathrm{d}B^1_t = \mathrm{sgn}(\theta B^1_t - B^2_t)\mathrm{d}B^2_t$ has a unique solution B^1_t and it is also a Brownian Motion.*

Proof. For simplicity, we let : $H_t = \theta B^1_t - B^2_t$.
For proving the uniqueness, we assume B^1_t satisfies this SDE, so we have

$$\begin{aligned} \mathrm{d}|H_t| &= \mathrm{sgn}(H_t)dH_t + \mathrm{d}L_t(H) \\ &= \theta\mathrm{sgn}(H_t) \times \mathrm{sgn}(H_t)\mathrm{d}B^2_t - \mathrm{sgn}(H_t)\mathrm{d}B^2_t + \mathrm{d}L_t(H) \\ &= \theta\mathrm{d}B^2_t - \mathrm{d}B^1_t + \mathrm{d}L_t(H). \end{aligned} \tag{211}$$

The first equality is by applying Tanaka's formula, where $L_t(H)$ is the local time of H_t at 0. The second and the third equalities are obtained by applying $\mathrm{d}B^1_t = \mathrm{sgn}(H_t)\mathrm{d}B^2_t$. Furthermore, we get

$$\mathrm{d}|H_t| + \mathrm{d}B^1_t - \frac{1}{\theta}\mathrm{d}B^2_t = (\theta - \frac{1}{\theta})\mathrm{d}B^2_t + \mathrm{d}L_t(H). \tag{212}$$

We let

$$W_t =: \frac{|H_t| + \frac{1}{\theta}H_t}{\frac{1}{\theta} - \theta},$$

then $L_t(W)$ is the local time of W_t at 0. By applying Tanaka's formula

$$\begin{aligned} \mathrm{d}|W_t| &= \mathrm{sgn}(W_t)\mathrm{d}W_t + \mathrm{d}L_t(W) \\ &= \mathrm{sgn}(W_t) \times \frac{\mathrm{d}|H_t| + \frac{1}{\theta}\mathrm{d}H_t}{\frac{1}{\theta} - \theta} + \mathrm{d}L_t(W) \\ &= \mathrm{sgn}(W_t) \times \frac{\mathrm{sgn}(H_t)\mathrm{d}H_t + \mathrm{d}L_t(H) + \frac{1}{\theta}\mathrm{d}H_t}{\frac{1}{\theta} - \theta} + \mathrm{d}L_t(W). \end{aligned} \tag{213}$$

On the other hand, since $\frac{1}{\theta} > 1$, which means the sign of $|H_t| + \frac{1}{\theta}H_t$ is dominated by $\frac{1}{\theta}H_t$, so

$$|W_t| = \frac{H_t + \frac{1}{\theta}|H_t|}{\frac{1}{\theta} - \theta}.$$

We can write $d|W_t|$ as

$$\begin{aligned} \mathrm{d}|W_t| &= \frac{\mathrm{d}H_t + \frac{1}{\theta}\mathrm{d}|H_t|}{\frac{1}{\theta} - \theta} \\ &= \frac{\mathrm{d}H_t + \frac{1}{\theta}(\mathrm{sgn}(H_t)\mathrm{d}H_t + \mathrm{d}L_t(H))}{\frac{1}{\theta} - \theta}. \end{aligned} \tag{214}$$

By comparing (213) with (214), since W_t has the same sign as H_t and

$$\mathrm{sgn}(W_t)\mathrm{d}L_t(H) = \mathrm{sgn}(0)\mathrm{d}L_t(H) = -\mathrm{d}L_t(H),$$

so we obtain

$$L_t(W) = \frac{1 + \frac{1}{\theta}}{\frac{1}{\theta} - \theta}L_t(H).$$

Therefore we can write (212) as

$$\mathrm{d}W_t = -\mathrm{d}B_t^2 + \frac{1}{1 + \frac{1}{\theta}}\mathrm{d}L_t(W). \tag{215}$$

As we have already discussed in Theorem 16, for the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + \beta\mathrm{d}L_t(X), \tag{216}$$

that if $\beta < 1/2$, with given Brownian motion B_t, the SDE has a unique solution X_t. Actually X_t is a skew Brownian motion with parameter $\frac{1}{2(1-\beta)}$. In our case, $\beta = \frac{1}{1+\frac{1}{\theta}} < 1/2$, so W_t is unique. Furthermore, W_t is a skew Brownian motion with parameter $\frac{1+\theta}{2}$. By simple algebra, it is easy to see

$$\begin{aligned} \theta B_t^1 - B_t^2 &= (1-\theta)W_t, \qquad \text{when } W_t > 0 \\ \theta B_t^1 - B_t^2 &= (1+\theta)W_t, \qquad \text{when } W_t \le 0. \end{aligned} \tag{217}$$

Since W_t is unique and B_t^2 is given, so B_t^1 is unique.

By giving B_t^2 and assuming B_t^1 satisfies the SDE, we have constructed a skew Brownian motion W_t. Via the uniqueness of W_t, we have proved B_t^1 is the only solution.

Now for proving the existence, we are going to construct B_t^1, such that it satisfies the SDE $\mathrm{d}B_t^1 = \mathrm{sgn}(\theta B_t^1 - B_t^2)\mathrm{d}B_t^2$. Note that with given B_t^2, the equation

$$\mathrm{d}\hat{W}_t + (-\frac{1}{1 + \frac{1}{\theta}})\mathrm{d}L_t(\hat{W}) = \mathrm{d}(-B_t^2)$$

has a unique solution $\hat{W}_t$ by applying Theorem 16. Here $L_t(\hat{W})$ is the local time of $\hat{W}_t$ at 0. We are going to construct B_t^1 by using $\hat{W}_t$. Let $\theta B_t^1 - B_t^2 = \hat{W}_t - \theta|\hat{W}_t|$, namely

$$B_t^1 = \frac{\hat{W}_t - \theta|\hat{W}_t| + B_t^2}{\theta},$$

then we have

$$\begin{aligned}
\mathrm{d}B_t^1 &= \frac{\mathrm{d}\hat{W}_t - \theta \mathrm{d}|\hat{W}_t| + \mathrm{d}B_t^2}{\theta} \\
&= \frac{(-\mathrm{d}B_t^2 + \frac{1}{1+\frac{1}{\theta}}\mathrm{d}L_t(\hat{W})) - \theta(\mathrm{sgn}(\hat{W}_t)\mathrm{d}\hat{W}_t + \mathrm{d}L_t(\hat{W})) + \mathrm{d}B_t^2}{\theta} \\
&= \frac{(-\mathrm{d}B_t^2 + \frac{1}{1+\frac{1}{\theta}}\mathrm{d}L_t(\hat{W})) - \theta(-\mathrm{sgn}(\hat{W}_t)\mathrm{d}B_t^2 + \mathrm{sgn}(\hat{W}_t)\frac{1}{1+\frac{1}{\theta}}\mathrm{d}L_t(\hat{W}) + \mathrm{d}L_t(\hat{W})) + \mathrm{d}B_t^2}{\theta} \\
&= \frac{\theta \mathrm{sgn}(\hat{W}_t)\mathrm{d}B_t^2}{\theta} = \mathrm{sgn}(\hat{W}_t)\mathrm{d}B_t^2. \qquad (218)
\end{aligned}$$

The fourth equality comes from $\mathrm{sgn}(\hat{W}_t)\mathrm{d}L_t(\hat{W}) = \mathrm{sgn}(0)\mathrm{d}L_t(\hat{W}) = -\mathrm{d}L_t(\hat{W})$ and simple algebra. Since $0 < \theta < 1$, $\theta B_t^1 - B_t^2$ and $\hat{W}_t$ have the same sign, so finally we obtain

$$\mathrm{d}B_t^1 = \mathrm{sgn}(\theta B_t^1 - B_t^2)\mathrm{d}B_t^2. \qquad (219)$$

□

Here in Figure7, we would like to show one simulated path of the Brownian motion (B_t^1, B_t^2) with such correlation. We let (B_t^1, B_t^2) start from $(1,1)$, in this path the Brownian motion stops while B_t^2 hits 0. By observing equation (217), we can obtain such conclusions: while $W_t > 0$, we have $\theta B_t^1 - B_t^2 > 0$, which means (B_t^1, B_t^2) is below the line $y = \theta x$; while $W_t < 0$, we have $\theta B_t^1 - B_t^2 < 0$, which means (B_t^1, B_t^2) is above the line $y = \theta x$. Furthermore, W_t is a skew Brownian motion with parameter $\frac{1+\theta}{2}$, this tells us when the Brownian motion is on the line $y = \theta x$, it has the probability $\frac{1+\theta}{2}$ moves below the line and it has the probability $\frac{1-\theta}{2}$ moves above the line. Since we have $\frac{1+\theta}{2} > 1/2$, then the Brownian motion has more excursions below the line $y = \theta x$ than the above. However, the excursions above the line $y = \theta x$ in average have greater magnitude than the below. These conclusions agree with Figure7.

Remark 11. In the proof we provide a new way to construct skew Brownian motion: Given Brownian motion (B_t^1, B_t^2) with correlation $\rho_t = \mathrm{sgn}(\theta B_t^1 - B_t^2)$, then

$$W_t := \frac{|\theta B_t^1 - B_t^2| + \frac{1}{\theta}(\theta B_t^1 - B_t^2)}{\frac{1}{\theta} - \theta}$$

is a skew Brownian motion.

