Solutions to Module 7 Practice Problems: Discrete Random Variables

1. Jim is worried about whether he has gotten infected with Covid. His initial belief about being infected is 0.5. So he comes home and takes a Rapid test, which gives a negative result. However, rapid tests are only somewhat accurate. Their probability of showing negative even when one is infected (i.e. a false negative) is 0.15; on the other hand, their probability of showing positive when one is not infected (i.e. a false positive) is 0.02.

(a) What is Jim's updated probability about being infected given the negative result from the rapid test?

$\frac{0.15*0.5}{0.15*0.5+0.98*0.5} = 0.133$

(b) Jim now decides to get a PCR test, which is much more accurate. Their probability of showing a false negative is 0.05, while their probability of saying positive when one is not infected (i.e. a false positive) is 0.005. Suppose the PCR test shows positive. What is Jim's updated probability given that he had a negative rapid test and a positive PCR test result?

$\frac{0.95*0.133}{0.95*0.133+0.005*0.867} = 0.97$

(c) What would have been Jim's updated probability if his PCR test result too came out negative?

$\frac{0.05*0.133}{0.05*0.133+0.995*0.867} = 0.0076$

2. The police pick up a suspect whom they suspect of using a drug. Suppose the prior probability that this person is a drug-user is 10%. He is subjected to a drug test that detects drug users with 97% accuracy. However, the test also returns a positive result occasionally for non-drug-users. This probability is 5%.

(a) Suppose the test gives a positive result. What is the updated probability that this person is in fact a drug user? The police and courts often have to make this sort of decisions.

$\frac{0.97*0.1}{0.97*0.1+0.05*0.9} = 0.68$

(b) Suppose this person had a previous conviction for drug use. Thus the prior on him being a drug user is higher, say 50%. Given a positive test result, what is the updated probability that this

person is still using drugs? See how this compares with your updated probability in (a), which is for a person without any previous conviction.

$\frac{0.97*0.5}{0.97*0.5+0.05*0.5} = 0.95$

(c) For this previously convicted person, what would be the updated probability if the test returned a negative result?

$\frac{0.03*0.5}{0.03*0.5+0.95*0.5} = 0.03$

3. A stock analyst gives three types of recommendations for stocks: "Buy", "Sell" or "Hold". Typically, he gives a "Buy" recommendation for stocks which are hot, "Sell" for those which are cold, and "Hold" when he is uncertain. In particular, based on his history, you see that his probabilities of recommendations are: $P(Buy \mid Hot) = 0.6, P(Hold \mid Hot) = 0.2, P(Sell \mid Hot) = 0.2,$
$P(Buy \mid Cold) = 0.1, P(Hold \mid Cold) = 0.4, P(Sell \mid Cold) = 0.5.$

You currently own a stock which you think has 40% chance of being hot.

(a) Suppose the analyst gives a "Buy" recommendation for the stock. What will be your updated probability that the stock is hot based on this recommendation?

$\frac{0.6*0.4}{0.6*0.4+0.1*0.6} = 0.8$

(b) What would be your updated probability that the stock is hot if instead the analyst made the recommendation of "Hold"?

$\frac{0.2*0.4}{0.2*0.4+0.4*0.6} = 0.25$

(c) What would be your updated probability that the stock is hot if instead the analyst made the recommendation of "Sell"?

$\frac{0.2*0.4}{0.2*0.4+0.5*0.6} = 0.21$

4. A company operates in China and in India. The amount of profits it makes depends on how the Chinese and Indian economies are doing. If the Chinese economy does well (the probability of which is 0.7), it makes profits of $\$100,000$ in China; if not, it makes profits of $\$10,000$ there. If the Indian economy does well (the probability of which is 0.5), it makes profits of $\$50,000$ in India; if not, it makes profits of $\$5,000$ there. Assume that the performance of the Chinese and Indian economies are independent of each other.

(a) Let Z denote the total profits for this company. Derive the probability mass function (p.m.f.) and the cumulative distribution function (c.d.f.) for Z.

