A TOPOLOGICAL THEORY OF $(\mathbb{T}, V)$-CATEGORIES

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Abstract

Lawvere's notion of completeness for quantale-enriched categories has been extended to the theory of lax algebras under the name of L-completeness. In this work we introduce the corresponding morphism concept and examine its properties. We explore some important relativized topological concepts like separation, density, compactness and compactification with respect to L-complete morphisms. We show that separated L-complete morphisms belong to a factorization system. Moreover, we investigate relativized topological concepts with respect to maps that preserve L-closure which is the natural symmetrized closure for lax algebras. We provide concrete characterizations of Zariski closure and Zariski compactness for approach spaces.

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1 Introduction

The concepts of completeness, separation and injectivity has been introduced and studied in the context of $(\mathbb{T}, V)$-categories under the names L-completeness [15], L-separation and L-injectivity [36]. In this work we introduce the morphism versions of these notions, the most important of which is L-completeness for morphisms. We investigate properties of L-complete morphisms and explore relativized topological concepts like compactness, compactification, separation and denseness with respect to this class. We show that L-complete and L-separated morphisms belong to a factorization system. We provide a concrete characterization of this factorization system in the categories of ordered sets, metric spaces and topological spaces. These results has been published by the author in [53].

The other focus of our investigation is the symmetrized closure for $(\mathbb{T}, V)$-categories called the L-closure [36]. We give a novel analysis of L-closure as a combination of the natural closure and the dual closure. We show that the topology induced by the L-closure is the join of the topologies induced by these closure operators. We explore relativized topological concepts like compactness, separation,denseness, openness with respect to morphisms that preserve L-closure. Our investigations lead to concrete characterizations of the Zariski closure and the Zariski compactness for approach spaces [26].

1.1 $(\mathbb{T}, V)$-categories

$(\mathbb{T}, V)$-categories arise from the marriage of two lines of research, one of which is enriched category theory [41]. A category X can be described in terms of its hom-sets. The specification of identity morphisms and the composition law are the mappings

$$\{\star\} \to hom(x,x) \qquad \& \qquad hom(x,y) \times hom(y,z) \to hom(x,z)$$

subject to associativity and unity axioms. An enriched category generalizes this idea by replacing hom-sets with objects from a general monoidal category. An important example comes from Lawvere's 1973 paper [42] where he describes (pre)metric spaces as enriched categories over the nonnegative extended real numbers $\mathrm{P}_+ = [0, \infty]$. Lawvere interprets a metric $d : X \times X \to \mathrm{P}_+$ as a hom-functor where P_+ is a complete, symmetric, monoidal closed category with arrows given by $\geq$, tensor given by $+$. So a P_+-category is nothing but a pair (X, d) that satisfies

$$0 \geq d(x,x) \qquad \& \qquad d(x,y) + d(y,z) \geq d(x,z).$$

Replacing P_+ with $2 = \{\text{false} \vdash \text{true}, \wedge, \text{true}\}$, hom-objects simply affirm or deny a binary relation " $\leq$ " between elements of X which is subject to

$$\text{true} \vdash x \leq x \qquad \& \qquad x \leq y \ \wedge \ y \leq z \ \vdash \ x \leq z.$$

These conditions express reflexivity and transitivity of the binary relation $\leq$. Hence a 2-category is a (pre)ordered set. In general one can consider V-categories where $V = (V, \otimes, k)$ is a (commutative and unital) quantale. In other words, V is a complete lattice with an associative and commutative operation $\otimes : V \times V \to V$ and a unit element k where $\otimes$ preserves supremum.

The other line of research that motivates the introduction of $(\mathbb{T}, V)$-categories is the axiomatization of topological structures via convergence. The works of Hausdorff [28], Fréchet [25], Moore and Smith [46], Cartan [7],[8] and Choquet [11] have been important milestones in this regard. Probably the most remarkable result is Manes' presentation of compact Hausdorff spaces as Eilenberg-Moore algebras of the ultrafilter monad $\mathbb{U} = (U, e, m)$ [44]. Barr extended this result to topological spaces by relaxing the conditions of the Eilenberg-Moore construction [4]. He showed that a topological space X can be completely characterized by an ultrafilter convergence relation " $\to$ " satisfying

$$\dot{x} \to x \qquad \& \qquad \mathfrak{X} \to \mathfrak{x},\ \mathfrak{x} \to x \implies \sum \mathfrak{X} \to x \qquad (\dagger)$$