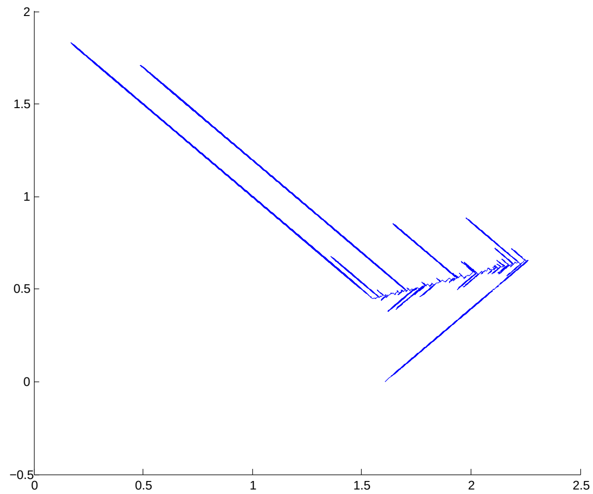


Figure 7: The Brownian motion path

7.3.2 Minimum case

The minimum case for the asymmetric payoff function is similar to the case for the symmetric payoff function.

Theorem 25. *For $0 < \alpha < 1$, we have*

$$\min_{\rho_t} \mathbb{E}[(B^1_\tau + bB^2_\tau)^\alpha] = (B^1_0 + B^2_0)^{\alpha-1}(B^1_0 - B^2_0)^+ + b^\alpha(B^1_0 + B^2_0)^{\alpha-1}(B^1_0 - B^2_0)^-,$$

and the optimal choice to obtain the minimum is $\rho_t = -1$ when $B^1_t \neq B^2_t$, $\rho_t \equiv 1$ once $B^1_t = B^2_t$.

Proof. Let $M_t := B^1_t + B^2_t$, $N_t := B^1_t - B^2_t$. By applying Tanaka's formula

$$\begin{aligned} \mathrm{d}(M_t^{\alpha-1}N_t^+ + b^\alpha M_t^{\alpha-1}N_t^-) &= M_t^{\alpha-1}(\mathbb{1}_{(N_t>0)} - b^\alpha \mathbb{1}_{(N_t\leq 0)})\mathrm{d}N_t + (\alpha-1)M_t^{\alpha-2}(N_t^+ + b^\alpha N_t^-)\mathrm{d}M_t \\ &\quad +(\alpha-1)(\alpha-2)M_t^{\alpha-3}(N_t^+ + b^\alpha N_t^-)(1+\rho_t)\mathrm{d}t \\ &\quad +1/2M_t^{\alpha-1}(1+b^\alpha)\mathrm{d}L_t. \end{aligned} \tag{220}$$

We have $1+\rho_t \geq 0$, $M_t^{\alpha-3} > 0$, $(\alpha-1)(\alpha-2) > 0$, $M_t^{\alpha-1} > 0$ and $dL_t \geq 0$, so $M_t^{\alpha-1}N_t^+ + b^\alpha M_t^{\alpha-1}N_t^-$ is a local submartingale for any choice of ρ_t. Under the optimal choice of ρ_t, by denoting $\tau^* = \inf\{t : B_t^1 = B_t^2\}$, we can observe that $N_t \equiv 0$ for $t \geq \tau^*$, thus we have

$$\mathrm{d}(M_t^{\alpha-1}N_t^+ + b^\alpha M_t^{\alpha-1}N_t^-) = \begin{cases} M_t^{\alpha-1}(\mathbb{1}_{(N_t>0)} - b^\alpha \mathbb{1}_{(N_t\leq 0)})\mathrm{d}N_t, & t < \tau^*, \\ 0, & t \geq \tau^*, \end{cases} \tag{221}$$

which can be written as

$$\mathrm{d}(M_t^{\alpha-1}N_t^+ + b^\alpha M_t^{\alpha-1}N_t^-) = M_t^{\alpha-1}(\mathbb{1}_{(N_t>0)} - b^\alpha \mathbb{1}_{(N_t\leq 0)})\mathbb{1}_{\{\tau^*>t\}}\mathrm{d}N_t.$$

By applying Theorem 9, we find $M_t^{\alpha-1}N_t^+ + b^\alpha M_t^{\alpha-1}N_t^-$ is a local martingale provided $\rho_t = -1$ when $B_t^1 \neq B_t^2$ and $\rho_t \equiv 1$ once $B_t^1 = B_t^2$.

We define the sequence of increasing stopping times $T_n = \inf\{t : B_t^1 \leq 1/n \text{ or } B_t^2 \leq 1/n\} \wedge n$. It is easy to see that $T_n < \tau$, and $T_n \to \tau$ as $n \to \infty$. By applying the same technique used in the proof of Theorem 21, we know $M_{t\wedge T_n}^{\alpha-1}N_{t\wedge T_n}^+ + b^\alpha M_{t\wedge T_n}^{\alpha-1}N_{t\wedge T_n}^-$ is a submartingale. Since T_n is bounded, optional sampling theorem implies that

$$\mathbb{E}\big(M_{T_n}^{\alpha-1}N_{T_n}^+ + b^\alpha M_{T_n}^{\alpha-1}N_{T_n}^-\big) \geq M_0^{\alpha-1}N_0^+ + b^\alpha M_0^{\alpha-1}N_0^- \ . \tag{222}$$

Same as in the proof of Theorem 21, we have

$$\begin{aligned} \mathbb{E}\big[\sup_n \big(M_{T_n}^{\alpha-1}N_{T_n}^+ + b^\alpha M_{T_n}^{\alpha-1}N_{T_n}^-\big)^{1/\alpha}\big] &\leq \Big(\frac{1}{1-\alpha}\Big)^{1/\alpha} \sup_n \mathbb{E}\big[\big(M_{T_n}^{\alpha-1}N_{T_n}^+ + b^\alpha M_{T_n}^{\alpha-1}N_{T_n}^-\big)^{1/\alpha}\big] \\ &< \Big(\frac{1}{1-\alpha}\Big)^{1/\alpha} \sup_n \mathbb{E}\big[\big((1+b^\alpha)M_{T_n}^{\alpha}\big)^{1/\alpha}\big] \\ &= \Big(\frac{1+b^\alpha}{1-\alpha}\Big)^{1/\alpha} M_0 \ . \end{aligned} \tag{223}$$

The first inequality comes from Doob's $\mathbb{L}^p-$inequality, the second one comes from $|N_{T_n}| < M_{T_n}$, and the final equality comes from $\mathbb{E}(M_{T_n}) = M_0$ by applying the optional sampling theorem.

Same as in the proof of Theorem 21, for $0 < \alpha < 1$ we find the fact

$$\mathbb{E}\big[\sup_n \big(M_{T_n}^{\alpha-1}N_{T_n}^+ + b^\alpha M_{T_n}^{\alpha-1}N_{T_n}^-\big)\big] < 1 + \mathbb{E}\big[\sup_n \big(M_{T_n}^{\alpha-1}N_{T_n}^+ + b^\alpha M_{T_n}^{\alpha-1}N_{T_n}^-\big)^{1/\alpha}\big],$$

the Dominated Convergence Theorem implies

$$\mathbb{E}\big(M_\tau^{\alpha-1}N_\tau^+ + b^\alpha M_\tau^{\alpha-1}N_\tau^-\big) = \lim_n \mathbb{E}\big(M_{T_n}^{\alpha-1}N_{T_n}^+ + b^\alpha M_{T_n}^{\alpha-1}N_{T_n}^-\big) \geq M_0^{\alpha-1}N_0^+ + b^\alpha M_0^{\alpha-1}N_0^- \ . \tag{224}$$

Since $(B_\tau^1 + bB_\tau^2)^\alpha = (B_\tau^1 + B_\tau^2)^{\alpha-1}(B_\tau^1 - B_\tau^2)^+ + b^\alpha(B_\tau^1 + B_\tau^2)^{\alpha-1}(B_\tau^1 - B_\tau^2)^-$, we get

$$\min_{\rho_t} \mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha] \geq (B_0^1 + B_0^2)^{\alpha-1}(B_0^1 - B_0^2)^+ + b^\alpha(B_0^1 + B_0^2)^{\alpha-1}(B_0^1 - B_0^2)^-. \tag{225}$$

Now we take the strategy $\rho_t = -1$ when $B_t^1 \neq B_t^2$ and $\rho_t \equiv 1$ once $B_t^1 = B_t^2$, we obtain $\mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha] = (B_0^1 + B_0^2)^{\alpha-1}(B_0^1 - B_0^2)^+ + b^\alpha(B_0^1 + B_0^2)^{\alpha-1}(B_0^1 - B_0^2)^-$.
□

α	b^α	B_0^1	B_0^2	max	min	Variation((max − min)/ min)
1	0.7	13	7	17.9	6	198.3%
1	0.7	7	13	16.1	4.2	283.3%
0.7	0.01	13	7	5.3458	2.4425	118.9%
0.7	0.01	7	13	2.9161	0.0244	11839%
0.5	0.3	13	7	3.3865	1.3416	152.4%
0.5	0.3	7	13	2.4427	0.4025	506.9%

Table 5: Range of option prices for asymmetric payoff function

Theorem 26. *For $\alpha \geq 1$, we have*

$$\min_{\rho_t} \mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha] = ((B_0^1 - B_0^2)^+ + b(B_0^1 - B_0^2)^-)^\alpha,$$

and the optimal choice to achieve the minimum is $\rho_t \equiv 1$.

Proof. Same as in the proof of Theorem 22, we get $\min_{\rho_t} \mathbb{E}[B_\tau^1 + bB_\tau^2] \geq (B_0^1 - B_0^2)^+ + b(B_0^1 - B_0^2)^-$. If we take $\rho_t \equiv 1$, then $\mathbb{E}[B_\tau^1 + bB_\tau^2] = (B_0^1 - B_0^2)^+ + b(B_0^1 - B_0^2)^-$.

For $\alpha > 1$, we have $\mathbb{E}(B_\tau^1 + bB_\tau^2)^\alpha \geq (\mathbb{E}(B_\tau^1 + bB_\tau^2))^\alpha \geq ((B_0^1 - B_0^2)^+ + b(B_0^1 - B_0^2)^-)^\alpha$. Now if we take $\rho_t \equiv 1$, then $\mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha] = ((B_0^1 - B_0^2)^+ + b(B_0^1 - B_0^2)^-)^\alpha$.

