TOPOLOGICAL DYNAMICAL SYSTEMS METHODS IN EARLY-UNIVERSE COSMOLOGIES

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A DISSERTATION SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

DOCTORATE OF PHILOSOPHY

GRADUATE PROGRAM IN PHYSICS AND ASTRONOMY
YORK UNIVERSITY
TORONTO, ONTARIO
JULY 2014

Abstract

This dissertation describes some important problems that we have tried to solve with respect to the state of the early universe, that is, the universe shortly after the Big Bang. Standard early-universe cosmological approaches almost always assume a perfect-fluid isotropic and spatially homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) model to study the universe's evolution. The problem is that the early universe was in a hot, dense, and unstable state. Hence, perfect-fluid models which assume no dissipation may not be accurate at such early epochs in the universe's evolution. Our approach is to introduce terms in the Einstein field equations that allow the representation of these dissipative/viscous effects. In addition, we relax the condition of isotropy to obtain a class of anisotropic and spatially homogeneous cosmological models, known as the Bianchi models. Our research then is largely focused on studying the dynamics of these Bianchi models in the presence of viscous effects. We feel that studying the early universe in this context is more fruitful than the standard approaches mainly because our models are more realistic representations of the conditions of the early universe. Our technique for studying these models is also quite different than the standard approaches in the literature, in that, we use topological dynamical systems theory to study the early and late-time asymptotic behaviour of the cosmological model under consideration. Our work in this regard has been quite successful, and has led to a number of publications in the Physical Review which are listed in the main dissertation document.

Acknowledgements

This dissertation represents my final endeavour in the field of formal education, it is the final piece of schoolwork that I will submit, and I am extremely grateful for that. Just for the sake of perspective, the dissertation is the final result of twelve years of elementary and high school, forty-two courses at The University of Toronto which led to an Honours Bachelor of Science degree in the Physics specialist program, with a minor in Mathematics, four graduate courses (including a formidable PHYS 5000 - Quantum Mechanics course), a Master's thesis in theoretical and mathematical physics at York, three publications in academic journals and now a Doctorate degree in theoretical and mathematical physics at York University. It has been a long journey with many bumps and triumphs along the way, but it was certainly worth it, and many individuals helped me along the way, and this section is dedicated to acknowledging them.

First, I would like to sincerely thank the Almighty God and Sri Guru Granth Sahib Ji. Without their guidance and teachings, strength and wisdom, I simply would not be here today.

I would like to acknowledge the tremendous support given to me by my mother, father, brother and sister. My supervisor, Michael Haslam deserves a great deal of acknowledgement, firstly, for agreeing to supervise me for both my M.Sc. and Ph.D. degrees, and more importantly, to let me work on projects that I personally found interesting. I do not think I could even fathom the idea of graduate school if I could not pursue my own interests, and I think he recognized that, to which I am quite grateful for. I would like to also acknowledge Randy Lewis, who no doubt showed tremendous patience while listening to my many complaints of almost every single thing in the world. I would also like to thank members of my committee, Matt Johnson, Wendy Taylor, and Alexey Kuznetsov for their helpful comments and suggestions in improving my dissertation. I would also like to thank Michael De Robertis for several engaging discussions over the years, these will be missed. I also offer my gratitude to Marshall McCall for building a fantastic physics and astronomy program at York that has cemented its future for many years to come without doubt. I would also like to thank Paul Delaney, Carl Wolfe, and Sampa Bhadra for making teaching assistantship positions as smooth as possible. Finally, I would very much like to acknowledge Marlene Caplan and Lauren O'Brien who in my opinion, and I am sure many will agree *really* run the entire department! Marlene and Lauren spent countless hours addressing a countless number of issues and problems I brought before them, and showed nothing but grace. I and many others would be very lost at York, and half the time, would have no idea what to do.

Outside of York, I would also like to thank Charles Dyer, Peter Krieger, Joe Repka, Stephen Morris, Dylan Jones, Dan Miller, Teresa Baptista, Natalia Krasnopolskaia, Man-Duen Choi, Yuri Burda and George Elliott. My special thanks goes to George Ellis of The University of Cape Town, John D. Barrow of Cambridge, and Sigbjorn Hervik of The University of Stavanger who answered a great deal of questions I had on theoretical cosmology and helped me along the way by providing a wealth of resources for me to pursue my various research endeavours, I especially acknowledge George in this regard.

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President Josiah Bartlet*: Sweden has a 100 % literacy rate, Leo. 100%! How do they do that?*
Leo McGarry: *Well, maybe they don't and they also can't count.*

The West Wing

1 Introduction

This dissertation was largely inspired by the author's immense interests in dynamical systems theory, general relativity, and cosmology, and a deep desire to combine all three to produce something of interest and of value to researchers and students in all three fields.

Modern-day cosmology is largely focused on combining particle physics with general relativity to try to understand the very early universe and has resulted in various prescriptions to try to develop a quantum theory of gravity, inflation, string theory in the cosmology setting, and loop quantum gravity amongst other theories.

In contrast, our research is based on using ideas from topological dynamical systems theory to study spatially homogeneous anisotropic models in the presence of anisotropic matter. The motivation for this choice is as follows. Although Friedmann-LeMaître-Robertson-Walker (FLRW) models are indeed spatially homogeneous, they are a very special/restricted subclass of such models because of their isotropy [WE97]. By contrast, spatially homogeneous, and anisotropic models, which are known as the Bianchi models, reduce the Einstein field equations to ordinary differential equations, as do the FLRW models, but in the former, one can investigate much more general behaviour. One can have a representation of anisotropic modes, including vorticity and tilt, shear viscosity, and global magnetic fields, which could all occur in the real universe. One also obtains a new class of singularities. There is also some suggestion that the Bianchi models would be a good approximation to spatially inhomogeneous models where the spatial gradients are small [EMM12].

The next question is why we used a dynamical systems approach, as compared to the standard metric approach, which involves using the Einstein field equations directly to obtain evolution equations and constraints for a specific Bianchi type. The first reason is that the metric approach has been taken several times in the scientific literature, and in some sense, much of the work that has been done involves the assumption that metric potentials are related by some factor, which may or may not have physical significance. One then just proceeds through arduous pages of algebraic simplifications to obtain solutions to the field equations. Examples of this can be found in [SR06], [SB09], [PRS09], [SK09b], and references therein. Indeed, a cleaner metric approach was taken by [BKL70], [BK70], [BK76], and [LL80] in their study of the Mixmaster singularity, but we wished to used an approach that was more intuitive and frankly, mathematically more beautiful. Dynamical systems theory provides such a route. Since the Bianchi models reduce the Einstein field equations to ordinary differential equations, one can use dynamical systems theory to obtain some powerful *analytical* results. Unlike the metric approach, fixed points of the resulting field equations end up being solutions to the full Einstein field equations, which is obviously very useful.

We will first give a brief overview of the different techniques of dynamical systems theory that we have used throughout this work. These notes have been added to this dissertation for completion, but in no way represent the author's original work, rather, they are a compilation of years of study of dynamical systems theory taken mostly from [WE97], [AM78], [AAA$^+$97a], [BD05], and the various theorems and proofs in [LKW95], [CW92], and [HW93]. To aid in the discussion, we have included a glossary of terms at the end of this document in Section 8.2. Much of the foundation of my interest in dynamical systems theory was laid in the MAT244 - Ordinary Differential Equations course I took many years ago at The University of Toronto, followed by graduate-level courses in Nonlinear Physics, PHY460/1460 also at The University of Toronto at MATH6002 - Advanced Nonlinear Dynamics at York University. The main ideas are as follows.

1.1 Differential Equations and Flows

1.1.0.1 Autonomous Differential Equations and State Space

The theory of dynamical systems is used to study physical systems whose state at an instant of time t can be described by an element $\mathbf{x}$ of a *state space* X, and whose evolution is governed by an autonomous ODE on X,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{f} : X \to X$. We usually will deal with the case $X = \mathbb{R}^n$, $\mathbf{x} = (x^1, \ldots, x^n)$, such that Eq. (1) represents a system of n *ordinary differential equations*. So, for brevity we will refer to Eq. (1) as a *differential equation* (DE) on $\mathbb{R}^n$ and write it as:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{1.2}$$

The function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is interpreted as a *vector field* on $\mathbb{R}^n$,

$$\mathbf{f}(\mathbf{x}) = (f^1(\mathbf{x}), \ldots, f^n(\mathbf{x})).$$

Note that, we will assume that $\mathbf{f}$ is (at least) of class C^1 on $\mathbb{R}^n$.

There are certain types of DE that are of particular interest:

- *Linear DE:* If $\mathbf{f}$ is linear function, that is, $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, where A is a $n \times n$ matrix of real numbers, then the DE, Eq. (2) is linear:

$$\mathbf{x}' = A\mathbf{x}. \tag{1.3}$$

- *Gradient DE:* If $\mathbf{f}$ is the gradient of a C^2 scalar field $Z : \mathbb{R}^n \to \mathbb{R}$, that is, $\mathbf{f}(\mathbf{x}) = -\nabla Z(\mathbf{x})$, then Eq. (2) is a gradient DE:

$$\mathbf{x}' = -\nabla Z(\mathbf{x}). \tag{1.4}$$

- *Hamiltonian DE:* If $\mathbf{x} = (q^1, \ldots, q^n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$, and $\mathbf{f}(\mathbf{x}) = \left(\frac{\partial H}{\partial p_A}, -\frac{\partial H}{\partial q^A}\right)$, where $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a C^1 function called the *Hamiltonian*, then Eq. (2) is a Hamiltonian DE,

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}, \quad \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A}, \tag{1.5}$$

where $A = 1, \ldots, n$. The evolution of conservative mechanical systems can be described by a Hamiltonian DE, with H of the form

$$H(q^A, p_A) = \frac{1}{2} g^{AB}(q^C) p_A p_B + V(q^C),$$

where $g^{AB} = g^{BA}$. The first term then is clearly the kinetic energy, while $V(q^C)$ is the potential energy.

1.1.1 Basic Theorems

A *solution* of the DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ is a function $\psi : \mathbb{R} \to \mathbb{R}^n$ which satisfies:

$$\psi'(t) = \mathbf{f}(\psi(t)),$$

for all $t \in \mathbb{R}$, (or possibly only for t in a finite interval). The image of solution in $\mathbb{R}^n$ is called the *orbit of the DE*. Then, the evolution of the physical system is described by the motion of the state vector $\mathbf{x} \in \mathbb{R}^n$ along an orbit of the DE. The DE implies that the vector field $\mathbf{f}(\mathbf{x})$ is tangent to the orbit, and that it can be thought of as the velocity of the moving point in *state space*.

Theorem 1. *(Existence-Uniqueness) Consider the initial value problem*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(\mathbf{0}) = \mathbf{a} \in \mathbb{R}^n.$$

If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is of class* $C^1(\mathbb{R}^n)$*, then for all* $\mathbf{a} \in \mathbb{R}^n$*, there exists an interval* $(-\delta, \delta)$ *and an unique function* $\psi_{\mathbf{a}} : (-\delta, \delta) \to \mathbb{R}^n$ *such that*

$$\psi'_{\mathbf{a}}(t) = \mathbf{f}(\text{-}_{\mathbf{a}}(t)), \quad \psi_{\mathbf{a}}(0) = 0.$$

Proof. See Hirsch and Smale (1974, page 162). $\square$

The existence-uniqueness theorem is a local result, it says that there is a solution in some interval $(-\delta, \delta)$ centred at $t = 0$. But, in cosmological applications, we are interested in long-term betaviour of solutions, so we would like the solutions to be valid for all $t \in \mathbb{R}$. We can then extend this interval by successively reapplying the theorem, and obtain a *maximal interval of definition* of the solution $\psi_{\mathbf{a}}(t)$, (t_{min}, t_{max}).

Theorem 2. *(Maximality) Let $\psi_{\mathbf{a}}(t)$ be the unique solution of the DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} \in C^1(\mathbb{R}^n)$, which satisfies $\psi_{\mathbf{a}}(0) = \mathbf{a}$, and let (t_{min}, t_{max}) denote the maximal interval on which $\psi_{\mathbf{a}}(t)$ is defined. If t_{max} is finite, then:*

$$\lim_{t \to t_{max}^-} ||\psi_{\mathbf{a}}(t)|| = +\infty.$$

Proof. See Hirsch and Smale (1974, pages 171-2). □

Corollary 1. *Consider the DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^1(\mathbb{R}^n)$, and let $D \subset \mathbb{R}^n$ be a compact set. If a maximally extended solution $\psi_{\mathbf{a}}(t)$ lies in D, then the solution is defined for all $t \in \mathbb{R}$.*

Theorem 3. *(Extendibility) If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and there exists a constant M such that $||\mathbf{f}(\mathbf{x})|| \leq M||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$, then any solution of the DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is defined for all $t \in \mathbb{R}^n$.*

Proof. See Nemytskii and Stepanov (1960, Theorem 1.31, page 9) □

Theorem 3 is important as it implies that one can change a given DE $\mathbf{x}' = f(x)$, $\mathbf{x} \in \mathbb{R}^n$ such that the orbits are unchanged, but such that all solutions are defined for all $t \in \mathbb{R}^n$. What one has to do is to re-scale the vector field $\mathbf{f}$ to make it bounded,

$$\mathbf{f}(\mathbf{x}) \to \lambda(\mathbf{x})\mathbf{f}(\mathbf{x}),$$

where $\lambda(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is C^1 and positive on $\mathbb{R}^n$, in order to preserve the direction of time. An example is to choose $\lambda(\mathbf{x}) = [1 + ||\mathbf{f}(\mathbf{x})||]^{-1}$.

We then can state a corollary of the theorem.

Corollary 2. *If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1(\mathbb{R}^n)$, and $\lambda : \mathbb{R}^n \to \mathbb{R}$ is C^1 and positive, then $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and $\mathbf{x}' = \lambda(\mathbf{x})\mathbf{f}(\mathbf{x})$ have the same orbits, and λ can be chosen such that all solutions of the second DE are defined for all $t \in \mathbb{R}$.*

1.1.2 The Flow of a DE

If solutions of a DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ can be extended for all times, one can the define the flow of the DE.

Definition 1. *Consider a DE* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, *where* $\mathbf{f}$ *is of class* $C^1(\mathbb{R}^n)$, *whose solutions are defined for all* $t \in \mathbb{R}$. *Let* $\psi_{\mathbf{a}}(t)$ *be the unique maximal solution which satisfies* $\psi_{\mathbf{a}}(0) = \mathbf{a}$. *The* flow of the DE *is defined to be the one-parameter family of maps* $\{\phi_t\}_{t\in\mathbb{R}}$, *of* $\mathbb{R}^n$ *into itself such that*

$$\phi_t(\mathbf{a}) = \psi_{\mathbf{a}}(t) \quad \forall \mathbf{a} \in \mathbb{R}^n. \tag{1.6}$$

Note that, if the solutions of the DE are extendible for $t \to +\infty$, but not for $t \to -\infty$, we can define what is known as a *positive semi-flow* ϕ_t^+ of the DE by simply replacing '$t \in \mathbb{R}$' by '$t \in \mathbb{R}^+$' in the definition of flow, and vice-versa.

Theorem 4. *(Group Property of a Flow) Let* $\{\phi_t\}$ *be the flow of a DE* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. *Then*

$$\begin{aligned} \phi_0 &= I \quad \textit{The Identity Map}, \\ \phi_{t_1+t_2} &= \phi_{t_1} \circ \phi_{t_2}, \quad \forall t_1, t_2 \in \mathbb{R} \quad \textit{The Law of Composition}, \\ \phi_{-t} &= (\phi_t)^{-1}, \quad \forall t \in \mathbb{R} \quad \textit{The Inverse Map.} \end{aligned} \tag{1.7}$$

Note that: The flow ϕ_t is defined in terms of the solution functions $\psi_{\mathbf{a}}(t)$ of the DE by Eq. (6). Conceptually, the differences between the two are:

- *For a specific* $\mathbf{a} \in \mathbb{R}^n$, the solution $\psi_{\mathbf{a}} : \mathbb{R} \to \mathbb{R}^n$ gives the state of the system $\psi_{\mathbf{a}}(t) \in \mathbb{R}^n$ $\forall t \in \mathbb{R}$, with $\psi_{\mathbf{a}}(0) = \mathbf{a}$ initially.
- *For a specific* $t \in \mathbb{R}$, the flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ gives the state of the system $\phi_t(\mathbf{a}) \in \mathbb{R}^n$ at time t *for all initial states* $\mathbf{a}$.

Theorem 5. *(Smoothness of a Flow) If* $\mathbf{f} \in C^1(\mathbb{R}^n)$, *then the flow* $\{\phi_t\}$ *of the DE* $\mathbf{x}' = f(\mathbf{x})$ *consists of* C^1 *maps.*

In this section, we have thus seen that a DE on $\mathbb{R}^n$ determines a flow on $\mathbb{R}^n$, and vice-versa:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \leftrightarrow \{\phi_t\}_{t\in\mathbb{R}}.$$

For a linear DE (Eq.(3)), one can write the flow down explicitly:

$$\phi_t(\mathbf{x}) = \exp(At)\mathbf{x}, \quad \forall t \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{1.8}$$

The explicit flow for a nonlinear system is usually impossible to write down. To make the concept of flow clearer, we will allude to some examples from linear systems.

Let A be an $n \times n$ real matrix. Then the initial value problem,

$$x' = Ax, \quad x(0) = a \in \mathbb{R}^n$$

has the unique solution

$$x(t) = e^{tA}a, \quad \forall t \in \mathbb{R}.$$

We can prove this using the standard requirements of existence and uniqueness. First, with respect to existence, we see that if let $x(t) = e^{tA}a$, then

$$\frac{dx}{dt} = \frac{d(e^{tA}a)}{dt} = Ae^{tA}a = Ax,$$

$$x(0) = e^0 a = Ia = a,$$

which shows that the solution $x(t)$ satisfies the IVP.

With respect to uniqueness, we let $x(t)$ be any solution of the IVP. It follows that

$$\frac{d}{dt}\left[e^{-tA}x(t)\right] = 0,$$

which shows that $e^{-tA}x(t) = C$, a constant. The initial condition then implies that $C = a$ and hence $x(t) = e^{tA}a$.

The unique solution of the IVP is given by $x(t) = e^{tA}a$ for all t. Thus, for each $t \in \mathbb{R}$, the matrix e^{tA} maps

$$a \to e^{tA}a,$$

where a is the state at time $t = 0$, and e^{tA} the state at time t. The set $\{e^{tA}\}_{t \in \mathbb{R}}$ is a 1-parameter family of linear maps of $\mathbb{R}^n \to \mathbb{R}^n$, and is called the *linear flow* of the DE. We usually write

$$g^t = e^{tA}$$

to denote the flow. *The flow describes the evolution in time of the physical system for all possible initial states.*

1.1.3 Orbits and Invariant Sets

1.1.3.1 Classification of Orbits

Definition 2. *Given a DE, Eq. (2) and the associated flow ϕ_t, the* orbit *through $\mathbf{x}_0$, denoted by $\gamma(\mathbf{x}_0)$, is defined by*

$$\gamma(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \phi_t(\mathbf{x}_0), \quad t \in \mathbb{R}\}. \tag{1.9}$$

Given a flow ϕ_t, the points in the state space can be divided into two different categories:

1. equilibrium points

2. ordinary points.

Definition 3. *An* equilibrium point $\mathbf{x}_0$ *of the DE* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ *satisfies* $\mathbf{f}(\mathbf{x_0}) = 0$ *or equivalently,* $\phi_t(\mathbf{x}_0) = \mathbf{x}_0$*, for all $t \in \mathbb{R}$.*

Note that, if an orbit connects distinct equilibrium points, it is denoted a *heteroclinic orbit*. If an orbit connects an equilibrium point to itself, it is known as a *homoclinic orbit*.

1.1.4 Invariant sets

The concept of an invariant set is one of the most important concepts in the theory of dynamical systems.

Some examples of invariant sets are:

- Single orbits, such as equilibrium points and periodic orbits

- Stable, unstable, and centre manifolds

- $\alpha-$ and ω-limit sets

- Attractors

- Heteroclinic sequences and cycles

Definition 4. *A set $S \subset \mathbb{R}^n$ is an* invariant set *of the flow ϕ_t on $\mathbb{R}^n$ (or of the corresponding DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$) means that for all $\mathbf{x} \in S$ and for all $t \in \mathbb{R}$, $\phi_t(\mathbf{x}) \in S$.*

There is a simple way in which invariant sets arise. Given the DE (2) on $\mathbb{R}^n$ and a C^1 function $Z : \mathbb{R}^n \to \mathbb{R}$, the derivative of Z along solutions of (2) (that is, along the flow) is given by

$$Z' = \nabla Z \cdot \mathbf{f}(\mathbf{x}), \tag{1.10}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^n$.

Proposition 1. *Consider a DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ with flow ϕ_t. Let $Z : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function with satisfies $Z' = \alpha Z$, where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Then the subsets of $\mathbb{R}^n$ defined by $Z > 0$, $Z = 0$, $Z < 0$ are invariant sets of the flow ϕ_t.*

1.1.5 Monotone Functions

Definition 5. *Let ϕ_t be a flow on $\mathbb{R}^n$, let S be an invariant set of ϕ_t, and let $Z : S \to \mathbb{R}$ be a continuous function. Z is a* monotone decreasing (increasing) function for the flow on S *means that for all $\mathbf{x} \in S$, $Z(\phi_t(\mathbf{x}))$ is a monotone decreasing (increasing) function of t.*

Consider a DE (2), and the corresponding flow ϕ_t, and suppose that Z is C^1. If

$$Z' \equiv \nabla Z \cdot \mathbf{f} < 0, \quad \text{on } S, \tag{1.11}$$

then Z is monotone decreasing on S. If, on the other hand we consider the weaker condition

$$Z' \equiv \nabla Z \cdot \mathbf{f} \leq 0, \quad \text{on } S, \tag{1.12}$$

one can also conclude that Z is monotone decreasing on S. We usually use this definition to prove that a function is monotone.

We make the following proposition with respect to the existence of a monotone function on an invariant set S that that simplifies the orbits in S significantly.

Proposition 2. *Let $S \subset \mathbb{R}^n$ be an invariant set of a flow ϕ_t. If there exists a monotone function $Z : S \to \mathbb{R}$ on S, then S contains no equilibrium points, periodic points, recurrent orbits, or homoclinic orbits. Recall, that a system exhibits oscillatory behaviour if it has a period orbit, or it can return arbitrarily close to an earlier state if it has a recurrent orbit.*

1.1.6 Dulac Functions

Another method for excluding periodic orbits for a DE in the plane is provided by the classical Dulac's theorem, which involves finding a *Dulac function*, which we denote by λ in the theorem.

Theorem 6. *(Dulac) Consider a DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. If there is a C^1 function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla \cdot (\lambda \mathbf{f}) > 0$ (or < 0) in a simply-connected open set $S \subset \mathbb{R}^2$, then there are no periodic orbits in S.*

1.1.7 Behaviour Near Equilibrium Points

1.1.7.1 Linear DEs in $\mathbb{R}^n$

Given a linear DE $\mathbf{x}' = A\mathbf{x}$ on $\mathbb{R}^n$, we consider the eigenvalues of A (complex in general, and not necessarily distinct) and the associated generalized eigenvectors. We define three subspaces of $\mathbb{R}^n$,

- *stable subspace*, $E^s = span(\mathbf{s_1}, \ldots, \mathbf{s_{n_s}})$,
- *unstable subspace*, $E^u = span(\mathbf{u_1}, \ldots, \mathbf{u_{n_u}})$,
- *centre subspace*, $E^c = span(\mathbf{c_1}, \ldots, \mathbf{c_{n_c}})$,

where $\mathbf{s_1}, \ldots, \mathbf{s_{n_s}}$ are the generalized eigenvectors whose eigenvalues have negative real parts, $\mathbf{u_1}, \ldots, \mathbf{u_{n_u}}$ are those whose eigenvalues have positive real parts, and $\mathbf{c_1}, \ldots, \mathbf{c_{n_c}}$ are those whose eigenvalues have zero real parts. It is well-known then that

$$E^s \oplus E^u \oplus E^c = \mathbb{R}^n,$$

and that

$$\begin{aligned}
\mathbf{x} \in E^s \Rightarrow \lim_{t\to+\infty} e^{\mathbf{A}t}\mathbf{x} &= \mathbf{0}, \\
\mathbf{x} \in E^u \Rightarrow \lim_{t\to-\infty} e^{\mathbf{A}t}\mathbf{x} &= \mathbf{0}.
\end{aligned}$$

These statements basically describe the asymptotic behaviour of the system: All initial states in the stable subspace are attracted to the equilibrium point $\mathbf{0}$, while all initial states in the unstable subspace are repelled by 0. In particular, if $\dim E^s = n$, *all* initial states are attracted to $\mathbf{0}$, which is referred to as a *linear sink*, while if $\dim E^u = n$, *all* initial states are repelled by $\mathbf{0}$, which is referred to as a *linear source*.

1.1.8 Linearization and The Hartman-Grobman Theorem

We will consider a DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$, where $\mathbf{f}$ is of class C^1. If $\mathbf{a}$ is an equilibrium point ($\mathbf{f}(\mathbf{a}) = \mathbf{0}$), the linear approximation of $\mathbf{f}$ at $\mathbf{a}$ becomes

$$\mathbf{f}(\mathbf{x}) \approx D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}),$$

where

$$D\mathbf{f}(\mathbf{a}) = \left(\frac{\partial f_i}{\partial x_j}\right)_{\mathbf{x}=\mathbf{a}} \tag{1.13}$$

is the derivative matrix of $\mathbf{f}$. Thus, with the given DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, we associate the linear DE

$$\mathbf{u}' = D\mathbf{f}(\mathbf{a})\mathbf{u}, \tag{1.14}$$

where $\mathbf{u} = \mathbf{x} - \mathbf{a}$, called the linearization of the DE at the equilibrium point $\mathbf{a}$. The hope is that the solutions of Eq. (14) will approximate the solutions of the non-linear DE in a neighbourhood of the equilibrium point $\mathbf{a}$. This will be true provided that the equilibrium point is hyperbolic.

Definition 6. *Given a DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$, an equilibrium point $\mathbf{x} = \mathbf{a}$ is* hyperbolic *means that all eigenvalues of $D\mathbf{f}(\mathbf{a})$ have non-zero real part.*

Definition 7. *Two flows ϕ_t and $\tilde{\phi}_t$ on $\mathbb{R}^n$ are* topologically equivalent *means that there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ which maps orbits of ϕ_t onto orbits of $\tilde{\phi}_t$, preserving their orientation. Recall, a map $h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism on $\mathbb{R}^n$ iff*

1. *h is one-to-one and onto,*

2. *h is continuous,*

3. *h^{-1} is continuous.*

A function h is one-to-one, also known as injective *if $\forall a, b \in A, \quad h(a) = h(b) \rightarrow a = b$. A function h is onto, also known as surjective if we suppose that $h : X \rightarrow Y$, and that $\forall y \in Y, \exists x \in X, \quad h(x) = y$, in other words h is a bijective function.*

Theorem 7. *(Hartman-Grobman) Consider a DE $\mathbf{x}' = (\mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$, where $\mathbf{f}$ is of class C^1, with flow ϕ_t. If $\mathbf{a}$ is a hyperbolic equilibrium point of the DE, then there exists a neighbourhood N of $\mathbf{a}$ on which ϕ_t is topologically equivalent to the flow of the linearization of the DE at $\mathbf{a}$.*

The theorem says that in the neighbourhood N of the equilibrium point $\mathbf{a}$, the orbits of the DE can be continuously deformed into the orbits of the linearization, that is, qualitatively the orbits are the same. This motivates the next definition.

Definition 8. *An equilibrium point $\mathbf{a}$ of a DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^n$ is called a* local sink *(*local source*) whenever the eigenvalues of the derivative matrix $D\mathbf{f}(\mathbf{a})$ satisfy $Re(\lambda_i) < 0 (Re(\lambda_i) > 0)$ for $i = 1, \ldots, n$. A hyperbolic equilibrium point that is neither a local source or sink is called a* saddle point.

1.1.9 Stable, Unstable, and Centre Manifolds

If $\mathbf{a}$ is an equilibrium point of the DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ it is useful to know which orbits are attracted to $\mathbf{a}$ as $t \rightarrow +\infty$, and which are repelled. We can generalize the idea of the stable, unstable, and centre subspaces defined from before. The *stable manifold* W^s of an equilibrium point $\mathbf{a}$ is a differentiable manifold that is tangent to the stable subspace E^s at $\mathbf{a}$ and such that all orbits in W^s are asymptotic to $\mathbf{a}$ as $t \rightarrow +\infty$. Similarly, if the *unstable manifold* W^u of an equilibrium point $\mathbf{a}$ is a differentiable manifold that is tangent to the unstable subspace E^u at $\mathbf{a}$ and such that all orbits in W^u are asymptotic to $\mathbf{a}$ as $t \rightarrow -\infty$. Finally, a *centre manifold* W^c of an equilibrium point $\mathbf{a}$ is a differentiable manifold that is tangent to the centre subspace E^c at $\mathbf{a}$. The orbits in W^c are not determined by linearization. The dimension of these manifolds is determined by the dimension of the corresponding subspace.

Assuming that the equilibrium point for convenience has been translated to the origin, we can state the following theorem.

Theorem 8. *(Invariant Manifolds) Let $\mathbf{x} = \mathbf{0}$ be an equilibrium point of the DE $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ and let E^s, E^u, and E^c denote the stable, unstable, and centre subspaces of the linearization at $\mathbf{0}$. Then there exists*

- *W^s tangent to E^s at $\mathbf{0}$,*
- *W^u tangent to E^u at $\mathbf{0}$,*
- *W^c tangent to E^c at $\mathbf{0}$.*

1.1.10 Non-Isolated Equilibrium Points

We are also interested in the case where the DE admits *non-isolated* equilibrium points, for example, a curve C of equilibrium points, called an *equilibrium set*. In the case of a curve of equilibrium points, the $n \times n$ matrix $D\mathbf{f}(\mathbf{x}_0)$ necessarily has one zero eigenvalue at each point $\mathbf{x}_0$ of the equilibrium set (r zero eigenvalues for an r-dimensional equilibrium set).

Although these equilibrium points are necessarily non-hyperbolic, we can still apply the Invariant Manifold theorem. Suppose that each point of the equilibrium set C, assumed to be a curve, has a stable manifold of dimension n_s. The union of these manifolds forms an $(n_s + 1)$-dimensional set whose orbits approach a point C as $t \to +\infty$, this is referred to as the *stable set* of the equilibrium set C. Similarly, one can define the *unstable set* of C, of dimension $n_u + 1$ say, with $n_u + n_s \leq n - 1$. If $n_s = n - 1$, we say that the equilibrium set C is a *local sink* for the DE, and if $n_u = n - 1$, we say that C is a local source. In general, if $n_s + n_u = n - 1$, that is, the equilibrium set only has one zero eigenvalue at each point, and all other eigenvalues have $Re(\lambda) \neq 0$, the equilibrium set is called *normally hyperbolic*.

1.1.11 Asymptotic Evolution and Intermediate Evolution

One of the main goals of dynamical systems theory is to determine future asymptotic behaviour ($t \to +\infty$), because on is interested in the long-term evolution of the corresponding physical system. In cosmology, one is also interested in the asymptotic behaviour near the initial singularity, that is, into the past. To apply dynamical systems theory to cosmology, we need to introduce a dimensionless time variable which tends to $-\infty$ at the initial singularity, and to $+\infty$ at late time.

1.1.11.1 Limits Sets and Attractors

The simplest type of asymptotic behaviour is that the physical system, starting in state $\mathbf{a}$, approaches an equilibrium state as $t \to +\infty$, that is, the orbit through $\mathbf{a}$ approaches an equilibrium point $\mathbf{x}^*$. We say that the ω-limit set of $\mathbf{a}$ is $\mathbf{x}^*$ and write $\omega(\mathbf{a}) = \{\mathbf{x}^*\}$.

Definition 9. *Let ϕ_t be a flow on $\mathbb{R}^n$, and let $\mathbf{a} \in \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is an* ω-limit point of $\mathbf{a}$ *means that $\exists$ a sequence $t_n \to +\infty$ such that $\lim_{n\to\infty} \phi_{t_n}(\mathbf{a}) = \mathbf{x}$. The set of all ω-limit points of $\mathbf{a}$ is called* the ω-limit set of $\mathbf{a}$, denoted $\omega(\mathbf{a})$. *Note that, the α-limit set is defined similarly, but with $t_n \to -\infty$.*

Theorem 9. *(ω-limit sets) Let ϕ_t be a flow on $\mathbb{R}^n$. For all $\mathbf{a}$, the ω-limit set $\omega(\mathbf{a})$ is a closed, invariant set. If the positive orbit through $\mathbf{a}$ is bounded, then $\omega(\mathbf{a})$ is non-empty and connected.*

Definition 10. *Given a flow ϕ_t on $\mathbb{R}^n$, the* future attractor *A^+ is the smallest closed invariant set such that $\omega(\mathbf{a}) \subset A^+$ for all $\mathbf{a} \in \mathbb{R}^n$ apart from a set of measure zero.*

1.1.12 Asymptotic Behaviour in Higher Dimensions

Theorem 10. *(LaSalle Invariance Principle) Consider a DE with flow as usual. Let S be a closed, bounded, and positively invariant set of ϕ_t and let Z be a C^1 monotone function. Then $\forall \mathbf{x}_0 \in S$,*

$$\omega(\mathbf{x}_0) \subseteq \{\mathbf{x} \in S | Z' = 0\},$$

where $Z' = \nabla Z \cdot \mathbf{f}$.

Theorem 11. *(The Monotonicity Principle) This follows from Proposition A1, [LKW95] and says that if ϕ_t is a flow on $\mathbb{R}^n$ with S an invariant set, and if $Z : S \to \mathbb{R}$ is a C^1 function whose range is the interval (a, b), where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $a < b$, then if Z is monotone decreasing on orbits in S, for all $\mathbf{x} \in S$ we have that $\omega(\mathbf{x}) \subseteq \left\{\mathbf{s} \in \bar{S}\backslash S : \lim_{\mathbf{y}\to\mathbf{s}} Z(\mathbf{y}) \neq b\right\}$, $\alpha(\mathbf{x}) \subseteq \left\{\mathbf{s} \in \bar{S}\backslash S : \lim_{\mathbf{y}\to\mathbf{s}} Z(\mathbf{y}) \neq a\right\}$.*

Theorem 12. *(Lyapunov Functions) Following Pages 24 and 25 of [AAA+97a], we note that a differentiable function Z is called a* Lyapunov function *for a singular point $\mathbf{x}_0$ of a vector field $(\mathbf{f}(\mathbf{x})$ if Z is defined on a neighborhood of $\mathbf{x_0}$ and has a local minimum at this point, and the derivative of Z along $\mathbf{f}(\mathbf{x})$ is nonpositive. Then a singular point of a differentiable vector field for which a Lyapunov function exists is stable.*

Theorem 13. *(Chetaev Functions) Following pages 24 and 25 of [AAA+97b], we note that a differentiable function Z is called a* Chetaev function *for a singular point $\mathbf{x_0}$ of a vector field $\mathbf{f}(\mathbf{x})$ if Z is defined on a domain W whose boundary contains $\mathbf{x}_0$, the part of the boundary of W is strictly contained in a sufficiently small ball with its center $\mathbf{x}_0$ removed is a piecewise-smooth, C^1 hypersurface along which $\mathbf{f}(\mathbf{x})$ points into the interior of the domain, that is,*

$$Z(\mathbf{x}) \to 0, \text{ as } \mathbf{x} \to \mathbf{x}_0, \quad \mathbf{x} \in W; \quad Z > 0, \quad \nabla Z \cdot \mathbf{f}(\mathbf{x}) > 0 \in W. \tag{1.15}$$

A singular point of a C^1 vector field for which a Chetaev function exists is unstable.

1.1.13 Intermediate Behaviour

Definition 11. *A* finite heteroclinic sequence *is a set of equilibrium points $E_0, E_1, \ldots, E_n$, where E_0 is a local source, E_n is a local sink, and the rest are saddles, such that there is a heteroclinic orbit which joins E_{i-1} to E_i, for $i = 1, \ldots, n$, which is an orbit that connects distinct equilibrium points.*

1.1.14 Dynamical Systems in Cosmology

We discuss some of the history of dynamical systems techniques in cosmology in what follows below.

Before continuing, to illustrate the power of dynamical systems methods in cosmology, we discuss their application to FLRW universes. In the standard approach in cosmology, one begins with the FLRW metric

$$ds^2 = -dt^2 + l^2(t)d\Omega^2, \tag{1.16}$$

where

$$d\Omega^2 = dr^2 + f^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.17}$$

with

$$f(r) = \sin r, \quad r, \quad \text{or} \quad \sinh r, \tag{1.18}$$

depending on whether the FLRW model under consideration is of $k = +1$, $k = 0$, or $k = -1$ respectively. In this context, k refers to the spatial curvature of the model. That is, $k = +1$ refers to spatially closed FLRW models, $k = 0$ refers to spatially flat FLRW models, and $k = -1$ refers to FLRW models with hyperbolic spatial sections. The FLRW models, necessarily due to their symmetry, imply that the energy-momentum tensor takes the form of a perfect fluid:

$$T_{ab} = \mu u_a u_b + p\left(g_{ab} + u_a u_b\right). \tag{1.19}$$

One then uses the Einstein field equations,

$$R_{ab} - \frac{1}{2}g_{ab}R = kT_{ab}, \tag{1.20}$$

to obtain the energy-density evolution equation,

$$\dot{\mu} = -3\frac{\dot{l}}{l}\left(\mu + p\right), \tag{1.21}$$

the Raychaudhuri equation [WE97],

$$\frac{\ddot{l}}{l} = -\frac{1}{6}\left(\mu + 3p\right), \tag{1.22}$$

and the Friedmann equation [WE97],

$$3\frac{\dot{l}^2}{l^2} = \mu - \frac{3k}{l^2}. \tag{1.23}$$

These equations determine the evolution of all single-fluid FLRW models. However, let us continue and define the Hubble scalar,

$$H = \frac{\dot{l}}{l}, \tag{1.24}$$

the deceleration parameter,

$$q = -\frac{\ddot{l}l}{\dot{l}^2}, \tag{1.25}$$

and the density parameter,

$$\Omega = \frac{\mu}{3H^2}. \tag{1.26}$$

It can be shown that q and Ω are now dimensionless, while H has dimensions of inverse time. Let us also define a dimensionless time variable τ, such that

$$l = l_0 e^{\tau}. \tag{1.27}$$

One then obtains that

$$\frac{dt}{d\tau} = \frac{1}{H}, \tag{1.28}$$

$$\frac{dH}{d\tau} = -(1+q)H, \tag{1.29}$$

and

$$q = \frac{1}{2}\left[3(w+1) - 2\right]\Omega, \tag{1.30}$$

where $-1 \leq w \leq 1$ is an equation of state parameter corresponding to the type of fluid. One then obtains the key equation as

$$\frac{d\Omega}{d\tau} = -\left[3(w+1) - 2\right]\left[1 - \Omega\right]\Omega. \tag{1.31}$$

This *single* equation describes the evolution of all single-fluid FLRW models with barotropic equation of state $p = w\mu$. One can use rather simple methods from dynamical systems theory to first note that this equation has two fixed points: $\Omega = 1$, which corresponds to the flat FLRW universe, and $\Omega = 0$, which corresponds to the Milne universe, which is a vacuum $k = -1$ perfect-fluid solution to the Einstein field equations, and is typically given (in homothetic form) [WE97] as

$$ds^2 = -dt^2 + t^2\left[dr^2 + \sinh^2 r\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{1.32}$$

Upon analyzing the phase space defined by $\Omega \geq 0$, the point $\Omega = 1$ is a source of the system if $w > -1/3$ and a sink of the system if $w < -1/3$. Indeed, $\Omega = 1$ is a saddle if and only if $w = -1/3$. Therefore, all single-fluid

FLRW models will evolve away from a flat FLRW universe if $w > -1/3$, but will be asymptotically close to it if $w < -1/3$. When $w = -1/3$ exactly, some orbits of the dynamical system will be attracted to the flat FLRW solution, while others will be repelled by it. As shown in [WE97], one can generalize Eq. (1.31) further by writing it independently of any assumptions about the source terms as

$$\frac{d\Omega}{d\tau} = -2q\left(1-\Omega\right). \tag{1.33}$$

Also, in the case of n fluids, one can write for Ω,

$$\Omega = \sum_{A=1}^{n} \Omega_A, \tag{1.34}$$

and obtain

$$\frac{d\Omega_A}{d\tau} = \left[2q - 3\left(w_A + 1\right) - 2\right]\Omega_A, \quad A = 1, \ldots, n, \tag{1.35}$$

where

$$q = \frac{1}{2}\sum_{A=1}^{n}\left[3\left(w+1\right)-2\right]\Omega_A, \tag{1.36}$$

and w_A are the equation of state parameters. One also sees from a dynamical systems analysis that $\Omega < 1$, $\Omega = 1$, and $\Omega > 1$ all define invariant sets of the flow. As one can see, dynamical systems methods in this case provide a much richer framework even in the special case of FLRW models compared to the standard metric approach.

An important issue with respect to Bianchi models is their connection to inflation. This is deeply related to whether the specific Bianchi model under investigation isotropizes at early or late times. Indeed, isotropization may occur (as we show below) regardless of an early inflationary phase/de Sitter-like epoch [EMM12]. As discussed in [RE86], inflation can only occur in Bianchi models if there is not a significant amount of anisotropy to begin with, and it is not entirely clear that both shear and spatial deviations from flatness are removed in all inflating cases [RM88]. Therefore, it can be said that some Bianchi models isotropize due to inflation, and others do not. In general, we have the following two theorems related to the evolution of Bianchi models [EMM12].

1. *Bianchi Evolution Theorem (I):* Consider a family of Bianchi models that allow intermediate isotropization. Define an ϵ-neighbourhood of an FLRW model as a region in state space where all geometrical and physical quantities are closer than ϵ to their values in an FLRW model. Choose a time scale L. Then, no matter how small ϵ and how large L, there is an open set of Bianchi models in the state space such that each model spends longer than L within the corresponding ϵ-neighbourhood of the FLRW model.

2. *Bianchi Evolution Theorem (II):* In each set of Bianchi models of a type admitting intermediate isotropization, there will be spatially homogeneous models that are linearizations of these Bianchi models about FLRW models. These perturbation modes will occur in any almost-FLRW model that is generic rather than fine-tuned; however, the exact models approximated by these linearizations will be quite unlike FLRW models at very early and very late times.

With respect to the first theorem, one can therefore conclude that there exist many Bianchi models that are compatible with observations and therefore viable models of the real universe. With respect to the second theorem, the perturbation modes can exist as linearizations of the FLRW model. If they are found not to occur, then very special initial conditions have been chosen to set these modes to vanish. So, if one wishes to pursue generality arguments, such perturbation modes will occur in the real universe. When they occur, they will only occur at early and late times, but the catch is, they will grow until the model has deviated very far from an FLRW geometry.

In this work, which was completed with Professor M.C. Haslam, we used all of the aforementioned as motivation to study and develop four cosmological models, which we now list:

1. *Exploring Vacuum Energy in a Two-Fluid Bianchi Type I Universe*, Ikjyot Singh Kohli and Michael C. Haslam, Submitted. March 2014, arXiv: 1402.1967. In this work, we considered a Bianchi Type I model, which is spatially flat and anisotropic in the presence of a fluid with bulk viscosity and a fluid representing vacuum energy. We showed that what we consider to be vacuum energy in the present-day universe could actually be a function of bulk viscosity. We also showed that it is possible for

such models to isotropize without going through a de Sitter-like epoch. We used a dynamical systems approach based on the method of orthonormal frames to study the dynamics of a two-fluid, non-tilted Bianchi Type I cosmological model. In our model, one of the fluids is a fluid with bulk viscosity, while the other fluid assumes the role of a cosmological constant and represents non-negative vacuum energy. We began by completing a detailed fixed-point analysis of the system, which gave information about the local sinks, sources and saddles. We then proceeded to analyze the global features of the dynamical system by using topological methods such as finding Lyapunov and Chetaev functions [AAA+97b], and finding the α- and ω-limit sets using the LaSalle invariance principle. The fixed points found were a flat Friedmann-LeMaître-Robertson-Walker (FLRW) universe with no vacuum energy, a de Sitter universe, a flat FLRW universe with both vacuum and non-vacuum energy, and a Kasner quarter-circle universe. We also showed in this chapter that the vacuum energy we observe in our present-day universe could actually be a result of the bulk viscosity of the ordinary matter in the universe, and proceeded to calculate feasible values of the bulk viscous coefficient based on observations reported in the Planck data [A+13]. We concluded the paper with some numerical experiments that shed further light on the global dynamics of the system.

2. *Dynamics of a Closed Viscous Universe*, Ikjyot Singh Kohli and Michael C. Haslam, Physical Review D, vol. 89 Issue 4, id 043518 (2014), arXiv: 1311.0389. In this work, we considered a Bianchi Type IX model, which is spatially closed and anisotropic. By far, this was our most detailed work, as it is the first article in the scientific literature to describe fully the dynamics of a closed universe. We discovered two new solutions to the Einstein field equations. This work can have potentially far-reaching applications as it showed that it is plausible that in spite of living in a spatially flat universe, our universe can be closed, hence, being of a torus-like, S^3 topology. We used a dynamical systems approach based on the method of orthonormal frames to study the dynamics of a non-tilted Bianchi Type IX cosmological model with a bulk and shear viscous fluid source. We began by completing a detailed fixed-point analysis which gave the local sinks, sources and saddles of the dynamical system.

We then analyzed the global dynamics by finding the α-and ω-limit sets which gave an idea of the past and future asymptotic behaviour of the system. The fixed points were found to be a flat Friedmann-LeMaître-Robertson-Walker (FLRW) solution [WE97], Bianchi Type II solution [WE97], Kasner circle [WE97], Jacobs disc [WE97], Bianchi Type VII_0 solutions [WE97], and several closed FLRW solutions in addition to the Einstein static universe solution. Each equilibrium point was described in both its expanding and contracting epochs. We concluded the paper with some numerical experiments that shed light on the global dynamics of the system along with its heteroclinic orbits. With respect to past asymptotic states, we were able to conclude that the Jacobs disc in the expanding epoch was a source of the system along with the flat FLRW solution in a contracting epoch. With respect to future asymptotic states, we were able to show that the flat FLRW solution in an expanding epoch along with the Jacobs disc in the contracting epoch were sinks of the system. We were also able to demonstrate a new result with respect to the Einstein static universe. Namely, we gave certain conditions on the parameter space such that the Einstein static universe has an associated stable subspace. We were however, not able to conclusively say anything about whether a closed FLRW model could be a past or future asymptotic state of the model.

3. *Dynamical systems approach to a Bianchi type I viscous magnetohydrodynamic model*, Ikjyot Singh Kohli and Michael C. Haslam, Physical Review D, vol. 88, Issue 6, id 063518 (2013), arXiv: 1304.8042. In this work, we described in detail a spatially flat anisotropic universe in the presence of a primordial magnetic field and viscous fluid with both bulk and shear viscosity. We gave precise conditions for the generation of a primordial magnetic field in the early universe from a Kasner-like state in terms of values of the equation-of-state parameter, and the bulk and shear viscous coefficients. In addition, we also discovered a new solution to the Einstein field equations. We used the expansion-normalized variables approach to study the dynamics of a non-tilted Bianchi Type I cosmological model with both a homogeneous magnetic field and a viscous fluid. In our model the perfect magnetohydrodynamic approximation was made, and both bulk and shear viscous effects were retained. The dynamical

system was studied in detail through a fixed-point analysis, which determined the local sink and source behaviour of the system. We showed that the fixed points may be associated with Kasner-type solutions, a flat universe FLRW solution, and interestingly, a new solution to the Einstein field equations involving non-zero magnetic fields, and non-zero viscous coefficients. It was further shown that for certain values of the bulk and shear viscosity and equation of state parameters, the model isotropizes at late times.

4. *Future asymptotic behaviour of a nontilted Bianchi type IV viscous model*, Ikjyot Singh Kohli and Michael C. Haslam, Physical Review D, vol. 87, Issue 6, id 063006 (2013), arXiv: 1207.6132. In this chapter, we studied a complicated Bianchi type model of class B (the three discussed previously were all of class A). Bianchi models of class B are more complex because they contain non-diagonal shear components as well as additional constraints on the dynamical state space. It is of interest to note that Bianchi Type IV models belong to the plane-wave class of solutions to the Einstein field equations and can therefore be of significance with respect to the the recent discovery of gravitational waves [BIC14]. The future asymptotic behaviour of a non-titled Bianchi Type IV viscous fluid model was analyzed. In particular, we considered the case of a viscous fluid without heat conduction, and constant expansion-normalized bulk and shear viscosity coefficients. We showed using dynamical systems theory that the only future attracting equilibrium points are the flat Friedmann-Lemaître (FL) solution [WE97], the open FL solution [WE97] and the isotropic Milne universe solution [WE97]. We also showed that bifurcations exist with respect to an increasing expansion-normalized bulk viscosity coefficient. It was finally shown through an extensive numerical analysis, that the dynamical system isotropizes at late times.

Therefore, much of the work in this dissertation has been reproduced from published work. Due to not wanting the dissertation document to extend several hundred pages, the dissertation is not completely self-contained. However, the interested reader is asked to consult the very detailed list of references at the end of this document, and the prerequisite references in the introduction of each chapter. However, in the next section we also give a brief introduction to some concepts from differential geometry and orthonormal frames that are paramount to the foundations of dynamical systems methods applied to the Bianchi cosmologies.

1.2 The Orthonormal Frame Formalism

In this section, we briefly introduce the orthonormal frame formalism as pioneered by Ellis and MacCallum [EM69]. However, to do that, we first need to introduce some concepts from differential geometry specialized to a non-coordinate basis. This is largely based on the descriptions from [EM69], [WE97], [GH07], and Chapter 1 of [Ste93].

We begin by noting that we usually describe cosmological models in terms of a vector basis introduced in the tangent space in the neighbourhood of a point in the space-time manifold, usually denoted as $\{\mathbf{e}_a\}$, with the corresponding dual basis of 1-forms $\{\omega^a\}$, where $a = 0, 1, 2, 3$. The metric tensor is then given as

$$g_{ab} = \mathbf{g}(\mathbf{e}_a, \mathbf{e}_b), \tag{1.37}$$

with the line element given as

$$ds^2 = g_{ab}\omega^a\omega^b. \tag{1.38}$$

An *orthonormal frame* is a type of basis such that

$$\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab} = \text{diag}(-1, 1, 1, 1), \tag{1.39}$$

that is, the basis vectors are mutually orthonormal, with $\mathbf{e}_0$ timelike. Corresponding to an orthonormal frame, we then have that

$$ds^2 = \eta_{ab}\omega^a\omega^b. \tag{1.40}$$

It is also important to note some properties of differential geometry corresponding to a general set of basis vector fields that are very important for the orthonormal frame formalism. Namely, for a basis of vector fields $\mathbf{e}_a$, we write the commutator relationship as

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c_{ab}\mathbf{e}_c, \tag{1.41}$$

where the γ^c_{ab} are the *commutation functions* of the basis.

We also use the Levi-Civita/Koszul connection ∇ to define covariant derivatives. We typically have

$$\nabla_{\mathbf{e}_b}\mathbf{e}_a = \Gamma^c_{ab}\mathbf{e}_c, \tag{1.42}$$

where the Γ^c_{ab} are the components of the connection relative to the basis $\{\mathbf{e}_a\}$. We also assume that this connection is torsion-free and metric. The implication of the former is the relation

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}], \tag{1.43}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are any two vector fields. The implication of the latter is

$$\Gamma_{abc} = \frac{1}{2}\left[\mathbf{e}_b(g_{ab}) + \mathbf{e}_c(g_{ba}) - \mathbf{e}_a(g_{cb}) + \gamma^d_{cb}g_{ad} + \gamma^d_{ac}g_{bd} - \gamma^d_{ba}g_{cd}\right]. \tag{1.44}$$

From Eq. (1.42), we note that the components of $\nabla_{\mathbf{X}}\mathbf{Y}$ relative to $\{\mathbf{e}_c\}$ are denoted by $\mathbf{Y}^c_{;b}\mathbf{X}^b$, where

$$\mathbf{Y}^c_{;b} = \mathbf{e}_b\left(Y^c\right) + \Gamma^c_{ab}Y^a. \tag{1.45}$$

Now, in the orthonormal frame formalism, because of Eqs. (1.39) and (1.40), Eq. (1.44) turns into

$$\Gamma_{abc} = \frac{1}{2}\left[\gamma^d_{cb}\eta_{ad} + \gamma^d_{ac}\eta_{bd} - \gamma^d_{ba}\eta_{cd}\right]. \tag{1.46}$$

The spatial frame vectors denoted by $\{\mathbf{e}_\alpha\}$, $(\alpha = 1, 2, 3)$ are spatial differential operators. The time-dependent commutation functions with one index zero, can be expressed entirely in terms of the kinematic quantities

$$\theta = u^a_{;a}, \tag{1.47}$$

$$\dot{u}_a = u_{a;b}u^b, \tag{1.48}$$

$$\sigma_{ab} = u_{(a;b)} - \frac{1}{3}\theta h_{ab} + \dot{u}_{(a}u_{b)}, \tag{1.49}$$

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a}u_{b]}, \tag{1.50}$$

(where θ is the expansion scalar, u^a is the four-velocity of the congruence, $\dot{u}_a$ is the four-acceleration, σ_{ab} is the shear tensor, and ω_{ab} is the vorticity tensor), and the local angular velocity of the spatial frame which is typically denoted as

$$\Omega^\alpha = \frac{1}{2}\epsilon^{\alpha\mu\nu}e^i_\mu e_{\nu i;j}u^j. \tag{1.51}$$

Note that Eqs. (1.47) - (1.50) follow immediately from the decomposition of the four-velocity vector:

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b. \tag{1.52}$$

It then follows that from Eqs. (1.47) - (1.50), Eq. (1.51), (1.45), (1.52), and (1.43) that the time-dependent commutation functions become

$$\begin{aligned}
\gamma^{\alpha}_{0\beta} &= -\sigma^{\alpha}_{\beta} - \frac{1}{3}\theta\delta^{\alpha}_{\beta} - \epsilon^{\alpha}_{\beta\mu}\left(\omega^{\mu} + \Omega^{\mu}\right), && (1.53)\\
\gamma^{0}_{0\alpha} &= \dot{u}_{\alpha}, && (1.54)\\
\gamma^{0}_{\alpha\beta} &= -2\epsilon^{\mu}_{\alpha\beta}\omega_{\mu}. && (1.55)
\end{aligned}$$

where $\epsilon^{\alpha\beta}_{\mu}$ is the standard Levi-Civita permutation symbol. We can also decompose the spatial commutation functions $\gamma^{\mu}_{\alpha\beta}$ into symmetric and anti-symmetric parts as

$$\gamma^{\mu}_{\alpha\beta} = \epsilon_{\alpha\beta\nu}n^{\mu\nu} + a_{\alpha}\delta^{\mu}_{\beta} - a_{\beta}\delta^{\mu}_{\alpha}. \tag{1.56}$$

This decomposition is important as the sign of the eigenvalues of $n^{\mu\nu}$ and the sign of a_{α} classify the different Bianchi types. Applying the Jacobi identity to Eq. (1.56), we see that we get the eigenvalue problem

$$n^{\alpha\beta}a_{\beta} = 0. \tag{1.57}$$

Now, when $a_{\beta} = 0$, Eq. (1.57) is automatically satisfied and this condition classifies Bianchi models of class A. When $a_{\beta} \neq 0$, this denotes Bianchi models of class B, and in order to satisfy Eq. (1.57), we note that a_{β} must lie in the kernel of $n^{\alpha\beta}$. Therefore, without loss of generalization, we can take

$$n^{\alpha\beta} = \mathrm{diag}\left(n_1, n_2, n_3\right), \quad a_{\beta} = \left(a, 0, 0\right). \tag{1.58}$$

The signs of n_1, n_2, n_3 determine the different types of Bianchi algebras, which we display in Table 1.1 for reference purposes.

Bianchi class	Bianchi type	$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$	Contains FLRW?
A ($a = 0$)	I	0	0	0	$k = 0$
A ($a = 0$)	II	+	0	0	
A ($a = 0$)	VI_0	0	+	-	
A ($a = 0$)	VII_0	0	+	+	$k = 0$
A ($a = 0$)	VIII	-	+	+	
A ($a = 0$)	IX	+	+	+	$k = +1$
B ($b \neq 0$)	V	0	0	0	$k = -1$
B ($b \neq 0$)	IV	0	0	+	
B ($b \neq 0$)	VI_h	0	+	-	
B ($b \neq 0$)	VII_h	0	+	+	$k = -1$

Table 1.1: Classification of the Bianchi algebras

Note that in Table 1.1, the parameter h is defined such that $a^2 = hn_2n_3$, so that h is well-defined if and only if $n_2n_3 \neq 0$ in class B models. In addition, as can be seen from the table and the aforementioned definition of h, $h < 0$ in type VI_h models, while $h > 0$ in type VII_h models.

The orthonormal frame formalism then implies that the Einstein field equations, Jacobi identities, and contracted Bianchi identities can all be written in terms of the basic variables

$$(H, \sigma_{\alpha\beta}, \dot{u}_\alpha, \omega_\alpha, \Omega_\alpha, n_{\alpha\beta}, a_\alpha)\,, \tag{1.59}$$

where we have defined $H = (1/3)\theta$ to be the Hubble parameter. In our work, we assume that the Bianchi model under consideration is hypersurface orthogonal, that is, the fluid four-velocity is orthogonal to the spatial hypersurfaces, and so we have that all the basic variables in Eq. (1.59) are functions of time only, and that $\dot{u}_\alpha = \omega_{\alpha\beta} = 0$. The Einstein field equations with respect to a group-invariant orthonormal frame

$\{\mathbf{n}, \mathbf{e}_\alpha\}$, (where $\mathbf{n}$ is the unit normal to the group orbits) become

$$\begin{aligned}
\dot{H} &= -H^2 - \frac{2}{3}\sigma^2 - \frac{1}{6}\left(\mu + 3p\right), && (1.60)\\
\dot{\sigma}_{\alpha\beta} &= -3H\sigma_{\alpha\beta} + 2\epsilon^{\mu\nu}_{(\alpha}\sigma_{\beta)\mu}\Omega_\nu -{}^3 S_{\alpha\beta} + \pi_{\alpha\beta}, && (1.61)\\
\mu &= 3H^2 - \sigma^2 + \frac{1}{2}{}^3R, && (1.62)\\
q_\alpha &= 3\sigma^\mu_\alpha a_\mu - \epsilon^{\mu\nu}_\alpha \sigma^\beta_\mu n_{\beta\nu}, && (1.63)
\end{aligned}$$

where μ and p denotes the energy density and pressure of the matter content respectively, q_α is the heat flux vector, while $\pi_{\alpha\beta}$ denotes the anisotropic matter in the universe. The spatial curvature terms are given by

$$\begin{aligned}
{}^3S_{\alpha\beta} &= b_{\alpha\beta} - \frac{1}{3}\left(b^\mu_\mu\right)\delta_{\alpha\beta} - 2\epsilon^{\mu\nu}_{(\alpha}n_{\beta)\mu}a_\nu, && (1.64)\\
{}^3R &= -\frac{1}{2}b^\mu_\mu - 6a_\mu a^\mu, && (1.65)
\end{aligned}$$

where

$$b_{\alpha\beta} = 2n^\mu_\alpha n_{\mu\beta} - \left(n^\mu_\mu\right)n_{\alpha\beta}. \tag{1.66}$$

The Jacobi identities give the evolution of the spatial curvature terms and a constraint equation as

$$\begin{aligned}
\dot{n}_{\alpha\beta} &= -Hn_{\alpha\beta} + 2\sigma^\mu_{(\alpha}n_{\beta)\mu} + 2\epsilon^{\mu\nu}_{(\alpha}n_{\beta)\mu}\Omega_\nu, && (1.67)\\
\dot{a}_\alpha &= -Ha_\alpha - \sigma^\beta_\alpha a_\beta + \epsilon^{\mu\nu}_\alpha a_\mu\Omega_\nu, && (1.68)\\
0 &= n^\beta_\alpha a_\beta. && (1.69)
\end{aligned}$$

The contracted Bianchi identities imply that

$$\begin{aligned}
\dot{\mu} &= -3H\left(\mu + p\right) - \sigma^\beta_\alpha \pi^\alpha_\beta + 2a_\alpha q^\alpha, && (1.70)\\
\dot{q}_\alpha &= -4Hq_\alpha - \sigma^\beta_\alpha q_\beta - \epsilon^{\mu\nu}_\alpha\Omega_\mu q_\nu + 3a^\beta\pi_{\alpha\beta} + \epsilon^{\mu\nu}_\alpha n^\beta_\mu \pi_{\beta\nu}, && (1.71)
\end{aligned}$$

which are evolution equations for the matter content in the universe. In particular, the $\dot{\mu}$ equation is the standard energy conservation equation in general relativity.

By now, one can hopefully see the power of the orthonormal frame formalism: The Einstein field equations, normally a coupled system of ten, nonlinear, hyperbolic partial differential equations are reduced to a system of *first-order* ordinary differential equations for the basic variables in Eq. (1.59). This first-order system is ripe for a dynamical systems analysis which is at the heart of much of the work in this dissertation.

When the NIH was founded, the idea was simple: Identify the best research as defined by scientists themselves. And now these people who are not even scientists have answered all the questions before any science has been done? Have you ever seen one of these grant applications? We're lucky Einstein didn't have to fill one out or God knows what 'E' would equal.

President Bartlet, The West Wing

2

Future Asymptotic Behaviour of a Nontilted Bianchi Type IV Viscous Model

2.1 Introduction

Spatially homogeneous and anisotropic models of the universe have undergone great study and continue to remain amongst the most popular areas of research in cosmology. Early-universe cosmological models for the most part have assumed the universe to be spatially homogeneous and anisotropic, with the important exception being the case of inhomogenous models such as the Lemaître-Tolman-Bondi and "Swiss-cheese" models [EMM12]. However, if one begins with the idea of the early universe being spatially homogeneous and anisotropic, then to transition to the present-day Friedmann-Lemaître-Robertson-Walker models requires the anisotropy in the former models to decay. The process by which this anisotropic decay occurs is arguably the most fundamental property of any early-universe model that aims to transition to the present-day models. For example, Belinskii, Khalatnikov, and Lifshitz [BKL70] studied the oscillatory approach to a singular point in relativistic cosmologies. Misner [Mis94] studied the anisotropic decay of the vacuum Bianchi Type IX/Mixmaster models of the universe. A very general approach to describing the isotropization of Bianchi models was described by Salucci and Fabbri [SF83]. Coley and van den Hoogen [vdHC95] studied causal anisotropic viscous fluid models and described conditions for such models to isotropize. As for very recent work on this subject, Pradhan, Rai, and Singh [PRS09] studied the Bianchi Type V bulk viscous models and showed that such models do isotropize for specific functional forms of the anisotropic scale factors.

Viscous models have become of general interest in early-universe cosmologies largely in two contexts. Grøn and Hervik (Chapter 13, [GH07]) discuss these in some detail. The first relates to the idea of inflation through bulk viscosity. In models where bulk viscous terms are permitted to dominate, they drive the universe into a de Sitter-like state. Because of these processes, the models isotropize indirectly through the massive expansion. Shear viscosity is found to play an important role in universe models with dissipative fluids. The dissipative processes that result from shear viscous terms are thought to be highly effective during the early stages of the universe. In particular, neutrino viscosity is considered to be one of the most important factors in the isotropization of our universe.

