A Short and Readable Proof of Cut Elimination for Two 1st Order Modal Logics

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Abstract

Since 1960s, logicians, philosophers, AI people have cast eyes on modal logic. Among various modal logic systems, propositional provability logic which was established by Gödel modeling provability in axiomatic Peano Arithmetic (PA) was the most striking application for mathematicians. After Gödel, researchers gradually explored the predicate case in provability logic. However, the most natural application QGL for predicate provability logic is not able to admit cut elimination. Recently, a potential candidate for the predicate provability logic ML^3 and its precursors BM and M^3 introduced by Toulakis, Kibedi, Schwartz are cut free. Whereas, the cut elimination proof with the unfriendly nested induction of high multiplicity is difficult to understand. In this thesis, I will show a cut elimination proof for all (Gentzenisations) of BM, M^3 and ML^3, with much more readable inductions of lower multiplicity.

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1 Introduction

Unlike classical logic, modal logic —developed in the 1960s but arguably having its roots in Aristotle's works just as classical logic does— can reason about necessity, possibility, temporality (Temporal Logic), morality (Deontic Logic), and beliefs (Doxastic Logic) and finds various applications in Artificial Intelligence and Philosophy.

People have thought about logic modally in ancient times. Aristotle already considered a calculus for reasoning with modal syllogistic forms. The topic continued in the Middle Ages, and we still can find modality (i.e., what is possible, necessary, or impossible), the core concept of modern modal logic in great philosopher Kant's work, even though Kant's logical thinking mentioned in the books is different from today's modal logic. In Kant's book —his magnum opus— Critique of Pure Reason [18] (A, pp.70–71/B, pp.95–96) [1] "Modality" was considered as a form of a judgment [2] along with "Quantity", "Quality" and "Relation" defined in Chapter III [3] Under Each "head" [4] Kant identifies three "moments" which jointly express all possible forms under that head. The definition of three "moments" of "Modality":

[1] There exist two primary publications of Critique of Pure Reason. The first publication is referred to as the "A" edition and was published in 1781, and the second, or "B" edition appeared in 1787. Most contemporary editions of the work combine the two editions.

[2] Kant defined that a judgment is the thought that a thing is known to have a certain quality or attribute.

[3] In Chapter III of the Transcendental Analytic, namely, the first division of Transscendental Logic in Critique of Pure Reason.

[4] i.e., quantity, quality, relation, and modality. Kant called them "heads".

problematic “A may be B”

assertoric “It is true that A is B”

apodeictic “A must be B”

has been fairly close to some important features of modern modal logic. [16]

Modern modal logic begins with the books of C. I. Lewis, (cf. [9] and [22]), the second co-authored with C. H. Langford, which included several different axiomatic systems of modal propositional logic. Lewis (cf. [43]) defined five systems in the attempt to axiomatize the implication relation: S1 - S5. Two of these systems, S4 and S5 are still in use today. Lewis's (cf. [5]) works are in fact on modal logic, which syntactically explicate (through axiomatic systems) the notions of logical necessity and possibility.

At this point in history, modal was treated as “a tool for analyzing a wide range of philosophical arguments” [20] about various modal notions. But non-philosophical applications were never far away, starting with mathematics. Gödel showed how to embed Heytings intuitionistic propositional logic faithfully into the modal logic (more precisely, into the system S4) in 1933 and briefly mentions that provability can be viewed as a modal operator. Moreover, Saul Kripke introduced the now-standard “Kripke semantics” (cf. [45]) for modal logics (later adapted to intuitionistic logic and other non-classical systems) in 1959. The weakest normal modal logic K is named after him, [5].

Although the origins of the study lie in philosophy, modal logic has developed equally intensive contacts with computer science, linguistics, and economics besides mathematics. This circle of contacts is still expanding.

In topology, Tarski showed how to axiomatize modal structures in topological spaces in 1938 [43].

Since 1970, modal logic has come to flourish at interfaces with linguistics. Montague focused on comparing the treatment of intensional operators and verbs in 1974.

It has also thrived in computer science with dynamic or temporal logics of programs, logics of spatial structures, or modal description logics for knowledge. For example, in artificial intelligence area, Dov M. Gabbay, C. J. Hogger and J. A. Robinson introduced temporal logic in their book "Handbook of Logic in artificial Intelligence and Logic Programming" [17] in 1994.

As to economics, modal logic also has a great impact on many important economic theories. For example, Leyton-Brown and Shoham applied modal logic to the foundation of game theory in 2008 [23].

The traditional modal logic, or modalities of truth, include possibility ("Possibly p", "It is possible that p", $\Diamond p$), necessity ("Necessarily p", "It is necessary that p", $\Box p$), and impossibility ("Impossibly p", "It is impossible that p"). However, the term modal logic is used more broadly to cover a family of logics with similar rules and a variety of different symbols. For example, Arthur Prior introduced a particular modal logic-based system of temporal logic in the late 1950s (cf. [25] and [26]). Temporal logic has the ability to "represent and reason propositions qualified in terms of time so the extra modal operators always express the meaning about time" (cf. [14]).

A list of "modal" symbols of various modal logics and their expected meaning is displayed in the following table:

Modal Logic Type	Symbol	Modal Symbol's semantic meaning
Deontic Logic	O	It is obligatory that ...
	P	It is permitted that ...
	F	It is forbidden that ...
Temporal Logic	H	It has always been the case that ...
	P	It was the case that ...
	F	It will be the case that ...
	G	It will always be the case that ...
Doxastic Logic	B	It believes that ...

Perhaps the most striking application of modal logic, to mathematical logic, is in the modal symbol's modeling of "provability" in arithmetised axiomatic Peano Arithmetic (PA). This was something that Gödel noticed early on, and researchers such as Magari, Kripke, Boolos, Solovay, Sambin, Valentini and Smoryński and others pursued this area further. The propositional modal logic GL was central in this development (G for Gödel, L for Löb). In next section, provability logic will be discussed more.

G. Boolos and other logicians have stated their belief that a predicate provability logic would provide new insights in the study of arithmetised provability in PA [5]. However, the road to finding an appropriate axiomatized predicate provability logic has twists and turns. If one "cheats" and takes as a predicate provability logic one over the language of QGL, and one starts by defining the set of its theorems as precisely those modal formulae all of whose arithmetical interpretations

are PA theorems, then one arrives at a logic that often is denoted by QML (Quantified Modal Logic). Unfortunately, QML —as Vardanyan proved [44, 5]— cannot be axiomatised, hence is unusable! In section 3, we will talk a bit more about attempts toward discovering a predicate provability logic (for PA).

More recently, work of Tourlakis, Kibedi, and Schwartz [40, 41, 19, 30, 32] applied predicate modal logic in order to model provability in a general —that is, "pure"— predicate calculus setting (not only in a particular theory such as PA). Among these logics, ML^3 a common first-order extension of M^3 and GL is most likely a predicate (PA-) arithmetised provability logic —or, as such logics are usually called, is arithmetically complete. Moreover, a very similar logic with a different approach to necessitation rule than ML^3 is known to be arithmetically complete ([27]) as we already noted. We observe that ML^3, like QGL, extends GL, but the particular language chosen and the fact that $\Box A$ is always closed give it properties that QGL lacks, as we already noted in the Abstract. In section 4, we will take a closer look at M^3 and ML^3.

Proof Theory is the syntactic study and analysis of the properties and limitations of proofs in a logic, just as Computability is an study and analysis of the properties and limitations of computations. Cut elimination is "a major powerful tool for proof-theory" [36], it is related to Gentzen style logic systems, and the proof-theorist owes it to her/himself to learn what the tool says/does, and how one proves that a given Gentzen system admits cut elimination. The Gentzenisation of GL admits cut elimination as originally proved by [42, 28]. In section 5, the syntactic proof of cut freedom for GLS —the equivalent Gentzen sequent calculus of

propositional provability logic GL will be briefly discussed. Moreover, Gentzenisations of some predicate modal logics such as M^3 and ML^3 —and unlike QGL— admit cut elimination as well; we will outline the proof of admissibility for them too.

Jude Brighton [8] gave a new proof of cut admissibility for GLS with lower complexity of induction (double induction) than was the case in Valentini's proof [42] (triple induction). His key ideas were 1) to state the cut admissibility result in a very simple (equivalent to the traditional) form, and 2) to do away with the "width" concept of Valentini and do the (double) induction along the complexity of formulae (primary) and (essentially) proof tree height (secondary).

The main results of this thesis are two cut admissibility proofs, one each for the Gentzenisations of M^3 and ML^3, where we have adapted Brighton's proof to the predicate case. Inevitably, our induction is triply nested, an unavoidable complication due to the presence, and syntactic properties, of quantifiers. Yet, the new proofs are substantially more readable than the originals [30, 32] and this simplicity is not only due to the reduction of induction nesting from 4 to 3.

The new proofs are much shorter and more readable than the originals. In the second half of this thesis, I will more thoroughly explain the motivation of my work and then present in full detail the two new proofs.

2 Provability logic

The term "provability logic" has been applied to modal logics where the modal box operator $\Box$ is interpreted as "it is provable that". The predominant version of "it is provable that" that these logics are called upon to model is the arithmetised provability within Peano Arithmetic (PA) in that $\Box A$ is (approximately) interpreted as $Pr(\ulcorner A \urcorner)$, where $Pr(.)$ is Gödel's arithmetised provability predicate: $Pr(x)$ "says" that the formula with Gödel number x is provable in PA.[5] Thus, "arithmetised" in "arithmetised provability logic" qualifies "provability".

By Gödel number of an expression we mean an arithmetical code for the expression. Coding of strings (e.g., formulae), and sequences of strings (e.g., proofs), by numbers is an idea and technique due to Gödel, which he employed in his proof of his first Incompleteness theorem. It has been adopted subsequently in the study of computability, coding computation snapshots ("configurations") and entire computations by numbers. Computability is predominantly using "prime power coding" but other more sophisticated codings have been used as well, most notably by Bennett in his PhD thesis On Spectra (cf. [37], Chapter 5).

What is arithmetisation good for? It allows a theory (e.g., PA, Computability —also axiomatised set theory) that contains natural numbers among its objects of study to talk and reason about itself, and its processes, whether they are proofs or computations, since each such process has an alter ego: a number.

[5] The notation $\ulcorner A \urcorner$ is standard and means the "Gödel number of expression A".

Such provability logics have been introduced in response to developments in metamathematics such as Gödel's Incompleteness theorems of 1931 and Löb's theorem of 1953 [13].

More recently, general provability logics where introduced [40, 41, 19, 32], where $\Box A$ for classical A means "A is provable within the pure first-order classical logic".

2.1 Propositional Provability logic

In 1931, Gödel's Incompleteness theorems introduced in [13] put an end to Hilbert's Program towards consistently formalising any "real" (one that may deal with infinity informally, that is) mathematical theory. Thus, one has a "real" theory $\mathcal{R}$ and a formal theory $\mathcal{F}$ and the latter conserves or verifies by finite means (i.e., by formal proofs) the former's theorems: If A is provable in $\mathcal{R}$, then it is provable in $\mathcal{F}$ as well —in symbols, if $\vdash_{\mathcal{R}} A$ then $\vdash_{\mathcal{F}} A$.

Moreover, Hilbert's Program required that we are able to metamathematically prove the consistency of the formalised theory, $\mathcal{F}$, by finite means. This is reasonable! If $\mathcal{F}$ can simulate $\mathcal{R}$ in a manner that we avoid the informal concept of infinity, we cannot use this "suspect" concept to assess the trustworthiness (read: consistency) of $\mathcal{F}$!

This refutation of Hilbert's Program was achieved by Gödel adapting the semantic liar's paradox[6] into a syntactic version. He produced a sentence ϕ that says

[6]Due to Epimenides. In a subsequently modified form it states, "I am stating a lie". Well, if this

"I am not a PA-theorem". Here is in outline how he constructed ϕ:

- He arithmetised PA.
- He then defined, within PA, a "search and replace" function, $sub(x, y)$, which returns the Gödel number of the formula obtained from the one that has Gödel number x if we substitute all free occurrences of the variable v_0[7] in the original formula by the term y.