for all $\mathfrak{X} \in U^2X$, $\mathfrak{x} \in UX$, $x \in X$ where $\dot{x}$ is the principal ultrafilter on x and $\sum$ is the "Kowalsky diagonal operation". Using the ultrafilter monad $\mathbb{U} = (U, e, m)$ and the quantale 2, one can express this information more formally. Denote the ultrafilter convergence relation by $a : UX \times X \to 2$. Then (†) can be written as

$$\text{true} \vdash a(e_X(x), x) \qquad \& \qquad \overline{U}a(\mathfrak{X}, \mathfrak{x}) \wedge a(\mathfrak{x}, x) \vdash a(m_X(\mathfrak{X}), x)$$

where $\overline{U}$ represents Barr's extension of the ultrafilter functor U to the category of relations.

Generalizing this idea, Clementino, Hofmann and Tholen introduced the notion of a $(\mathbb{T}, V)$-category [17], [16]. It is obtained by replacing the ultrafilter monad $\mathbb{U}$ with an arbitrary **Set**-monad $\mathbb{T}$ and the quantale 2 with an arbitrary quantale V. In concrete terms, a $(\mathbb{T}, V)$-category is a pair (X, a) where $a : TX \nrightarrow X$ is a V-relation (given by a function $a : TX \times X \to V$) that satisfies

$$k \le a(e_X(x), x) \qquad \& \qquad \overline{T}a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x)$$

for all $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$. Here $\overline{T}$ represents a suitable extension of the **Set**-functor T to the category of V-relations. Using V-relational composition, these conditions are written as

$$1_X \le a.e_X \qquad \& \qquad a.\overline{T}a \le a.m_X.$$

Hence a $(\mathbb{T}, V)$-category (X, a) is a lax Eilenberg-Moore algebra in the category of sets and V-relations.

Taking $\mathbb{T}$ as the identity monad, this framework captures ordered sets and metric spaces for $V = 2$ and $V = \mathrm{P}_+$ respectively. When $\mathbb{T}$ is the ultrafilter monad, one obtains topological spaces for $V = 2$. The question is what will be obtained if one replaces 2 with P_+. The answer to this question was given by Clementino and Hofmann in their 2003 paper [14] where they gave a lax algebraic description of approach spaces [42] by using numerical convergence relations.

1.2 Summary of the work

Lawvere's 1973 paper [42] describes Cauchy completeness of metric spaces by adjoint (bi)modules. A corresponding concept for $(\mathbb{T}, V)$-categories was introduced under the name of L-completeness in [15], which was followed by the development of the concepts of L-separation, L-density and L-closure [36]. In this context, to a large extent, L-completeness behaves similarly to compactness. To give a couple of examples, L-completeness is inherited by the L-closed subsets; secondly, for any subset of an L-separated $(\mathbb{T}, V)$-category L-completeness implies L-closedness. In topology the morphism notion for compactness leads to proper maps. Inspired by the interplay between compactness and L-completeness at the level of objects, we introduce a morphism notion for L-completeness which will be the counterpart of proper maps in this context. To establish the analogy between compactness and L-completeness further and rather rigourously we choose to explore topological concepts for $(\mathbb{T}, V)$-categories using this class of maps.

Early instances of the development of topological concepts in a category appear in [47],[45],[31]. More recently, as presented in [13], given a finitely complete category equipped with a proper factorization system, one can pursue topological notions in that category by using a distinguished class of "closed morphisms". In fact many of these notions can be expressed by using "proper maps" which are stably "closed" [37]. Having developed an analogue of proper maps in the context of completeness, we put L-complete morphisms to work in a topological framework. Our investigation reveals that the topological concepts, like separation and density can be recovered by L-complete maps, while compactness and compactification naturally translate into L-completeness and L-completion. For example, it is known that any continuous map of topological spaces with compact domain and Hausdorff codomain is proper. For $(\mathbb{T}, V)$-categories any morphism with an L-complete domain and an L-separated codomain is L-complete. Likewise, the (Antiperfect, Perfect) factorization of continuous maps of Tychonoff spaces [30], [54], [12], [55] is obtained with the help of the left adjoint Stone-Čech compactification functor. Here "antiperfect maps" are the maps which are sent to isomorphisms by the compactification functor. Replacing the notion of compactification by L-completion, we obtain a similar factorization system for $(\mathbb{T}, V)$-categories, where perfect maps are replaced by L-complete and L-separated maps. Instead of the antiperfect maps we now have the morphisms which are sent to isomorphisms by the left adjoint L-completion functor.