As for Bianchi Type IV models specifically, Hervik, van den Hoogen, and Coley [HvdHC05] studied future asymptotic behaviour of tilted vacuum Bianchi Type IV models, and found that such models do not necessarily isotropize at late times. Uggla and Rosquist [UR88] studied the orthogonal Bianchi Type IV model near the initial singularity with a vacuum or perfect-fluid source. We chose to study the isotropization behaviour of the Bianchi Type IV viscous model largely because such a study has not been taken on extensively in the literature, and perhaps such a study will add to the already rich landscape of spatially homogeneous and anisotropic models of the early universe.

We will use the Hubble-normalized dynamical systems approach based upon the theory of orthonormal frames pioneered by Ellis and MacCallum [EM69], which reduces the Einstein field equations, a coupled set of ten hyperbolic nonlinear partial differential equations, to a system of autonomous nonlinear first-order ordinary differential equations. We will also provide a fixed-point analysis of the dynamical system and make connections with the global dynamics through sophisticated numerical experiments.

2.2 The Energy-Momentum Tensor for a Viscous Fluid

In this section, we will derive the form of the energy-momentum tensor under concern, namely, for that of a viscous fluid without heat conduction. Recall that the energy-momentum tensor for a perfect fluid takes the form

$$T^{ab} = (\mu + p)u^a u^b - u^c u_c g^{ab} p. \tag{2.1}$$

For the moment, letting $\mu + p = \mathcal{W}$, we obtain

$$T^{ab} = \mathcal{W} u^a u^b - u^c u_c g^{ab} p. \tag{2.2}$$

Denoting the viscous contributions by $\mathcal{V}_{ab}$, we seek a modification of Eq. (2.2) such that

$$T_{ab} = \mathcal{W} u_a u_b - u_c u^c g_{ab} p + \mathcal{V}_{ab}. \tag{2.3}$$

To obtain the form of this additional tensor term, we note that from classical fluid mechanics, the Euler equation is given as

$$(\rho u_i)_{,t} = -\Pi_{ik,k}, \tag{2.4}$$

where Π_{ik} is the momentum flux tensor. Also, recall that for a non-viscous fluid, one has the fundamental relationship

$$\Pi_{ik} = p\delta_{ik} + \rho u_i u_k. \tag{2.5}$$

We simply add a term to Eq. (2.5) that represents the viscous momentum flux, $\tilde{\Sigma}_{ik}$, to obtain

$$\Pi_{ik} = p\delta_{ik} + \rho u_i u_k - \tilde{\Sigma}_{ik} = -S_{ik} + \rho u_i u_k. \tag{2.6}$$

It is important to note that

$$S_{ik} = -p\delta_{ik} + \tilde{\Sigma}_{ik} \tag{2.7}$$

is the stress tensor, while, $\tilde{\Sigma}_{ik}$ is the *viscous* stress tensor. Note that, in what follows below, the viscous stress tensor, $\tilde{\Sigma}_{ik}$, is not to be confused with Σ_{ik}, the Hubble-normalized shear tensor. The general form of the viscous stress tensor can be formed by recalling that viscosity is formed when the fluid particles move with respect to each other at different velocities, so this stress tensor can only depend on spatial components of the fluid velocity. We assume that these gradients in the velocity are small, so that the momentum tensor only depends on the first derivatives of the velocity in some Taylor series expansion. Therefore, $\tilde{\Sigma}_{ik}$ is some function of the $u_{i,k}$. In addition, when the fluid is in rotation, no internal motions of particles can be occurring, so we consider linear combinations of $u_{i,k} + u_{k,i}$, which clearly vanish for a fluid in rotation with some angular velocity, Ω_i. The most general viscous tensor that can be formed is given by

$$\tilde{\Sigma}_{ik} = \eta \left(u_{i,k} + u_{k,i} - \frac{2}{3}\delta_{ik} u_{l,l} \right) + \xi \delta_{ik} u_{l,l}, \tag{2.8}$$

where η and ξ are the coefficients of shear and bulk/second viscosity, respectively [LL11] [KC08]. In Eq. (2.8), we note that $\delta_{ik} u_{l,l}$ is an expansion rate tensor, and $\left(u_{i,k} + u_{k,i} - \frac{2}{3}\delta_{ik} u_{l,l}\right)$ represents the shear rate tensor. Since we would like to generalize this expression to the general relativistic case, we replace the partial derivatives above with covariant derivatives, and the Kroenecker tensor with a more general metric tensor, that is, $\delta_{ik} \to g_{ik}$. We thus have that

$$\tilde{\Sigma}_{ik} = \eta \left(u_{i;k} + u_{k;i} - \frac{2}{3} g_{ik} u_{l;l} \right) + \xi g_{ik} u_{l;l}. \tag{2.9}$$

Denoting the shear rate tensor as σ_{ab}, and the expansion rate scalar as $\theta \equiv u^a_{;a}$, Eq. (2.9) becomes

$$\mathcal{V}_{ab} = -2\eta\sigma_{ab} - \xi\theta h_{ab}. \tag{2.10}$$

Since we are interested in the Hubble-normalized approach, we will make use of the definition $\theta \equiv 3H$, where H is the Hubble parameter. This means that Eq. (2.9) becomes

$$\mathcal{V}_{ab} = -2\eta\sigma_{ab} - 3\xi H h_{ab}. \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (2.3) we finally obtain the required form of the energy-momentum tensor as

$$T_{ab} = (\mu + p)\, u_a u_b - u_c u^c g_{ab} p - 2\eta\sigma_{ab} - 3\xi H h_{ab}. \tag{2.12}$$

For simplicity, we shall let $\pi_{ab} = -2\eta\sigma_{ab}$ denote the anisotropic stress tensor, and commit to the metric signature $(-1, +1, +1, +1)$ such that Eq. (2.12) takes the form

$$T_{ab} = (\mu + p)\, u_a u_b + g_{ab} p - 3\xi H h_{ab} + \pi_{ab}. \tag{2.13}$$

2.3 Bianchi Type IV Universe Dynamics

With the required energy-momentum tensor in hand, we will now derive the Bianchi Type IV dynamical equations. The general evolution equations for any Bianchi type have already been derived in [HBW01], [HLSU10] and [EMM12], and we will simply make use of their results in this section.

The general evolution equations in the expansion-normalized variables are

$$\begin{aligned}
\Sigma'_{ij} &= -(2-q)\Sigma_{ij} + 2\epsilon^{km}_{\ \ (i}\Sigma_{j)k}R_m - \mathcal{S}_{ij} + \Pi_{ij}, \\
N'_{ij} &= qN_{ij} + 2\Sigma^k_{\ (i}N_{j)k} + 2\epsilon^{km}_{\ \ (i}N_{j)k}R_m, \\
A'_i &= qA_i - \Sigma^j_i A_j + \epsilon_i^{\ km}A_k R_m, \\
\Omega' &= (2q-1)\Omega - 3P - \frac{1}{3}\Sigma^j_i \Pi^i_j + \frac{2}{3}A_i Q^i, \\
Q'_i &= 2(q-1)Q_i - \Sigma^j_i Q_j - \epsilon_i^{\ km} R_k Q_m + 3A^j \Pi_{ij} + \epsilon_i^{\ km} N^j_k \Pi_{jm}.
\end{aligned} \tag{2.14}$$

These equations are subject to the constraints

$$
\begin{aligned}
N_i^j A_j &= 0, \\
\Omega &= 1 - \Sigma^2 - K, \\
Q_i &= 3\Sigma_i^k A_k - \epsilon_i^{km} \Sigma_k^j N_{jm}.
\end{aligned}
\tag{2.15}
$$

In Eqs. (2.14) and (2.15) we have made use of the following notation:

$$
\left(\Sigma_{ij}, R^i, N^{ij}, A_i\right) = \frac{1}{H}\left(\sigma_{ij}, \Omega^i, n^{ij}, a_i\right), \quad \left(\Omega, P, Q_i, \Pi_{ij}\right) = \frac{1}{3H^2}\left(\mu, p, q_i, \pi_{ij}\right). \tag{2.16}
$$

In the expansion-normalized approach, Σ_{ab} denotes the kinematic shear tensor, and describes the anisotropy in the Hubble flow, A_i and N^{ij} describe the spatial curvature, while Ω^i describes the relative orientation of the shear and spatial curvature eigenframes. In addition, μ and p denote the *total* energy density and total effective pressure, and are found by evaluating

$$
\mu = u^a u^b T_{ab}, \quad p = \frac{1}{3} h^{ab} T_{ab}, \tag{2.17}
$$

where, $h_{ab} = u_a u_b + g_{ab}$ denotes the projection tensor, and u^a, the fluid four-velocity [HLSU10]. Since we are interested in a *non-tilted* cosmology, the fluid is taken to be geodesic and irrotational, and thus has four-velocity $u^a = (1, 0, 0, 0)$. We first note that the total energy density is indeed just μ, as can be seen by applying the definition above. In addition, the total effective pressure is found from Eqs. (2.17) and (2.13) to be

$$
p = \frac{1}{3} h^{ab} T_{ab} = \tilde{p} - 3\xi H, \tag{2.18}
$$

where $\tilde{p}$ denotes the fluid pressure in the barotropic equation of state, such that $\tilde{p} = w\mu$. This implies that

$$
P = w\Omega - 3\xi_0, \tag{2.19}
$$

where we have defined the equation of state

$$
\frac{\xi}{H} \equiv 3\xi_0, \tag{2.20}
$$

with ξ_0 being a non-negative constant. Similarly, we find that

$$\Pi_{ab} = -2\eta_0 \Sigma_{ab}, \tag{2.21}$$

where η_0 is a non-negative constant, as defined by the equation of state

$$\frac{\eta}{H} = 3\eta_0. \tag{2.22}$$

From these definitions of the expansion-normalized shear and bulk viscosity parameters, $\eta_0 and \xi_0$, we would like to stress that throughout the proceeding analysis, we consider these parameters to be *non-negative constants*.

Since the fluid four-velocity is taken to be $u^a = (1,0,0,0)$, the quantity $q_a = Q_a 3H^2$ vanishes by definition:

$$q^a \equiv -h^b_a u^c T_{bc} = -\left(u_a u^0 + \delta^a_0\right) T_{00} = 0. \tag{2.23}$$

Our dynamical system evolves according to a dimensionless time variable, τ, such that

$$\frac{dt}{d\tau} = \frac{1}{H}, \tag{2.24}$$

where H is the Hubble parameter. The deceleration parameter q is very important in the expansion-normalized approach, and through the evolution equation for H,

$$H' = -(1+q)H, \tag{2.25}$$

one can show that q is defined as

$$\begin{aligned} q &\equiv 2\Sigma^2 + \frac{1}{2}\left(\Omega + 3P\right) \\ &= \frac{1}{3}\left(\Sigma_{ab}\Sigma^{ab}\right) + \Omega\left[\frac{1}{2} + \frac{3}{2}w\right] - \frac{9}{2}\xi_0, \end{aligned} \tag{2.26}$$

where we have made use of Eq. (2.19) and the definition $\Sigma^2 \equiv \frac{1}{6}\Sigma^{ab}\Sigma_{ab}$.

Following the convention in [GH07], for the Bianchi Type IV models, we have

$$A^i = A\delta^i_3 \neq 0, \quad N_{11} \neq 0, \quad N_{22} = N_{33} = 0. \tag{2.27}$$

Computing the evolution equations requires one to first compute the Hubble-normalized spatial curvature variables, $\mathcal{S}_{ij}$ and K. According to (A.7) in [HBW01], we have that

$$\mathcal{S}_{ab} = B_{ab} - \frac{1}{3}B_u^u\delta_{ab} - 2\epsilon^{uv}_{(a}N_{b)u}A_v, \quad K = \frac{1}{12}B_u^u + A_uA^u, \tag{2.28}$$

where $B_{ab} \equiv 2N_a^uN_{ub} - N_u^uN_{ab}$. Evaluating these expressions for the Bianchi IV model, we obtain

$$\mathcal{S}_{11} = \frac{2}{3}N_{11}, \quad \mathcal{S}_{12} = N_{11}A, \quad \mathcal{S}_{22} = \mathcal{S}_{33} = -\frac{N_{11}^2}{3}, \quad K = A^2 + \frac{N_{11}^2}{12}. \tag{2.29}$$

The constraints in Eq. (2.15) imply that

$$\Sigma_{31} = \Sigma_{32} = 0, \quad 3A\Sigma_{33} + N_{11}\Sigma_{21} = 0, \tag{2.30}$$

that is, that $\Sigma_{21} \neq 0$. In addition, the Σ_{13}' and Σ_{23}' equations from Eqs. (2.14) imply that $R_1 = R_2 = 0$. Looking at the N_{12}' equation from the same set implies that $R_3 = \Sigma_{12}$. We have therefore uniquely determined R_i in terms of Σ_{ij}, and can see that the independent expansion-normalized variables are: $\Sigma_{22}, \Sigma_{33}, \Sigma_{12}, A$, and N_{11}. Taking advantage of the fact that the shear tensor Σ_{ab} is trace-free, we will define new variables as follows:

$$\begin{aligned}
\Sigma_+ &= (\Sigma_{22} + \Sigma_{33}), \\
\Sigma_- &= \frac{1}{\sqrt{3}}(\Sigma_{33} - \Sigma_{22}), \\
N_1 &= N_{11}, \\
\Sigma_3 &= \frac{1}{\sqrt{3}}\Sigma_{12}.
\end{aligned} \tag{2.31}$$

As mentioned in [HBW01] and [EMM12], the off-diagonal shear component Σ_3 determines the angular velocity of the spatial frame. The set of independent expansion-normalized variables is then

$$(\Sigma_+, \Sigma_-, N_1, A, \Sigma_3). \tag{2.32}$$

The evolution equations for these variables are

$$
\begin{aligned}
\Sigma_+' &= \frac{2N_1^2}{3} - 4\Sigma_3^2 + \Sigma_+\left[-2+q-2\eta_0\right], \\
\Sigma_-' &= \frac{4\Sigma_3^2}{\sqrt{3}} + \Sigma_-\left[q-2\left(1+\eta_0\right)\right], \\
\Sigma_3' &= \Sigma_3\left[-(2-q)+\frac{3}{2}\Sigma_+ - \frac{\sqrt{3}}{2}\Sigma_- - 2\eta_0\right] - \frac{1}{\sqrt{3}}N_1, \\
N_1' &= N_1\left[q-2\Sigma_+\right], \\
A' &= A\left[q-\frac{\sqrt{3}\Sigma_-}{2}-\frac{\Sigma_+}{2}\right],
\end{aligned}
\tag{2.33}
$$

where

$$
q = 2\left(\Sigma_3^2+\Sigma_-^2+\Sigma_+^2\right) - \frac{1}{24}\left[12A^2+N_1^2+12\left(-1+\Sigma_3^2+\Sigma_-^2+\Sigma_+^2\right)\right](1+3w) - \frac{9\xi_0}{2}, \tag{2.34}
$$

with the constraint given by

$$
g(\mathbf{x}) = \frac{3}{2}A\left(\sqrt{3}\Sigma_- + \Sigma_+\right) + \sqrt{3}N_1\Sigma_3 = 0. \tag{2.35}
$$

The state space is the subset of $\mathbb{R}^5$ defined by the physical inequality $\Omega \geq 0$, which is equivalent to

$$
\Sigma_+^2 + \Sigma_-^2 + \Sigma_3^2 + A^2 + \frac{1}{12}N_1^2 \leq 1, \tag{2.36}
$$

This restriction indeed implies that the state space is bounded.

We additionally find that the evolution equations (2.33) have a transformation invariance such that

$$
\left[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A\right] \to \left[\Sigma_+, \Sigma_-, \Sigma_3, \pm N_1, \pm A\right]. \tag{2.37}
$$

One can therefore assume without loss of generality that $N_1 \geq 0$ and $A \geq 0$.

2.4 A Local Stability Analysis

In this section, we consider the local stability of the equilibrium points of the system (2.33) and (2.35). The critical points are points $\mathbf{x} = \mathbf{a}$ that simultaneously satisfy

$$
\mathbf{f}(\mathbf{a}) = 0, \quad g(\mathbf{a}) = 0, \tag{2.38}
$$

where $\mathbf{f}$ denotes the right-hand-side of the system (2.33). The local stability is determined by linearizing the system (2.33) at $\mathbf{x} = \mathbf{a}$, which leads to the relationship $\mathbf{x}' = D\mathbf{f}(\mathbf{a})\mathbf{x}$. The stability of the system is then determined by finding the eigenvalues and eigenvectors of the derivative matrix $D\mathbf{f}(\mathbf{a})$. Because of the constraint, we are to only consider physical eigenvalues, that is, eigenvalues whose corresponding eigenvectors are orthogonal to $\nabla g(\mathbf{a})$. The gradient of the constraint Eq. (2.35) is found to be

$$\nabla g(\mathbf{x}) = \left[\frac{3A}{2}, \frac{3\sqrt{3}}{2}A, \sqrt{3}N_1, \sqrt{3}\Sigma_3, \frac{3}{2}\left(\sqrt{3}\Sigma_- + \Sigma_+\right)\right]. \tag{2.39}$$

2.4.1 Equilibrium Point 1

The first equilibrium point is found to be

$$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0, 0, 0, 0, 0]. \tag{2.40}$$

The cosmological parameters at this point take the form:

$$\Omega = 1, \quad q = \frac{1}{2}(1 + 3w - 9\xi_0), \quad \Sigma^2 = 0, \tag{2.41}$$

where

$$\xi_0 \geq 0, \quad \eta_0 \geq 0, \quad -1 \leq w \leq 1. \tag{2.42}$$

The eigenvalues corresponding to this critical point are:

$$\lambda_1 = \lambda_2 = \frac{1}{2}[1 + 3w - 9\xi_0], \quad \lambda_3 = \lambda_4 = \lambda_5 = \frac{1}{2}[-3 + 3w - 4\eta_0 - 9\xi_0]. \tag{2.43}$$

This equilibrium point is a local sink (all of the eigenvalues have negative real parts) if $\eta_0 \geq 0$, and

$$\left\{\left[0 \leq \xi_0 \leq \frac{4}{9}\right] \bigcap \left[-1 \leq w < \frac{1}{3}(-1 + 9\xi_0)\right]\right\} \bigcup \left\{\left[\xi_0 > \frac{4}{9}\right] \bigcap [-1 \leq w < 1]\right\}. \tag{2.44}$$

Based on the cosmological parameters (2.41), we see that this equilibrium point represents a non-vacuum, flat Friedmann-LeMaître (FL) universe [HBW01] [WE97]. An important point to note is that $q = -1$ when $0 \leq \xi_0 \leq \frac{2}{3}$ and $w = 3\xi_0 - 1$, thus, the equilibrium point in the domain defined by these values of η_0, ξ_0 ,and w does not correspond to a self-similar solution. In particular, if one chooses $\xi_0 = 0$ such that $w = -1$, the corresponding model is locally the de Sitter solution [EMM12].

2.4.2 Equilibrium Point 2

The second equilibrium point is found to be:

$$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = \left[0, 0, 0, 0, \frac{\sqrt{1 + 3w - 9\xi_0}}{\sqrt{1+3w}}\right]. \tag{2.45}$$

The cosmological parameters at this point take the form:

$$\Omega = \frac{9\xi_0}{1 + 3w}, \quad q = 0, \quad \Sigma^2 = 0. \tag{2.46}$$

This equilibrium point represents a Bianchi Type V model if

$$\eta_0 \geq 0, \quad \left\{0 \leq \xi_0 < \frac{4}{9}\right\} \bigcap \left\{\frac{1}{3}\left[-1 + 9\xi_0\right] < w < 1\right\}. \tag{2.47}$$

The eigenvalues corresponding to this critical point are:

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = -2\left(1 + \eta_0\right), \quad \lambda_4 = -1 - 3w + 9\xi_0. \tag{2.48}$$

We see that this equilibrium point is a non-isolated equilibrium point in general, because $\lambda_1 = 0$. Whether it is a sink or a source depends on the signs of the other eigenvalues. Note that, $\lambda_2 = \lambda_3 = -2(1 + \eta_0) < 0, \quad \forall\ \eta_0 \geq 0$, so the equilibrium classification depends on λ_4 alone. In particular,

$$\lambda_4 = -1 - 3w + 9\xi_0 < 0 \Leftrightarrow \left\{0 \leq \xi_0 < \frac{4}{9}\right\} \bigcap \left\{\frac{1}{3}\left[-1 + 9\xi_0\right] < w < 1\right\}. \tag{2.49}$$

Therefore, we see that, if and only if $\lambda_4 < 0$, this equilibrium point is a local sink (Section 4.3.4 [WE97]). One can also show that given the restrictions in (2.47), $\lambda_4 \neq 0$, and $\lambda_4 \ngtr 0$.

An interesting point is that if one chooses $\xi_0 = 0$, then $A = 1$, $\Omega = q = \Sigma^2 = 0$, and this equilibrium point represents the Milne universe [WE97] [HvdHC05].

2.4.3 Other Possible Equilibrium Points

We should note that in addition to the equilibrium points found above, there are additional ones that are purely mathematical. We are forced to ignore these points on physical grounds, because in order for $\Omega \geq 0$, we would have to have either $N < 0$, $A < 0$, $\xi_0 < 0$, $\eta_0 < 0$, $w < -1$ or $w > 1$. The first pair violate the

Bianchi Type IV requirements, the second pair violate the requirement that any fluid must have non-negative viscosity coefficients, and the last pair violate the well-known equation of state restrictions in cosmology. As a particular example, a fluid having an equation of state for which $w > 1$ implies that the matter under consideration has the speed of sound exceeding the speed of light, which would violate relativity theory.

As argued by Hervik et. al. [HvdHC05], since we are only concerned with the future asymptotic behaviour of the Bianchi Type IV model, we will not be concerned with Type I vacuum equilibrium points for which $N_1 = A = \Omega = 0$, since for these models all of the equilibrium points are Kasner circles of which none are stable in the future, that is, they all represent local sources [HvdHC05] [WE97].

2.4.4 Bifurcation Behaviour

The physical equilibria found above are related to each other by a sequence of bifurcations, which can be understood as follows. The linearizations of the equations for N_1 and A at the flat FL point are

$$\begin{aligned} N_1' &= \frac{1}{2}\left(1+3w-9\xi_0\right)N_1, \\ A' &= \frac{1}{2}\left(1+3w-9\xi_0\right)A, \end{aligned} \tag{2.50}$$

which show that N_1 and A destabilize the flat FL point if $\xi_0 = \frac{1}{9}\left(1+3w\right)$, and that there is a bifurcation from the Bianchi Type V / Open FL point to the Bianchi Type I / Flat FL point, which can been seen from the ranges of ξ_0 in Table I. Further, the linearization of the A' equation at the Bianchi Type V point is found to be

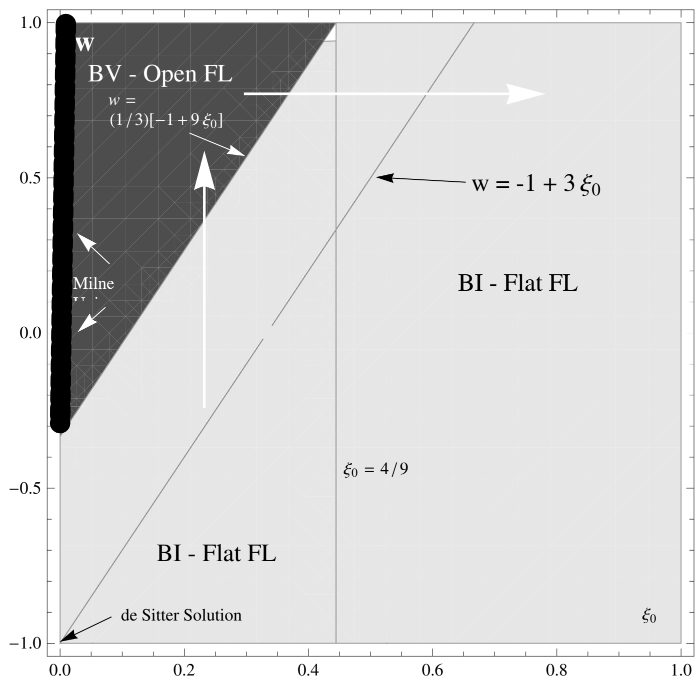
$$A' = \left(-\frac{\sqrt{1+3w-9\xi_0}}{2\sqrt{1+3w}}\right)\Sigma_+ - \left(-\frac{\sqrt{3+9w-27\xi_0}}{2\sqrt{1+3w}}\right)\Sigma_- + \left(-1-3w+9\xi_0\right)A, \tag{2.51}$$

which destabilizes the Bianchi Type V point if $\xi_0 = \frac{1}{9}\left(1+3w\right)$ and $-\frac{1}{3} < w < 1$. It is clear then that with respect to this analysis, the line

$$\xi_0 = \frac{1}{9}\left(1+3w\right) \tag{2.52}$$

is very important as it governs the bifurcations of the system. We give in Fig. 2.1 a useful summary diagram of the bifurcation regions in terms of the viscosity coefficients.

Figure 2.1: This figure gives a schematic view of the viscosity coefficients as related to specific Bianchi type regions. The large white arrows indicate bifurcation transitions in terms of increasing expansion-normalized bulk-viscosity coefficient, where η_0 is assumed to be non-negative in general. Note how the different regions are bounded by the lines $w = \frac{1}{3}[-1 + 9\xi_0]$ and $\xi_0 = \frac{4}{9}$. Also indicated on the diagram by a thick black line $\xi_0 = 0$ in the BV region, which indicates Milne universe solutions. For completeness, we have included the line $w = -1 + 3\xi_0$, which for $0 \leq \xi_0 \leq \frac{2}{3}$ represents non-self-similar solutions. In particular, the point $\xi_0 = 0, w = -1$ represents the de Sitter solution.



2.5 Late-Time Asymptotic Behaviour

The goal of this section is to complement the preceding stability analysis of the equilibrium points with extensive numerical experiments in order to confirm that the local results are in fact global in nature.

By the Hartman-Grobman theorem for hyperbolic equilibrium points, and the invariant manifold theorem for non-isolated equilibrium points [WE97], we know that for any local sink, any orbit that enters a sufficiently small neighbourhood of the sink approaches the sink as $\tau \to \infty$ [HBW01]. The numerical solutions to the dynamical system presented below also provide strong evidence that for the given values of ξ_0 and w, the local sinks are the future attractors of the evolution equations. For each numerical solution, we chose the initial conditions such that the constraints (2.15), (2.35) were satisfied, and are indicated by asterisks in the figures below.

Although numerical integrations were done from $0 \leq \tau \leq 1000$, for demonstration purposes, we presented solutions for shorter time intervals. We completed numerical integrations of the dynamical system for physically interesting cases of w equal to 0 (dust), 0.325 (a dust/radiation mixture), and $\frac{1}{3}$ (radiation).

2.5.1 $\xi_0 = \frac{4}{9}$, $\eta_0 = 1$, $w = 0$ (Dust)

Figure 2.2: This figure shows the dynamical system behaviour for $\xi_0 = \frac{4}{9}$, $\eta_0 = 1$, and $w = 0$. The plus sign indicates the equilibrium point. Notice how the equilibrium point in this case, the Bianchi Type I / Flat FL point, is indeed the local sink. The model also isotropizes, as can be seen from the last figure, where $\Sigma_\pm \to 0$ as $\tau \to \infty$.

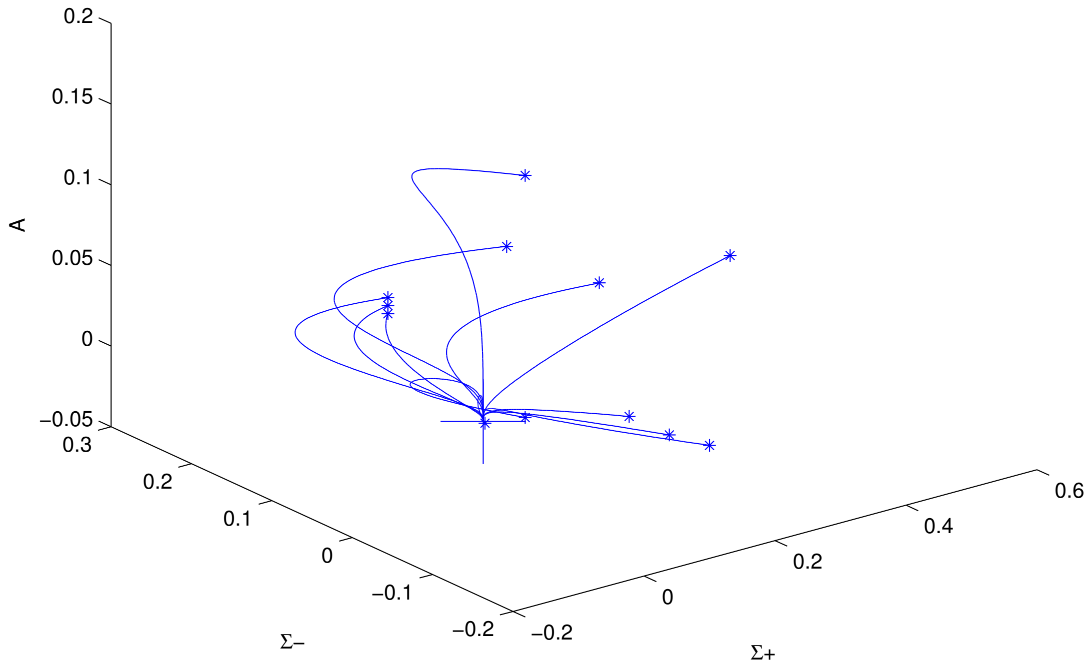


Fig. 2.2 continued.

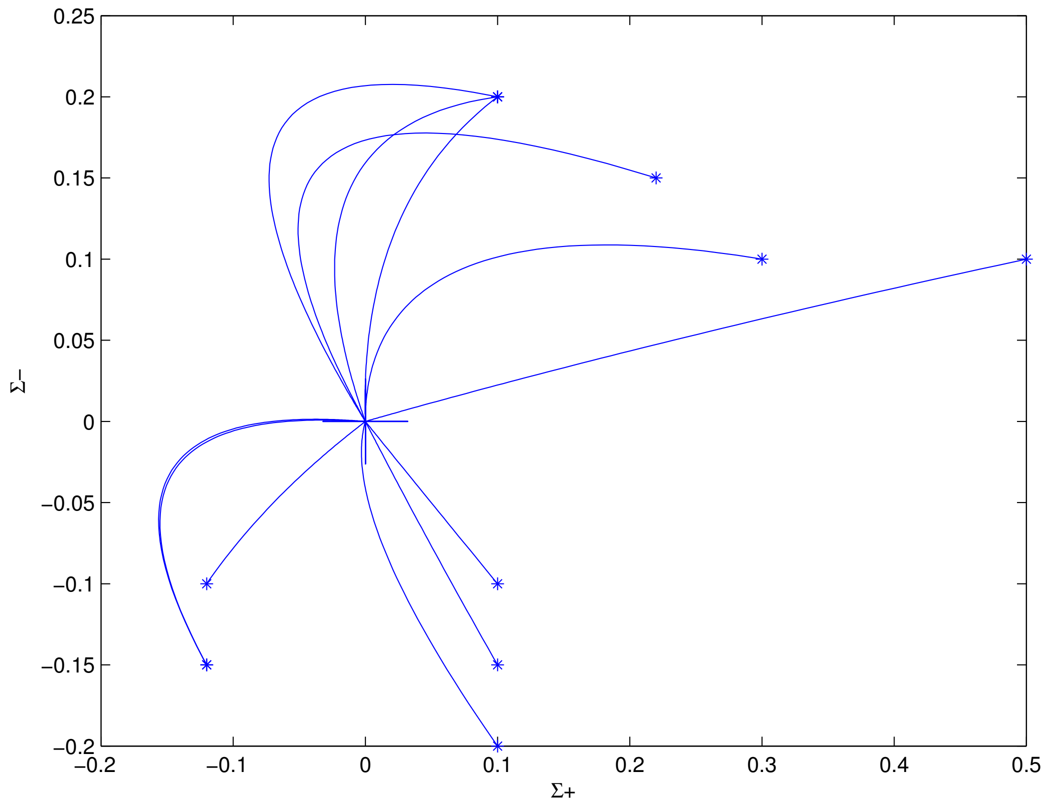


Fig. 2.2 continued.

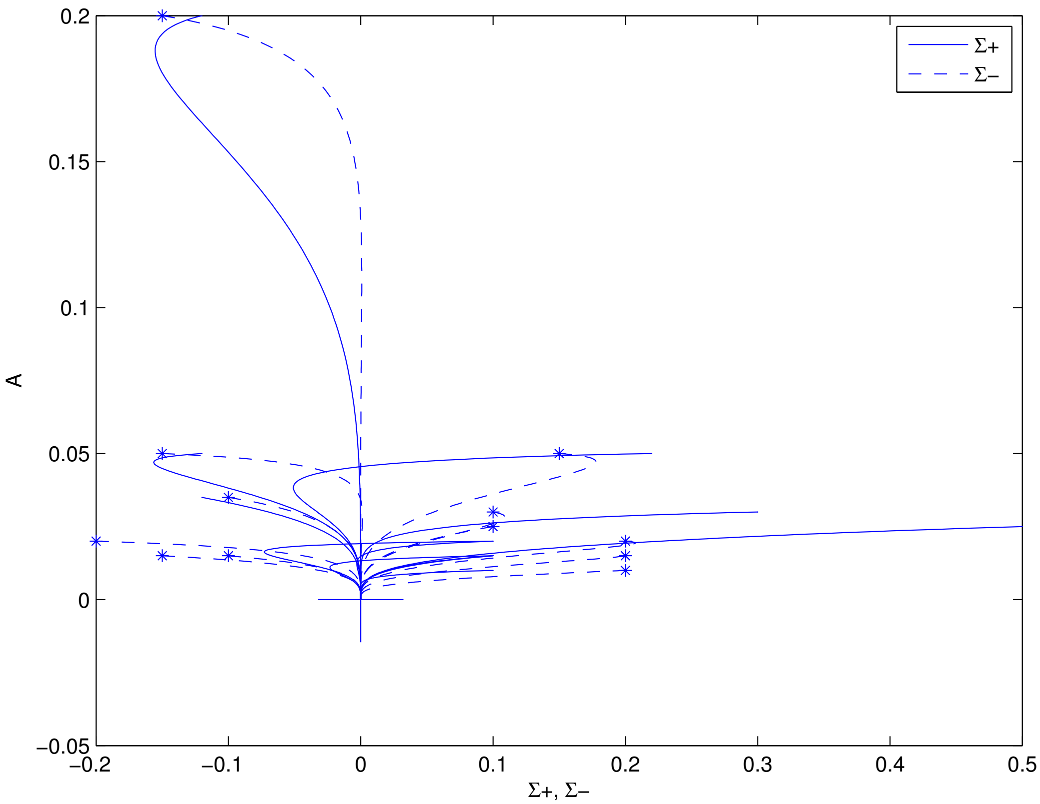
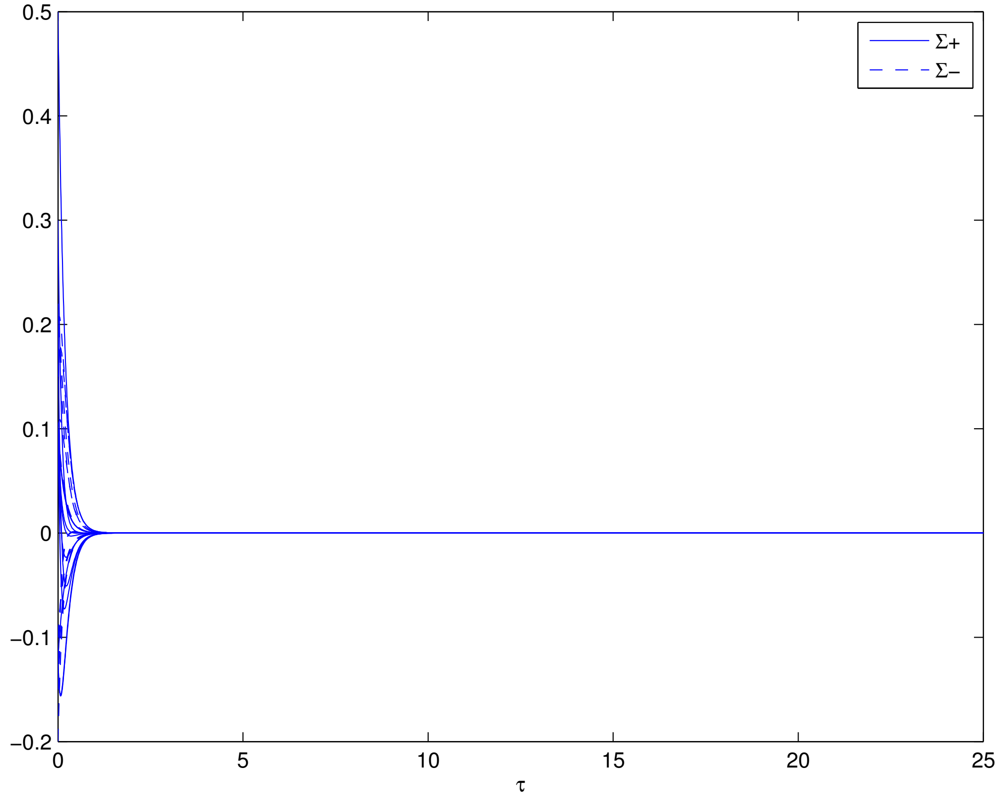


Fig. 2.2 continued.



2.5.2 $\xi_0 = 0.30$, $\eta_0 = 1$, $w = 0.325$ (Dust/Radiation Mixture)

Figure 2.3: This figure shows the dynamical system behaviour for $\xi_0 = 0.30$, $\eta_0 = 1$, and $w = 0.325$. The plus sign indicates the equilibrium point. Notice how the equilibrium point in this case, the Bianchi Type I / Flat FL point is indeed the local sink. The model also isotropizes as can be seen from the last figure, where $\Sigma_{\pm} \to 0$ as $\tau \to \infty$.

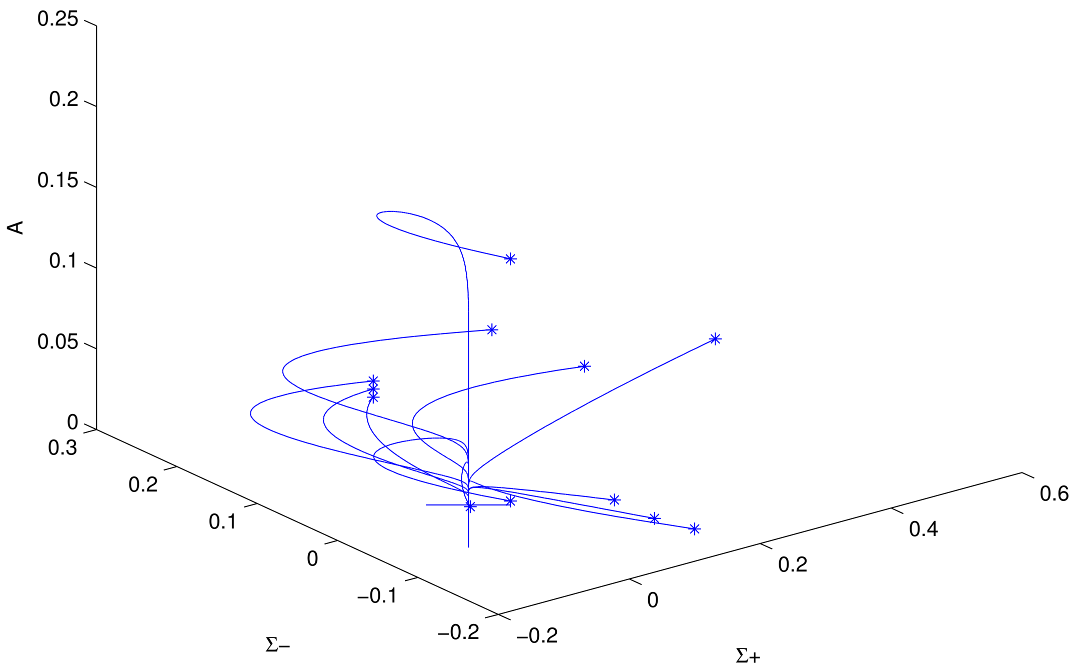


Fig. 2.3 continued.

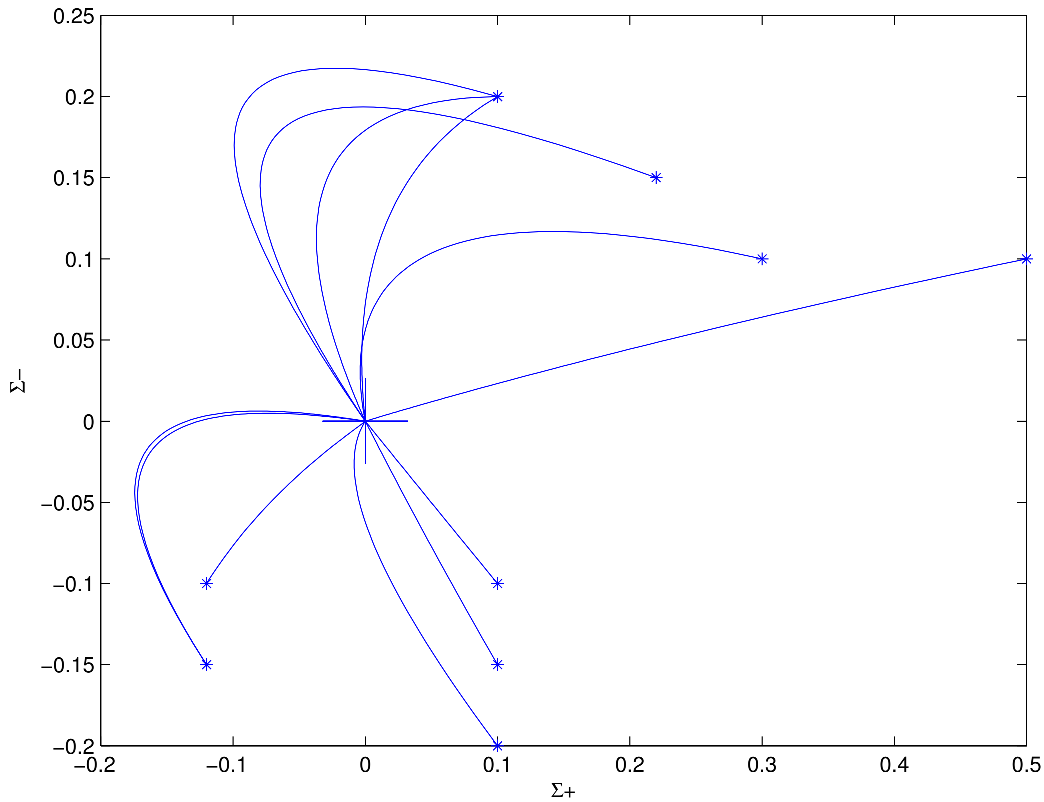


Fig. 2.3 continued.

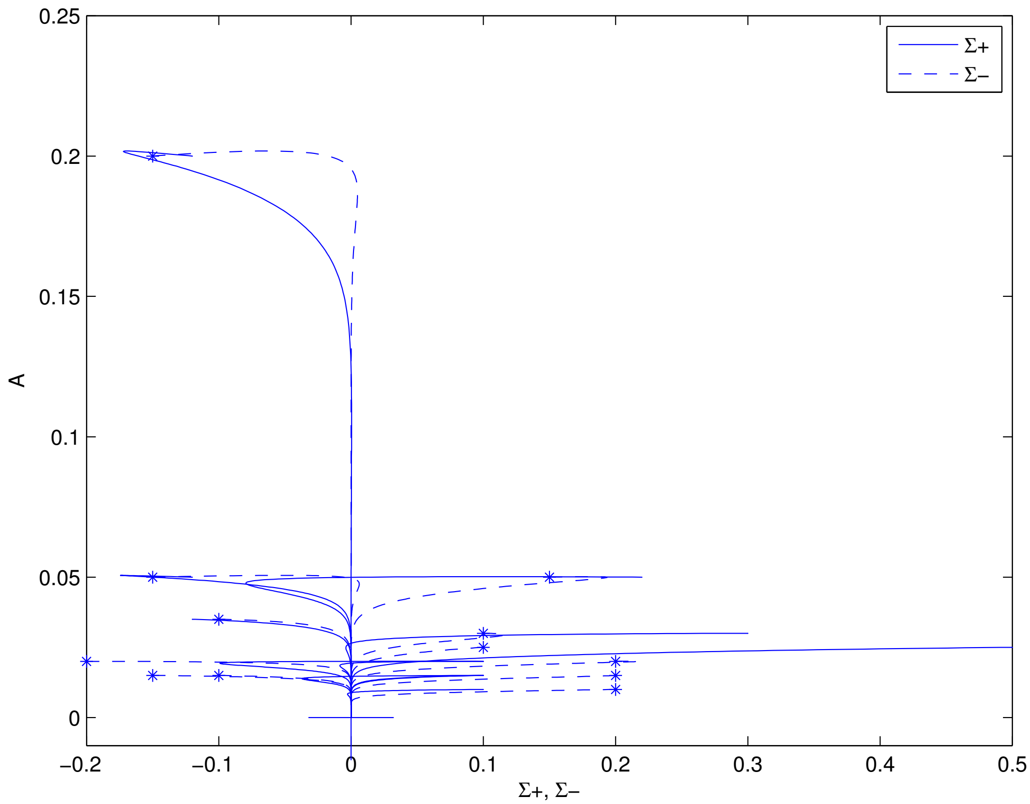
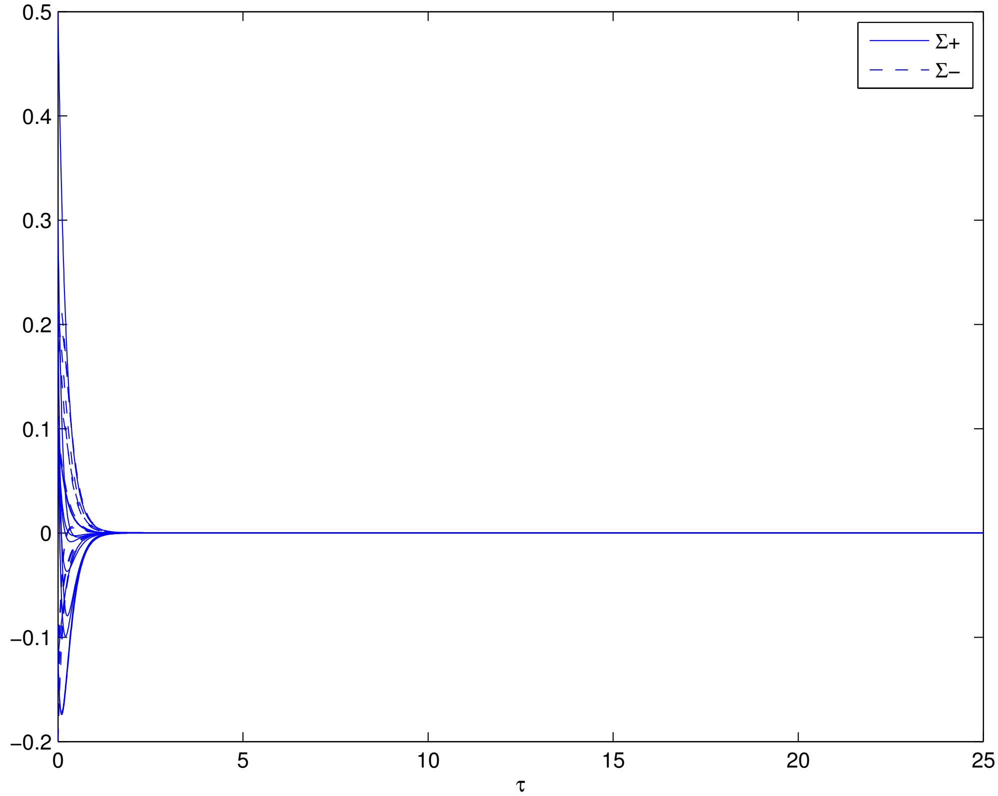


Fig. 2.3 continued.



2.5.3 $\xi_0 = 2$, $\eta_0 = \frac{1}{2}$, $w = \frac{1}{3}$ (Radiation)

Figure 2.4: This figure shows the dynamical system behaviour for $\xi_0 = 2$, $\eta_0 = \frac{1}{2}$, and $w = \frac{1}{3}$. The plus sign indicates the equilibrium point. Notice how the equilibrium point in this case, the Bianchi Type I / Flat FL point is indeed the local sink. The model also isotropizes as can be seen from the last figure, where $\Sigma_{\pm} \to 0$ as $\tau \to \infty$.

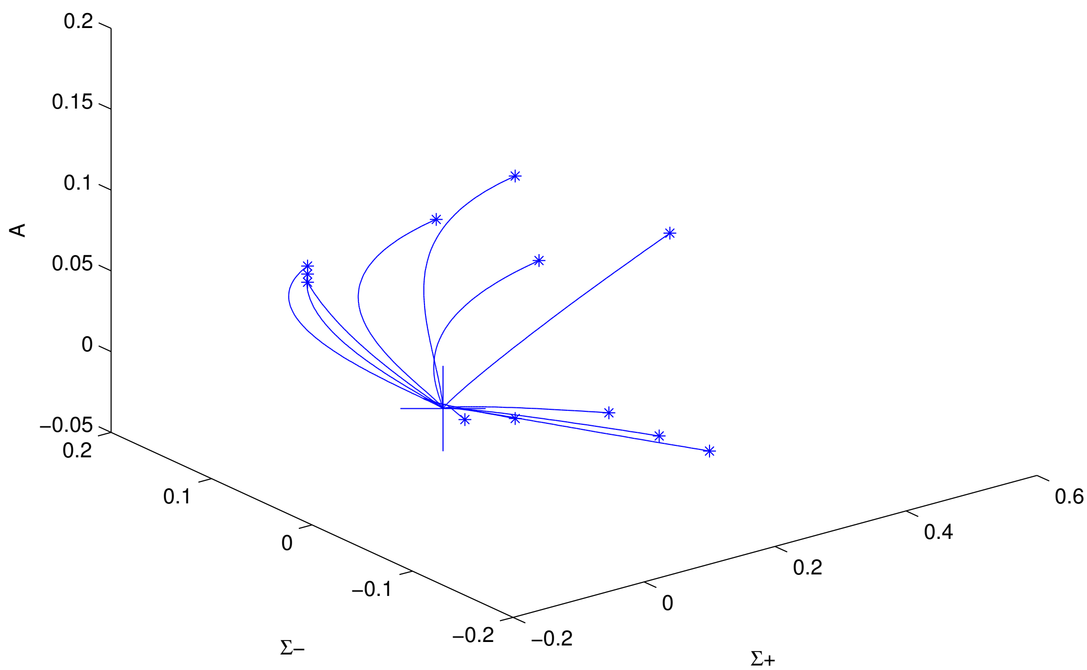


Fig. 2.4 continued.

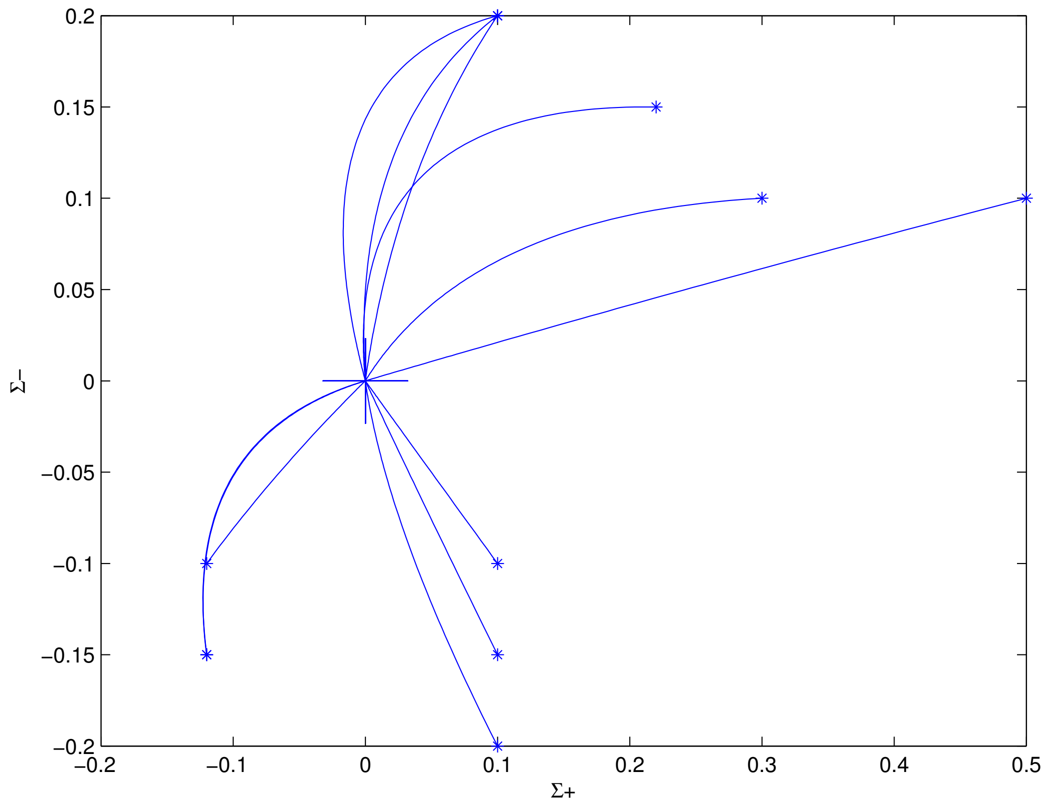


Fig. 2.4 continued.

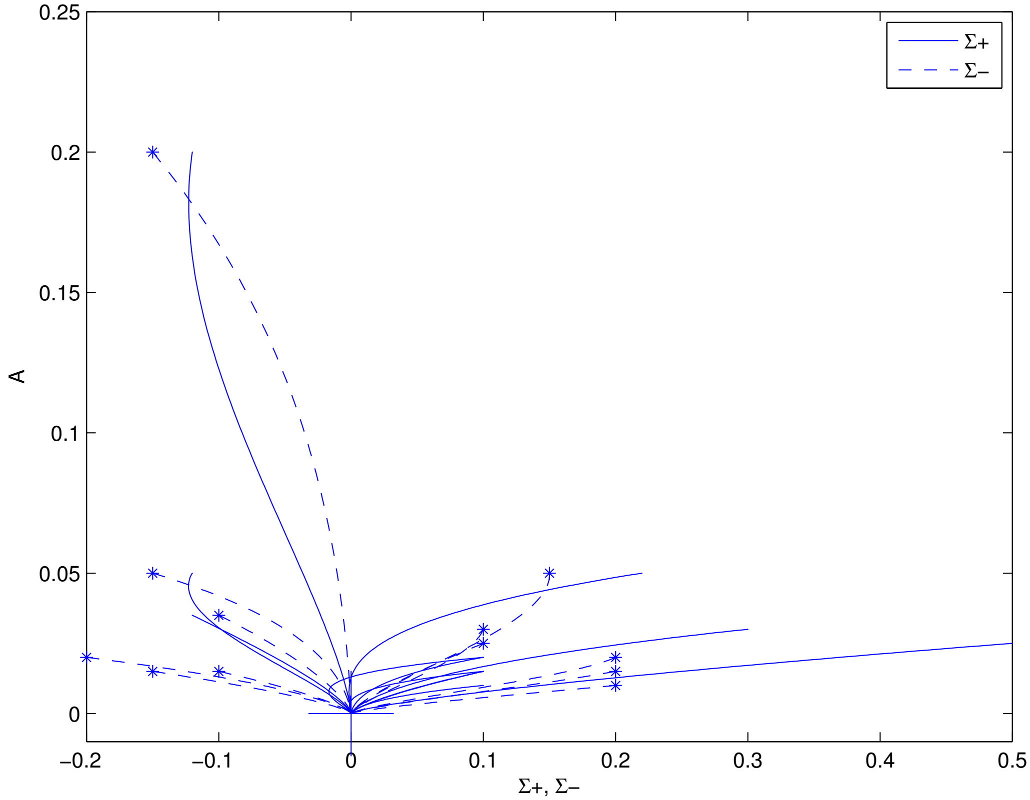
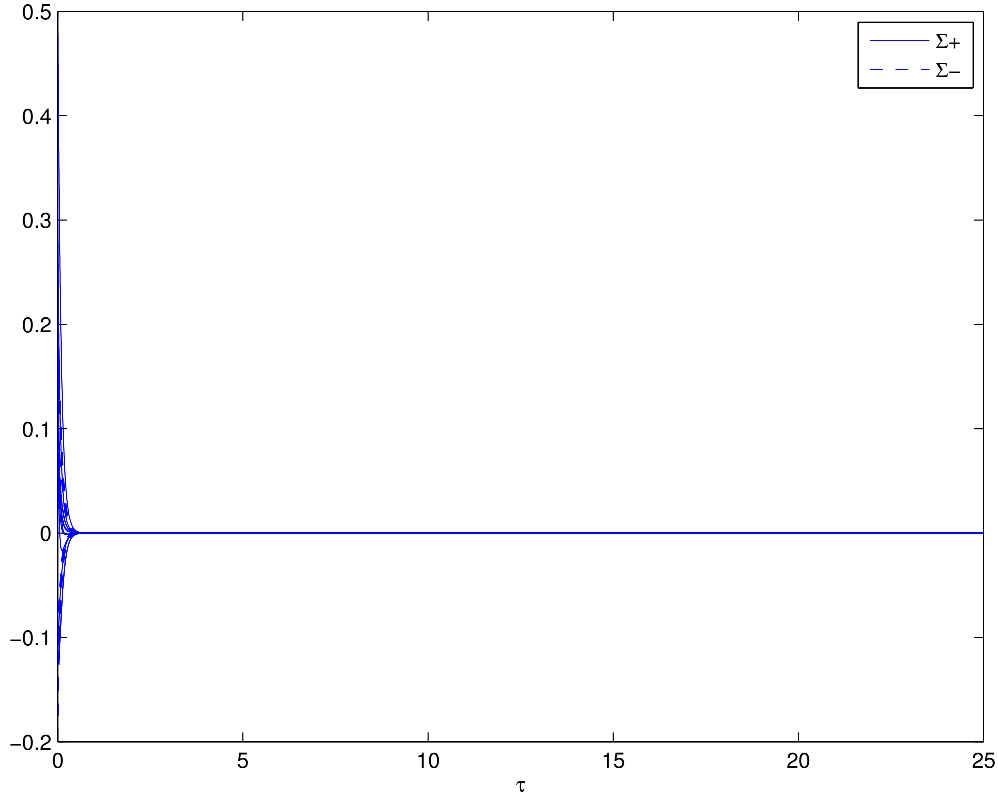


Fig. 2.4 continued.



2.5.4 $\xi_0 = 0.05$, $\eta_0 = 1$, $w = 0$ (Dust)

Figure 2.5: This figure shows the dynamical system behaviour for $\xi_0 = 0.05$, $\eta_0 = 1$, and $w = 0$. The plus sign indicates the equilibrium point. Notice how the equilibrium point in this case, the Bianchi Type V / Open FL point is indeed the local sink. The model also isotropizes as can be seen from the last figure, where $\Sigma_\pm \to 0$ as $\tau \to \infty$.

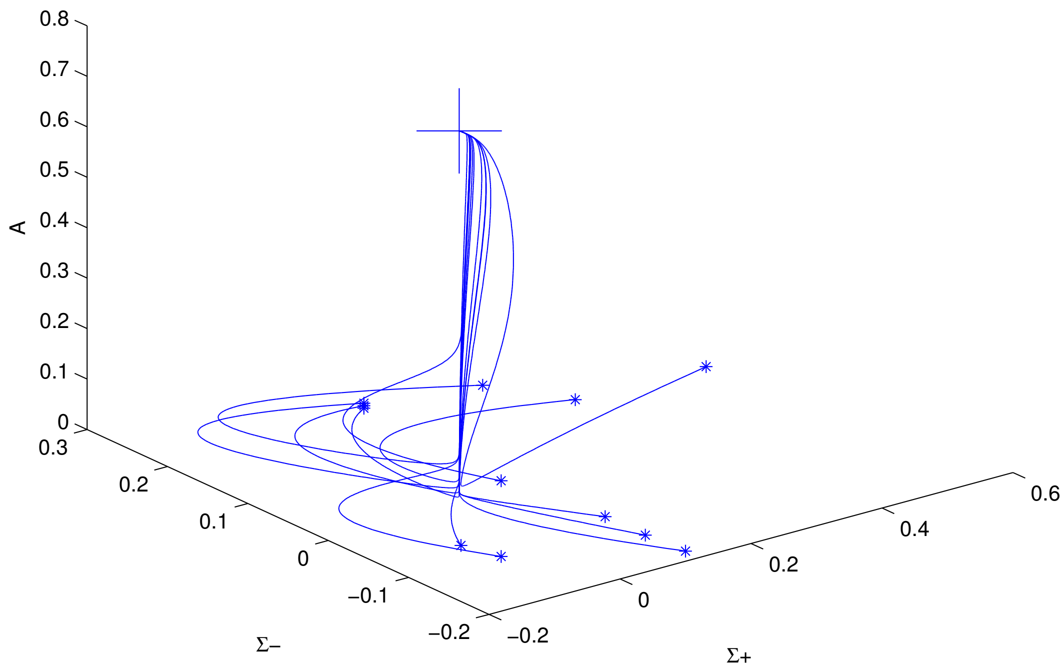


Fig. 2.5 continued.

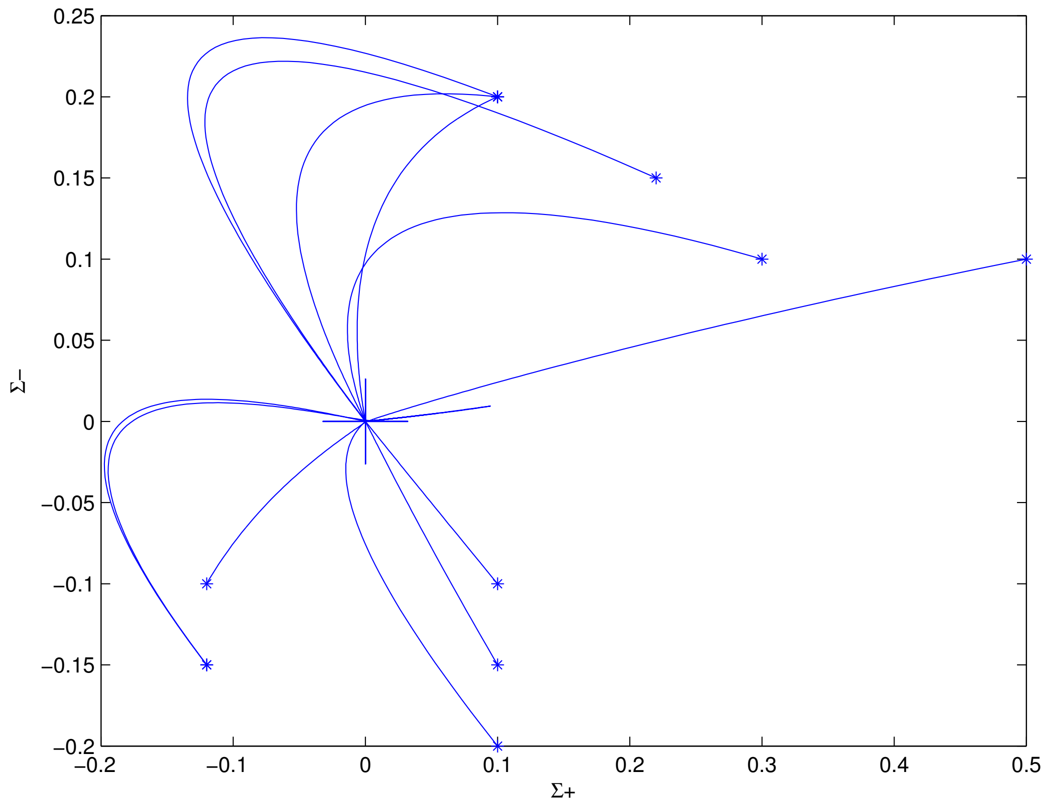


Fig. 2.5 continued.