Overall, $\min_{\rho_t} \mathbb{E}[(B_\tau^1 + bB_\tau^2)^\alpha] = ((B_0^1 - B_0^2)^+ + b(B_0^1 - B_0^2)^-)^\alpha$ for $\alpha \geq 1$. □

7.3.3 Conclusions

We have constructed closed form solutions for special boundary conditions, we are going to demonstrate the spread of the maximum and minimum in Table 5. For example, by setting the parameters $\alpha = 0.5, \beta^\alpha = 0.3, B_0^1 = 13, B_0^2 = 7$, we obtain that the maximum price max $= 3.3865$ and the minimum price min $= 1.3416$, which have a variation of 152.4%. This is large, so we conclude that uncertainty in the correlation ρ_t yields a wide range of option prices. In that sense, together with the results in Table 4, these compares with the results of Avallaneda, Levy and Parás [1], who show that volatility uncertainty implies a wide range of call option prices.

7.4 Numerical techniques

As we mentioned in Section 7.1, we will demonstrate the correlation's evolution according to different choice of the payoff function $f(x)$. Since our main goal is to show some insight into the correlation's evolution, we will use the explicit method as the numerical scheme. The advantage of such scheme is straightforward and easy to handle.

Let us recall the maximum case, our value function is $v(x, y, T-t)$, which satisfies

$$-v_t + \frac{1}{2}v_{xx} + \frac{1}{2}v_{yy} + |v_{xy}| = 0, \tag{226}$$

with the terminal condition

$$v(x,y,0) = \begin{cases} 0 & \text{if } \ x > 0 \text{ and } y > 0, \\ f(x) & \text{if } \ y = 0, \\ f(y) & \text{if } \ x = 0, \end{cases} \tag{227}$$

and the boundary condition

$$v(x,y,T-t) = \begin{cases} f(x) & \text{if } \ y = 0 \text{ and } 0 \le t \le T, \\ f(y) & \text{if } \ x = 0 \text{ and } 0 \le t \le T. \end{cases} \tag{228}$$

To solve a PDE by finite difference methods, we must set up a discrete grid, in our case with respect to time and two asset prices. Since we know the domain for the PDE is unbounded with respect to asset prices, so we need to bound it for computational purposes. We set them as $x_{\max}$ and $y_{\max}$.

In order to have more discretization points close to 0 and fewer near the boundary, we will change variable for x and y,

$$x = (1+\xi)^{\gamma} - 1, \quad y = (1+\eta)^{\gamma} - 1.$$

Therefore

$$\xi_{\max} = (1 + x_{\max})^{\frac{1}{\gamma}} - 1, \quad \eta_{\max} = (1 + y_{\max})^{\frac{1}{\gamma}} - 1.$$

The equation (226) will change after change of variable, first we have

$$\begin{aligned}
\frac{\partial v}{\partial \xi} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} = \frac{\partial v}{\partial x} \times \gamma(1+\xi)^{\gamma-1}, \\
\frac{\partial v}{\partial \eta} &= \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\partial v}{\partial y} \times \gamma(1+\eta)^{\gamma-1}, \\
\frac{\partial^2 v}{\partial \xi^2} &= \frac{\partial^2 v}{\partial x^2}(\frac{\partial x}{\partial \xi})^2 + \frac{\partial v}{\partial x}\frac{\partial^2 x}{\partial \xi^2} = \frac{\partial^2 v}{\partial x^2} \times \gamma^2(1+\xi)^{2\gamma-2} + \frac{\partial v}{\partial x} \times \gamma(\gamma-1)(1+\xi)^{\gamma-2}, \\
\frac{\partial^2 v}{\partial \eta^2} &= \frac{\partial^2 v}{\partial y^2}(\frac{\partial y}{\partial \eta})^2 + \frac{\partial v}{\partial y}\frac{\partial^2 y}{\partial \eta^2} = \frac{\partial^2 v}{\partial y^2} \times \gamma^2(1+\eta)^{2\gamma-2} + \frac{\partial v}{\partial y} \times \gamma(\gamma-1)(1+\eta)^{\gamma-2}, \\
\frac{\partial^2 v}{\partial \xi \partial \eta} &= \frac{\partial^2 v}{\partial x \partial y}\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} = \frac{\partial^2 v}{\partial x \partial y} \times \gamma^2(1+\xi)^{\gamma-1}(1+\eta)^{\gamma-1}.
\end{aligned}$$

By expressing $\frac{\partial^2 v}{\partial x^2}$, $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial^2 v}{\partial x \partial x}$ in (226) with the form of $\frac{\partial v}{\partial \xi}$, $\frac{\partial v}{\partial \eta}$, $\frac{\partial^2 v}{\partial \xi^2}$, $\frac{\partial^2 v}{\partial \eta^2}$, $\frac{\partial^2 v}{\partial \xi \partial \eta}$, we obtain

$$-v_t + \frac{(1+\xi)^{2-2\gamma}}{2\gamma^2}\Big(v_{\xi\xi} - \frac{\gamma-1}{1+\xi}v_{\xi}\Big) + \frac{(1+\eta)^{2-2\gamma}}{2\gamma^2}\Big(v_{\eta\eta} - \frac{\gamma-1}{1+\eta}v_{\eta}\Big) + \frac{(1+\xi)^{1-\gamma}(1+\eta)^{1-\gamma}}{\gamma^2}|v_{\xi\eta}| = 0.$$

As one can see above that v_{xy} and $v_{\xi\eta}$ have the same sign, therefore the sign of $v_{\xi\eta}$ will be enough to decide the behavior of the correlation.

The grid consists of points (ξ, η, t) such that

$$\begin{aligned} \xi &= 0, \delta_\xi, 2\delta_\xi, \cdots, M\delta_\xi \equiv \xi_{\max}, \\ \eta &= 0, \delta_\eta, 2\delta_\eta, \cdots, M\delta_\eta \equiv \eta_{\max}, \\ t &= 0, \delta_t, 2\delta_t, \cdots, N\delta_t \equiv T. \end{aligned}$$

We will use the grid notation $v_{i,j,k} = v\left(i\delta_\xi, j\delta_\eta, k\delta_t\right)$.

The partial derivatives will be approximated by the finite differences:

$$\begin{aligned} v_t &= \frac{v_{i,j,k+1} - v_{i,j,k}}{\delta_t}, \\ v_\xi &= \frac{v_{i+1,j,k} - v_{i-1,j,k}}{2\delta_\xi}, \\ v_\eta &= \frac{v_{i,j+1,k} - v_{i,j-1,k}}{2\delta_\eta}, \\ v_{\xi\xi} &= \frac{v_{i+1,j,k} - 2v_{i,j,k} + v_{i-1,j,k}}{(\delta_\xi)^2}, \\ v_{\eta\eta} &= \frac{v_{i,j+1,k} - 2v_{i,j,k} + v_{i,j-1,k}}{(\delta_\eta)^2}, \\ v_{\xi\eta} &= \frac{v_{i+1,j+1,k} - v_{i+1,j-1,k} - v_{i-1,j+1,k} + v_{i-1,j-1,k}}{4(\delta_\xi)(\delta_\eta)}. \end{aligned}$$

Before we present the results from the numerical approach, we need to discuss about the stability of the solution obtained from such numerical scheme. Note that, by changing variable, the PDE (226) is transformed into the PDE (229). Therefore, under the same numerical scheme, the PDE (226) has stable solution is equivalent to the PDE (229) has stable solution. Also note that, since we do not have complete boundary condition, we only can provide the necessary condition for the stability.

We define

$$v_{k,j,n} = a^n(w)e^{\mathrm{i}kw\delta_x + \mathrm{i}jw\delta_y} \tag{229}$$

and

$$G(w) = \frac{a^n(w)}{a^{n+1}(w)},$$

the Von Neumann stable condition states that: If the scheme needs to be stable, then $|G(w)| \leq 1$ for $0 \leq w\delta_x \leq \pi$ and $0 \leq w\delta_y \leq \pi$.(see Chapter 3 in [66])

For the numerical scheme, by substituting (229) into the PDE (226), we have

$$\begin{aligned} a^{n+1}(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y} &= a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y} \\ &+\frac{\delta_t}{2\delta_x^2}\Big(a^n(w)e^{\mathrm{i}(k+1)w\delta_x+\mathrm{i}jw\delta_y}-2a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}+a^n(w)e^{\mathrm{i}(k-1)w\delta_x+\mathrm{i}jw\delta_y}\Big) \\ &+\frac{\delta_t}{2\delta_y^2}\Big(a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}(j+1)w\delta_y}-2a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}+a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}(j-1)w\delta_y}\Big) \\ &+\frac{\delta}{4\delta_x\delta_y}\Big|a^n(w)e^{\mathrm{i}(k+1)w\delta_x+\mathrm{i}(j+1)w\delta_y}-a^n(w)e^{\mathrm{i}(k+1)w\delta_x+\mathrm{i}(j-1)w\delta_y} \\ &\qquad -a^n(w)e^{\mathrm{i}(k-1)w\delta_x+\mathrm{i}(j+1)w\delta_y}+a^n(w)e^{\mathrm{i}(k-1)w\delta_x+\mathrm{i}(j-1)w\delta_y}\Big|, \end{aligned} \tag{230}$$

then we can compute

$$\begin{aligned} G(w) &= \frac{a^{n+1}(w)}{a^n(w)} \\ &= 1+\frac{\delta_t}{2\delta_x^2}\Big(e^{\mathrm{i}w\delta_x}-2+e^{-\mathrm{i}w\delta_x}\Big)+\frac{\delta_t}{2\delta_y^2}\Big(e^{\mathrm{i}w\delta_y}-2+e^{-\mathrm{i}w\delta_y}\Big) \\ &+\frac{\delta_t}{4\delta_x\delta_y}\Big|e^{\mathrm{i}w\delta_x+\mathrm{i}w\delta_y}-e^{\mathrm{i}w\delta_x-\mathrm{i}w\delta_y}-e^{-\mathrm{i}w\delta_x+\mathrm{i}w\delta_y}+e^{-\mathrm{i}w\delta_x-\mathrm{i}w\delta_y}\Big|\frac{|a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}|}{a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}} \\ &= 1+\frac{\delta_t}{\delta_x^2}\Big(\cos(w\delta_x)-1\Big)+\frac{\delta_t}{\delta_y^2}\Big(\cos(w\delta_y)-1\Big)+\frac{\delta_t}{\delta_x\delta_y}|\sin(w\delta_x)\sin(w\delta_y)|\frac{|a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}|}{a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}}, \end{aligned} \tag{231}$$

where the last equality is obtained by applying Euler's formula and trigonometric identities.