Having L-closure at hand, we also develop topological concepts via the maps that preserve L-closure, which we call L-closed. To characterize these concepts concretely, we study L-closure in detail. Under reasonable assumptions L-closure satisfies the Kuratowski closure axioms and induces a topology on $(\mathbb{T}, V)$-categories. Hence one obtains a functor L from the category of $(\mathbb{T}, V)$-categories to the category of topological spaces [36]. By breaking L-closure into two parts we obtain two closure operators called the natural closure and the dual closure. We find that the topology induced by L-closure is the join of the topologies induced by the natural closure and the dual closure in our main examples. In other words the functor L going to the category of topological spaces decomposes through the category of bitopological spaces. Furthermore, we explore the conditions under which the functor L preserves finite products. We then show that compactness of a $(\mathbb{T}, V)$-category with respect to L-closed maps can be equivalently characterized by compactness of its image under the functor L. A comparable result also holds for openness. Separation and density with respect to L-closed maps turn out to be the original notions of L-separation and L-density respectively.

Our work on L-closure and L-closed maps leads to two new results for approach spaces. In his 2006 paper [26] Giuli introduces a closure operator called Zariski closure. Furhermore, he studies compactness with respect to maps that preserve Zariski closure, called Zariski compactness. Concrete characterizations of these notions for approach spaces have been open problems stated by Giuli. Our investigation reveals that L-closure coincides with Zariski closure for approach spaces. By our findings on L-closure and compactness with respect to L-closed maps we provide concrete characterizations of Zariski closure and Zariski compactness for approach spaces.

1.3 Outline

Now we briefly describe the contents of the following chapters:

Chapter 2

This chapter provides the basic notions and results concerning $(\mathbb{T}, V)$-categories which appeared originally in [14], [17], [16], [36], [33], [15].

We start by reviewing ordered sets from a categorical perpective. We examine adjointness and completeness for ordered sets. We give the definition of a quantale and provide its main examples in our context. Other notions discussed for ordered sets are complete distributivity and its choice free version constructive complete distributivity. As ultrafilters play an important role in our examples, we give a brief summary of facts about filters and ultrafilters.

We present V-relations. These are relations $r : X \nrightarrow Y$ given by functions $r : X \times Y \to V$. Sets and V-relations (with V-relational composition) form the category V-**Rel**. We review some important notions like order and adjointness in this category.

A basic component of the theory of $(\mathbb{T}, V)$-categories is the monad $\mathbb{T} = (T, e, m)$. We give the definition of a monad with its main examples in our context. We present the Beck-Chevalley condition which will be one of the assumptions of our setting. In particular we show that the ultrafilter monad satisfies this condition.

In order to define a $(\mathbb{T}, V)$-category, one has to extend the **Set** functor T to the category V-**Rel**. This is achieved through certain assumptions on the monad $\mathbb{T}$ and the quantale V. In this regard, we adopt the framework called "strict topological theory" introduced by Hofmann [33]. We show that our main examples provide instances of strict topological theories. The extension $\overline{T}$ of T is defined following [33].

We present $(\mathbb{T}, V)$-relations and their composition rule: Kleisli convolution. A $(\mathbb{T}, V)$-relation $r : X \rightharpoonup Y$ is a V-relation $r : TX \nrightarrow Y$. $(\mathbb{T}, V)$-relations inherit the order on V-relations.

We define $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors. The category they form is called $(\mathbb{T}, V)$-**Cat**. When $\mathbb{T}$ is the identity monad one simply calls the resulting category V-**Cat**. We provide the main examples of $(\mathbb{T}, V)$-categories which are ordered sets, metric spaces, topological spaces, approach spaces. We present some important functors between $(\mathbb{T}, V)$-**Cat** and V-**Cat**, namely the adjoint pair $\mathsf{A}^\circ \dashv \mathsf{A} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ and $\mathsf{M} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$. Given a $(\mathbb{T}, V)$-category

X, its free Eilenberg-Moore algebra $|X|$ and its dual X^{op} are defined according to [15]. We show that the quantale V itself is a $(\mathbb{T}, V)$-category. Following [33], we review $\otimes$-exponentiability in $(\mathbb{T}, V)$-**Cat** as well as some basic limits.