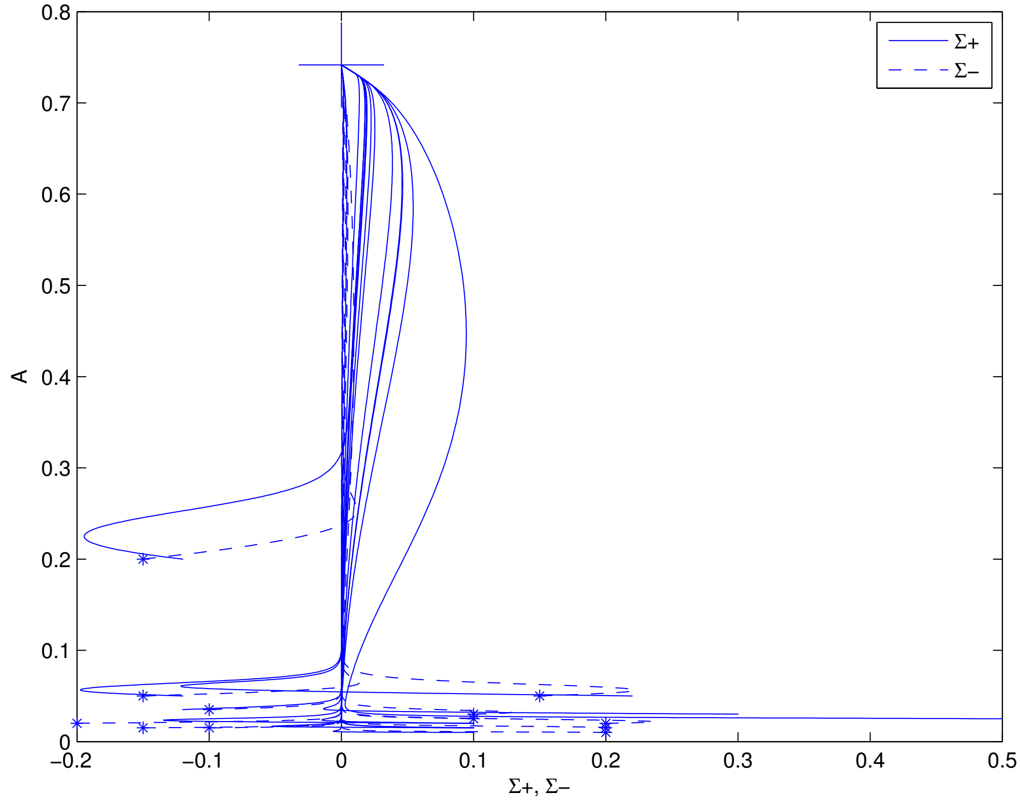
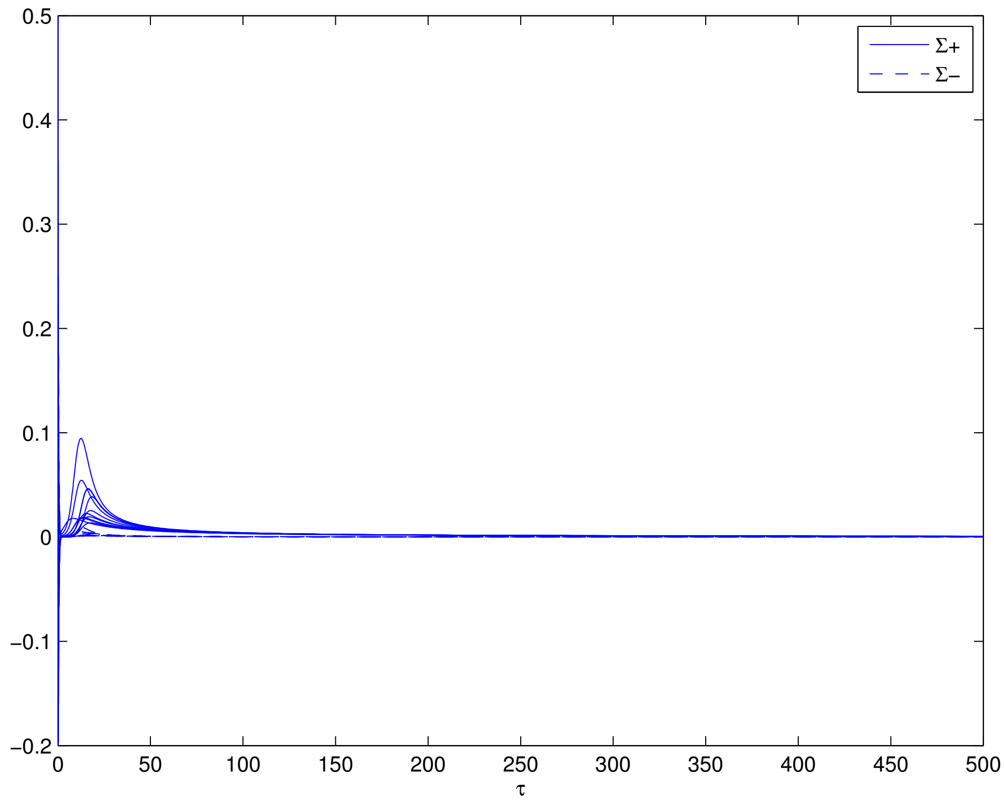


Fig. 2.5 continued.



2.5.5 $\xi_0 = 0.15$, $\eta_0 = 1$, $w = 0.325$ (Dust/Radiation Mixture)

Figure 2.6: This figure shows the dynamical system behaviour for $\xi_0 = 0.15$, $\eta_0 = 1$, and $w = 0.325$. The plus sign indicates the equilibrium point. Notice how the equilibrium point in this case, the Bianchi Type V / Open FL point is indeed the local sink. The model also isotropizes as can be seen from the last figure, where $\Sigma_\pm \to 0$ as $\tau \to \infty$.

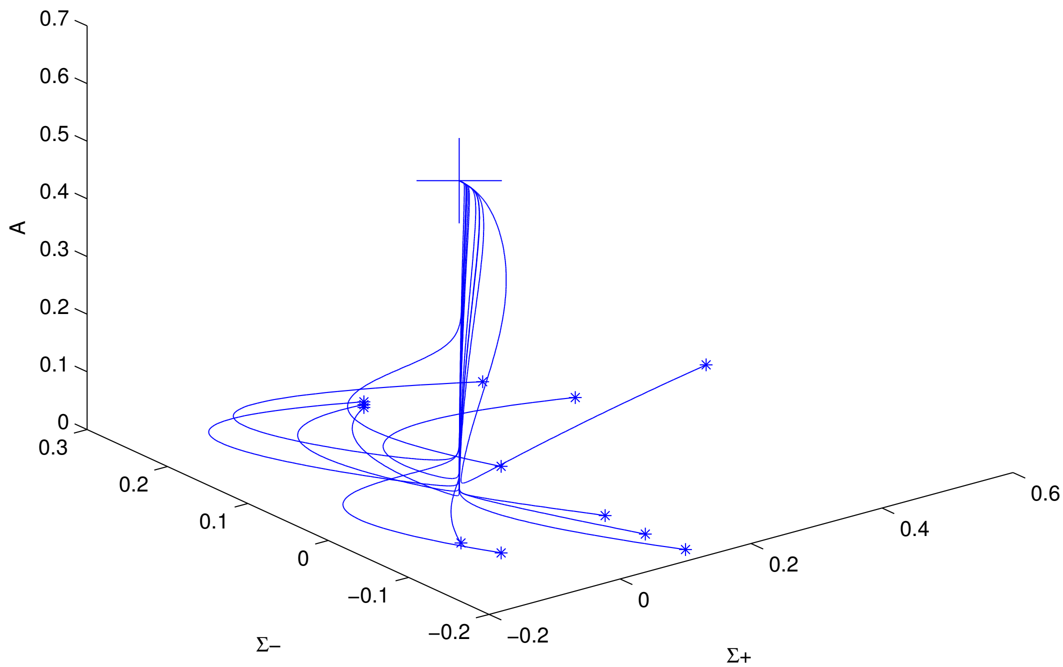


Fig. 2.6 continued.

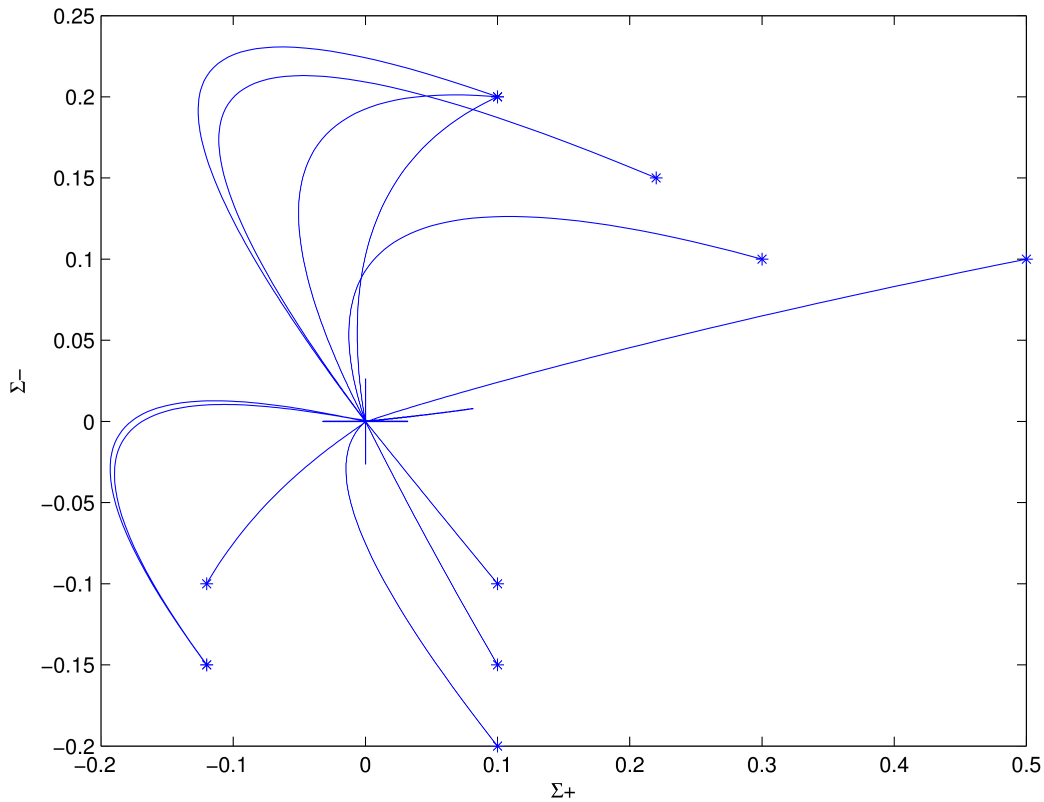


Fig. 2.6 continued.

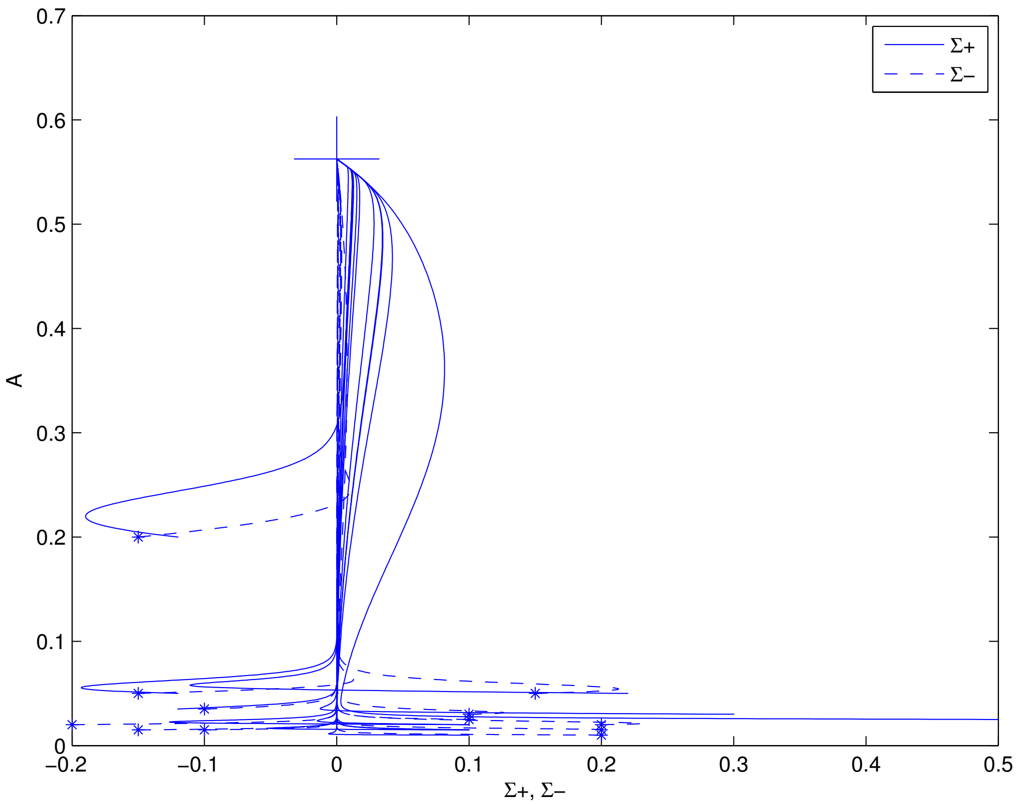
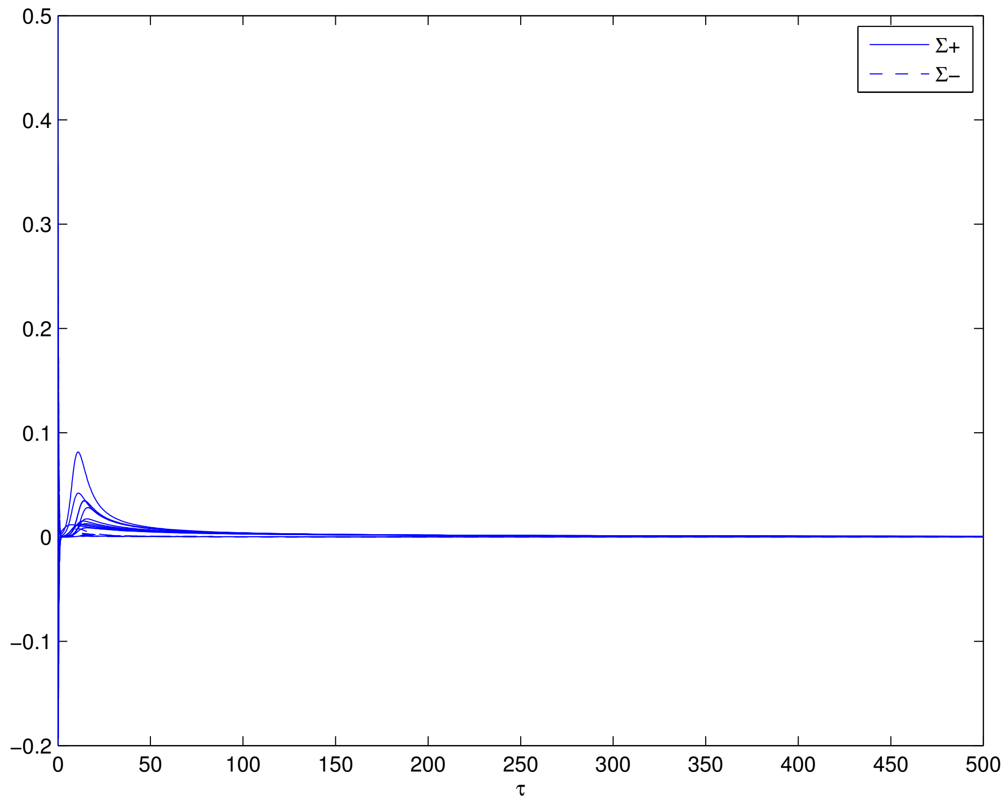


Fig. 2.6 continued.



2.5.6 $\xi_0 = 0$, $\eta_0 = 1$, $w = \frac{1}{3}$ (Radiation)

Figure 2.7: This figure shows the dynamical system behaviour for $\xi_0 = 0$, $\eta_0 = 1$, and $w = \frac{1}{3}$. The plus sign indicates the equilibrium point. Notice how the equilibrium point in this case, the isotropic Milne universe, is indeed the local sink. The model also isotropizes as can be seen from the last figure, where $\Sigma_{\pm} \to 0$ as $\tau \to \infty$.

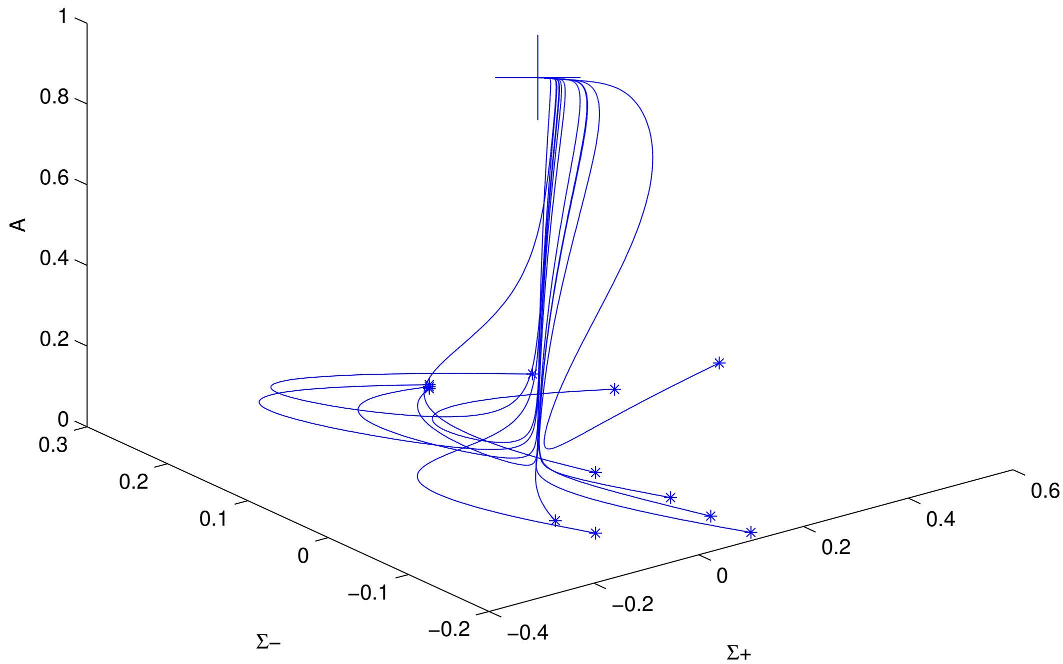


Fig. 2.7 continued.

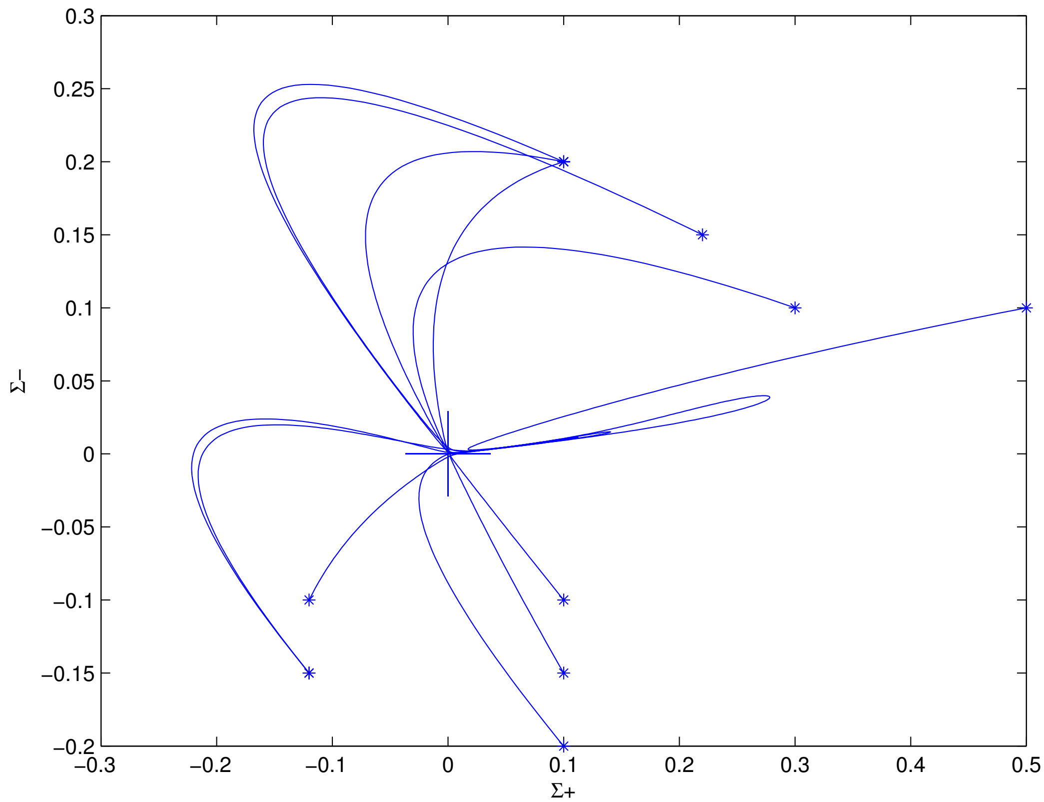


Fig. 2.7 continued.

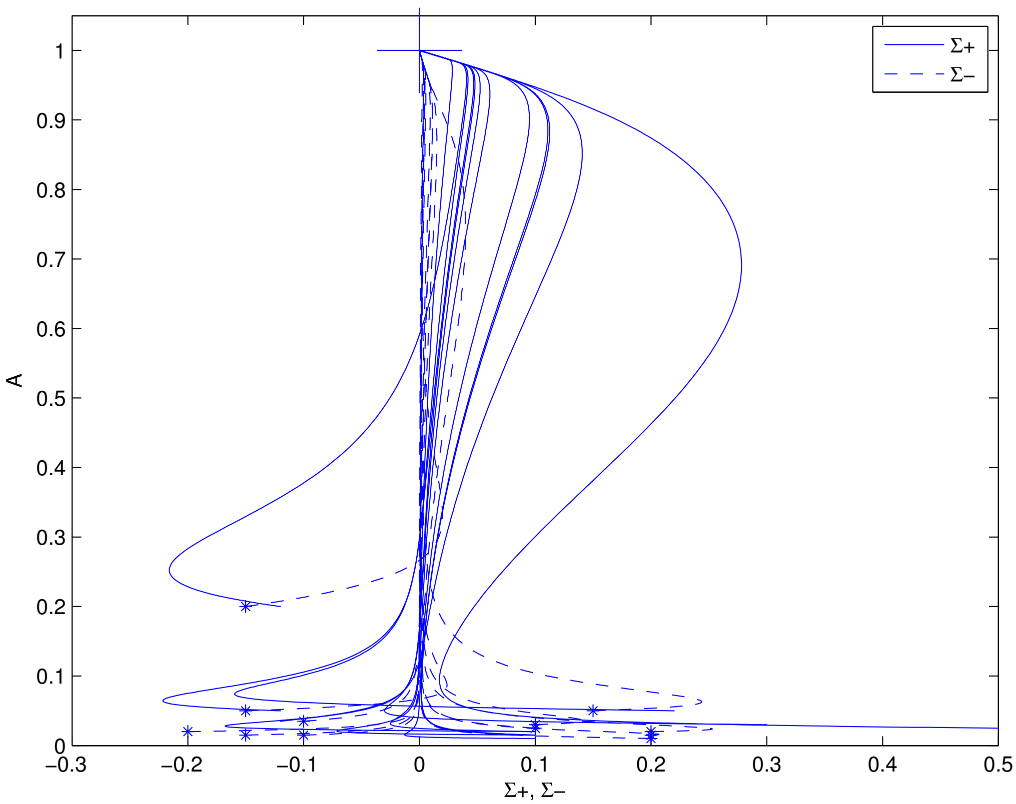
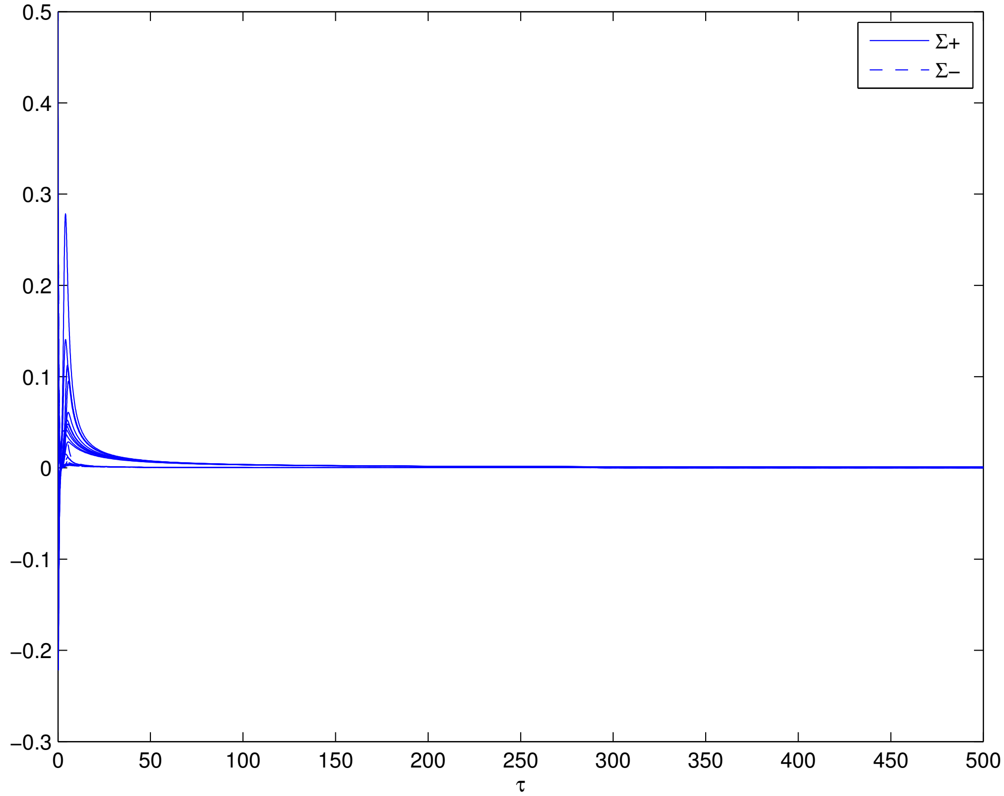


Fig. 2.7 continued.



2.5.7 Interpretation of Numerical Results

In Figs. 2.2, 2.3, and 2.4 we chose values for ξ_0, η_0, and w that satisfied Eq. (2.44), such that we could model physically interesting situations of dust, a radiation/dust mixture, and radiation. It was clear from these figures that the dynamical system had a local sink at the origin, corresponding to the flat FL solution. It is also of interest to note that the models isotropized asymptotically, as one would expect from any models that asymptotically approach the FL solutions.

In Figs. 2.5, 2.6, and 2.7 we chose values for ξ_0, η_0, and w that satisfied Eq. (2.49), such that we could model the physically interesting situations of dust, a radiation/dust mixture, and radiation. It is clear from the numerical simulations, which were done over sufficiently long time scales, that the models had the Bianchi Type V / open FL solution as a local sink. The models also were found to asymptotically isotropize. Interestingly, in Fig. 2.7, where we chose $\xi_0 = 0$, then set $A = 1$, the equilibrium point represented the isotropic Milne universe. As one can see from the numerical simulations, it is clear that the isotropic Milne universe is a local sink, which we will elaborate on further in what follows.

In their detailed study of the asymptotic behaviour of Bianchi Type Class B models, Hewitt and Wainwright [HW93] state two interesting conjectures that apply to our present work. Both conjectures have to do with asymptotic stability in a global sense. The first conjecture states that a Bianchi Type IV perfect fluid model with equation of state parameter w satisfying $-1 < w < -\frac{1}{3}$, is asymptotic at late times to the flat FL model. Looking at our inequality in Eq. (2.44), we said that the viscous fluid Bianchi Type IV model has the flat FL solution as a local sink if $0 \leq \xi_0 \leq \frac{4}{9}$ and $-1 \leq w < \frac{1}{3}(-1 + 9\xi_0)$. It is our assumption that Hewitt and Wainwright's conjecture only considered *inviscid* perfect fluid models, such that $\xi_0 = \eta_0 = 0$, as it is well-known that a perfect fluid can indeed include a bulk viscous pressure. If one sets $\xi_0 = \eta_0 = 0$, the inequality in Eq. (2.44) becomes $-1 \leq w < -\frac{1}{3}$, which would match the findings of Hewitt and Wainwright. Of course, the second inequality in Eq. (2.44), where $\xi_0 > \frac{4}{9}$, is not considered by Hewitt and Wainwright, but the flat FL model is a local sink in this region as well.

Hewitt and Wainwright's second conjecture is that a Bianchi Type IV perfect fluid model with equation of state parameter w satisfying $-\frac{1}{3} < w < 1$ is asymptotic at late times to a plane-wave model. Once again, setting $\eta_0 = \xi_0 = 0$ in Eq. (2.49), the inequality reduces to $-\frac{1}{3} < w < 1$, which is precisely the domain under consideration and Hewitt and Wainwright's work. Our model, according to Eq. (2.49) has a Bianchi Type V model as a local sink in this domain. Interestingly, the local sink represents an open FL model if $0 < A < 1$ which is true if $0 < \xi_0 < \frac{4}{9}$, and represents the Milne model if $\xi_0 = 0$, where $A = 1$. The Milne model is the isotropic limit of the Bianchi Type IV plane wave solutions [HvdHC05], and so Hewitt and Wainwright's second conjecture is satisfied in this case as well.

Therefore, there is strong evidence that Hewitt and Wainwright's conjectures for perfect, inviscid fluids can be extended to models having viscous fluids, at least for the viscous fluids considered here with constant expansion-normalized bulk and shear viscosity coefficients. Indeed, looking at the auxiliary equation for Ω' in Eqs. (2.14), we obtain

$$\Omega' = (2q-1)\Omega - 3w\Omega + 9\xi_0 + \frac{2}{3}\eta_0 \left[3\Sigma_3^2 + \frac{9}{2}\Sigma_-^2 + \frac{1}{2}\Sigma_+^2\right]. \tag{2.53}$$

Applying Eq. (2.26), we see that our equation for Ω' becomes

$$\Omega' = \frac{1}{3}\left[\eta_0\left(6\Sigma_3^2 + 9\Sigma_-^2 + \Sigma_+^2\right) + 27\xi_0 + \Omega\left(-3 + 12\Sigma_3^2 + 12\Sigma_-^2 + 12\Sigma_+^2 - 9w - 27\xi_0\right) + \Omega^2\left(3+9w\right)\right]. \tag{2.54}$$

Let us consider for the time being inflationary models where $-1 \leq w < -\frac{1}{3}$ and $\Omega > 0$, as is done in [HW93]. We will also make the general assumption that $\xi_0 \geq 0$, $\eta_0 \geq 0$. We see from Eq. (2.54) that

$$\frac{1}{3}\left[\eta_0\left(6\Sigma_3^2 + 9\Sigma_-^2 + \Sigma_+^2\right) + 27\xi_0 + \Omega\left(-3 + 12\Sigma_3^2 + 12\Sigma_-^2 + 12\Sigma_+^2 - 9w - 27\xi_0\right) + \Omega^2\left(3+9w\right)\right] = 0, \tag{2.55}$$

if $\Omega = 1$. We therefore can extend Hewitt and Wainwright's conjecture for inflationary models as follows. If $-1 \leq w < \frac{1}{3}$, $\xi_0 \geq 0$, $\eta_0 \geq 0$, and $\Omega > 0$, then for any orbit Γ, $\omega(\Gamma) = P(I)$, where $P(I)$ characterizes the flat FL point. We see from the calculation above that the right side of Eq. (2.54) vanishes if $\Omega = 1$, which is precisely the conclusion reached by Hewitt and Wainwright for inviscid perfect fluids. Therefore, by the LaSalle invariance principle, $\omega(\Gamma) \subseteq \{\Omega = 1\}$. Since Ω was assumed to be strictly increasing and $\Omega = 1$ denotes $P(I)$, it follows that the non-vacuum Bianchi IV model under consideration here is asymptotic in the future to the flat FL model and hence, isotropizes.

For the case where $-\frac{1}{3} < w < 1$, the proof of the existence of asymptotic sinks is much more difficult. All we have been able to do is provide some strong evidence for a local sink through our computations of the eigenvalues in Eqs. (2.47), (2.48), (2.49), which is further supported by the long-time numerical solutions presented in Figs. 2.5, 2.6, and 2.7.

As mentioned in the introduction, Hervik, van den Hoogen, and Coley studied the future asymptotic behaviour of tilted Bianchi Type IV models with a perfect fluid. We therefore find it appropriate to compare our results to theirs in the limits of no viscosity and tilt. In this regard, they also found as equilibrium points: the Bianchi Type I/flat FL solution, the Bianchi Type V / open FL solution, and as a special case of the Bianchi Type IV, the isotropic Milne solution. Indeed, they also found that for $-\frac{1}{3} < w < 1$, the isotropic Milne solution is a stable future attractor in the isotropic limit of the plane-wave equilibrium points of Bianchi Type IV. They also confirmed that for inflationary fluids, where $-1 < w < -\frac{1}{3}$, the flat Friedmann solution is indeed a stable future attractor [HvdHC05], which was also a conclusion reached by Hewitt, Bridson, and Wainwright in their study of the titled Bianchi Type II models [HBW01].

2.6 Conclusions

We have used a dynamical systems approach combined with a sophisticated numerical analysis to analyze the future asymptotic behaviour of a non-tilted Bianchi Type IV viscous fluid model with constant non-negative expansion-normalized shear and bulk viscosity coefficients. After deriving the equations of motion, we proceeded with a fixed-point analysis and found the corresponding equilibrium points. The future asymptotic behaviour of the non-titled Bianchi IV viscous fluid models can be summarized as follows:

1. $\eta_0 \geq 0, \quad \left\{\left[0 \leq \xi_0 \leq \frac{4}{9}\right] \bigcap \left[-1 \leq w < \frac{1}{3}\left(-1+9\xi_0\right)\right]\right\} \bigcup \left\{\left[\xi_0 > \frac{4}{9}\right] \bigcap \left[-1 \leq w < 1\right]\right\}$: Asymptotically flat FL.

2. $\eta_0 \geq 0, \quad \left\{0 < \xi_0 < \frac{4}{9}\right\} \bigcap \left\{\frac{1}{3}\left[-1+9\xi_0\right] < w < 1\right\}$: Asymptotically open FL.

3. $\eta_0 \geq 0, \quad \left\{\xi_0 = 0\right\} \bigcap \left\{-\frac{1}{3} < w < 1\right\}$: Asymptotically isotropic Milne universe (which is an isotropic limit of the plane-wave equilibrium points of Bianchi Type IV [HvdHC05]).

We also established that bifurcations exist such that the spatial curvature destabilizes the flat FL point if $\xi_0 = \frac{1}{9}\left(1+3w\right)$, and destabilizes the Bianchi Type V point if $\xi_0 = \frac{1}{9}\left(1+3w\right)$ and $-\frac{1}{3} < w < 1$.

Finally, we showed that for each numerical solution, our Bianchi Type IV model with constant viscous coefficients isotropized at late times for the regions corresponding to the flat FL, open FL, and isotropic Milne equilibrium points.

2.7 Table of Initial Conditions

For completeness, we present in this section a table of the initial conditions used in the preceding numerical experiments.

Initial Conditions	$g(\mathbf{x})$
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.1, 0.2, -0.386603, 0.01, 0.01]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.1, 0.2, -0.773205, 0.01, 0.02]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.1, -0.2, 0.426795, 0.01, 0.02]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.1, 0.2, -0.579905, 0.01, 0.015]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.1, -0.1, 0.0475481, 0.02, 0.015]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.1, -0.15, 0.103798, 0.02, 0.015]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.3, 0.1, -0.614711, 0.02, 0.03]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.5, 0.1, -0.145753, 0.1, 0.025]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [0.22, 0.15, -0.831051, 0.025, 0.05]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [-0.12, -0.15, 0.657846, 0.025, 0.05]$	0
$[\Sigma_+, \Sigma_-, \Sigma_3, N_1, A] = [-0.12, -0.1, 0.177746, 0.05, 0.035]$	0

Table 2.1: Initial conditions used in the numerical simulations.

No. Try not. Do... or do not. There is no try.

Master Yoda, The Empire Strikes Back

3

A Dynamical Systems Approach to a Bianchi Type I Viscous Magnetohydrodynamic Model

3.1 Introduction

The current standard model of cosmology based on the Friedmann-Lemaître-Robertson-Walker (FLRW) metric assumes that the present-day universe is spatially homogeneous and isotropic, and indeed this assumption strongly concurs with empirical observation. As a result of the symmetry of this spacetime, related models must be treated within the framework of perfect fluids, in which case the shear and rotational terms in the energy-momentum tensor vanish (page 52, [WE97]).

If one wishes to formulate a cosmological model of the early universe, however, at a minimum it is necessary to include viscous (shear) terms in the energy-momentum tensor. As discussed by Grøn and Hervik (Chapter 13, [GH07]), viscous models have become of general interest in early-universe cosmologies largely in two contexts. Firstly, in models where bulk viscous terms dominate over shear terms, the universe expands to a de Sitter-like state, which is a spatially flat universe neglecting ordinary matter, and including only a cosmological constant. Such models isotropize indirectly through the massive expansion. Secondly, in the absence of any significant heat flux, shear viscosity is found to play an important role in models of the universe at its early stages. In particular, neutrino viscosity is considered to be one of the most important factors in the isotropization of our universe.

Magnetic fields have also been thought to play a major role in the early universe. Grasso and Rubinstein [GR01] reviewed in great detail the origin and possible effects of magnetic fields in the early universe. In recent work, Ando and Kusenko [AK10], examined intergalactic magnetic fields and discussed how these magnetic fields originated from primordial seed fields created shortly after the Big Bang, which relates to our understanding of the origin of cosmic magnetic fields in the early universe. In addition, Gregori et al. [Gea12] also studied the origin of galactic magnetic fields through the amplification of primordial seed fields. Schlickeiser [Sch12] described a new process by which the primordial magnetic fields arose in the universe before the emergence of the first stars.

After inflation, the early universe was a good conductor: even though the number density of free electrons dropped dramatically during recombination, its residual value was enough to maintain high conductivity in baryonic matter. As a result, cosmic magnetic fields have remained frozen into the expanding baryonic fluid during most of their evolution. In this situation, one can analyze the magnetic effects on the dynamics of the early universe through ideal magnetohydrodynamics (hereafter referred to as MHD), in which case the magnetic field source is considered to be a perfect conductor and related terms in the energy momentum tensor are simply those corresponding to a classical magnetic field (Page 115, [EMM12]).

Hughston and Jacobs [HJ70] showed that in the case of a pure magnetic field, only Bianchi Types I, II, VI($h = -1$) (which is the same as Type III), and VII ($h = 0$) admit field components, whereas Types IV, V, VI ($h = -1$), VII ($h \neq 0$), VIII, and IX admit no field components. These results led to a number of papers on Bianchi models with a perfect-fluid magnetic field source; we discuss these works briefly below. Using a dynamical systems approach, LeBlanc [LeB98] studied Bianchi Type II magnetic cosmologies in which he provided an analysis on the future and past asymptotic states of the resulting dynamical system. In a separate work, LeBlanc [LeB97] also studied the asymptotic states of magnetic perfect-fluid Bianchi Type I cosmologies. Using phase plane analysis techniques, Collins [Col72] studied the behaviour of a class of perfect-fluid anisotropic cosmological models, and established a correspondence between magnetic models of Bianchi Type I and perfect fluid models of Bianchi Type II. In addition, LeBlanc, Kerr, and Wainwright [LKW95] studied the asymptotic states of magnetic Bianchi Type VI cosmologies and showed that there is a non-zero probability that an arbitrarily selected model will be close to isotropy during some time interval in its evolution. We also note that Barrow, Maartens, and Tsagas [BMT07] did significant work in the reformulation of a $1 + 3$ covariant description of the magnetohydrodynamic equations that has provided further understanding and clarity on the role of large-scale electromagnetic fields in the perturbed Friedmann-Lemaître-Robertson-Walker models.

Viscous MHD Bianchi models treated using a metric approach have appeared in the literature on a number of occasions. van Leeuwen and Salvati [vLS85] studied the dynamics of general Bianchi class A models containing a magneto-viscous fluid and a large-scale magnetic field. Banerjee and Sanyal [BS86] presented some exact solutions of Bianchi Types I and III cosmological models consisting of a viscous fluid and axial magnetic field. Benton and Tupper [BT86] studied Bianchi Type I models with a “powers-of-t” metric under the influence of a viscous fluid with a magnetic field . Salvati, Schelling, and van Leeuwen [SSvL87] numerically analyzed the evolution of the Bianchi type I universe with a viscous fluid and large-scale magnetic field. Ribeiro and Sanyal [SR87] studied a Bianchi Type VI_0 viscous fluid cosmology with an axial magnetic field in which they obtained exact solutions to the Einstein field equations assuming linear relations among the square root of matter density and the shear and expansion scalars. van Leeuwen, Miedema, and Wiersma [vLMW89] proved that a non-rotating Bianchi model of class A containing a viscous fluid and magnetic field can only be of Type I or IV_0. Pradhan and Pandey [PP03] studied the Bianchi Type I model with a bulk viscous fluid in addition to a varying cosmological constant. Pradhan and Singh [PS04] studied the Bianchi Type I model in the presence of a magnetic field and shear and bulk viscosity, but assumed that the shear tensor was proportional to the expansion tensor. Bali and Anjali [BA04] studied a Bianchi Type I magnetized fluid model with a bulk viscous string dust fluid, in which they compared their results in the presence and absence of large-scale magnetic fields.

Here we examine a viscous MHD Bianchi Type I non-tilted viscous magnetohydrodynamic model. In contrast to the references cited above, which use a metric approach, we use the Hubble-normalized dynamical systems approach based upon the theory of orthonormal frames pioneered by Ellis and MacCallum [EM69]. In treating a problem with the method of Ellis and MacCallum, the Einstein field equations (a coupled set of ten hyperbolic nonlinear partial differential equations) are reduced to a system of autonomous nonlinear first-order ordinary differential equations. In a previous work [KH13b], we employed such an approach to treat a Bianchi Type IV viscous model in the absence of magnetic sources. To the best of our knowledge, a treatment of a viscous MHD model along these lines has not yet appeared in the literature. In the present work, we examine the important role of the fixed points of the dynamical system. In particular we show that the fixed points may be associated with Kasner-type solutions, a flat universe FLRW solution, and interestingly, a new solution to the Einstein field equations involving non-zero magnetic fields, and non-zero viscous coefficients. We examine several features of the dynamical system, including its early and late time asymptotic behaviour, and its bifurcation behaviour. Finally, numerical results are presented that illustrate the behaviour of the system over long times with several initial configurations. In several cases of interest, it is shown that the dynamical model isotropizes asymptotically; that is, the spatial anisotropy and the anisotropic magnetic field decay to negligible values giving a close approximation to the present-day universe. Throughout this work, we assume that the signature of the metric tensor is $(-,+,+,+)$, and the use of *geometrized units*, where $G = c = 1$.

3.2 The Matter Sources

In the absence of heat conduction, the energy-momentum tensor corresponding to a viscous fluid cosmological model with fluid velocity four-vector u_a is given by [KH13b]

$$\mathcal{V}_{ab} = (\mu_f + p_f)u_a u_b + g_{ab}p_f - 3\xi H h_{ab} - 2\eta\sigma_{ab}, \tag{3.1}$$

where μ_f, p_f, and σ_{ab} denote the fluid's energy density, pressure, and shear tensor, respectively. In addition, the quantities ξ and η denote the bulk and shear viscosity coefficients of the fluid, respectively, H denotes the Hubble parameter, and $h_{ab} \equiv u_a u_b + g_{ab}$ denotes the projection tensor corresponding to the metric signature $(-,+,+,+)$.

The energy-momentum tensor corresponding to an electromagnetic field is given by [Ell73]

$$\mathcal{T}_{ab} = \frac{1}{2}u_a u_b(E^2 + B^2) + 2u_{(a}n_{b)}^{\ \ cgd}u_c E_g B_d - E_a E_b - B_a B_b + \frac{1}{2}h_{ab}\left(E^2 + B^2\right), \tag{3.2}$$

where n^{abcd} is the standard skew pseudo-tensor, and E_a and B_a are the electric and magnetic field three-vectors, respectively. Note that in an orthonormal frame, where $g_{ab} = n_{ab} = diag(-1,1,1,1)$, the E^2 and B^2 terms in Eq. (3.2), take the form $E^2 \equiv E^a E_a = E_1^2 + E_2^2 + E_3^2$, and $B^2 \equiv B^a B_a = B_1^2 + B_2^2 + B_3^2$. In this work, we assume that the cosmological model is non-tilted, and thus in both Eqs. (3.1) and (3.2) we take u_a as the four-velocity of a comoving observer: $u^a = (1,0,0,0)$. We also assume the ideal MHD approximation, in which case the early universe behaves as a *perfect* conductor. The electric field (whose magnitude is inversely proportional to the conductivity) approaches zero, even in the presence of a non-zero electric current. In other words, we assume that after recombination, the universe is such a good conductor that the cosmic electric fields required to drive a current in it are negligible. Under these conditions, the energy-momentum tensor in Eq. (3.2) simplifies to

$$\mathcal{T}_{Bab} = \frac{1}{2}u_a u_b\left(B^2\right) - B_a B_b + \frac{1}{2}h_{ab}B^2. \tag{3.3}$$

The total energy-momentum tensor, denoted $\mathbb{T}_{ab}$, for our cosmological model is then given by

$$\mathbb{T}_{ab} = \mathcal{V}_{ab} + \mathcal{T}_{Bab}. \tag{3.4}$$

In order to formulate the evolution equations corresponding to our model, we compute from Eq. (3.4) the total energy density $\tilde{\mu}$, the total pressure $\tilde{p}$, and total anisotropic stress $\tilde{\pi}_{ab}$. Using the definitions

$$\tilde{\mu} = \mathbb{T}_{ab} u^a u^b, \quad \tilde{p} = \frac{1}{3} h^{ab} \mathbb{T}_{ab}, \quad \tilde{\pi}_{ab} = h_a^c h_b^d \mathbb{T}_{cd} - \tilde{p} h_{ab}, \tag{3.5}$$

we find that

$$\tilde{\mu} = \mu_f + \frac{1}{2}\left(B_1^2 + B_2^2 + B_3^2\right), \tag{3.6}$$

$$\tilde{p} = w\mu_f - 3\xi H + \frac{1}{6}\left(B_1^2 + B_2^2 + B_3^2\right), \tag{3.7}$$

and

$$\tilde{\pi}_{ab} = -2\eta\sigma_{ab} - B_a B_b + \frac{1}{3} h_{ab}\left(B_1^2 + B_2^2 + B_3^2\right). \tag{3.8}$$

Note that in obtaining the expression for the pressure in Eq. (3.7), we assumed that the fluid obeys the barotropic equation of state, $p_f = w\mu_f$, where $-1 \leq w \leq 1$.

It is advantageous to re-express the above quantities as expansion-normalized variables [HBW01] and we thus introduce the definitions

$$\tilde{\Omega} = \frac{\tilde{\mu}}{3H^2}, \quad \tilde{P} = \frac{\tilde{p}}{3H^2}, \quad \tilde{\Pi}_{ab} = \frac{\tilde{\pi}_{ab}}{H^2}. \tag{3.9}$$

We will also define the expansion-normalized magnetic field vector as

$$\mathcal{B}_a = \frac{B_a}{3H}. \tag{3.10}$$

The relevant expressions for the expansion-normalized variables are then given by

$$\tilde{\Omega} = \Omega_f + \frac{3}{2}\left(\mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2\right), \tag{3.11}$$

$$\tilde{P} = w\Omega_f - 3\xi_0 + \frac{1}{2}\left(\mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2\right), \tag{3.12}$$

and

$$\tilde{\Pi}_{ab} = -2\eta_0\Sigma_{ab} - 9\mathcal{B}_a\mathcal{B}_b + 3\delta_{ab}\left(\mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2\right). \tag{3.13}$$

In Eqs. (3.11), (3.12) and (3.13), $\Omega_f = \mu_f/(3H^2)$ is Hubble-normalized fluid energy density, and $\xi_0 = \xi/(3H)$ and $\eta_0 = \eta/(3H)$ are the expansion-normalized bulk and shear viscosity coefficients, respectively; these quantities are assumed to be *non-negative constants* throughout this paper. In Eq. (3.13) we also denote $\Sigma_{ab} = \sigma_{ab}/H$ as the expansion-normalized shear tensor.

3.3 Bianchi Type I Universe Dynamics

With the required energy-momentum tensor in Eq. (3.4), and the expansion-normalized source variables (Eqs. (3.11) - (3.13)) in hand, we now derive the Bianchi Type I dynamical equations. The general evolution equations for any Bianchi type are presented in [HBW01] and [HLSU10]. The general evolution equations in the expansion-normalized variables using our notation are:

$$\begin{aligned}
\Sigma'_{ij} &= -(2-q)\Sigma_{ij} + 2\epsilon^{km}_{(i}\Sigma_{j)k}R_m - \mathcal{S}_{ij} + \tilde{\Pi}_{ij}, \\
N'_{ij} &= qN_{ij} + 2\Sigma^k_{(i}N_{j)k} + 2\epsilon^{km}_{(i}N_{j)k}R_m, \\
A'_i &= qA_i - \Sigma^j_i A_j + \epsilon^{km}_i A_k R_m, \\
\tilde{\Omega}' &= (2q-1)\tilde{\Omega} - 3\tilde{P} - \frac{1}{3}\Sigma^j_i\tilde{\Pi}^i_j + \frac{2}{3}A_iQ^i, \\
Q'_i &= 2(q-1)Q_i - \Sigma^j_iQ_j - \epsilon^{km}_i R_kQ_m + 3A^j\tilde{\Pi}_{ij} + \epsilon^{km}_i N^j_k\tilde{\Pi}_{jm}.
\end{aligned} \tag{3.14}$$

These equations are subject to the constraints

$$\begin{aligned}
N^j_iA_j &= 0, \\
\tilde{\Omega} &= 1 - \Sigma^2 - K, \\
Q_i &= 3\Sigma^k_iA_k - \epsilon^{km}_i\Sigma^j_kN_{jm}.
\end{aligned} \tag{3.15}$$

As in Eq. (3.9), we have made use of the following notation:

$$\left(\Sigma_{ij}, R^i, N^{ij}, A_i\right) = \frac{1}{H}\left(\sigma_{ij}, \Omega^i, n^{ij}, a_i\right), \quad \left(\tilde{\Omega}, \tilde{P}, Q_i, \tilde{\Pi}_{ij}\right) = \frac{1}{3H^2}\left(\tilde{\mu}, \tilde{p}, q_i, \tilde{\pi}_{ij}\right). \tag{3.16}$$

In the expansion-normalized approach, the kinematic shear tensor Σ_{ab} describes the anisotropy in the Hubble flow, A_i and N^{ij} describe the spatial curvature, while Ω^i describes the relative orientation of the shear and spatial curvature eigenframes. The Bianchi Type I model is a flat anisotropic model and is Abelian, and therefore has the property that

$$A^i = 0, \quad N_{11} = N_{22} = N_{33} = 0. \tag{3.17}$$

The dynamical system (3.14) evolves according to a dimensionless time variable, τ such that

$$\frac{dt}{d\tau} = \frac{1}{H}, \tag{3.18}$$

where H is the Hubble parameter with evolution equation

$$H' = -(1+q)H. \tag{3.19}$$

The deceleration parameter q is very important in the expansion-normalized approach: when $q < -1$ the universe expansion is accelerating, when $q > -1$ the universe expansion is decelerating, and when $q = -1$ the universe is static, that is, it is not self-similar. From Eq. (1.90) in [WE97], and using Eq. (3.16), the parameter q may be written as

$$\begin{aligned} q &\equiv 2\Sigma^2 + \frac{1}{2}\left(\tilde{\Omega} + 3\tilde{P}\right) \\ &= 2\Sigma^2 + \Omega_f\left(\frac{1}{2} + \frac{3w}{2}\right) - \frac{9}{2}\xi_0 + \frac{3}{2}\left(\mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2\right), \end{aligned} \tag{3.20}$$

where $2\Sigma^2 \equiv \left(\Sigma_{ab}\Sigma^{ab}\right)/3$.

In the case of a magnetic field source, one must also include an evolution equation for the magnetic field, which is the orthonormal frame analog of the standard Maxwell-Faraday equation. According to Eq. (71) in [vEU97], Eq. (2.4) in [LKW95], and Eqs. (3.10), (3.16), (3.18), and (3.19) above, the magnetic field evolution is given by

$$\mathcal{B}_a' = \mathcal{B}_a\left(-1+q\right) + \Sigma_{ab}\mathcal{B}^b + \epsilon_{abv}R^v\mathcal{B}^b. \tag{3.21}$$

For convenience, we introduce the notation

$$\Sigma_+ = \frac{1}{2}\left(\Sigma_{22} + \Sigma_{33}\right), \quad \Sigma_- = \frac{1}{2\sqrt{3}}\left(\Sigma_{22} - \Sigma_{33}\right), \tag{3.22}$$

such that $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2$. In the evolution equations (3.14), the expansion-normalized angular velocity variables R_a can be found from the non-diagonal shear equations, $\Sigma_{12}', \Sigma_{23}'$, and Σ_{13}'. From these equations, we get that

$$R_1 = -\frac{3\sqrt{3}\mathcal{B}_2\mathcal{B}_3}{2\Sigma_-}, \quad R_2 = \frac{9\mathcal{B}_1\mathcal{B}_3}{\sqrt{3}\Sigma_- - 3\Sigma_+}, \quad R_3 = \frac{9\mathcal{B}_1\mathcal{B}_2}{\sqrt{3}\Sigma_- + 3\Sigma_+}. \tag{3.23}$$

To avoid situations where R_1, R_2, or R_3 become singular, we will set $\mathcal{B}_1 = \mathcal{B}_3 = 0$, and keep $\mathcal{B}_2 \neq 0$, hence assuming that the magnetic field acts in a single spatial direction, as is done in [TM00], [Dor65], [Col72], and [Tho67]. Then, $R_1 = R_2 = R_3 = 0$, and according to Eqs. (3.23), (3.22), (3.17), and (3.20), the evolution equations (3.14) become:

$$\begin{aligned}
\Sigma_+' &= -\frac{3}{2}\mathcal{B}_2^2 + \Sigma_+\left[q - 2\left(1+\eta_0\right)\right], && (3.24)\\
\Sigma_-' &= -\frac{3\sqrt{3}}{2}\mathcal{B}_2^2 + \Sigma_-\left[q - 2\left(1+\eta_0\right)\right], && (3.25)\\
\mathcal{B}_2' &= \mathcal{B}_2\left(-1 + q + \sqrt{3}\Sigma_- + \Sigma_+\right), && (3.26)
\end{aligned}$$

where the deceleration parameter is now given by

$$q = 2\left(\Sigma_+^2 + \Sigma_-^2\right) + \Omega_f\left(\frac{1}{2} + \frac{3w}{2}\right) - \frac{9}{2}\xi_0 + \frac{3}{2}\mathcal{B}_2^2. \tag{3.27}$$

In Eq. (3.27) we have defined the energy density as

$$\Omega_f = 1 - \frac{3}{2}\mathcal{B}_2^2 - \Sigma_-^2 - \Sigma_+^2 \geq 0, \tag{3.28}$$

which, as indicated in Eq. (3.28), is restricted to be non-negative on physical grounds. After some algebra, the auxiliary equation in (3.14) becomes

$$\Omega_f' = \Omega_f\left(2q - 1 - 3w\right) + 4\eta_0\left(\Sigma_+^2 + \Sigma_-^2\right) + 9\xi_0. \tag{3.29}$$

In seeking solutions to (3.24), (3.25) and (3.26), we further enforce the physical restrictions

$$-1 \leq w \leq 1, \quad \xi_0 \geq 0, \quad \eta_0 \geq 0, \tag{3.30}$$

on the state parameter, bulk and shear viscosity coefficients, respectively. Any combinations of these parameters must additionally satisfy $\Omega_f \geq 0$, $\Sigma_+ \in \mathbb{R}, \Sigma_- \in \mathbb{R}$, and $\mathcal{B}_2 \geq 0$.

3.4 A Fixed-Point Analysis

We now consider the local stability of the equilibrium points of the system (3.24)-(3.26), which we abbreviate as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{3.31}$$

Here $\mathbf{x} = [\Sigma_+, \Sigma_-, \mathcal{B}_2] \in \mathbf{R}^3$, and the vector function $\mathbf{f}(\mathbf{x})$ denotes the right-hand-side of the dynamical system. The state space of the system is the subset of $\mathbb{R}^3$ defined by the inequality in Eq. (3.28), which is equivalent to

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{2}\mathcal{B}_2^2 \leq 1, \tag{3.32}$$

so the state space is clearly bounded. This inequality also is a constraint for the initial conditions of the dynamical system. There is only one symmetry of the dynamical system, given by

$$[\Sigma_+, \Sigma_-, \mathcal{B}_2] \to [\Sigma_+, \Sigma_-, -\mathcal{B}_2]\,. \tag{3.33}$$

The system is therefore invariant with respect to spatial inversions in the function $\mathcal{B}_2$, and we can take $\mathcal{B}_2 \geq 0$. In most cases, we examine the stability of the critical points $\mathbf{a}$, where $\mathbf{f}(\mathbf{a}) = 0$, by locally linearizing the system leading to the relationship $\mathbf{x}' = D\mathbf{f}(\mathbf{a})\mathbf{x}$. The stability of the system is then determined by the sign of the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{a})$. In the work that follows, we will denote eigenvalues of the dynamical system by λ_i, where $i = 1, 2, 3, ...$

3.4.1 Kasner Equilibrium Points

We now discuss a set of equilibrium points which are known as the Kasner solutions to the system [WE97]. Each such equilibrium point corresponds to a vacuum solution and is unstable for our model. These equilibrium points, the set of which we denote $\mathcal{K}$, lie on the *Kasner circle*

$$\Sigma_-^2 + \Sigma_+^2 = 1 \tag{3.34}$$

in the plane $\mathcal{B}_2 = 0$ for parameter values $\xi_0 = \eta_0 = 0$, and $-1 \leq w \leq 1$. The cosmological parameters at every point on the Kasner circle are

$$\Omega_f = 0, \quad q = 2, \quad \Sigma^2 = 1. \tag{3.35}$$

The eigenvalues of the Jacobian matrix at each point are

$$\lambda_1 = 0, \quad \lambda_2 = 3(1-w), \quad \lambda_3 = 1 + \Sigma_+ - \sqrt{3(1-\Sigma_+^2)}. \tag{3.36}$$

As can be seen from Eq. (3.36) when $w = 1$ two of the eigenvalues are zero, and these equilibrium points are not normally hyperbolic. One therefore cannot use linearization methods to determine the local asymptotic behaviour. In the following discussion we restrict our attention to the parameter region defined by $-1 \leq w < 1$.

Let us parametrize the Kasner circle points using the polar angle ψ as is done in [WE97]:

$$\Sigma_+ = \cos\psi, \quad \Sigma_- = \sin\psi, \quad -\pi < \psi \leq \pi. \tag{3.37}$$

The Kasner exponents p_1, p_2, and p_3 of the Kasner metric

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{3.38}$$

are then given by

$$p_1 = \frac{1}{3}\left(1 - 2\cos\psi\right), \quad p_{2,3} = \frac{1}{3}\left(1 + \cos\psi \pm \sqrt{3}\sin\psi\right). \tag{3.39}$$

It is well known that the Taub points occur for $\psi = -\pi/3$, π, and $\pi/3$. We use these Taub points to subdivide the circle $\mathcal{K}$ into three open arcs. Along the arc $\mathcal{K}_1$ defined by

$$-\frac{\pi}{3} < \psi < \frac{\pi}{3}, \tag{3.40}$$

the eigenvalue λ_3 is positive, and hence each point on the arc corresponds to a source. Furthermore, on $\mathcal{K}_1$ we have $p_1 < 0$, $p_2 > 0$, and $p_3 > 0$, which implies that each of these equilibrium points represent a cigar-type past singularity of the system. Along the arcs $\mathcal{K}_2$ and $\mathcal{K}_3$, defined by

$$-\pi < \psi < -\frac{\pi}{3} \quad \text{and} \quad \frac{\pi}{3} < \psi < \pi, \tag{3.41}$$

respectively, the eigenvalue λ_3 is negative and each Kasner point on these arcs corresponds to a local saddle point. On both these arcs we also have $p_1 > 0$, $p_2 > 0$, and $p_3 < 0$, which corresponds to a cigar-type singularity as well. In the case of a cigar singularity, matter collapses in along one spatial direction from infinity, halts, and then begins to re-expand, while in the other spatial directions, the matter expands monotonically at all times. Each Taub point, on the other hand, corresponds to a pancake singularity, where matter is found to expand monotonically in all directions, starting from a very high expansion rate in one spatial direction, but from zero expansion rates in the other spatial directions (Page 144, [HE06]).

3.4.2 Flat Universe Equilibrium Point

This equilibrium point, which we denote as $\mathcal{F}$, occurs for

$$\Sigma_+ = 0, \quad \Sigma_- = 0, \quad \mathcal{B}_2 = 0, \tag{3.42}$$

and represents the *flat FLRW* universe. The cosmological parameters at this point take the form

$$\Omega_f = 1, \quad q = \frac{1}{2}\left(1 + 3w - 9\xi_0\right), \quad \Sigma^2 = 0. \tag{3.43}$$

The eigenvalues of the Jacobian matrix of the dynamical system at $\mathcal{F}$ are given by

$$\lambda_1 = \frac{1}{2}(-1 + 3w - 9\xi_0), \quad \lambda_2 = \lambda_3 = \frac{1}{2}(-3 + 3w - 4\eta_0 - 9\xi_0), \tag{3.44}$$

where in Eqs. (3.43) and (3.44) we require that $\eta_0 \geq 0$, $\xi_0 \geq 0$, and $-1 \leq w \leq 1$.

The point $\mathcal{F}$ represents a local sink if

$$\eta_0 \geq 0, \quad \xi_0 \geq 0, \quad -1 \leq w < \frac{1}{3}, \tag{3.45}$$

or

$$\eta_0 \geq 0, \quad \frac{1}{3} \leq w \leq 1, \quad \xi_0 > \frac{1}{9}\left(-1 + 3w\right). \tag{3.46}$$

In Fig. (3.1), we have denoted the region defined by (3.45) and (3.46) as $S1(F)$.

The point $\mathcal{F}$ represents a saddle point if

$$\eta_0 = 0, \quad \frac{1}{3} < w < 1, \quad 0 \leq \xi_0 < \frac{1}{9}(-1 + 3w), \tag{3.47}$$

or

$$\eta_0 = 0, \quad w = 1, \quad 0 < \xi_0 < \frac{2}{9}, \tag{3.48}$$

or

$$\eta_0 > 0, \quad \frac{1}{3} < w \leq 1, \quad 0 \leq \xi_0 < \frac{1}{9}\left(-1 + 3w\right), \tag{3.49}$$

where in each case $\lambda_1 > 0$ and $\lambda_2 = \lambda_3 < 0$. We will subsequently denote the region defined by (3.47) - (3.49) as SA(F).

The point $\mathcal{F}$ can also represent a local source if

$$\eta_0 = 0, \quad w = 1, \quad \xi_0 = 0, \tag{3.50}$$

where in this case, $\lambda_1 > 0$ and $\lambda_2 = \lambda_3 = 0$. An analysis nearly identical to that presented in the classification of the Kasner point $\mathcal{K}_1$ does confirm this is a source point. We will subsequently denote the region defined by Eq. (3.50) as U(F).

