

FROM THE PP-GRAPHICS TO THE FINITENESS PART OF HILBERT'S 16TH PROBLEM FOR QUADRATIC SYSTEMS

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This is part of the effort, the program launched by Dumortier, Roussarie and Rousseau, in proving the finiteness part of Hilbert's 16th problem. In this paper, we highlight the ideas of proving the finite cyclicity of pp-graphics in quadratic systems.

1. Introduction

In 1994, Dumortier, Roussarie and Rousseau ⁴ launched a program aiming to prove the finiteness part of Hilbert's 16th problem for quadratic system: **The finiteness part of Hilbert's 16th problem for quadratic vector fields:** There exists a natural number N such that any quadratic system $P_2(x, y) \frac{\partial}{\partial x} + Q_2(x, y) \frac{\partial}{\partial y}$, with $P_2(x, y)$ and $Q_2(x, y)$ are real quadratic polynomials, has at most N limit cycles, or written as $H(2) < \infty$.

The program consists in proving the 121 graphics listed in ⁴ to have finite cyclicity among quadratic systems. It has been progressing well since the 1991 when the program first started. Several papers have permitted to prove the finite cyclicity of nearly all elementary graphics ^{1,2,5}. The latest development of the program is summarized by Rousseau in ¹⁰. Nilpotent graphics are among the open cases.

There are two types of nilpotent graphics for quadratic systems: elliptic and saddle type. The nilpotent graphics and related limit periodic sets were left unknown in ⁹. The ideas in ⁷ for dealing with the cuspidal loop have been refined by Zhu and Rousseau ¹² and extended to prove the finite cyclicity of several graphics of codimension 3 or 4 passing through a nilpotent point of saddle or elliptic type for any analytic vector fields. A machinery was built

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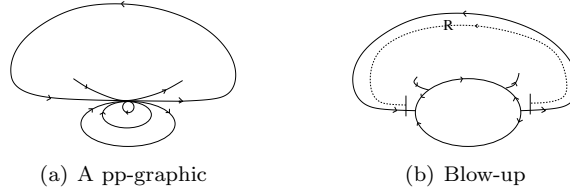


Figure 1. pp-graphics with a nilpotent elliptic point having finite cyclicity

which can be applied to study the cyclicity of other graphics. Typically, there are three types of elliptic graphics: PP, HP and HH. The following theorem was proved in ¹².

Theorem 1.1. *A pp-graphic with a triple nilpotent elliptic point (Epp) of any codimension with 2 parabolic and 2 hyperbolic sectors (Fig. 1) has cyclicity $\leq n$ ($\text{Cycl}(Epp) \leq n$) if the regular transition map R calculated using normalizing coordinates has its n -th derivative non-vanishing.*

To apply the above theorem, one always needs to check the hypothesis on nonlinearity. By using the theorem, for all the pp-graphics of quadratic systems, the following theorem was proved in ¹¹, (I_{10a}^1) was also proved in ³:

Theorem 1.2. *All the 16 pp-graphics of the quadratic systems have finite cyclicity.*

In this survey paper, we will use two typical examples of PP-graphics to explore the ideas and techniques in proving the finite cyclicity of nilpotent graphics for quadratic systems.

2. Normal forms of quadratic systems with PP-graphics and Global blow-up

Theorem 2.1. *Any quadratic system with a graphic of the form (H_6^1) , (F_{6a}^1) , (H_7^3) and (I_{17a}^2) (Fig. 2) is affine equivalent to*

$$\begin{cases} \dot{x} = y + ax^2 + cxy - y^2 \\ \dot{y} = xy. \end{cases}, \quad a \in (0, \frac{1}{2}) \quad (1)$$

with $0 < c < 2\sqrt{1-a}$ for the first two cases and $c = 2\sqrt{1-a}$ for the last two cases.