By letting $\delta_x=\delta_y$ and using formula (231), we have

$$\begin{aligned} |G(w)| &\le \left|1+\frac{\delta_t}{\delta_x^2}\Big(\cos(w\delta_x)-1\Big)+\frac{\delta_t}{\delta_y^2}\Big(\cos(w\delta_y)-1\Big)\right| \\ &\quad +\left|\frac{\delta_t}{\delta_x\delta_y}|\sin(w\delta_x)\sin(w\delta_y)|\frac{|a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}|}{a^n(w)e^{\mathrm{i}kw\delta_x+\mathrm{i}jw\delta_y}}\right| \\ &= \left|1+2\frac{\delta_t}{\delta_x^2}\Big(\cos(w\delta_x)-1\Big)\right|+\left|\frac{\delta_t}{\delta_x^2}\sin^2(w\delta_x)\right| \\ &= |1-4\frac{\delta_t}{\delta_x^2}\sin^2(w\delta_x/2)|+4\frac{\delta_t}{\delta_x^2}\sin^2(w\delta_x/2)\cos^2(w\delta_x/2), \end{aligned} \tag{232}$$

where the last equality is obtained by applying trigonometric identities.

Let $\lambda:=\delta_t/\delta_x^2$ and $a:=\sin^2(w\delta_x/2)$, we can write (232) as

$$|G(w)|\le|1-4\lambda a|+4\lambda a(1-a),$$

where $0\le a\le 1$. When $1-4\lambda a\ge 0$, we have $1-4\lambda a+4\lambda a(1-a)\le 1$; when $1-4\lambda a<0$, we have $4\lambda a-1+4\lambda a(1-a)\le 4\lambda-1$; above all, if $4\lambda-1\le 1$, we can guarantee $|G(w)|\le 1$ for all the $a \in [0,1]$. Therefore, if $\delta_t/\delta_x^2 \leq 1/2$, we can guarantee the stability of the numerical solution to the PDE (226).

Note that, during the process of changing variable $x = (1+\xi)^\gamma - 1$ and $y = (1+\eta)^\gamma - 1$, we have $\delta_x = \gamma(1+\xi)^{(\gamma-1)}\delta_\xi \geq \gamma\delta_\xi$ for $\gamma \geq 1$, thus we can find

$$\frac{\delta_t}{\delta_x^2} \leq \frac{\delta_t}{\gamma^2\delta_\xi^2}.$$

It is the same for δ_y and δ_η. By letting $\frac{\delta_t}{\gamma^2\delta_\xi^2} < \frac{1}{2}$ and $\delta_\eta = \delta_\xi$, we have the necessary condition for the stability of the numerical solution to the PDE (226), then such necessary condition also applies to the PDE (229) under the same numerical scheme.

Now we need to take care of the boundary condition. As we mentioned, the boundary condition is partially known. For computational purpose, we need to give values to the unknown boundary condition.

We take $f(x,y) = (x+by)^\alpha$, for the known boundary condition, in grid notation we have

$$v_{i,j,0} = \begin{cases} 0 & \text{if } i \neq 0 \text{ and } j \neq 0, \\ \big((1+i\delta_\xi)^\gamma - 1\big)^\alpha & \text{if } j = 0, \\ b^\alpha\big((1+j\delta_\eta)^\gamma - 1\big)^\alpha & \text{if } i = 0, \end{cases} \tag{233}$$

and

$$\begin{aligned} v_{i,0,k} &= \big((1+i\delta_\xi)^\gamma - 1\big)^\alpha, \\ v_{0,j,k} &= b^\alpha\big((1+j\delta_\eta)^\gamma - 1\big)^\alpha. \end{aligned}$$

For the unknown boundary condition, we will do interpolation between $v_{0,M,k}$ and $v_{M,0,k}$:

$$\begin{aligned} v_{i,M,k} &= \frac{2M-i}{2M} \times v_{0,M,k} + \frac{i}{2M} \times v_{M,0,k}, \\ v_{M,j,k} &= \frac{2M-j}{2M} \times v_{M,0,k} + \frac{j}{2M} \times v_{0,M,k}. \end{aligned}$$

Because of giving value to the unknown part of the boundary condition, the obtained correlation may be unreliable for the location near such boundary. In order to control the impact on the correlation from the unknown boundary value, we will only investigate the correlation constraint in the area $[0, x_{\max}/8] \times [0, y_{\max}/8]$.

The parameters we have used in the numerical scheme are:

$$x_{\max} = y_{\max} = 40,\ M = 200,\ T = 150,\ N = 120000,\ b^\alpha = 0.01,\ \alpha = 0.7,\ \gamma = 2.$$

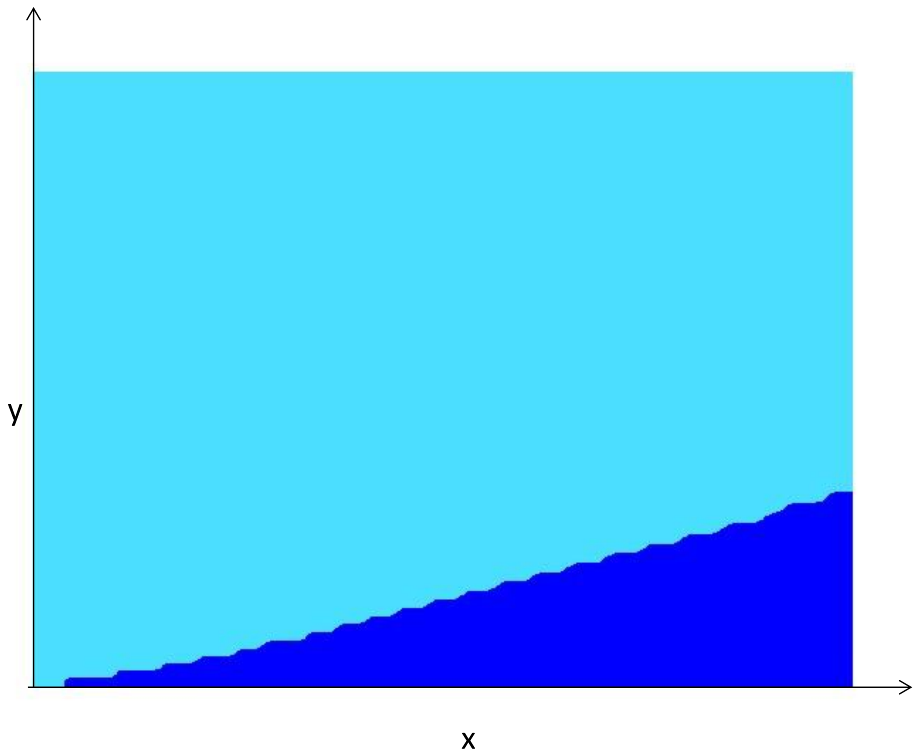


Figure 8: The correlation for the asymmetric payoff function

We demonstrate the correlation with respect to the location (x, y) in Figure 8, the light blue region represents $\rho = -1$, the dark blue region represents $\rho = 1$. It can be seen that the correlation switches when the two dimensional Brownian motion moves across a line.

In the following example, we will show a more complicated switch pattern of the correlation. We set the payoff function as $f(x, y) = 1 + \sin(8(x + y))$, which is non-negative and periodic. The parameters we have used in the numerical scheme are:

$$x_{\max} = y_{\max} = 40,\ M = 200,\ T = 150,\ N = 120000,\ \gamma = 2.$$

We demonstrate the correlation with respect to the location (x, y) in Figure 9, the light blue region represents $\rho = -1$, the dark blue region represents $\rho = 1$. It is easy to observe that the