It is important to note that $q = -1$ when $0 \leq \xi_0 \leq \frac{2}{3}$ and $w = 3\xi_0 - 1$, and thus the equilibrium point in the domain defined by these values of η_0, ξ_0 , and w does not correspond to a self-similar solution. In particular, if one chooses $\xi_0 = 0$ such that $w = -1$, the corresponding model is locally the de Sitter solution [EMM12].

3.4.3 A New Equilibrium Point

We will denote this equilibrium point as $\mathcal{BI}_{\mathcal{MV}}$. For brevity in our presentation, we introduce the condensed notation for the fixed points

$$\Sigma_+ = -\frac{1}{16\alpha}(\beta_1 + \gamma), \quad \Sigma_- = -\frac{\sqrt{3}}{16\alpha}(\beta_1 + \gamma), \quad \mathcal{B}_2 = \frac{1}{4\sqrt{3}}\left(\beta_2 - \gamma\right)^{1/2}, \tag{3.51}$$

where

$$\alpha = 5 - 6\eta_0 + 3w(1 + 2\eta_0), \tag{3.52}$$

$$\beta_1 = 9w^2(1 + 2\eta_0)^2 + 12w(1 + 2\eta_0)(3 - 2\eta_0) + (7 - 2\eta_0)(5 - 6\eta_0), \tag{3.53}$$

$$\beta_2 = -9w^2(1 + 2\eta_0)^2 + 12w(1 - 2\eta_0)^2 - (17 - 6\eta_0)(3 - 2\eta_0) - 144\xi_0, \tag{3.54}$$

and

$$\gamma = |\alpha|\left[9w^2(1 + 2\eta_0)^2 - 6w(3 - 2\eta_0)^2 + (7 - 2\eta_0)^2 + 32(1 + 9\xi_0)\right]^{1/2}. \tag{3.55}$$

Similarly, the cosmological parameters at this point take the form

$$\Omega_f = -\frac{1}{16\alpha}\left(\beta_3 + (1 + 2\eta_0)\gamma\right), \quad q = \frac{1}{4\alpha}(\beta_4 + \gamma), \quad \Sigma^2 = \frac{1}{64\alpha^2}\left(\beta_1 + \gamma\right)^2, \tag{3.56}$$

where

$$\beta_3 = 9w^2(1+2\eta_0)^3 - 12w(1-2\eta_0)^2(1+2\eta_0) - (5-6\eta_0)(3-2\eta_0)^2 \tag{3.57}$$

and

$$\beta_4 = 9w^2(1+2\eta_0)^2 + 24w(1+2\eta_0)(2-\eta_0) + (5-6\eta_0)(11-2\eta_0). \tag{3.58}$$

The restrictions require us to set

$$\eta_0 > \frac{3}{2}, \quad \frac{1}{3} \le w < \frac{-5+6\eta_0}{3+6\eta_0}, \quad 0 \le \xi_0 \le \frac{1}{9}\left(-1+3w\right). \tag{3.59}$$

We will subsequently denote the parameter region defined by (3.59) as S2(BIMV). We were not able to obtain exact expressions for the eigenvalues in this region due to overwhelming algebraic complexity; however, comprehensive numerical experiments demonstrate that the eigenvalues in this region are either zero or negative thus corresponding to a sink. Interestingly, for a fixed value of $\eta_0 > 3/2$, the dependence of the largest eigenvalue λ_1 on the parameters w and ξ_0 is very nearly linear on S2(BIMV). For several values of η_0 a planar approximation for the λ_1 surface was constructed in our numerical experiments using computed values in the (w, ξ_0) domain. The planar approximation with equation $\lambda_1 = 1 - 3w + 9\xi_0$ agreed with numerically-computed values of λ_1 everywhere in S2(BIMV) to within 3 to 5 digits accuracy, depending on the value of η_0 chosen in the range $3/2 < \eta_0 \le 500$; the best agreement was obtained for larger values of η_0. Despite the algebraic complexity, we can show analytically that the equilibrium point corresponding to parameters in S2(BIMV) is indeed a sink by the following considerations. For convenience we have included the Jacobian matrix $D\mathbf{f}(\mathbf{a})$ (where $\mathbf{a}$ is the equilibrium point under consideration) in Appendix A. As we discuss in the following section, the surface $\xi_0 = (3w-1)/9$, which forms one boundary of the domain S2(BIMV), corresponds to bifurcations in the solution; on this surface the Jacobian matrix is diagonal and its eigenvalues are seen to be

$$\lambda_1 = 0, \lambda_2 = \lambda_3 = -1 - 2\eta_0. \tag{3.60}$$

We seek to characterize the equilibrium point slightly inside the region S2(BIMV), and thus in what follows we find expressions for the eigenvalues corresponding to $\xi_0 = (3w-1)/9 - \varepsilon$, where $\varepsilon > 0$ is a small parameter chosen to ensure that indeed $\xi_0 \geq 0$ and $w < (6\eta_0 - 5)/(6\eta_0 + 3)$. Expanding the elements of the Jacobian matrix in a series in ε to first order allows simple expressions for the eigenvalues to be obtained:

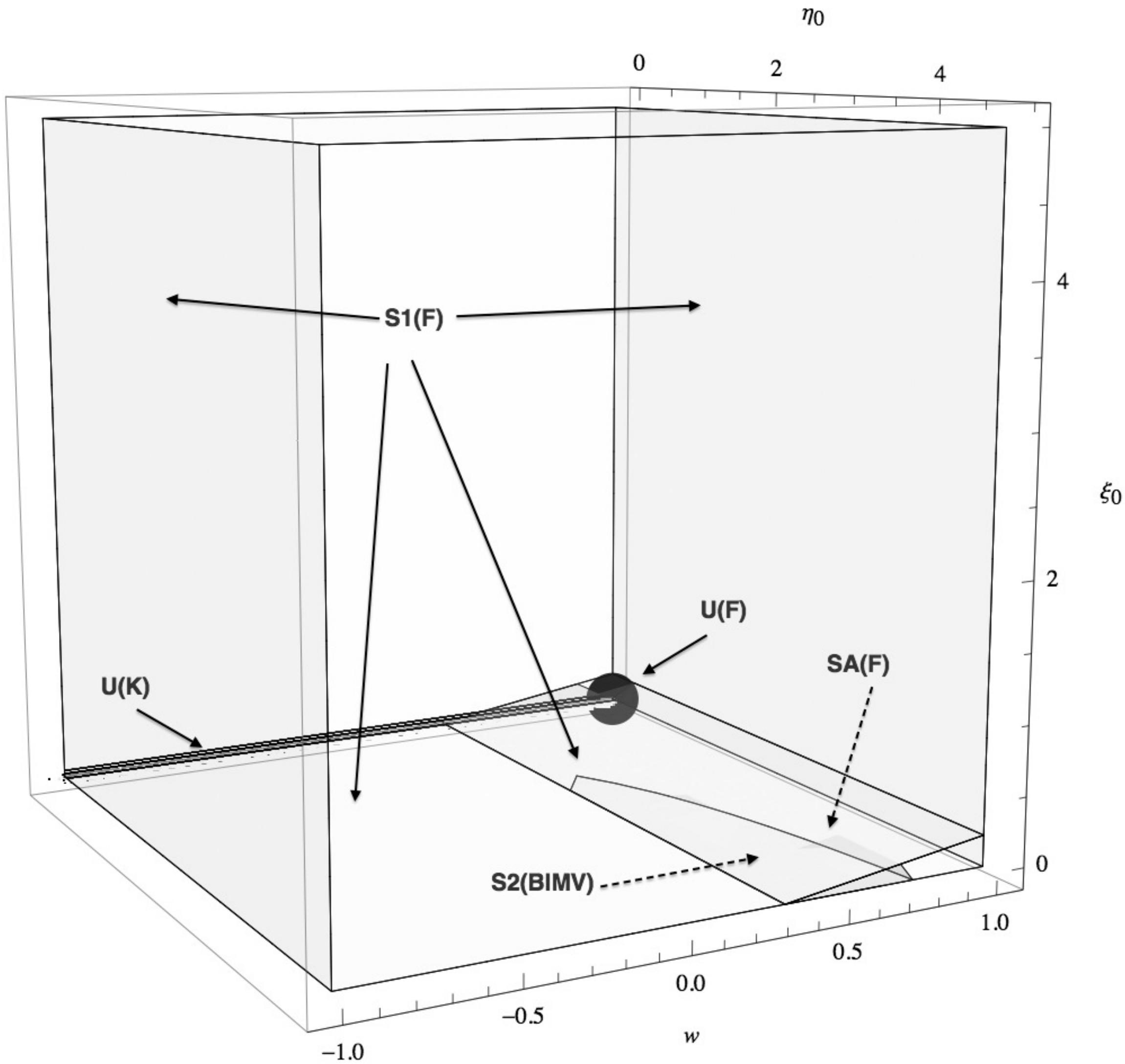
$$\lambda_1 = -9\varepsilon, \lambda_2 = -1 - 2\eta_0 + \frac{108\varepsilon}{(7 - 2\eta_0) + 3w(1 + 2\eta_0)}, \lambda_3 = -1 - 2\eta_0 + \frac{36\varepsilon}{(7 - 2\eta_0) + 3w(1 + 2\eta_0)}. \tag{3.61}$$

We note that all the terms in (3.61) have error of order $\mathcal{O}(\varepsilon^2)$. The quantity ε may be taken arbitrarily small, and thus all the eigenvalues corresponding to parameters slightly inside the bifurcation boundary are negative; i.e., the equilibrium point is a local sink. Since the solution does not bifurcate inside the region S2(BIMV) – it does so only across its boundaries – it follows that *all* parameter values inside the region correspond to a local sink. In addition, the results (3.61) indicate that $\partial\lambda_1/\partial\xi_0 \approx 9$ at the boundary $\xi_0 = (3w - 1)/9$; this approximation for the ξ_0-slope of the λ_1 surface agreed to several digits with the same quantity which was numerically computed and used to form the planar approximation for this surface discussed above.

To the best knowledge of the authors, the equilibrium point $\mathcal{BI}_{\mathcal{MV}}$ has not previously been reported in the literature, and represents a new solution to the Einstein field equations. Interestingly, the model with parameter values in S2(BIMV) will not isotropize, since this equilibrium point is a local source with $\Sigma_+, \Sigma_-, \mathcal{B}_2 \neq 0$.

For convenience, we have summarized the results of this section in Fig. (3.1) which depicts the different regions of sinks, saddles, and sources of the dynamical system.

Figure 3.1: A depiction of the different regions of sinks, sources, and saddles of the dynamical system as defined by the aforementioned restrictions on the values of the expansion-normalized bulk and shear viscosities, ξ_0, η_0 and equation of state parameter, w.



3.5 Bifurcations

We note that the equilibria found above are related to each other by sequences of bifurcations. We will now give in this section the details of these bifurcations. The method we use involves determining for what values of η_0, ξ_0, and w do the different equilibrium points destabilize. A similar method was employed in [HBW01].

The linearized system for points on $\mathcal{K}$, where $\Sigma_- = \pm\sqrt{1-\Sigma_+^2}$ becomes:

$$\Sigma_+' = -3(-1+w)\Sigma_+^3 - 3(-1+w)\Sigma_+^2\sqrt{1-\Sigma_+^2}, \tag{3.62}$$

$$\Sigma_-' = -3(-1+w)\Sigma_+^2\sqrt{1-\Sigma_+^2} + 3\Sigma_-(-1+w)(-1+\Sigma_+^2), \tag{3.63}$$

$$\mathcal{B}_2' = (1+\Sigma_+ + \sqrt{3(1-\Sigma_+^2}). \tag{3.64}$$

We can therefore see that Σ_+ destabilizes $\mathcal{K}$ when $\Sigma_+ = 0, -1 \leq w < 1$, Σ_- destabilizes $\mathcal{K}$ when $\Sigma_+ = \pm 1, -1 \leq w < 1$, and $\mathcal{B}_2$ destabilizes $\mathcal{K}$ when $\Sigma_+ = -1, -1 \leq w < 1$, where in each case $\xi_0 = \eta_0 = 0$.

We next consider the linearized system at $\mathcal{F}$, given by

$$\Sigma_+' = \frac{1}{2}\left(-3+3w-4\eta_0-9\xi_0\right)\Sigma_+, \tag{3.65}$$

$$\Sigma_-' = \frac{1}{2}\left(-3+3w-4\eta_0-9\xi_0\right)\Sigma_-, \tag{3.66}$$

$$\mathcal{B}_2' = \frac{1}{2}\left(-1+3w-9\xi_0\right)\mathcal{B}_2. \tag{3.67}$$

It may be seen that both Σ_+ and Σ_- destabilize $\mathcal{F}$ when $\xi_0 = \eta_0 = 0$, and $w = 1$, while $\mathcal{B}_2$ destabilizes $\mathcal{F}$ when $\eta_0 \geq 0$, $1/3 \leq w \leq 1$, and $\xi_0 = (3w-1)/9$.

We now turn to the final equilibrium point of the system, $\mathcal{BI}_{\mathcal{MV}}$, whose corresponding Jacobian matrix is given in Appendix A. It may be seen that the Jacobian is in fact diagonal when $\eta_0 > 3/2$ and $\xi_0 = (3w-1)/9$ in which case $\mathcal{B}_2' = 0$. Thus $\mathcal{B}_2$ destabilizes this equilibrium point along the surface $\xi_0 = (3w-1)/9$, which is a shared boundary with the region SA(F). Across this boundary, the source point in SA(F) becomes a sink in S2(BIMV). Extensive numerical experiments indicated that there were no other destabilizations for this equilibrium point.

We thus see that the system destabilizes either on the line in parameter space $\xi_0 = \eta_0 = 0$ or it destabilizes on the parameter surface $\xi_0 = (3w-1)/9$. Given this information on the destabilizations, we see that some possible bifurcation sequences can be obtained as follows. First, let us set $w = 1/3, \xi_0 = 0$. Then, we have that:

$$\mathcal{K}_{(\eta_0=0)} \rightarrow \mathcal{F}_{(0<\eta_0\leq 3/2)} \rightarrow \mathcal{BI}_{\mathcal{MV}(\eta_0>3/2)}. \tag{3.68}$$

Another possible bifurcation sequence is obtained when:

$$\mathcal{K}_{(-1\leq w<1/3,\xi_0=\eta_0=0)} \rightarrow \mathcal{BI}_{\mathcal{MV}(\eta_0>3/2,w=1/3,\xi_0=0)} \rightarrow \mathcal{F}_{(\eta_0>3/2,1/3<w\leq 1,0<\xi_0\leq 2/9)}. \tag{3.69}$$

One can also have that

$$\mathcal{K}_{(-1\leq w<1/3,\xi_0=\eta_0=0)} \rightarrow \mathcal{BI}_{\mathcal{MV}(\eta_0>3/2,w=1/3,\xi_0=0)} \rightarrow \mathcal{F}_{(\eta_0=\xi_0=0,w=1)}. \tag{3.70}$$

As discussed previously, the surface $\xi_0 = (3w-1)/9$ governs bifurcations of the dynamical system. It is constructive to display this bifurcation behaviour for cases where first $\xi_0 < (3w-1)/9$, then $\xi_0 = (3w-1)/9$, and finally, $\xi_0 > (3w-1)/9$. For the purposes of this numerical experiment, we specifically chose $w = 1/2$, $\eta_0 = 5$, therefore requiring that the three aforementioned cases reduce to $\xi_0 < 1/18$, $\xi_0 = 1/18$, and $\xi_0 > 1/18$. The outcomes of this experiment are shown in Fig. 3.2.

Figure 3.2: These figures show bifurcation behaviour for a varying expansion-normalized bulk viscosity coefficient, ξ_0, while w and η_0 were held fixed. The circles indicate the $\mathcal{BI}_{\mathcal{MV}}$ equilibrium points, while the diamond indicates the FLRW equilibrium point. For the first figure, $\xi_0 = 0.05$, for the second figure, $\xi_0 = 1/18$, and for the last figure, $\xi_0 = 0.6$. Note how the increasing values of ξ_0 first result in a slight shift of the equilibrium point position of $\mathcal{BI}_{\mathcal{MV}}$, and then finally a transition to a new state, namely the FLRW equilibrium, which was predicted by our fixed-point analysis.

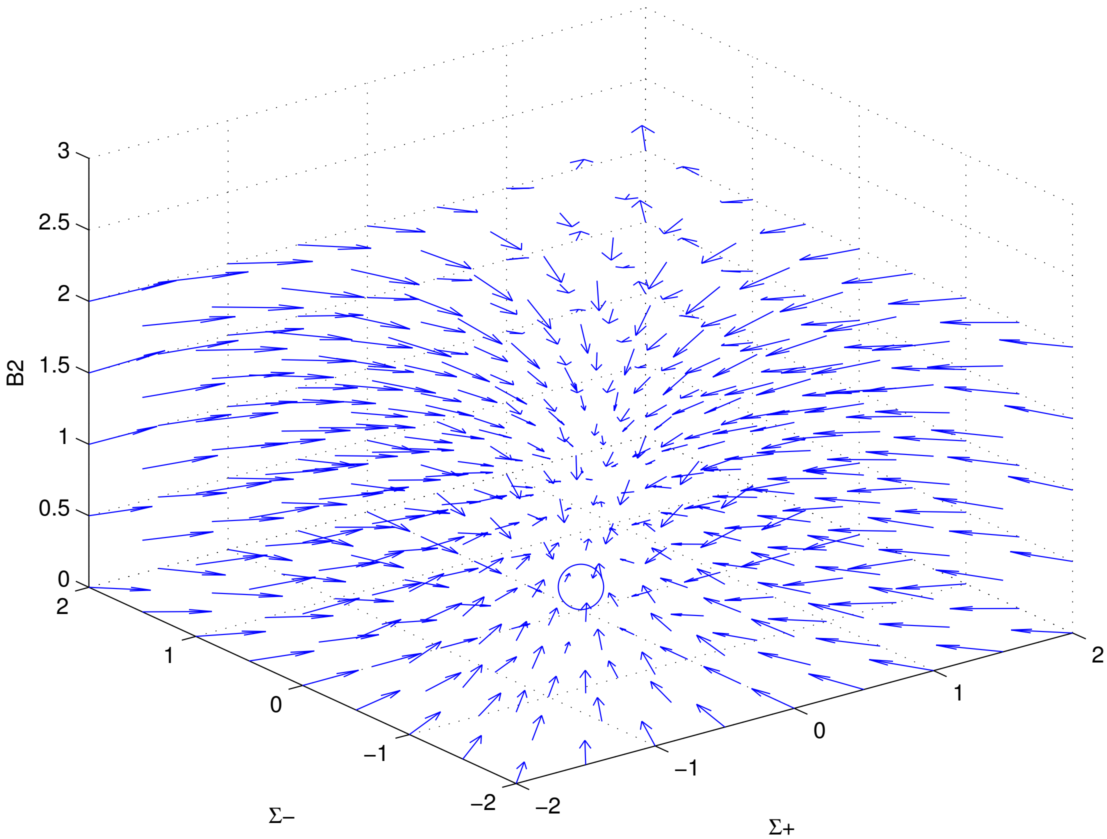


Fig. 3.2 continued.

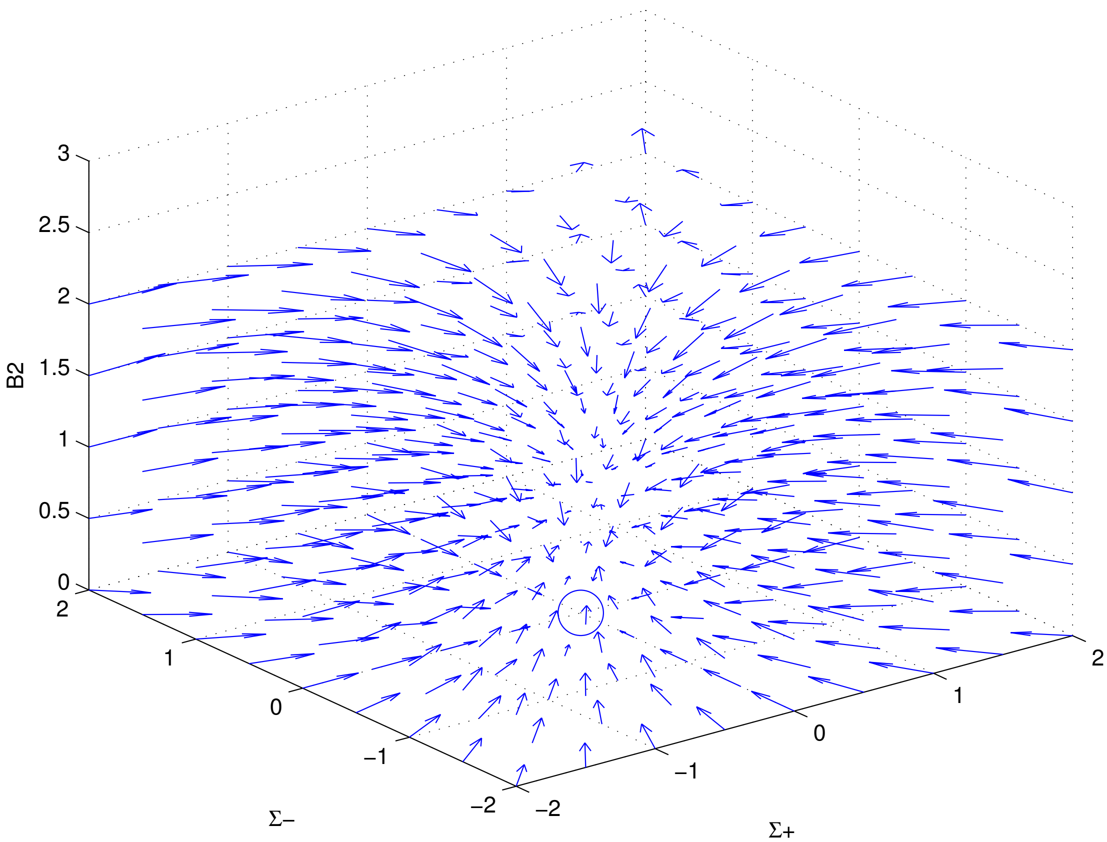
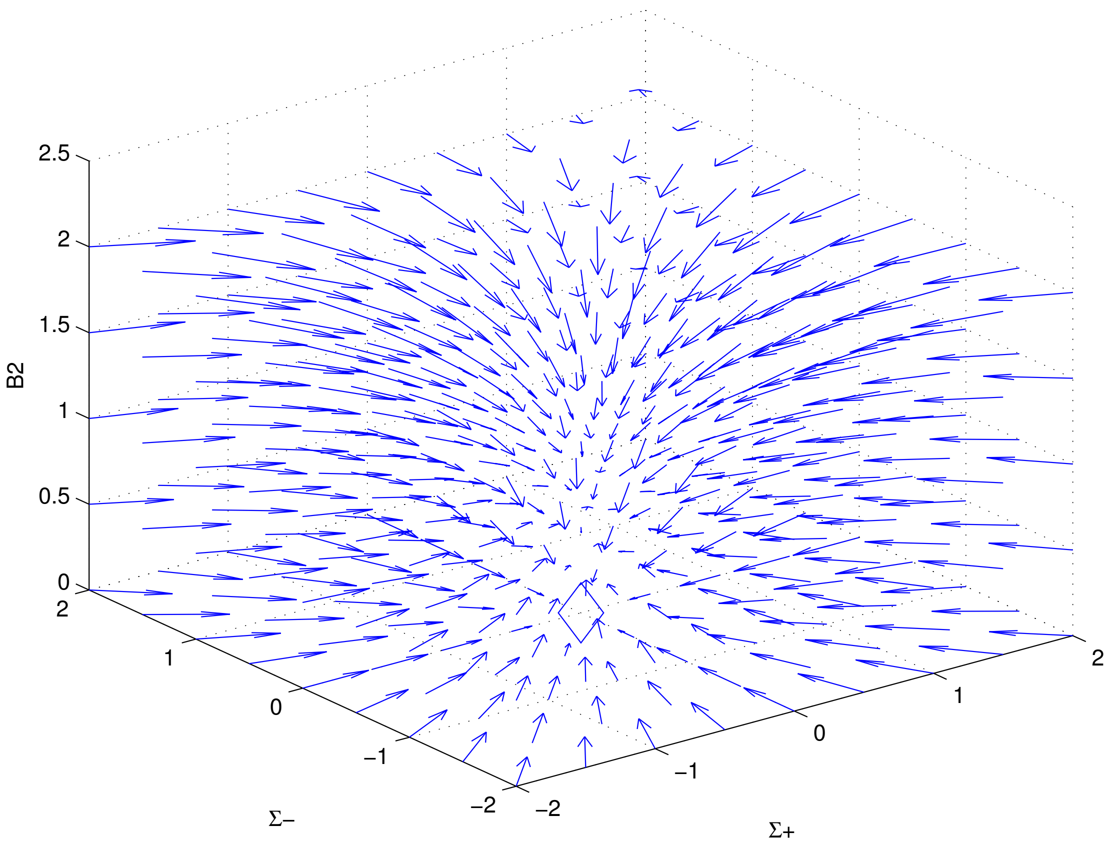


Fig. 3.2 continued.



3.6 Qualitative Properties of the System

3.6.1 A Further Analysis of the Asymptotic Behavior

An important question to ask in analyzing some qualitative properties of the dynamical system is whether there are any invariant sets of the dynamical system. A very useful proposition in this regard is given by Proposition 4.1 in reference [WE97], which states that for a dynamical system of type (3.31), if $Z : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function such that $Z' = \alpha Z$, where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then the subsets of $\mathbb{R}^n$ defined by $Z > 0$, $Z = 0$, and $Z < 0$ are invariant sets of the flow of the system of differential equations. From Eq. (3.26), we see that this proposition applies with $Z = \mathcal{B}_2$, and thus $\mathcal{B}_2 = 0$ and $\mathcal{B}_2 > 0$ are invariant sets of the system. We also note that if one sets $\eta_0 = \xi_0 = 0$ in Eq. (3.29), then the proposition also applies with $Z = \Omega_f$, and hence $\Omega_f \geq 0$ is an invariant set of the system.

With respect to the existence of limit sets, we first make a proposition about the late-time dynamics of the system:

Proposition 3. *Consider the dynamical system* (3.14) *with parameters in the region* $S1(F)$ *defined by* $-1 \leq w < \frac{1}{3}$, $\xi_0 = 0$ *and* $\eta_0 = 0$. *Then, as* $\tau \to +\infty$, $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2 \to 0$ *and* $\mathcal{B}_2^2 \to 0$, *and hence the model isotropizes.*

Proof. The details of the proof essentially follow the arguments given in the appendix of reference [CW92]. Substitution of Eq. (3.28) in (3.27) results in the expression

$$q = \Sigma^2 \left(\frac{3-3w}{2}\right) + \mathcal{B}_2^2 \left(\frac{3-9w}{4}\right) + \frac{3w+1}{2} - \frac{9}{2}\xi_0, \tag{3.71}$$

and hence the Ω_f' evolution equation (3.29) may be written as

$$\Omega_f' = \Omega_f \left[\Sigma^2 (3-3w) + \mathcal{B}_2^2 \left(\frac{3-9w}{2}\right) - 9\xi_0\right] + 4\eta_0 \Sigma^2 + 9\xi_0. \tag{3.72}$$

In addition, from the generalized Friedmann equation, Eq. (3.28), we have that $\Omega_f \leq 1$. Therefore, to prove the proposition it remains to show that the function Ω_f is monotonically increasing, i.e., $\Omega_f' > 0$. Then,

$$\left[\Sigma^2 (3-3w) + \mathcal{B}_2^2 \left(\frac{3-9w}{2}\right)\right] + 4\eta_0 \Sigma^2 > 0 \Leftrightarrow -1 \leq w < \frac{1}{3}. \tag{3.73}$$

Therefore, in the region where $\eta_0 \geq 0$, $\xi_0 = 0$, and $-1 \leq w < \frac{1}{3}$, Ω_f is monotonically increasing. In can therefore be said that in this region,

$$\lim_{\tau \to +\infty} \Omega_f = 1. \tag{3.74}$$

Using this result in the Friedmann equation (3.28), we have that

$$\lim_{\tau \to +\infty} \Omega_f = 1 \Rightarrow \lim_{\tau \to +\infty} \left(-\frac{3}{2}\mathcal{B}_2^2 - \Sigma^2 \right) = 0. \tag{3.75}$$

The latter then implies that

$$\lim_{\tau \to +\infty} \Sigma^2 = \lim_{\tau \to +\infty} \mathcal{B}_2^2 = 0. \tag{3.76}$$

□

In order to gain some insight into the asymptotic behaviour of the system as $\tau \to -\infty$, we use the extended LaSalle principle for negatively invariant sets; see Proposition B.3 in reference [HW93]. In particular, suppose $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is an autonomous system of first-order differential equations and let $Z : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. If $S \subset \mathbb{R}^n$ is a closed, bounded, and negatively invariant set, and $Z'(\mathbf{x}) \equiv \nabla Z \cdot \mathbf{f}(\mathbf{x}) \leq 0$, $\forall\ \mathbf{x} \in S$, then the extended LaSalle principle states that $\forall\ \mathbf{a} \in S$, $\alpha(\mathbf{a}) \subseteq \{\mathbf{x} \in S | Z'(\mathbf{x}) = 0\}$. That is, the α-limit set $\alpha(\mathbf{a})$ contains the local sources in the system at $\tau \to -\infty$. We use this principle to establish past asymptotic behaviour in the following proposition.

Proposition 4. *For the dynamical system (3.14),* $\alpha(\mathbf{a}) \subseteq \{\Omega_f = 0\} = \{\mathcal{K}\}$.

Proof. The set $\{\Omega_f = 0\}$ is negatively invariant, closed, and bounded. From Eq. (3.72) when $\Omega_f = 0$ it follows that $\Omega_f' \leq 0$ if and only if $\eta_0 = \xi_0 = 0$. Therefore, $\Omega_f' = 0$ if $\Omega_f = \xi_0 = \eta_0 = 0$, which is precisely the region defining the Kasner circle, so $\alpha(\mathbf{a}) \subseteq \{\Omega_f = 0\}$. □

3.6.2 Heteroclinic Orbits

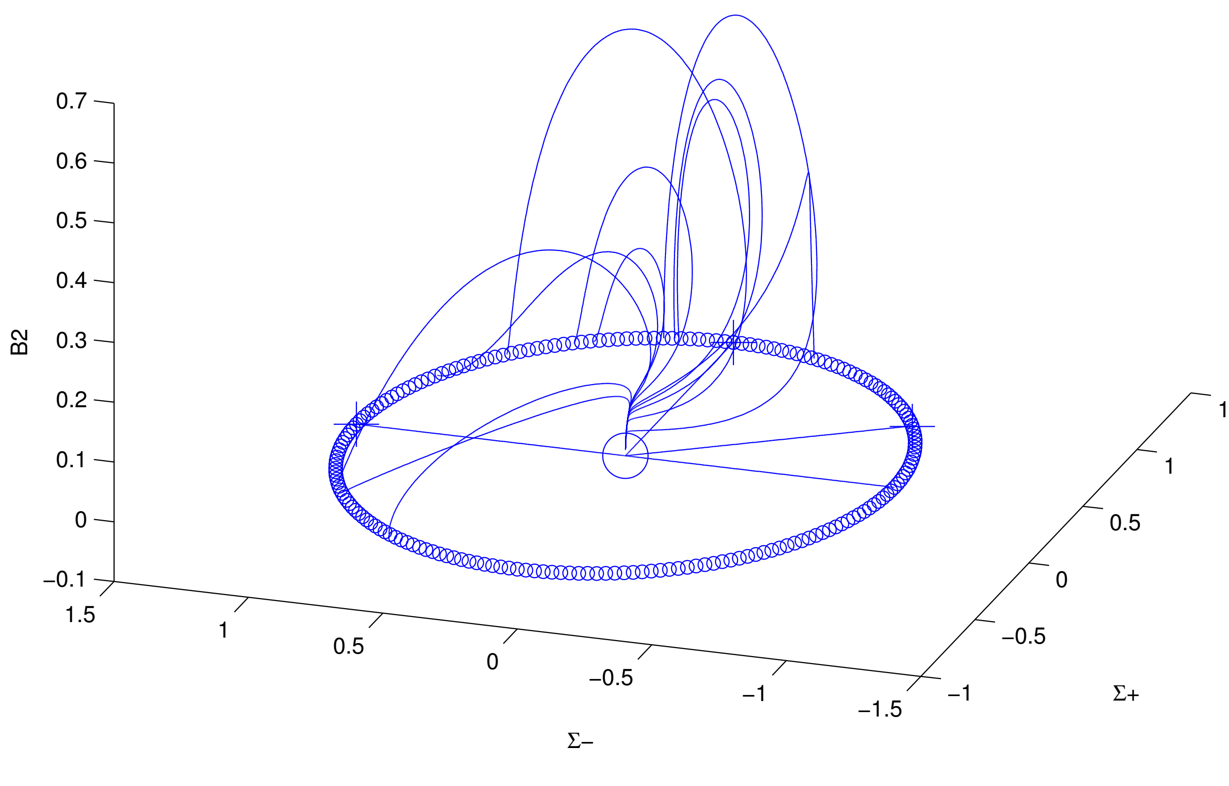
It is interesting to note that for the cosmological model under consideration in this chapter, no finite heteroclinic sequences exist. The reason is that every heteroclinic sequence has an initial point that represents a local source, intermediate points that represent saddles, and a terminal point that represents a local sink.

The caveat, however, is that each equilibrium point and its corresponding asymptotic behaviour must belong to the same region of the parameter space (η_0, ξ_0, w), which is not possible for our dynamical system. There are, however, several heteroclinic orbits which connect distinct equilibrium points, of which some have been plotted in Figs. 3.3, 3.4, and 3.5.

For the region defined by $\{(\eta_0, \xi_0, w) | (\eta_0, \xi_0, w) \in U(K) \cup S1(F)\}$, we have:

$$\mathcal{K} \rightarrow \mathcal{F}. \tag{3.77}$$

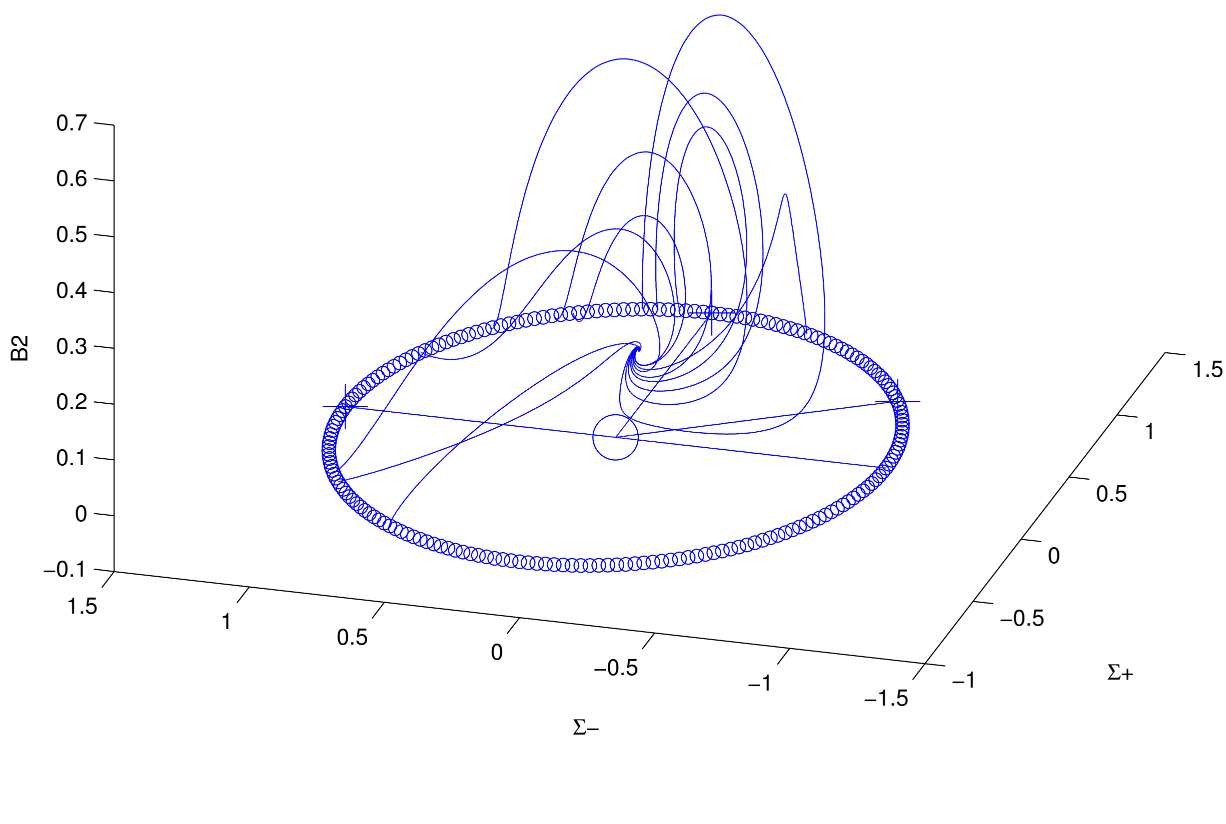
Figure 3.3: The heteroclinic orbits joining the $\mathcal{K} \rightarrow \mathcal{F}$. The plus signs indicate the Kasner equilibrium points that we found above, while the large circle indicates the FLRW equilibrium point. The numerical integration was completed with $\eta_0 = \xi_0 = 0$, $w = 0.325$.



For the region defined by $\{(\eta_0, \xi_0, w) | (\eta_0, \xi_0, w) \in U(K) \cup SA(F)\}$, we have:

$$\mathcal{K} \to \mathcal{F}. \tag{3.78}$$

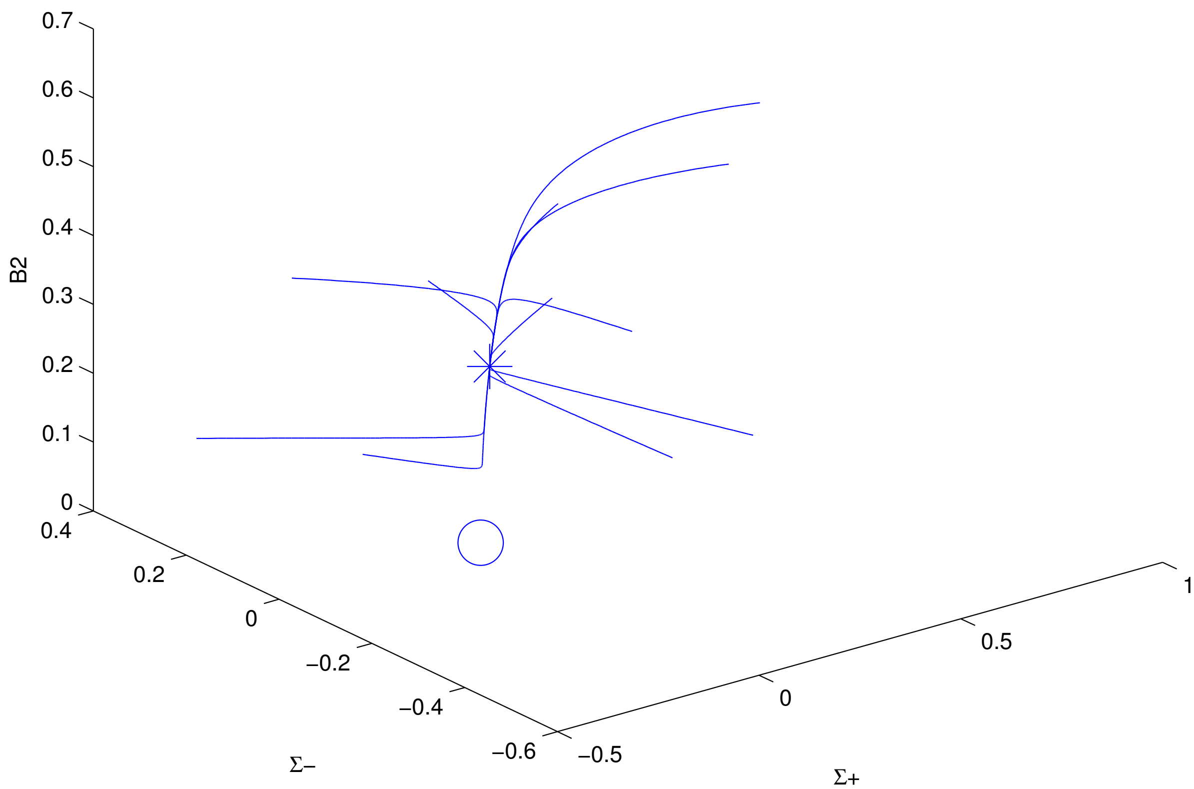
Figure 3.4: The heteroclinic orbits joining the $\mathcal{K} \to \mathcal{F}$. The plus signs indicate the Kasner equilibrium points that we found above, while the large circle indicates the FLRW equilibrium point. The numerical integration was completed with $\eta_0 = \xi_0 = 0$, $w = \frac{1}{2}$.



For the region defined by $\{(\eta_0, \xi_0, w) | (\eta_0, \xi_0, w) \in SA(F) \cup S2(BIMV)\}$, we have

$$\mathcal{F} \rightarrow \mathcal{BI}_{\mathcal{MV}}. \tag{3.79}$$

Figure 3.5: The heteroclinic orbits joining SA(F) to S2(BIMV). The circle represents the FLRW equilibrium point, while the star represents the $\mathcal{BI}_{\mathcal{MV}}$ equilibrium point. The numerical integration was completed with $\eta_0 = 2$, $\xi_0 = 0$, and $w = 0.40$.



3.6.3 The General Case - Extending the Phase Space

As discussed earlier, our work up to this point has assumed that the magnetic field is aligned along the shear eigenvector. The result of this approach was seen in Eq. (3.23), where to avoid R_1, R_2 or R_3 becoming singular we set $\mathcal{B}_1 = \mathcal{B}_3 = 0$, and $\mathcal{B}_2 \neq 0$. Of course, this is not the most general case.

For a Bianchi Type I universe with a magnetic field source, one can also consider the case for which the magnetic field is not a shear eigenvector as was done for the perfect fluid case by LeBlanc [LeB97]. The result of this approach is that the dynamical system is six-dimensional to accommodate additional non-diagonal shear components compared to just three dimensions with no non-diagonal shear components as is the case in our work. This extension of the phase space leads to dynamical equations that are indeed smooth over all phase space, with R_1, R_2, R_3 being continuous in general.

With respect to qualitative behaviour, in LeBlanc's extended approach, he also obtains a flat FLRW equilibrium point, a new solution the Einstein field equations (via a previously undiscovered equilibrium point) and the Kasner vacuum (Page 2287, [LeB97]). He also concludes that a possible late-time future asymptotic state is a flat FLRW model (Page 2290, [LeB97]). Finally, LeBlanc also concludes that the Kasner circle is a past attractor (Page 2292, [LeB97]). Although LeBlanc obtains additional equilibrium points which is natural given the extension of the phase space dimension, the asymptotic qualitative behaviour he finds is the same as we have found in our work.

3.7 A Numerical Analysis

The goal of this section is to complement the preceding stability analysis of the equilibrium points with extensive numerical experiments in order to confirm that the local results are in fact global in nature. For each numerical simulation, we chose the initial conditions such that the constraint Eq. (3.28) in addition to $\mathcal{B}_2 \geq 0$ were satisfied. Although numerical integrations were done from $0 \leq \tau \leq 3000$, for demonstration purposes we present solutions for shorter time intervals. We completed numerical integrations of the dynamical system for physically interesting cases of w equal to 0 (dust), 0.325 (a dust/radiation mixture), and 1/3 (radiation).

Also note that in the subsequent plots, asterisks denote initial conditions.

3.7.1 Dust Models: $w = 0$

3.7.1.1 $\xi_0 = 0.1, \eta_0 = 0.2$

Figure 3.6: This figure shows the dynamical system behaviour for $\xi_0 = 0.1$, $\eta_0 = 0.2$, and $w = 0$. The diamond indicates the FLRW equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model also isotropizes, as can be seen from the last figure, where $\Sigma_\pm, \mathcal{B}_2 \to 0$ as $\tau \to \infty$.

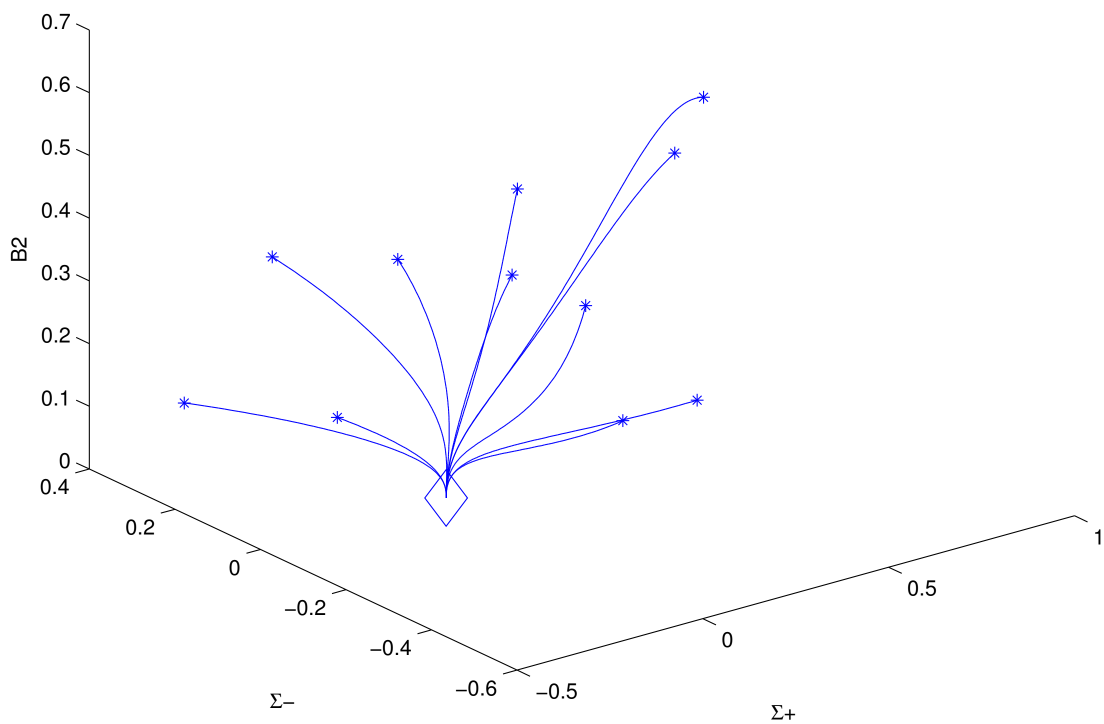
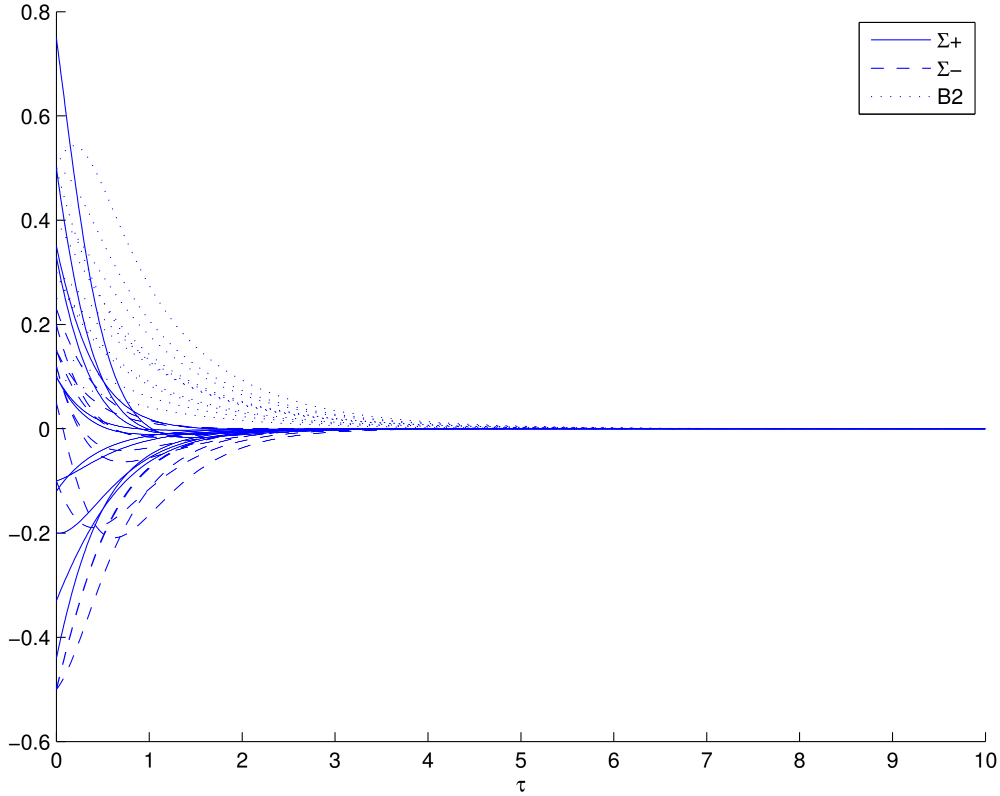


Fig. 3.6 continued.



3.7.1.2 $\xi_0 = 1, \eta_0 = 0.5$

Figure 3.7: This figure shows the dynamical system behaviour for $\xi_0 = 1$, $\eta_0 = 0.5$, and $w = 0$. The diamond indicates the FLRW equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model also isotropizes as can be seen from the last figure, where $\Sigma_{\pm}, \mathcal{B}_2 \to 0$ as $\tau \to \infty$.

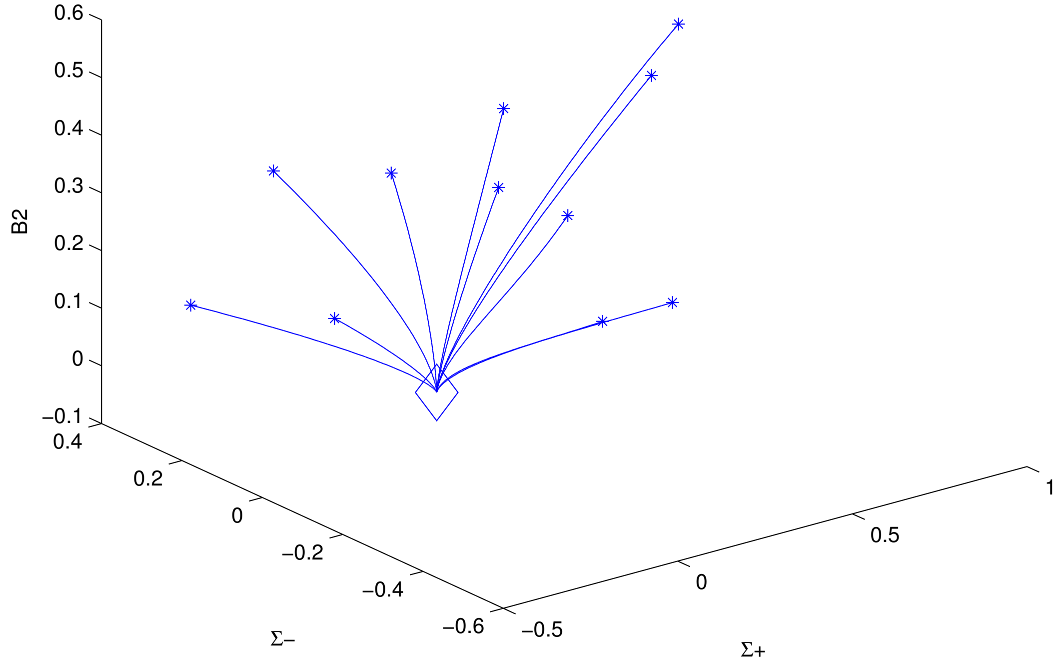
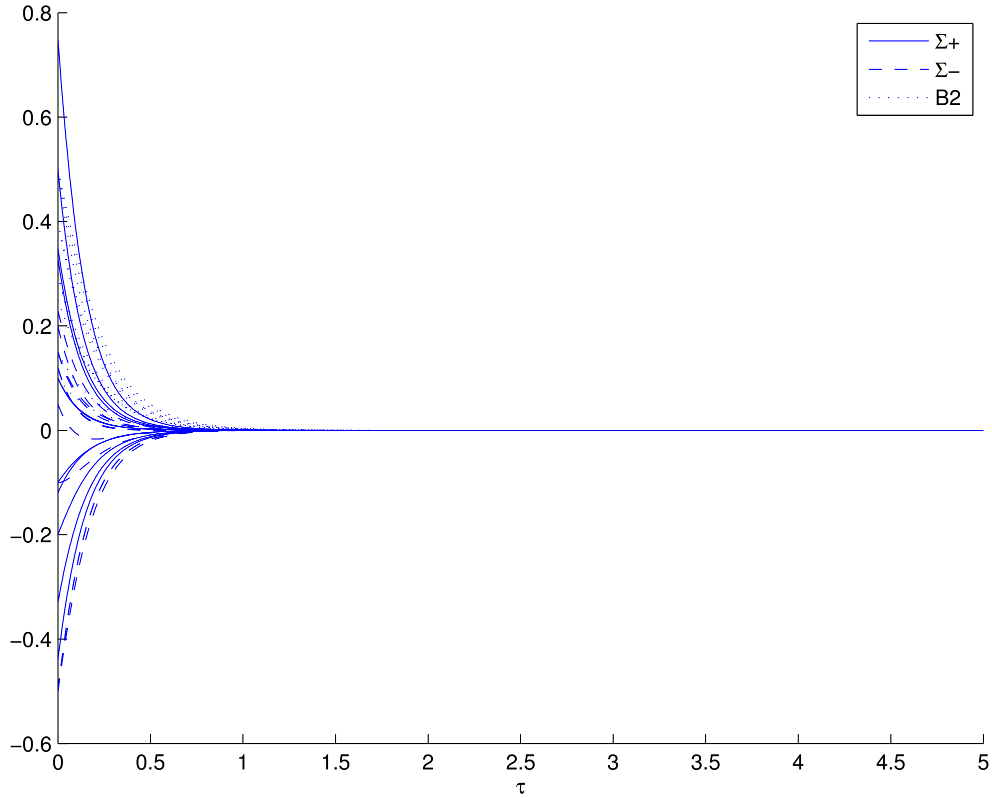


Fig. 3.7 continued.



3.7.2 Radiation Models: $w = 1/3$

3.7.2.1 $\xi_0 = 1.5, \eta_0 = 0$

Figure 3.8: This figure shows the dynamical system behaviour for $\xi_0 = 1.5$, $\eta_0 = 0$, and $w = 1/3$. The diamond indicates the FLRW equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model also isotropizes as can be seen from the last figure, where $\Sigma_{\pm}, \mathcal{B}_2 \to 0$ as $\tau \to \infty$.

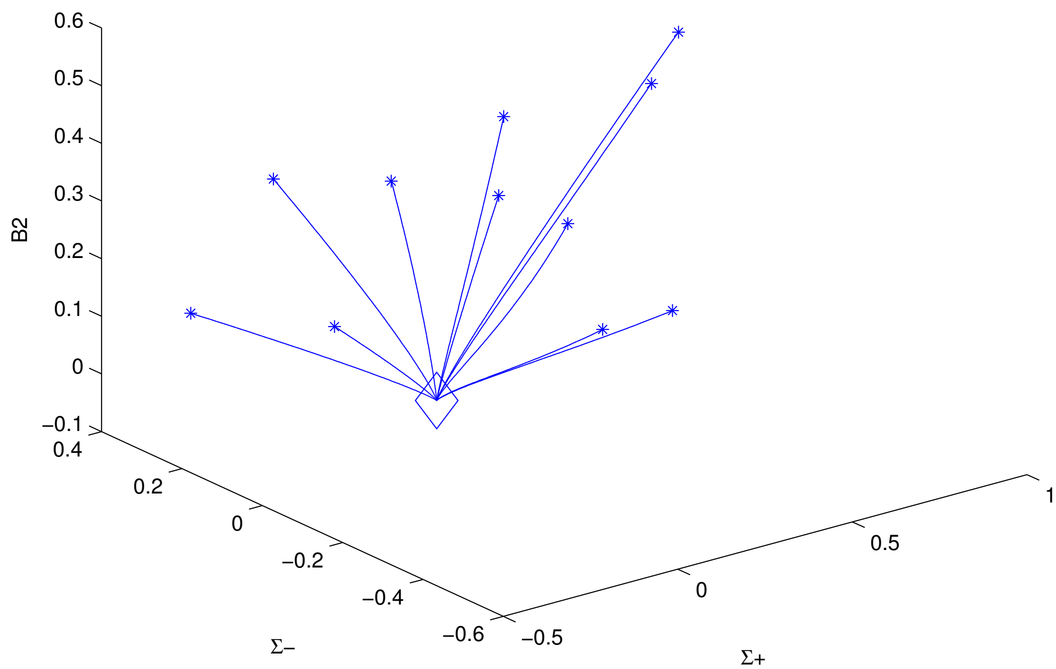
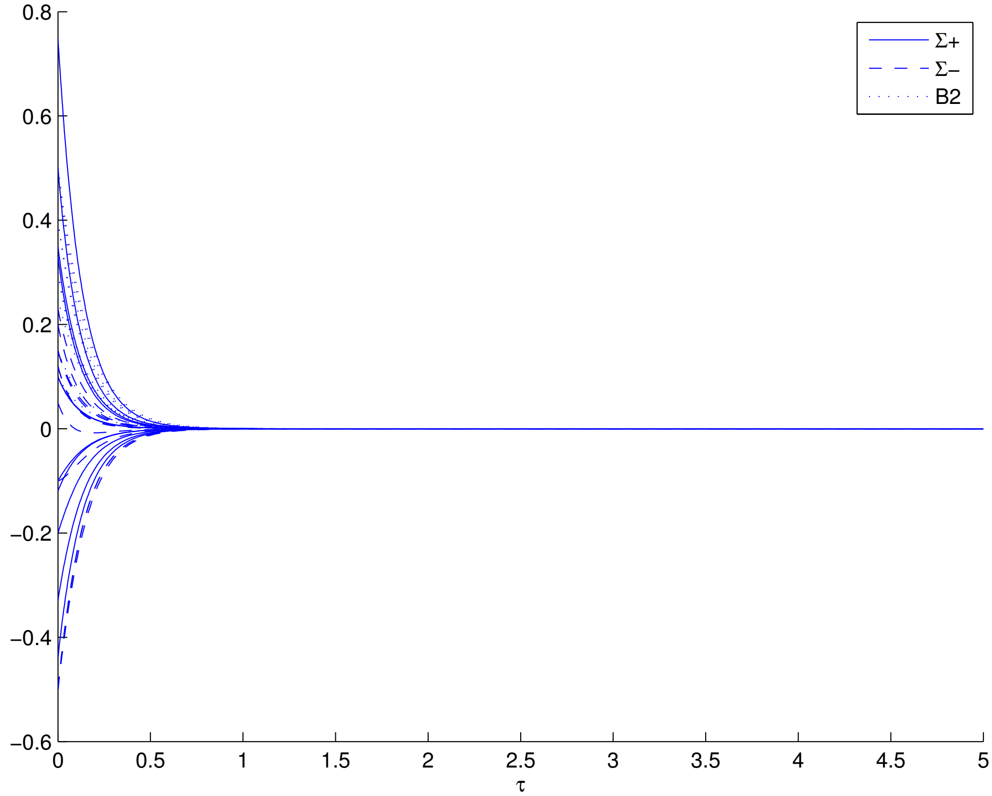


Fig. 3.8 continued.



3.7.2.2 $\xi_0 = 1.5, \eta_0 = 0.5$

Figure 3.9: This figure shows the dynamical system behaviour for $\xi_0 = 1.5$, $\eta_0 = 0.5$, and $w = 1/3$. The diamond indicates the FLRW equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model also isotropizes as can be seen from the last figure, where $\Sigma_{\pm}, \mathcal{B}_2 \to 0$ as $\tau \to \infty$.

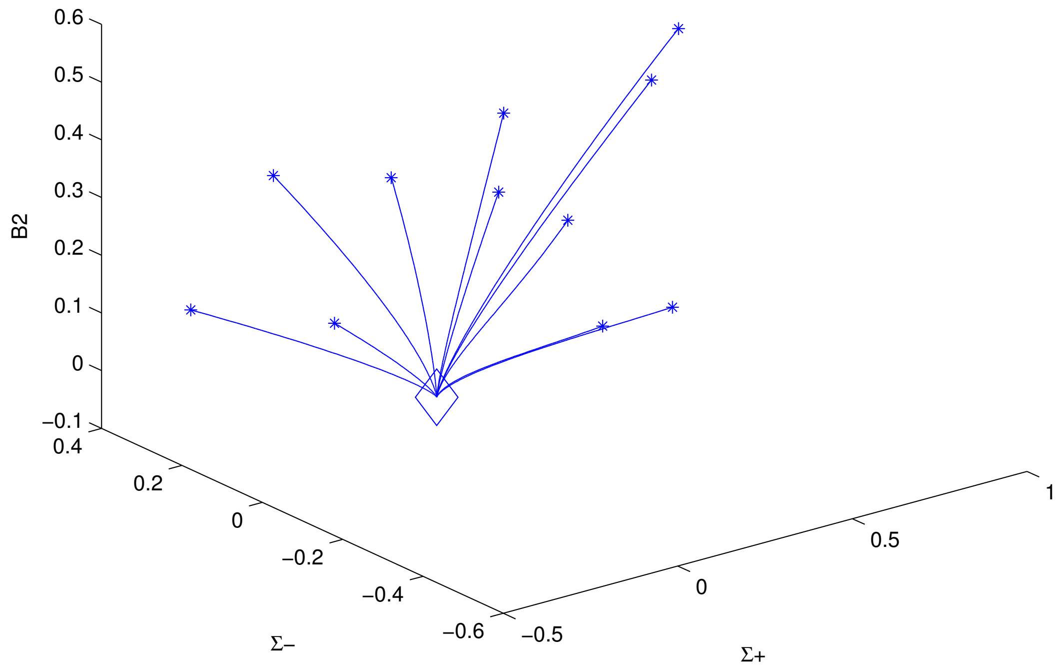
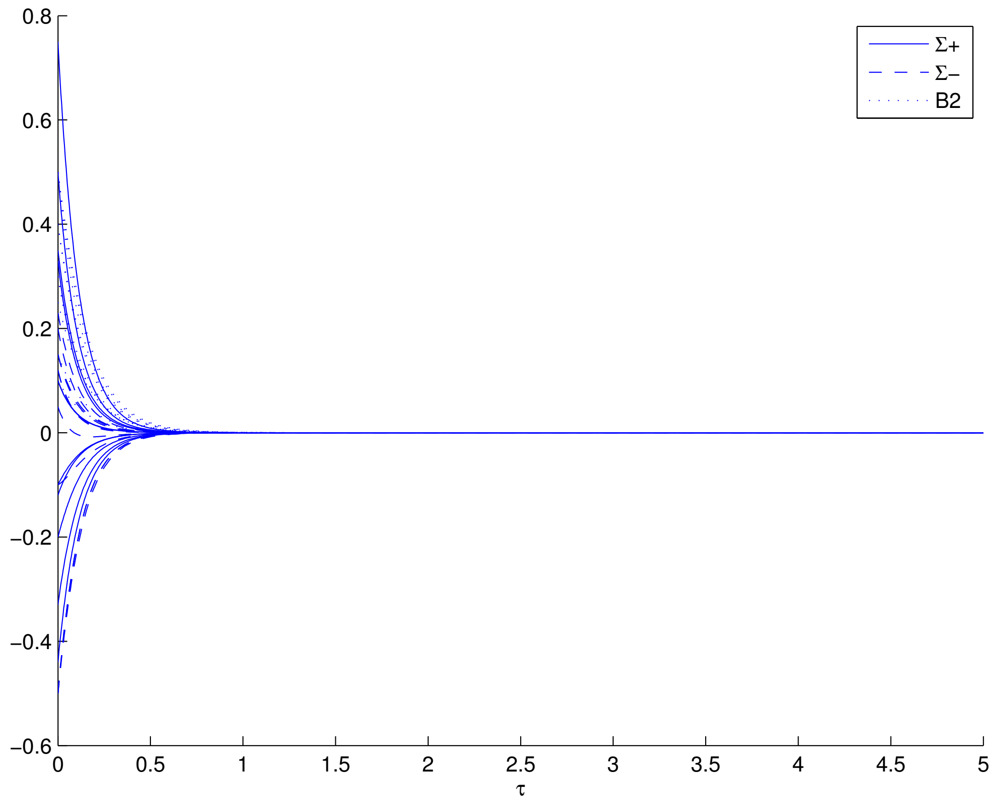


Fig. 3.9 continued.