$(\mathbb{T}, V)$-modules (also known as bimodules, profunctors or distributors in literature) play an important role for developing the notions of separation and completeness. A $(\mathbb{T}, V)$-module $\psi : (X, a) \rightsquigarrow (Y, b)$ is a $(\mathbb{T}, V)$-relation $\psi : X \nrightarrow Y$ which is compatible with the structure maps of its domain and codomain. For each $(\mathbb{T}, V)$-category (X, a), its structure map $a : X \nrightarrow X$ is a $(\mathbb{T}, V)$-module and serves as an identity morphism with respect to Kleisli convolution. $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-modules (with Kleisli convolution) form the category $(\mathbb{T}, V)$-**Mod**. $(\mathbb{T}, V)$-modules inherit the order on $(\mathbb{T}, V)$-relations making $(\mathbb{T}, V)$-**Mod** a 2-category. Hence one can consider adjointness for $(\mathbb{T}, V)$-modules. We present the lower star functor $(\_)_* : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Mod}$ and the upper star functor $(\_)^* : (\mathbb{T}, V)\text{-}\mathbf{Cat}^{\mathrm{op}} \to (\mathbb{T}, V)\text{-}\mathbf{Mod}$. We examine the notions of full faithfulness, L-density and L-equivalence for $(\mathbb{T}, V)$-functors.

There is a close relationship between $(\mathbb{T}, V)$-modules and $(\mathbb{T}, V)$-functors. Following [15], we show that a $(\mathbb{T}, V)$-relation $\psi : X \nrightarrow Y$ is a $(\mathbb{T}, V)$-module if and only if both $\psi : |X| \otimes Y \to V$ and $\psi : X^{\mathrm{op}} \otimes Y \to V$ are $(\mathbb{T}, V)$-functors. As a result, given a $(\mathbb{T}, V)$-category $X = (X, a)$, the $(\mathbb{T}, V)$-module $a : X \rightsquigarrow X$ gives rise to the Yoneda $(\mathbb{T}, V)$-functor $y : X \to V^{|X|}$. Here $V^{|X|}$ is the $(\mathbb{T}, V)$-category whose elements are $(\mathbb{T}, V)$-functors $\psi : |X| \to V$. We provide the Yoneda lemma for $(\mathbb{T}, V)$-categories in accordance with [15].

Chapter 3

This chapter provides the notions of L-completeness [15], L-separation, L-injectivity and L-closure [36], as well as their interactions. The results presented on these notions originally belong to [15] and [36]. In addition to this known material, we introduce two new closure operators on $(\mathbb{T}, V)$-categories: the natural closure and the dual closure. We show that the Zariski closure for approach spaces coincides with L-closure and provide a concrete characterization for it.

Lawvere describes Cauchy completeness for metric spaces categorically using adjoint modules [42]. Given a metric space (X, d), there is a bijective correspondance between equivalence classes of Cauchy sequences in X and pairs of adjoint modules $\varphi \dashv \psi$ between X and the singleton set. A Cauchy sequence (x_n) converges to a point x in X if and only if the corresponding pair of adjoint modules $\varphi \dashv \psi$ is representable by x as $\varphi = d(x, \_)$ and $\psi = d(\_, x)$. Hence a metric space$X$ is Cauchy complete if and only if each pair of adjoint modules between X and the singleton set is representable. Lawvere's result motivates the introduction of completeness for $(\mathbb{T}, V)$-categories under the name L-completeness where one now asks representability of adjoint $(\mathbb{T}, V)$-modules. After defining L-completeness, we examine it in our main examples.

We present tight $(\mathbb{T}, V)$-functors. These are $(\mathbb{T}, V)$-functors $\psi : |X| \to V$ where $\psi : X^{\mathrm{op}} \to V$ is a $(\mathbb{T}, V)$-functor and as a $(\mathbb{T}, V)$-module $\psi : X \rightsquigarrow E$ is a right adjoint. For any $(\mathbb{T}, V)$-category X, one denotes the collection of tight $(\mathbb{T}, V)$-functors by $\tilde{X}$ and considers it as a subcategory of $V^{|X|}$.

We define L-injectivity for $(\mathbb{T}, V)$-categories which is the generalization of the concept of an injective object in a category (or an injective module in algebra). In this context one replaces monomorphisms with L-equivalences and demands commutativity of diagrams only up to "equivalence".

We define the natural closure, the dual closure and the L-closure. We consider these closures in our main examples. When the functor T preserves finite sums, i.e $T(M \uplus N) = TM \uplus TN$, and the unit element k is $\vee$-irreducible, i.e. $k \leq u \vee w$ implies $k \leq u$ or $k \leq w$, the L-closure defines a topology on $(\mathbb{T}, V)$-categories. Hence L-closure induces a functor $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$.