  In plain language, think of the number equivalent (say, in the 0-1 computer representation) of a computer-stored text file as the file's "Gödel number", x. Let us use the symbol "x" ambiguously to denote both a text (string of keyboard symbols) and the text's Gödel number. Now let us think of one such text file, x, and let us imagine using a word processor's "search/replace" function to replace all occurrences of a substring v_0 of x by a string y. The resulting text is what we call $sub(x, y)$ here.
- He proved the fixed point lemma in PA: For any formula $A(v_0)$ there is a sentence ψ such that PA proves $\psi \equiv A(\ulcorner \psi \urcorner)$, in symbols $\vdash_{\mathrm{PA}} \psi \equiv A(\ulcorner \psi \urcorner)$. A proof goes like this: Let the formula $B(v_0)$ stand for $A(sub(v_0, v_0))$ and let the closed[8] PA term t be the the Gödel number of $B(v_0)$. Then the Gödel number of $B(t)$ is $sub(t, t)$. So, we can prove in PA that $A(sub(t, t)) \equiv$

sentence is true, then it cannot be stating a lie; but this realisation has just falsified the statement! If on the other hand, it is false, then I am not stating a lie; I am stating a true statement; contradiction again! The original statement of Epimenides, him a Cretan, was "all Cretans are liars".

[7] We assume that our formal object variables in PA are $v_0, v_1, v_2, \ldots$

[8] Variable-free.

$A(\ulcorner B(t) \urcorner)$ and $B(t) \equiv A(sub(t,t))$, hence also that $B(t) \equiv A(\ulcorner B(t) \urcorner)$ by transitivity of $\equiv$. We can then take $\psi = B(t)$.

- Gödel, having arithmetised PA and the proof processes in it, he then defined within PA a provability predicate $Pr(.)$ such that $Pr(\ulcorner A \urcorner)$ means "A is a PA-theorem".

  Clarification: We said "means" above; in what sense? In the syntactic sense that $\vdash_{PA} A$ iff $\vdash_{PA} Pr(\ulcorner A \urcorner)$.

  He then applied the fixed point lemma in PA to define a sentence of PA, ϕ, such that

  $$\vdash_{PA} \phi \equiv \neg Pr(\ulcorner \phi \urcorner) \tag{1}$$

  Thus, ϕ "says" that "$\neg Pr(\ulcorner \phi \urcorner)$". That is, that ϕ is not a PA-theorem. For short, it says, "I am not a theorem".

- One can now see that if PA is a consistent formal system, then: One, $\nvdash_{PA} \phi$, for otherwise we would have (by (1)) $\vdash_{PA} \neg Pr(\ulcorner \phi \urcorner)$ and, by the Clarification (only-if part), also $\vdash_{PA} Pr(\ulcorner \phi \urcorner)$, contradicting consistency of PA. Two, can it be that $\vdash_{PA} \neg\phi$? No, for if so, then by (1) we get $\vdash_{PA} Pr(\ulcorner \phi \urcorner)$, and by the Clarification (if part) we get $\vdash_{PA} \phi$, again contradicting consistency.

- Why is this fatal for Hilbert's Program? ϕ being a sentence, either it or $\neg\phi$

must hold true in the "real" arithmetic. So one of the two sentences ought to be provable in the formalised counterpart, PA, as Hilbert's "conservation" requires. Neither is!

But wait! Maybe we did not put enough axioms in PA to formalise completely the "real arithmetic"?

That is not so, as Gödel showed (but this is beyond the aims of this Thesis to retell —cf. however [35, 38]): All consistent extensions of PA that continue to have a recognisable (technically speaking, recursive) set of axioms will each fail to "verify by finite means (syntactic proofs)" infinitely many valid sentences of "real arithmetic"!

The second part of Hilbert's Program, that the formalised theory $\mathcal{F}$ (which conserves the theorems of the real theory $\mathcal{R}$) can be proved consistent (if it is, of course) by "finite means" also fails, if we take as $\mathcal{R}$ the real arithmetic, and for $\mathcal{F}$ we take PA. Gödel, via his Second Incompleteness Theorem, showed that if PA is consistent, then it cannot not prove its own consistency. [9] The relevance can be seen at once: Finitary means to prove the consistency of PA can be arithmetised, and hence carried out within PA; the 2nd Incompleteness theorem makes this impossible.

Although the idea of provability as a modality is hardly novel, the serious study of the modal logic of (arithmetised) provability in PA did not get underway until 1970s. About two years after Gödel's Incompleteness Theorem was

[9] Using infinitary techniques in the metatheory of PA, we know that PA is consistent.

published, he introduced a translation from intuitionistic propositional logic into modal logic (more precisely, into the system nowadays called S4), and briefly mentions that provability can be viewed as a modal operator. After him, Richard Montague, Angus Macintyre and Harry Simmons [35] started to investigate the relationship between provability and modality gradually in 1960s – 1970s.

Smoryński ([35]) gives a very detailed discussion about the relationship between the modal $\Box$ operator and Peano arithmetised provability, where, roughly, $\Box$ is arithmetically interpreted as $Pr(.)$. The arithmetical interpretation of $\Box$ extends to arithmetical interpretations of all modal formulae in a, now standard, manner. One traditionally uses the superscript * to ambiguously denote any such interpretation from the modal language to the language of PA, indicating by the classical A^* in the language of PA, the interpretation of the modal A. Specifically, one sets arbitrarily —and that is why we have infinitely many different arithmetical interpretations— $A^* = A$ for all modal atomic A. We then let * to commute with Boolean connectives, and we set $(\Box A)^* = Pr(\ulcorner A^* \urcorner)$. In one word, the modal $\Box$ is able to stand for arithmetised provability in PA.

Besides Gödel's theorems, there is another result which was recognised early on as one that leads abstractly to the proof of the 2nd Incompleteness theorem. It is Löb's Theorem. It states:

Let ψ be any sentence. Then: $\vdash_{\text{PA}} Pr(\ulcorner \psi \urcorner) \rightarrow \psi$ iff $\vdash_{\text{PA}} \psi$, where, of course, the if-part is logically trivial (cf. [35, 38]).

No doubt, including the counterpart of Löb's theorem in modal logic systems

we would be able to express more properties of PA arithmetised provability.

Before looking at the exact manner of embedding Löb's theorem to a modal logic system, let's take a look at the modal logic system K^4, which is defined below:

Definition 2.1 Axioms:

1. All(boolean) tautologies

2. $\Box(A \to B) \to \Box A \to \Box B$

3. $\Box A \to \Box\Box A$

Rules:

Modus Ponens: $A, A \to B \vdash B$

Necessitation: $A \vdash \Box A$

where the informal provability symbol $\vdash$, in its use as a binary operator, means that the right hand side is provable from the left hand side "assumption(s)". ■

Historically, the system K^4 was not designed to capture all properties of $Pr(.)$. The propositional provability logic GL obtained by adding a single axiom —the formalised Löb's axiom $\Box(\Box A \to A) \to \Box A$— to the earlier modal logic K^4, this one introduced to express necessity and possibility, has a fundamental relationship with PA: All arithmetical interpretations of the theorems of GL —in the sense of $*$ above— are theorems of PA, which we name it as the arithmetical soundness. But this is also true of the sublogic K^4. Significantly, there is a converse proved by Solovay [36] for the logic GL: for any formula A of GL, if all its arithmetical

interpretations are PA theorems, then A is a GL theorem. This is known as the arithmetical completeness of GL. This landmark result of arithmetical completeness is retold in [5, 35].

For this ability to mirror proof activities of PA, GL has been called an "Arithmetised Provability Logic".

Kripke semantics introduced by Saul Kripke is useful for the metamathematical investigation of a modal logic. In Kripke semantics, sentences are true or false in various possible worlds. These worlds are nodes of a graph, with graph relation R, and the graph relation properties are induced by the modal axioms. For example the $\Box A \to \Box\Box A$ axiom schema of K^4 forces the the R to be transitive, i.e., whenever $aRbRc$, then also aRc is true. Arithmetised provability logic has suitable possible worlds semantics [5] that lead to finite Kripke structures, whose R is transitive and reverse well-founded, that is, there are no infinite ascending chains along R: $aRbRcRdRe\ldots$

A Kripke model for propositional modal logic is a triple $M = \langle W, R, V\rangle$, where W is a set of possible worlds (possibly empty), R is a binary accessibility relation on W, and V is a valuation that assigns a truth value to each propositional variable for each world in W. A formula $\Box A$ of the propositional modal logic under study is true on the world w iff A is true on all worlds x such that wRx. For Boolean connectives $\neg, \vee$ truth is local: $\neg A$ (respectively, $B \vee C$) is false on w iff A is true on w (respectively, B and C are false on w). A formula A is valid in a model $\langle W, R, V\rangle$ iff it is true on all worlds in W.

A Kripke frame, or modal frame, is just a pair $\langle W, R\rangle$. A frame $\langle W, R\rangle$ is called transitive if R is transitive. It is reflexive, if R (aRa holds for all $a \in W$) is.

A frame $\langle W, R\rangle$ is called reverse well-founded iff R is. Note that reverse well-founded R are also irreflexive, for otherwise we would have the infinite chain $aRaRaRa\ldots$.

The theorems of K (that is, K^4 with axiom 3 removed; cf. Definition 2.1) are valid in every Kripke structure, while those of K^4 are valid in every transitive Kripke structure [42]. It is quite obvious that GL is modally sound with respect to the class of possible worlds models on transitive reverse well-founded frames, because all axioms and rules of GL are valid on such models. The question is whether completeness also holds. Unaware of the arithmetical significance of GL, K. Segerberg proved in 1971 (cf. [34]) that GL is indeed complete with respect to transitive reverse well-founded structures.

2.2 Quantified GL

Since the time of Solovay's arithmetical completeness theorem [36], logicians wondered whether it was possible to have a first-order (predicate) arithmetised provability logic for Peano Arithmetic as it was assumed that such a logic, being first-order like the language of PA, would provide more insights in the proof theory of PA. Boolos wrote in his book [5]: "...after expressing regret at not being able to include a treatment of the application of quantified modal logic to proof theory, I mentioned that one major question then (1979) open was whether quan-

tified provability logic could be axiomatised". By "quantified provability logic" he meant the set of all formulae over a first-order modal language that are PA-provable under all their arithmetical interpretations *. A result of Vardanyan [44], proves that this set is not recursively enumerable, and hence cannot be (recursively) axiomatised.

The question:

Does a first-order arithmetised provability logic exist?

was posed by Boolos. It was answered in the affirmative by Yavorsky [27] who proved that a variant of QGL —QGL^b— is such a logic, where, unlike the "full" QGL, formulae of the type $\Box A$ are closed. He also needed the presence of the axiom schema $\Box A \to \Box \forall x A$ presumably to make his Kripke models work, but he did not discuss this axiom at all. Thus, except for the rule necessitation, which is primary in QGL^b but only "admissible" (derivable) in ML^3, the two logics are very similar and the techniques of loc. cit. should apply to ML^3 as well to establish it as another example of a first-order arithmetised provability logic.

Earlier attempts to build a first-order arithmetised provability logic used predicate extensions of GL, notably QGL (quantified GL) (see [1, 21, 24]).

QGL is the predicate version of the propositional provability logic GL [1]; the formulae of QGL are those of the modal predicate calculus; the axioms are those of the classical predicate calculus together with the following two

$$\Box(A \to B) \to \Box A \to \Box B \quad \text{and} \quad \Box(\Box A \to A) \to \Box A$$

The inference rules are those (Modus Ponens and Generalization) of predicate calculus plus the rule called "Necessitation": $A \vdash \Box A$. As in the propositional case, one can show that QGL contains the schema $\Box A \rightarrow \Box\Box A$ and is closed under Lob's rule: if $\vdash_{QGL} \Box A \rightarrow A$, then we have also $\vdash_{QGL} A$.

By above features, one might have a reason to expect that QGL can be the eligible candidate of predicate provability logic.

However, they stumbled upon Vardanyan's result, the latter of which, to draw attention to one important parameter in its proof (bolded here), should read:

"If the first-order modal language is such that the free variables of A are also free in $\Box A$, then the set of of all formulae over the language that are provable in PA under all interpretations is not axiomatizable." [44, 5] Thus, it is a corollary of Vardanyan's result that QGL —a straightforward first-order extension of GL with no new axioms beyond those of pure classical predicate calculus— which is recursively axiomatised, cannot possibly be a provability logic! (This was also proved in [24] independently of Vardanyan's result).

Researchers reasonably expected that QGL can provide more insights in the study of Peano provability, being itself a first-order logic. QGL does have some of GL's interesting properties. For example, QGL satisfies the so-called disjunction property (as does GL) [1, 39]:

$$QGL \vdash \Box\phi_1 \vee \Box\phi_2 \vee ... \vee \Box\phi_k \quad \text{iff} \quad QGL \vdash \phi_i, \quad \text{for some} \quad i, 1 \leq i \geq k$$

Unfortunately, however, many critical properties of GL are not inherited by its predicate version, QGL. Avron [1] shows in his paper that the most natural sequent calculus for QGL does not admit cut-elimination. Subsequently, Montagna [24] proves that QGL is not complete with respect to any class of Kripke frames; moreover, QGL is not even arithmetically complete (as we know since [44]), and does not enjoy the fixed point property.