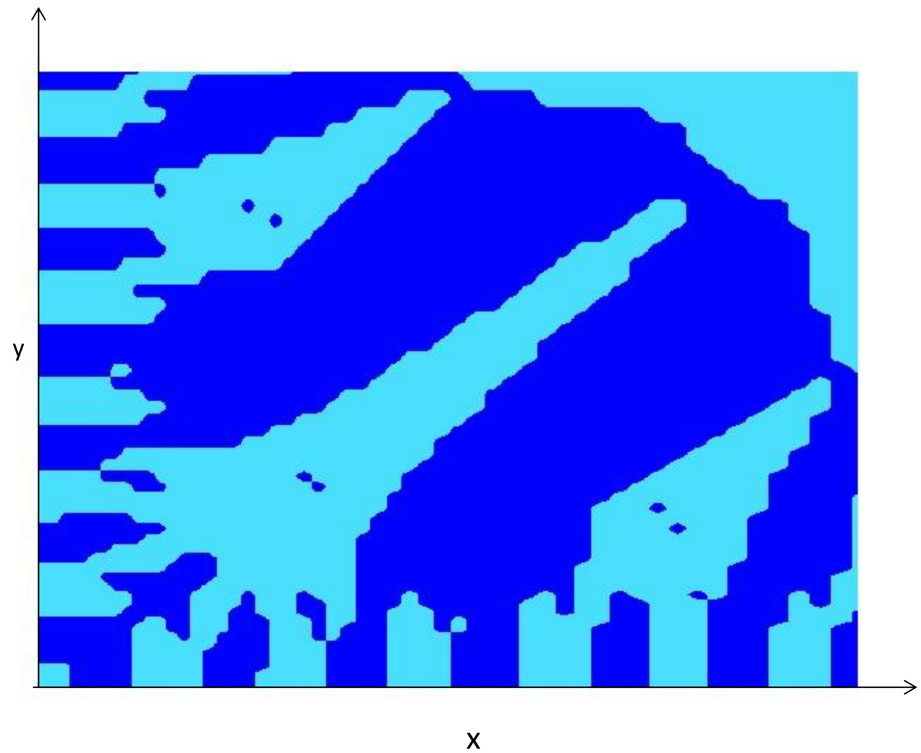


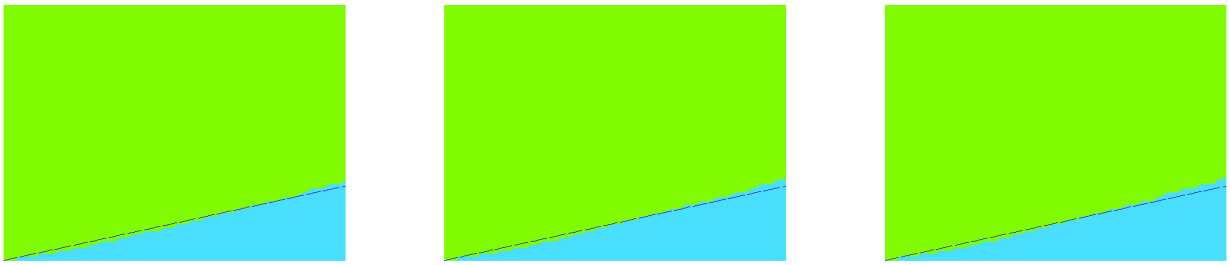
Figure 9: The correlation for the periodic payoff function

switch pattern of the correlation has some periodic property. However, other parts of the switch pattern are too complicated to explain, which motivate us to work on some simple payoff functions to obtain analytic results for the correlation switch.

At the end we compare the numerical results with the theoretical result obtained in Section 7.3. The payoff function is given as $f(x, y) = (x + by)^\alpha$, where we take $\alpha = 0.7$ and $b^\alpha = 0.01$. For the numerical results, we set the T as 75, 150 and 300 respectively; for the theoretical result, given $b^\alpha = 0.01$, we can obtain $\theta = 0.2909$ from equation (196). The parameters we have used in the numerical scheme are:

Figure 10(a)

$$x_{\max} = y_{\max} = 40,\ M = 200,\ T = 75,\ N = 60000,\ b^\alpha = 0.01,\ \alpha = 0.7,\ \gamma = 2,\ \theta = 0.2909.$$



(a) T=75 (b) T=150 (c) T=300

Figure 10: The correlation (numerical result vs theoretical result)

Figure 10(b)

$$x_{\max} = y_{\max} = 40,\ M = 200,\ T = 150,\ N = 120000,\ b^{\alpha} = 0.01,\ \alpha = 0.7,\ \gamma = 2,\ \theta = 0.2909.$$

Figure 10(c)

$$x_{\max} = y_{\max} = 40,\ M = 200,\ T = 300,\ N = 240000,\ b^{\alpha} = 0.01,\ \alpha = 0.7,\ \gamma = 2,\ \theta = 0.2909.$$

We demonstrate the correlation with respect to the location (x, y) in Figure 10, the green region represents $\rho = -1$, the light blue region represents $\rho = 1$, which are numerical results; the dash line (the slope $\theta = 0.2909$) is the theoretical result, which represents the boundary where the correlation switches. Note that as T becomes large, the critical boundary becomes linear, as our closed form solution to the infinite horizon problem implies.

8 Duality relations for hypergeometric functions

At the beginning of this chapter, we introduce the approach which is used to derive the following duality relations. By taking one simple identity as an example, we demonstrate how to apply such approach. After that, we present the main results: duality relations for hypergeometric functions and duality relations for basic hypergeometric functions.

8.1 Introduction

As we mentioned in Chapter 6, we will present a more generalized identity with new proof and we have already showed there that the identity $H(x) = 0$ in (145) is a special case of Theorem 27. Although this identity arises from the problem in Chapter 6, it has independent interest in the special functions area.

Duality relations for hypergeometric functions refer to identities involving finite sums of products of two such functions. There is also a similar notion of duality for basic hypergeometric functions. It seems that the first instances of such formulas have appeared in 1932 in the paper [16] by Darling. These results have been expanded by Bailey [2] in 1933, and they have been greatly generalized recently by Beukers and Jouhet [6], who have used the theory of D-modules of general linear differential (or difference) equations. Our goal in this chapter is to present a different approach to derive duality relations. As we will demonstrate, our approach is elementary and it is based on the generalization of a simple fact that the sum of residues of a rational function is zero when the degree of the denominator is greater than one plus the degree of numerator.

In the following one lemma will be presented which is exactly what our approach based on. This lemma is stated using notions of *a set* and *a multiset*. We remind the reader that the only difference in the definition of a set $A = \{a_1, \ldots, a_n\}$ and a multiset $B = \{b_1, \ldots, b_n\}$ is that all elements a_i of the set must be distinct ($a_i \neq a_j$ for $i \neq j$) whereas the elements b_i of a multiset may be repeated several times (b_i may be equal to b_j for some $i \neq j$).

Lemma 5. *Assume that A is a set of n_A complex numbers and B is a multiset of n_B complex numbers (possibly empty). For each element $a \in A$ we define*

$$\gamma(a) = \gamma(a; A, B) = \frac{\prod\limits_{x \in B} (a - x)}{\prod\limits_{y \in A \setminus \{a\}} (a - y)}, \tag{234}$$

with the convention that the product over an empty set is equal to one. Then we have

$$\sum_{a \in A} \gamma(a) = \begin{cases} 0, & \text{if } \; n_A > n_B + 1, \\ 1, & \text{if } \; n_A = n_B + 1, \\ \sum\limits_{a \in A} a - \sum\limits_{b \in B} b, & \text{if } \; n_A = n_B. \end{cases} \tag{235}$$

Proof. We define the rational function

$$f(z) := \frac{\prod_{b\in B}(z-b)}{\prod_{a\in A}(z-a)}. \tag{236}$$

Since A is a set, all the numbers $a \in A$ are distinct, therefore $f(z)$ has only simple poles. This fact and the condition $n_A \geq n_B$ allow us to write the partial fraction expansion of $f(z)$ in the form

$$f(z) = \delta_{n_A,n_B} + \sum_{a\in A} \frac{\gamma(a)}{z-a}. \tag{237}$$

Here $\delta_{m,n} = 1$ if $m = n$, otherwise $\delta_{m,n} = 0$.

Here we are going to give a short proof to show why equation (237) holds. We let

$$g(z) := f(z) - \delta_{n_A,n_B} - \sum_{a\in A} \frac{\gamma(a)}{z-a}, \tag{238}$$

so $g(z)$ at most can have simple poles. By applying residue theory, from (236) we have

$$\mathrm{Res}(f(z) : z = a) = \gamma(a); \tag{239}$$

Cauchy residue theory also tells us

$$\mathrm{Res}(\delta_{n_A,n_B} + \sum_{a\in A} \frac{\gamma(a)}{z-a} : z = a) = \gamma(a); \tag{240}$$

therefore, we have $\mathrm{Res}(g(z) : z = a) = 0$, combining with the fact $g(z)$ at most can have simple poles, we find $g(z)$ is analytic in the whole complex plane.

Furthermore, as $|z| \to \infty$,

$$f(z) = \begin{cases} 0, & \text{if } n_A > n_B, \\ 1, & \text{if } n_A = n_B. \end{cases} \tag{241}$$

Thus we have $g(z) = 0$ as $|z| \to \infty$. Liouville's theorem tells us: if a function is analytic in the whole complex plane and bounded, then it is clearly a constant. Therefore $g(z)$ is a constant, and since $g(z) = 0$ as $|z| \to \infty$, we obtain $g(z) \equiv 0$. This means equation (237) holds.

From the equation (237) we obtain an asymptotic expansion of $f(z)$ as $|z| \to \infty$:

$$f(z) = \delta_{n_A,n_B} + z^{-1} \sum_{a\in A} \gamma(a) + O(z^{-2}). \tag{242}$$

We can obtain another asymptotic expansion of $f(z)$ if we start from (236):

$$f(z) = z^{n_B - n_A} \frac{\prod\limits_{b\in B} (1 - bz^{-1})}{\prod\limits_{a\in A} (1 - az^{-1})} \tag{243}$$
$$= z^{n_B-n_A} + z^{n_B-n_A-1}\left[\sum_{a\in A} a - \sum_{b\in B} b\right] + O(z^{n_B-n_A-2}),$$

where $\frac{1}{1-az^{-1}} = 1 + az^{-1} + O(z^{-2})$ as $|z| \to \infty$ by applying Taylor series.

The desired result (235) follows by comparing the coefficients in front of the term z^{-1} in the two formulas (242) and (243). $\square$

Remark 12. The result (235) in the case $n_A = n_B$ is equivalent to *the nonlocal derangement identity* (see formula (1.20) in [28]). In fact, the case $n_A = n_B$ is really the main one – the other two cases can be deduced from it by a simple limiting procedure. For example, the result in the case $n_A = n_B + 1$ can be deduced from the case $n_A = n_B$ as follows: take an element $b_1 \in B$, divide both sides of (235) by b_1 and then let $b_1 \to \infty$. In a similar way one can derive the result in case $n_A > n_B + 1$.

We will take the simple identity we obtained from the research in exponential functional as an example to show how we prove it in a very simple way by using Lemma 5.