We discuss the connections between L-completeness, L-closure and L-separation. As indicated earlier, L-completeness behaves similarly to compactness in this context: L-completeness is inherited by L-closed subsets and in an L-separated $(\mathbb{T}, V)$-category L-completeness implies L-closedness. Futhermore, one sees that L-completeness is equivalent to L-injectivity. We show that the full subcategory of L-complete and L-separated $(\mathbb{T}, V)$-categories, $(\mathbb{T}, V)\text{-}\mathbf{Cat}_{\mathrm{cpl}\,\&\,\mathrm{sep}}$, is a reflective subcategory of $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ where the reflection maps are the Yoneda functors $y : X \to \tilde{X}$ for each $(\mathbb{T}, V)$-category X.

Chapter 4

In this chapter we introduce the morphism notions for L-completeness, L-separation and L-injectivity.

We define L-completeness for $(\mathbb{T}, V)$-functors. Considering the lower star functor as $(\_)_* : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Mod}_l$, where $(\mathbb{T}, V)\text{-}\mathbf{Mod}_l$ is the subcategory of $(\mathbb{T}, V)$-**Mod** whose morphisms are the left adjoint $(\mathbb{T}, V)$-modules, we see that L-complete morphisms are $(\_)_*$-quasi cartesian morphisms in the sense of fibrational category theory [27]. Equivalently, L-complete morphisms are $(\_)^*$-quasi cartesian morphisms with respect to $(\_)^* : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Mod}_r$ where $(\mathbb{T}, V)\text{-}\mathbf{Mod}_r$ is the subcategory of $(\mathbb{T}, V)$-**Mod** whose morphisms are the right adjoint $(\mathbb{T}, V)$-modules.

We examine L-complete morphisms in our examples. We investigate their properties. To name a few, L-complete morphisms have cancellation properties with respect to monomorphism and L-equivalences, one can characterize them via the naturality squares induced by the Yoneda functors. Most importantly, L-complete morphisms are pullback stable.

We define L-separation and L-injectivity for $(\mathbb{T}, V)$-functors. In the language of abstract homotopy theory L-injective morphisms turn out to be the ones which have the weak right lifting property with respect to L-equivalences. We show that

L-completeness and L-injectivity are equivalent notions at the level of morphisms as well.

Chapter 5

In this chapter we examine the functor $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ induced by the L-closure. Our findings support the results of Chapter 7 where we explore functional topology with respect to L-closed morphisms.

Firstly, we consider preservation of finite products for L. Incompatibility of the product structures in $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ and $\mathbf{Top}$ poses some technical difficulties in this context. To remedy these problems, we assume that the quantale V is constructively completely distributive and the functor $A^\circ : V\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Cat}$, presented in Chapter 2, preserves finite products. These assumptions hold in our main examples and imply that L preserves finite products in these cases.

Secondly, we try to characterize the functor L for approach spaces. We examine the natural closure, the dual closure and the L-closure in our main examples. We find that the topology induced by the L-closure is the join of the topologies induced by the natural closure and the dual closure for approach spaces. Here the join of two topologies is the smallest topology that contains both topologies. Expressing this result in categorical terms, one can write $\mathsf{L} = \mathsf{J}.\mathsf{B}$ where the functor B takes an approach space to the bitopological space whose topologies are the ones induced by the natural closure and the dual closure and the functor J takes the join of two topologies of a bitopological space.

Chapter 6

As presented in [13], one can develop topological notions in a category (which is finitely complete and comes with a proper factorization system) via a distinguished class $\mathcal{F}$ of morphisms of which one thinks as "closed". This chapter provides the basics of the framework of [13] and some of its important results.

We review the notions of a proper factorization system and a subobject in a category. We give the axioms that the distinguished class $\mathcal{F}$ of morphisms has to satisfy. We present density, properness, compactness, separation, perfectness, compactifications and openness with respect to the class $\mathcal{F}$. To express the relativized nature of these notions we use $\mathcal{F}$ as a prefix.

Chapter 7

In this chapter we explore topological notions in $(\mathbb{T}, V)$-**Cat** with respect to L-closed morphisms.

We define L-closed morphisms and examine its properties. The collection of L-closed morphisms is denoted by $\mathcal{C}$. We define $\mathcal{C}$-compactness and characterize it for ordered sets and metric spaces. To characterize $\mathcal{C}$-compactness for topological spaces and approach spaces, we use the functor L induced by the L-closure. We show that a topological space or an approach space is $\mathcal{C}$-compact if and only if its image under L is a compact topological space. For topological spaces $\mathcal{C}$-compactness corresponds to b-compactness which is characterized by Salbany [51]. For approach spaces this notion coincides with Zariski compactness [26] which has not been characterized in concrete terms yet. To do this, we take advantage of the factorization $\mathsf{L} = \mathsf{J}.\mathsf{B}$ given in Chapter 5. We formulate Zariski compactness of an approach space X equivalently as 2-compactness of the bitopological space X whose topologies are induced by its natural closure and its dual closure. We provide a concrete characterization for this notion.