The technical reasons for the rather disappointing (proof theoretic) behaviour of QGL point to the particular way $\Box A$ is built in QGL: It has the same free variables as A.

3 Two 1st-order modal logics

The presence of free variables in $\Box A$ —in the language of QGL— ruined expectations of it being an arithmetised provability logic. Not only QGL, but QML (Quantified Modal Logic), the latter a first-order modal logic which is by construction arithmetically complete will not work. QGL, because it is not arithmetically complete. QML, because it is not recursively axiomatisable ([44]), hence is unusable to write deductions in it! But logicians do not give up. Contrary to QGL and QML, the predicate modal logics M^3, BM and ML^3 introduced in [40, 41, 19, 30, 32] have a (common) language where $\Box A$ is closed. All three have Gentzen equivalents that support cut elimination; all three are provability logics for pure classical predicate calculus; all three are complete with respect to appropriate Kripke structures. In fact ML^3 is complete for finite, transitive and reverse well-founded Kripke structures, just as GL is, and also being so similar (except for a rule of inference) to Yavorsky's QGL^b [27], is most likely an arithmetised provability predicate logic too.

3.1 M^3 logic

In a series of Tourlakis and Kibedi's papers ([40, 41, 19]), M^3 and BM were introduced as a means toward formalising some of the metatheory of classical equational proofs to make them, as a result, more "mechanical". In particular, some informal proof techniques of classical equational logic can be made to proceed formally and entirely "calculationally" eliminating, for example, the need to

split equational proofs into disconnected pieces, and thus eliminating a source of "human" error.

Apropos "human error", suffice it to note one example of extremely experienced users and researchers in logic who fell victims to it, because they were mixing formal with informal statements: In below case it was the distinction between the informal (metamathematical) $\vdash$ and the formal $\rightarrow$.

One temptation for error is presented by the generalisation rule: Classical equational proofs become discontinuous where generalisation must be applied, because the formula

$$A \leftrightarrow \forall x A \tag{1}$$

is not a theorem. We have $A \leftarrow \forall x A$ (axiom) but not $A \rightarrow \forall x A$. But we do have $A \vdash \forall x A$.

[15] erroneously invoke (1) above, confusing it with

$$\vdash A \text{ iff } \vdash \forall x A \text{ (this is their "9.16")}$$

in an ϵ/δ proof on p.173.

Equational proofs can handle the metasymbol $\vdash$ formally within any of the logics M^3, ML^3 or BM [40, 41, 19, 32] because the informal "A is a classical theorem iff $\forall x A$ is a classical theorem" is formalisable in them as we outline below:

The above references showed that for any classical formula B and any set of

closed [10] classical formulae $\Gamma \cup \{A\}$ we have

$$\Gamma, A \vdash_{CL} B \quad \text{iff} \quad \Gamma, \Box\Gamma \vdash_{BM \text{ or } M^3} \Box A \to \Box B \qquad (C)$$

where $\Box\Gamma = \{\Box X : X \in \Gamma\}$, while the subscript CL of $\vdash$ to the left of "iff" indicates a classical proof —that is, one where no modal axioms are used.[11]

They called the statement (C) the conservation property (of proofs). It has the special case (C_1) below

$$A \vdash_{CL} B \text{ iff } \vdash \Box A \to \Box B \qquad (C_1)$$

Because of (C) or (C_1), any of M^3, ML^3 or BM bypass the difficulty encountered in an equational invalid proof like

$$\ldots \longleftrightarrow Q \longleftrightarrow A \underbrace{\longleftrightarrow}_{\substack{\uparrow \\ \text{invalid}}} \forall x A \longleftrightarrow R \longleftrightarrow \ldots \qquad (invalid)$$

that we must split into two proofs in order to make it valid:

$$\ldots \longleftrightarrow Q \longleftrightarrow A \quad \text{and} \quad \forall x A \longleftrightarrow R \longleftrightarrow \ldots \qquad (val)$$

We must begin a new (branch) of the proof at $\forall x A$, as in (val) above! However, because of (C_1) the two proof chains in (val) can be recast in BM and M^3 —and

[10] In the cases of M^3 and ML^3 we do not need the restriction that all formulae in $\Gamma \cup \{A\}$ be closed.

[11] [40, 41, 19, 30, 32, 31] employ only classical rules in BM, M^3, ML^3, and even QGL, by turning Necessitation into an derived rule.

as shown in [32], also in ML^3— as a single (modal) valid chain with no need to spawn a new chain:

$$\ldots \longleftrightarrow \Box Q \longleftrightarrow \Box A \longleftrightarrow \Box \forall x A \longleftrightarrow \Box R \longleftrightarrow \ldots \qquad (valid)$$

Reference [41, 19] introduces appropriate Kripke semantics for M^3 and BM respectively and prove soundness and completeness: That is, roughly, all theorems of these logics are valid in all the —appropriate for the logics— Kripke structures (soundness) and conversely, all valid (in the appropriate Kripke structures) formulae of each logic are theorems (completeness).

The reader will recall that QGL is not compete with respect to any Kripke structures [24] and admits no cut elimination [1]. By contrast, BM and M^3 are much nicer, as additionally to being complete with respect to appropriate Kripke structures, their Gentzen equivalent (GTKS, cf. [30]) admits cut elimination.

Logics BM and M^3 were designed to support the conservation property (C) on p.21, and thus these logics formalise the classical $\vdash$, as $\Box$, by properties (C) or (C_1). Reference [31] (Theorem 6.4) proves the unexpected result that QGL too has a weak form of the conservation property for classical proofs, namely:

$$\text{If } \Gamma \cup \{B\} \text{ is classical, then } \Gamma \vdash_{CL} B \quad \text{iff} \quad \Gamma, \Box\Gamma \vdash \Box B \qquad (C')$$

In fact, a somewhat more general result was proved, where the formula B need not be classical. Of course, in this more general form (where B is not classical) the proof $\Gamma \vdash_{CL} B$ will not be classical and one must drop the "CL"subscript.

Note that even though the quoted result (C') from loc. cit. is expressed there as an "if-then", the converse (the "only-if") trivially holds by Necessitation.

The reader can now appreciate the motivation that led to the requirement that $\Box A$ be always closed:

The BM, as was also the case for the earlier M^3, were defined as first-order extensions of K^4 (see Definition 2.1, p.13) with all the pure classical predicate axioms, plus one new modal axiom schema: $\Box A \rightarrow \Box \forall x A$. The intended behaviour of $\Box$, encapsulated by the conservation property (C) above, was forced by the axioms and the language structure. In particular, by:

- Ensuring that $\Box A$ is closed in the 1st-order modal language: Because, given that "$\vdash A$ iff $\vdash \forall x A$" holds in classical logic —that is, $\vdash$ does not care about free variables— then $\Box$ must be blind to them too.

- Adding the axiom schema $\Box A \rightarrow \Box \forall x A$ to guarantee [12] the logics BM and M^3 can formalise generalisation (the only-if of the quoted statement above).

In classical predicate logic, generalisation is a valid inference rule. It states that "if A has been derived, then $\forall x A$ can be derived". There are two flavours of generalisation:

(i) M^3 uses unrestricted generalisation as above.

(ii) BM uses restricted generalisation, where there is a condition on "A has been

[12] A recent result shows that $\Box A \rightarrow \Box \forall x A$ is independent of the remaining axioms, so it must be included as one! [39]

derived"; that is, "the hypotheses from which A was derived have no formulae with a free x".

In both M^3 and BM, the schema $\Box A \rightarrow \Box \forall x\, A$ formalises classical generalisation "if $\vdash\ A$, then $\vdash\ \forall x A$" as long as the context sorts out the presence or not of a condition on "if $\vdash A$".

3.2 ML^3 logic

Retaining the main design criterion of each of BM and M^3 —that is, the validity of conservation property (C)— the authors of [32] introduced a further extension of M^3 over the same modal language, obtained by adding Löb's axiom schema $\Box(\Box A \rightarrow A) \rightarrow \Box A$. This common extension of M^3 and GL was named ML^3. As it retains the syntax of $\Box A$ it was expected, and this proved to be the case, that is "well behaved", meaning that at a minimum it satisfies

- the conservation property (C),
- it is complete with respect to appropriate Kripke structures,
- its Gentzen version admits cut elimination.

The language and some key terminology of ML^3 will be outlined here (cf. [32]) since we will focus our mind on the equivalent Gentzen-style logic for ML^3 later:

The set of logical axioms of ML^3 is $\Lambda \cup \Box\Lambda \cup \Box\Box\Lambda$, where Λ consists of all instances of the following basic schemata:

(1) All tautologies

(2) $\forall x A[x] \to A[a]$

(3) $A[a] \to \forall x A[x]$, provided a does not occur in A.

(4) $\forall x(A \to B) \to (\forall x A \to \forall x B)$

(5) $\Box(A \to B) \to \Box A \to \Box B$

(6) $\Box(\Box A \to A) \to \Box A$

(7) $\Box A \to \Box \forall x A$

There are two primary rules of inference. Modus Ponens (MP) "from A and $A \to B$ infer B", and (unconditional) generalisation (Gen) "from A, infer $(\forall x)A$". In symbols, these rules are denoted by

$$A, A \to B \vdash B \quad \text{and} \quad A \vdash \forall x A$$

Compared to the precursor M^3, ML^3 doesn't include the axiom schema $\Box A \to \Box\Box A$ explicitly, but we can derive it as a theorem in ML^3 due to the "trick" of adding $\Box\Box\Lambda$ to its logical axiom set: $\Lambda \cup \Box\Lambda \cup \Box\Box\Lambda$ —by comparison, M^3 does not have the $\Box\Box\Lambda$ part. It is however often convenient to think of ML^3 in a slightly alternative manner: We start with exactly the same axiom set as M^3, including $\Box A \to \Box\Box A$, and we simply add Löb's axiom schema and also remember to add the boxed Löb's axiom schema in order to update $\Box\Lambda$. In this approach we drop $\Box\Box\Lambda$ as redundant. Finally, if we were to add necessitation $A \vdash \Box A$ as primary

rule as [27] does for QGL^b, then we do not need $\Box\Lambda$ either. The role of $\Box\Lambda$ in proving necessitation as admissible is fully expanded in [40, 41].

Not having necessitation as primary rule has many technical advantages, in particular in the application of the "formulator" tool (introduced in [32] and further used in [39]) in the metatheory of ML^3.

ML^3 shares the same language as BM and its predecessor M^3, thus it too requires its formulae of the type $\Box A$ to be always closed. This enables ML^3 to formalise and simulate the classical pure predicate calculus $\vdash$ —the logic supports the exact same version of (C) (p.21) as M^3 does, that is, there is no requirement on Γ, A to be closed [32]. Notably, the Gentzen equivalent of ML^3 called GLTS in loc. cit., is cut free but the cut rule is admissible (provable) [32] —this is a cut elimination result in reverse (like the ones for BM and M^3 [30]). Moreover, it is complete with respect to finite transitive and reverse well-founded Kripke structures.

It is also arithmetically sound for arithmetical interpretations, $*$, over PA, just like M^3 and QGL. Arithmetical soundness states, "If $\vdash_{ML^3} A$, then for every arithmetical interpretation mapping $*$, we have $\vdash_{PA} A^*$" [32]. We conjecture that ML^3, differing from QGL^b of [27] only in how necessitation is handled, is arithmetically complete as well. We do not explore this question in this thesis where our objective is to improve cut elimination proofs in modal 1st-order Gentzen-style logics.

4 Cut Elimination

Users of mathematical logic in mathematics are usually using a Hilbert-style logical calculus that relies on a lot of axioms but only one or two rules of inference. For example, in propositional calculus, there are several axioms such as the famous one —Excluded Middle— $A \vee \neg A$, but only one rule of inference exists, namely, Modus Ponens $A, A \to B \vdash B$.

Many who study, rather than do, logic and proofs prefer to use syntactic tools in their study, or as we say are doing "proof theory" — which is a particular aspect of "logic metatheory". Proof theory is a research tool for the syntactic study of a logic system, in contrast to model theory, which is semantic in nature. In Proof theory, proofs are represented as formal mathematical objects that obey certain rules of construction (the rules of inference of the logical system) [29]. Not only proof theory has been applied in logic research mainly, but also more and more practical methods rooted from proof theory such as proof mining and automated theorem proving has been applied in computer science, linguistics and philosophy widely. The direction of my research is to continue —after my Masters— working in the area of proof theory of first-order modal logic.

For those who do proof theory, the most appropriate formulation of the logic they study is Gentzen's Sequent Calculus that has a lot of rules of inference but no axioms!