3.7.2.3 $\xi_0 = 0, \eta_0 = 2$

Figure 3.10: This figure shows the dynamical system behaviour for $\xi_0 = 0$, $\eta_0 = 2$, and $w = 1/3$. The circle indicates the BIMV equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model does not isotropize with respect to the anisotropic magnetic field as can be seen from the last figure, where $B_2 > 0$ as $\tau \to \infty$, but does isotropize with respect to the spatial anisotropic variables, $\Sigma_{\pm}, \to 0$ as $\tau \to \infty$. This state is also special, since according to our fixed-point analysis, this behaviour is only exhibited for $w = 1/3, \eta_0 > 3/2$, and $\xi_0 = 0$.

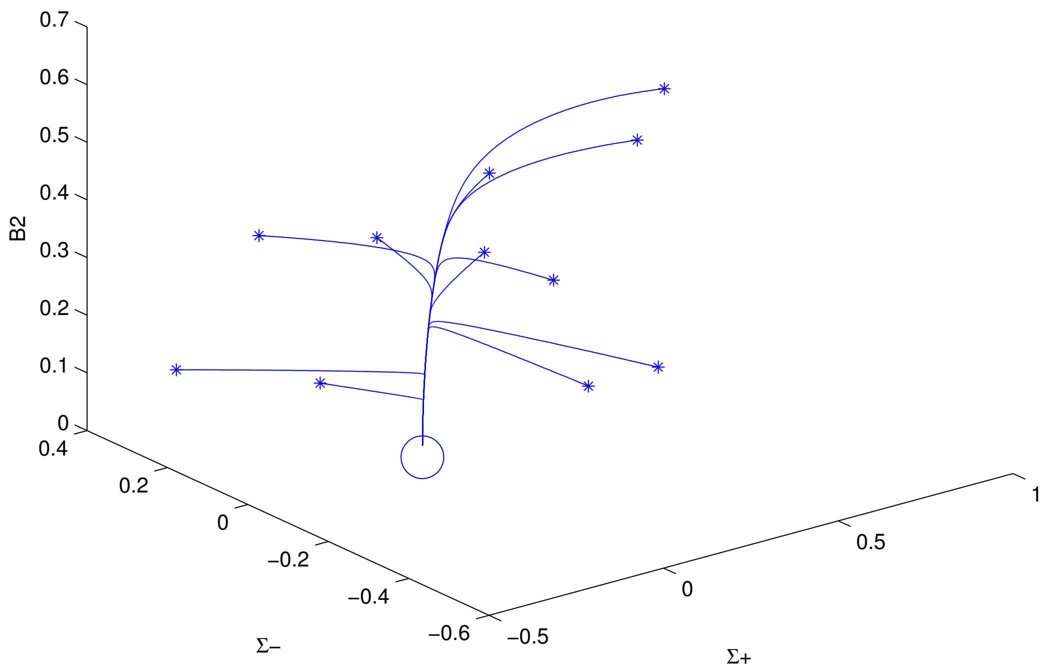
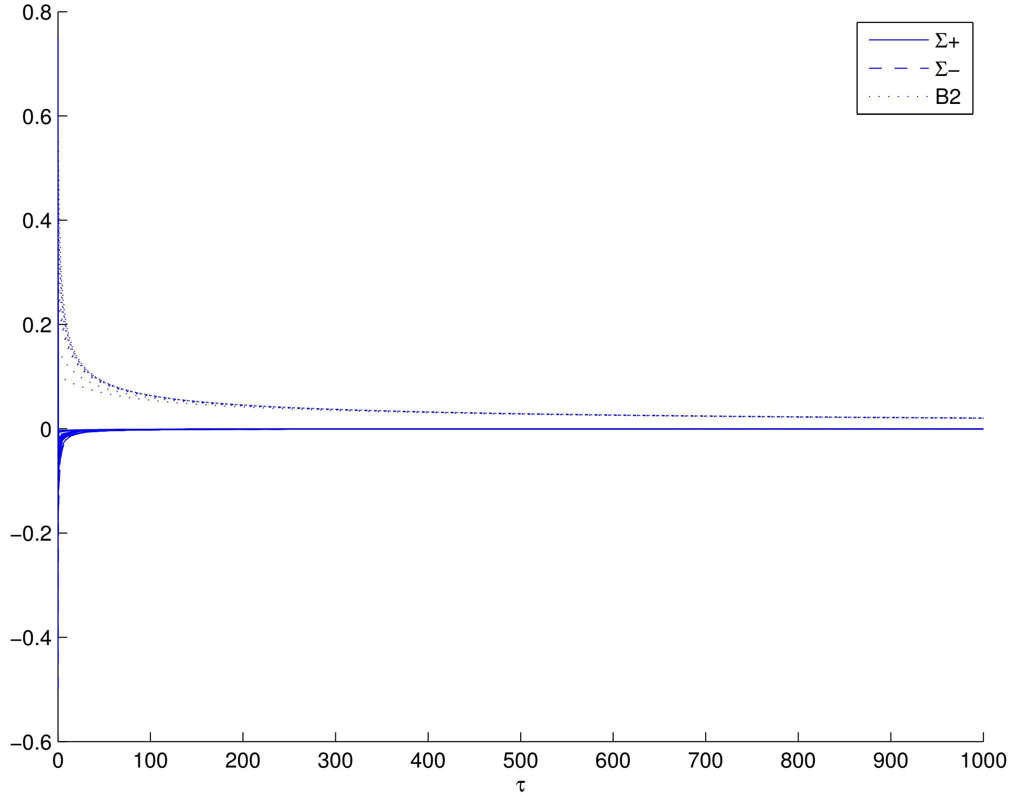


Fig. 3.10 continued.



3.7.2.4 $\xi_0 = 0, \eta_0 = 10$

Figure 3.11: This figure shows the dynamical system behaviour for $\xi_0 = 0$, $\eta_0 = 10$, and $w = 1/3$. The circle indicates the BIMV equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model does not isotropize with respect to the anisotropic magnetic field as can be seen from the last figure, where $B_2 > 0$ as $\tau \to \infty$, but does isotropize with respect to the spatial anisotropic variables, $\Sigma_\pm, \to 0$ as $\tau \to \infty$. This state is also special, since according to our fixed-point analysis, this behaviour is only exhibited for $w = 1/3, \eta_0 > 3/2$, and $\xi_0 = 0$.

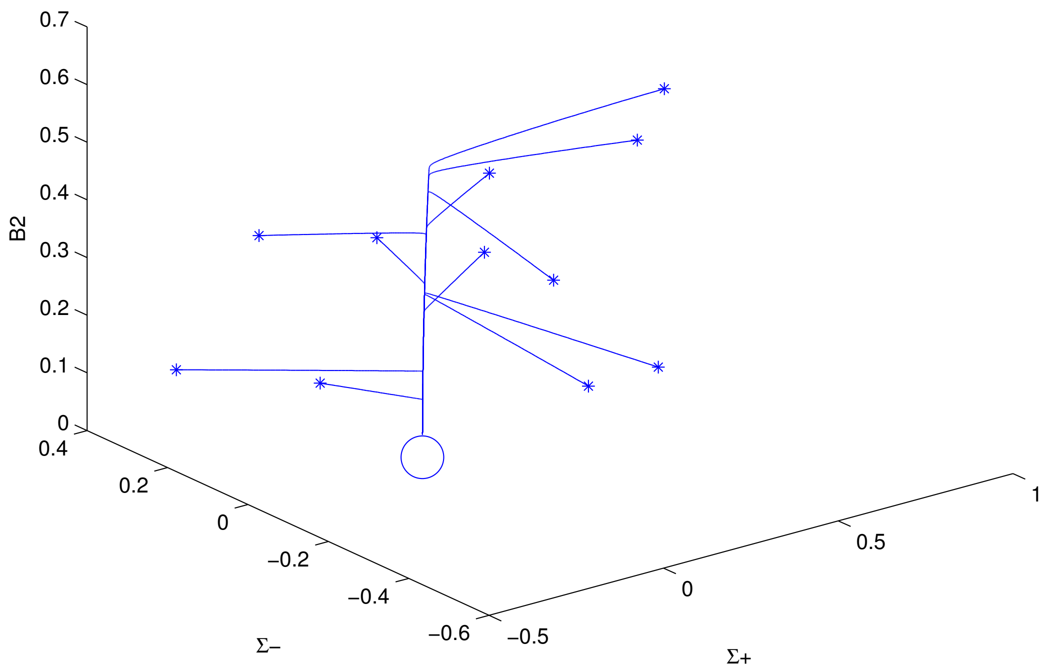
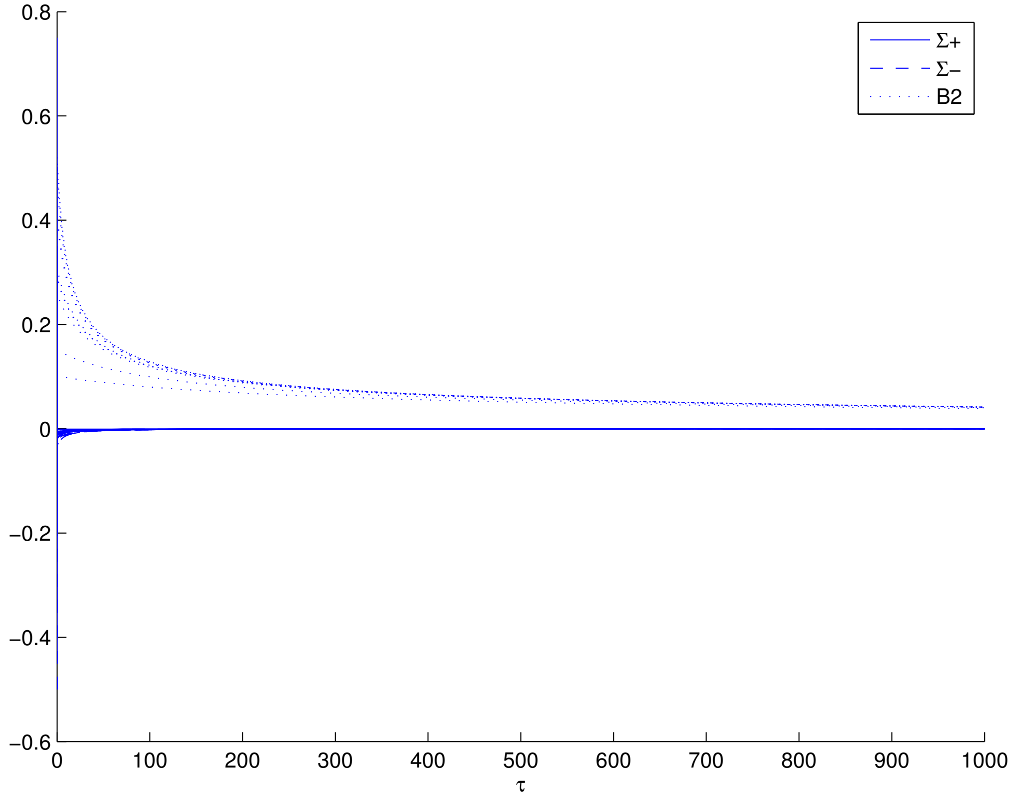


Fig. 3.11 continued.



3.7.3 Dust/Radiation Models: $w = 0.325$

3.7.3.1 $\xi_0 = 0.5, \eta_0 = 0.5$

Figure 3.12: This figure shows the dynamical system behaviour for $\xi_0 = 0.5$, $\eta_0 = 0.5$, and $w = 0.325$. The diamond indicates the FLRW equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model also isotropizes as can be seen from the last figure, where $\Sigma_\pm, \mathcal{B}_2 \to 0$ as $\tau \to \infty$.

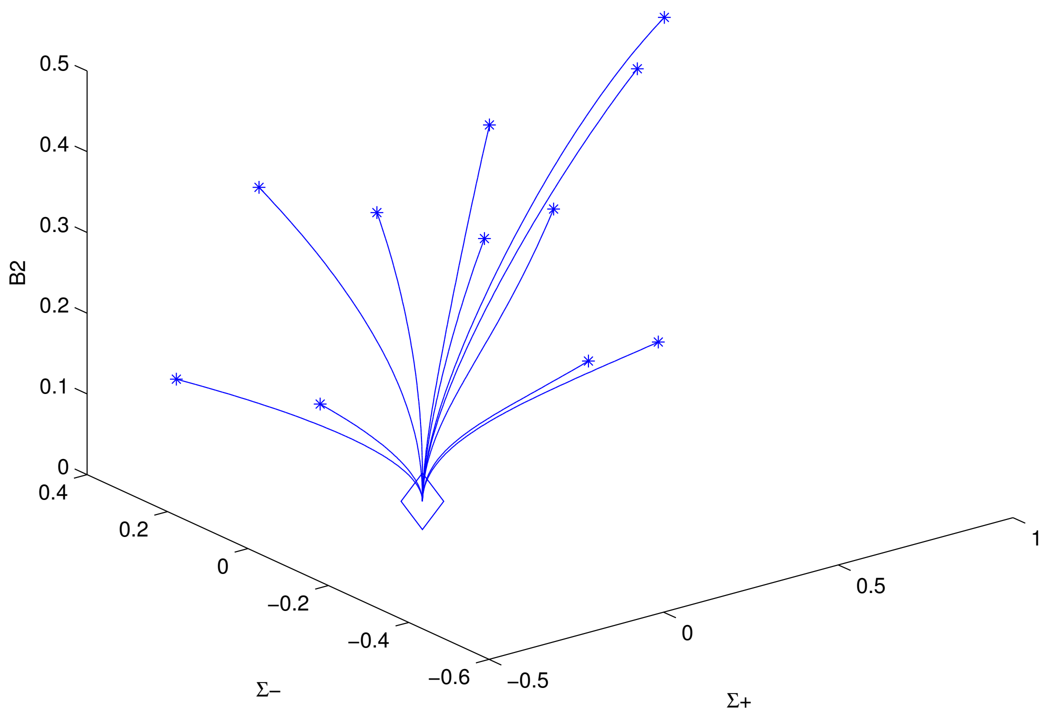
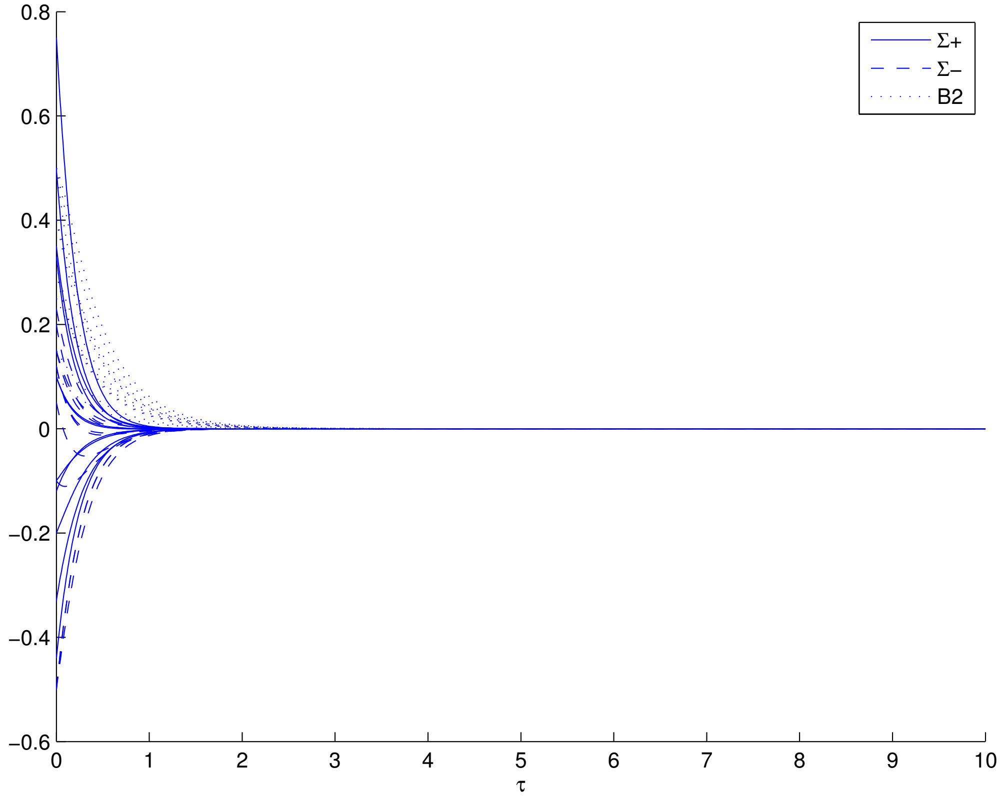


Fig. 3.12 continued.



3.7.4 Dust/Radiation Models: $w = 0.325$

3.7.4.1 $\xi_0 = 1, \eta_0 = 2$

Figure 3.13: This figure shows the dynamical system behaviour for $\xi_0 = 1$, $\eta_0 = 2$, and $w = 0.325$. The diamond indicates the FLRW equilibrium point, and this numerical solution shows that it is a local sink of the dynamical system. The model also isotropizes as can be seen from the last figure, where $\Sigma_\pm, \mathcal{B}_2 \to 0$ as $\tau \to \infty$.

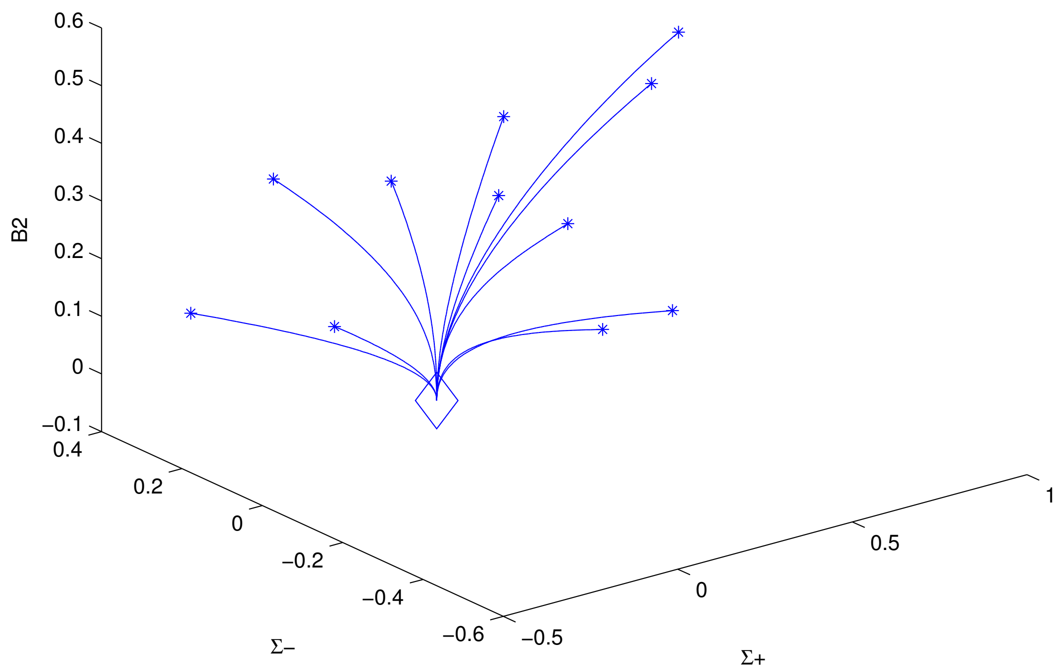
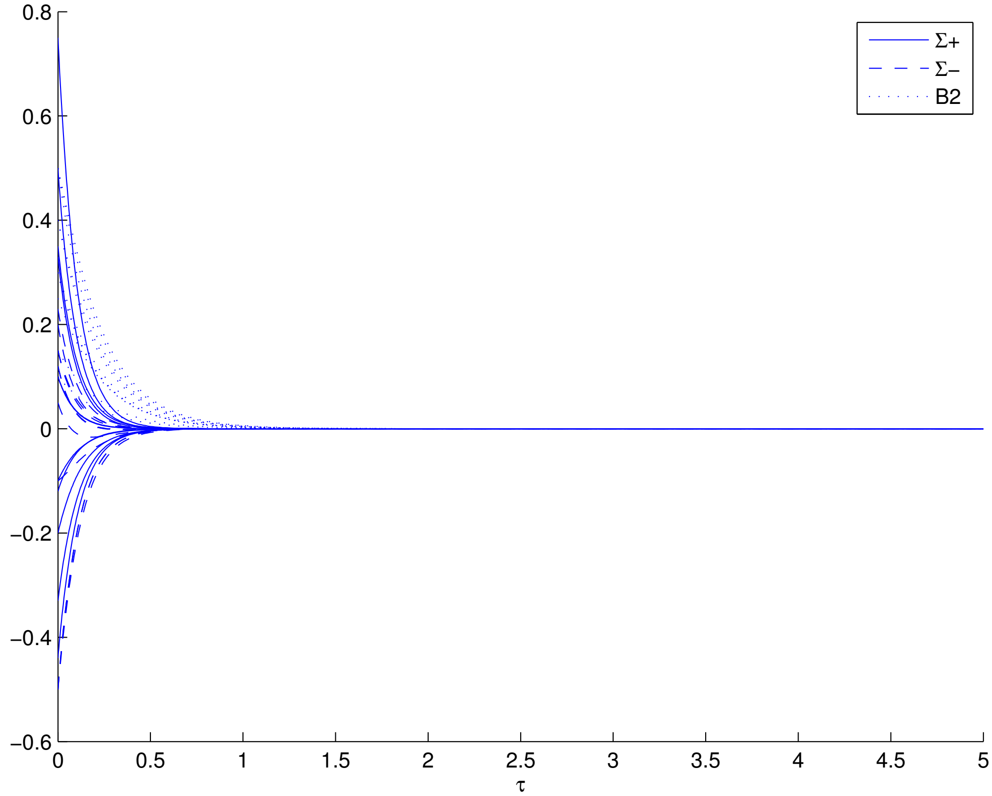


Fig. 3.13 continued.



3.8 Conclusions

We have presented in this chapter a comprehensive analysis of the dynamical behaviour of a Bianchi Type I viscous magnetohydrodynamic cosmology, using a variety of techniques ranging from a fixed-point analysis to analyzing asymptotic behaviour using standard dynamical systems theory combined with numerical experiments. We have shown that the fixed points may be associated with Kasner-type solutions, a flat universe FLRW solution, and interestingly, a new solution to the Einstein field equations involving non-zero magnetic fields, and non-zero viscous coefficients.

For cases in which $\eta_0 \geq 0, \quad \xi_0 \geq 0, \quad -1 \leq w < 1/3$ or $\eta_0 \geq 0, \quad 1/3 \leq w \leq 1, \quad \xi_0 > (3w-1)/9$, the dynamical model isotropizes asymptotically; that is, the spatial anisotropy and the anisotropic magnetic field decay to negligible values giving a close approximation to the present-day universe. We were also able to show that for regions in which $\eta_0 > 3/2, \xi_0 = 0, w = 1/3$ or $\eta_0 > 3/2, \quad 1/3 < w < (6\eta_0 - 5)/(6\eta_0 + 3), \quad 0 \leq \xi_0 \leq (3w-1)/9$, the model does not isotropize, rather at late times goes into a stable equilibrium in which there is a non-zero magnetic field.

The flat FLRW model whose associated equilibrium point was denoted by $\mathcal{F}$, is of primary importance with respect to models of the present day universe. Through our fixed point analysis, we showed that $\mathcal{F}$ represents a saddle point if $\eta_0 = 0, \quad 1/3 < w < 1, \quad 0 \leq \xi_0 < (3w-1)/9$, $\eta_0 = 0, \quad w = 1, \quad 0 < \xi_0 < 2/9$, or $\eta_0 > 0, \quad 1/3 < w \leq 1, \quad 0 \leq \xi_0 < (3w-1)/9$, (which was denoted above by SA(F)). In these regions, $\mathcal{F}$ attracts along its stable manifold and repels along its unstable manifold. More precisely, the stable manifold W^s of the equilibrium point $\mathcal{F}$, is tangent to the stable subspace E^s at $\mathcal{F}$ such that all orbits in W^s approach $\mathcal{F}$ as $\tau \to \infty$. Similarly, there exists an unstable manifold W^u of $\mathcal{F}$ such that it is tangent to the unstable subspace E^u at $\mathcal{F}$ and such that all orbits in W^u will approach $\mathcal{F}$ as $\tau \to -\infty$. Therefore, in the region denoted by SA(F), some orbits will have an initial attraction to $\mathcal{F}$, but will eventually be repelled by it. In the region denoted by S1(F), the point $\mathcal{F}$ is a local sink, and as such $\mathcal{F}$ attracts along its stable manifold, where the stable manifold W^s of the equilibrium point $\mathcal{F}$, is tangent to the stable subspace E^s at $\mathcal{F}$ such that all orbits in W^s approach $\mathcal{F}$ as $\tau \to \infty$. There is therefore a time period, and two possible configurations for which the cosmological model will asymptotically isotropize, and be compatible with present-day observations of high-degree isotropy.

3.9 Appendix

3.9.1 Jacobian Matrix for $\mathcal{BI}_{\mathcal{MV}}$

The Jacobian matrix for equilibrium point 3 is

$$J = \frac{1}{128\alpha}\begin{bmatrix} -(\alpha\mu_1+\mu_2\gamma) & -\sqrt{3}\mu_3(\beta_1+\gamma)^2/(2\alpha) & \sqrt{3}(\beta_2-\gamma)^{1/2}(\alpha\mu_4+\mu_5\gamma) \\ -\sqrt{3}\mu_3(\beta_1+\gamma)^2/(2\alpha) & -(\alpha\mu_6+\mu_7\gamma) & 3(\beta_2-\gamma)^{1/2}(\alpha\mu_4+\mu_5\gamma) \\ 2\sqrt{3}(\beta_2-\gamma)^{1/2}(\alpha^2\mu_5+\mu_3\gamma)/3 & 2(\beta_2-\gamma)^{1/2}(\alpha^2\mu_5+\mu_3\gamma) & 4\alpha\mu_5(\gamma-\beta_2) \end{bmatrix}, \tag{3.80}$$

where, in addition to the definition of parameters in equations (3.52), (3.53), (3.54) and (3.55), we define

$$\mu_1 = 2\beta_1-3(w-1)\beta_2-144w(1+2\eta_0)-16(13-22\eta_0), \quad \mu_2 = 9w^2(1+2\eta_0)+12w(1-2\eta_0)-53+6\eta_0, \tag{3.81}$$

$$\mu_3 = 3(w-1), \quad \mu_4 = 9w^2(1+2\eta_0)+6w(3-2\eta_0)-39+2\eta_0, \quad \mu_5 = 3w-1, \tag{3.82}$$

$$\mu_6 = 6\beta_1-9(w-1)\beta_2-240w(1+2\eta_0)-16(27-26\eta_0), \quad \mu_7 = 27w^2(1+2\eta_0)+36w(1-2\eta_0)-95+18\eta_0. \tag{3.83}$$

On the bifurcation surface $\xi_0 = (3w-1)/9$ we have the simplifications $\gamma = \beta_2 = -\beta_1$ and $\alpha\mu_1+\mu_2\gamma = \alpha\mu_6+\mu_7\gamma = 128\alpha(1+2\eta_0)$, and thus the matrix J is diagonal.

3.9.2 Initial Values for Numerical Experiments

Σ_+	Σ_-	$\mathcal{B}_2$	Ω_f
0.1	0.2	0.3	0.8150
0.1	-0.5	0.3	0.6050
-0.1	-0.5	0.3	0.6050
-0.2	-0.5	0.5	0.3350
0.5	-0.1	0.5	0.3650
0.75	0.05	0.5	0.0600
0.33	0.12	0.4	0.6367
-0.33	0.12	0.4	0.6367
-0.44	0.32	0.15	0.7198
-0.12	0.15	0.1	0.9481
0.35	0.15	0.25	0.7613
0.99	0	0	0.0199
0.499	-0.855	0	0.0200
0	-0.99	0	0.0199
0	0.99	0	0.0199

Table 3.1: Initial conditions used in the numerical experiments. Note that in each case, $0 \leq \Omega_f \leq 1$ and $\mathcal{B}_2 \geq 0$ as required.

I could write shorter sermons but when I get started I'm too lazy to stop.

Lincoln

4

On The Dynamics of a Closed Viscous Universe

4.1 Introduction

In this chapter, we analyze a non-tilted Bianchi Type IX cosmological model with a viscous fluid source containing constant bulk and shear viscous coefficients, while neglecting the effects of heat conduction. Such a cosmological model is homogeneous on three-dimensional spacelike orbits of the isometry group G_3, and is therefore spatially homogeneous (Page 22-23, [WE97]). We take the dimension of the isotropy subgroup to be $d = 0$, hence considering a spatially homogeneous and anisotropic model of the universe. In this case, the Einstein field equations reduce to a coupled system of nonlinear first-order ordinary autonomous differential equations. One can then use methods of analyzing dynamical systems to obtain important information about the dynamical evolution of such a universe model, with particular emphasis on past and future asymptotic states.

The Bianchi Type IX model is perhaps among the most well-known and well-studied models in cosmology. Belinskii and Khalatnikov [BK70] investigated the evolutionary dynamics of the Bianchi Type IX metric as it approached an initial singularity. Belinksii, Lifshitz, and Khalatnikov [BKL70] showed that near the initial singularity, the Bianchi IX model exhibits oscillatory behaviour represented by a series of Kasner-like "bounces". Misner [Mis69b] [Mis69a] applied Hamiltonian methods to show that the dynamics of a Bianchi IX model is equivalent to the classical problem of a particle in a potential well. In the former paper, he formulated a quantum theory based on this geometry by setting up canonical commutation relations on the independent canonical variables. In the latter paper, Misner introduced the well-known Mixmaster universe as an attempt to describe the present-day spatially homogeneous and isotropic universe as a result of a "smoothing-out" process of early-universe anisotropy. Matzner, Shepley, and Warren [MSW70] performed a detailed analytical and numerical analysis of Bianchi Type IX models containing dust. They were able to prove the existence of regions of infinite density and a time of maximum expansion. Ryan [Rya71a] [Rya71b] [Rya72] using the Hamiltonian methods of Arnowitt, Deser, and Misner [ADM08] analyzed Bianchi Type IX universes which simultaneously exhibited expansion, rotation, and shear, and placed particular emphasis on the dynamics near the initial singularity in his analysis. Ryan also showed that the dynamical equations simplify in the region near the initial singularity, and used the concept of a point moving in a set of potentials to analyze the dynamical behaviour near the singularity. Barrow and Matzner [BM80] analyzed the evolution of a massive scalar field in Bianchi Type IX model. They were able to show that the probability of a bouncing epoch occurring at very early times is infinitesimally small.

The problem of recollapse is one of the central themes of this paper, and we will revisit it in the next section, when deciding how to normalize our dynamical variables to allow for the possibility of our model expanding or contracting. Barrow and Tipler [BT85] first showed that the existence of a maximal hypersurface is a necessary and sufficient condition for the existence of a final singularity in a universe with a compact Cauchy surface. They further showed that a cosmological model with topology S^3 can admit such maximal hypersurfaces. Barrow, Galloway and Tipler [BGT86] formulated the closed-universe recollapse conjecture, where they showed that if the positive pressure criterion, dominant energy and matter regularity conditions hold, then a FLRW universe with topology S^3 must recollapse. They also considered a number of Bianchi Type IX universes with various matter tensors, and provided a new recollapse conjecture for such matter-filled universes. Barrow [Bar88a] discussed in detail the question of whether closed universes can avoid recollapsing before an inflationary period ensues. It was shown that closed universes in an extreme initial anistropic state cannot recollapse until they are close to isotropy. Barrow made the point that even if a universe possesses a S^3 topology and the strong energy condition holds, it is not known whether all anisotropic closed universes recollapse. Related to this, Calogero and Heinzle [CH10] proved that there exists a class of Bianchi Type IX models that obey the strong energy condition but do not recollapse, rather, they expand for all times. Lin and Wald [LW89] [LW90] showed that for matter which satisfies the dominant energy condition in addition to having non-negative average principal pressures, there is no corresponding Bianchi Type IX model that expands for an infinite time. Wald [Wal83] examined the future asymptotic behaviour of initially expanding spatially homogeneous models containing a positive cosmological constant. It was shown that if the cosmological constant, Λ is sufficiently large compared with spatial-curvature terms, the Bianchi Type IX model exhibits stable future asymptotic behaviour only in the case of recollapse and an asymptotic late-time approach to a de Sitter spacetime.

Burd, Buric, and Ellis [BBE90] performed a detailed study of the chaotic behaviour of the Bianchi Type IX model. They numerically calculated the Lyapunov exponent and showed that it decreases steadily. In contradiction to this result, Rugh and Jones [RJ90] showed that the maximal Lyapunov exponent for the phase flow is zero with respect to the time variable used in previous studies for the Bianchi IX model, but concluded that the deterministic model is unpredictable due to a large non-negative entropy. Uggla and Zur-Muhlen [UZM90] investigated locally rotationally symmetric Bianchi Type IX models with a perfect fluid source. By considering a locally rotationally symmetric model, they obtained a reduced first-order system of differential equations that allowed them so see the full set of solutions from the initial big bang singularity to the final big crunch singularity. Cornish and Levin [CL97b] [CL97a] used coordinate-invariant fractal methods to show that the Bianchi Type IX model is indeed chaotic, independent of any analysis using the methods of calculating Lyapunov exponents. Rendall [Ren94] showed that for Bianchi IX models whose observers have worldlines orthogonal to the spatial hypersurfaces, no singularity can occur in finite time. Rendall [Ren97] also performed a detailed analysis of the asymptotic behaviour of the Bianchi Type IX model. It was shown that there are infinitely many oscillations near the singularity and that the Kretschmann scalar is unbounded in that region. Van Den Hoogen and Olasagasti [VDHO99] investigated the isotropization of the Bianchi Type IX model with an exponential potential field. They found conditions on the potential exponent that classified inflationary and isotropization behaviour. Ringström [Rin01] used dynamical system methods to investigate the asymptotic behaviour of the Bianchi Type IX model with an orthogonal perfect fluid close to the initial singularity. It was found that in the case of a stiff fluid, the solution converges to a point. For other types of matter, the solutions converge to an attractor consisting of Bianchi Type II vacuum orbits. De Oliveira et.al. [DOOdADoST02] analyzed the dynamics of a Bianchi Type IX model with comoving dust and a cosmological constant, in which they found evidence of homoclinic chaos in the dynamical evolution. Barrow, Ellis, Maartens, and Tsagas [BEMT03] showed that spatially homogeneous Bianchi Type IX models destabilize an Einstein static universe. Heinzle, Röhr, and Uggla [HRU05] investigated a locally rotationally symmetric Bianchi Type IX model with an orthogonal perfect fluid source. They showed that when the perfect fluid obeys the strong energy condition, such a model expands from an initial singularity and recollapses to a singularity. Heinzle and Uggla [HU09b] considered the past asymptotic dynamics of both a Bianchi Type IX vacuum and an orthogonal perfect fluid model. They formulated precise conjectures regarding the past asymptotic states using a dynamical systems approach. They also made detailed comparisons with previous metric and Hamiltonian approaches that also analyzed dynamical behaviour in the neighborhood of the initial singularity. Heinzle and Uggla [HU09a] also gave a new proof of the Bianchi type IX attractor theorem which states that the past-time asymptotic behaviour of the Bianchi Type IX solutions is determined by

Bianchi Type I and II vacuum states. Calogero and Heinzle [CH11] considered among other models a locally rotationally symmetric Bianchi Type IX model containing Vlasov/collisionless matter, elastic matter, and magnetic fields. They discovered that generic type IX solutions oscillate toward the initial singularity. Barrow and Yamamoto [BY12] studied the stability of the Einstein static universe as a non-locally rotationally symmetric, that is, a general Bianchi Type IX model with both non-tilted and tilted perfect fluids. They showed that the Einstein static universe is unstable to homogeneous perturbations of the Bianchi Type IX model to both the future and the past. Uggla [Ugg13] has described recent developments with respect to oscillatory spacelike singularities in Bianchi Type IX models.

As discussed by Grøn and Hervik (Chapter 13, [GH07]), viscous models have become of general interest in early-universe cosmologies largely in two contexts. Firstly, in models where bulk viscous terms dominate over shear terms, the universe expands to a de Sitter-like state, which is a spatially flat universe neglecting ordinary matter, and including only a cosmological constant. Such models isotropize indirectly through the massive expansion. Secondly, in the absence of any significant heat flux, shear viscosity is found to play an important role in models of the universe at its early stages. In particular, neutrino viscosity is considered to be one of the most important factors in the isotropization of our universe. Parnovskii [Par77] for example, investigated the influence of viscosity on the dynamics of a Bianchi type II universe, where it was shown that at late times such a universe approaches an FLRW type singularity or an anisotropic solution. It was further shown that the bulk viscosity has an important influence in creating entropy per particle throughout the future evolution of such a model. Misner [Mis68] considered solutions of the Einstein field equations with flat homogeneous spacelike hypersurfaces with anisotropic expansion rates in which effects of viscosity were included in the associated radiation. Misner [Mis67] also studied the effect of neutrino viscosity on the homogeneous anisotropy in relation to the expansion of the early universe.

Bianchi Type IX models containing viscous fluids have also been studied in some detail. Caderni and Fabbri [CF78] [CF79] investigated the isotropization of the Bianchi Type IX model due to neutrino viscosity. Banerjee and Santos [BS84] studied the dynamical effects of viscous fluids on the Bianchi Type IX model. Banerjee, Sanyal, and Chakraborty [BSC90] found exact solutions to the Einstein field equations for a Bianchi

Type IX model with a viscous fluid distribution. Chakraborty and Chakraborty [CC01] investigated the dynamics of a Bianchi Type IX model with a bulk viscous fluid and variable gravitational and cosmological constants. Pradhan, Srivastav, and Yadav [PSY05a] examined the dynamics of a Bianchi Type IX model with a varying cosmological constant and both bulk and shear viscosities. Bali and Yadav [BY05] investigated a Bianchi Type IX model with a viscous fluid containing both bulk and shear viscosities.

All of the aforementioned methods that consider viscous fluids employ the metric formalism of general relativity and assume supplemental conditions between the different metric components in order to obtain exact solutions. In this chapter, we will use dynamical systems methods built upon the pioneering framework of orthonormal frames initiated by Ellis and MacCallum [EM69] to analyze the behaviour of the Bianchi Type IX model with a viscous fluid with respect to early times, late times, and intermediate times. Dynamical systems methods have been used to study viscous cosmologies by van den Hoogen and Coley [vdHC95], and Kohli and Haslam [KH13b] [KH13a]. As we mentioned above, and explicitly state below, in our model, the bulk and shear viscosity coefficients are taken to be non-negative constants. The case where these coefficients are not constant has been studied in some detail. In particular, Barrow [Bar88b] showed that models of an inflationary universe driven by Witten strings in the very early universe are equivalent to the addition of bulk viscosity to perfect fluid cosmological models with zero curvature. In this work, Barrow considered the case where the bulk viscosity has a power-law dependence upon the matter density. It was shown that if the exponent is greater than 1/2, there exist deflationary solutions which begin in a de Sitter state and evolve away from it asymptotically in the future. On the other hand, if this exponent is less than 1/2 (which includes the case considered in our present work), then solutions expand from an initial singularity towards a de Sitter state. Barrow [Bar82] estimated the entropy production associated with anisotropy damping in the early universe by considering a Bianchi type I metric with an equilibrium radiation gas and anisotropic stresses produced by shear viscosity. It was shown that the shear viscosity based on kinetic theory has the general form of being proportional to the matter density and that the entropy production due to collisional transport is negligible in such a model.

Our approach will allow one to fully ascertain the effects of the constant bulk and shear viscous coefficients on the dynamics of the Bianchi Type IX model, and will therefore be more general than the metric approaches taken so far. We note that to the best of the authors' knowledge at the time of writing this article, such an approach based on dynamical systems theory has not been investigated in the literature.

4.2 The Viscous Fluid Matter Source

In the absence of heat conduction, the energy-momentum tensor corresponding to a viscous fluid with fluid velocity four-vector is given by [KH13b]

$$T_{ab} = (\mu + p)\, u_a u_b + g_{ab} p - 3\xi H h_{ab} - 2\eta\sigma_{ab}, \tag{4.1}$$

where μ, p, and σ_{ab} denote the fluid's energy density, pressure, and shear respectively, while ξ and η denote the bulk and shear viscosity coefficients of the fluid. Throughout this chapter, both coefficients are taken to be *non-negative constants*. H denotes the Hubble parameter, and $h_{ab} \equiv u_a u_b + g_{ab}$ is the standard projection tensor corresponding to the metic signature $(-,+,+,+)$.

We additionally assume that this fluid obeys a barotropic equation of state, $p = w\mu$, where $w : \{w \in \mathbb{R} : -1 \leq w \leq 1\}$, is an equation of state parameter. Some typical values for w are $w = 0$ (dust), $w = -1$ (cosmological constant), $w = 1/3$ (radiation), and $w = 1$ (stiff fluid). In order to derive a set of evolution equations for the model, we will need expressions for the total energy density, $\bar{\mu}$, total pressure $\bar{p}$, and total anisotropic stress $\bar{\pi}_{ab}$. Using the definitions,

$$\bar{\mu} = T_{ab} u^a u_b, \quad \bar{p} = \frac{1}{3} h^{ab} T_{ab}, \quad \bar{\pi}_{ab} = h_a^c h_b^d T_{cd} - \bar{p} h_{ab}, \tag{4.2}$$

we find that

$$\bar{\mu} = \mu, \quad \bar{p} = w\mu, \quad \text{and} \quad \bar{\pi}_{ab} = -2\eta\sigma_{ab}. \tag{4.3}$$

4.3 The Dynamical Equations

The Bianchi cosmologies are described by a four-dimensional pseudo-Riemannian manifold $\mathcal{M}$, a corresponding metric tensor $\mathbf{g}$ defined on $\mathcal{M}$, and a fundamental four-velocity $\mathbf{u}$ that we will take to be orthogonal to the group orbits. Denoting the orthonormal basis vectors by $\mathbf{e}_\alpha$ and the unit vector normal to the orbits of G_3 by $\mathbf{n}$, and using the quantities in Eq. (4.3), the Einstein field equations take the form (Page 39, [WE97]):

$$\dot{H} = -H^2 - \frac{2}{3}\sigma^2 - \mu\left(\frac{1}{6} + \frac{1}{2}w\right), \tag{4.4}$$

$$\dot{\sigma}_{ab} = -3H\sigma_{ab} + 2\epsilon^{uv}_{(a}\sigma_{b)u}\Omega_v - S_{ab} - 2\eta\sigma_{ab}, \tag{4.5}$$

$$\mu = 3H^2 - \sigma^2 + \frac{1}{2}R, \tag{4.6}$$

$$0 = 3\sigma^u_a a_u - \epsilon^{uv}_a \sigma^b_u n_{bv}, \tag{4.7}$$

where S_{ab} and R are the three-dimensional spatial curvature and Ricci scalar and are defined as:

$$S_{ab} = b_{ab} - \frac{1}{3}b^u_u\delta_{ab} - 2\epsilon^{uv}_{(a}n_{b)u}a_v, \tag{4.8}$$

$$R = -\frac{1}{2}b^u_u - 6a_u a^u, \tag{4.9}$$

where $b_{ab} = 2n^u_a n_{ub} - (n^u_u)\, n_{ab}$. We have also denoted by Ω_v the angular velocity of the spatial frame. The matrix n_{ab} and vector a_c are used in decomposing the structure constants of G_3 and classify the Bianchi cosmologies (See Page 36, [WE97] for more details). Using the Jacobi identities, one obtains evolution equations for these variables as well:

$$\dot{n}_{ab} = -Hn_{ab} + 2\sigma^u_{(a}n_{b)u} + 2\epsilon^{uv}_{(a}n_{b)u}\Omega_v, \tag{4.10}$$

$$\dot{a}_a = -Ha_a\sigma^b_a a_b + \epsilon^{uv}_a a_u\Omega_v, \tag{4.11}$$

$$0 = n^b_a a_b. \tag{4.12}$$

The contracted Bianchi identities give the evolution equation for μ as (Page 40, [WE97])

$$\dot{\mu} = -3H(\mu + p) - \sigma^b_a\pi^a_b + 2a_a q^a. \tag{4.13}$$

The algebraic constraints for the Bianchi Type IX model are

$$a_a = 0, \quad n_{ab} = \mathrm{diag}(n_{11}, n_{22}, n_{33}), \text{ where } n_{11} > 0, \quad n_{22} > 0, \quad n_{33} > 0. \tag{4.14}$$

Note that, for the Bianchi class A models, it can also be shown that $\Omega_v = 0$.

The standard way to proceed from this point is to use expansion-normalized variables, for which one reduces the dimension of the state space by introducing a dimensionless time variable τ. In this approach, the Raychaudhuri equation Eq. (4.4) decouples from the system of differential equations, yielding a reduced system of autonomous first-order ordinary differential equations. The problem with using this method for the Bianchi Type IX model is that as we discussed in the introduction, the Bianchi Type IX model has the potential to recollapse. The notion of recollapse has been investigated for cosmological models whose spatial sections have topology S^3 or $S^2 \times S^1$ (see the references in the introduction of this article). It should be noted that such models do not always recollapse, but have the potential to do so. Therefore, in order to get a complete picture of the dynamics of the system, one needs to employ a different normalization than the standard expansion-normalized variables.

We will make use of the approach outlined by Hewitt, Uggla, and Wainwright (Chapter 8, [WE97]) in which H assumes all real values so as to include a re-collapsing epoch ($H < 0$). The evolving state vector has the form $\mathbf{x} = (\sigma_+, \sigma_-, n_1, n_2, n_3)$, where we have defined:

$$\sigma_+ \equiv \frac{1}{2}\left(\sigma_{22} + \sigma_{33}\right), \quad \sigma_- \equiv \frac{1}{2\sqrt{3}}\left(\sigma_{22} - \sigma_{33}\right), \tag{4.15}$$

and

$$n_{11} \equiv n_1, \quad n_{22} \equiv n_2, \quad n_{33} \equiv n_3. \tag{4.16}$$

We will normalize this state vector by a normalization factor D, defined by

$$D \equiv \sqrt{H^2 + \frac{1}{4}\left(n_1 n_2 n_3\right)^{2/3}}. \tag{4.17}$$

The resulting state vector is given by

$$\tilde{\mathbf{x}} = \left(\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3\right), \tag{4.18}$$

where

$$\tilde{H} = \frac{H}{D}, \quad \tilde{\Sigma}_\pm = \frac{\sigma_\pm}{D}, \quad \tilde{N}_\alpha = \frac{n_\alpha}{D}. \tag{4.19}$$

These variables satisfy the constraint

$$\tilde{H}^2 + \frac{1}{4}\left(\tilde{N}_1 \tilde{N}_2 \tilde{N}_3\right)^{2/3} = 1. \tag{4.20}$$

We will additionally define a dimensionless time variable $\tilde{\tau}$ such that

$$\frac{dt}{d\tilde{\tau}} = \frac{1}{D}. \tag{4.21}$$

Hewitt, Uggla, and Wainwright then obtain the evolution equation for D as

$$\frac{dD}{d\tilde{\tau}} = -\left(1+\tilde{q}\right)\tilde{H}D, \tag{4.22}$$

where

$$\tilde{q} = \tilde{H}^2 q. \tag{4.23}$$

We will also define several quantities in addition to Eq. (4.19) that will be needed in deriving the full set of evolution equations and their corresponding constraints. Analogous to the case of expansion-normalized variables as found in the Appendix of [HBW01], we have

$$\tilde{S}_{ij} = \frac{R_{\langle ij \rangle}}{D^2}, \quad \tilde{\Omega} = \frac{\bar{\mu}}{3D^2}, \quad \tilde{P} = \frac{\bar{p}}{3D^2}, \quad \tilde{\Pi}_{ij} = \frac{\bar{\pi}_{ij}}{D^2}, \quad 3\tilde{\xi}_0 = \frac{\xi}{D}, \quad 3\tilde{\eta}_0 = \frac{\eta}{D}, \quad \hat{\Sigma}^2 = \frac{\sigma^2}{3D^2}, \tag{4.24}$$

where expressions for $\bar{\mu}, \bar{p}$, and $\bar{\pi}_{ij}$ were derived in Eq. (4.3). Note that we have additionally used the abbreviation $\hat{\Sigma}^2 = \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2$. The angled brackets indicate that the projected symmetric trace-free components are to be taken. Note that for clarity in notation, we have left off the standard "three" superscript on $R_{\langle ij \rangle}$. In this chapter, it is to be assumed that R_{ij} indicates the three-dimensional Ricci curvature, and R the corresponding three-dimensional Ricci scalar.

First applying the Bianchi Type IX algebraic constraints as given in Eq. (4.14) to Eqs. (4.4), (4.5), (4.11), and (4.13), and then normalizing these equations according to Eqs. (4.17), (4.19), (4.24), (4.21) and

(4.22) we obtain the full set of evolution equations as

$$\tilde{H}' = -(1-\tilde{H}^2)\tilde{q}, \tag{4.25}$$

$$\tilde{\Sigma}'_+ = \tilde{\Sigma}_+\tilde{H}\left(-2+\tilde{q}\right) - 6\tilde{\Sigma}_+\tilde{\eta}_0 - \tilde{S}_+, \tag{4.26}$$

$$\tilde{\Sigma}'_- = \tilde{\Sigma}_-\tilde{H}\left(-2+\tilde{q}\right) - 6\tilde{\Sigma}_-\tilde{\eta}_0 - \tilde{S}_-, \tag{4.27}$$

$$\tilde{N}'_1 = \tilde{N}_1\left(\tilde{H}\tilde{q} - 4\tilde{\Sigma}_+\right), \tag{4.28}$$

$$\tilde{N}'_2 = \tilde{N}_2\left(\tilde{H}\tilde{q} + 2\tilde{\Sigma}_+ + 2\sqrt{3}\tilde{\Sigma}_-\right), \tag{4.29}$$

$$\tilde{N}'_3 = \tilde{N}_3\left(\tilde{H}\tilde{q} + 2\tilde{\Sigma}_+ - 2\sqrt{3}\tilde{\Sigma}_-\right), \tag{4.30}$$

$$\tilde{\Omega}' = \tilde{\Omega}\tilde{H}\left(-1+2\tilde{q}-3w\right) + 9\tilde{H}^2\tilde{\xi}_0 + 12\tilde{\eta}_0\left(\tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2\right), \tag{4.31}$$

where

$$\tilde{q} = 2\left(\tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2\right) + \frac{1}{2}\tilde{\Omega}\left(1+3w\right) - \frac{9}{2}\tilde{\xi}_0\tilde{H}. \tag{4.32}$$

The variables $\tilde{S}_\pm$ were obtained by normalizing the components of the trace-free spatial Ricci tensor given by Eq. (6.6) in [WE97], and computed to be

$$\tilde{S}_+ = \frac{1}{6}\left[\left(\tilde{N}_2 - \tilde{N}_3\right)^2 - \tilde{N}_1\left(2\tilde{N}_1 - \tilde{N}_2 - \tilde{N}_3\right)\right], \tag{4.33}$$

$$\tilde{S}_- = \frac{1}{2\sqrt{3}}\left[\left(\tilde{N}_3 - \tilde{N}_2\right)\left(\tilde{N}_1 - \tilde{N}_2 - \tilde{N}_3\right)\right]. \tag{4.34}$$

The additional constraint on the dynamical system Eqs. (4.25)-(4.31) is given by the generalized Friedmann equation, (4.6). We first note that the Ricci scalar as defined in Eq. (4.9), upon applying the Bianchi Type IX algebraic constraints Eq. (4.14) takes the form

$$R = -\frac{1}{2}\left[n_1^2 + n_2^2 + n_3^2 - 2\left(n_1n_2 + n_2n_3 + n_3n_1\right)\right]. \tag{4.35}$$

Applying Eqs. (4.19), (4.24), and (4.35) to Eq. (4.6), we obtain

$$\tilde{\Omega} = \tilde{H}^2 - \hat{\Sigma}^2 + \frac{R}{6D^2}. \tag{4.36}$$

Upon further applying the constraint Eq. (4.20) to Eq. (4.36), we obtain

$$\tilde{\Omega} + \hat{\Sigma}^2 + \tilde{V} = 1, \tag{4.37}$$

where

$$\tilde{V} = \frac{1}{12}\left[\tilde{N}_1^2 + \tilde{N}_2^2 + \tilde{N}_3^2 - 2\tilde{N}_1\tilde{N}_2 - 2\tilde{N}_2\tilde{N}_3 - 2\tilde{N}_3\tilde{N}_1 + 3\left(\tilde{N}_1\tilde{N}_2\tilde{N}_3\right)^{2/3}\right]. \tag{4.38}$$

An important point to note is that from Eq. (4.20) we have that

$$-1 \leq \tilde{H} \leq 1. \tag{4.39}$$

We also can see from Eq. (4.38) that

$$\tilde{V} \geq 0. \tag{4.40}$$

These subsequent conditions were also noted on Page 181 in [WE97].

Before we proceed, we feel that a technical point is in order. In the standard expansion-normalized variables approach to analyzing the Bianchi cosmologies, one typically reduces the dimension of the dynamical system state space by using the generalized Friedmann equation to eliminate $\tilde{\Omega}$ from the system of equations, thereby making $\tilde{\Omega}'$ an auxiliary equation. The problem with this is that in our approach because the constraint equation for $\tilde{\Omega}$ Eq. (4.37) contains $\tilde{V}$, the Jacobian matrix will not be defined at all equilibrium points. For example, if we were to use Eq. (4.37) to eliminate $\tilde{\Omega}$ from the system of equations, each term would be replaced with a term that contained $\tilde{V}$ as given by Eq. (4.38), in particular, a factor of $\left(\tilde{N}_1\tilde{N}_2\tilde{N}_3\right)^{-1/3}$ would enter into each equation, which of course is not defined when any one of $\tilde{N}_{1,2,3} = 0$, even if both constraint equations (4.20) (4.37) above are satisfied. Therefore, we will not in this chapter eliminate $\tilde{\Omega}$ from the dynamical system of equations, as we wish to ascertain the dynamical behaviour of all possible equilibrium points. This same methodology was employed by Barrow and Yamamoto [BY12].

4.4 A Qualitative Analysis of the Dynamical System

With the dynamical equations (4.25)-(4.31) and their constraints (4.20) and (4.37) in hand, we are in position to perform a detailed analysis of the fixed points of the system. Before we proceed, however, we would like to note some important qualitative properties that can be deduced from the dynamical system.

4.4.1 Symmetries and Invariant Sets

We note that the dynamical system given by Eqs. (4.25)-(4.31) has three symmetries given by

$$\left[\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{\Omega}\right] \rightarrow \left[\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, -\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{\Omega}\right], \tag{4.41}$$

$$\left[\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{\Omega}\right] \rightarrow \left[\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, \tilde{N}_1, -\tilde{N}_2, \tilde{N}_3, \tilde{\Omega}\right], \tag{4.42}$$

$$\left[\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{\Omega}\right] \rightarrow \left[\tilde{H}, \tilde{\Sigma}_+, \tilde{\Sigma}_-, \tilde{N}_1, \tilde{N}_2, -\tilde{N}_3, \tilde{\Omega}\right]. \tag{4.43}$$

The dynamical system is therefore invariant with respect to spatial inversions in the functions $\tilde{N}_1, \tilde{N}_2$, and $\tilde{N}_3$, which implies that we can take $\tilde{N}_1 \geq 0$, $\tilde{N}_2 \geq 0$, and $\tilde{N}_3 \geq 0$. We can also see that these correspond to the invariant sets of the system. Recall that if we let M be phase space of the flow of the dynamical system, then an invariant set is a set $A \subset M$ such that $g^t A = A$, $\forall\, t$, where $\{g^t\}$ represents the dynamical system on the phase space M, and $t \in \mathbb{R}$. In other words, the invariant set consists of entire trajectories [AAA$^+$97b]. Tavakol (Chapter 4, [WE97]) discusses a simple way to obtain the invariant sets of a dynamical system. Let us consider a dynamical system $\dot{x} = v(x), \quad x \in \mathbb{R}^7$. Let $Z : \mathbb{R}^7 \to \mathbb{R}$ be a C^1 function such that $Z' = \alpha Z$, where $\alpha : \mathbb{R}^7$ is a continuous function. Then the subsets of $\mathbb{R}^7$ defined by $Z > 0$, $Z = 0$, and $Z < 0$ are invariant sets of the flow of the dynamical system. Applying this proposition to our dynamical system in combination with the symmetries found above, we see that $\tilde{N}_i > 0$ and $\tilde{N}_i = 0$, where $i = 1, 2, 3$ are invariant sets of the system.

Combinations of $\tilde{N}_i > 0$ and $\tilde{N} = 0$ determine various Bianchi types of Class A. However, because of the constraint (4.20), these combinations necessarily restrict the value of $\tilde{H}$ as well. We list these Bianchi invariant sets based on the description given on Page 126 of [WE97] in Table 4.1.

Table 4.1: The various Bianchi invariant sets, with $\alpha = 1, 2, 3$.

Notation	Restrictions on $\tilde{N}_i \geq 0$	Restriction on $\tilde{H}$
$B(I)$	All zero	$\tilde{H}^2 = 1$
$B_\alpha(II)$	One non-zero	$\tilde{H}^2 = 1$
$B_\alpha(VII_0)$	Two non-zero	$\tilde{H}^2 = 1$
$B(IX)$	All non-zero	$\tilde{H}^2 + \frac{1}{4}\left(\tilde{N}_1\tilde{N}_2\tilde{N}_3\right)^{2/3} = 1$

The dynamical system also admits *shear invariant sets* which arise from enforcing certain restrictions on the shear variables. It follows from Eqs. (4.25)-(4.31) that

$$\tilde{\Sigma}_- = 0 \quad \Rightarrow \quad \tilde{N}_2 = \tilde{N}_3 > 0, \quad \tilde{N}_1 = 0, \quad \tilde{H} = \pm 1, \quad \text{for Bianchi Types } VII_0,\ IX, \tag{4.44}$$

$$\tilde{\Sigma}_- = 0 \quad \Rightarrow \quad \tilde{N}_2 = \tilde{N}_3 = 0, \quad \tilde{N}_1 > 0, \quad \tilde{H} = \pm 1, \quad \text{for Bianchi Type II.} \tag{4.45}$$

We have summarized these shear invariant sets with the corresponding cosmological model and notation in Table 4.2, and refer the reader to Page 127 in [WE97] for further details.

Table 4.2: The various shear invariant sets, with $\alpha = 1, 2, 3$.

Notation	Class of Models
$S_\alpha(II)$	LRS Bianchi II
$S_\alpha(VII_0)$	LRS Bianchi VII_0
$S_\alpha(IX)$	LRS Bianchi IX

Note that in Table 4.2, LRS stands for locally rotationally symmetric. Therefore, all the models corresponding to the shear invariant sets are the locally rotationally symmetric Bianchi models. These models are still homogeneous on spacelike orbits, but the dimension of the isotropy subgroup is one greater than the non-locally rotationally symmetric Bianchi models. That is, the LRS Bianchi models belong to the isometry group G_4 (Pages 22 and 23, [WE97]).

4.5 A Fixed-Point Analysis

In this section we list the equilibrium points of the dynamical system (4.25)-(4.31). This is an autonomous system, and can be written in the form

$$\dot{x} = v(x), \quad x \in \mathbb{R}^7. \tag{4.46}$$

An equilibrium point of the system is a point at which the vector field, $v(x) \in \mathbb{R}^7$ vanishes. In our analysis of the stability of these equilibrium points, we first note that an equilibrium point of a differential equation is *hyperbolic* if no eigenvalue of the linear part of the equation at this singular point lies on the imaginary axis (Page 47, [AAA+97b]). We then make use of the Grobman-Hartman theorem (Page 48, [AAA+97b], Pages 95-96 [WE97]) which says that a C^1 vector field is topologically equivalent to its linear part in a neighborhood of a hyperbolic equilibrium point. As a consequence of this theorem, if the eigenvalues of the linear part of the system evaluated at the hyperbolic equilibrium point are strictly negative, the equilibrium point will be a local sink of the system. Similarly, if the eigenvalues of the linear part of the system evaluated at the hyperbolic equilibrium point are strictly positive, the equilibrium point will be a local source of the system. A hyperbolic equilibrium point which is neither a source nor a sink is termed a saddle point. We will then make use of the invariant manifold theorem which will allow us to classify orbits that are either attracted to or repelled by certain hyperbolic equilibrium points as $\tau \to \pm\infty$.

4.5.1 Flat Friedmann-LeMaître-Robertson-Walker (FLRW) Equilibrium Points: $F_\pm$

4.5.1.1 The Expanding Epoch

The flat FLRW equilibrium point describing the expanding epoch is given by

$$F_+ : \tilde{\Sigma}_+ = \tilde{\Sigma}_- = 0, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = 0, \quad \tilde{H} = 1, \quad \tilde{\Omega} = 1. \tag{4.47}$$

The eigenvalues are found to be

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}\left(1 + 3w - 9\tilde{\xi}_0\right), \quad \lambda_4 = \lambda_5 = 1+3w-9\tilde{\xi}_0, \quad \lambda_6 = \lambda_7 = \frac{3}{2}\left(-1 + w - 4\tilde{\eta}_0 - 3\tilde{\xi}_0\right). \tag{4.48}$$

F_+ is a local sink of the system if

$$\tilde{\eta}_0 \geq 0 \bigcap \left[\left(0 \leq \tilde{\xi}_0 \leq \frac{4}{9} \bigcap -1 \leq w < \frac{1}{3}(-1 + 9\tilde{\xi}_0)\right) \bigcup \left(\tilde{\xi}_0 > \frac{4}{9} \bigcap -1 \leq w \leq 1\right)\right]. \tag{4.49}$$

There regions where F_+ corresponds to a saddle point of the system are:

$$\tilde{\eta}_0 = 0 \bigcap \left[\left(\tilde{\xi}_0 = 0 \bigcap -\frac{1}{3} < w < 1\right) \bigcup \left(0 < \tilde{\xi}_0 < \frac{4}{9} \bigcap \frac{1}{3}(-1 + 9\tilde{\xi}_0) < w \leq 1\right)\right], \tag{4.50}$$

and

$$\tilde{\eta}_0 > 0 \bigcap 0 \leq \tilde{\xi}_0 < \frac{4}{9} \bigcap \frac{1}{3}(-1 + 9\tilde{\xi}_0) < w \leq 1. \tag{4.51}$$

We note that there exist no $w, \tilde{\xi}_0, \tilde{\eta}_0$ corresponding to $-1 \leq w \leq 1$, $\tilde{\xi}_0 \geq 0$ and $\tilde{\eta}_0 \geq 0$ such that the eigenvalues presented in Eq. (4.48) are strictly positive. Hence, the equilibrium point, F_+ corresponding to a flat expanding FLRW solution is not a local source of the system.