We examine $\mathcal{C}$-separation and $\mathcal{C}$-density. These concepts correspond to L-separation and L-density respectively. We define $\mathcal{C}$-openness. Given that the functor L preserves finite products, we show that a $(\mathbb{T}, V)$-functor is $\mathcal{C}$-open if and only if its image under the functor L is an open map. We also examine $\mathcal{C}$-discrete objects and $\mathcal{C}$-local homeomorphisms.

Chapter 8

In this chapter we explore topological notions in $(\mathbb{T}, V)$-**Cat** with respect to L-complete morphisms.

We start with a comparison between L-closed morphisms and L-complete morphisms. By providing examples we show that these are essentially different classes of morphisms with the following exception: for a fully faithful $(\mathbb{T}, V)$-functor L-completeness is equivalent to L-closedness.

The collection of L-complete morphisms is denoted by $\mathcal{L}$. We examine $\mathcal{L}$-density and $\mathcal{L}$-separation. These notions correspond to L-separation and L-density respectively. $\mathcal{L}$-compactness turns out to be L-completeness. This puts the initial analogy between compactness and L-completeness on a firm ground. By taking advantage of the findings of Chapter 6 we obtain more results on L-completeness and L-separation. We define functorial $\mathcal{L}$-compactification for $(\mathbb{T}, V)$-categories. An example of this concept is given by L-completion which corresponds to soberification for topological spaces. As analogous to the (Antiperfect, Perfect) factorization system of continuous maps of Tychonoff spaces obtained via Stone-Čech compactification, we obtain the ($\mathcal{L}$-antiperfect, $\mathcal{L}$-perfect) factorization for $(\mathbb{T}, V)$-categories via L-completion. Here $\mathcal{L}$-perfect morphisms are L-complete and L-separated morphisms, $\mathcal{L}$-antiperfect morphisms are the ones which are sent to isomorphisms by the L-completion functor. We find that $\mathcal{L}$-antiperfect morphisms are precisely L-equivalences.

2 Preliminaries

This chapter reviews basic mathematical concepts and provides the preliminaries for $(\mathbb{T}, V)$-categories which originally appearred in [14], [17], [16], [36], [33], [15]. An excellent source for more details is [50].

2.1 Ordered sets

In this section we review some basic notions and results about ordered sets. We recall the category of ordered sets and adjunctions in Subsection 2.1.1. Subsection 2.1.2 provides a categorical formulation of completeness of ordered sets as well as adjointness criteria for monotone maps. The theory of $(\mathbb{T}, V)$-categories assumes the presence of a quantale V. We review the notion of a quantale and provide its main examples in our context in Subsection 2.1.3. The last two subsections are devoted to distributivity of complete lattices. Constructive complete distributivity, which is the choice free version of complete distributivity, is explored in Subsection 2.1.4. We recall complete distributivity in Subsection 2.1.5.

2.1.1 Basic notions

A *preordered set* $(X, \leq)$ is a set X together with a reflexive and transitive relation $\leq$, i.e.

$$x \leq x \qquad \& \qquad (x \leq y,\ y \leq z \Rightarrow x \leq z)$$

for all $x, y, z \in X$. We usually omit the prefix "pre" and call a preordered set $(X, \leq)$ an *ordered set* (as in [57]). If $x \leq z$ and $z \leq x$, we write $x \simeq z$ and say that x and z are *equivalent.* X is said to be *separated* (or *antisymmetric*) if equivalent elements are equal to each other, i.e. $x \simeq z$ implies $x = z$.

Given an ordered set $(X, \leq)$ one can consider its dual $X^{\mathrm{op}} = (X, \geq)$. A map $f : (X, \leq) \to (Y, \leq)$ is called *monotone* if

$$x \leq z \quad \Longrightarrow \quad f(x) \leq f(z)$$

for all $x, z \in X$. Ordered sets and monotone maps form the category **Ord**. The hom-sets of **Ord** carry the pointwise order: given monotone maps $f, f' : X \to Y$, one has $f \leq f'$ if $f(x) \leq f'(x)$ for all $x \in X$.