Gerhard Gentzen gave birth to the sequent calculus. The first sequent calculi, systems LK and LJ were introduced in 1934–1935 by his [10] and [11] as a tool for studying natural deduction in first-order logic (see also [5]). According to his paper [10], the presence of sequents is the predominant characteristic of sequent calculus. An expression like $\Gamma \vdash \Delta$ is called a sequent, where the symbols Γ and Δ represent sets of formulae. A sequent stands for the informal statement "the conjunction of the formulae in Γ proves the disjunction of those in Δ". Γ in a sequent $\Gamma \vdash \Delta$ is given the name "antecedent" and correspondingly, Δ is the "succedent" where Γ and Δ are finite sets of formulae [28]. A (formal) proof in the sequent calculus is a sequence of sequents, where each of the sequents is derivable from sequents appearing earlier in the sequence by using one of the rules of inference such as $\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$. All rules, except two "initial" rules, are written as fractions that are applied as follows: If we have proved all the sequents of the numerator, then we have also proved (and therefore can write down) the sequent of the denominator. The initial rules have no numerators but are single sequents; these can be written outright, essentially, exactly like the axioms in a Hilbert-style proof.

Gentzen-style systems have significant practical and theoretical advantages: On the practice side, in Software Engineering area, "Event-B", the popular formal method for system-level modelling and refinement takes advantage of sequent calculus to validate models and verify consistency between refinement levels within a safety-critical system, such as Railway signalling and control systems and Emergency services dispatch systems. Concretely speaking, in the "Rodin" platform —

an Eclipse-style integrated development environment designed for Event-B programming— In each "machine" (an ioslated system) or "context" (the underlying environment of all systems), users are able to manually apply Gentzen style deductions, or Equational style deductions for "invariants proof" [13] when they're discharging "Proof Obligation" [14] in "ProB" [15] model checker view, which is helpful for model validation.

Another instance of practicality is in resolution proofs.

Prolog is a programming language in which the underlying computation mechanism is logical deduction. The deduction process used by Prolog has been explained as a form of resolution. Michael Besson showed that Prolog is intimately connected with Gentzen's cut-free sequent calculus. In his paper, Prolog computation can be viewed as the construction of certain cut-free derivations.[3] (pp.101–116)

The Gentzen formulation of logic is particularly suited for metatheoretical, in particular proof theoretical (i.e., syntactic metatheory) study, especially if the system admits cut elimination as we say, that is, if it can be formulated without the cut rule without loss of deductive power because we were able to show that the cut rule is derivable from the remaining rules!

[13] "Invariants proof" is the proof provided by users that every system invariant is always maintained during the change of all system or local variables.

[14] Proof Obligation means the initial values of the variables declared in a machine or context must satisfy the invariants of the machine or context.

[15] ProB is an animator, constraint solver and model checker for the "Event-B" description language. It allows fully automatic animation of B specifications, and can be used to systematically check a specification for a wide range of errors.

Note that not all Gentzen systems admit cut elimination! The system for QGL does not, as we have noted earlier.

What is the Cut Rule? It is a generalisation of modus ponens, and the Gentzen style formulation is this:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Sigma \vdash \Phi}{\Gamma, \Sigma \vdash \Delta, \Phi}$$

A is the cut formula and is "cut out" after applying the rule since it does not appear in the conclusion sequent (denominator).

Why is the absence of the cut rule as a primary rule good?[16]

Actually, the presence of cut rule as primary is a big headache for proof theoretic study of proofs, as it hinders application of induction because the "numerator" (hypothesis) of the rule contains a formula, A, that does not occur in the "denominator" (conclusion) —in fact the conclusion is less complex than the hypothesis and this would ruin the induction step if induction relies on the complexity! Without cut rule, the sequent on "denominator" of any other rule gets more complex than the sequent on "numerator" so that induction on the complexity of sequents works well.[17] In fact, cut-free proofs in a Gentzen style logic have the subformula property, that is, all formulae appearing prior to the proved theorem

[16] All rules given in the definition of a logic are called primary. The ones that were not included definitionally, but we can prove as theorem-schemata are called "derived" or "admissible".

[17] This claim depends on the type of the remaining rules, but is supported by the usual rules one employs in a Gentzen logic.

are subformulae of the theorem.[18]

The cut rule can be eliminated in many systems of interest —that is, it is provable from the other rules of the sequent calculus— as Gentzen proved for LJ and LK [10] and [11]. The (meta)proof that the cut rule is derivable or admissible is called cut elimination proof. In 1964, Gentzen in his landmark paper "Investigations into Logical Deduction" [12] provided the complete details of cut elimination proofs for the systems LK (classical) and LJ (intuitionistic).

After Gentzen's work, several logicians turned attention to the cut elimination for various known modal logic systems. As these systems were invariably defined as Hilbert style, a technique emerged (e.g., [42, 28, 21]) to build a Gentzen equivalent —a "Gentzenisation"— of the under metamathematical study logic and study instead its Gentzenisation. For example, as to obtain cut elimination proofs of KS system (KS = LK + the modal rule KR) —where KR is $\dfrac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$— [28] mentioned that it is necessary to change the definition of "rank"[19] of cut formulae and in the same paper they presented a cut elimination proof for system K^4S —that is, the Gentzenisation ("S" for Sequent) of the modal logic K^4— (K^4S = LK + the modal rule TR[20]). Generally speaking, the cut-elimination theorem states that any provable sequent can also be proved by a cut-free proof, that is, a proof that does not make use of the cut rule. The cut-elimination theorem is also called "Gentzen's

[18] A is a subformula of $\Gamma \vdash \Delta$ iff it is a subformula of at least one of the formulae appearing in the sequent.

[19] A concept of Gentzen's that our approach does not use, and we will not define.

[20] See p.37.

Hauptsatz" and has become a significant proof-theoretic tool. The article of [33] shows many applications including some for PA. Whether the Gentzenisation of a logic system admits cut elimination is a critical measurement of our ability to study the logic proof theoretically.

4.1 Cut-elimination for (the Gentzenisation of) GL

Prior to finding syntactic proofs of cut elimination for the Gentzenisation of GL, logicians discovered model theoretic proofs at first. Logicians Sambin and Valentini successfully provided a cut elimination proof for propositional provability logic GL by using a semantic method in the paper "The Modal Logic of Provability: the Sequential Approach" [28]. At first, a Gentzen style system called GLS, which consists of system LK plus GLR modal rule,

$$\frac{\Gamma, \Box\Gamma, \Box A \vdash A}{\Box\Gamma \vdash \Box A}$$

was described in loc. cit. and was proved to have exactly the same theorems as GL, therefore GLS is the Gentzenisation of GL. They then offered a semantic proof of cut elimination for GLS by a method of searching proofs backwards (from the theorem, back towards the axioms that were used in its proof.)

However, the path to a cut elimination (syntactic) proof in proof theory is a zigzag.

Leivant [21] proposed a syntactic Gentzen-like proof of cut elimination for the Gentzenisation of GL in 1981. Although some statements is correct, the "alleged proof" was wrong. Arnon Avron [1] pointed out Leivant's mistake —applying the induction hypothesis incorrectly— in his paper "on Modal Systems Having Arithmetical Interpretations" . In the meantime, Sambin and Valentini believed that Leivant's proof was untenable [28]. Consequently, in the later paper [42] Valentini with Sambin's help constructed a counterexample against Leivant's proof according to Leivant's definition of "secondary grade". In that counterexample, if someone needs to eliminate two sequential cut rules at the bottom of the proof, Leivant's reduction would bring them to an infinite loop.

Not only a counterexample, but also a wonderful —but quite complex— syntactic proof of cut elimination for GL was given by [42]. Since Gianluigi Bellin in his paper "A system of natural deduction for GL" [4] suggested a rather complicated proof in which no measure of the complexity of the used reductions was provided, Vanlentini introduced a very clear definition of the essential parameters, degree δ, rank r and width n —the latter a rather complicated notion! The induction was a triple induction on the ordinal

$$\delta(A)\omega^2 + n(A)\omega + r(A)$$

associated with the cut formula A. In simple words, the triple induction views the triple of numbers (δ, n, r) lexicographically where δ is in the most significant

position and r is in the least one.

There have been several successful attempts to give a syntactical proof of cut-elimination for (the Gentzenisation of) GL with different approaches after Vanlentini's proof. Among them, Marco Borga's proof [6] is a very impressive one. In Borga's paper, merely a double induction on grade and rank was used just as in Gentzen's paper. Unsurprisingly, proof was based on a modification of Gentzen's approach [6].

Entering 21 century, J. Brighton ([8]), presented a double induction proof of cut elimination for the Gentzenisation, GLS. In his paper, Brighton came up with a fairly different method compared to Valentini's [42] or Borga's.

In [28], Sambin and Valentini showed that, given Γ, Δ, sets of formulae, it is possible to find out whether $\Gamma \vdash \Delta$ is derivable or not in GLS' (which is equivalent to GLS, but has no cut rule) by inverting its inference rules and thus obtaining a finite number of "backwards" (along the proof) search trees, and then checking whether any of them is a legal proof (tree) in GLS'. They also showed that the height of each search tree is finite. Brighton converts this idea into a simple syntactic proof by a double induction on formula complexity (primary induction) and search tree height (secondary induction). He calls his search trees "regress trees" and retains the backwards rules of [28]. The breakthrough in his proof are two simple facts:

- The Gentzen cut rule is equivalent to the rule: $\dfrac{\Gamma, A \to A \vdash \Delta}{\Gamma \vdash \Delta}$

- All rules of the Gentzenisation (except the GLR rule) of GLS' are reversible, that is, if the denominator is a theorem, then so are the sequents in the numerator.

Each fact contributes tremendously to the simple proof!

This thesis adapts Brighton's proof to the predicate modal case for the Gentzenisations of M^3 and ML^3.

4.2 Cut elimination proof for Predicate Modal Logic

After Valentini's success on cut elimination of GL, naturally more and more logicians turn to seek cut elimination proof for Gentzenisations of predicate modal logic.

Unfortunately, the most natural predicate extension for provability logic, QGL, doesn't admit cut elimination at all [1]. Valentini admitted that his proof for GL couldn't work well for the predicative case because of one induction hypothesis that cannot be always valid. Later, Avron proved that cut elimination fails for QGL [1]. Montagna also added to the negative results about QGL [24]. Its failure at cut elimination and the results in the last reference illustrate again that QGL cannot offer a complete description of the logic of (arithmetised) provability. There is a clear need for alternative approaches to first-order modal logic!

How about the predicate modal logics BM, M^3 and ML^3? These are not given as Gentzen style systems, but nor is QGL. They are defined as Hilbert-style logics

as is the custom for logics meant to be used rather than studied. However, as we learnt from the work of [42, 28, 21, 1, 24] and others, a rich proof theory can be developed for them if one can find equivalent Gentzen logics (Gentzenisations) for them, for which the cut rule can be proved to be redundant.

Such Gentzenisations exist.

The Gentzen style first-order modal logic system named GTKS —equivalent to BM and, with a tweak,[21] also to M^3— was defined and studied in the paper "On the Proof-Theory of two Formalisations of Modal First-Order Logic" [30]. These two logics, as I mentioned earlier, were originally introduced and studied in [40, 41, 19], but this study was carried in the semantic metatheory via Kripke models. [30] focused on the syntactic metatheory —the proof-theory— and set sights primarily on cut-elimination for these logics. Instead of including the cut rule in system explicitly like LK did, the authors of loc. cit. introduced GTKS as a cut-free system (in the style of [29]) and proved that the cut is an admissible rule —a "backwards" cut elimination result. Following [28] they also added a "modified TR" modal rule in GTKS

$$\frac{\forall\Gamma, \Box\Gamma \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi} \tag{1}$$

[21] Theorems 4.1 and 4.2 in [30] state respectively: GTKS proves $\Gamma \vdash \Delta$ iff $\Gamma \vdash_{\mathrm{BM}} \bigvee \Delta$ —where $\bigvee \Delta$ is the disjunction of all formulae in Δ.

But, for M^3:

If GTKS proves $\Gamma \vdash \Delta$, then $\Gamma \vdash_{\mathrm{M^3}} \bigvee \Delta$. Conversely, if $\Gamma \vdash_{\mathrm{M^3}} A$, then $\forall\Gamma \vdash_{\mathrm{GTKS}} A$. Note the "$\forall\Gamma$". The difference is because BM has the weak generalisation rule (derived) while M^3 has strong generalisation as a primary rule. The generalisation in GTKS is weak. See also rulle (6), p.44.