4.5.1.2 The Contracting Epoch

The flat FLRW equilibrium point describing the contracting epoch is given by

$$F_- : \tilde{\Sigma}_+ = \tilde{\Sigma}_- = 0, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = 0, \quad \tilde{H} = -1, \quad \tilde{\Omega} = 1. \tag{4.52}$$

The eigenvalues are found to be

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}\left(-1 - 3w - 9\tilde{\xi}_0\right), \quad \lambda_4 = \lambda_5 = -\frac{3}{2}\left(-1 + w + 4\tilde{\eta}_0 + 3\tilde{\xi}_0\right), \quad \lambda_6 = \lambda_7 = -1 - 3w - 9\tilde{\xi}_0. \tag{4.53}$$

F_- is a local sink of the system in three separate regions of the parameter space. These are given by

$$\tilde{\eta}_0 = 0 \bigcap \left[\left(0 < \tilde{\xi}_0 \leq \frac{2}{3} \bigcap 1 - 3\tilde{\xi}_0 < w \leq 1\right) \bigcup \left(\tilde{\xi}_0 > \frac{2}{3} \bigcap -1 \leq w \leq 1\right)\right], \tag{4.54}$$

$$\begin{aligned} &0 < \tilde{\eta}_0 \leq \frac{1}{3} \bigcap \left[\left(0 \leq \tilde{\xi}_0 < \frac{1}{3}(2 - 4\tilde{\eta}_0) \bigcap 1 - 4\tilde{\eta}_0 - 3\tilde{\xi}_0 < w \leq 1\right)\right] \\ &\bigcup \left[\left(\tilde{\xi}_0 = \frac{1}{3}(2 - 4\tilde{\eta}_0) \bigcap -1 < w \leq 1\right) \bigcup \left(\tilde{\xi}_0 > \frac{1}{3}(2 - 4\tilde{\eta}_0) \bigcap -1 \leq w \leq 1\right)\right], \end{aligned} \tag{4.55}$$

and

$$\tilde{\eta}_0 > \frac{1}{3} \bigcap \left[\left(0 \le \tilde{\xi}_0 \le \frac{2}{9} \bigcap \frac{1}{3}(-1 - 9\tilde{\xi}_0) < w \le 1 \right) \bigcup \left(\tilde{\xi}_0 > \frac{2}{9} \bigcap -1 \le w \le 1 \right) \right]. \tag{4.56}$$

F_- is a local source in two separate regions of the parameter space. These are given by

$$0 \le \tilde{\eta}_0 \le \frac{1}{3} \bigcap 0 \le \tilde{\xi}_0 < \frac{2}{9} \bigcap -1 \le w < \frac{1}{3}(-1 - 9\tilde{\xi}_0), \tag{4.57}$$

and

$$\frac{1}{3} < \tilde{\eta}_0 < \frac{1}{2} \bigcap 0 \le \tilde{\xi}_0 < \frac{1}{3}(2 - 4\tilde{\eta}_0) \bigcap -1 \le w < 1 - 4\tilde{\eta}_0 - 3\tilde{\xi}_0. \tag{4.58}$$

F_- also represents a saddle point if the following regions of the parameter space:

$$\begin{aligned} \tilde{\eta}_0 = 0 \bigcap \left[\left(\tilde{\xi}_0 = 0 \bigcap -\frac{1}{3} < w < 1 \right) \bigcup \left(0 < \tilde{\xi}_0 \le \frac{2}{9} \bigcap \frac{1}{3}(-1 - 9\tilde{\xi}_0) < w < 1 - 3\tilde{\xi}_0 \right) \right] \bigcup \\ \left[\left(\frac{2}{9} < \tilde{\xi}_0 < \frac{2}{3} \bigcap -1 \le w < 1 - 3\tilde{\xi}_0 \right) \right], \end{aligned} \tag{4.59}$$

$$\begin{aligned} 0 < \tilde{\eta}_0 < \frac{1}{3} \bigcap \left[\left(0 \le \tilde{\xi}_0 \le \frac{2}{9} \bigcap \frac{1}{3}(-1 - 9\tilde{\xi}_0) < w < 1 - 4\tilde{\eta}_0 - 3\tilde{\xi}_0 \right) \right] \\ \bigcup \left[\left(\frac{2}{9} < \tilde{\xi}_0 < \frac{1}{3}(2 - 4\tilde{\eta}_0) \bigcap -1 \le w < 1 - 4\tilde{\eta}_0 - 3\tilde{\xi}_0 \right) \right], \end{aligned} \tag{4.60}$$

$$\begin{aligned} \frac{1}{3} < \tilde{\eta}_0 < \frac{1}{2} \bigcap \left[\left(0 \le \tilde{\xi}_0 \le \frac{1}{3}(2 - 4\tilde{\eta}_0) \bigcap 1 - 4\tilde{\eta}_0 - 3\tilde{\xi}_0 < w < \frac{1}{3}(-1 - 9\tilde{\xi}_0) \right) \right] \bigcup \\ \left[\left(\frac{1}{3}(2 - 4\tilde{\eta}_0) < \tilde{\xi}_0 < \frac{2}{9} \bigcap -1 \le w < \frac{1}{3}(-1 - 9\tilde{\xi}_0) \right) \right], \end{aligned} \tag{4.61}$$

$$\tilde{\eta}_0 = \frac{1}{2} \bigcap \left[\left(\tilde{\xi}_0 = 0 \bigcap -1 < w < -\frac{1}{3} \right) \bigcup \left(0 < \tilde{\xi}_0 < \frac{2}{9} \bigcap -1 \le w < \frac{1}{3}(-1 - 9\tilde{\xi}_0) \right) \right], \tag{4.62}$$

and

$$\tilde{\eta}_0 > \frac{1}{2} \bigcap 0 \le \tilde{\xi}_0 < \frac{2}{9} \bigcap -1 \le w < \frac{1}{3}(-1 - 9\tilde{\xi}_0). \tag{4.63}$$

4.5.2 Bianchi Type II Equilibrium Points: $B(II)$

4.5.2.1 The Expanding Epoch

The Bianchi Type II equilibrium point corresponding to the expanding epoch, $\tilde{H} = 1$, shall be denoted by $P_+(II)$. This point is given by

$$
\begin{aligned}
\tilde{\Sigma}_+ &= \frac{1}{16}\left[17 + 3w + 3\tilde{\eta}_0 + 9w\tilde{\eta}_0 - \gamma\right], \\
\tilde{\Sigma}_- &= 0, \\
\tilde{N}_1 &= \frac{1}{4}\sqrt{\frac{3}{2}}\left[-3\left(63 - 38\tilde{\eta}_0 - 9\tilde{\eta}_0^2 + 3\left(w + 3w\tilde{\eta}_0\right)^2 - 2w\left(1 - 42\tilde{\eta}_0 + 9\tilde{\eta}_0^2\right)\right)\right. \\
&+ \left.\gamma\left(13 + 3w - 9\tilde{\eta}_0 + 9w\tilde{\eta}_0\right) - 288\tilde{\xi}_0\right]^{1/2}, \\
\tilde{N}_2 &= \tilde{N}_3 = 0, \tilde{H} = 1, \\
\tilde{\Omega} &= \frac{1}{32}\left[15 - 3w - 54\tilde{\eta}_0 - 18w\tilde{\eta}_0 - 9\tilde{\eta}_0^2 - 27w\tilde{\eta}_0^2 + \left(1 + 3\tilde{\eta}_0\right)\gamma\right],
\end{aligned}
\tag{4.64}
$$

where

$$
\gamma = \left[\left(17 + 3\tilde{\eta}_0 + w\left(3 + 9\tilde{\eta}_0\right)\right)^2 - 64\left(1 + 3w - 9\tilde{\xi}_0\right)\right]^{1/2}. \tag{4.65}
$$

The regions of the parameter space that correspond to this point are

$$
\tilde{\eta}_0 = 0 \bigcap \left[\left(\tilde{\xi}_0 = 0 \bigcap -\frac{1}{3} < w < 1\right) \bigcup \left(0 < \tilde{\xi}_0 < \frac{4}{9} \bigcap \frac{1}{3}\left(-1 + 9\tilde{\xi}_0\right) < w \leq 1\right)\right], \tag{4.66}
$$

and

$$
\tilde{\eta}_0 > 0 \bigcap 0 \leq \tilde{\xi}_0 < \frac{4}{9} \bigcap \frac{1}{3}\left(-1 + 9\tilde{\xi}_0\right) < w \leq 1. \tag{4.67}
$$

Under the first set of inequalities in Eq. (4.66), namely, $\tilde{\xi}_0 = \tilde{\eta}_0 = 0$, $-1/3 < w < 1$, the point $P_+(II)$ takes the form

$$
\tilde{\Sigma}_+ = \frac{1}{8}\left(1 + 3w\right), \quad \tilde{\Sigma}_- = 0, \quad \tilde{N}_1 = \frac{3}{4}\left(1 + 2w - 3w^2\right)^{1/2}, \quad \tilde{N}_2 = \tilde{N}_3 = 0, \quad \tilde{H} = 1, \quad \tilde{\Omega} = -\frac{3}{16}\left(-5 + w\right), \tag{4.68}
$$

where $-\frac{1}{3} < w < 1$. The eigenvalues corresponding to Eq. (4.68) are given by

$$
\begin{aligned}
\lambda_1 = \frac{3}{2}\left(-1 + w\right), \quad \lambda_2 = \lambda_3 = \frac{3}{4}\left(1 + 3w\right), \quad \lambda_4 = \lambda_5 = \left(1 + 3w\right) \\
\lambda_6 = -\frac{3}{8}\left(2 - 2w + \beta\right), \quad \lambda_7 = \frac{3}{8}\left(-2 + 2w + \beta\right),
\end{aligned}
\tag{4.69}
$$

where

$$\beta = \left(-6 - 26w + 38w^2 - 6w^3\right)^{1/2}.$$

By examining these eigenvalues, one sees that the equilibrium point $P_+(II)$ can only be a saddle point of the system. The corresponding parameter space region is given by

$$-\frac{1}{3} < w \leq \frac{1}{3}\left(8 - \sqrt{73}\right), \quad \tilde{\xi}_0 = \tilde{\eta}_0 = 0. \tag{4.70}$$

To analyze the stability of the equilibrium point as defined in Eq. (4.64) in the rest of the parameter space as given in Eqs. (4.66) and (4.67), we must resort to numerical techniques. The reason is that the characteristic polynomial of the Jacobian matrix in each of the reasons admits roots that cannot be written down in closed form. We conducted a variety of numerical experiments that demonstrated that indeed $P_+(II)$ is a saddle point of the system. The results of some of these experiments can be seen in Figs. 4.3 and 4.4.

4.5.2.2 The Contracting Epoch

The Bianchi Type II equilibrium point corresponding to the contracting epoch, $\tilde{H} = -1$, which we denote by $P_-(II)$ is given by

$$\begin{aligned}
\tilde{\Sigma}_+ &= \frac{1}{16}\left[-17 - 3w + 3\tilde{\eta}_0 + 9w\tilde{\eta}_0 + \epsilon\right], \\
\tilde{\Sigma}_- &= 0, \\
\tilde{N}_1 &= \frac{1}{4}\sqrt{\frac{3}{2}}\left[-189 + 6w - 9w^2 - 114\tilde{\eta}_0 + 252w\tilde{\eta}_0 + 54w^2\tilde{\eta}_0 + 27\tilde{\eta}_0^2 + 54w\tilde{\eta}_0^2 - 81w^2\tilde{\eta}_0^2 + 288\tilde{\xi}_0 + \epsilon j\right]^{1/2}, \\
\tilde{N}_2 &= \tilde{N}_3 = 0, \quad \tilde{H} = -1, \\
\tilde{\Omega} &= \frac{1}{32}\left[15 - 3w + 54\tilde{\eta}_0 + 18w\tilde{\eta}_0 - 9\tilde{\eta}_0^2 - 27w\tilde{\eta}_0^2 + \epsilon\left(1 - 3\tilde{\eta}_0\right)\right],
\end{aligned} \tag{4.71}$$

where

$$\epsilon = \left[\left(-17 + 3\tilde{\eta}_0 + w\left(-3 + 9\tilde{\eta}_0\right)\right)^2 - 64\left(1 + 3w + 9\tilde{\xi}_0\right)\right]^{1/2}, \tag{4.72}$$

and

$$j = \left(13 + 3w + 9\tilde{\eta}_0 - 9w\tilde{\eta}_0\right).$$

There are many regions of the parameter space that correspond to this point. The majority of them are too complex to write out in this chapter. As examples, we have listed two of the simpler ones below:

$$\begin{aligned}
&\tilde{\eta}_0 = 0 \quad \bigcap \\
&\qquad \left(\tilde{\xi}_0 = 0 \bigcap -\frac{1}{3} < w < 1\right) \bigcup \left(0 < \tilde{\xi}_0 \leq \frac{2}{9} \bigcap \frac{1}{3}(-1 - 9\tilde{\xi}_0) < w < \frac{1}{3}(3 - 12\tilde{\xi}_0)\right) \bigcup \\
&\qquad\qquad \left(\frac{2}{9} < \tilde{\xi}_0 < \frac{1}{2} \bigcap -1 \leq w < \frac{1}{3}(3 - 12\tilde{\xi}_0)\right)
\end{aligned} \tag{4.73}$$

$$\begin{aligned}
&\left(0 < \tilde{\eta}_0 < \frac{1}{3}\left(5 - 2\sqrt{6}\right)\right) \bigcap \\
&\qquad \left(0 \leq \tilde{\xi}_0 \leq \frac{2}{9} \bigcap \frac{1}{3}(-1 - 9\tilde{\xi}_0) < w < \frac{-3 + 10\tilde{\eta}_0 + 9\tilde{\eta}_0^2 + 12\tilde{\xi}_0}{-3 - 6\tilde{\eta}_0 + 9\tilde{\eta}_0^2}\right) \bigcup \\
&\qquad\qquad \left(\frac{2}{9} < \tilde{\xi}_0 < \frac{1}{6}\left(3 - 2\tilde{\eta}_0 - 9\tilde{\eta}_0^2\right) \bigcap -1 \leq w < \frac{-3 + 10\tilde{\eta}_0 + 9\tilde{\eta}_0^2 + 12\tilde{\xi}_0}{-3 - 6\tilde{\eta}_0 + 9\tilde{\eta}_0^2}\right)
\end{aligned} \tag{4.74}$$

Finding the eigenvalues for this general case is clearly very difficult to do since the characteristic polynomial has no closed-form solutions. However, based on an extensive numerical analysis, we conjecture that this equilibrium point is in fact a saddle, and is hence unstable. We will in fact show later using the method of Chetaev functions that for the case when $0 < w < 1$ and $\tilde{\xi} \geq 0$, this point is unstable.

We note that to the best of the authors' knowledge, the Bianchi Type II solutions as presented in Eqs. (4.64) and (4.71) have not been appeared before in the literature, and hence are new solutions to the Einstein field equations. Both equilibrium points $P_{\pm}(II)$ belong to the invariant set $S_{\alpha}(II)$ as listed in Table 4.2, which corresponds to the class of locally rotationally symmetric Bianchi Type II cosmological models.

4.5.3 Kasner Equilibrium Points

There are two possible Kasner equilibrium points, differing only by the value of $\tilde{H}$:

$$\mathcal{K}_{\pm}: \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 = 1, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = 0, \quad \tilde{\Omega} = 0, \qquad \tilde{H} = \pm 1, \tag{4.75}$$

where in both cases we have

$$\tilde{\xi}_0 = \tilde{\eta}_0 = 0, \quad -1 \leq w < 1. \tag{4.76}$$

Following [WH89], we note that the constant values of $\tilde{\Sigma}_\pm$ at $\mathcal{K}_\pm$ are related to the Kasner exponents of the Kasner solution:

$$p_1 = \frac{1}{3}\left(1 - 2\tilde{\Sigma}_+\right), \quad p_2 = \frac{1}{3}\left(1 + \tilde{\Sigma}_+ + \sqrt{3}\tilde{\Sigma}_-\right), \quad p_3 = \frac{1}{3}\left(1 + \tilde{\Sigma}_+ - \sqrt{3}\tilde{\Sigma}_-\right). \tag{4.77}$$

4.5.3.1 The Expanding Epoch

The eigenvalues corresponding to $\mathcal{K}_+$ are given by

$$\lambda_1 = \lambda_2 = 4, \quad \lambda_3 = 0, \quad \lambda_4 = 3\left(-1 + w\right), \quad \lambda_5 = 6p_1, \quad \lambda_6 = 6p_3, \quad \lambda_7 = 6p_2. \tag{4.78}$$

The zero eigenvalue in Eq. (4.78) indicates that $\mathcal{K}_+$ is a one-dimensional family of equilibrium points. Additionally, this zero eigenvalue implies the existence of a one-dimensional center manifold. The Kasner exponents p_1, p_2, p_3 obey the relations

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1, \tag{4.79}$$

which implies that exactly one of λ_5, λ_6 or λ_7 in Eq. (4.78) is negative except when

$$(p_i) = (1,0,0), (0,1,0), (0,0,1) \equiv (T_i), \quad (i = 1,2,3). \tag{4.80}$$

The points T_i are the Taub points corresponding to Taub flat spacetime metric (Page 132, [WE97]). We see that in the region $-1 \leq w < 1$, $\lambda_4 < 0$, and therefore, $\mathcal{K}_+$ is a (normally hyperbolic) saddle point. If on the other hand, $w = 1$, then $\lambda_4 = 0$, which leads to the creation of a two-dimensional center manifold, and the stability behaviour in this case cannot be determined by linearization.

4.5.3.2 The Contracting Epoch

The eigenvalues corresponding to $\mathcal{K}_-$ are found to be

$$\begin{aligned} \lambda_1 &= \lambda_2 = -4, \quad \lambda_3 = 0, \quad \lambda_4 = 3(-1+w), \quad \lambda_5 = -4 + 6p_1, \quad \lambda_6 = -1 - 3p_1 - 3p_2 + 3p_3, \\ \lambda_7 &= -1 - 3p_1 + 3\sqrt{3}p_2 - 3\sqrt{3}p_3. \end{aligned} \tag{4.81}$$

In general, since the Kasner exponents $p_i, (i = 1, 2, 3)$ must obey the Kasner relations as given in Eq. (4.79), λ_5, λ_6 and λ_7 in Eq. (4.81) will in general have alternating signs. Therefore, in the full state space, $\mathcal{K}_-$ is also a saddle point.

4.5.4 Jacobs Disc

We see that two Jacobs disc solutions, corresponding to expanding and contracting epochs are equilibrium points of the system as well:

$$\mathcal{J}_{\pm} : \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 < 1, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = 0, \quad 0 < \tilde{\Omega} < 1, \quad \tilde{H} = \pm 1, \tag{4.82}$$

where $\tilde{\eta}_0 = \tilde{\xi}_0 = 0$, and $w = 1$.

4.5.4.1 The Expanding Epoch

The eigenvalues corresponding to $\mathcal{J}_+$ are found to be

$$\lambda_1 = \lambda_2 = 4, \quad \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = 6p_1, \quad \lambda_6 = 6p_3, \quad \lambda_7 = 6p_2, \tag{4.83}$$

where $p_i, (i = 1, 2, 3)$ are the Kasner exponents as given in Eq. (4.77) and satisfy the Kasner relations as given in Eq. (4.79). The two zero eigenvalues in Eq. (4.83) indicate that $\mathcal{J}_+$ is a two-dimensional set of equilibrium points. As can be shown, the eigenspaces associated with λ_5, λ_6 and λ_7 in Eq. (4.83) are parallel to the $\tilde{N}_1, \tilde{N}_2$ and $\tilde{N}_3$ axes. We can therefore conclude that the subset for which $\lambda_{5,6,7} > 0$ is a source in the interior of the Kasner circle $\mathcal{K}_+$, belonging to the Jacobs disc $\mathcal{J}_+$.

4.5.4.2 The Contracting Epoch

The eigenvalues corresponding to $\mathcal{J}_-$ are found to be

$$\begin{aligned} \lambda_1 &= \lambda_2 = -4, \quad \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = -4 + 6p_1, \quad \lambda_6 = -1 - 3p_1 - 3p_2 + 3p_3, \\ \lambda_7 &= -1 - 3p_1 + 3\sqrt{3}p_2 - 3\sqrt{3}p_3, \end{aligned} \tag{4.84}$$

where $p_i, (i = 1, 2, 3)$ are the Kasner exponents as given in Eq. (4.77) and satisfy the Kasner relations as given in Eq. (4.79). The two zero eigenvalues in Eq. (4.84) indicate that $\mathcal{J}_-$ is a two-dimensional set of equilibrium points. As in the expanding epoch case, we still have that the eigenspaces associated with λ_5, λ_6 and λ_7 in Eq. (4.84) are parallel to the $\tilde{N}_1, \tilde{N}_2$ and $\tilde{N}_3$ axes. However, we find that $\lambda_{5,6,7} < 0$ in Eq. (4.84) if $p_1 + p_2 + p_3 = 1$, that is, the first of the Kasner relations in Eq. (4.79) is satisfied and

$$-\frac{1}{3} < p_1 < \frac{2}{3}, \quad p_1 + p_2 > \frac{1}{3}, \quad 9 + \sqrt{3} + 3\left(-3 + \sqrt{3}\right) p_1 > 18 p_2. \tag{4.85}$$

Therefore, we conclude that $\mathcal{J}_-$ is a local sink of the system. One can also show that there exists no real values for $p_{1,2,3}$ such that the eigenvalues $\lambda_{5,6,7}$ in Eq. (4.84) are greater than zero. Hence, $\mathcal{J}_-$ is never a source of the dynamical system.

4.5.5 Bianchi Type VII_0 Equilibrium Points

4.5.5.1 Line of Equilibrium Points originating on $\mathcal{K}_+$

The line of equilibrium points originating on $\mathcal{K}_+$ belonging to Bianchi Type VII_0 is given by

$$\mathcal{L}_1^+ : \tilde{\Sigma}_+ = -1, \quad \tilde{\Sigma}_- = 0, \quad \tilde{N}_1 = 0, \quad \tilde{N}_2 = \tilde{N}_3 = k > 0, \quad \tilde{H} = 1, \quad \tilde{\Omega} = 0, \tag{4.86}$$

with restrictions $k \in \mathbb{R}$, $\tilde{\xi}_0 = \tilde{\eta}_0 = 0$, $-1 \leq w < 1$. The eigenvalues corresponding to $\mathcal{L}_1^+$ are found to be:

$$\lambda_1 = 6, \quad \lambda_2 = \lambda_3 = 4, \quad \lambda_4 = 0, \quad \lambda_5 = -2ik, \quad \lambda_6 = 2ik, \quad \lambda_7 = 3 - 3w, \tag{4.87}$$

where $k > 0 \in \mathbb{R}$.

In general, we see that $\mathcal{L}_1^+$ is not hyperbolic because three of its eigenvalues lie entirely on the imaginary axis. In addition, in the case where $-1 \leq w < 1$, there are four eigenvalues that are positive, so that $\mathcal{L}_1^+$ has a three-dimensional unstable set. However, because of the non-hyperbolic nature of this point, its stability in the full state space cannot be determined by linearization methods. We also note that $\mathcal{L}_1^+$ is a line of equilibrium points originating from the Taub point T_1 on $\mathcal{K}_+$. We can however, restrict the dynamical system to the shear invariant set $S_1(IX)$ as described in Eq. (4.44). Within this shear invariant set, only the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_7 in Eq. (4.87) arise. Therefore, we have that within $S_1(IX)$, $\mathcal{L}_1^+$ is a local source.

4.5.5.2 Line of Equilibrium Points originating from $F_\pm$

Expanding Epoch There are three lines of Bianchi type VII_0 equilibrium points originating from $F_\pm$ in the expanding epoch. These are found to be

$$\begin{aligned}
\mathcal{F}_1^+(VII_0) &: \quad \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_1 = 0, \quad \tilde{N}_2 = \tilde{N}_3 = d > 0 \in \mathbb{R}, \quad \tilde{H} = 1, \quad \tilde{\Omega} = 1, \\
\mathcal{F}_2^+(VII_0) &: \quad \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_2 = 0, \quad \tilde{N}_1 = \tilde{N}_3 = d > 0 \in \mathbb{R}, \quad \tilde{H} = 1, \quad \tilde{\Omega} = 1, \\
\mathcal{F}_3^+(VII_0) &: \quad \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_3 = 0, \quad \tilde{N}_1 = \tilde{N}_2 = d > 0 \in \mathbb{R}, \quad \tilde{H} = 1, \quad \tilde{\Omega} = 1,
\end{aligned} \tag{4.88}$$

where the eigenvalues for each point described by Eq. (4.88) are found to be

$$\begin{aligned}
\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = -2 - 6\tilde{\eta}_0, \quad \lambda_6 = -1 - 3\tilde{\eta}_0 - \sqrt{-4d^2 + (1 + 3\tilde{\eta}_0)^2}, \\
\lambda_7 &= -1 - 3\tilde{\eta}_0 + \sqrt{-4d^2 + (1 + 3\tilde{\eta}_0)^2},
\end{aligned} \tag{4.89}$$

with

$$\tilde{\eta}_0 \geq 0, \quad 0 \leq \tilde{\xi}_0 \leq \frac{4}{9}, \quad w = \frac{1}{3}\left(-1 + 9\tilde{\xi}_0\right). \tag{4.90}$$

One can see that from Eq. (4.89) the four zero eigenvalues indicate that $\mathcal{F}_i^+ (i = 1, 2, 3)$ each represent a four-dimensional set of equilibrium points, which imply the existence of a four-dimensional center manifold. It is not possible to determine the stability of these equilibrium points by linearization methods because they are clearly non-hyperbolic. However, an interesting feature to note is that these equilibrium points are only defined in a very specific region of parameter space described by Eq. (4.90). Moreover, these lines of equilibrium points originate from the flat equilibrium point F_+ and determine the destabilization of F_+ at $w = (1/3)\left(-1 + 9\tilde{\xi}_0\right)$.

Contracting Epoch There are three lines of Bianchi type VII_0 equilibrium points originating from $F_\pm$ in the contracting epoch. These are found to be

$$
\begin{aligned}
\mathcal{F}_1^-(VII_0) &: \quad \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_1 = 0, \quad \tilde{N}_2 = \tilde{N}_3 = d > 0 \in \mathbb{R}, \quad \tilde{H} = -1, \quad \tilde{\Omega} = 1, \\
\mathcal{F}_2^-(VII_0) &: \quad \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_2 = 0, \quad \tilde{N}_1 = \tilde{N}_3 = d > 0 \in \mathbb{R}, \quad \tilde{H} = -1, \quad \tilde{\Omega} = 1, \\
\mathcal{F}_3^-(VII_0) &: \quad \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_3 = 0, \quad \tilde{N}_1 = \tilde{N}_2 = d > 0 \in \mathbb{R}, \quad \tilde{H} = -1, \quad \tilde{\Omega} = 1,
\end{aligned}
\tag{4.91}
$$

where the eigenvalues for each point described by Eq. (4.91) are found to be

$$
\begin{aligned}
\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = 2 - 6\tilde{\eta}_0, \quad \lambda_6 = 1 - \sqrt{-4d^2 + (1 - 3\tilde{\eta}_0)^2} - 3\tilde{\eta}_0, \\
\lambda_7 &= 1 + \sqrt{-4d^2 + (1 - 3\tilde{\eta}_0)^2} - 3\tilde{\eta}_0,
\end{aligned}
\tag{4.92}
$$

with

$$
\tilde{\eta}_0 \geq 0, \quad 0 \leq \tilde{\xi}_0 \leq \frac{2}{9}, \quad w = \frac{1}{3}\left(-1 - 9\tilde{\xi}_0\right). \tag{4.93}
$$

One can see that from Eq. (4.92) the four zero eigenvalues indicate that $\mathcal{F}_i^-(i = 1, 2, 3)$ each represent a four-dimensional set of equilibrium points, which imply the existence of a four-dimensional center manifold. It is also not possible, as in the case of the expanding epoch, to determine the stability of these equilibrium points by linearization methods, because they are clearly non-hyperbolic. Moreover, these lines of equilibrium points originate from the flat equilibrium point F_- and determine the destabilization of F_- at $w = (1/3)\left(-1 - 9\tilde{\xi}_0\right)$.

4.5.6 Bianchi Type IX Equilibrium Points

The equilibrium points in the interior of $B(IX)$ are generally described by the following values of the dynamical variables and normalized shear viscosity parameter $\tilde{\eta}_0$:

$$
F_c : \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = f > 0\mathbb{R}, \quad \tilde{\Omega} = 1, \quad \tilde{\eta}_0 \geq 0, \tag{4.94}
$$

where F_c denotes a closed FLRW universe, of which the Einstein static universe is a special case. With the definitions in Eq. (4.94), there are four possibilities involving the values of the other dynamical variables and parameters w and $\tilde{\xi}_0$, which we list below in succession. Before continuing, an important point must be made. Upon observing Eq. (4.94), one will notice an *apparent* contradiction between $\tilde{\Omega} = 1$, and that $\tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = f > 0, f \in \mathbb{R}$, since this seems to imply that there is a solution to the Einstein field equations that has constant positive curvature, but with unit matter density. This confusion arises because in standard cosmology theory (which includes the expansion-normalized variables approach to the Bianchi cosmologies based on the theory of orthonormal frames), one relates the curvature of the universe to the matter density in that universe via the generalized Friedmann equation (Page 114, [WE97])

$$\Omega = 1 - \Sigma^2 - K, \tag{4.95}$$

which is obtained by normalizing Eq. (4.6) with the Hubble parameter, H (See [WE97] [WH89] [EMM12] and references therein for further details). In Eq. (4.95), Ω is the expansion-normalized density parameter, Σ^2 is the expansion-normalized shear scalar parameter, and K is the negated expansion-normalized three-dimensional Ricci scalar. For simplicity, let us assume that the universe we are considering is isotropic so that Σ^2 vanishes. It is then clear from Eq. (4.95) that for a positively curved universe, $K < 0$ which implies that $\Omega > 1$. For a negatively curved universe, we have that $K > 0$, which implies that $\Omega < 1$. For a flat universe, $K = 0$: which implies that $\Omega = 1$. It is from this point that the confusion arises.

In our work, because we have normalized our variables with powers of D and not H, we have a slightly different analog of the Friedmann equation as given in Eq. (4.37). Considering the definitions in Eq. (4.94), we have from Eq. (4.38) that $\tilde{V} = 0$, and so $\tilde{\Omega} = 1$. However, the three dimensional Ricci scalar as defined in Eq. (4.9) for $n_1 = n_2 = n_3 = f > 0, f \in \mathbb{R}$ evaluates to

$$R = \frac{3}{2} f^2, \tag{4.96}$$

which is always positive for $f > 0$. Therefore, the apparent confusion arises due to our choice of normalization variable, and is therefore of no real concern with respect to our analysis of the interior of the $B(IX)$ equilibrium points.

4.5.6.1 Case 1: The Einstein Static Universe

The line element for the Einstein static universe is given by (Page 55, [WE97])

$$ds^2 = -dt^2 + l^2 \left[dr^2 + \sin^2 r \left(d\theta^2 + \sin^2\theta\phi^2\right)\right], \tag{4.97}$$

where $l > 0$ is a constant. It can be shown that the energy density and pressure corresponding to this line element are given by

$$\mu = \frac{3}{l^2}, \quad p = -\frac{1}{l^2}, \tag{4.98}$$

which implies that the equation of state parameter has the value $w = -1/3$. Therefore, it is not necessary to describe the Einstein static universe using a two-fluid description as done in [BY12] for example. It is however, the more popular choice to consider two non-interacting fluids that have separate equations of state described by two separate equation of state parameters w_1 and w_2. Einstein himself took $w_1 = 0$, and $w_2 = -1$, with the latter being equivalent to the cosmological constant (Page 55, [WE97]). However, as discussed by Ellis and Wainwright (Page 55, [WE97]), a spatially homogeneous and isotropic universe with constant positive curvature as we have described in Eq. (4.94) with equation of state parameter $w = -1/3$ is indeed the Einstein static universe.

This equilibrium point is described by

$$\tilde{H} = 0, \quad w = -\frac{1}{3}, \quad f = 2, \quad \tilde{\xi}_0 \geq 0. \tag{4.99}$$

We find that the eigenvalues are given by

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = -3\tilde{\eta}_0 - \sqrt{-8 + 9\tilde{\eta}_0^2}, \quad \lambda_5 = \lambda_6 = -3\tilde{\eta}_0 + \sqrt{-8 + 9\tilde{\eta}_0^2}, \quad \lambda_7 = \frac{9\tilde{\xi}_0}{2}. \tag{4.100}$$

We note that $\lambda_{3,4,5,6}$ in Eq. (4.100) are strictly negative if and only if $\tilde{\eta}_0 \geq 2\sqrt{2}/3$. That is, if $\tilde{\eta}_0 \geq 2\sqrt{2}/3$, the static universe under consideration admits a four-dimensional stable subset. The stability of the Einstein static universe has been a major topic of study in cosmology ever since Einstein introduced the idea [Ein52]. The stability properties were first studied by Lemaître [Lem13] [Lem31] and Eddington [Edd30]. More recent studies of the stability of the Einstein static universe were completed by Barrow, Ellis, Maartens and Tsagas [BEMT03] and Barrow and Yamamoto [BY12] as mentioned in the introduction of this chapter.

4.5.6.2 Case 2: A Set of Closed FLRW Universes in a Contracting Epoch

This equilibrium point is described by

$$-1 < \tilde{H} < 0, \quad 0 \leq \tilde{\xi}_0 \leq -\frac{2}{9\tilde{H}}, \quad w = \frac{1}{3}\left(-1 + 9\tilde{H}\tilde{\xi}_0\right), \quad f = \sqrt{4 - 4\tilde{H}^2}, \tag{4.101}$$

where there is a unique solution for each value of $\tilde{H}$ for $-1 < \tilde{H} < 0$. The eigenvalues are found to be

$$\begin{aligned}
\lambda_1 &= \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = -\tilde{H} - 3\tilde{\eta}_0 - \sqrt{-8 + 9\tilde{H}^2 + 6\tilde{H}\tilde{\eta}_0 + 9\tilde{\eta}_0^2}, && (4.102)\\
\lambda_5 &= \lambda_6 = -\tilde{H} - 3\tilde{\eta}_0 + \sqrt{-8 + 9\tilde{H}^2 + 6\tilde{H}\tilde{\eta}_0 + 9\tilde{\eta}_0^2}, \\
\lambda_7 &= -\frac{9}{2}\left(-1 + \tilde{H}^2\right)\tilde{\xi}_0. && (4.103)
\end{aligned}$$

The two zero eigenvalues in Eq. (4.102) indicate that the equilibrium point admits a two-dimensional center manifold. However, because of these two zero eigenvalues, the equilibrium point is non-hyperbolic, and its stability cannot be determined by linearization methods. We also note that there are no values for $\tilde{H}$ and $\tilde{\eta}_0$ that satisfy Eqs. (4.101) and (4.94) such that $\lambda_{3,4,5,6,7} < 0$ simultaneously. Indeed, $\lambda_{3,4,5,6,7} > 0$ simultaneously if and only if

$$0 < \tilde{\xi}_0 \leq \frac{2}{9}, \quad -1 < H < 0, \quad \tilde{\eta}_0 \leq -\frac{H}{3} - \frac{2}{3}\sqrt{2}\sqrt{1 - \tilde{H}^2} \tag{4.104}$$

or

$$\tilde{\xi}_0 > \frac{2}{9}, \quad -\frac{2}{9\tilde{H}} \leq \tilde{H} < 0, \quad \tilde{\eta}_0 \leq -\frac{\tilde{H}}{3} - \frac{2}{3}\sqrt{2}\sqrt{1 - \tilde{H}^2}. \tag{4.105}$$

Therefore, there is strong evidence to suggest that if one could find some subset of the domain $\mathbb{R}^7$ such that $\lambda_{3,4,5,6,7} > 0$ while satisfying the parameter conditions in Eqs. (4.104) and (4.105), then this equilibrium point would represent a local source at least within this subset.

4.5.6.3 Case 3: A Set of Closed FLRW Universes in an Expanding Epoch

This equilibrium point is described by

$$0 < \tilde{H} < 1, \quad 0 \leq \tilde{\xi}_0 < \frac{4}{9\tilde{H}}, \quad w = \frac{1}{3}\left(-1 + 9\tilde{H}\tilde{\xi}_0\right), \quad f = \sqrt{4 - 4\tilde{H}^2}, \tag{4.106}$$

where there is a unique solution for each value of $\tilde{H}$ for $0 < \tilde{H} < 1$. The eigenvalues are found to be

$$\begin{aligned}
\lambda_1 &= \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = -\tilde{H} - 3\tilde{\eta}_0 - \sqrt{-8 + 9\tilde{H}^2 + 6\tilde{H}\tilde{\eta}_0 + 9\tilde{\eta}_0^2}, \qquad (4.107)\\
\lambda_5 &= \lambda_6 = -\tilde{H} - 3\tilde{\eta}_0 + \sqrt{-8 + 9\tilde{H}^2 + 6\tilde{H}\tilde{\eta}_0 + 9\tilde{\eta}_0^2},\\
\lambda_7 &= -\frac{9}{2}\left(-1 + \tilde{H}^2\right)\tilde{\xi}_0. \qquad (4.108)
\end{aligned}$$

The two zero eigenvalues in Eq. (4.107) indicate that the equilibrium point has associated with it a two-dimensional center manifold. However, because of these two zero eigenvalues, the equilibrium point is non-hyperbolic, and its stability cannot be determined by linearization methods. We also note that there are no values for $\tilde{H}$ and $\tilde{\eta}_0$ that satisfy Eqs. (4.106) and (4.94) such that $\lambda_{3,4,5,6,7} < 0$ or $\lambda_{3,4,5,6,7} > 0$ simultaneously.

4.5.6.4 Case 4: A Set of Closed FLRW Universes in an Expanding Epoch

This equilibrium point is described by

$$0 < \tilde{H} < 1, \quad \tilde{\xi}_0 = \frac{4}{9\tilde{H}}, \quad w = 1, \quad f = \sqrt{4 - 4\tilde{H}^2}, \qquad (4.109)$$

where there is a unique solution for each value of $\tilde{H}$ for $0 < \tilde{H} < 1$. This equilibrium point only arises for $w = 1$, which corresponds to a stiff fluid. The eigenvalues are found to be

$$\begin{aligned}
\lambda_1 &= \lambda_2 = 0, \quad \lambda_3 = \frac{2}{\tilde{H}} - 2\tilde{H}, \quad \lambda_4 = \lambda_5 = -H - 3\tilde{\eta}_0 - \sqrt{-8 + 9\tilde{H}^2 + 6\tilde{H}\tilde{\eta}_0 + 9\tilde{\eta}_0^2},\\
\lambda_6 &= \lambda_7 = -\tilde{H} - 3\tilde{\eta}_0 + \sqrt{-8 + 9\tilde{H}^2 + 6\tilde{H}\tilde{\eta}_0 + 9\tilde{\eta}_0^2}. \qquad (4.110)
\end{aligned}$$

The two zero eigenvalues in Eq. (4.110) indicate that the equilibrium point has associated with it a two-dimensional center manifold. However, because of these two zero eigenvalues, the equilibrium point is non-hyperbolic, and its stability cannot be determined by linearization methods. We also note that there are no values for $\tilde{H}$ and $\tilde{\eta}_0$ that satisfy Eqs. (4.109) and (4.94) such that $\lambda_{3,4,5,6,7} < 0$ or $\lambda_{3,4,5,6,7} > 0$ simultaneously.

4.5.7 Global Behavior

Complementing the preceding fixed-point analysis, we wish to obtain some information about the asymptotic behaviour of the dynamical system as $\tau \to \pm\infty$. To accomplish this, we make use of both the LaSalle Invariance Principle and Monotonicity Principle. According to Theorem 4.11 in [WE97], the LaSalle Invariance Principle for ω-limit sets is stated as follows. Consider a dynamical system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$, with flow ϕ_t. Let S be a closed, bounded and positively invariant set of ϕ_t and let Z be a C^1 monotone function. Then $\forall$ $\mathbf{x}_0 \in S$, we have that $\omega(\mathbf{x}_0) \subseteq \{\mathbf{x} \in S | Z' = 0\}$, where $Z' = \nabla Z \cdot \mathbf{f}$. The extended LaSalle Invariance Principle for α-limit sets can be found in Proposition B.3. in [HW93]. To use this principle, one simply considers S to be a closed, bounded, and negatively invariant set. Then $\forall$ $\mathbf{x}_0 \in S$, we have that $\alpha(\mathbf{x}_0) \subseteq \{\mathbf{x} \in S | Z' = 0\}$, where $Z' = \nabla Z \cdot \mathbf{f}$.

The Monotonicity Principle (Proposition A1, [LKW95]) says if ϕ_t is a flow on $\mathbb{R}^n$ with S an invariant set, and if $Z : S \to \mathbb{R}$ is a C^1 function whose range is the interval (a, b), where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $a < b$, then if Z is monotone decreasing on orbits in S, for all $\mathbf{x} \in S$ we have that $\omega(\mathbf{x}) \subseteq \left\{\mathbf{s} \in \bar{S} \backslash S : \lim_{\mathbf{y} \to \mathbf{s}} Z(\mathbf{y}) \neq b\right\}$, $\alpha(\mathbf{x}) \subseteq \left\{\mathbf{s} \in \bar{S} \backslash S : \lim_{\mathbf{y} \to \mathbf{s}} Z(\mathbf{y}) \neq a\right\}$.

Following pages 24 and 25 of [AAA+97b], we note that a differentiable function Z is called a *Chetaev function* for a singular point $\mathbf{x_0}$ of a vector field $\mathbf{f}(\mathbf{x})$ if Z is defined on a domain W whose boundary contains $\mathbf{x}_0$, the part of the boundary of W is strictly contained in a sufficiently small ball with its center $\mathbf{x}_0$ removed is a piecewise-smooth, C^1 hypersurface along which $\mathbf{f}(\mathbf{x})$ points into the interior of the domain, that is,

$$Z(\mathbf{x}) \to 0, \text{ as } \mathbf{x} \to \mathbf{x}_0, \quad \mathbf{x} \in W; \quad Z > 0, \quad \nabla Z \cdot \mathbf{f}(\mathbf{x}) > 0 \in W. \tag{4.111}$$

A singular point of a C^1 vector field for which a Chetaev function exists is unstable.

Let us first consider the function

$$Z_1 = \tilde{\Omega}. \tag{4.112}$$

Upon using Eqs. (4.31), (4.32), (4.37) and (4.38) , we see that

$$Z_1' = -\tilde{\eta}_0 \left(-12 + \delta + 12\tilde{\Omega}\right) - \frac{1}{3}\tilde{H} \left[27\tilde{H}\tilde{\xi}_0 \left(-1 + \tilde{\Omega}\right) + \tilde{\Omega} \left(-9 + \delta + 9w + 9\tilde{\Omega} - 9w\tilde{\Omega}\right)\right], \tag{4.113}$$

where

$$\delta = \tilde{N}_1^2 + \tilde{N}_2^2 - 2\tilde{N}_2\tilde{N}_3 + \tilde{N}_3^2 + 3\left(\tilde{N}_1\tilde{N}_2\tilde{N}_3\right)^{2/3} - 2\tilde{N}_1\left(\tilde{N}_2 + \tilde{N}_3\right). \tag{4.114}$$

In the Bianchi I invariant set $B(I)$ with $\tilde{H} = 1$, Eq. (4.113) becomes

$$Z_1' = -3\left(-1 + \tilde{\Omega}\right)\left[4\tilde{\eta}_0 + \left(3\tilde{H}\tilde{\xi}_0 + \tilde{\Omega} - w\tilde{\Omega}\right)\right]. \tag{4.115}$$

Clearly, Z_1 is monotone decreasing in two cases. First, if $\tilde{\eta}_0 = \tilde{\xi}_0 = 0$, then Z_1 is monotone decreasing if

$$\left(w = 1, 0 < \tilde{\Omega} < 1\right) \bigcup \left(\tilde{\Omega} = 0, -1 \leq w \leq 1\right). \tag{4.116}$$

Second, in the general viscous case where $\tilde{\xi}_0 \geq 0, \tilde{\eta}_0 \geq 0$, Z_1 is monotone decreasing if

$$-1 \leq w \leq 1, \quad \tilde{\Omega} \geq 1. \tag{4.117}$$

On the other hand, considering the $B(I)$ set with $\tilde{H} = -1$, Eq. (4.113) becomes

$$Z_1' = -3\left(-1 + \tilde{\Omega}\right)\left[4\tilde{\eta}_0 + 3\tilde{\xi}_0 + (-1 + w)\,\tilde{\Omega}\right]. \tag{4.118}$$

So, Z_1 is monotone decreasing if

$$\tilde{\xi}_0 = 0, \quad \tilde{\eta}_0 = 0, \quad 0 \leq \tilde{\Omega} \leq 1, \quad -1 \leq w \leq 1. \tag{4.119}$$

By the LaSalle invariance principle and the preceding fixed-point analysis, we conclude that for any orbit Γ

$$\alpha(\Gamma) \in F_- \in B(I), \quad \omega(\Gamma) \in F_+ \in B(I). \tag{4.120}$$

We can also conclude that in the expanding epoch, where $\tilde{H} = 1$,

$$\alpha(\Gamma) \in \mathcal{J}_+ \in B(I), \quad \tilde{\eta}_0 = \tilde{\xi} = 0, w = 1. \tag{4.121}$$

For the contracting epoch, where $\tilde{H} = -1$, we have that

$$\omega(\Gamma) \in \mathcal{J}_- \in B(I), \quad \tilde{\eta}_0 = \tilde{\xi} = 0, w = 1. \tag{4.122}$$

Let us now consider the function

$$Z_2 = \left(\tilde{N}_1\tilde{N}_2\tilde{N}_3\right)^2 \tag{4.123}$$

as suggested on Page 149 of [WE97]. Upon using Eqs. (4.28), (4.29), (4.30) and (4.32), we see that

$$Z_2' = 3\tilde{H}Z_2\left(-9\tilde{H}\tilde{\xi}_0 + 4\hat{\Sigma}^2 + \tilde{\Omega} + 3w\tilde{\Omega}\right), \tag{4.124}$$

Therefore, Z_2 is strictly monotone decreasing in the invariant set

$$S_1 = \left\{\mathbf{x} : -1 < \tilde{H} < 0 \bigcap \tilde{N}_{1,2,3} > 0 \bigcap \hat{\Sigma}^2 > 0 \bigcap \tilde{\Omega} > 0\right\}, \tag{4.125}$$

where $-1/3 \leq w \leq 1$. The boundary of S_1 is the invariant set

$$\bar{S}_1 \backslash S_1 = \left\{\mathbf{x} : \tilde{H} = -1\right\} \cup \left\{\mathbf{x} : \tilde{H} = 0\right\} \cup \left\{\mathbf{x} : \tilde{N}_{1,2,3} = 0\right\} \cup \left\{\mathbf{x} : \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 = 0\right\} \cup \left\{\mathbf{x} : \tilde{\Omega} = 0\right\}. \tag{4.126}$$

Therefore, by the monotonicity principle,

$$\omega(\mathbf{x}) \subseteq \left\{\mathbf{x} : \tilde{N}_{1,2,3} = 0\right\} \tag{4.127}$$

What this shows is that the future asymptotic state of the $B(IX)$ invariant set belongs to the set $\left\{\mathbf{x} : \tilde{N}_{1,2,3} = 0\right\}$, which according to our fixed-point analysis can either correspond to the Jacobs disc $\mathcal{J}_-$ or the flat FLRW universe F_-.

Considering the function

$$Z_3 = \tilde{\Sigma}_-^2, \tag{4.128}$$

which was suggested as a monotone function on orbits of $B(II)$ by Wainwright (Page 150, [WE97]). Upon using Eqs. (4.32), (4.27), (4.37) and (4.38), and restricting to the $B(II)$ invariant set (with $\tilde{H} = 1$), we see that

$$Z_3' = Z_3\left[-12\tilde{\eta}_0 - 9\tilde{\xi}_0 - \frac{1}{3}\tilde{N}_1^2 - 3\tilde{\Omega}(1-w)\right]. \tag{4.129}$$

Therefore, Z_3 is strictly monotone decreasing in the invariant set

$$S_2 = \left\{\mathbf{x} : \tilde{\Sigma}_- > 0 \bigcap \tilde{N}_1 > 0 \bigcap \tilde{\Omega} > 0\right\}. \tag{4.130}$$

The boundary of S_2 is the invariant set

$$\bar{S}_2 \backslash S_2 = \left\{\mathbf{x} : \tilde{\Sigma}_- = 0\right\} \cup \left\{\mathbf{x} : \tilde{N}_1 = 0\right\} \cup \left\{\mathbf{x} : \tilde{\Omega} = 0\right\}. \tag{4.131}$$

Therefore, by the monotonicity principle,

$$\omega(\mathbf{x}) \subseteq \left\{\mathbf{x} : \tilde{\Sigma}_- = 0\right\}, \tag{4.132}$$

$$\alpha(\mathbf{x}) \subseteq \left\{\mathbf{x} : \tilde{N}_1 = 0, \tilde{\Omega} = 0\right\}. \tag{4.133}$$

This result shows that, although in the full state space the point $P_+(II)$, according to our fixed point analysis, represented a saddle point, restricting to the Bianchi II shear invariant set, the point $P_+(II)$ is a local sink by the monotonicity principle.

Finally, let us consider the function

$$Z_4 = \frac{(1+\tilde{H})^2}{(1-\tilde{H})^2}. \tag{4.134}$$

We will define a domain W as

$$\begin{aligned} W &= \\ \tilde{\Sigma}_+ > \frac{1}{16}\left[-17 - 3w + 3\tilde{\eta}_0 + 9w\tilde{\eta}_0 + \epsilon\right] &\cup \\ \tilde{\Sigma}_- > 0 &\cup \\ \tilde{N}_1 > \frac{1}{4}\sqrt{\frac{3}{2}}\left[-189 + 6w - 9w^2 - 114\tilde{\eta}_0 + 252w\tilde{\eta}_0 + 54w^2\tilde{\eta}_0 + 27\tilde{\eta}_0^2 + 54w\tilde{\eta}_0^2 - 81w^2\tilde{\eta}_0^2 + 288\tilde{\xi}_0\right] + & \\ \epsilon\left(13 + 3w + 9\tilde{\eta}_0 - 9w\tilde{\eta}_0\right)^{1/2} &\cup \\ \tilde{N}_2 > 0 \quad \cup \quad \tilde{N}_3 > 0 \quad \cup \quad \tilde{H} < -1 &\cup \\ \tilde{\Omega} > \frac{1}{32}\left[15 - 3w + 54\tilde{\eta}_0 + 18w\tilde{\eta}_0 - 9\tilde{\eta}_0^2 - 27w\tilde{\eta}_0^2 + \epsilon\left(1 - 3\tilde{\eta}_0\right)\right], & \end{aligned} \tag{4.135}$$

where ϵ is given in Eq. (4.72). The boundary of this domain, ∂W then contains the equilibrium point $P_-(II)$ as described in Eq. (4.71). One can then show that

$$Z_4' = -\frac{\left(1+\tilde{H}\right)\left(9\tilde{H}\tilde{\xi}_0 - 4\hat{\Sigma}^2 - \tilde{\Omega} - 3w\tilde{\Omega}\right)}{-1+\tilde{H}}. \tag{4.136}$$

Therefore, we have that

$$Z_4(\mathbf{x}) \to 0, \text{ as } \mathbf{x} \to P_-(II), \quad \mathbf{x} \in W; \quad Z_4 > 0, \quad \nabla Z_4 \cdot \mathbf{f}(\mathbf{x}) > 0 \in W, \quad \tilde{\xi}_0 > 0, \quad 0 < w < 1. \tag{4.137}$$

This implies that for the case when $\tilde{\xi}_0 > 0, \quad 0 < w < 1$, the point $P_-(II)$ is unstable.

4.5.8 Bifurcations

The dynamical system under study admits some bifurcations. That is, some of the equilibrium points found above change their stability behaviour for different values of the equation of state parameter w, and the bulk and shear viscosity parameters $\tilde{\xi}_0$ and $\tilde{\eta}_0$. A local bifurcation occurs when the Jacobian matrix of the corresponding equilibrium point has at least one eigenvalue with zero real part. If this eigenvalue lies entirely on the real axis, the bifurcation is known as a steady-state bifurcation.

With respect to local bifurcations, we consider only hyperbolic equilibrium points. Looking first at the equilibrium point F_+, local bifurcations can occur if

$$0 \leq \tilde{\xi}_0 \leq \frac{4}{9}, \quad w = \frac{1}{3}\left(-1 + 9\tilde{\xi}_0\right) \tag{4.138}$$

or

$$\tilde{\xi}_0 = 0, \quad \tilde{\eta}_0 = 0, \quad w = 1. \tag{4.139}$$

At the equilibrium point F_-, local bifurcations can occur if

$$\tilde{\eta}_0 \geq 0, \quad 0 \leq \tilde{\xi}_0 \leq \frac{2}{9}, \quad w = \frac{1}{3}\left(-1 - 9\tilde{\xi}_0\right), \tag{4.140}$$

or

$$\tilde{\xi}_0 = 0, \quad \tilde{\eta}_0 = 0, \quad w = 1. \tag{4.141}$$

The only other purely hyperbolic points are $P_\pm(II)$, which as can be seen from the corresponding eigenvalues, never admit local bifurcations.

4.5.9 Heteroclinic Orbits

From the preceding fixed-points analysis, one can obtain information about heteroclinic orbits produced by the dynamical system. Heteroclinic orbits are simply orbits that connect *distinct* equilibrium points. A very interesting heteroclinic orbit is generated via the equilibrium points $\mathcal{J}_\pm$. In the interior of $\mathcal{K}_\pm$, we have

$$\mathcal{J}_+ \longleftrightarrow \mathcal{J}_-$$

where $\mathcal{J}_+$ was found to be a local source within $\mathcal{K}_+$ and $\mathcal{J}_-$ was found to be a local sink within $\mathcal{K}_-$.

Another interesting heteroclinic orbit is one that connects the equilibrium points $\mathcal{K}_\pm$. That is,

$$\mathcal{K}_+ \longleftrightarrow \mathcal{K}_-$$

We can also have

$$\begin{tikzcd} F_- \arrow[r] \arrow[d] & \mathcal{K}_- \arrow[dl, leftrightarrow] \\ \mathcal{K}_+ & \end{tikzcd}$$

where F_- is a local source, and $\mathcal{K}_\pm$ are saddle points.

4.5.10 Mixmaster Attractor

We now briefly describe the famous Mixmaster attractor that generically appears in the study of the dynamics of $B(IX)$ models. One typically observes very complex dynamical behaviour, albeit chaotic behaviour as such models are evolved in the past towards $\mathcal{K}_\pm$. As our fixed point analysis demonstrated, one can only hope to evolve towards $\mathcal{K}_\pm$ if $\tilde{\eta}_0 = \tilde{\xi}_0 = 0$. There are numerous studies in the literature of Mixmaster dynamics and chaotic behaviour in perfect-fluid $B(IX)$ models, many of which we have mentioned in the introduction of this paper. The interested reader should refer to the papers listed there for further elaboration on the points we make in this subsection.

We begin by noting that in the interior of $B(IX)$, there exists no equilibrium point that is a well-defined local source. Following Section 6.4 and the references therein in [WE97], we attempt to construct a *compact* invariant set in $\overline{B(IX)} = B(IX) \cup \partial B(IX)$ that is conjectured to be a past attractor. We know from our analysis of the point $\mathcal{K}_+$, that there are six families of Taub orbits. Let us consider Eqs. (4.37) and (4.38) in the vacuum boundary, $\tilde{\Omega} = 0$. Each family lies on a half-ellipsoid. The closures of these six half-ellipsoids are defined as

$$E_1^+ \quad : \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \frac{1}{12}\tilde{N}_1^2 = 1, \quad \tilde{N}_1 > 0, \quad \tilde{N}_2 = \tilde{N}_3 = 0, \tag{4.142}$$

$$E_1^- \quad : \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \frac{1}{12}\tilde{N}_1^2 = 1, \quad \tilde{N}_1 < 0, \quad \tilde{N}_2 = \tilde{N}_3 = 0, \tag{4.143}$$

$$E_2^+ \quad : \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \frac{1}{12}\tilde{N}_2^2 = 1, \quad \tilde{N}_2 > 0, \quad \tilde{N}_1 = \tilde{N}_3 = 0, \tag{4.144}$$

$$E_2^- \quad : \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \frac{1}{12}\tilde{N}_2^2 = 1, \quad \tilde{N}_1 < 0, \quad \tilde{N}_1 = \tilde{N}_3 = 0, \tag{4.145}$$

$$E_3^+ \quad : \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \frac{1}{12}\tilde{N}_3^2 = 1, \quad \tilde{N}_3 > 0, \quad \tilde{N}_2 = \tilde{N}_1 = 0, \tag{4.146}$$

$$E_3^- \quad : \quad \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \frac{1}{12}\tilde{N}_3^2 = 1, \quad \tilde{N}_3 < 0, \quad \tilde{N}_2 = \tilde{N}_1 = 0. \tag{4.147}$$

The Taub orbits $T_i, (i = 1, 2, 3)$ and equilibrium points on $\mathcal{K}_+$ imply the existence of infinite heteroclinic sequences that map $\mathcal{K}_+$ onto itself. The chaotic dynamical behaviour can be seen from the basic notions that first, $\overline{B(IX)}$ is conjectured to be an attractor. Second, since orbits have to be confined within the region of the attractor and $\mathcal{K}_+$ is a saddle, orbits will indefinitely leave $\mathcal{K}_+$ and then approach $\mathcal{K}_+$ via the Taub points. This suggests that in the case where $-1 \leq w < 1$, and $\tilde{\xi}_0 = \tilde{\eta}_0 = 0$, as $\tau \to -\infty$ the union of $\mathcal{K}_+$ and the family of Taub orbits is the past attractor of the dynamical system. Specifically, we have that

$$M^+ : E_1^+ \cup E_2^+ \cup E_3^+, \tag{4.148}$$

where M^+ denotes the *Mixmaster attractor*. Let us additionally define the scalar quantity

$$\Delta = \left(\tilde{N}_1\tilde{N}_2\right)^2 + \left(\tilde{N}_2\tilde{N}_3\right)^2 + \left(\tilde{N}_3\tilde{N}_1\right)^2. \tag{4.149}$$

Showing that M^+ is indeed an attractor requires one to show that both Δ and $\tilde{\Omega}$ vanish as $\tau \to -\infty$. In the next section, we perform some numerical experiments to test this hypothesis.

4.6 Numerical Solutions

In this section, we perform numerical experiments to complement the analysis of the dynamical system in the previous sections. For all of the numerical experiments, initial conditions denoted by asterisks in the figures were chosen such that Eqs. (4.20) and (4.37) were satisfied. The goal of this section is to complement the preceding stability analysis of the equilibrium points with extensive numerical experiments in order to confirm that the local results are in fact global in nature. The details of the parameters used are described in the captions of the respective figures and are based on the fixed-point analysis that characterized the stability of the equilibrium points in the different regions of the parameter space $\left\{\tilde{\eta}_0, \tilde{\xi}_0, w : \tilde{\eta}_0 \geq 0, \tilde{\xi}_0 \geq 0, -1 \leq w \leq 1\right\}$.

We display in Figs. 4.1 and 4.2 the results of a numerical experiment that show that $F_{\pm}$ are local sinks of the system.

Figure 4.1: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 1/2$, and $w = -1/3$. The plus sign denotes the equilibrium point F_+. The model also isotropizes as can be seen from the last figure, where $\tilde{\Sigma}_\pm \to 0$ as $\tau \to \infty$. Numerical solutions were computed for $0 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.

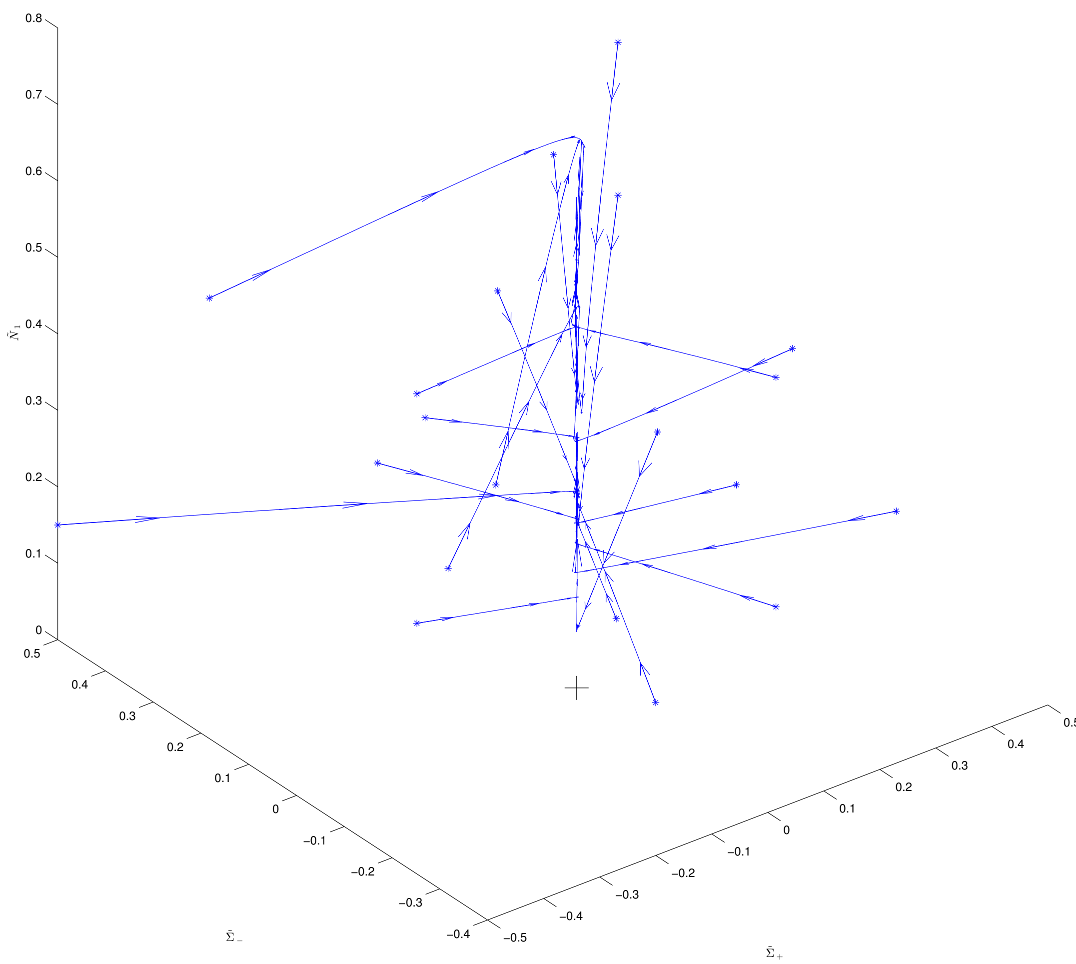


Fig. 4.1 continued.