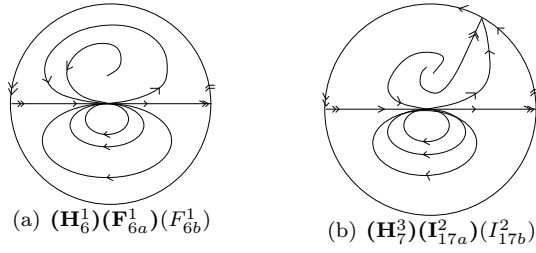


Figure 2. 4 pp-graphics in quadratic vector fields

Table 1. Limit periodic sets of pp-type for elliptic graphics

graphic Epp1	graphic Epp2	graphic Epp3

By using the classical normal form in⁶ for a family containing a triple nilpotent singularity of elliptic type with two parabolic sectors, a new normal form was developed in¹². Applying the refined global blow-up techniques to the new normal form, we get the list of all limit periodic sets for which finite cyclicity must be proved. For any pp-graphic, there three possible passages in the blown-up neighborhood of the elliptic point, Table. 1. To verify theorem 1.1 is true for any of the three limit periodic sets, as shown in Fig. 3, the ingredient is that the transition map from Σ_1 to Σ_2 is “almost linear”, so some nonlinearity is needed to conclude the finite cyclicity.

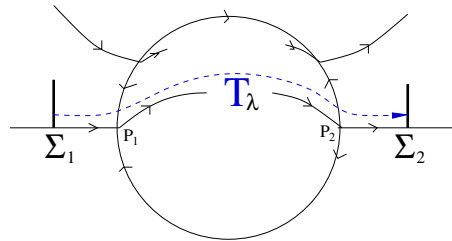


Figure 3. The transition map along the pp-passage: “funnelling effect”.

Denote T_λ the map along the passage from P_1 to P_2 (in fact its second component). Let V_i be the subset of parameters in which the pp limit periodic set Eppi exists ($i = 1, 2, 3$). Then if $\lambda \in V_3$ (resp. $\lambda \in V_2$) all the derivatives of T_λ (resp. T_λ^{-1}) are sufficiently small, while for EPP1 with $\lambda \in V_1$, $T_\lambda(y_1)$ is C^k and moderate, has the funnelling effect:

Theorem 2.2. ¹² *There exists $\varepsilon_0 > 0$ such that $\forall k \in \mathbb{N}$, $a_0 \in (0, \frac{1}{2})$, $\exists A_0 \subset (0, \frac{1}{2})$, a neighborhood of a_0 such that $\forall (a, \bar{\mu}) \in A_0 \times V_1$ and $\|\lambda\|$ sufficiently small, T_λ is C^k , and*

$$T_\lambda(y) = \gamma_0(\lambda) + \gamma_1(\lambda)y + h_\lambda(y) \quad (2)$$

with $h_\lambda(y) = o(y)$ and $h_\lambda(y) = O(\nu)$

3. Finite cyclicity of the hemicycles (H_6^1) and (H_7^3) Analytic extension principle and its application

The hemicycle (H_6^1) , Fig. 2(a), is a graphic with a typical pp-connection and two extra saddle points at infinity, .

Theorem 3.1. *The hemicycle (H_6^1) has finite cyclicity.*

Proof. The (H_6^1) occurs in (1) with $a \in (0, \frac{1}{2})$ and $0 < c < 2\sqrt{1-a}$. Take sections Σ_1 and Σ_2 in the normal form coordinates at the entrance and exit parabolic sectors in the neighborhood of P_1 and P_2 respectively, Fig. 4(a). Let $R_\lambda : \Sigma_2 \rightarrow \Sigma_1$ be the transition map shown in Fig. 4(a). Then the cyclicity of (H_6^1) is determined by the number of roots of the map $L_\lambda := R_\lambda - T_\lambda(y)$. By Theorem 2.2, $T_\lambda(y)$ is almost a affine map, hence we expect the nonlinearity of R will give the finite cyclicity.

As shown in Fig. 4(a), discompose R_λ as $R_\lambda = R_1 \circ D_l \circ R_0 \circ D_r \circ R_2$, here D_l and D_r are the Dulac maps in the normal form coordinates in the neighborhood of the saddle in the infinity, R_1 and R_2 are regular maps along x -axis, and $R_0 : \Pi_r \rightarrow \Pi_l$ is regular along the equator in the normal form coordinates on Π_r and Π_l . It follows from a long and very technical calculation (details in ¹¹), that

$$R_\lambda(y) = \beta_1 y + \beta_2 y^{1+\sigma_r} + o(y^{1+\sigma_r}) \quad (3)$$

where $\sigma_r = \frac{a}{1-a} < 1$, the hyperbolicity of the saddle point at positive infinity, $\beta_1 > 0$ $\beta_2 \neq 0$.