The original TR of [28] was

$$\frac{\Gamma, \Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad (TR)$$

introduced by these authors for the Gentzenisation of the propositional modal logic K[4]. What TR (and (1)) does in particular, is it proves $\Box A \to \Box\Box A$.

The reason [30] called rule (1) a "modified TR" was

- They added universal closure [22] $\forall$ in the numerator of the original TR rule, needed in the new first-order setting in the theorem that proves the equivalence of GTKS and M[3]

- What is most worth mentioning here is that, in GTKS system, weakening and strengthening rules are not primary —unlike the LK case— to facilitate metatheoretical inductions. To recover weakening and strengthening as admissible rules, one of the tricks used was to integrate weakening (Φ) and strengthening (Ψ) into the modal rule (1).

Four years later, the paper "On the Proof-Theory of a First-Order Extension of GL" [32] introduces the common first-order extension of GL and M[3] that they name ML[3] and develop its proof theory. Again, the proof theory is developed in an equivalent brand new Gentzen system they introduce, GLTS. This again is

[22] For a formula A, $\forall A$ is shorthand, where the prefix $\forall$ indicates a string of $\forall x$ components, one for each free variable x of A. Thus we derive a new formula which is provable iff the original was, but the new one is rendered closed via repeated universal quantification; "universal closure". For a set Γ, $\forall\Gamma = \{\forall A : A \in \Gamma\}$.

influenced by [28] in that the modified "GLR" modal rule (2) below is used:

$$\frac{\forall\Gamma, \Box\Gamma, \Box A \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi} \tag{2}$$

Just as for GTKS and TR, the "modification" here adds weakening and strengthening in the denominator, and replaces Γ by $\forall\Gamma$ in the numerator. [32] tell us how the equivalence between ML^3 and its Gentzenlation is formulated in detail: If $\Gamma \vdash \Delta$ is provable in GLTS, then $\Gamma \vdash_{\mathrm{ML}^3} \bigvee \Delta$. Conversely, if $\Gamma \vdash_{\mathrm{ML}^3} A$, then $\forall\Gamma \vdash A$ is provable in GLTS [32].

The definition of the Gentzen-style sequent calculi GTKS and GLTS is influenced by Schütte's book "Proof Theory" [29] in two important aspects. No primitive weakening/strengthening rules, and no primitive cut-rule are included in these two logic systems. Recall that the classic systems LK and LJ do include the weakening/strengthening rules ("structural rules" as Schütte calls them) and the cut rule explicitly as primitives [28]. As in Schütte, the "cut elimination theorem" is more like a "cut introduction theorem": cut is an admissible rule [29]. One thing needs to be said: except for the modal rule (TR in GTKS, GLR in GLTS), the other primitive rules are common in GTKS and GLTS systems. The precise description of GLTS and GTKS, including all their rules, will be introduced in the last section of the thesis.

The good news is that both admit cut elimination whose proof was presented in two papers of Schwartz and Tourlakis [30, 32]. Strengthening and weakening are proved to be admissible within both GTKS and GLTS systems, and they, of

course, help in the proof of the cut-introduction ("cut is admissible") theorem for these logics. Equally helpful is the "inversion theorem", which says that certain rules are reversible, that is, if the denominator is provable in GTKS (resp. GLTS), then the numerator(s) is/are too.

The cut admissibility proof for GTKS is principally by induction on the modified complexity of the cut formula A and secondarily on the derivation orders for $\Gamma \vdash A, \Delta$ (order $\leq m$) and $\Theta, A \vdash \Phi$ (order $\leq n$). The "modified complexity" is the name given to the ordinal $\omega \cdot k + r$ where k counts $\Box$ occurrences and r counts the total of all $\rightarrow, \forall$ occurrences in A [30]. This neat concept means that of two formulae A and B, the one with more boxes $\Box$ is more complex than the other, even if the latter has more occurrences of $\rightarrow$ and $\forall$. The proof is a relatively complex triple induction modelled after Schütte's classical Gentzen predicate calculus cut admissibility proof [29].

The cut admissibility for GLTS is quite complex, imagine adding a layer of complexity to the [42] proof, required because of the quantifier rules. It is a quadruple induction with primary component the modified complexity C of the cut formula A. The secondary induction is on the width W of the sequent $\Gamma \vdash \Delta, A$.[23] Moreover, the tertiary and the quaternary inductions respectively are on the derivation orders for $\Gamma \vdash A, \Delta$ (order $\leq m$) and $\Theta, A \vdash \Phi$ (order $\leq n$) [31]. Obviously, a quadruple induction and the width concept (see [42, 32]) make the proof difficult to understand!

[23]The admissibility statement to prove is "if GLTS proves $\Gamma \vdash \Delta, A$ and $\Theta, A \vdash \Phi$, then it can also prove $\Gamma, \Theta \vdash \Delta, \Phi$."

5 New simplified proofs for cut elimination

Cut-elimination/simulation proofs in general have been long, but also extremely complicated, especially in the presence of the modal operator. In this thesis I want to contribute to the work of [40, 41, 19, 30, 32] and make an improvement on the cut elimination proofs for the Gentzenisations of M^3 and ML^3 contained in the last two references.

The cut elimination proofs for GTKS and GLTS in [30, 32] will likely tax all but the most expert readers because of their relatively high complexity. It will be useful to have alternative simple and readable proofs with significantly less complexity.

So, The improvement we want to effect will be in the area of reducing the proof's complexity, and improving its readability.

Complexity is a general and somewhat arbitrary term for a proof. It can be length of proof, but I also chose as additional metric in induction proofs: the depth of nesting of the induction hypotheses (I.H.) as dictated by the multiplicity of the induction: double, triple, etc., with 2, 3, etc., nested I.H. Such inductions are common in cut elimination or cut admissibility proofs. Gentzen's original induction proofs for LK and LJ were via double induction. Higher induction nesting level, length of proof, and complex concepts employed, such as width, contribute to a very complex proof. For example, Leivant's incorrect proof of cut

elimination for GL in [21] was the victim of complexity, where he did not notice that he misapplied one of the multiple I.H.!

Let's review the complexity of cut elimination proofs in the literature:

The original cut elimination proof of Gentzen's for LK was long and hard to read in his [12], even though it was only a double induction! The fact that weakening/strengthening were primary rules made things messy! In GTKS and GLTS these rules are derived.

Valentini [42] employs a triple induction to obtain cut freedom for (the Gentzenisation of) the propositional GL introducing a very hard to understand concept of sequent $\Gamma \vdash \Delta, A$ width (where A is the cut formula) as a parameter for the secondary induction!

For (the Gentzenisation of) M^3, GTKS, [30] uses a triple induction and is based on ideas of logician Schütte [29].

Logicians Schwartz and Tourlakis use in [32] four induction hypotheses in a quadruple induction —one more nesting level than in [42] because of quantifier rules— to obtain cut elimination for their ML^3 logic! They also used width, as in Valentini, and that made their proof very hard to understand!

Looking at above inductions, one may ask if it is possible that a cut elimination proof with less complexity exists for GTKS and GLTS. The answer is "yes". As mentioned above, J. Brighton [8], presented a double induction proof of cut elimination for GLS logic system. Brighton's method inspires the people working not only on GL and similar propositional modal logics but also in the predicate modal logic case. The excellent extensibility and transferability of his method

suggests that people should be able to devise much more readable inductions of lower multiplicity that present a significant simplification from the originals. I wanted to apply this methodology to (Gentzenisations) of M^3 and ML^3! I worked on this simplification throughout my Masters research and study and eventually succeeded. In next section, I will demonstrate two such simple and readable cut admissibility proofs.

5.1 Two Gentzen-style modal first-order logics

The rules for the Gentzenisations of M^3 and ML^3 are given in the next two definitions (cf. [30, 32]). Upper case Latin letters stand for formulae while upper case Greek letters $\Gamma, \Delta, \Psi, \Sigma$ (and other choices that are not also Latin capital letters) stand for finite sets of formulae; so do primed such letters. The expression $\Gamma \vdash \Delta$ is called a sequent and intuitively says that the set of hypotheses (formulae) in Γ proves the disjunction of the formulae in Δ. Γ is the antecedent part of the sequent, while Δ is the succedent. "Γ, A" and "A, Γ" mean $\Gamma \cup \{A\}$.

We will not repeat the description of the common language of all four logics (M^3, ML^3, GTKS and GLTS) in detail, but we will revisit the less standard points here. In fact, we will not define M^3 or ML^3 in this section (but see Section 3.2), since the sole purpose of this section is to offer simplified cut admissibility proofs for GTKS and GLTS; the latter two logics we define here in detail.

The primary connectives are $\bot, \rightarrow, \forall, \Box$. There are two types of (object) variables, free ($a, b, c, a', c'', a_0, b_{12}$, etc.), and bound ($x, y, z, x', y'', z_0$, x_{12}, etc.). The

syntax of formulae ensures that $\Box A$ is a sentence, for all formulae A.[24] The expression, $\Box A$ is metanotation for the expression obtained from A as follows: (1) Replace all free variables that occur in A by the lexicographically smallest[25] unused (in A) bound variables $x_{j_1}, \ldots, x_{j_k}$; this results to an expression we will call A'. (2) Let α represent the string formed by arranging the used in (1) bound variables in their lexicographic order. (3) Then "$\Box A$" names the string $\Box \alpha A'$. Note that if A has no free variables, then the meta name $\Box A$ names itself (that is, A is the same string as A' and α is empty).[26]

For any expression F,[27] $F[a]$ or $F[x]$ indicates that we want to pay attention to the free variable a or bound variable x that possibly occur in F. In the context of the notation $F[a]$, $F[t]$ denotes the result of replacing a by t, everywhere in F — an operation on the expression F that we will on occasion denote more explicitly by "$F[a := t]$". The $[a := t]$ operation has the highest priority, so, for example, $A \to B[a := x]$ stands for $A \to (B[a := x])$.

$\forall x A[x]$ (or just $\forall x A$) are metanotation for our familiar "for all values of x, $A[x]$ holds". Thus, provided that x does not occur in A, $\forall x A[x]$ names $\forall x A[a := x]$, for some a known from the context. Note that in the last expression $[a := x]$ applies to A before $\forall x$ does.

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[24] The motivation and rationale for this choice of an "opaque" $\Box$ vs. the "transparent" one in the case of QGL has been explained elsewhere (in this thesis —p.23, first bullet— and at length in [40, 41, 30, 32, 31]) and will not be repeated here.

[25] The infinite set of bound variables is finitely generated as suggested above from the alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ', x, y, z\}$, whose members we list here in the intended increasing order.

[26] This description and use of $\Box A$ as metanotation parallels the one in Bourbaki ([7]) for the meta-expression $\tau_x A$.

[27] This expression could be a formula A, a set of formulae Σ, or a sequent $\Psi \vdash \Omega$.

Definition 5.1 (GTKS Rules [30])

(1) Initial rules: $\Gamma, A \vdash \Delta, A$ and $\Gamma, \bot \vdash \Delta$, where A is atomic.

(2) $\rightarrow$- left rule: $\dfrac{\Gamma, A \rightarrow \bot \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}$, where B is not $\bot$.

(3) $\rightarrow$- right rule: $\dfrac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$

(4) $\bot$- right rule: $\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A \rightarrow \bot}$

(5) $\bot$- left rule: $\dfrac{\Gamma \vdash \Delta, A}{\Gamma, A \rightarrow \bot \vdash \Delta}$

(6) $\forall$- right rule: $\dfrac{\Gamma \vdash \Delta, A\,[a]}{\Gamma \vdash \Delta, \forall x A\,[x]}$ —as long as a, the eigenvariable of the rule, does not occur in the conclusion ("denominator") of the rule.

(7) $\forall$- left rule: $\dfrac{\Gamma, A\,[a] \vdash \Delta}{\Gamma, \forall x A\,[x] \vdash \Delta}$

(8) The modified "TR" modal rule: $\dfrac{\forall\Gamma, \Box\Gamma \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi}$ ■

Definition 5.2 (GLTS Rules [32])

(1)–(7) As for GTKS, but instead of TR see GLR below:

(8) The modified "GLR" modal rule: $\dfrac{\forall\Gamma, \Box\Gamma, \Box A \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi}$

The Γ and Δ in the rules are called the "side formulae" (s.f.); the resulting single formula in the "denominator" in rules (2)–(8) is the "principal formula" (p.f.) of the rule (for example, formula $A \rightarrow B$ is the p.f. of rule (3)); rule (1) has A as principal formula. The single formulae displayed in the "numerators" of

(2)–(8) are the "minor formulae" (m.f.) of the rule (for example, formula A and B are the m.f. of rule (3)). A numerator sequent is a premise while the denominator sequent is the conclusion of the rule. Φ and Ψ in rule (8) of both definitions are weakening and strengthening parts respectively. $\Box A$ in 5.2(8) is the diagonal formula. We call the rule (2) "Y-type" (as adjective) because of its shape. All the other rules are "I-type". ■

Remark 5.3 The departure from [28, 42] in using here $\forall\Gamma$ in the premise of TR and GLR, rather than using Γ, permits a central part of the proof (given in [30, 32]) —that GTKS and GLTS are equivalent to M^3 and ML^3 respectively— to conclude successfully. That is the part of the proof that derives in GTKS (and GLTS) the common axiom schema $\Box A \rightarrow \Box\forall xA$ of M^3 and ML^3. ■

Definition 5.4 (Theorems) A $theorem$, or $derivedsequent$, is defined recursively to be one of:

(1) A sequent of one of the two types in rule (1). We say it is derived with $order$ 0, or that it is an $axiom$.