Proposition 5.

$$b \times {}_1F_1\left(\begin{matrix} a-c \\ 1+b-c \end{matrix}\middle| z\right){}_2F_2\left(\begin{matrix} 1+c-a, c \\ 1+c-b, 1+c \end{matrix}\middle| -z\right) \tag{244}$$
$$- c \times {}_1F_1\left(\begin{matrix} a-b \\ 1+c-b \end{matrix}\middle| z\right){}_2F_2\left(\begin{matrix} 1+b-a, b \\ 1+b-c, 1+b \end{matrix}\middle| -z\right) = (b-c){}_2F_2\left(\begin{matrix} a, 1 \\ 1+b, 1+c \end{matrix}\middle| z\right).$$

Here a, b, c are complex numbers satisfying $b \notin \mathbb{Z}$, $c \notin \mathbb{Z}$ and $b - c \notin \mathbb{Z}$.

Proof. We transform (244) to the form of

$$\frac{1}{(-b-(-c))(-b-0)}{}_2F_2\left(\begin{matrix}1-b-(1-a),1-b-0\\1-b-(-c),1-b-0\end{matrix}\middle|z\right){}_2F_2\left(\begin{matrix}1-a-(-b),0-(-b)\\1-c-(-b),1+0-(-b)\end{matrix}\middle|(-1)^{2+2+1}z\right)$$
$$+\frac{1}{(-c-(-b))(-c-0)}{}_2F_2\left(\begin{matrix}1-c-(1-a),1-c-0\\1-c-(-b),1-c-0\end{matrix}\middle|z\right){}_2F_2\left(\begin{matrix}1-a-(-c),0-(-c)\\1-b-(-c),1+0-(-c)\end{matrix}\middle|(-1)^{2+2+1}z\right)$$
$$+\frac{1}{(0-(-b))(0-(-c))}{}_2F_2\left(\begin{matrix}1+0-(1-a),1+0-0\\1+0-(-b),1+0-(-c)\end{matrix}\middle|z\right){}_2F_2\left(\begin{matrix}1-a-(0),0-(0)\\1-b-0,1-c-0\end{matrix}\middle|(-1)^{2+2+1}z\right)=0. \tag{245}$$

We are going to prove the identity by expand the hypergeometric function by definition, so the left-hand side of (245) can be written as $\sum_{n=0}^{\infty} c_n z^n$. The method here is to prove for each term z^n, its coefficient $c_n = 0$.

For the constant term z^0

$$c_0 = \frac{1}{(-b-(-c))(-b-0)} + \frac{1}{(-c-(-b))(-c-0)} + \frac{1}{(0-(-b))(0-(-c))}.$$

By applying Lemma 5, we set $A = \{-b,-c,0\}$, $B = \{\}$, then we get $c_0 = 0$.

For the term z^1. First let us denote $\{a_i\}_{1\le i\le 3} = \{-b,-c,0\}$, $\{b_i\}_{1\le i\le 2} = \{1-a,0\}$ for notion purposes. We have

$$c_1 = \sum_{i=1}^{3} \frac{1}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(a_i-a_j)} \times \left(\frac{\prod\limits_{l=1}^{2}(1+a_i-b_l)}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(1+a_i-a_j)} + \frac{\prod\limits_{l=1}^{2}(b_l-a_i)}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(1-a_i+a_j)}(-1)^5 \right). \tag{246}$$

Since $1+a_i-a_i=1$, we can rewrite (246) as

$$c_1 = \sum_{i=1}^{3} \frac{1}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(a_i-a_j)} \times \left(\frac{\prod\limits_{l=1}^{2}(1+a_i-b_l)}{\prod\limits_{1\le j\le 3}(1+a_i-a_j)} + \frac{\prod\limits_{l=1}^{2}(a_i-b_l)(-1)^2}{\prod\limits_{1\le j\le 3}(a_i-(1+a_j))(-1)^3}(-1)^5 \right) \tag{247}$$

By applying Lemma 5, we set $A = \{a_1,a_2,a_3,a_1+1,a_2+1,a_3+1\}$, $B = \{b_1,b_2\}$, then we get $c_1 = 0$.

For the general term z^n,

$$c_n = \sum_{i=1}^{3} \frac{1}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(a_i-a_j)} \times \left(\sum_{k=0}^{n} \frac{1}{k!} \frac{\prod\limits_{l=1}^{2}(1+a_i-b_l)_k}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(1+a_i-a_j)_k} \times \frac{(-1)^{5(n-k)}}{(n-k)!} \frac{\prod\limits_{l=1}^{2}(b_l-a_i)_{n-k}}{\prod\limits_{\substack{1\le j\le 3\\ j\ne i}}(1-a_i+a_j)_{n-k}} \right)$$

By the definition and some simple algebra, we obtain

$$\begin{aligned}(1+a_i-b_l)_k &= (1+a_i-b_l)(1+a_i-b_l+1)\cdots(a_i-b_l+k)\\ &= (a_i+k-(b_l+k-1))(a_i+k-(b_l+k-2))\cdots(a_i+k-b_l),\end{aligned} \tag{248}$$

$$\begin{aligned}(b_l-a_i)_{n-k} &= (b_l-a_i)(b_l-a_i+1)\cdots(b_l-a_i+n-k-1)\\ &= (-1)^{n-k}(a_i+k-(b_l+k))(a_i+k-(b_l+k+1))\cdots(a_i+k-(b_l+n-1)),\end{aligned}$$

with the same idea, we can get

$$(1+a_i-a_j)_k = (a_i+k-(a_j+k-1))(a_i+k-(a_j+k-2))\cdots(a_i+k-a_j),$$

$$(a_i-a_j) = (a_i+k-(a_j+k)), \quad k! = (a_i+k-(a_i+k-1))(a_i+k-(a_i+k-2))\cdots(a_i+k-a_i),$$

$$(1-a_i+a_j)_{n-k} = (-1)^{n-k}(a_i+k-(a_j+k+1))(a_i+k-(a_j+k+2))\cdots(a_i+k-(a_j+n)),$$

$$(n-k)! = (-1)^{n-k}(a_i+k-(a_i+k+1))(a_i+k-(a_i+k+2))\cdots(a_i+k-(a_i+n)).$$

By setting

$$A = \{a_1,a_2,a_3,a_1+1,a_2+1,a_3+1,\cdots,a_1+n,a_2+n,a_3+n\},$$
$$B = \{b_1,b_2,b_1+1,b_2+1,\cdots,b_1+n-1,b_2+n-1\},$$

we can rewrite c_n as

$$c_n = \sum_{a\in A}\frac{\prod_{b\in B}(a-b)}{\prod_{y\in A\setminus\{a\}}(a-y)}.$$

By applying Lemma 5, then we get $c_n = 0$. □

8.2 Hypergeometric functions duality

In what follows we will be working with functions represented by power series in z, and we will use notation $F(z)\equiv G(z)$ to mean that $F(z)=G(z)$ for all z in some neighbourhood of zero. Let $\mathcal{P}_n$ be the set of polynomials of degree n. We say that $F(z)\equiv G(z) \pmod{\mathcal{P}_n}$ if $F(z)-G(z)\in\mathcal{P}_n$.

Theorem 27. *Assume that $p\le r+1$, $\{a_i\}_{1\le i\le r+1}$ are complex numbers satisfying $a_i-a_j\notin\mathbb{Z}$ for $1\le i<j\le r+1$, $\{b_i\}_{1\le i\le p}$ are complex numbers and $\{m_i\}_{1\le i\le p}$ are integers. Define $M := \sum_{i=1}^{p} m_i$,*

$$c_i := \frac{\prod_{j=1}^{p}(1+a_i-b_j)_{m_j}}{\prod_{\substack{1\le j\le r+1\\ j\ne i}}(a_i-a_j)} \quad \text{for } 1\le i\le r+1, \tag{249}$$

and

$$H(z) := \sum_{i=1}^{r+1} c_i \times {}_pF_r\left(\begin{matrix} 1+a_i+m_1-b_1, \dots, 1+a_i+m_p-b_p \\ 1+a_i-a_1, \dots, *, \dots, 1+a_i-a_{r+1} \end{matrix}\middle| z\right) \tag{250}$$
$$\times {}_pF_r\left(\begin{matrix} b_1-a_i, b_2-a_i, \dots, b_p-a_i \\ 1+a_1-a_i, \dots, *, \dots, 1+a_{r+1}-a_i \end{matrix}\middle| (-1)^{p+r+1} z\right),$$

where the asterisk means that the term $1+a_i-a_i$ *is omitted. Assuming that* $m_i \ge 0$ *for* $1 \le i \le r+1$, *the following is true:*

(i) If $M < r$ *then* $H(z) \equiv 0$;

(ii) If $M = r$ *then* $H(z) \equiv 1$ *in the case* $p \le r$, *and* $H(z) \equiv 1/(1-z)$ *in the case* $p = r+1$;

(iii) If $M = r+1$ *then* $H(z) \equiv C$ *in the case* $p \le r-1$, *and* $H(z) \equiv C + z$ *in the case* $p = r$, *and*

$$H(z) \equiv (\alpha - \beta + p)\frac{z}{(1-z)^2} + \frac{C}{1-z} \quad \text{in the case } p = r+1,$$

where $\alpha = \sum_{i=1}^{r+1} a_i$, $\beta = \sum_{i=1}^{p} b_i$ *and* $C = \alpha + \sum_{i=1}^{p} m_i(m_i+1-2b_i)/2$.

In the case when some of m_i *are negative, the above results in (i)-(iii) hold modulo* $\mathcal{P}_{-\hat{m}}$, *where* $\hat{m} = \min_{1 \le i \le p} m_i$.

Proof. Let us prove the first part of Theorem 27: we assume that $m_i \ge 0$ for $1 \le i \le p$. Let k be a non-negative integer. We define

$$A = \bigcup_{1 \le i \le r+1} \{a_i + j \ : \ 0 \le j \le k\}. \tag{251}$$

Note that the condition $a_i - a_j \notin \mathbb{Z}$ for $1 \le i < j < r+1$ implies that the set A has $n_A = (r+1)(k+1)$ elements. Similarly, we define a multiset

$$B = \biguplus_{1 \le i \le p} \{b_i + j \ : \ -m_i \le j \le k-1\}. \tag{252}$$

The symbol "$\uplus$" means that we are taking union of multisets; in other words, one complex number may be repeated several times in B. It is clear that the multiset B has $n_B = M + kp$ elements (recall that $M = m_1 + \cdots + m_p$).