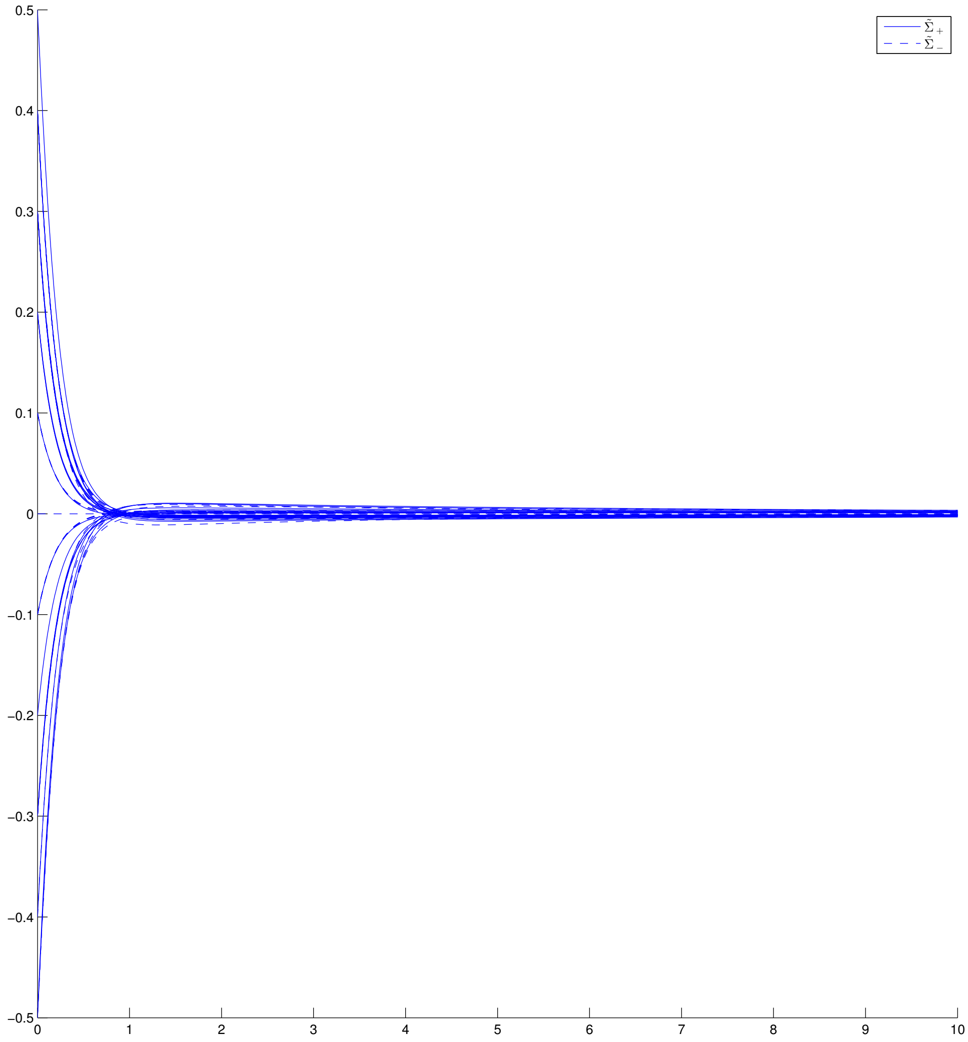


Figure 4.2: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 1/3$, and $w = 1$. The plus sign denotes the equilibrium point F_-. The model also isotropizes as can be seen from the last figure, where $\tilde{\Sigma}_\pm \to 0$ as $\tau \to \infty$. Numerical solutions were computed for $0 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.

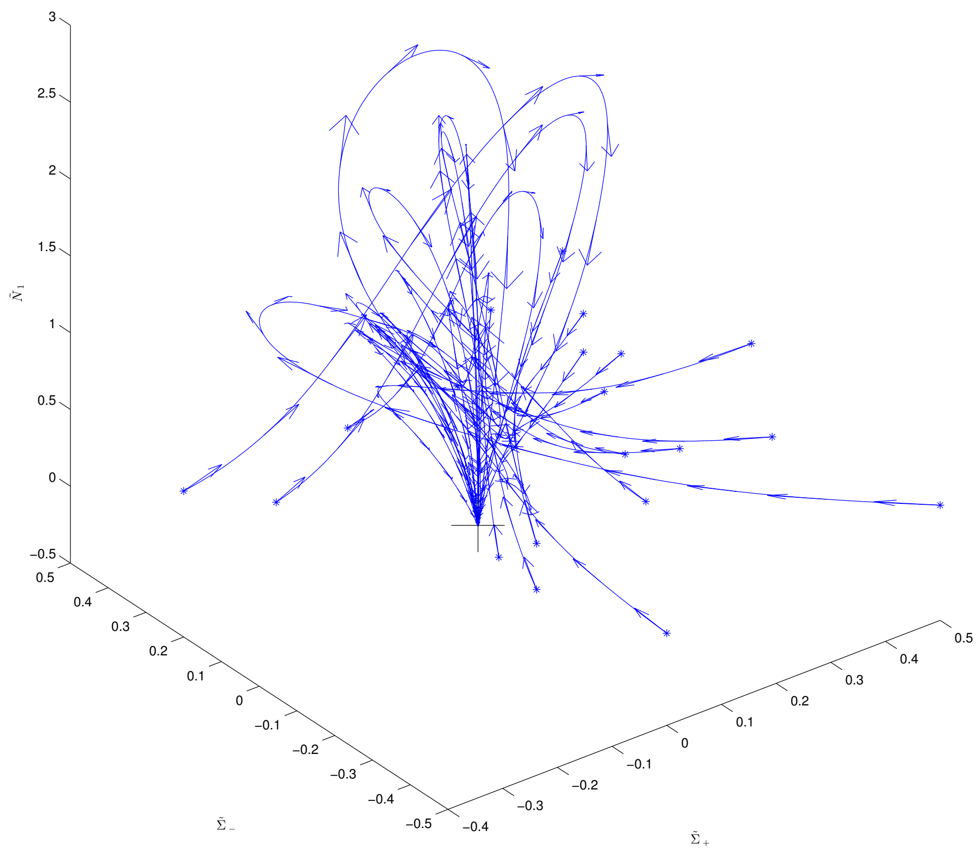
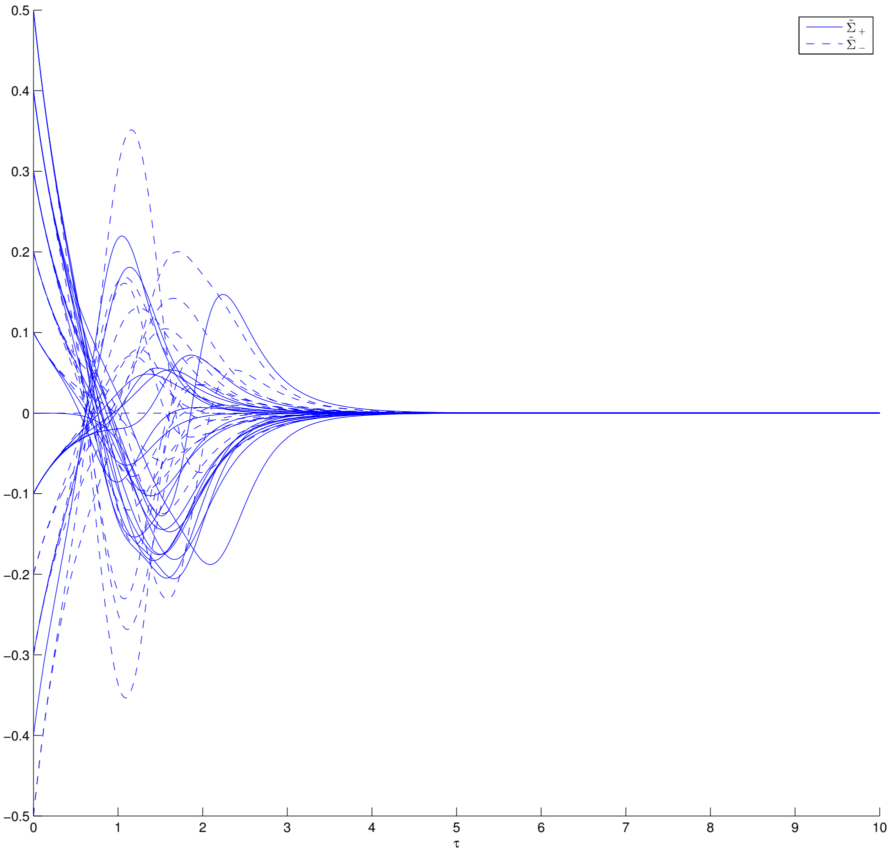


Fig. 4.2 continued.



In Figs. 4.3 and 4.4, we display the results of a numerical experiment that show that the points $P_{\pm}(II)$ correspond to saddles of the system.

Figure 4.3: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 1/2$, and $w = 1/3$. The plus sign denotes the equilibrium point $P_+(II)$. Numerical solutions were computed for $0 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.

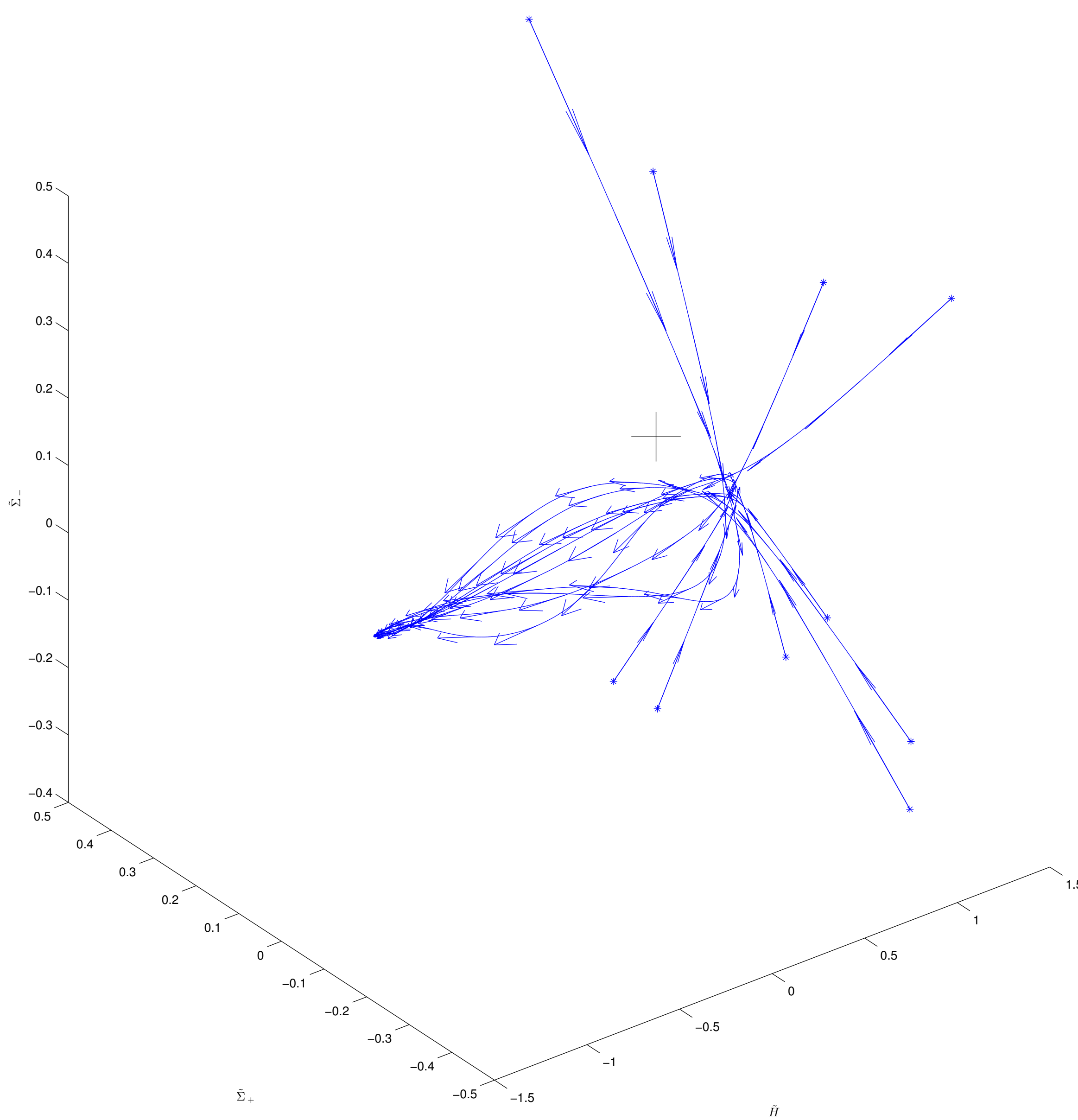
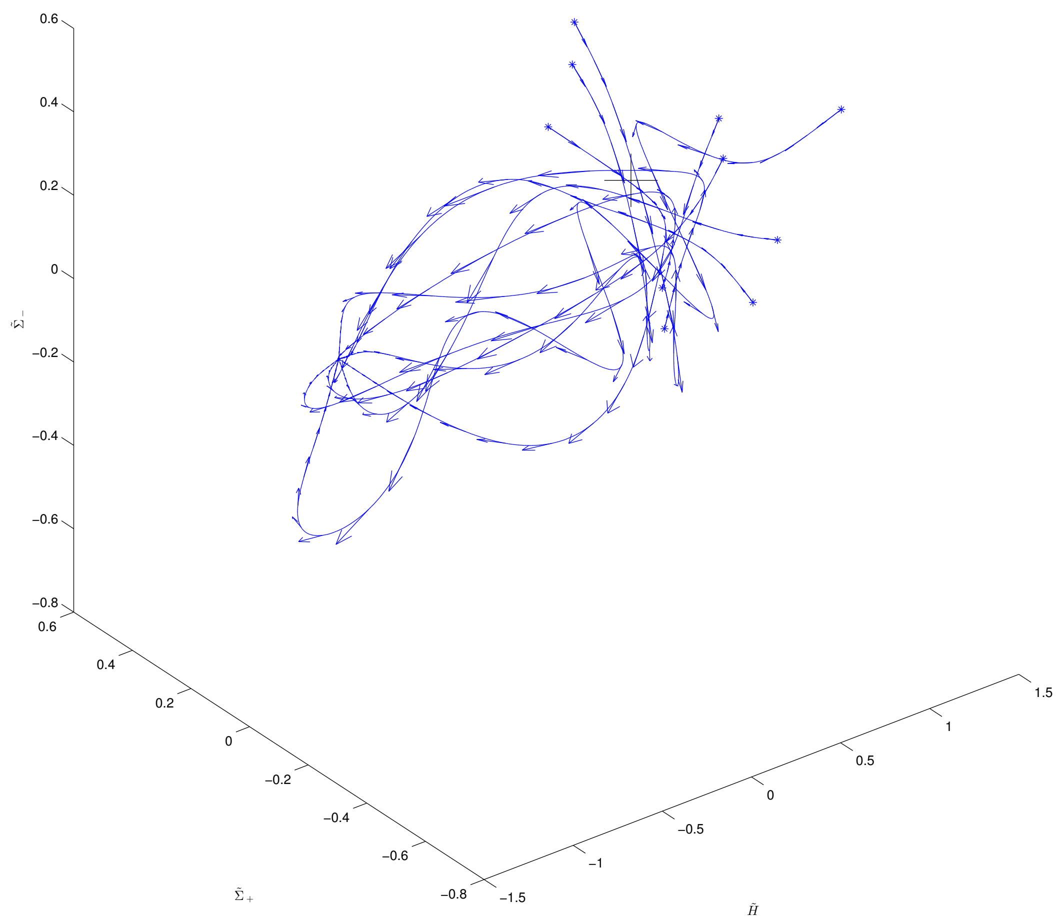
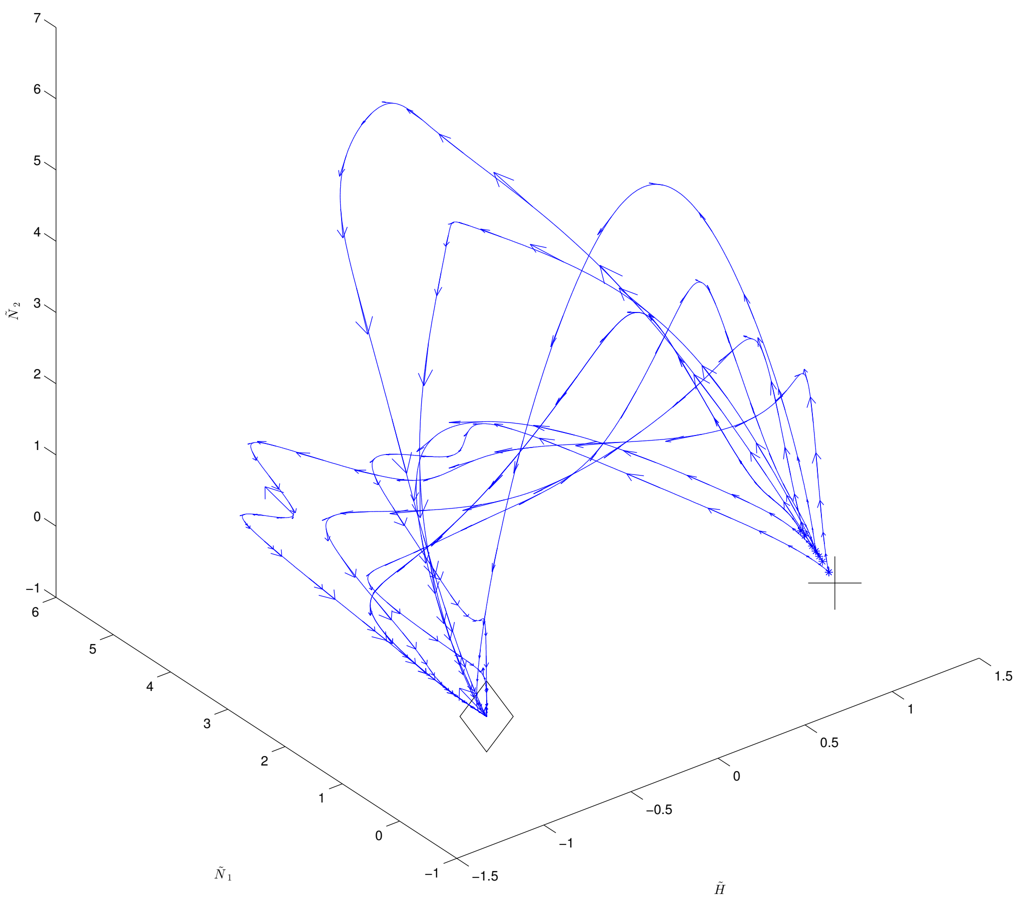


Figure 4.4: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 2/9$, $\tilde{\eta}_0 = 0$, and $w = 1$. The plus sign denotes the equilibrium point $P_+(II)$. Numerical solutions were computed for $0 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.



In Fig. 4.5, we display the results of a numerical experiment that show that the Jacobs disc set of equilibrium points $\mathcal{J}_{\pm}$ correspond to a local source and sink respectively. The same figure also shows the heteroclinic orbit behaviour between $\mathcal{J}_{+}$ and $\mathcal{J}_{-}$.

Figure 4.5: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 0$, and $w = 1$. The plus sign denotes the equilibrium point $\mathcal{J}_+$, while the diamond denotes the equilibrium point $\mathcal{J}_-$. Notice how all of the orbits are repelled by $\mathcal{J}_+$ but attracted by $\mathcal{J}_-$. Numerical solutions were computed for $0 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.



In Figs. 4.6 and 4.7 we display the results of a numerical experiment that show the Mixmaster oscillatory behaviour as the past orbits approach $\mathcal{K}_+$.

Figure 4.6: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 0$, and $w = 1/3$. The circular boundary defines the Kasner circle $\mathcal{K}_+$. In the last image, our numerical solutions for Δ as defined in Eq. (4.149) and $\tilde{\Omega}$ are displayed. Based on the conjecture discussed above, these results provide strong evidence that M^+ is indeed a past attractor for the dynamical systems. Numerical solutions were computed for $0 \leq \tau \leq -1000$. For clarity, we have displayed solutions for shorter timescales. Note how in the first image half-ellipsoids form in the vertical direction as was predicted by the preceding analysis.

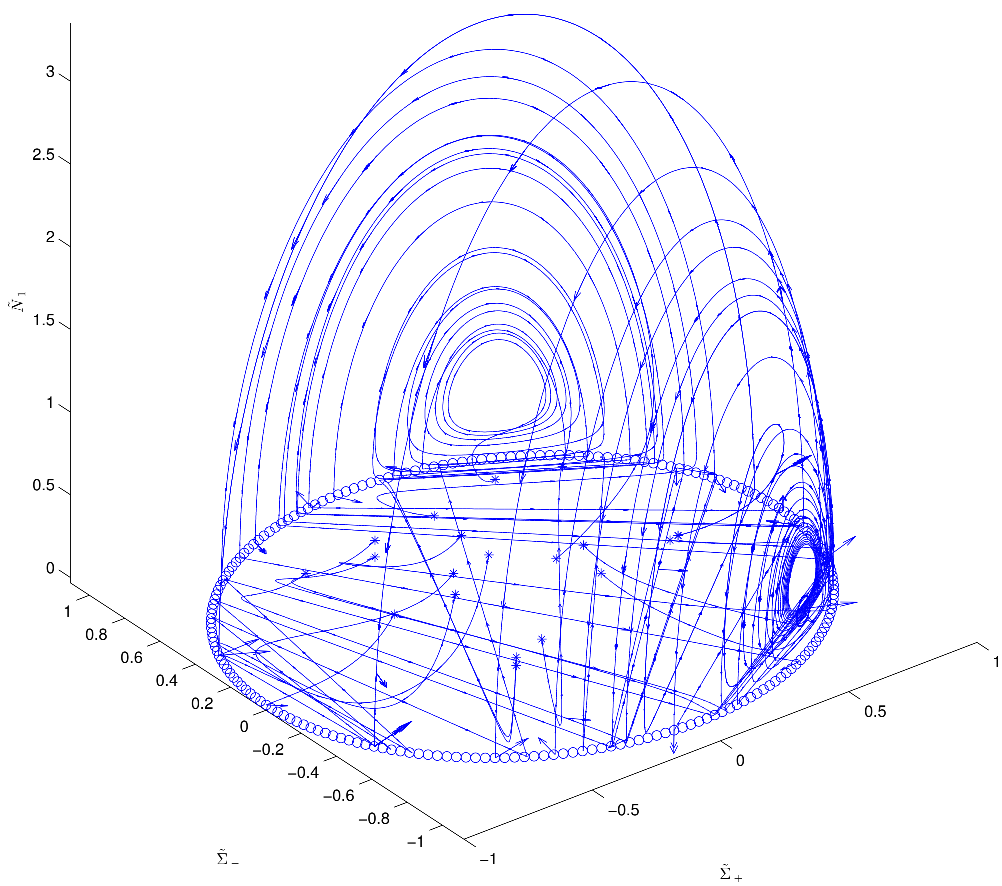


Fig. 4.6 continued.

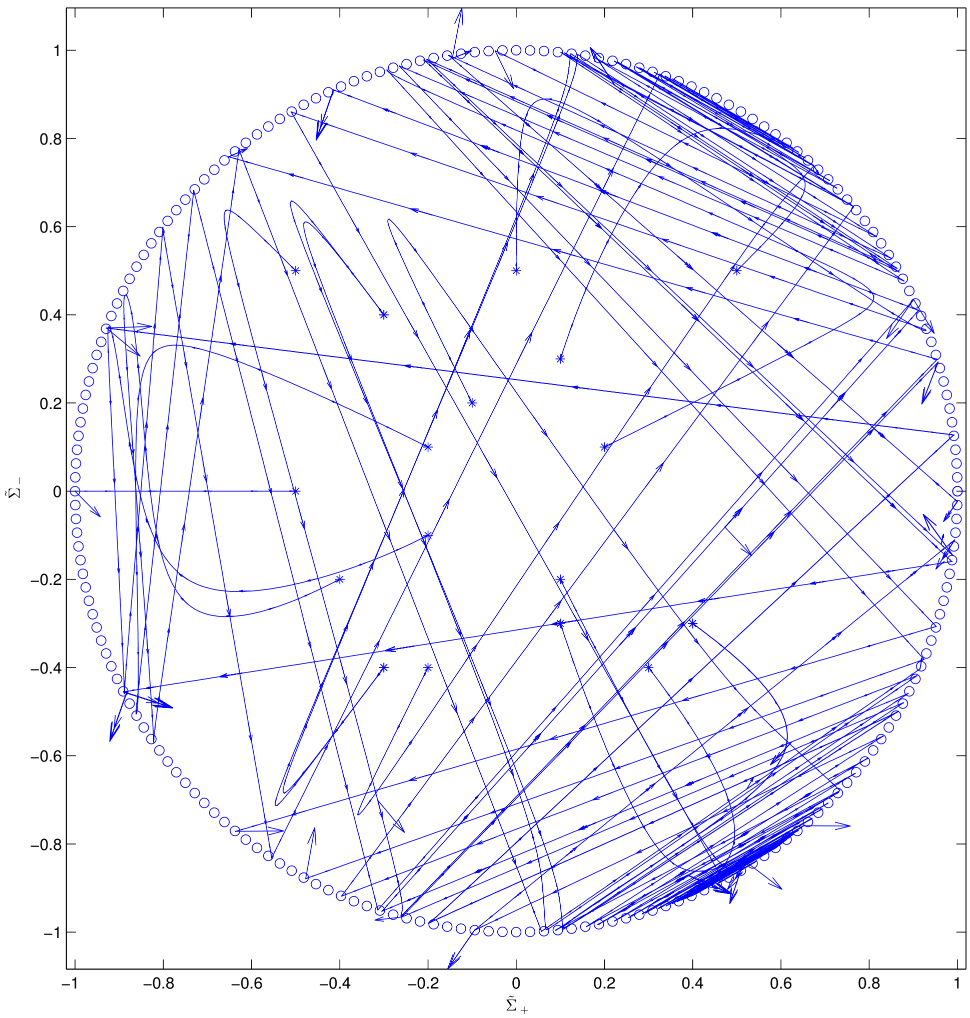
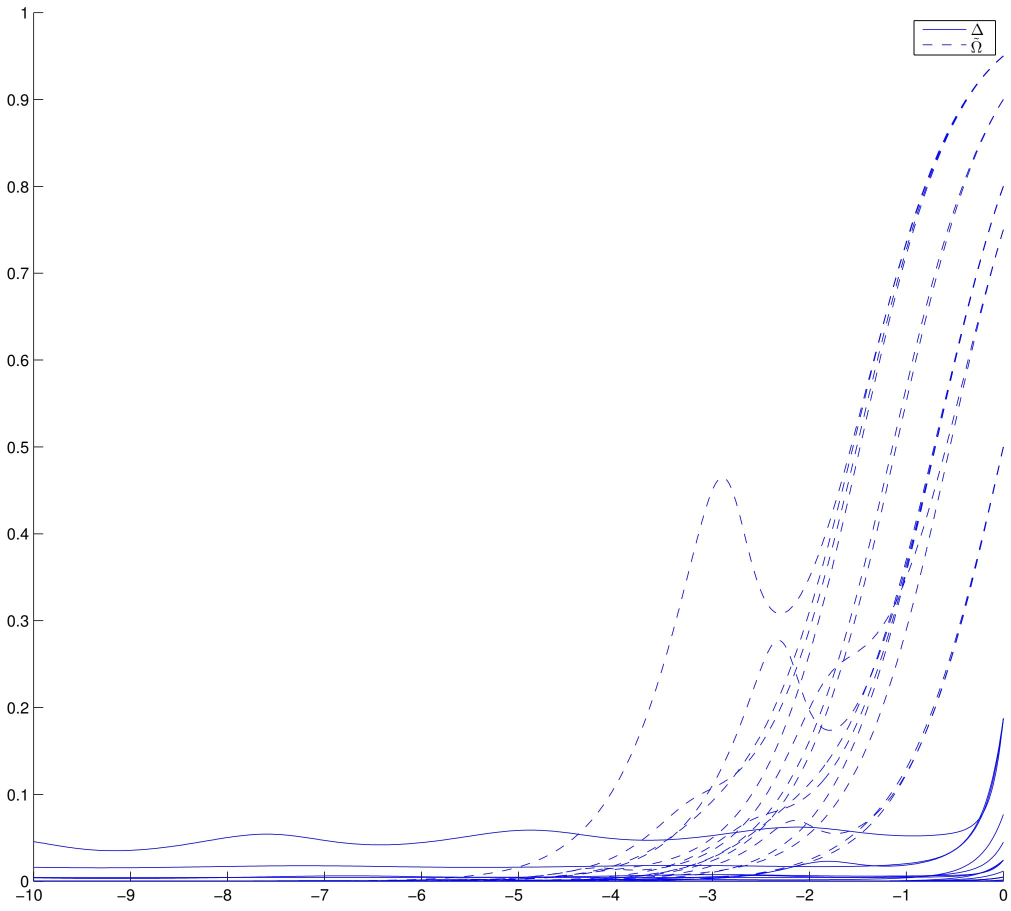
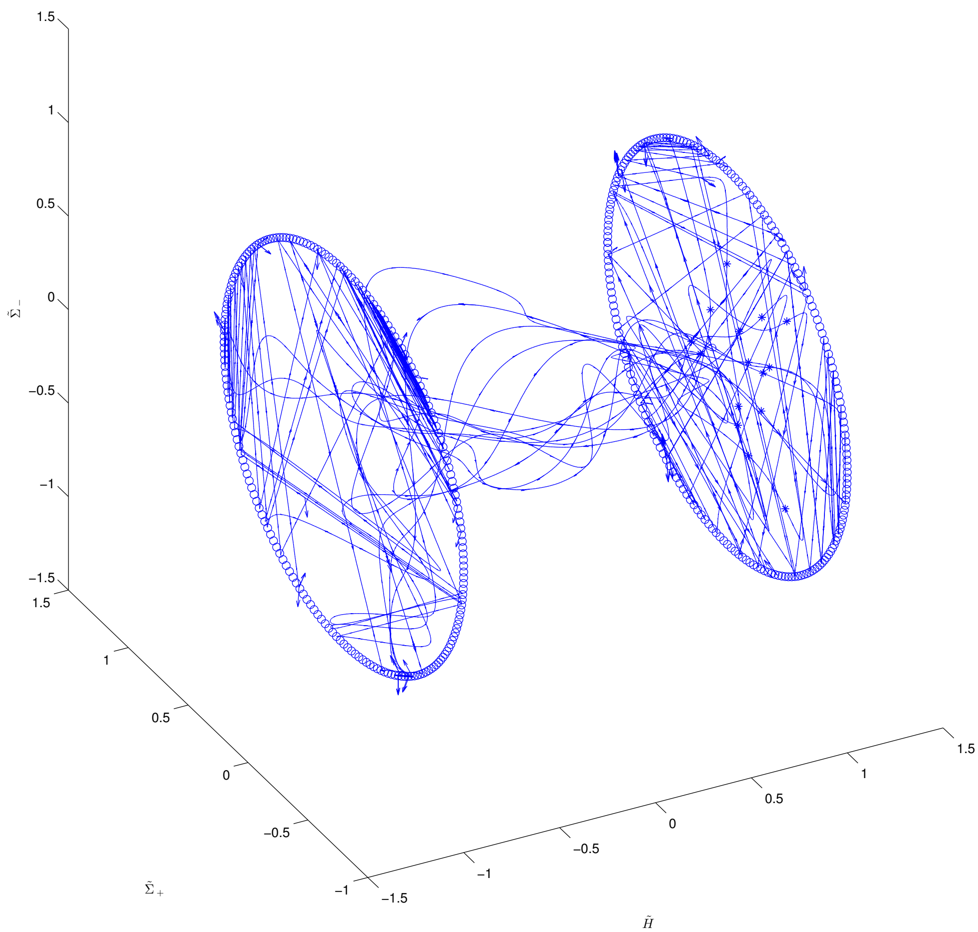


Figure 4.7: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 0$, and $w = 1/3$. Our numerical solutions for Δ as defined in Eq. (4.149) and $\tilde{\Omega}$ are displayed. Based on the conjecture discussed above, these results provide strong evidence that M^+ is indeed a past attractor for the dynamical systems. Numerical solutions were computed for $0 \leq \tau \leq -1000$ For clarity, we have displayed solutions for shorter timescales.



In Fig. 4.8 we display the results of a numerical experiment that shows the heteroclinic orbits between $\mathcal{K}_+$ and $\mathcal{K}_-$.

Figure 4.8: This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 0$, and $w = 1/3$. In particular, it displays the heteroclinic orbits joining $\mathcal{K}_+$ to $\mathcal{K}_-$, where $\mathcal{K}_+$ is located at $\tilde{H} = 1$, and $\mathcal{K}_-$ is located at $\tilde{H} = -1$ in the figure. Numerical solutions were computed for $-1000 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.



4.7 Conclusions

We have presented in this chapter a comprehensive analysis of the dynamical behaviour of a Bianchi Type IX viscous cosmology. We began by completing a detailed fixed-point analysis, which gave the local sinks, sources and saddles of the dynamical system. We then proceeded to analyze the global dynamics by finding the α- and ω-limit sets, which gave an idea of the past and future asymptotic behaviour of the system. The fixed points found were a flat FLRW solution, Bianchi Type II solution, Kasner circle, Jacobs disc, Bianchi Type VII_0 solutions, and several closed FLRW solutions in addition to the Einstein static universe solution. Each equilibrium point was described in both its expanding and contracting epochs.

With respect to past asymptotic states, we were able to conclude that the Jacobs disc in the expanding epoch was a source of the system along with the flat FLRW solution in a contracting epoch. With respect to future asymptotic states, we were able to show that the flat FLRW solution in an expanding epoch along with the Jacobs disc in the contracting epoch were sinks of the system. We were also able to demonstrate a new result with respect to the Einstein static universe. Namely, we gave certain conditions on the parameter space such that the Einstein static universe has a stable subspace. We were however, not able to conclusively say anything about whether a closed FLRW model could be a past or future asymptotic state of the model.

The flat FLRW solution is clearly of primary importance with respect to modeling the present-day universe, which is observed to be very close to flat. We gave conditions in the parameter space for which this solution represents a saddle and a sink. When it is a saddle, the equilibrium point attracts along its stable manifold and repels along its unstable manifold. Therefore, some orbits will have an initial attraction to this point, but will eventually be repelled by it. In the case when it was found to be a sink, all orbits approach the equilibrium point in the future. Therefore, there exists a time period and two separate configurations for which our cosmological model will isotropize and be compatible with present-day observations of a high degree of isotropy in the cosmic microwave background.

Introduce a little anarchy. Upset the established order, and everything becomes chaos. I'm an agent of chaos. Oh, and you know the thing about chaos? It's fair!

The Joker - The Dark Knight

5

Exploring Vacuum Energy in a Two-Fluid Bianchi Type I Universe

5.1 Introduction

in this chapter, we use a dynamical systems approach to investigate in detail the dynamics of a Bianchi Type I universe with a bulk viscous fluid and cosmological constant. Such a universe is spatially flat, spatially homogeneous, and anisotropic. Such a model may have considerable importance in present studies of cosmology given the recent results of the Planck measurements [A+13], which suggest that the curvature of the spatial sections of the present-day universe is in agreement with spatial flatness. Moreover, Bianchi models are more general than the Friedmann-Lemaître-Robertson-Walker (FLRW) models and therefore can provide better descriptions of the early universe where viscous effects may have been dominant [GH07]. One can then study the effects of viscosity on the dynamical evolution of the universe which we observe to be of FLRW-type today. As discussed by Coley and Wainwright [CW92], cosmological models with single fluids are necessarily a simplification in the sense that they can only describe one epoch during the evolution of the universe. More general models can be constructed using two fluids with barotropic equations of state that are also comoving. One can then use these models to describe the transitions between different epochs in the universe's evolution, such as going from a radiation-dominated phase ($w = -1/3$) to a matter-dominated phase ($w = 0$). As discussed by Grøn and Hervik (Chapter 13, [GH07]), viscous models have become of general interest in early-universe cosmologies largely in two contexts. Firstly, in models where bulk viscous terms dominate over shear terms, the universe expands to a de Sitter-like state, which is a spatially flat universe neglecting ordinary matter, and including only a cosmological constant. Such models isotropize indirectly through the massive expansion. Secondly, in the absence of any significant heat flux, shear viscosity is found to play an important role in models of the universe at its early stages. In particular, neutrino viscosity is considered to be one of the most important factors in the isotropization of our universe. By also including a non-negative cosmological constant in our model, and interpreting it to represent vacuum energy, we are also able to give a detailed description of the roles played by both viscosity and vacuum energy in the isotropization of our universe.

Bianchi cosmological models which contain a viscous fluid matter source in addition to a cosmological constant have been studied in detail several times. Lorenz-Petzold [LP89] examined Bianchi Type I and V models in the the presence of perfect fluid matter with bulk viscosity and a nonzero cosmological constant. Pradhan and Pandey [PP03] studied Bianchi Type I magnetized cosmological models in the presence of a bulk viscous fluid in addition to a monotonically decreasing cosmological constant. Saha [Sah05] studied the evolution of a Bianchi Type I universe with a viscous fluid and a cosmological constant. Pradhan, Srivastav, and Yadav [PSY05b] studied Bianchi Type IX viscous models with a time-dependent positive cosmological constant. Belinchón [Bel05] investigated the dynamics of a locally rotationally symmetric (LRS) Bianchi Type I universe with a bulk viscous fluid and a time-dependent cosmological constant. Pradhan, Jotania, and Rai [PJR06] studied Bianchi Type V cosmological models with bulk viscous fluid and a time-dependent cosmological constant. They also discussed some physical and geometrical aspects of such models. Pradhan and Pandey [PP06] studied Bianchi Type I cosmological models with both shear and bulk viscosity and a monotonically decreasing cosmological constant. The authors considered the special case in which the expansion tensor only had two components. Saha and Rikhvitsky [SR06] analyzed a Bianchi Type I universe with a cosmological constant and dissipative processes due to viscosity. They showed that a positive cosmological constant leads to an ever-expanding universe. Singh and Kale [SK09a] studied Bianchi Type I, Kantowski-Sachs, and Bianchi Type III cosmological models containing as matter sources a bulk viscous fluid and non-constant gravitational and cosmological constants. Pradhan and Kumhar [PK09] studied LRS Bianchi Type II models with bulk viscous fluid and a decaying cosmological constant. Mostafapoor and Grøn [MGn13] studied a Bianchi Type I universe with a cosmological constant and nonlinear viscous fluid. Sadeghi, Amani, and Tahmasbi [SAT13] investigated a Bianchi Type-VI cosmological model with a cosmological constant and viscous fluid. Barrow [Bar88b] showed that models of an inflationary universe driven by Witten strings in the very early universe are equivalent to the addition of bulk viscosity to perfect fluid cosmological models with zero curvature. In this work, Barrow considered the case where the bulk viscosity has a power-law dependence upon the matter density. It was shown that if the exponent is greater than 1/2, there exist deflationary solutions which begin in a de Sitter state and evolve away from it asymptotically in the future. On the other hand, if this exponent is less than 1/2, then solutions expand from an initial singularity towards a de Sitter state. Barrow [Bar82] also estimated the entropy production associated with anisotropy damping in the early universe by considering a Bianchi type I metric with an equilibrium radiation gas and anisotropic stresses produced by shear viscosity. It was shown that the shear viscosity based on kinetic theory has the general form of being proportional to the matter density and that the entropy production due to collisional transport is negligible in such a model.

All of the aforementioned papers use the metric approach (Page 39, [WE97]) to obtain the dynamical evolution of the Bianchi model under consideration. The alternative approach which is based on the method of orthonormal frames pioneered by Ellis and MacCallum [EM69] in conjunction with dynamical systems theory is the path we take in this chapter. Belinkskii and Khalatnikov [BK76] used phase-plane techniques to study a Bianchi Type I model under the influence of both shear and bulk viscosity. Goliath and Ellis [GE99] used dynamical systems methods to study FLRW, Bianchi Type I, Bianchi Type II, and Kantoswki-Sachs models with a positive cosmological constant. Coley and van den Hoogen [CH94] analyzed in detail a Bianchi Type V model with viscosity, heat conduction, and a cosmological constant. They showed that all models that satisfy the weak energy condition isotropize. Coley, van den Hoogen, and Maartens [CvdH96] examined the full Israel-Stewart theory of bulk viscosity applied to dissipative FLRW models. Coley and Dunn [CD92] used dynamical systems methods to study the evolution of a Bianchi Type V model with both shear and bulk viscosity. Burd and Coley [BC94] examined using dynamical systems methods the effects of both bulk and shear viscosities upon the FLRW, Bianchi Type I, Bianchi Type V, and Kantowski-Sachs models. They found that these models were structurally stable under the introduction of bulk viscosity. Kohli and Haslam [KH13b] used dynamical systems methods to study the future asymptotic behaviour of a Bianchi Type IV model containing both bulk and shear viscosity. Kohli and Haslam [KH13a] used dynamical systems methods to study a Bianchi Type I model containing bulk and shear viscosity in addition to a homogeneous magnetic field.

With respect to dynamical systems methods in multi-fluid models, Stabell and Refsdal [SR66] considered the dynamics of a two-fluid FLRW system consisting of dust and a cosmological constant. Phase plane methods were used by Madsen, Mimoso, Butcher, and Ellis [MMBE92] to study the evolution of FLRW models in the presence of an arbitrary mixture of perfect fluids. Coley and Wainwright [CW92] examined orthogonal Bianchi and FLRW models in the presence of a two-fluid system. Ehlers and Rindler [ER89] studied in great detail three-fluid models, containing radiation, dust, and a cosmological constant. Recently, Barrow and Yamamoto[BY12] considered a two-fluid system with one of the fluids representing a cosmological constant in their study of the instabilities of Bianchi Type IX Einstein static universes. For more details on the history of multi-fluid models, the interested reader should see Pages 53-55, 60-62, 171-172 and references therein of [WE97].

Despite all of the important aforementioned contributions, we feel it will be of considerable value to consider the dynamics of a Bianchi Type I universe with a viscous fluid and cosmological constant with respect to dynamical systems theory following the methods outlined in [WE97] and [BY12]. To the best of the authors' knowledge at the time of writing this paper, such an investigation has not been carried out in the literature.

Throughout this chapter, we assume a metric signature of $(-,+,+,+)$ and use geometrized units, where $8\pi G = c = 1$.

5.2 The Evolution Equations

We begin by describing the physical constituents of our two-fluid model. It can be shown [KH13b] that in the absence of heat conduction, the energy-momentum tensor of a fluid with both bulk and shear viscosity is given by

$$\mathcal{V}_{ab} = (\mu_m + p_m)\, u_a u_b + g_{ab} p_m - 3\xi H h_{ab} - 2\eta\sigma_{ab}, \tag{5.1}$$

where μ_m, p_m, σ_{ab}, and u_a represent the matter density, pressure, shear tensor, and fluid four-velocity respectively. Further, the quantities ξ and η denote the bulk and shear viscosity coefficients of the fluid matter source, H denotes the Hubble parameter, and $h_{ab} \equiv u_a u_b + g_{ab}$ is the standard projection tensor corresponding to our assumed metric signature.

The second fluid in our model represents a cosmological constant, which can be modelled as a perfect fluid with barotropic equation of state $p_\Lambda = -\mu_\Lambda$. That is, the equation of state parameter is $w = -1$. The energy-momentum tensor for such a cosmological constant takes the simple form

$$\Lambda_{ab} = -g_{ab}\mu_\Lambda, \tag{5.2}$$

where μ_Λ in this case represents the vacuum energy density corresponding to the cosmological constant.

Assuming the matter in our model described by Eq. (5.1) assumes a barotropic equation of state $p_m = w\mu_m$, where in general, $-1 \leq w \leq 1$, using the definitions

$$\mu_m = \mathcal{V}_{ab}u^a u^b, \quad p_m = \frac{1}{3}h^{ab}\mathcal{V}_{ab}, \quad \pi_{ab} = h_a^c h_b^d \mathcal{V}_{cd} - p h_{ab}, \tag{5.3}$$

we find that

$$p_m = w\mu_m - 3\xi H, \quad \pi_{ab} = -2\eta\sigma_{ab}, \tag{5.4}$$

where π_{ab} represents the total anisotropic stress of the fluid.

To write down the Einstein field equations as a dynamical system, it is necessary that we express the above variables in their expansion-normalized form [WE97], thus introduce the definitions

$$\Omega_m = \frac{\mu_m}{3H^2}, \quad \Omega_\Lambda = \frac{\mu_\Lambda}{3H^2}, \quad P_m = \frac{p_m}{3H^2}, \quad P_\Lambda = \frac{p_\Lambda}{3H^2}, \quad \Pi_{ab} = \frac{\pi_{ab}}{H^2}. \tag{5.5}$$

Following [CvdH96], [BK76], and [Bar88b], we define the expansion-normalized form of the bulk and shear viscosity coefficients as

$$\frac{\xi}{3H} = \xi_0\Omega_m^a, \quad \frac{\eta}{3H} = \eta_0\Omega_m^b, \tag{5.6}$$

where ξ_0 and η_0 denote the bulk and shear viscosity parameters and are taken to be non-negative. In addition, the exponents a and b are also assumed to be non-negative. We will discuss the problem of choosing values for these exponents in the next section when deriving equilibrium points of the dynamical system.

In deriving the evolution equations, we essentially follow [WE97] and note that we consider Bianchi models relative to a group-invariant orthonormal frame $\{\mathbf{n}, \mathbf{e}_k\}$, $(k = 1, 2, 3)$ where $\mathbf{n}$ is the the unit normal to the group orbits. Since $\mathbf{n}$ is tangent to a hypersurface-orthogonal congruence of geodesics, these equations are obtained by assuming that all variables are only functions of time, the motion of the matter is along geodesics, and there is no vorticity. The basic dynamical variables are then

$$\left(H, \sigma_{ab}, n_{ab}, a_a\right), \tag{5.7}$$

where n_{ab} and a_a classify and represent the spatial curvature of the specific Bianchi model under question.

If we now apply the definitions in Eq. (5.6) to the basic variables in Eq. (5.7), we obtain the expansion-normalized evolution equations as given in [HBW01] and [EMM12] as:

$$
\begin{aligned}
\Sigma'_{ij} &= -(2-q)\Sigma_{ij} + 2\epsilon^{km}_{(i}\Sigma_{j)k}R_m - \mathcal{S}_{ij} + \Pi_{ij}, \\
N'_{ij} &= qN_{ij} + 2\Sigma^k_{(i}N_{j)k} + 2\epsilon^{km}_{(i}N_{j)k}R_m, \\
A'_i &= qA_i - \Sigma^j_i A_j + \epsilon^{km}_i A_k R_m, \\
\Omega' &= (2q-1)\Omega - 3P - \frac{1}{3}\Sigma^j_i \Pi^i_j + \frac{2}{3}A_i Q^i, \\
Q'_i &= 2(q-1)Q_i - \Sigma^j_i Q_j - \epsilon^{km}_i R_k Q_m + 3A^j \Pi_{ij} + \epsilon^{km}_i N^j_k \Pi_{jm}.
\end{aligned} \tag{5.8}
$$

These equations are subject to the constraints

$$
\begin{aligned}
N^j_i A_j &= 0, \\
\Omega &= 1 - \Sigma^2 - K, \\
Q_i &= 3\Sigma^k_i A_k - \epsilon^{km}_i \Sigma^j_k N_{jm}.
\end{aligned} \tag{5.9}
$$

In the expansion-normalized approach, Σ_{ab} denotes the kinematic shear tensor, and describes the anisotropy in the Hubble flow, A_i and N^{ij} describe the spatial curvature, while Ω^i and R^i describe the relative orientation of the shear and spatial curvature eigenframes and energy flux respectively. Further the *prime* denotes differentiation with respect to a dimensionless time variable τ such that

$$
\frac{dt}{d\tau} = \frac{1}{H}. \tag{5.10}
$$

Considering a Bianchi Type I model, by definition, we have that

$$
N_{ij} = \mathrm{diag}\,(0,0,0)\,, \quad A_i = R_i = Q_i = 0. \tag{5.11}
$$

in this chapter, we only consider the case where the fluid matter source has nonzero bulk viscosity, and therefore set $\eta_0 = 0$. The importance of this assumption has been discussed in for example, [Bar88b]. Therefore, upon considering Eqs. (5.4), (5.5), (5.6), (5.8), and (5.9), we obtain the evolution equations for our system as

$$\Sigma'_+ = \Sigma_+ (q-2), \tag{5.12}$$

$$\Sigma'_- = \Sigma_- (q-2), \tag{5.13}$$

$$\Omega'_m = \Omega_m (2q - 1 - 3w) + 9\xi_0 \Omega_m^a, \tag{5.14}$$

$$\Omega'_\Lambda = 2(q+1)\Omega_\Lambda, \tag{5.15}$$

where q is the deceleration parameter which can be obtained by setting the Raychaudhuri equation (Eq (1.90) in [WE97]) to the general evolution equation for H (Eq. (5.8) in [WE97]) and then solving for q. Proceeding in this manner gives

$$q = 2\left(\Sigma_+^2 + \Sigma_-^2\right) + \frac{1}{2}\left[\Omega_m (1+3w)\right] - \frac{9}{2}\xi_0 \Omega_m^a - \Omega_\Lambda. \tag{5.16}$$

The equations (5.12)-(5.15) are subject to the constraint

$$\Omega_m + \Omega_\Lambda + \Sigma_+^2 + \Sigma_-^2 = 1, \tag{5.17}$$

which is just the generalized Friedmann equation (Eq. (1.92) in [WE97]) in expansion-normalized form. Also, note that in equations (5.12)-(5.15) we have made use of the notation

$$\Sigma_+ = \frac{1}{2}(\Sigma_{22} + \Sigma_{33}), \quad \Sigma_- = \frac{1}{2\sqrt{3}}(\Sigma_{22} - \Sigma_{33}), \tag{5.18}$$

such that $\Sigma^2 \equiv \Sigma_+^2 + \Sigma_-^2$.

5.3 Stability Analysis of the Dynamical System

With the evolution and constraint equations in hand, we will now perform a detailed analysis of the equilibrium points of the dynamical system. The system of equations (5.12)-(5.15) is a nonlinear, autonomous system of ordinary differential equations, and can be written as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \tag{5.19}$$

where $\mathbf{x} = [\Sigma_+, \Sigma_-, \Omega_m, \Omega_\Lambda] \in \mathbf{R}^4$, and the vector field $\mathbf{f}(\mathbf{x})$ denotes the right-hand-side of the dynamical system. The dynamical system also exhibits some symmetries, specifically ones that leave the system invariant with respect to spatial inversions of the dynamical variables. These are given by

$$\phi_1 \quad : \quad [\Sigma_+, \Sigma_-, \Omega_m, \Omega_\Lambda] \to [-\Sigma_+, \Sigma_-, \Omega_m, \Omega_\Lambda]\,, \tag{5.20}$$

$$\phi_2 \quad : \quad [\Sigma_+, \Sigma_-, \Omega_m, \Omega_\Lambda] \to [\Sigma_+, -\Sigma_-, \Omega_m, \Omega_\Lambda]\,. \tag{5.21}$$

These symmetries imply that we can take

$$\Sigma_\pm \geq 0. \tag{5.22}$$

In addition, based on the physical constraints of having non-negative energy density, we make the assumption that

$$\Omega_m \geq 0, \quad \Omega_\Lambda \geq 0. \tag{5.23}$$

Tavakol (Chapter 4, [WE97]) discusses a simple way to obtain the invariant sets of a dynamical system. Let us consider a dynamical system $\dot{x} = v(x), \quad x \in \mathbb{R}^4$. Let $Z : \mathbb{R}^4 \to \mathbb{R}$ be a C^1 function such that $Z' = \alpha Z$, where $\alpha : \mathbb{R}^4$ is a continuous function. Then the subsets of $\mathbb{R}^4$ defined by $Z > 0$, $Z = 0$, and $Z < 0$ are invariant sets of the flow of the dynamical system. Applying this proposition to our dynamical system in combination with the symmetries found above, we see that $\Sigma_\pm \geq 0$ and $\Omega_\Lambda \geq 0$ constitute invariant sets of the dynamical system.

Following [AM78], we first note that the vector field $\mathbf{f}(\mathbf{x})$ is clearly at least C^1 on $M = \mathbb{R}^4$. We call a point $\mathbf{m_0}$ an equilibrium point of $\mathbf{f}(\mathbf{x})$ if $\mathbf{f}(\mathbf{m_0}) = 0$. Let (U, ϕ) be a chart on M with $\phi(\mathbf{m_0}) = \mathbf{x}_0 \in \mathbb{R}^4$, and let $\mathbf{x} = (\Sigma_+, \Sigma_-, \Omega_m, \Omega_\Lambda)$ denote coordinates in $\mathbb{R}^4$. Then, the linearization of $\mathbf{f}(\mathbf{x})$ at $\mathbf{m_0}$ in these coordinates is given by

$$\left(\frac{\partial \mathbf{f}(\mathbf{x})^i}{\partial x^j} \right)_{\mathbf{x}=\mathbf{x}_0} \tag{5.24}$$

It is a remarkable fact of dynamical systems theory that if the point $\mathbf{m_0}$ is hyperbolic, then there exists a neighborhood N of $\mathbf{m_0}$ on which the flow of the system F_t is topologically equivalent to the flow of the linearization Eq. (5.24). This is the theorem of Hartman and Grobman [WE97]. That is, in N, the orbits of the dynamical system can be deformed continuously into the orbits of Eq. (5.24), and the orbits are therefore topologically equivalent. We use the following convention when discussing the stability properties of the dynamical system. If all eigenvalues λ_i of Eq. (5.24) satisfy $Re(\lambda_i) < 0 (Re(\lambda_i) > 0)$, $\mathbf{m_0}$ is local sink (source) of the system. If the point $\mathbf{m_0}$ is neither a local source or sink, we will call it a saddle point.

Solving for the equilibrium points, we first obtain three types of flat FLRW-type solutions:

$$\mathcal{F}_1 \quad : \quad \Sigma_+ = 0, \quad \Sigma_- = 0, \quad \Omega_m = 1, \quad \Omega_\Lambda = 0, \tag{5.25}$$

$$\mathcal{D} \quad : \quad \Sigma_+ = 0, \quad \Sigma_- = 0, \quad \Omega_m = 0, \quad \Omega_\Lambda = 1, \tag{5.26}$$

$$\mathcal{F}_2 \quad : \quad \Sigma_+ = 0, \quad \Sigma_- = 0, \quad \Omega_m = \left(\frac{1+w}{3\xi_0}\right)^{\frac{1}{a-1}}, \quad \Omega_\Lambda = 1 - \Omega_m, \tag{5.27}$$

where $\mathcal{D}$ is the de Sitter solution. For $\mathcal{F}_2$, based on the physical constraints of having non-negative bulk viscosity and energy densities, we must have additionally that

$$-1 < w \leq 1, \quad 0 < \xi_0 \leq \frac{1+w}{3}. \tag{5.28}$$

It is also interesting to see that for $\mathcal{F}_2$,

$$\lim_{a \to \pm\infty} \Omega_m = 1 \Rightarrow \Omega_\Lambda = 0. \tag{5.29}$$

Therefore, it can be said that the point $\mathcal{F}_1$ exists in the extreme limit with respect to the exponent a of the point $\mathcal{F}_2$.

In the special case where we additionally have that $\xi_0 = 0$, we obtain a Kasner circle equilibrium point:

$$\mathcal{K} \quad : \quad \Sigma_+^2 + \Sigma_-^2 = 1, \quad \Omega_m = 0, \quad \Omega_\Lambda = 0. \tag{5.30}$$

We should note that this is a special type of Kasner circle, in that, it is actually a Kasner quarter-circle. This is evident due to the restrictions described in Eq. (5.22).

Analyzing the stability of $\mathcal{F}_1$, we note that the eigenvalues of Eq. (5.24) at this point are found to be

$$\lambda_1 = \lambda_2 = \frac{3}{2}(-1 + w - 3\xi_0), \quad \lambda_3 = 1 + 3w - 9\xi_0, \quad \lambda_4 = 3(1 + w - 3\xi_0). \tag{5.31}$$

Therefore, the point $\mathcal{F}_1$ is a local sink if

$$(-1 \leq w \leq 1) \bigcap \left(\xi_0 > \frac{1+w}{3}\right), \tag{5.32}$$

and is a saddle point if

$$\left(-1 < w < -\frac{1}{3} \bigcap 0 \leq \xi_0 < \frac{1+w}{3}\right) \bigcup \left(-\frac{1}{3} \leq w \leq 1 \bigcap \frac{1}{9}(1+3w) < \xi_0 < \frac{1+w}{3}\right) \tag{5.33}$$

or

$$\left(-\frac{1}{3} < w < 1 \bigcap 0 \leq \xi_0 < \frac{1}{9}(1+3w)\right) \bigcup \left(w = 1 \bigcap 0 < \xi_0 < \frac{4}{9}\right). \tag{5.34}$$

As can be shown the point $\mathcal{F}_1$ is never a source of the system.

Analyzing the stability of $\mathcal{F}_2$, we note that the eigenvalues of Eq. (5.24) at this point are found to be

$$\lambda_1 = -2 + 3(1+w) z^{\frac{1}{-1+a}} - 9\left(z^{\frac{1}{-1+a}}\right)^a \xi_0, \tag{5.35}$$

$$\lambda_2 = \frac{3}{2}\left[-2 + (1+w) z^{\frac{1}{-1+a}} - 3\left(z^{\frac{1}{-1+a}}\right)^a \xi_0\right], \tag{5.36}$$

$$\lambda_3 = \lambda_2, \tag{5.37}$$

$$\lambda_4 = z^{\frac{1}{1-a}}\left[-3(1+w) z^{\frac{1}{-1+a}} + 6(1+w) z^{\frac{2}{-1+a}} - 9\left(z^{\frac{1}{-1+a}}\right)^a \left(z^{\frac{1}{-1+a}} + a\left(-1 + z^{\frac{1}{-1+a}}\right)\right)\xi_0\right], \tag{5.38}$$

where we have defined

$$z \equiv \frac{1+w}{3\xi_0}. \tag{5.39}$$

Clearly, finding equilibrium points of the system by solving $\mathbf{f}(\mathbf{m_0}) = 0$ is a difficult task for general a in Eqs. (5.12)-(5.15). We must therefore choose values for a beforehand, and then perform the fixed-point analysis. Our choices for these values are not completely arbitrary. Belinksii and Khalatnikov [BK76] and Barrow [Bar88b] have both presented arguments for physically relevant choices for these exponents. In particular, following [BK76] and [Bar88b], we note that it is reasonable to consider $a \leq 1/2$ for the early universe, where we expect viscous effects to play a significant role in its dynamical evolution.

5.3.1 The case: $a = 0$

Setting $a = 0$ in Eq. (5.27), we obtain:

$$\mathcal{F}_3 \quad : \quad \Sigma_+ = 0, \quad \Sigma_- = 0, \quad \Omega_m = \frac{3\xi_0}{1+w}, \quad \Omega_\Lambda = \frac{1+w-3\xi_0}{1+w}. \tag{5.40}$$

Analyzing the stability of $\mathcal{F}_3$, the eigenvalues in Eqs. (5.35) - (5.38) are found to be

$$\lambda_1 = -2, \quad \lambda_2 = -3\left(1 + w - 3\xi_0\right), \quad \lambda_3 = \lambda_4 = -3. \tag{5.41}$$

Therefore, $\mathcal{F}_3$ is a local sink if

$$(-1 < w \leq 1) \bigcap \left(0 \leq \xi_0 < \frac{1+w}{3}\right), \tag{5.42}$$

and is a saddle point if

$$(-1 < w \leq 1) \bigcap \left(0 \leq \xi_0 > \frac{1+w}{3}\right). \tag{5.43}$$

Clearly, the region corresponding to $\mathcal{F}_3$ being a saddle point is unphysical since it violates Eq. (5.28). Therefore, $\mathcal{F}_3$ can never be a saddle point. Additionally, $\mathcal{F}_3$ can never be a source of the system.

5.3.2 The case: $a = 1/2$

Setting $a = 1/2$ in Eq. (5.27), we obtain:

$$\mathcal{F}_4 \quad : \quad \Sigma_+ = 0, \quad \Sigma_- = 0, \quad \Omega_m = \frac{9\xi_0^2}{(1+w)^2}, \quad \Omega_\Lambda = \frac{1 + 2w + w^2 - 9\xi_0^2}{(1+w)^2}. \tag{5.44}$$

The eigenvalues in Eqs. (5.35) - (5.38) are found to be

$$\lambda_1 = \lambda_2 = -3, \quad \lambda_3 = -2, \quad \lambda_4 = -\frac{3\left(1 + 2w + w^2 - 9\xi_0^2\right)}{2(1+w)}. \tag{5.45}$$

It is easy to show that this point is always a local sink if

$$-1 < w \leq 1, \quad 0 \leq \xi_0 < \frac{1+w}{3}. \tag{5.46}$$

It is in fact true that even if $\xi_0 = (1+w)/3$ such that $\lambda_4 = 0$, $\mathcal{F}_4$ will still be a local sink, since it will be a normally hyperbolic point (Chapter 4, [WE97]).

We note that Eq. (5.24) is not defined at $\mathcal{D}$ or $\mathcal{K}$, so linearization techniques will not help us in determining the stability of these points. However, in the next section, we describe some other techniques that will help us determine the global stability of these points.

5.4 Topological Results

Complementing the preceding fixed-point analysis, we wish to obtain some information about the asymptotic behaviour of the dynamical system as $\tau \to \pm\infty$. To accomplish this, we make use of the LaSalle Invariance Principle, and the methods of finding Lyapunov and Chetaev functions. According to Theorem 4.11 in [WE97], the LaSalle Invariance Principle for ω-limit sets is stated as follows. Consider a dynamical system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$, with flow ϕ_t. Let S be a closed, bounded and positively invariant set of ϕ_t and let Z be a C^1 monotone function. Then $\forall\ \mathbf{x}_0 \in S$, we have that $\omega(\mathbf{x}_0) \subseteq \{\mathbf{x} \in S | Z' = 0\}$, where $Z' = \nabla Z \cdot \mathbf{f}$. The extended LaSalle Invariance Principle for α-limit sets can be found in Proposition B.3. in [HW93]. To use this principle, one simply considers S to be a closed, bounded, and negatively invariant set. Then $\forall\ \mathbf{x}_0 \in S$, we have that $\alpha(\mathbf{x}_0) \subseteq \{\mathbf{x} \in S | Z' = 0\}$, where $Z' = \nabla Z \cdot \mathbf{f}$.

Following Pages 24 and 25 of [AAA+97a], we note that a differentiable function Z is called a *Lyapunov function* for a singular point $\mathbf{x}_0$ of a vector field ($\mathbf{f}(\mathbf{x})$ if Z is defined on a neighborhood of $\mathbf{x_0}$ and has a local minimum at this point, and the derivative of Z along $\mathbf{f}(\mathbf{x})$ is nonpositive. Then a singular point of a differentiable vector field for which a Lyapunov function exists is stable. Further, a differentiable function Z is called a *Chetaev function* for a singular point $\mathbf{x_0}$ of a vector field $\mathbf{f}(\mathbf{x})$ if Z is defined on a domain W whose boundary contains $\mathbf{x}_0$, the part of the boundary of W is strictly contained in a sufficiently small ball with its center $\mathbf{x}_0$ removed is a piecewise-smooth, C^1 hypersurface along which $\mathbf{f}(\mathbf{x})$ points into the interior of the domain, that is,

$$Z(\mathbf{x}) \to 0, \text{ as } \mathbf{x} \to \mathbf{x}_0, \quad \mathbf{x} \in W; \quad Z > 0, \quad \nabla Z \cdot \mathbf{f}(\mathbf{x}) > 0 \in W. \tag{5.47}$$

A singular point of a C^1 vector field for which a Chetaev function exists is unstable.

Let us first consider the invariant set

$$S_1 = \{\Sigma_+ = \Sigma_- = \Omega_\Lambda = 0\}. \tag{5.48}$$

We will also define the function

$$Z_1 = \Omega_m, \tag{5.49}$$

such that within S_1 we have

$$Z_1' = (-1 + \Omega_m)\left[\Omega_m (1 + 3w) - 9\xi_0 \Omega_m^a\right]. \tag{5.50}$$

This function is monotone if for example $\Omega_m = 1$, and strictly monotone decreasing if

$$\Omega_m < 1, \quad w > 3\xi_0 - \frac{1}{3}. \tag{5.51}$$

By the LaSalle invariance principle, for any orbit $\Gamma \in S_1$, we have that

$$\omega(\Gamma) \subseteq \{\Omega_m = 1\}, \tag{5.52}$$

which corresponds precisely to the FLRW equilibrium point as given in Eq. (5.25). Therefore, the global future asymptotic state of the system corresponding to the invariant set Eq. (5.48) is a flat FLRW universe with $\Omega_m = 1$.