(2) A sequent of the same type as in the denominator of rule (2) provided the two corresponding sequents in the numerator are also theorems. If the latter two are derived with orders m and n, then the former is derived with order $1 + \max(m, n)$.

(3) A sequent of the same type as in the denominator of rules (3–8) provided the corresponding sequent in the numerator is also a theorem. If the latter is derived with order m, then the former is derived with order $1 + m$. ■

Remark 5.5 The above recursive definition of theorems implicitly defines a tree —a proof tree— with root labeled by the theorem. This root has one (case where an I-rule was the last one applied) or two subtrees (case where the Y-rule was the last one applied), which have root(s) labeled by the premise(s) of the last rule used to derive the theorem. The leafs of the proof tree are labeled by the axioms.

A derivable sequent may be derived with many different proof trees, and therefore with many different orders as the latter depend on the particular proof we have in mind. Thus the sentence "$\Gamma \vdash \Delta$ is (a theorem) provable (or derivable) with order m" simply means that it is possible to derive said sequent with order m.

We note the absence of weakening/strengthening rules, unlike the original formulation of Gentzen's in the case of classical logic. This is so because it is desirable to introduce weakening and strengthening as admissible rather than as primary rules in a Gentzen logic, of which we aim to develop the proof theory. For example, proofs by induction on the height of proof trees are much simpler in the absence of such primary rules. This approach was earlier followed in [29],[28] where his weakening/strengthening "structural rules" are admissible, and was also present in [33, 28]. The second of the last two references incorporates weakening and strengthening parts in TR and GLR and in the rules under (1), just as we do. See also the proof of 5.7 below. ■

The following theorems and corollaries hold in both GTKS and GLTS. The

[28] Schütte, loc. cit. uses a generalisation of sequent calculus for first-order classical and Intuitionistic logic, where his "negative" and "positive" parts generalise Gentzen's "antecedent" and "succedent" formulae. Nevertheless, his techniques adapted to Gentzen's setting make it a straightforward matter to not require weakening/strengthening as primary rules.

proofs are indebted to [29] and were adapted in the sequent setting in [30, 32]. We include the proofs for 5.6, 5.7 and 5.9 and omit the others as being similar.

Theorem 5.6 (cf. [30, 32, 29]) If $(\Gamma \vdash \Delta)\,[a]$ is provable with order m and b is some other free variable, then $(\Gamma \vdash \Delta)\,[b]$ is provable with order $\leq m$.

Proof By induction on the order of derivation, m, of $(\Gamma \vdash \Delta)[a]$. For $m = 0$, $(\Gamma \vdash \Delta)[a]$ is an axiom. Then so is $(\Gamma \vdash \Delta)[b]$. For the induction step we prove the case for $m > 0$. Cases (2)–(8) are numbered by the rule number (Definition 5.1) of the last rule applied in deriving $(\Gamma \vdash \Delta)[a]$.

(2) $(\Gamma \vdash \Delta)[a] = \Gamma[a], A[a] \to B[a] \vdash \Delta[a]$. Thus the premises of the rule,[29] $\Gamma[a], A[a] \to \bot \vdash \Delta[a]$ and $\Gamma[a], B[a] \vdash \Delta[a]$, are each derived with orders $< m$. By the I.H., $\Gamma[b], A[b] \to \bot \vdash \Delta[b]$ and $\Gamma[b], B[b] \vdash \Delta[b]$ are derived with orders $< m$, thus $\Gamma[b], A[b] \to B[b] \vdash \Delta[b]$ is derived with order $\leq m$.

(3) $(\Gamma \vdash \Delta)[a] = \Gamma[a] \vdash \Delta[a], A[a] \to B[a]$. Then the premise $\Gamma[a], A[a] \vdash \Delta[a], B[a]$ is derived with order $< m$. By I.H. so is $\Gamma[b], A[b] \vdash \Delta[b], B[b]$, from which $\Gamma[b] \vdash \Delta[b], A[b] \to B[b]$ is derived with order $\leq m$.

(4), (5) We omit the similar cases for these rules.

(6) $(\Gamma \vdash \Delta)[a] = \Gamma[a] \vdash \Theta[a], \forall x A[a, x]$. The premise $\Gamma[a] \vdash \Theta[a], A[a, a_0]$ is derivable with order $< m$, where a_0 is the eigenvariable used. Let a_1 be a new variable that does not occur in $(\Gamma \vdash \Delta)[a]$ and is distinct from

[29] In each case of propagating the claim from order $< m$ to order m, we indicate without comment the p.f. for each rule considered, for example, $A[a] \to B[a]$ here.

b. Applying the I.H. twice —first changing a_0 into a_1 and then a into b— we get $\Gamma[b] \vdash \Theta[b], A[b, a_1]$, which is derivable with order $< m$.[30] Thus, $\Gamma[b] \vdash \Theta[b], \forall x A[b, x]$ —i.e., $(\Gamma \vdash \Delta)[b]$— is derivable with order $\leq m$, with eigenvariable a_1.

(7) $(\Gamma \vdash \Delta)[a] = \Theta[a], \forall x A[a, x] \vdash \Delta[a]$. The premise $\Theta[a], A[a, c] \vdash \Delta[a]$ is derivable with order $< m$ and so is $\Theta[b], A[b, c] \vdash \Delta[b]$ by I.H.

Applying the rule to the latter we derive $\Theta[b], \forall x A[b, x] \vdash \Delta[b]$ with order $\leq m$.

(8) $(\Gamma \vdash \Delta)[a] = \Phi[a], \Box\Omega[a] \vdash \Box A[a], \Psi[a]$. The premise $\forall\Omega[a], \Box\Omega[a] \vdash A[a]$ is derivable with order $< m$. By the I.H. so is $\forall\Omega[b], \Box\Omega[b] \vdash A[b]$. Thus $\Phi[b], \Box\Omega[b] \vdash \Box A[b], \Psi[b]$ is derivable with order $\leq m$ by an application of the same rule. It is noted that, since boxed formulae have no free variables, $\Box A[a] = \Box A[b]$ and $\Box\Omega[a] = \Box\Omega[b]$; moreover $\forall\Omega[a] = \forall\Omega[b]$ since $\forall\Omega$ has no free variables either.

The last case was argued based on rule TR, but the proof based on rule GLR is entirely similar, the presence of the diagonal formula $\Box A$ in the antecedent of the premise not adding any complexity. ■

Theorem 5.7 (Weakening; cf. [30, 32, 33, 29]) For either GTKS or GLTS, if $\Gamma \vdash \Delta$ is derived with order m then $\Phi, \Gamma \vdash \Delta$ is derivable with order $\leq m$.

[30] Since a_0 does not occur in $\Gamma[a] \vdash \Theta[a], \forall x A[a, x]$, $(\Gamma[a] \vdash \Theta[a], A[a, a_0])[a_0 := a_1] = \Gamma[a] \vdash \Theta[a], A[a, a_1]$.

Proof By induction on the order of derivation, m, of $\Gamma \vdash \Delta$. For $m = 0$, $\Gamma \vdash \Delta$ is an axiom. Then so is $\Phi, \Gamma \vdash \Delta$. For the induction step we prove the case for $m > 0$. Cases (2)–(8) are numbered by the rule number of the last rule applied in deriving $\Gamma \vdash \Delta$.

(2) Suppose $\Gamma, A \to B \vdash \Delta$ is derived with order m. Thus the premises of the rule,[31] $\Gamma, A \to \bot \vdash \Delta$ and $\Gamma, B \vdash \Delta$, are each derived with orders $< m$. By the I.H., $\Phi, \Gamma, A \to \bot \vdash \Delta$ and $\Phi, \Gamma, B \vdash \Delta$ are derived with orders $< m$, thus $\Phi, \Gamma, A \to B \vdash \Delta$ is derived with order $\leq m$ using the same rule.

(3), (4), (5) We omit the similar cases for these rules.

(6) Let $\Gamma \vdash \Delta = \Gamma \vdash \Theta, \forall x A[x]$. The premise $\Gamma \vdash \Theta, A[a_0]$ is derivable with order $< m$, where a_0 is the eigenvariable used. By 5.6, $\Gamma \vdash \Theta, A[a_1]$ is derivable with order $< m$, where a_1 is a new variable that does not occur in $\Phi, \Gamma \vdash \Delta$. By the I.H. $\Phi, \Gamma \vdash \Theta, A[a_1]$ is derivable with order $< m$ and thus so is $\Phi, \Gamma \vdash \Theta, \forall x A[x]$ with order $\leq m$ and eigenvariable a_1.

(7) We omit this case as it is similar to the previous.

(8) $\Gamma \vdash \Delta = \Theta, \Box\Omega \vdash \Box A, \Psi$. The premise $\forall\Omega, \Box\Omega \vdash A$ is derivable with order $< m$. Thus $\Phi, \Theta, \Box\Omega \vdash \Box A, \Psi$ is derivable with order $\leq m$ by an application of TR.

This case was argued about rule TR (and the I.H. was not used), but the

[31] Once again, we implicitly indicate the p.f. in each case considered, for example, $A \to B$ here.

proof for rule GLR is entirely similar, the presence of the diagonal formula $\Box A$ in the antecedent of the premise not adding any complexity. ■

Theorem 5.8 (Strengthening; cf. [30, 32]) For either GTKS or GLTS, if $\Gamma \vdash \Delta$ is derived with order m then $\Gamma \vdash \Delta, \Theta$ is derivable with order $\leq m$.

Proof Similar to the proof of 5.7. ■

Theorem 5.9 (Inversion rules; cf. [30, 32, 33, 29]) For either GTKS or GLTS, we have

(1) If $\Gamma, A \to B \vdash \Delta$ is derivable with order m, then each of $\Gamma, A \to \bot \vdash \Delta$ and $\Gamma, B \vdash \Delta$ are derivable with order $\leq m$.

(2) If $\Gamma \vdash \Delta, A \to B$ is derivable with order m, then $\Gamma, A \vdash \Delta, B$ is derivable with order $\leq m$.

(3) If $\Gamma \vdash \Delta, A \to \bot$ is derivable with order m, then $\Gamma, A \vdash \Delta$ is derivable with order $\leq m$.

(4) If $\Gamma, A \to \bot \vdash \Delta$ is derivable with order m, then $\Gamma \vdash \Delta, A$ is derivable with order $\leq m$.

(5) If $\Gamma \vdash \Delta, \forall x A\,[x]$ is derivable with order m, then $\Gamma \vdash \Delta, A\,[a]$ is derivable with order $\leq m$ (for any choice of a).

Proof By induction on the order of derivation m. We include the standard proof for a few cases and refer the reader to the literature for the ones we omit.

(1) If $\Gamma, A \to B \vdash \Delta$ is an axiom then so is $\Gamma \vdash \Delta$ since $A \to B$ is not atomic, and hence so are $\Gamma, A \to \bot \vdash \Delta$ and $\Gamma, B \vdash \Delta$.

For the induction step we have two cases:

- Case where $A \to B$ is the p.f. of rule (2) that derived $\Gamma, A \to B \vdash \Delta$. Then the rule premises, $\Gamma, A \to \bot \vdash \Delta$ and $\Gamma, B \vdash \Delta$ are each derived with order $< m$ by 5.4.

- Case where $A \to B$ is not the p.f. of the rule (k) (for $k = 2, 3, 4, 5, 6, 7, 8$) that derived $\Gamma, A \to B \vdash \Delta$.

  Consider the subcase where the Y-rule was the last applied with p.f. $X \to Y$ other than $A \to B$, that is, $\Gamma = \Gamma', X \to Y$. The premises $\Gamma', A \to B, X \to \bot \vdash \Delta$ and $\Gamma', A \to B, Y \vdash \Delta$ are derivable with order $< m$ each.