Let us fix i and j such that $1 \le i \le r+1$ and $0 \le j \le k$ and consider the element $a_i + j$ of the set A. From formula (234) we find

$$\gamma_{i,j}^k := \gamma(a_i + j; A, B) = \frac{\prod\limits_{x \in B} (a_i + j - x)}{\prod\limits_{y \in A \setminus \{a_i + j\}} (a_i + j - y)}$$

$$= \frac{\prod\limits_{l=1}^{p} \prod\limits_{s=-m_l}^{k-1} (a_i + j - b_l - s)}{\prod\limits_{\substack{0 \le s \le k \\ s \ne j}} (j - s) \prod\limits_{\substack{1 \le l \le r+1 \\ l \ne i}} \prod\limits_{s=0}^{k} (a_i + j - a_l - s)} \tag{253}$$

Now we will simplify the expression in (253). We check that

$$\prod_{\substack{0 \le s \le k \\ s \ne j}} (j - s) = (-1)^{k-j} j!(k-j)!$$

and for any $w \in \mathbb{C}$, $m \ge 0$, $k \ge 0$ and $0 \le j \le k$

$$\prod_{s=-m}^{k-1} (w + j - s) = (-1)^{k-j} (1+w)_m (1+m+w)_j (-w)_{k-j}. \tag{254}$$

The above two identities allow us to rewrite the expression in (253) as follows

$$\gamma_{i,j}^k = \frac{\prod\limits_{l=1}^{p} (1 + a_i - b_l)_{m_l}}{\prod\limits_{\substack{1 \le l \le r+1 \\ l \ne i}} (a_i - a_l)} \times \frac{1}{j!} \times \frac{\prod\limits_{l=1}^{p} (1 + m_l + a_i - b_l)_j}{\prod\limits_{\substack{1 \le l \le r+1 \\ l \ne i}} (1 + a_i - a_l)_j} \times \frac{(-1)^{(k-j)(p+r+1)}}{(k-j)!} \frac{\prod\limits_{l=1}^{p} (b_l - a_i)_{k-j}}{\prod\limits_{\substack{1 \le l \le r+1 \\ l \ne i}} (1 - a_i + a_l)_{k-j}}. \tag{255}$$

Using the above equation and formulas (19), (249) and (250) we see that

$$\sum_{i=1}^{r+1} \sum_{k \ge 0} z^k \sum_{j=0}^{k} \gamma_{i,j}^k = H(z). \tag{256}$$

At the same time, we can change the order of summation in (256) and write $H(z)$ as

$$H(z) = \sum_{k \ge 0} z^k \left[\sum_{i=1}^{r+1} \sum_{j=0}^{k} \gamma_{i,j}^k \right]. \tag{257}$$

Now the plan is to compute the sum in the square brackets by applying Lemma 5. Recall that we have denoted $\alpha = \sum\limits_{i=1}^{r+1} a_i$ and $\beta = \sum\limits_{i=1}^{p} b_i$. Definitions (251) and (252) easily give us

$$s_k := \sum_{x \in A} x - \sum_{y \in B} y = (k+1)\alpha + (r+1)\frac{k(k+1)}{2} - k\beta - p\frac{(k-1)k}{2} + \frac{1}{2} \sum_{i=1}^{p} m_i(m_i + 1 - 2b_i).$$

Then, using our earlier computations $n_A = (r+1)(k+1)$ and $n_B = M + kp$ and applying Lemma 5, we find

$$\sum_{i=1}^{r+1}\sum_{j=0}^{k}\gamma_{i,j}^{k} = \begin{cases} 0, & \text{if } (r+1-p)k > M-r, \\ 1, & \text{if } (r+1-p)k = M-r, \\ s_k, & \text{if } (r+1-p)k = M-r-1. \end{cases} \tag{258}$$

By combining (257) and (258) we finish the proof of Theorem 27 in the case when $m_i \geq 0$ for $1 \leq i \leq p$.

Let us consider the case when some m_i are negative. Note that formula (254) holds true when m is negative, as long as $k \geq |m|$. Thus formula (255) is also true, as long as $k \geq |m_i|$ for all negative m_i. Therefore, our result (258) remains true for all $k \geq -\hat{m}$ (recall that $\hat{m} = \min\{m_i \,:\, 1 \leq i \leq p\}$), which means that all results in Theorem 27 hold true modulo $\mathcal{P}_{-\hat{m}}$. □

8.3 Basic hypergeometric functions duality

Theorem 27 has an analogue given in terms of basic hypergeometric functions. The definition of the basic hypergeometric function can be found in Section 3.2.

Theorem 28. *Assume that q is a complex number satisfying $|q| < 1$, $\{a_i\}_{1\leq i\leq r+1}$ are non-zero complex numbers satisfying*

$$a_i/a_j \notin \{\dots, q^{-2}, q^{-1}, 1, q, q^2, \dots\},$$

$\{b_i\}_{1\leq i\leq r+1}$ are non-zero complex numbers and $\{m_i\}_{1\leq i\leq r+1}$ are integers. Define $M := \sum\limits_{i=1}^{r+1} m_i$, $M_2 := \sum\limits_{i=1}^{r+1} m_i(m_i+1)/2$ and

$$c_i := (-1)^M q^{-M_2} \frac{\prod\limits_{j=1}^{r+1} b_j^{m_j} (qa_i/b_j;q)_{m_j}}{\prod\limits_{\substack{1\leq j\leq r+1 \\ j\neq i}} (a_i - a_j)} \quad \text{for } 1 \leq i \leq r+1. \tag{259}$$

Let

$$G(z) := \sum_{i=1}^{r+1} c_i \times {}_{r+1}\phi_r\left(\begin{matrix} q^{1+m_1}a_i/b_1, \dots, q^{1+m_{r+1}}a_i/b_{r+1} \\ qa_i/a_1, \dots, *, \dots, qa_i/a_{r+1} \end{matrix}\middle| wz\right) \tag{260}$$
$$\times\, {}_{r+1}\phi_r\left(\begin{matrix} b_1/a_i, \dots, b_{r+1}/a_i \\ qa_1/a_i, \dots, *, \dots, qa_{r+1}/a_i \end{matrix}\middle| z\right),$$

where $w := q^{-r}\prod_{i=1}^{r+1} b_i/a_i$ *and the asterisk means that the term* qa_i/a_i *is omitted. Assuming that* $m_i \geq 0$ *for* $1 \leq i \leq r+1$*, the following is true:*

(i) If $M < r$ *then* $G(z) \equiv 0$*;*

(ii) If $M = r$ *then* $G(z) \equiv 1/(1-z)$*;*

(iii) If $M = r+1$ *then*

$$G(z) \equiv \frac{1}{1-q}\left[\frac{C}{1-z} - \frac{q\alpha - \beta}{1-qz}\right],$$

where $\alpha = \sum_{i=1}^{r+1} a_i$*,* $\beta = \sum_{i=1}^{r+1} b_i$ *and* $C = \alpha - \sum_{i=1}^{r+1} b_i q^{-m_i}$*.*

In the case when some of m_i *are negative, the above results in (i)-(iii) hold modulo* $\mathcal{P}_{-\hat{m}}$*, where* $\hat{m} = \min_{1\leq i\leq r+1} m_i$*.*

Proof. The proof is very similar to the proof of Theorem 27, thus we will present only the important steps and we will omit many details. Assume that $m_i \geq 0$ for $1 \leq i \leq r+1$ and $k \geq 0$ (or $k \geq -\hat{m}$ if some of m_i are negative). We define

$$A = \bigcup_{1\leq i\leq r+1} \{a_i q^j \ : \ 0 \leq j \leq k\}, \qquad B = \biguplus_{1\leq i\leq r+1} \{b_i q^j \ : \ -m_i \leq j \leq k-1\}.$$

It is clear that $n_A = (r+1)(k+1)$ and $n_B = M + (r+1)k$. Next, we fix indices i and j such that $1 \leq i \leq r+1$ and $0 \leq j \leq k$ and compute

$$\gamma_{i,j}^k := \gamma(a_i q^j; A, B) = \frac{\prod_{l=1}^{r+1}\prod_{s=-m_l}^{k-1}(a_i q^j - b_l q^s)}{\prod_{\substack{0\leq s\leq k\\ s\neq j}}(a_i q^j - a_i q^s)\prod_{\substack{1\leq l\leq r+1\\ l\neq i}}\prod_{s=0}^{k}(a_i q^j - a_l q^s).} \tag{261}$$

After some straighforward (though tedious) computations we rewrite the above expression in the form

$$\gamma_{i,j}^k = (-1)^M q^{-M_2}\frac{\prod_{l=1}^{r+1} b_l^{m_l}(qa_i/b_l;q)_{m_l}}{\prod_{\substack{1\leq l\leq r+1\\ l\neq i}}(a_i - a_l)} \times \frac{w^j \prod_{l=1}^{r+1}(q^{1+m_l}a_i/b_l;q)_j}{(q;q)_j \prod_{\substack{1\leq l\leq r+1\\ l\neq i}}(qa_i/a_l;q)_j} \tag{262}$$

$$\times \frac{\prod_{l=1}^{r+1}(b_l/a_i;q)_{k-j}}{(q;q)_{k-j}\prod_{\substack{1\leq l\leq r+1\\ l\neq i}}(qa_l/a_i;q)_{k-j}},$$

which shows that

$$\sum_{i=1}^{r+1} \sum_{k \geq 0} z^k \sum_{j=0}^{k} \gamma_{i,j}^k = G(z), \tag{263}$$

where the function $G(z)$ is defined in (260). We also compute

$$s_k := \sum_{x \in A} x - \sum_{y \in B} y = \frac{1}{1-q} \Big[\alpha - \sum_{l=1}^{r+1} b_l q^{-m_l} - (q\alpha - \beta) q^k \Big],$$

and Lemma 5 gives us

$$\sum_{i=1}^{r+1} \sum_{j=0}^{k} \gamma_{i,j}^k = \begin{cases} 0, & \text{if } \ M < r, \\ 1, & \text{if } \ M = r, \\ s_k, & \text{if } \ M = r+1. \end{cases} \tag{264}$$

The remaining steps of the proof are exactly the same as in the proof of Theorem 27 and we leave them to the reader. □

References

[1] M. Avellaneda, A. Levy and A. Parás. Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance*, vol(2): 73–88, 1995.