Consider now the function

$$Z_2 = \Omega_\Lambda^2 + 1. \tag{5.53}$$

Let us define the neighborhood of the de Sitter equilibrium point $\mathcal{D}$ as the open ball

$$D_r(\mathcal{D}) = \left[\Sigma_+^2 + \Sigma_-^2 + \Omega_m^2 + (\Omega_\Lambda - 1)^2\right]^{1/2} < r, \tag{5.54}$$

where $r > 0$ is the radius of this ball in $\mathbb{R}^4$. That this is an open neighborhood of $\mathcal{D}$ is proven in Section 2.2 in [MT03]. Clearly the function Z_2 is defined on $D_r(\mathcal{D})$. One can also show that the point $\Omega_\Lambda = 1$ is a local minimum of Z_2 in $D_r(\mathcal{D})$. Furthermore, from computing

$$Z_2' = -4\Omega_\Lambda^2 (-1 + \Omega_\Lambda), \tag{5.55}$$

one can see that on $D_r(\mathcal{D})$, $Z_2' \leq 0$ as long as $\Omega_\Lambda \leq 1$. Therefore, Z_2 is a Lyapunov function corresponding to $\mathcal{D}$, and as a result, $\mathcal{D}$ is stable.

Consider now the invariant set

$$S_2 = \{\Sigma_+ = 0, \quad \Sigma_- = 0 \quad 0 < \Omega_m < 1, \quad 0 < \Omega_\Lambda < 1\}. \tag{5.56}$$

We now define the function

$$Z_3 = -9\left[\frac{1}{3}(1+w)^2\Omega_\Lambda^3 - \frac{1}{2}(1+w)^2\Omega_\Lambda^4 + \frac{1}{5}(1+w)^2\Omega_\Lambda^5 + \frac{6(1+w)\xi_0(1-\Omega_\Lambda)^{2+a}\left(2+2(2+a)\Omega_\Lambda+\left(6+5a+a^2\right)\Omega_\Lambda^2\right)}{(2+a)(3+a)(4+a)} - \frac{9\xi_0^2(1-\Omega_\Lambda)^{1+2a}\left(1+\Omega_\Lambda+2a\Omega_\Lambda+\left(1+3a+2a^2\right)\Omega_\Lambda^2\right)}{(1+a)(1+2a)(3+2a)}\right]. \tag{5.57}$$

Noting that

$$\frac{dZ_3}{d\tau} = \frac{dZ_3}{d\Omega_\Lambda}\frac{d\Omega_\Lambda}{d\tau}, \tag{5.58}$$

we have that within the invariant set S_2,

$$Z_3' = 18(-1+\Omega_\Lambda)\Omega_\Lambda^3\left[-1+3\xi_0(1-\Omega_\Lambda)^a + w(-1+\Omega_\Lambda)+\Omega_\Lambda\right]^2, \tag{5.59}$$

which is strictly monotone decreasing along orbits in S_2, specifically, for $0 < \Omega_\Lambda < 1$. Note that based on Eq. (5.57), we must have that $a \neq -2, -3, -4, -1, -1/2, -3/2$. Also, in deriving Eq. (5.59), we used Eq. (5.17) to write $\Omega_m = 1 - \Omega_\Lambda$. The point $\mathcal{F}_2$ as given in Eq. (5.27) is contained in the set S_2. Furthermore, we have that

$$Z_3'(\mathcal{F}_2) = 0, \quad \mathcal{F}_2 \in S_2, \quad a > 1 \in \mathbb{Z}. \tag{5.60}$$

Therefore, by the LaSalle invariance principle, we must have that for all orbits $\Gamma \in S_2$,

$$\omega(\Gamma) \subseteq \{0 < \Omega_\Lambda < 1\}, \tag{5.61}$$

which is another possible future asymptotic state of the dynamical state and corresponds to the equilibrium point $\mathcal{F}_2$.

Let us now consider the domain

$$W = \left\{\Sigma_+^2 + \Sigma_-^2 < 1, \quad 0 < \Omega_m < 1, \quad 0 < \Omega_\Lambda < 1\right\}. \tag{5.62}$$

The boundary of this domain is given by

$$\bar{W}\backslash W = \left\{\Sigma_+^2 + \Sigma_-^2 = 1\right\} \cup \{\Omega_m = 0\} \cup \{\Omega_\Lambda = 0\} \cup \{\Omega_m = 1\} \cup \{\Omega_\Lambda = 1\}. \tag{5.63}$$

Clearly, the Kasner circle equilibrium point $\mathcal{K}$ is contained in $\bar{W}\backslash W$. Let us now define a function on W,

$$Z_4(\mathbf{x}) = \frac{\Omega_m}{1-\Omega_\Lambda}. \tag{5.64}$$

We note some important properties of this function. First,

$$\lim_{\mathbf{x}\to\mathcal{K}} Z_4(\mathbf{x}) = 0, \quad \mathbf{x} \in W. \tag{5.65}$$

Further, we have that

$$Z_4(\mathbf{x}) > 0, \quad \mathbf{x} \in W. \tag{5.66}$$

Using Eqs. (5.12)-(5.15) and computing $Z_4'(\mathbf{x})$, we obtain

$$Z_4'(\mathbf{x}) = \frac{\Omega_m \left[-1 + 4\Sigma_-^2 + 4\Sigma_+^2 + \Omega_\Lambda + \Omega_m + 3w\left(-1 + \Omega_\Lambda + \Omega_m\right)\right]}{\left(\Omega_\Lambda - 1\right)^2}. \tag{5.67}$$

Considering the special case of $w = -1/3$, this equation takes the form

$$Z_4'(\mathbf{x}) = \frac{4\Omega_m\left(\Sigma_+^2 + \Sigma_-^2\right)}{\left(\Omega_\Lambda - 1\right)^2}, \tag{5.68}$$

which is strictly positive everywhere in W. Therefore, we have just proven that for the special case $w = -1/3$, Z_4 is a Chetaev function and corresponds to $\mathcal{K}$. This implies that the Kasner equilibrium point corresponding to $w = -1/3$ is unstable. We were only able to prove instability for this case. However, we conjecture based on extensive numerical experiments (see Fig. 5.6) that $\mathcal{K}$ is unstable for all values $-1 \leq w \leq 1$ and $\xi_0 = 0$.

5.5 Bifurcations and Orbits

With the stability analysis completed in the previous section, we now will describe bifurcations that occur in the dynamical system. These occur by changing values of either the bulk viscosity coefficient ξ_0, the equation of state parameter w or both. These bifurcations are displayed in Fig. 5.1.

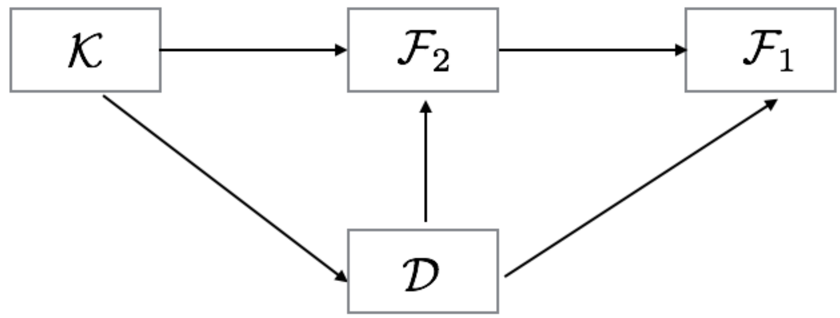


Figure 5.1: The possible bifurcations that occur in the dynamical system as a result of changing the bulk viscosity coefficient ξ_0, the equation of state parameter w or both.

There also exists a finite heteroclinic sequence (Page 104, [WE97]) when $\xi_0 = 0$ and $-1 < w < -1/3$, and is given by

$$\mathcal{K} \to \mathcal{F}_1 \to \mathcal{F}_2. \tag{5.69}$$

When $w = -1$ and $\xi_0 = 0$, there exists a heteroclinic orbit:

$$\mathcal{K} \to \mathcal{D}. \tag{5.70}$$

When $\xi_0 = 0$ and $-1 < w \leq 1$, we have the following heteroclinic orbit:

$$\mathcal{K} \to \mathcal{F}_2. \tag{5.71}$$

There are some interesting things to note about the bifurcations, heteroclinic sequence and orbits described in this section. The Kasner equilibrium point, $\mathcal{K}$ is a state according to Eq. (5.30) that has no matter/energy or vacuum energy whatsoever, that is, $\Omega_m = \Omega_\Lambda = 0$. Yet, from this empty state, we see that in the case of the bifurcations, an increase in the bulk viscosity causes the universe to evolve towards a FLRW universe either with no vacuum energy or with a mix of matter and vacuum energy, which represents our universe today. A question that may be related to this phenomenon is what mechanism generates matter or particle creation not only in generic Bianchi type I spacetimes, but near a Kasner-type state specifically. This problem was first studied by Zeldovich [Zel70], where he showed that spontaneous particle production could occur near a Kasner singularity. In related work, Parker [Par72] gave conditions for particle creation near an isotropic Friedmann-type singularity. Berger [Ber74] studied quantum graviton creation in a general spatially homogeneous, anisotropic three-torus solution of Einstein's equations. Ahmed and Mondal [AM92] studied the Hawking radiation of Dirac particles in a Kasner spacetime. They showed that the anisotropy gives rise to particle creation. Further, Harko and Mak [HM00] also investigated in some detail the effects of matter creation on the evolution and dynamics of a Bianchi type I model. Related to this, Barrow [Bar82] studied the entropy production associated with anisotropy damping in the era of grand unification in the early universe. The general consensus in all of these investigations is that particle creation leads to the isotropization of an anisotropic spacetime.

Further, according to the bifurcation sequences and finite heteroclinic sequences given in Eqs. (5.69) and (5.71), we see that it is completely possible to go from an empty state to an FLRW state bypassing the de Sitter state altogether, which can have some interesting implications with regard to cosmic inflation. These results show that it is a valid to ask the question whether the universe can go from an initial empty state to the universe we have today without undergoing the standard inflationary epoch. That is, it may be that what we consider to be inflation with respect to an inflaton field may actually be an expansionary epoch driven by bulk viscosity. Barrow [Bar88b] investigated this matter in quite some detail. He found that when the bulk viscous exponent is larger than 1/2 it is no longer guaranteed that an asymptotic de Sitter state will occur. In addition, when the exponent is greater than or equal to 1, the asymptotic de Sitter state is replaced with a Weyl curvature singularity. Belinskii and Khalatnikov [BK76] also concluded in their study of the Bianchi type I universe with viscosity, that viscous effects alone showed an "essential isotropizing action". The relationship between the presence of bulk viscosity and the existence of an asymptotically stable de Sitter universe was also investigated in [CJMM97], [Zim96], [Zim93], and [Bar87].

5.6 Connections with Observations

The Planck team [A+13] recently calculated based on observations that

$$\Omega_\Lambda = 0.6825, \quad \Omega_m = 0.3175. \tag{5.72}$$

This configuration is precisely modelled by our $\mathcal{F}_2$ equilibrium point given in Eq. (5.27), which we found to be a local sink:

$$\Sigma_+ = 0, \quad \Sigma_- = 0, \quad \Omega_m = \left(\frac{1+w}{3\xi_0}\right)^{\frac{1}{a-1}}, \quad \Omega_\Lambda = 1 - \Omega_m.$$

Given these facts and the fact that it is conjectured that the very early universe consisted of incoherent radiation with matter equation of state $w = 1/3$, or stiff matter with equation of state $w = 1$ (Page 99, [EMM12]), we can ask the question what values of ξ_0 in the early universe could have led to the state described by Eqs. (5.72) and (5.27) today. To answer this question, we follow [BK76] and note that for the early universe, $a \leq 1/2$. The interesting thing is that $\mathcal{F}_2$ is a future asymptotic state of the dynamical system as the eigenvalue computations show in Eqs. (5.41) and (5.44). Therefore, choices of w and ξ_0 in the early universe within the acceptable range of values as described in Eq. (5.42) would yield a future state that is similar to what we observe today.

As a consequence of the aforementioned arguments, we now present some solutions of the system of equations

$$\left(\frac{1+w}{3\xi_0}\right)^{\frac{1}{a-1}} = 0.3175, \quad 1 - \left(\frac{1+w}{3\xi_0}\right)^{\frac{1}{a-1}} = 0.6825, \tag{5.73}$$

where $-1 < w \leq 1, \quad 0 < \xi_0 \leq (1+w)/3$ as in Eq. (5.28).

For $a = 0$, we obtain the parametrized solution

$$0 < \xi_0 \leq \frac{127}{600}, \quad w = \frac{1}{127}\left(-127 + 1200\xi_0\right). \tag{5.74}$$

It is important to recall that in our two-fluid model, the equation of state parameter w describes the non-vacuum energy. It is conjectured that the majority of the non-vacuum energy/matter in the early universe consisted of radiation with equation of state parameter $w = 1/3$ (Page 98, [EMM12]). Solving for ξ_0 in Eq.

(5.74) with $w = 1/3$, yields

$$\xi_0 = \frac{127}{900} \approx 0.1411, \quad w = 1/3, \quad a = 0. \tag{5.75}$$

Solving for ξ_0 in Eq. (5.74) with $w = 1$ yields

$$\xi_0 = \frac{127}{600} \approx 0.2117, \quad w = 1, \quad a = 0. \tag{5.76}$$

In this case where $a = 1/2$, we obtain the parameterized solution

$$0 < \xi_0 \leq \frac{\sqrt{127}}{30}, \quad w = -1 + \frac{60\xi_0}{\sqrt{127}}. \tag{5.77}$$

Solving for ξ_0 in Eq. (5.77) with $w = 1/3$, yields

$$\xi_0 = \frac{\sqrt{127}}{45} \approx 0.2504, \quad w = 1/3, \quad a = \frac{1}{2}. \tag{5.78}$$

Solving for ξ_0 in Eq. (5.77) with $w = 1$ yields

$$\xi_0 = \frac{\sqrt{127}}{30} \approx 0.3756, \quad w = 1, \quad a = \frac{1}{2}. \tag{5.79}$$

Therefore, our model shows that an early-universe configuration with bulk viscosity values as computed in Eqs. (5.75), (5.76), (5.78), (5.79) could lead to the mixture of vacuum and non-vacuum energy that we observe today. Further, our calculation of the $\mathcal{F}_2$ equilibrium point further shows that it is quite possible that the vacuum energy that exists in our universe today could be the result of some bulk viscous effect of the ordinary matter in the early universe or at the present time. The connections between bulk viscosity and vacuum energy have been explored in [FLS13], [SSB12], [AAR96] [BG05] [RM06], [FGR06], [CFTZ07], [LB09], [DR03], [WMF07],[GL11], [VWM13], [DB12], and [MIK$^+$08], though the calculations presented above to the best of the authors' knowledge are new and have not been reported before in the literature.

5.7 Numerical Solutions

To complement both the fixed-point and abstract topological analysis in the previous section, we now present some numerical solutions to the dynamical system Eqs. (5.12)-(5.15). Initial conditions were chosen to satisfy the constraint equations (5.17), (5.22), and(5.23), and are represented in the numerical experiments by asterisks. Furthermore, we note that the numerical solutions were completed over sufficiently long time intervals ($0 \leq \tau \leq 1000$), but in some cases we present the solutions over shorter time intervals for clarity.

We display in Figs. 5.2 and 5.3 the results of numerical experiments that show that $\mathcal{F}_3$ is indeed a local sink of the system.

We display in Figs. 5.4 and 5.5 the results of numerical experiments that show that $\mathcal{F}_4$ is indeed a local sink of the system.

We display in Fig. 5.6 the results of several numerical experiments that show $\mathcal{K}$ as a source of the dynamical system.

We display in Figs. 5.7, 5.8, 5.9 the results of several numerical experiments that show $\mathcal{F}_1$ as a local sink of the dynamical system.

Figure 5.2: This figure shows the dynamical system behaviour for $\xi_0 = 127/900$, $w = 1/3$, and $a = 0$. The circle denotes the equilibrium point $\mathcal{F}_3$. This precise case corresponds to Eq. (5.75).

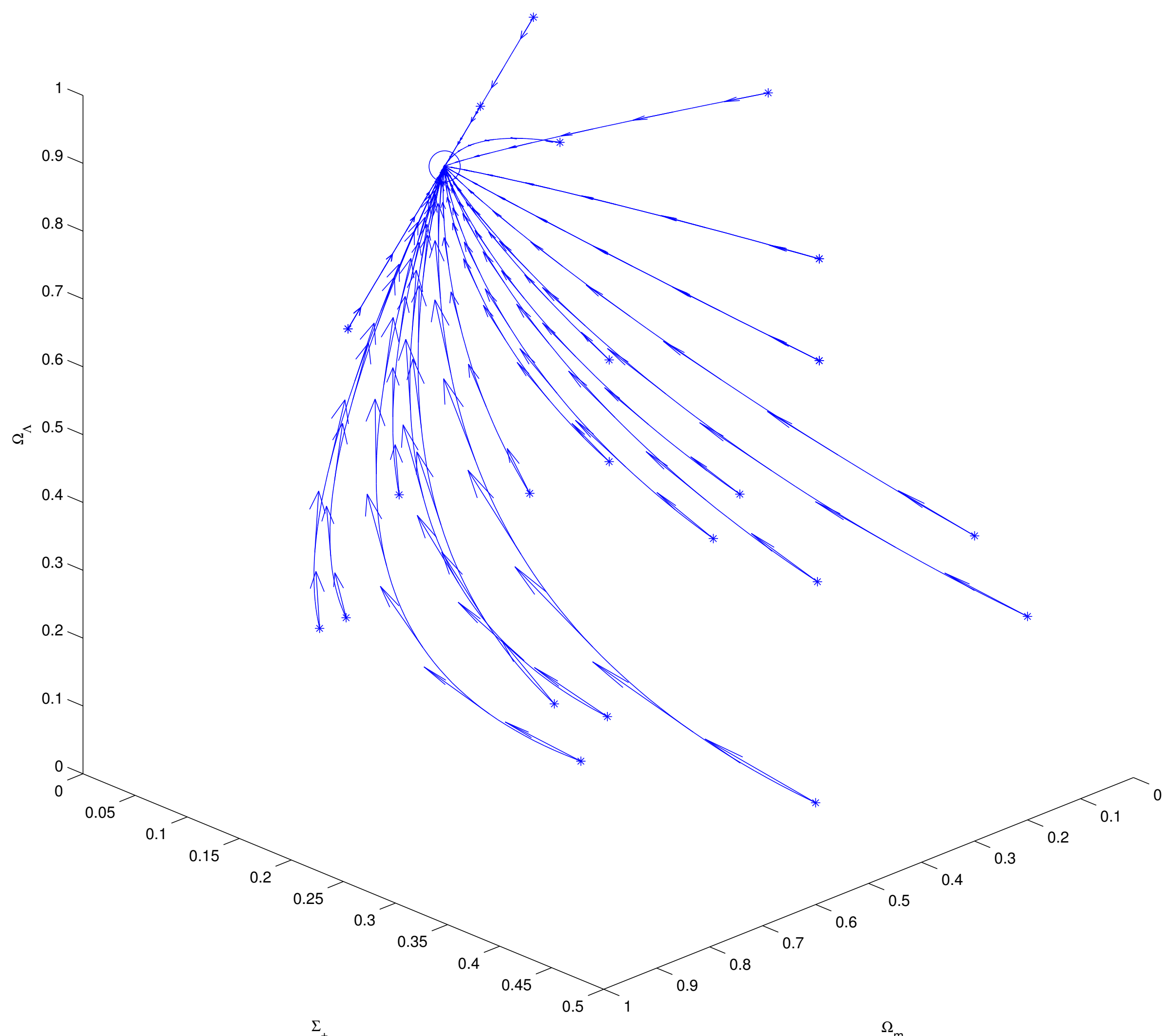


Figure 5.3: This figure shows the dynamical system behaviour for $\xi_0 = 127/600$, $w = 1$, and $a = 0$. The circle denotes the equilibrium point $\mathcal{F}_3$. This precise case corresponds to Eq. (5.76).

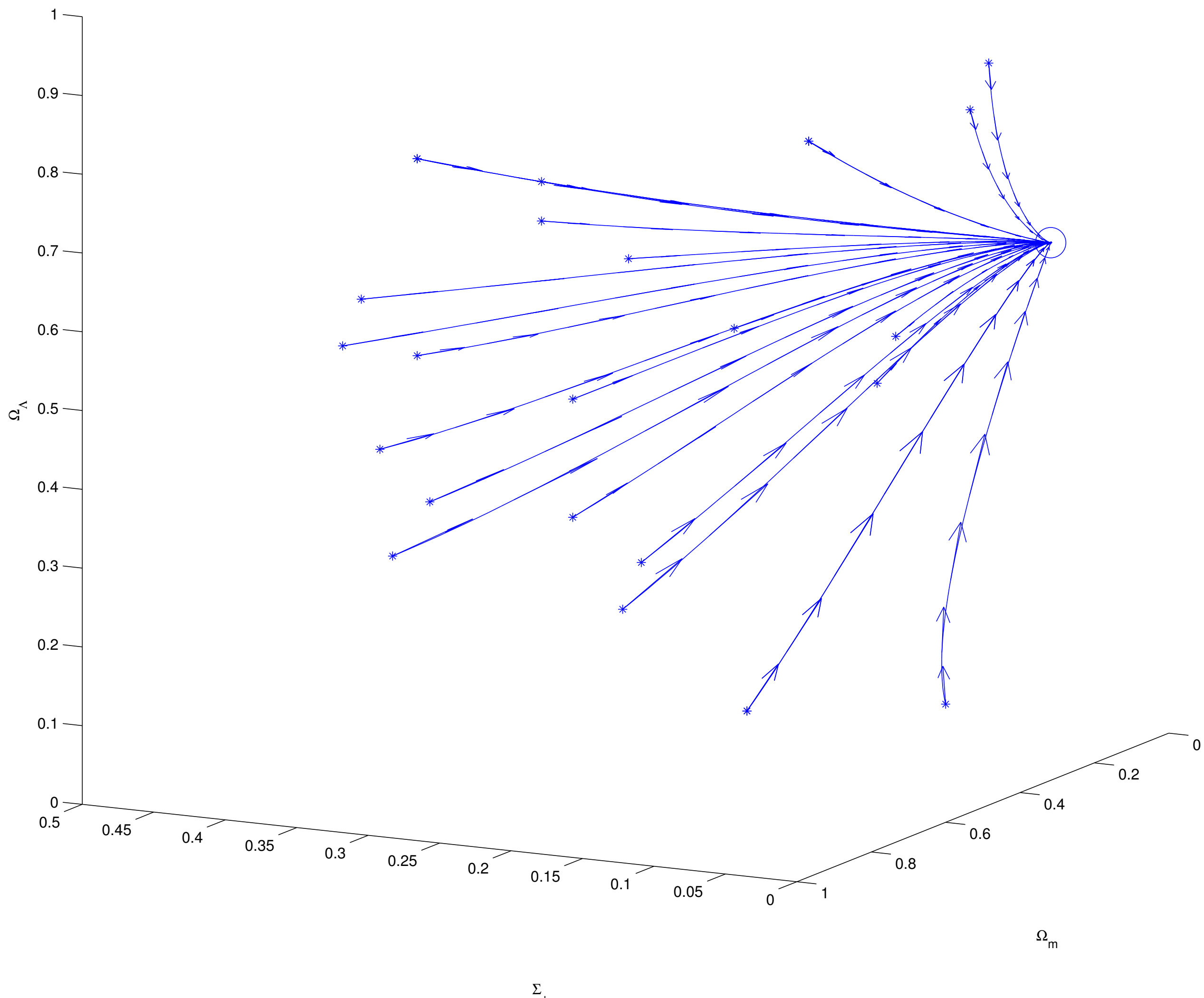


Figure 5.4: This figure shows the dynamical system behaviour for $\xi_0 = \sqrt{127}/45$, $w = 1/3$, and $a = 1/2$. The circle denotes the equilibrium point $\mathcal{F}_4$. This precise case corresponds to Eq. (5.78).

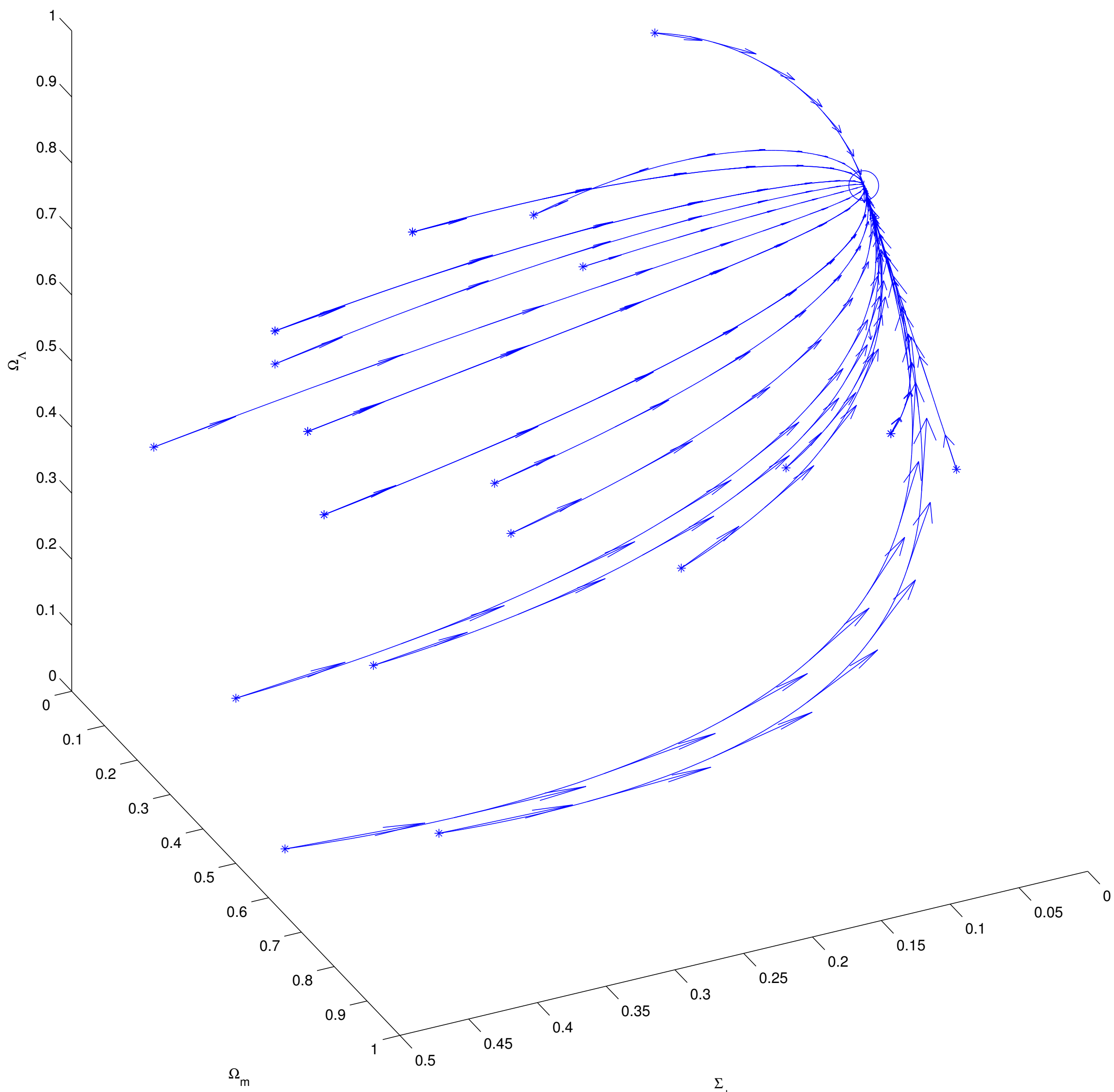


Figure 5.5: This figure shows the dynamical system behaviour for $\xi_0 = \sqrt{127}/30$, $w = 1$, and $a = 1/2$. The circle denotes the equilibrium point $\mathcal{F}_4$. This precise case corresponds to Eq. (5.79).

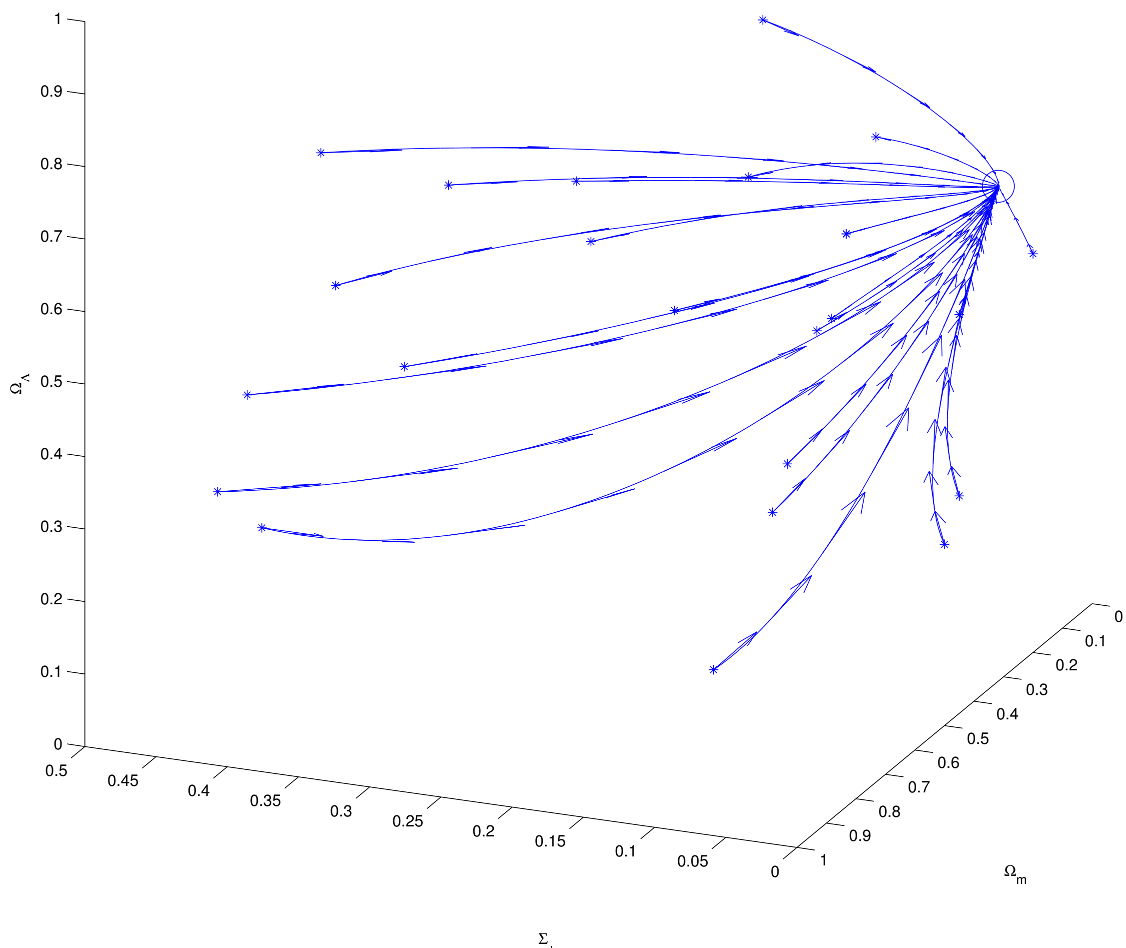


Figure 5.6: This figure shows the dynamical system behaviour for $\xi_0 = 0$, and $w = -1, -1/3, 0, 1/3, 1$. One can see that in each case, $\mathcal{K}$ is a source of the dynamical system. The case $w = -1/3$ clearly corresponds to our analysis at the end of Section IV. Note that the boundary of circles corresponds to the Kasner quarter-circle.

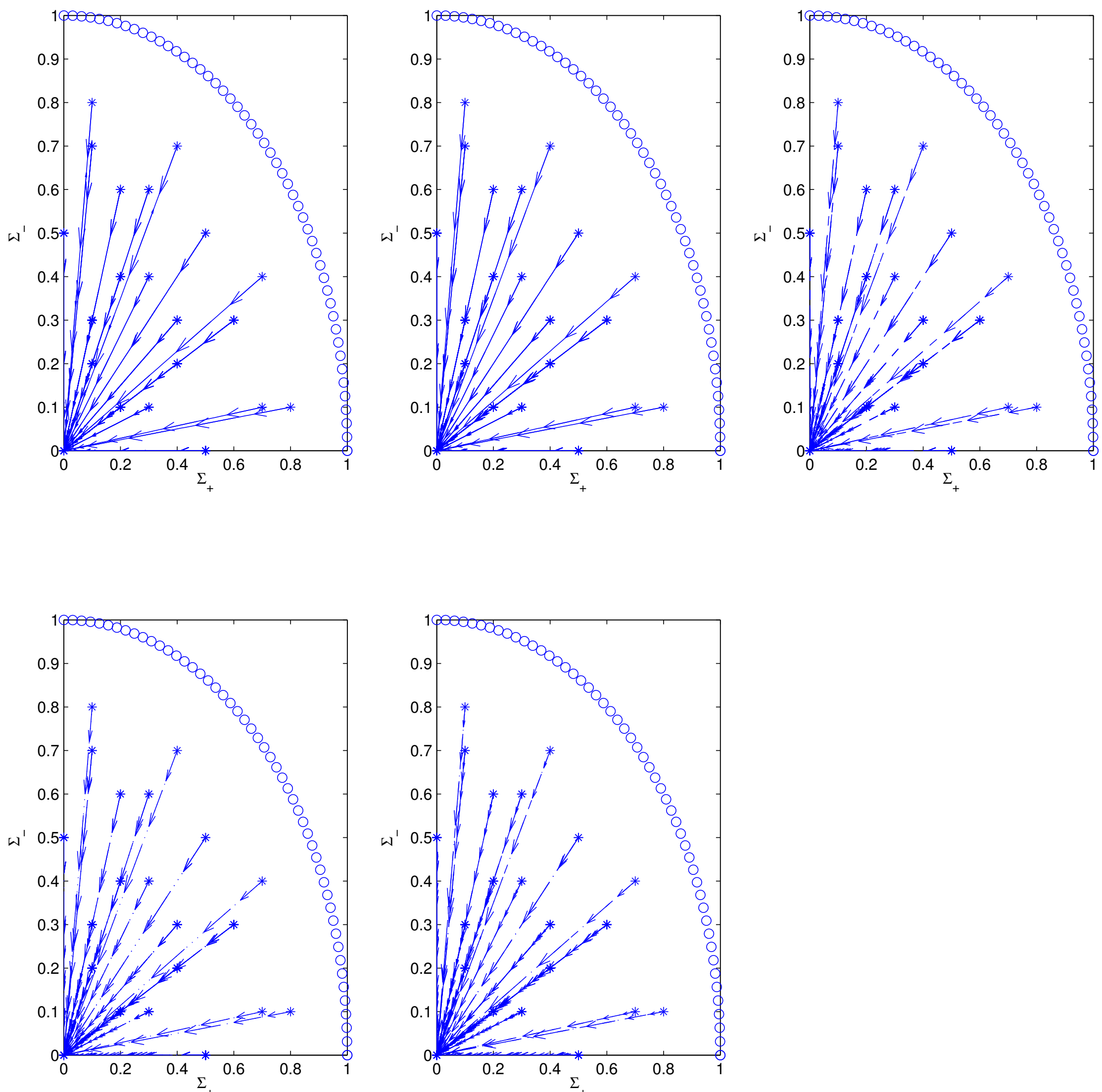


Figure 5.7: This figure shows the dynamical system behaviour for $\xi_0 = 1/2$ and $w = 1/3$, which denotes radiation. The circle denotes the equilibrium point $\mathcal{F}_1$. Clearly this point is a local sink of the dynamical system.

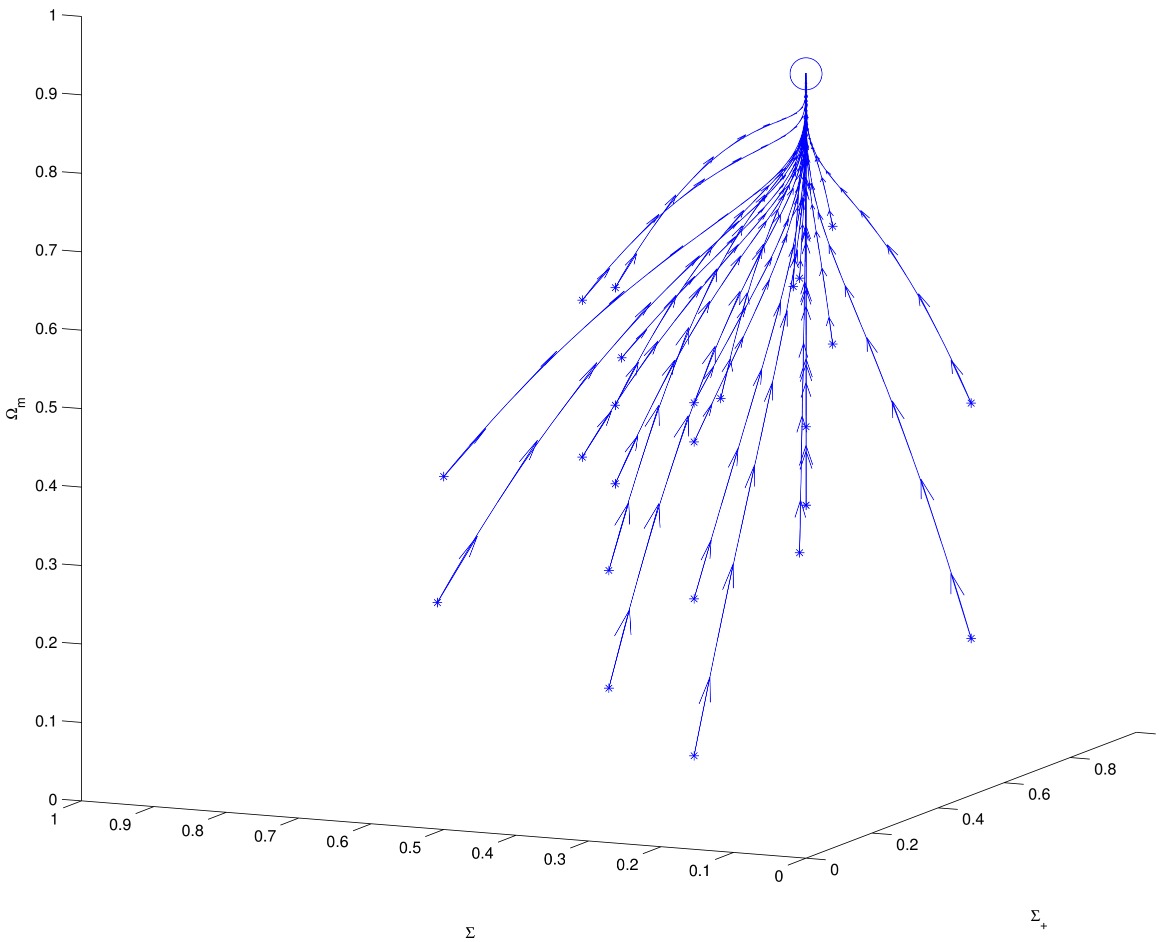


Figure 5.8: This figure shows the dynamical system behaviour for $\xi_0 = 0.34$ and $w = 0$, which denotes dust. The circle denotes the equilibrium point $\mathcal{F}_1$. Clearly this point is a local sink of the dynamical system.

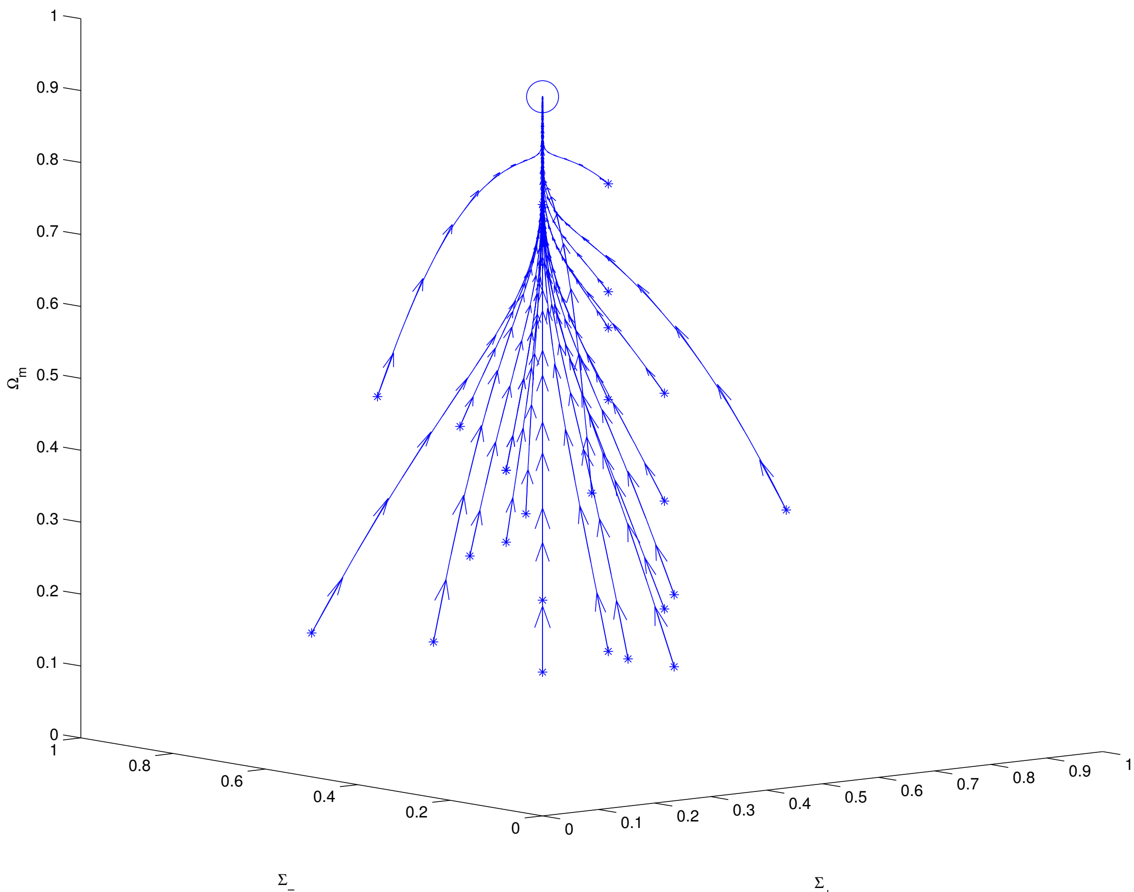
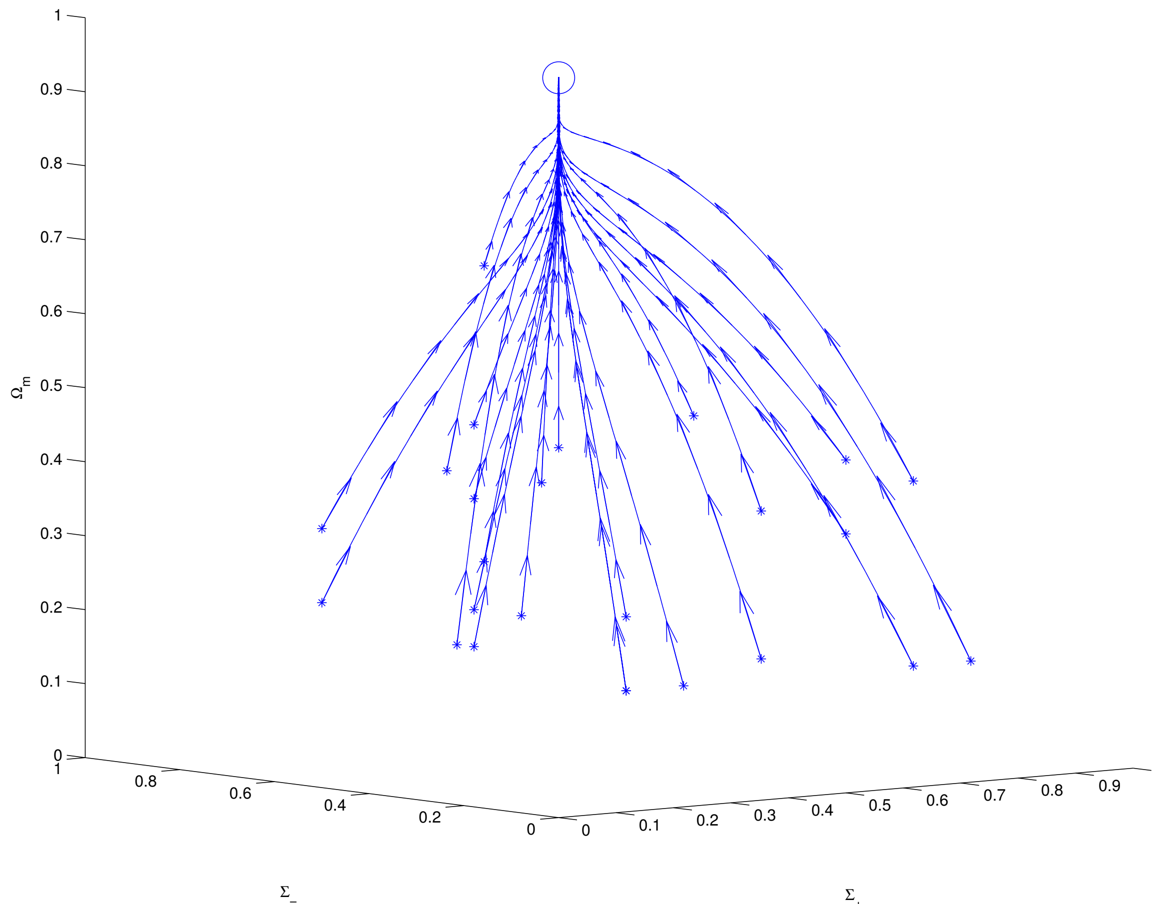


Figure 5.9: This figure shows the dynamical system behaviour for $\xi_0 = 0.45$ and $w = 0.325$, which denotes a dust-radiation mixture. The circle denotes the equilibrium point $\mathcal{F}_1$. Clearly this point is a local sink of the dynamical system.



5.8 Conclusions

We have presented in this chapter a comprehensive analysis of the dynamical behaviour of a Bianchi Type I two-fluid model with bulk viscosity and a cosmological constant. We began by completing a detailed fixed-point analysis of the system which gave information about the local sinks, sources and saddles. We then proceeded to analyze the global features of the dynamical system by using topological methods such as finding Lyapunov and Chetaev functions, and finding the α- and ω-limit sets using the LaSalle invariance principle.

The fixed points found were a flat FLRW universe with no vacuum energy and only energy due to ordinary matter, a de Sitter universe, a mixed FLRW universe with both vacuum and non-vacuum energy, and a Kasner quarter-circle universe. We found conditions for which the former three were local sinks of the system, that is, future asymptotic states, and where the latter was a source of the system, that is, a past asymptotic state.

The flat FLRW universe solution we found with both vacuum and non-vacuum energy is clearly of primary importance with respect to modelling the present-day universe, especially in light of the recently-released Planck data. In fact, using this Planck data we gave possible conditions for which a non-zero bulk viscosity in the early universe could have led to some of the conditions described in the Planck data in the present epoch. In particular, since we found that this equilibrium point is a local sink of the dynamical system, all orbits approach this equilibrium point in the future. Therefore, there exists a time period for which our cosmological model will isotropize and be compatible with present-day observations of a high degree of isotropy of the cosmic microwave background in addition to the existence of both vacuum and non-vacuum energy.

Are you ready to begin?

Henri Ducard - Batman Begins

6

On The Distribution of Prime Numbers

This chapter has no direct connection with any of the topics covered in this dissertation, although, a possible connection would certainly be very interesting! The purpose of this chapter is to complete a full circle and briefly write about the topic that got me interested in mathematics and physics. Throughout the majority of my elementary and high school education, mathematics and science were my *least* favourite subjects.There are many reasons for this, quality of the teachers, applicability of the material, and so on. However, while in the eleventh grade, and on a spare period in the library, I came across a lecture by Jeffrey Vaaler, a mathematics professor at the University of Texas on the distribution of prime numbers [Vaa]. I do not recall exactly why I watched this lecture, but I suppose it had something to do with wanting to learn more about the Riemann hypothesis after seeing the movie, A Beautiful Mind. Watching this lecture basically changed everything. In this concluding chapter, I will try to describe some of the topics I came across which I found extremely interesting. Much of this exposition is based on the details in [Wei].

One of the most beautiful results to come out of analytic number theory is the prime number theorem, which gives one the number of primes less than some $n \in \mathbb{Z}$. Legendre's approximation was to take

$$\pi(n) \approx \frac{n}{\ln n + B}, \tag{6.1}$$

where $\pi(n)$ is known as the prime counting function, and $B = -1.08366$, also known as Legendre's constant.

Alternatively, Gauss proposed that

$$\pi(n) \approx \int_2^n \frac{dx}{\ln x}. \tag{6.2}$$

The most important contributions arguably were made by Riemann, where he estimated that

$$\pi(n) \approx Li(n) - \frac{1}{2} Li(n^{1/2}), \tag{6.3}$$

where $Li(x)$ is the logarithmic integral, and is defined as

$$Li(x) = \int_2^x \frac{dx}{\ln x}. \tag{6.4}$$

The famous Riemann hypothesis is equivalent to the assertion that

$$|Li(x) - \pi(x)| \leq c\sqrt{x} \ln x, \tag{6.5}$$

where $c \in \mathbb{R}$. Following [Rie] and [Bom], we note that these conjectures are very interesting because the distribution of prime numbers among the natural numbers does not follow any regular pattern. Riemann was able to show that the frequency of prime numbers is closely related to the Riemann zeta function. Riemann's hypothesis is that the zeros of the zeta function lie on certain vertical straight line. More precisely, the Riemann zeta function for $s \in \mathbb{C}$ is defined for $\mathbb{R}(s) > 1$ as

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{6.6}$$

Riemann was able to show that $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{6.7}$$

Riemann in his very important 1859 memoir was able to obtain an analytic formula for the number of primes up to some preassigned limit, which was given as the zeros of the zeta function, that is, solutions $\rho \in \mathbb{C}$ of the equation

$$\zeta(\rho) = 0. \tag{6.8}$$

In this paper, Riemann introduced the following function

$$\xi(t) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{6.9}$$

where $s, t \in \mathbb{C}$, and $s = \frac{1}{2} + it$, and showed that all zeros of $\xi(t)$ have imaginary part between $-i/2$ and $i/2$. Riemann conjectured without proving that between 0 and T, the function $\xi(t)$ has $(T/2\pi)\log(T/2\pi) - T/2\pi$ zeros. Riemann then continues to conjecture that all zeros of the function $\xi(t)$ are *real*, and this is known as the Riemann hypothesis.

This to me was utterly remarkable. I did not understand the full details at the time, and I am by no means an expert in number theory, but the fact that someone could obtain an analytic formula that could shed light on the distribution of prime numbers, more precisely, their frequency amongst the natural numbers, placed me on a path to learn about all the mathematics I could. I had a deep fascination about Riemann, which eventually led me to his work on geometry, and the concept of a *Riemannian manifold*, which then propelled me straight towards learning about General Relativity. I can say with confidence that if it was not for this brief exposition on the distribution of prime numbers so many years ago, I would not be in the position I am today, and am thus very grateful for having come across these topics at an earlier age.

7

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8

Appendix

8.1 A Note on Constants and Units

As we stated in the introduction of this dissertation, we use geometrized units, which means we set $G = c = 1$. The implication of this is that all dynamical quantities are functions of length alone [Wal84], [EMM12]. Following [EMM12], we note that in units with the speed of light set to unity, length and time, mass and energy, and energy density and pressure, have the same dimensions:

$$c = 1 \Rightarrow [length] = [time] = L, \quad [mass] = [energy] = M, \quad [\mu] = [p] = ML^{-3}. \tag{8.1}$$

The Newton's constant has dimensions

$$[G] = LM^{-1} = 1. \tag{8.2}$$

The implication of this is that in the orthonormal frame formalism, we have that the line element, ds^2 has units of L^2, while η_{ab}, δ^a_b, u^a, and h_{ab} all have units of L^0, while $\mathbf{e}_a$, ∇_a, Γ^a_{bc}, γ^a_{bc}, $u_{b;a}$, $\dot{u}^a$, θ, σ_{ab}, and ω^{ab} have units of L^{-1}, and R_{abcd}, R_{ab}, R, T_{ab}, Λ, and μ all have units of L^{-2}.

8.2 Glossary of Terms

As requested, this section has been added to clarify some of the acronyms and cosmology terms used in this dissertation. These definitions were taken from Professor John F. Hawley's webpage [Haw], titled, *Cosmology Key Terms*.

- *Anthropic Principle:* The observation that, since we exist, the conditions of the universe must be such as to permit life to exist. This is the weak form of the anthropic principle.

  The so-called Strong Anthropic Principle holds that the universe must have those properties so that life will exist. In other words, life is a requirement for the universe.

- *Arrow of Time:* The direction, apparently inviolable, of the "flow" of time that distinguishes the past from the future.

  What is the physical origin of of the arrow of time? Most of physics does not distinguish between a forward and a reverse direction of time. The one exception is the second law of thermodynamics which states that entropy must increase with time. Perhaps the overall arrow of time in the universe is due to its beginning in a state of very low entropy.

- *Baryogenesis:* The creation of matter in excess of antimatter in the early universe. Only the relatively few unmatched matter particles survived to make up all subsequent structures.

- *Big Bang:* The state of extremely high (classically, infinite) density and temperature from which the universe began expanding. The beginning point of time and space for the universe.

- *Big Crunch:* The state of extremely high density and temperature into which a closed universe will recollapse in the distant future.

- *Bottom-up Structure Formation:* The idea that small structures, perhaps galaxies or even smaller substructures, form first in the universe, followed later by larger structures.

- *Causality:* The principle that a cause must precede its effect in time.

- *CBR/Cosmic Background Radiation:* The Cosmic Background Radiation (CBR) consists of relic photons left over from the very hot, early phase of the Big Bang. It now peaks in the microwave band, corresponding to blackbody radiation with a temperature of about 2.7 degrees Kelvin. The CBR is also sometimes called the Microwave Background, or the Cosmic Microwave Background (CMB).

- *Chaotic Inflation:* A highly speculative model in which many distinct universes form from different regions of a "mother" universe, with some inflating and others perhaps not. One reason for its possible appeal has to do with the Anthropic Principle. The multitude of universe resulting would have all possible values of the fundamental parameters. Given this, we would naturally find ourselves living in a baby universe that was compatible with life. This would appear to remove the need for "fine tuning" in making our universe compatible with life if it were the only universe rather than one of an infinite ensemble of universes.

- *Closed Universe:* A Friedmann model of the universe that has spherical geometry (hence is finite in space) and which will eventually stop expanding and recollapse (and hence is finite in time as well).

- *Cold Dark Matter:* Cold dark matter (CDM) refers to an exotic particle whose energy is low (hence cold) at the time it decouples from the ordinary (baryonic) matter. In a cold dark matter cosmological model of structure formation, the CDM is primarily responsible for structure formation. CDM cosmologies produce a Bottom-Up hierarchy of structure formation.

- *Comoving Coordinates:* A system of coordinates fixed with respect to the overall Hubble flow of the universe, so that a given galaxy's location in Comoving coordinates does not change as the Universe expands. This allows distances, locations, etc. in an expanding homogeneous and isotropic cosmology to be related solely in terms of the scale factor.

- *Coordinates:* Quantities which provide references for locations in space and time. A typical coordinate system consists of a point of reference (the origin), a set of directions (axes) that span space, and a set of labels that indicate how points are related to the origin. Coordinates in and of themselves are user defined and arbitrary, although certain simple, regular coordinate systems (e.g. Cartesian coordinates) are widely used.

  A Coordinate Singularity is a location at which a particular coordinate system fails, such as the Schwarzschild metric coordinates at the Schwarzschild radius of a black hole, or lines of longitude at the North pole. This failure doesn't indicate a breakdown in the underlying geometry. It is merely a failure of the coordinate system to give a unique well-defined label to a point in that geometry.

- *Cosmic Strings:* Long, stringlike concentrations of matter-energy that may have formed during symmetry breaking in the first moments of the big bang. If they exist, they would be candidates for the seed perturbations of structure formation.

- *Cosmic Time:* A time coordinate which can be defined for all frames in a homogeneous metric, representing the proper time of observers at rest with respect to the Hubble flow. In a Big Bang model, this coordinate marks the time elapsed since the initial singularity.

- *Cosmological Constant:* A constant introduced into Einstein's field equations of general relativity in order to provide a supplement to gravity. If positive (repulsive), it counteracts gravity, while if negative (attractive), it augments gravity. It can be interpreted physically as an energy density associated with space itself (see Vacuum Energy Density).

- *Cosmological Principle:* The principle that there is no center to the universe, that the universe is the same in all directions (that is, Isotropic) and the same everywhere (that is, homogeneous), when considered on the largest scales. This principle means that what we observe of the universe from our specific location will be representative of the true nature of the universe.

- *Critical Density:* The mass density of the universe which just stops the expansion of space, after infinite cosmic time has elapsed. The critical density is the boundary value between universe models that expand forever (open models) and those that recollapse (closed models). A measurement of the actual density of the universe could be compared to the critical density which would then, in principle, indicate the fate of the cosmos. The ratio of the actual density of the universe to the critical density is the cosmological parameter Omega. Closed Friedmann models have Omega greater than one, open Friedmann models have Omega less than one.

- *Curvature Constant:* A constant k appearing in the Robertson-Walker metric which determines the curvature of the spatial geometry of the universe. The three standard Friedmann models have

  - $k > 1$ for positive curvature (spherical geometry)
  - $k < 1$ for negative curvature (hyperbolic geometry)
  - $k = 1$ for zero curvature (flat geometry)

- *Dark Matter:* Term used to describe any astronomical mass that does not produce significant light and hence is hard to observe. Examples of dark matter include planets, black holes, white dwarfs (because they are low luminosity) and more exotic things like weakly interacting particles (WIMPs).

- *Deceleration Parameter:* A parameter q which denotes the rate of change with time of the Hubble constant.

- *Density Parameter:* The density parameter Ω is the ratio of the actual density of the universe to the critical density. A value greater than one indicates that the universe is denser than the critical value and this corresponds to a closed universe. A value less than one is an open universe.

- *de Sitter Model:* A model of the universe which contains no matter, but only a positive cosmological constant. It expands at an exponential rate forever, with no initial big bang, nor with a final big crunch.

- *Einstein-de Sitter Model:* The flat (curvature constant $k = 0$), pressureless standard Friedmann model of the universe.

- *Equivalence Principle:* The complete equality of gravitational and inertial mass, gravity and acceleration, and the identification of freefalling frames with inertial frames. The Equivalence Principle is the fundamental basis for the theory of general relativity.

  The Weak (or Newtonian) Equivalence Principle is the principle that the laws of mechanics are the same in inertial and freefalling frames of reference. This implies that gravitational mass and inertial mass are equivalent.

  The Strong (or Einstein) Equivalence Principle is the principle that all physical laws, not just those of mechanics, are the same in all inertial and freely falling frames of reference.

- *Euclidean Geometry:* Flat geometry based upon the geometric axioms of Euclid.

- *False Vacuum:* A metastable state in which a quantum field is zero, but its corresponding vacuum energy density is not zero.

- *Flat Geometry:* Geometry in which the curvature is zero; ordinary Euclidean geometry.

- *Flat Universe:* A model whose three-dimensional spatial geometry is flat, i.e. with a zero curvature constant k.

- *Flatness Problem:* The observed fact that the geometry of the universe is very nearly flat, a very special condition, without an explanation of why it should be flat.

- *Friedmann Equation:* The equation that describes the evolution of the cosmological scale factor of the Robertson-Walker metric.

- *FLRW/ Friedmann-Lemaître-Robertson-Walker Model:* A class of cosmological models which are isotropic and homogeneous, contain a specified matter-energy density and conserve matter.

- *Geodesic:* In geometry, that path between two points/events which is an extremum in length. In some geometries, such as Euclidean, the geodesics are the shortest paths, whereas in others, such as in the spacetime geometries appropriate to general relativity, the geodesics are the longest paths.

- *Homogeneity:* The property of a geometry in that all points in space are equivalent.

- *Hubble Constant:* The constant of proportionality (designated H) between recession velocity and distance in the Hubble law. It is a constant of proportionality but not a constant in time, because it can change over the history of the universe. Measuring the Hubble constant is difficult and remains and important task for astronomers. Present best values lie between approximately 50 km/sec/Mpc and 100 km/sec/Mpc, with a value around 70 km/sec/Mpc favoured.

- *Hubble Flow:* The separation of galaxies due to the expansion of space, not to their individual gravitational interactions.

- *Hubble Law:* A linear relationship between the distance to a galaxy R and the velocity with which that galaxy is receeding from us v due to the overall expansion of the universe.The present "best" value of the Hubble constant is about 70 kilometers per second per Megaparsec.

- *Hubble Length:* The distance r_H traveled by light along a straight geodesic in one Hubble time, $r_H = ct_H$.

- *Hubble Sphere:* A sphere centered about any arbitrary point whose radius is the Hubble length. The center of the Hubble sphere is not a "center" to the universe, because each point has its own Hubble sphere. The Hubble sphere approximately defines that portion of the universe which is observable from the specified point at a specified time.

- *Hubble Time:* The inverse of the Hubble constant. The Hubble time, also called the Hubble age or the

Hubble period, provides an estimate for the age of the universe by presuming that the universe has always expanded at the same rate as it is expanding today.

- *Hyperbolic Geometry:* A geometry which has negative constant curvature. Hyperbolic geometries cannot be fully visualized, because a two-dimensional hyperbolic geometry cannot be embedded in the three-dimensional Euclidean space. However, the lowest point of a saddle, that point at which curvature goes both "uphill" and "downhill," provides a local representation.

- *Inflation:* A period of exponential increase in the scale factor due to a nonzero vacuum energy density, that occurs early in the history of the universe in certain cosmological models.

- *Inflaton:* The generic name of the unidentified particle which may be responsible for an episode of inflation in the very early universe.

- *Isotropy:* The property of a geometry of being the same in all directions.

- *Lemaîtee Model:* The cosmological model developed by Georges Lemaitre, which contains a positive cosmological constant, uniform matter density, and spherical spatial geometry.

- *Metric Equation:* The expression which describes how to compute the distance between two infinitesimally-separated points (or events) in a given geometry. Also called simply the metric.

  Metric coefficients are the functions in the metric that multiply with the coordinate differentials (e.g., the change in x) to convert these differentials into physical distances.

- *Minkowskian Spacetime:* The geometrically flat, four-dimensional spacetime appropriate to special relativity.

- *Open Universe:* A Friedmann universe which expands forever and is infinite in space and time, although it begins with a Big Bang. Sometimes applied strictly to the hyperbolic Friedmann model, though both the hyperbolic and flat models are open in the sense of expanding forever. The total mass density of the universe is too small to cause recollapse.

- *Particle Horizon:* A surface beyond which we cannot see because the light from farther objects has not had time to reach us over the age of the universe.

- *Perfect Cosmological Principle:* The principle that the universe is unchanging, homogeneous and isotropic in time as well as in space. Refuted by the direct observation that the oldest objects in the universe are not like those in our immediate surroundings.

- *Scale Factor:* The quantity in the Robertson-Walker metric which describes how the distances (scales) change in an expanding or contracting universe.

- *Singularity:* In classical general relativity, a location at which physical quantities such as density become infinite. Another definition is a point in spacetime where timelike worldlines end (or begin). Singularities can be initial singularities (such as the big bang itself) or ending singularities, such as at the center of a black hole, or the big crunch.

- *Top-Down Structure Formation:* The formation of large structures, such as galaxy superclusters or perhaps even the vast filaments and voids, prior to the formation of smaller structures such as individual galaxies.

- *Vacuum Energy Density:* The amount of energy per unit volume associated with empty space itself.

  Although the idea of empty space having a nonzero energy associated with it seems a strange one, this idea is at the root of the cosmological constant and inflationary cosmologies.