  By the I.H., the sequents $\Gamma', A \to \bot, X \to \bot \vdash \Delta$ and $\Gamma', B, X \to \bot \vdash \Delta$, as well as $\Gamma', Y, A \to \bot \vdash \Delta$ and $\Gamma', B, Y \vdash \Delta$ are also derivable with orders $< m$, Therefore, applying rule (2) to the first and third, and then to the second and fourth, we derive (with order $\leq m$) $\Gamma, A \to \bot \vdash \Delta$ and $\Gamma, B \vdash \Delta$, respectively.

  Similar argument for the I-rules 3–7.

  Finally, consider the subcase where the TR or GLR was used to derive $\Gamma, A \to B \vdash \Delta$. Here the subcase that $A \to B$ is a side formula cannot apply, since the s.f. are of the form $\forall X$ or $\Box X$. If on the other hand $A \to B$ is a weakening formula, then $\Gamma, A \to B \vdash \Delta$ was obtained with

order m from a sequent of the form $\Gamma' \vdash C$ that was derived with order $< m$. Applying TR (or GLR) to the latter, but changing the weakening part $A \to B$ to B, we obtain $\Gamma, B \vdash \Delta$ with order $\leq m$. Then we invoke again the rule on the same premise, this time applying the weakening part $A \to \bot$ to obtain $\Gamma, A \to \bot \vdash \Delta$ also with order $\leq m$.

(5) If $\Gamma \vdash \Delta, \forall x A[x]$ is an axiom then so is $\Gamma \vdash \Delta, A[a]$ for any choice of a since $\forall x A[x]$ is not atomic.

For the induction step we consider first the case where $\forall x A[x]$ is the p.f. of rule (6) that derived $\Gamma \vdash \Delta, \forall x A[x]$. Then $\Gamma \vdash \Delta, A[a_0]$ is derived with order $< m$ by 5.4, where a_0 is the eigenvariable used. By 5.6, for any a, $\Gamma \vdash \Delta, A[a]$ is derived with order $< m$ as well.[32]

Say, on the other hand, that $\forall x A[x]$ is not the p.f. in the rule (k) ($k = 2, 3, 4, 5, 6, 7$) that derived $\Gamma \vdash \Delta, \forall x A[x]$.

If the Y-rule was used to derive the previous sequent, then $\Gamma = \Gamma', X \to Y$ and the premises used were $\Gamma', X \to \bot \vdash \Delta, \forall x A[x]$ and $\Gamma', Y \vdash \Delta, \forall x A[x]$, each being derivable with order $< m$. By the I.H. each of $\Gamma', X \to \bot \vdash \Delta, A[b]$ and $\Gamma', Y \vdash \Delta, A[b]$ is derivable with order $< m$ —using a b that does not occur in $\Gamma' \cup \Delta \cup \{\forall x A[x], X, Y\}$. By rule (2), $\Gamma \vdash \Delta, A[b]$ is derivable with order $\leq m$, and an application of 5.6 allows us to use any free variable a in the place of b.

[32] Recall that a_0 occurs nowhere in $\Gamma \vdash \Delta$, so the substitution $(\Gamma \vdash \Delta, A[a_0])[a_0 := a]$ will be localised to $A[a_0]$.

The case of I-rules 3–7 is argued similarly.

Finally, let TR (or GLR) be the rule applied last to derive $\Gamma \vdash \Delta, \forall x A[x]$, from premise $\Gamma' \vdash C$, itself derived with order $< m$. Thus $\forall x A[x]$ must be a strengthening formula and reapplying TR (or GLR) with a different strengthening, $A[a]$ for any a we may choose, derives $\Gamma \vdash \Delta, A[a]$ with order $\leq m$.

■

5.2 Reducibility

Definition 5.10 In either GTKS or GLTS we define that a sequent $\Gamma \vdash \Delta$ is irreducible if one of the following applies:

(1) $\bot \in \Gamma$.

(2) There exists an atomic formula, A, such that $A \in \Gamma \cap \Delta$.

(3) The members of Γ are atomic or boxed and Δ is atomic, $\bot \notin \Gamma$ and $\Gamma \cap \Delta = \emptyset$.

We say that a sequent $\Gamma \vdash \Delta$ is reducible if that sequent is not irreducible. ■

Definition 5.11 In either GTKS or GLTS, and in the case of a reducible sequent, at least one rule from Definitions 5.1 and 5.2 applies backwards to yield a predecessor sequent. The predecessor relation between so related sequents, $\Gamma' \vdash \Delta'$ (predecessor) and $\Gamma \vdash \Delta$ we will denote by $\prec$, that is, $\Gamma' \vdash \Delta' \prec \Gamma \vdash \Delta$. ■

Remark 5.12 In either GTKS or GLTS the relation $\prec$ is well-founded —that is, there can be no infinite "descending" $\prec$-paths because each rule, (2)–(8), when

applied "backwards" from $\Gamma \vdash \Delta$, reduces the number of occurrences of one of the connectives $\rightarrow, \forall, \Box$ in $\Gamma \vdash \Delta$. [33]

The case of TR/GLR calls for some more elaboration: Each backwards application reduces the number of occurrences of $\Box$ in the succedent and so after a finite number of (backwards) steps neither of the two will be applicable. Now, each of these two rules introduces a $\forall\Omega$ in the antecedent, which will be eventually depleted by reverse applications of rule (7). This latter rule does not introduce any new TR/GLR-specific p.f. to the right of $\vdash$ that were not already subformulae of $\Gamma \vdash \Delta$. Finally, we note that a reverse application of GLR introduces a $\Box A$ in the antecedent (diagonal formula), but this is not a p.f. for any rule, and causes no thread backwards.

Thus one can do induction along $\prec$ or on the reducibility rank —$RR(\Gamma \vdash \Delta)$— of $\Gamma \vdash \Delta$, that is, the path length upwards from this sequent to an irreducible sequent. The minimal elements of this order are the irreducible sequents.

We note that, for any Φ and Ψ, we have $RR(\Phi, \Gamma \vdash \Delta, \Psi) \leq RR(\Gamma \vdash \Delta)$. If $\Gamma \vdash \Delta$ is derivable, then this is what 5.7 and 5.8 say. If not, then adding weakening (strengthening) to $\Gamma \vdash \Delta$ is effected by introducing it via applications of TR/GLR along a reverse path along $\prec$, from this sequent to an irreducible (but not an axiom) $\Gamma' \vdash \Delta'$; or by modifying the side formulae of $\Gamma' \vdash \Delta'$. Neither of these actions lengthen the path. ■

In order to explain reducibility more concretely, a couple of examples with

[33] The Y-rule, applied backwards, still has the $\rightarrow$ connective in one of the predecessor sequents. However rule (5), applied backwards, will remove it.

sample reducibility trees will be shown:

$$\begin{array}{c} \Gamma, \bot \vdash \Delta \\ \Big| \\ \Gamma \vdash \Delta, \bot \to \bot \end{array}$$

Example 5.13

In this example, $\Gamma, \bot \vdash \Delta$ is a leaf of the tree,[34] namely, an irreducible sequent because of the 5.10 case (1) ($\bot$ is in the hypothesis). Applying 5.1 rule (4) backwards on the the node $\Gamma \vdash \Delta, \bot \to \bot$, we can get the leaf sequent. By the reducibility definition, $\Gamma, \bot \vdash \Delta \prec \Gamma \vdash \Delta, \bot \to \bot$.

$$\begin{array}{c} \Gamma, A \vdash \Delta, A \\ \Big| \\ \Gamma \vdash \Delta, A \to A \end{array}$$

Example 5.14

Let's assume that the formula A is atomic. In this example, $\Gamma, A \vdash \Delta, A$ is a leaf of the tree, namely, an irreducible sequent because of the 5.10 case (2). Applying 5.1 rule (3) backwards on the the node $\Gamma \vdash \Delta, A \to A$, we can get the leaf sequent. By the reducibility definition, $\Gamma, A \vdash \Delta, A \prec \Gamma \vdash \Delta, A \to A$.

$$\begin{array}{c} A, B[a] \vdash C[b] \\ \Big| \\ A, \forall x B \vdash C[b] \end{array}$$

Example 5.15

[34] Recall that proof trees have the root at the bottom, and leafs at the top

In this example, A, B and C are assumed to be all distinct atomic formulae and the variable a is the only variable in formula B. Compared with the last example, $A, B[a] \vdash C[b]$ is an irreducible sequent (5.10 case (3)) but not an axiom because of the 5.1 rule (1). Please note that $A, \forall x B \vdash C[b]$ is not derivable because the leaf sequent is not axiom at all.

Example 5.16

$$\begin{array}{c} A, B \vdash C[b] \\ \Big| \\ A, B \vdash \forall x C \end{array}$$

In this example, again A, B and C are assumed to be all distinct atomic formulae and b is the only variable in formula C. Provided that b does not occur in the formulas A and B, the node $A, B \vdash C[b]$ is an irreducible sequent (5.10 case (3)) is not an axiom because of the 5.1 rule (1). And then, applying the reverse application of 5.1 rule (6) on $A, B \vdash \forall x C$, we can get $A, B \vdash C[b]$. Please note again: $A, B \vdash \forall x C$ is not derivable because the leaf sequent above is not axiom at all.

Example 5.17

$$\begin{array}{c} \ldots \\ \Big| \\ A[a], \Box A, \vdash B \\ \Big| \\ \forall A, \Box A \vdash B \\ \Big| \\ C, \Box A \vdash \Box B, D \end{array}$$

In this subtree, applying the modified "TR" rule backwards on the node $C, \Box A \vdash B, D$, the node $\forall A, \Box A \vdash B$ will be obtained. Assume now, for simplicity, that A has one free variable only, a. Thus, $\forall A$ is $\forall x A$. As explained in the remark section, universal quantifier $\forall x A$ introduced by TR rule at the left side of $\vdash$ will be deleted (by 5.1 rule (7)). There are no infinite "descending" $\prec$ paths. Additionally, note that $C, \Box A \vdash B, D$ is of higher reducibility rank than $\forall A, \Box A \vdash B$.

6 Cut Derivability in GTKS and GLTS

Proposition 6.1 The following two statements are equivalent for every formula A and any $\Gamma, \Delta, \Theta, \Omega, \Phi, \Psi$:

a. If $\Gamma \vdash \Delta, A$ and $A, \Theta \vdash \Omega$ then $\Gamma, \Theta \vdash \Delta, \Omega$ (cut admissibility).

b. If $A \to A, \Phi \vdash \Psi$ then $\Phi \vdash \Psi$.

Proof $a. \to b.$ Derivability of $A \to A, \Phi \vdash \Psi$ entails that of $A, \Phi \vdash \Psi$ and $\Phi \vdash \Psi, A$ (5.9, cases 1 and 4) and we are done by a.

$b. \to a.$ The assumption in $a.$ entails (by weakening/strengthening) the derivability of $\Gamma, \Theta \vdash \Delta, \Omega, A$ and $A, \Gamma, \Theta \vdash \Delta, \Omega$. By rule (2) we get $A \to A, \Gamma, \Theta \vdash \Delta, \Omega$. We are done by b. ■

Lemma 6.2 (Cut admissibility Lemma for GTKS) For any formula A, if $A \to A, \Gamma \vdash \Delta$ is derivable, then so is $\Gamma \vdash \Delta$.

Proof The proof is by induction on the ordinal

$$\alpha = \omega^2 \cdot C + \omega \cdot RR + m \tag{1}$$

where C is the *modified complexity* of A.[35] —this is the primary (P.I.) or main induction. A secondary induction (S.I.) is done along the $\prec$ relation on the $\Gamma \vdash \Delta$ "companion" of $A \to A$ —more accurately on $RR(\Gamma \vdash \Delta)$— and on occasion we do a "local induction" (L.I.) on the order of derivation of $A, \Gamma \vdash \Delta$, which we typically call m in this proof. Thus we will embark on a triple induction, where C is not allowed to increase during the induction step of either the S.I. or L.I., and neither C nor RR are allowed to increase during the induction step of L.I.

Case 1. A is atomic.

By invertibility (Theorem 5.9, case (1) followed by case (4)), both $A, \Gamma \vdash \Delta$ and $\Gamma \vdash \Delta, A$ are derivable. By a L.I. on the order of derivation m of $A, \Gamma \vdash \Delta$ we prove the derivability of $\Gamma \vdash \Delta$.

(i) Basis. If $m = 0$, what if $\Gamma \vdash \Delta$ is not itself an axiom? Then $A \in \Delta$, so $\Gamma \vdash \Delta = \Gamma \vdash \Delta, A$ is derivable, contradicting our "what if".

(ii) Let us now take a L.I.H. and assume that $A, \Gamma \vdash \Delta$ is obtained by one of the rules (2)–(7) with order m. Note that A cannot be the p.f. in the application of such rules.