[2] W. N. Bailey. On certain relations between hypergeometric series of higher order. *Journal of the London Mathematical Society*, s1-8(2):100–107, 1933.

[3] J. Bak and D. J. Newman. Complex analysis. *Springer New York*, 2010.

[4] M. T. Barlow and E. A. Perkins. Strong existence, uniqueness and nonuniqueness in an equation involving local time. *Sem. Prob.*, XVII:32–66, 1983.

[5] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212(electronic), 2005.

[6] F. Beukers and F. Jouhet. Duality relations for hypergeometric series. *Bulletin of the London Mathematical Society*, 47(2):343–358, 2015.

[7] P. Boyle, M. Broadie, and P. Glasserman. Simulation methods for security pricing. *J. Econom. Dynam. Control* , 21:1267-1321, 1997.

[8] M. Caballero and V Rivero. On the asymptotic behaviour of increasing self-similar Markov processes. *Electron. J. Probab* , 14:865-894 (electronic), 2009.

[9] N. Cai and S. Kou. Pricing Asian options under a hyper-exponential jump diffusion model. *Oper. Res.*, 60(1):64–77, 2012.

[10] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997.

[11] P. Carr, H. Geman, D. B. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation. *J. Business*, 75(2):305–332, 2002.

[12] L. Chaumont, A. E. Kyprianou, and J. C. Pardo. Some explicit identities associated with positive self-similar Markov processes. *Stoch. Proc. Appl* , 119(3):980-1000, 2009.

[13] L. Chaumont, J. C. Pardo. the lower envelope of positive self-similar Markov processes. *Electron. J. Probab* , 11(11):1321-1341 (electronic), 2006.

[14] Z. Chen, P. Glasserman. Fast pricing of basket default swaps. *Operations Research* , 56(2):286-303, 2008.

[15] R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. 1:223–236, 2001.

[16] H. B.C. Darling. On certain relations between hypergeometric series of higher orders. *Proceedings of the London Mathematical Society*, s2-34(1):323–339, 1932.

[17] D. Duffie, J. Pan, and K. Singleton. Transform analysis and option pricing for affine jump-diffusions. *Econometrica* , 68:1343-1376, 2000.

[18] D. Duffie, K. Singleton. Credit risk: pricing, measurement, and management. *Princeton University Press* 2003.

[19] J. P. Fouque and B. Ren. Approximation for option prices under uncertain volatility. *Siam J. Financial Math. 5: 360–383*, 2014.

[20] R. Feng and X. Jing. Analytical valuation and hedging of variable annuity guaranteed lifetime withdrawal benefits. *Preprint*, 2016.

[21] R. Feng, A. Kuznetsov, and F. Yang. A short proof of duality relations for hypergeometric functions. *J. Math. Anal. Appl., 443(1), 116-122*, 2016.

[22] R. Feng, A. Kuznetsov, and F. Yang. Exponential functionals of Levy processes and variable annuity guaranteed benefits. *Submitted*, 2016.

[23] R. Feng and H. W. Volkmer. Analytical calculation of risk measures for variable annuity guaranteed benefits. *Insurance Math. Econom.*, 51(3):636–648, 2012.

[24] R. Feng and H. W. Volkmer. Spectral methods for the calculation of risk measures for variable annuity guaranteed benefits. *Astin Bull.*, 44(3):653–681, 2014.

[25] I. I. Gihman and A. V. Skorohod. Stochastic Differential Equations. *Springer, New York.*, 1972.

[26] L. M. Gorski and R. A. Brown. Recommended approach for setting regulatory risk-based capital requirements for variable annuities and similar products. Technical report, American Academy of Actuaries Life Capital Adequacy Subcommittee, Boston, June 2005.

[27] L. M. Gorski and R. A. Brown. C3 phase II risk-based capital for variable annuities: Pre-packaged scenarios. Technical report, American Academy of Actuaries, March 2005.

[28] R. W. Gosper, M. E. H. Ismail, and R. Zhang. On some strange summation formulas. *Illinois J. Math.*, 37(2):240–277, 1993.

[29] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[30] M. Jeannin and M. Pistorius. A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes. *Quantitative Finance*, 10:629644, 2010.

[31] J.M. Harrison and L.A Shepp. On Skew Brownian Motion. *The Annals of Probability*, 9(2): 309-313, 1981.

[32] H. Huang, M. A. Milevsky, and T. S. Salisbury. Optimal initiation of a GLWB in a variable annuity: no arbitrage approach. *Insurance: Mathematics and Economics*, 56: 102-111, 2014.

[33] J. Hull, A. White. Valuation of a CDO and an nth to default CDS without Monte Carlo Simulation. *Journal of Derivatives*, 12(2): 8-23, 2004.

[34] T. R. Hurd. Credit risk modeling using time-changed Brownian motion. *International Journal of Theoretical and Applied Finance*, 12(8): 1213-1230, 2009.

[35] M. Joshi and D. Kainth. Rapid and accurate development of prices and Greeks for nth to default credit swaps in the Li model. *Quantitative Finance*, 4: 266-275, 2004.

[36] S. Kou. A jump-diffusion model for option pricing. *Management Sci.*, 48(8):1086–1101, 2002.

[37] S. Kou and H. Wang First passage times of a jump diffusion process. *Adv.Appl.Prob.*, 35(8):504–531, 2003.

[38] S. Kou and H. Wang. Option pricing under a double exponential jump diffusion model. *Management Sci.*, 50(9):1178–1192, 2004.

[39] A. Kuznetsov. On the distribution of exponential functionals for Lévy processes with jumps of rational transform. *Stochastic Processes and their Applications*, 122(2):654 – 663, 2012.

[40] A. Kuznetsov and D. Hackmann. Asian options and meromorphic Lévy processes. *Finance and Stochastics*, 18:825 – 844, 2014.

[41] A. Kuznetsov and J. C. Pardo. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Applicandae Mathematicae*, 123(1):113139, 2013.

[42] A. Kuznetsov, J. C. Pardo, and M. Savov. Distributional properties of exponential functionals of Lévy processes. *Electron. J. Probab.*, 17:no. 8, 1–35, 2012.

[43] Life Practice Note Steering Committee. The application of C3 phase II and actuarial guideline XLIII. A public policy practice note, American Academy of Actuaries, 2009.

[44] A.E. Kyprianou. Fluctuations of Lévy Processes with Applications. Universitext, Springer-Verlag Berlin Heidelberg 2014.

[45] A. E. Kyprianou and J. Pardo. Continuous-state branching processes and self-similarity. *J. Appl. Probab*, 45(4):1140-1160, 2008.

[46] D. Li. On default correlation: A copula function approach. *J. Fixed Income*, 9:43–54, 2000.

[47] V. Linetsky. Spectral expansions for Asian (average price) options. *Oper. Res.*, 52(6):856–867, 2004.

[48] D. B. Madan and E. Seneta. The variance Gamma (V.G.) model for share market returns. *J. Business*, 63(4):511–524, 1990.

[49] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.*, 2:312–347, 2005.

[50] H. Matsumoto and M. Yor. Exponential functionals of Brownian motion. II. Some related diffusion processes. *Probab. Surv.*, 2:348–384, 2005.

[51] K. Maulik and B. Zwart. Tail asymptotics for exponential functionals of Lévy processes. *Stoch. Proc. Appl.*, 116(2):156–177., 2006.

[52] R. C.Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Financial*, 29(2):449–470, 1974.

[53] R. C.Merton. Option pricing when the underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

[54] R. C.Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

[55] S. Nakao. On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. *Osaka J. Math*, 9:513–518, 1972.

[56] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and W. C. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, 2010.

[57] J. Pardo. On the future infimum of positive self-similar Markov processes. *Stochastics* , 78(3):123-155, 2006.

[58] P. Patie. Exponential functionals of a new family of Lévy processes and self-similar continuous state branching processes with immigration. *Bull. Sci. Math.* , 133(4):355-382, 2009.

[59] P. Patie. Law of the absorption time of positive self-similar Markov processes. *Ann. Probab.* , 40(2):765-787, 2012.

[60] P. Patie and M. Savov. Exponential functional of Lévy processes: generalized Weierstrass products and Wiener-Hopf factorization. *Comptes Rendus Mathematique*, 351(9 - 10):393 – 396, 2013.

[61] P. Patie and M. Savov. Bernstein-gamma functions and exponential functionals of Lévy processes. *arXiv:1604.05960*, 2016.

[62] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and Series, Vol. 3: More Special Functions*. Gordon and Breach Science Publishers, New York, 1990.

[63] D. Revuz, M. Yor. *Continuous Martingales and Brownian Motion.* Springer-Verlag Berlin Heidelberg, New York, 1991.

[64] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli*, 11(3):471–509, 2005.

[65] L. C.G. Rogers. Optimal investment. *Springer-Verlag Berlin Heidelberg*, 2013.

[66] J. W. Thomas. Numerical partial difference equations: finite difference methods. *Springer-Verlag Berlin Heidelberg*, 2013.

[67] J. Vecer. A new pde approach for pricing arithmetic average Asian options. *J. Comput. Finance*, 4(4):105–113, 2001.

[68] M. Yor. On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.*, 24(3):509–531, 1992.

[69] A. H. Zemanian. Generalized integral transformations. *New York: Interscience Publishers*, 1968.