Since **Ord** is an ordered category, one can consider adjunctions. A monotone map $f : X \to Y$ is called a *left adjoint* if there exists a monotone map $g : Y \to X$ such that

$$1_X \leq g.f \quad \& \quad f.g \leq 1_Y.$$

In that case g is called a *right adjoint* and the adjunction is denoted by $f \dashv g$. A map $f : X \to Y$ is a left adjoint if and only if there exists a map $g : Y \to X$ such that

$$f(x) \leq y \quad \Longleftrightarrow \quad x \leq g(y) \tag{2.1.1}$$

for all $x \in X$, $y \in Y$. The monotonicity of f and g follows from this condition.

2.1.2 Completeness

Let $(X, \leq)$ be an ordered set and $S \subseteq X$. The *down-closure* of S is the set

$$\downarrow S = \{z \mid \exists x \in S : \ z \leq x\}.$$

S is said to be *down-closed* if $\downarrow S = S$. Down-closed sets of the form $\downarrow x = \{z \mid z \leq x\}$ are called *principal.* One has the monotone map

$$\downarrow : X \to \mathrm{Dn}\, X$$

where the lattice $\mathrm{Dn}\, X$ of down-closed sets ordered by inclusion.

An ordered set X is called *complete* if $\downarrow : X \to \mathrm{Dn}\, X$ has a left adjoint. Denote the left adjoint of $\downarrow$ by $\bigvee : \mathrm{Dn}\, X \to X$. By (2.1.1), one has

$$\forall x \in X \ (\bigvee S \leq x \iff S \subseteq \downarrow x) \tag{2.1.2}$$

for any down-closed set $S \subseteq X$. Observe that $x \in X$ is an upper bound for S if and only if $S \subseteq \downarrow x$. So (2.1.2) implies that

- $\bigvee S$ is an upper bound of S,
- for any upper bound x of S, one has $\bigvee S \leq x$.

Hence $\bigvee S$ is the supremum of S up to equivalence. Given any $T \subseteq X$, one puts

$$\bigvee T := \bigvee \downarrow T,$$

since both T and $\downarrow T$ have the same upper bounds. For any $x, z \in X$, $\bigvee\{x, z\}$ is simply denoted by $x \vee z$.

As X complete if and only if X^{op} is complete, one has $\bigvee_{X^{\mathrm{op}}} \dashv \downarrow_{X^{\mathrm{op}}} : \mathrm{Dn}\, X^{\mathrm{op}} \to X^{\mathrm{op}}$. Dualizing this adjunction yields $\uparrow \dashv \bigwedge : X \to \mathrm{Up}\, X$ where $\uparrow := (\downarrow_{X^{\mathrm{op}}})^{\mathrm{op}}$, $\bigwedge := (\bigvee_{X^{\mathrm{op}}})^{\mathrm{op}}$ and $\mathrm{Up}\, X := (\mathrm{Dn}\, X^{\mathrm{op}})^{\mathrm{op}}$. The dual notion of infimum is naturally describable by this adjunction. One denotes the infimum of any $T \subseteq X$ by $\bigwedge T$. For any $x, z \in X$, $\bigwedge\{x, z\}$ is simply denoted by $x \wedge z$.

A monotone map $f : X \to Y$ is said to *preserve supremum* of $S \subseteq X$ if

$$f(\bigvee S) = \bigvee f(S)$$

whenever $\bigvee S$ exists. f is called a *sup-map* if it preserves all existing suprema in X. Dually, f is called an *inf-map* if it preserves all existing infima in X.

Any left adjoint $f : X \to Y$ is a sup-map. In case X is complete, f is a left adjoint if and only if it is a sup-map. Its right adjoint $g : Y \to X$ is given by

$$g(y) \simeq \bigvee\{x \mid f(x) \leq y\}. \tag{2.1.3}$$

Dually, given Y complete, a monotone map $g : Y \to X$ is a right adjoint if and only if it is an inf-map. Its left adjoint $f : X \to Y$ is given by

$$f(x) \simeq \bigwedge\{y \mid x \leq g(y)\}. \tag{2.1.4}$$

A complete and seperated ordered set is called a *complete lattice*. We will denote the top and the bottom elements of a complete lattice by $\top$ and $\bot$ respectively.

2.1.3 Quantales

A commutative and unital *quantale* $V = (V, \otimes, k)$ is a complete lattice with a commutative and associative operation $\otimes : V \times V \to V$ and a unit element k where tensoring preserves suprema in each variable, i.e.

$$v \otimes (\bigvee_{i \in I} u_i) = \bigvee_{i \in I} (v \otimes u_i).$$

Throughout this work V will denote a commutative and unital quantale. We will drop the adjectives commutative and unital and simply call V a quantale. Furthermore, we assume that V is nontrivial, i.e. $V \neq \{k\}$ or, equivalently, $k \neq \bot$.