[35] By modified complexity we mean the ordinal $\omega \cdot k + r$ where k counts $\Box$ occurrences and r counts the total of all $\to, \forall$ occurrences in A. Thus $(k, r) < \left(k+1, r^{'}\right)$ and $(k, r) < (k, r+1)$ for all $k, r, r^{'}$.

- Case where the "Y-rule" derived $A, \Gamma \vdash \Delta$: Then some $A, \Gamma' \vdash \Delta$ and $A, \Gamma'' \vdash \Delta$ (the rule's premises) are derivable each with order $< m$, and the same is true, by weakening 5.7, for $A, \Gamma, \Gamma' \vdash \Delta$ and $A, \Gamma, \Gamma'' \vdash \Delta$.

  Since $\Gamma, \Gamma' \vdash \Delta, A$ and $\Gamma, \Gamma'' \vdash \Delta, A$ are also derivable by weakening, the local I.H. yields the derivability of each of $\Gamma, \Gamma' \vdash \Delta$ and $\Gamma, \Gamma'' \vdash \Delta$, and an application of the Y-rule derives $\Gamma, \Gamma \vdash \Delta = \Gamma \vdash \Delta$ as needed.[36]

- Case where one of the "I-rules" (3)–(7) derived $A, \Gamma \vdash \Delta$. This is similar to and slightly simpler than the Y-case.

  Note that $\Gamma' \vdash \Delta \prec \Gamma \vdash \Delta$ and $\Gamma'' \vdash \Delta \prec \Gamma \vdash \Delta$, hence RR did not increase during this induction step (cf. also concluding part of Remark 5.12).

(iii) $A, \Gamma \vdash \Delta$ is obtained by rule (8). Thus, A is a weakening formula, but then $\Gamma \vdash \Delta$ is also derivable by omitting the weakening A (the L.I.H. was not needed in this case).

Case 2. $A = B \to C$.

By 5.9, cases (1) and (4), we can also derive $B \to C, \Gamma \vdash \Delta$ and $\Gamma \vdash \Delta, B \to C$; and, again by invertibility, we can derive $S_1 = \Gamma \vdash \Delta, B$ —thus

[36] Recall that what we are proving via this "local" induction is that if both $A, \Gamma \vdash \Delta$ and $\Gamma \vdash \Delta, A$, are provable then so is $\Gamma \vdash \Delta$. Thus, the acrobatics involving weakening are needed to ensure that the "and" holds: Even though, e.g., $A, \Gamma' \vdash \Delta$ is provable, we cannot necessarily expect that so is $\Gamma' \vdash \Delta, A$. But $\Gamma, \Gamma' \vdash \Delta, A$ is provable!

also $S_1' = \Gamma \vdash \Delta, C, B$— and $S_2 = C, \Gamma \vdash \Delta$ and also $S_3 = B, \Gamma \vdash \Delta, C$ by case (2). Now, we can derive $S_4 = B \to B, \Gamma \vdash \Delta, C$ from S_1' and S_3 and rule (2); similarly, we can derive $C \to C, B \to B, \Gamma \vdash \Delta$ from S_4 and the obvious weakening of S_2. We can finally apply the P.I.H. twice to get $\Gamma \vdash \Delta$.

Case 3. $A = \forall x B$. Now, $S = \forall x B, \Gamma \vdash \Delta$ and $\Gamma \vdash \Delta, \forall x B$ are derivable by 5.9. We do a L.I. on the order of derivation m of S, as before, to show that $\Gamma \vdash \Delta$ is derivable.

(i) Now, if S is an axiom, then so is $\Gamma \vdash \Delta$ since $\forall x B$ is not atomic.

(ii) Otherwise, let first $A, \Gamma \vdash \Delta$ be obtained by one of the rules (2)–(7) with order m. We have the cases,

(a) A is the p.f. in the derivation of $A, \Gamma \vdash \Delta$. Then $B[a], \Gamma \vdash \Delta$ is derivable for some a and so is $\Gamma \vdash \Delta, B[a]$ by 5.9, last case. Hence $B[a] \to B[a], \Gamma \vdash \Delta$ is, by rule (2), and we are done by the P.I.H. (the L.I.H was not needed here).

(b) A is not p.f. in the derivation of $A, \Gamma \vdash \Delta$, and $A, \Gamma \vdash \Delta$ is obtained by one of the rules (2)–(7) with order m.

- Case where the Y-rule derived $A, \Gamma \vdash \Delta$: Exactly as in the corresponding case under (ii) of Case 1.
- Case where some I-rule among 3–7 derived $A, \Gamma \vdash \Delta$. Again as in (ii) of Case 1.

The same note regarding RR as in Case 1(ii) applies here as well.

(iii) $A, \Gamma \vdash \Delta$ is obtained by rule (8). Exactly as (iii) under Case 1.

Case 4. $A = \Box B$.

(I) $\Gamma \vdash \Delta$ is irreducible. Thus, $\Box B, \Gamma \vdash \Delta$ is derivable as an initial sequent, which means that $\Gamma \vdash \Delta$ is also an initial sequent.

(II) $\Gamma \vdash \Delta$ is reducible. We have two subcases:

(i) $A, \Gamma \vdash \Delta$ is obtained by one of the rules (2)–(7) with order m. As A cannot be p.f. in any of rules (2)–(7), sub-subcase iib of Case 3 applies, and we have nothing further to add here. The note inserted at the end of Case 1(ii) applies here as well: RR did not increase.

(ii) (Adapting Brighton's approach ([8]) in this case.) Case where the only applicable rule to $\Gamma \vdash \Delta$ is (8). By invertibility, $S = \Box B, \Gamma \vdash \Delta$ and $S' = \Gamma \vdash \Delta, \Box B$ are derivable. Now, if $\Box B \in \Gamma$ or $\Box B \in \Delta$, then $S = \Gamma \vdash \Delta$ or $S' = \Gamma \vdash \Delta$ respectively, and we are done. So let $\Box B \notin \Gamma \cup \Delta$, and let us also pay no attention to the possibility that $\Box B$ is weakening/strengthening introduced by the TR rule, as then we are done immediately.

Thus, S and S' were obtained by proofs ending as:

$$\frac{S1}{S} = \frac{\forall B, \Box B, \forall \Gamma', \Box \Gamma' \vdash D}{\Box B, \underbrace{\Phi, \Box \Gamma'}_{\Gamma} \vdash \underbrace{\Box D, \Psi}_{\Delta}}$$

and

$$\frac{S2}{S'} = \frac{\forall\Gamma', \Box\Gamma' \vdash B}{\Phi, \Box\Gamma' \vdash \Box D, \Psi, \Box B}$$

Now, the derivable $S1, S2$ above can also derive

$$S3 = \Box B, \Box\Gamma' \vdash \Box D$$

$$S4 = \Box\Gamma' \vdash \Box B$$

and

$$S2' = \Box B, \forall\Gamma', \Box\Gamma' \vdash B, \text{ this by weakening,}$$

respectively. Now, we can obtain the derivable sequent $S5 = \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma' \vdash B$ from $S2'$ and $S4$ (via (2) and (5)), and thus also $S5' = \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma' \vdash \forall B$ by repeated application of rule (6) —note that the left hand side of $\vdash$ in $S5$ is closed. We can also obtain $S6 = \forall B, \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma' \vdash D$ from S1 and S4. Thus we can next obtain $S7 = \forall B \rightarrow \forall B, \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma' \vdash D$ from $S5'$ and $S6$. We can now apply the P.I.H. to obtain $S8 = \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma' \vdash D$ from S7 (recall that $\forall B$ has lower (modified) complexity than $\Box B$).

But $\forall\Gamma', \Box\Gamma' \vdash D \prec \Phi, \Box\Gamma' \vdash \Box D, \Psi = \Gamma \vdash \Delta$, thus, by S.I.H., $\forall\Gamma', \Box\Gamma' \vdash D$ is derivable. This, via TR, derives $\Phi, \Box\Gamma' \vdash \Box D, \Psi = \Gamma \vdash \Delta$. ■

Lemma 6.3 (Cut admissibility Lemma for GLTS) For any formula A, if $A \rightarrow A, \Gamma \vdash \Delta$ is derivable, then so is $\Gamma \vdash \Delta$.

Proof As in the proof of 6.2 except for Case 4(IIii): The only applicable rule to $\Gamma \vdash \Delta$ is GLR. By invertibility, $S = \Box B, \Gamma \vdash \Delta$ and $S^{'} = \Gamma \vdash \Delta, \Box B$ are derivable. Now, if $\Box B \in \Gamma$ or $\Box B \in \Delta$, then $S = \Gamma \vdash \Delta$ or $S' = \Gamma \vdash \Delta$ respectively, and we are done. So let $\Box B \notin \Gamma \cup \Delta$, and let us also pay no attention to the possibility that $\Box B$ is weakening/strengthening introduced by the GLR rule, as then we are done immediately.

Thus, S and S' were obtained by proofs ending as:

$$\frac{S1}{S} = \frac{\forall B, \Box B, \forall \Gamma^{'}, \Box \Gamma^{'}, \Box D \vdash D}{\Box B, \underbrace{\Phi, \Box \Gamma^{'}}_{\Gamma} \vdash \underbrace{\Box D, \Psi}_{\Delta}}$$

and

$$\frac{S2}{S'} = \frac{\forall \Gamma^{'}, \Box \Gamma^{'}, \Box B \vdash B}{\Phi, \Box \Gamma' \vdash \Box D, \Psi, \Box B}$$

Now, the derivable $S1, S2$ above can also derive

$$S3 = \Box B, \Box \Gamma^{'} \vdash \Box D$$

$$S4 = \Box \Gamma^{'} \vdash \Box B$$

respectively. Now, we can obtain the derivable sequent $S5 = \Box B \rightarrow \Box B, \forall \Gamma^{'}, \Box \Gamma^{'} \vdash B$ from $S2$ and $S4$ (via (2) and (5)), and thus also $S5' = \Box B \rightarrow \Box B, \forall \Gamma^{'}, \Box \Gamma^{'} \vdash$

$\forall B$ by repeated application of rule (6) —note that the left hand side of $\vdash$ in $S5$ is closed. We can also obtain $S6 = \forall B, \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma', \Box D \vdash D$ from S1 and S4. Thus we can next obtain $S7 = \forall B \rightarrow \forall B, \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma', \Box D \vdash D$ from $S5'$ and $S6$. We can now apply the P.I.H. to obtain $S8 = \Box B \rightarrow \Box B, \forall\Gamma', \Box\Gamma', \Box D \vdash D$ from S7 (recall that $\forall B$ has lower (modified) complexity than $\Box B$). But $\forall\Gamma', \Box\Gamma', \Box D \vdash D \prec \Phi, \Box\Gamma' \vdash \Box D, \Psi = \Gamma \vdash \Delta$, thus, by S.I.H., $\forall\Gamma', \Box\Gamma', \Box D \vdash D$ is derivable. This, via GLR, derives $\Phi, \Box\Gamma' \vdash \Box D, \Psi = \Gamma \vdash \Delta$. ■

Theorem 6.4 (Cut admissibility for GTKS and GLTS) In each of GTKS and GLTS, if $\Gamma \vdash \Delta, A$ and $A, \Theta \vdash \Omega$, then $\Gamma, \Theta \vdash \Delta, \Omega$.

Proof By 6.2 and 6.3 via 6.1. ■

7 Conclusion

The proof I provided in last section (5.1) has three induction hypotheses —the modified complexity of cut formula A, the reducibility rank RR of $\Gamma \vdash \Delta$ and the order of derivation of $A, \Gamma \vdash \Delta$. No doubt, this proof is of lower complexity than the original GLTS proof with four Induction Hypotheses and benefiting from the concept of reducibility, it is much shorter and more readable than the original proof of [32]. Here the observation about the simple formulation of cut admissibility "if $\Gamma, A \rightarrow A \vdash \Delta$ is derivable, then so is $\Gamma \vdash \Delta$" also helped a lot!

I am interested in further working on proof-theory for first-order modal logics, in particular that of the very versatile ML^3, evidently more much more capable

than QGL, in that it preserves the "full" main conservation result (C) (p.21), and admits cut elimination that QGL does not. There are many interesting questions to be explored about it, such as its (almost certain) arithmetical completeness raised in section 3. Can an ML^3 that is allowed primary necessitation (and thus becomes the QGL^b of [27]) benefit from the formulator techniques of [31]? If not, can these techniques be adapted? [24] proves that QGL does not enjoy the fixpoint property; but how about ML^3?

More research on ML^3's metamathematical properties is required. After my Masters, I hope to do research about these and related questions on first-order modal proof theory in general.

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