The comparison of R_λ and $T_\lambda(y)$ gives that $\frac{\partial^2}{\partial y^2} L_\lambda(0) \neq 0$, by which we get $Cycl(H_6^1) \leq 2$. \square

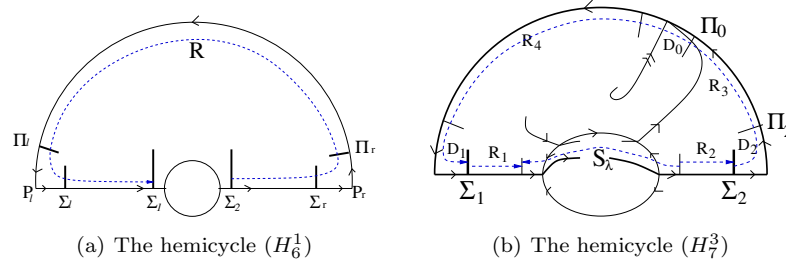


Figure 4. The transition maps of hemicycle in proving their finite cyclicity

The proof of the cyclicity of (H_6^1) not only relies on the understanding of the transition near the elliptic point, but depends in particular on the fact that the equator is invariant which leads to the calculation of R . While for any graphic in the family (H_{6a}^1) , this becomes impossible since we do not have the invariance and the expression of the regular orbit. As such, the following analytic extension principle becomes crucial.

Theorem 3.2. Analytic extension principle *Using Poincaré theorem stating that it is possible to bring a node to normal form via an analytic change of coordinates, it is possible to choose normalizing coordinates which are analytic in the coordinate plane where the singular point has a node behavior. The section is then analytic and parameterized by an analytic coordinate inside that plane.*

Using the sectorial normalizing theorem for a saddle-node it is possible to show that there exist a normalizing change of coordinates which is analytic in the node sector for the zero value of the parameters. Then the section parallel to the stable (unstable) manifold in the node sector is analytic for the zero value of the parameters and parameterized by an analytic coordinate (details in ³).

The transition map must be calculated in normalizing coordinates on sections parallel to the axes. If a regular transition appears for a graphic in a family of graphics and if the two sections on which it is defined are analytic and parameterized by analytic coordinates, then, if the transition is nonlinear in one point, it is nonlinear everywhere.

It is usually easy to prove the nonlinearity of the transition near one of the boundary graphics of the family. It follows immediately from Theorem 3.1 and the analytic extension principle that (F_{6a}^1) has finite cyclicity.

Theorem 3.3. *The hemicycle (H_7^3) has finite cyclicity.*

Proof. Compare to (H_6^1) , (H_7^3) has an extra saddle-node on the equator which changes the “balance” of the displacement map used for (H_6^1) . The presence of an additional saddle-node allows to conclude that Epp1 and Epp3 have cyclicity 1.

In case Epp3 some nonlinearity is needed to be able to conclude to cyclicity 3. As shown in Fig. 4(b), we consider $V_\lambda(y) = R_\lambda(y) - R_1^{-1} \circ S_\lambda \circ R_2^{-1}(y)$, where S_λ is the inverse of T_λ defined in (2), and $R_\lambda(y) = D_1 \circ R_4 \circ D_0 \circ R_3 \circ D_2$. It turns out that as one component of R , $R_3 : \Pi_2 \rightarrow \Pi_0$ in the normalizing coordinates contributes to the nonlinearity of $R_\lambda(y)$. One can verify through the standard differentiation-division technique that the number of roots of V_λ is at most one plus the number of roots of $W_\lambda(y) = a_0(\lambda) + a_1(\lambda)y^{\sigma_r(\lambda)} + O(y)$ where $a_1(\lambda) = *R_3''(0)$ and $\sigma_r(0) < 1$. Similar to the proof of the nonlinearity of R_λ in (3)¹², one can prove that $R_3''(0) \neq 0$. Hence $W'_\lambda(y)$ is large for small y and $\|\lambda\|$, yielding that (H_7^3) has cyclicity at most 2. \square

For (I_{17a}^2) , the nonlinearity follows again from the nonlinearity of R_3 since the hyperbolic saddle on the right has a hyperbolicity ratio $\sigma_r(0) < 1$, then the analytic extension principle ensures that (I_{17a}^2) has finite cyclicity.

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