(b) What is the expected profits of this company?

p.m.f.:

z	$15,000$	$60,000$	$105,000$	$150,000$
$p(z)$	0.15	0.15	0.35	0.35

c.d.f.:

z	$15,000$	$60,000$	$105,000$	$150,000$
$p(z)$	0.15	0.3	0.65	1

(b) What is the expected profits of this company?

$\$100,500$

5. Suppose you are due to visit a friend, who you know has 4 children, but you don't know the gender of the children. Let X be the random variable denoting the number of children who are girls. Derive the p.m.f. and the c.d.f. of X.

p.m.f.:

x	0	1	2	3	4
$p(x)$	0.0625	0.25	0.375	0.25	0.0625

c.d.f.:

x	0	1	2	3	4
$p(x)$	0.0625	0.3125	0.6875	0.9375	1

6. Derive the c.d.f.s of the random variables whose p.m.f.s are given below.

(a)

x	5	10	20	30	40	50
$p(x)$	0.1	0.2	0.1	0.2	0.3	0.1

c.d.f.:

x	5	10	20	30	40	50
$p(x)$	0.1	0.3	0.4	0.6	0.9	1

(b)

y	5	10	15	25	30	35	40	60
$p(y)$	0.05	0.2	0.05	0.3	0.1	0.05	0.05	0.2

c.d.f.:

y	5	10	15	25	30	35	40	60
$p(y)$	0.05	0.25	0.3	0.6	0.7	0.75	0.8	1

(c)

z	50	60	70	80	90	100	110	120	130	140
$p(z)$	0.15	0.2	0.1	0.15	0.04	0.06	0.05	0.05	0.08	0.12

c.d.f.:

z	50	60	70	80	90	100	110	120	130	140
$p(z)$	0.15	0.35	0.45	0.6	0.64	0.7	0.75	0.8	0.88	1

(d) Plot the c.d.f.s of (a) and (b).

7. Derive the p.m.f.s of the random variables whose c.d.f.s are given below.

(a)

x	-5	0	5	10	15	20
$p(y)$	0.1	0.3	0.4	0.7	0.8	1

p.m.f.:

x	-5	0	5	10	15	20
$p(y)$	0.1	0.2	0.1	0.3	0.1	0.2

(b)

y	-15	-10	-5	0	5	10	15	20
$p(y)$	0.05	0.25	0.35	0.50	0.55	0.6	0.85	1

p.m.f.:

y	-15	-10	-5	0	5	10	15	20
$p(y)$	0.05	0.2	0.1	0.15	0.05	0.05	0.25	0.15

(c)

z	500	550	700	750	800	1000	1100	1150	1300	1400
$p(z)$	0.15	0.2	0.25	0.38	0.4	0.6	0.75	0.82	0.98	1

p.m.f.:

z	500	550	700	750	800	1000	1100	1150	1300	1400
$p(z)$	0.15	0.05	0.05	0.13	0.02	0.2	0.15	0.07	0.16	0.02

8. (a) Derive the mean and variance for the random variables X given in 6(a) and 7(a). Check your answers using Excel/Google sheets.

6(a): Mean = 27.5, Variance = 216.25

7(a): Mean = 8.5, Variance = 65.25

(b) What is the mean and variance for $4X + 5$ in 6(a) and 7(a)?

6(a): Mean = 110, Variance = 3460

7(a): Mean = 34, Variance = 1044

(c) What is the mean for $6X^2 + 10$ in 6(a) and 7(a)?

6(a): Mean = 5845

7(a): Mean = 835

9. (a) Use Excel/Google sheets to derive the mean and variance for the random variables given in 6(b), 6(c), 7(b), 7(c).

See Excel (or .ods) solutions file in module for these and other computations in the problem set using Excel.

6(b): Mean = 29.25, Variance = 315.6875

6(c): Mean = 86.8, Variance = 963.76

7(b): Mean = 4.25, Variance = 138.1875

7(c): Mean = 932.5, Variance = 77168.75

(b) Use Excel/Google sheets to derive $E(Y^3)$ for the random variable Y given in 6(b).

6(b): Mean = 56306.25, Variance = 6.6 X 10^9

10. A polling firm calls 10 households. The probability that a random household will pick up the call and agree to take its survey is 0.2.

(a) Which common discrete distribution would be appropriate to model the (random) number of households that agree?

Binomial distribution with $n = 10$, $p = 0.2$

(b) What is the probability that exactly 5 households agree out of the 10? [After figuring out what you are looking for, you may wish to use Excel/Google Sheets to calculate your answers for this and the subsequent parts.]

$\binom{10}{5} (0.2)^5 (0.8)^5 = 0.026424$

(c) What is the probability that no households agree out of the 10?

$\binom{10}{0} (0.2)^0 (0.8)^{10} = 0.107374$

(d) What is the probability that 4 or fewer households agree out of the 10?

$F(4) = 0.967207$

(e) What is the probability that more than 6 households agree out of the 10?

$1 - F(6) = 0.000864$

11. A store sells extended warranties on its product. It has found from past experience that its product malfunctions and needs fixing in only 1 out of 10 cases. In such cases, the defective product will have to be fixed or replaced for the customers who have purchased extended warranty.

The firm sells extended warranty to 50 customers. In determining the (random) number of customers X who are going to make use of the extended warranty, the firm decides to use a Binomial distribution.

(a) What are the parameters of the Binomial distribution in this case?

$n = 50$, $p = 0.1$

(b) What is the probability that exactly 10 customers out of the 50 will return and avail of the extended warranty? [After figuring out what you are looking for, you may wish to use Excel/Google Sheets to calculate your answers for this and the subsequent parts.]

$\binom{50}{10} (0.1)^{10} (0.9)^{40} = 0.015183$

(c) What is the probability that 10 or fewer customers will return to have their product fixed?

$F(10) = 0.990645$

(d) What is the probability that between $10-20$ customers will return to have their product fixed?

$F(20) - F(9) = 0.024538$

(e) What is the mean and variance of X?

Mean $= np = 5$, Variance $= np(1-p) = 4.5$

12. For purposes of planning, the manager of a large store makes use of the Poisson distribution to model the (random) number of customers who visit the store in a given hour. From past experience, she has found that on average, 80 customers visit the store in the 9 – 10 am period.

(a) What is the probability that exactly 80 customers will visit the store that day between 9 and 10 am? [First, write down the expression for the probability. Then use Excel/Google Sheets to calculate your answers for this and the subsequent parts.]

Poisson: $Prob(\widetilde{X} = 80) = \frac{e^{-80}80^{80}}{80!} = 0.045$

(b) What is the probability that exactly 70 customers will visit the store that day between 9 and 10 am?

$Prob(\widetilde{X} = 50) = \frac{e^{-80}80^{70}}{70!} = 0.0248$

(c) What is the probability that 50 or fewer customers will visit the store that day between 9 and 10 am?

$F(50) = 0.00022$

(d) What is the probability that more than 80 customers will visit the store that day between 9 and 10 am?

$1 - F(80) = 0.47$

(e) The manager decides to staff the store with 10 employees that day, which are enough to serve 100 customers. But if more than 100 customers show up, the store will be understaffed. What is the probably that the store maybe understaffed that day between 9 and 10 am?

$1 - F(100) = 0.0132$

13. Cars pass through an intersection at the rate of 150 per hour on average. For planning purposes, the city officials decide to model the (random) number of cars C passing through this intersection every 10 minutes as a Poisson distribution.

(a) What will be the parameter of this Poisson distribution? What is the mean and variance of C?

$\lambda = 15$, $Mean = Variance = \lambda = 15$

(b) What is the probability that no cars pass through this intersection in the next 10 minutes? [First, write down the expression for the probability. Then use Excel/Google Sheets to calculate your answers for this and the subsequent parts.]

$Prob(\widetilde{X} = 0) = \frac{e^{-15}15^0}{0!} = 0.0000003$

(c) What is the probability that exactly 12 cars pass through this intersection in the next 10 minutes?

$Prob(\widetilde{X} = 12) = \frac{e^{-15}15^{12}}{12!} = 0.083$

(d) What is the probability that 12 or fewer cars pass through this intersection in the next 10 minutes? What is the probability that more than 12 cars pass?

$F(12) = 0.268, 1 - F(12) = 0.732$