Since V is complete and tensoring preserves suprema, for any $u \in V$ $u \otimes (\_)$ has a right adjoint $u \multimap (\_)$. Following (2.1.3), one has

$$u \multimap v = \bigvee \{w \mid u \otimes w \leq v\}. \tag{2.1.5}$$

Also

$$u \otimes w \leq v \iff w \leq u \multimap v$$

for any $v, w \in V$ by (2.1.1). Sometimes we also write $v \multimapinv u$ instead of $u \multimap v$.

The unit element k is called $\vee$-*irreducible* if $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$.

A *frame* is a complete lattice which satisifes the infinite distributive law:

$$v \wedge (\bigvee_{i \in I} u_i) = \bigvee_{i \in I} (v \wedge u_i).$$

So every frame is a quantale with $\otimes = \wedge$ and $k = \top$.

The main examples of quantales in our context are given below.

Examples 2.1.1. 1. Two element chain $2 = \{\text{false, true}\} = \{0,1\}$ where $\otimes = \wedge$ and $k = 1$. The binary operation $\multimap$ is the counterpart of the "implication", i.e.

$$u \multimap v = 0 \iff u = 1 \text{ and } v = 0.$$

2. Consider the extended nonnegative real numbers $[0,\infty]$ with its natural order. Reversing the order one has the complete lattice $[0,\infty]^{\mathrm{op}}$ where $0 = \top$ and $\infty = \bot$. $[0,\infty]^{\mathrm{op}}$ becomes a quantale with $k = 0$ and $\otimes = +$ where the addition is extended by $x + \infty = \infty + x = \infty$ for any $x \in [0,\infty]$. We will denote this quantale by

$$\mathrm{P}_{+} = ([0,\infty]^{\mathrm{op}}, +, 0).$$

In this context the operation $\multimap$ becomes truncated subtraction, i.e.

$$v \multimapinv u = \begin{cases} v - u & \text{if } v \geq u, \\ 0 & \textit{else}. \end{cases}$$

3. $[0,\infty]^{\mathrm{op}}$ is a frame, since it is a chain. So one can consider it as a quantale where the tensor is the binary operation meet. Since the order in $[0,\infty]^{\mathrm{op}}$ is the reversed natural order, one has $\otimes = \max$ and $k = \top = 0$. We will denote this quantale by

$$\mathrm{P}_{\max} = ([0,\infty]^{\mathrm{op}}, \max, 0).$$

One has

$$v \multimap u = \begin{cases} v & \text{if } v > u, \\ \\ 0 & \textit{else}. \end{cases}$$

2.1.4 Constructive complete distributivity

Let X be a complete lattice. Recall from section 2.1.2 that completeness of X entails the existence of a left adjoint $\bigvee : \mathrm{Dn}\, X \to X$ to $\downarrow : X \to \mathrm{Dn}\, X$.

X is called *constructively completely distributive* (ccd) [23] if $\bigvee : \mathrm{Dn}\, X \to X$ has itself a left adjoint $\Downarrow : X \to \mathrm{Dn}\, X$. This means that for any down-closed set $S \subseteq X$ and $x \in X$, one has

$$\Downarrow x \subseteq S \iff x \leq \bigvee S. \tag{2.1.6}$$

(2.1.4) implies

$$\Downarrow x = \bigcap \{S \in \mathrm{Dn}\, X \mid x \leq \bigvee S\}. \tag{2.1.7}$$

An element $u \in X$ is called *totally below* x, written $u \ll x$, if $u \in \Downarrow x$. So (2.1.7) implies that

$$u \ll x \iff \forall S \in \mathrm{Dn}\ X,\ x \leq \bigvee S \Rightarrow u \in S.$$

Letting S to be an ordinary subset of X, one obtains

$$u \ll x \iff \forall S \subseteq X,\ x \leq \bigvee S \Rightarrow u \in \downarrow S. \tag{2.1.8}$$

Remark 2.1.2. The complete lattices 2 and $[0,\infty]^{\mathrm{op}}$ are both ccd. In 2 one has $(u \ll x \iff x = 1)$, whereas in $[0,\infty]^{\mathrm{op}}$ one has $(u \ll x \iff u > x)$.

Lemma 2.1.3. Let X be a ccd lattice and u, v, x, z be arbitrary elements of X. The following assertions hold:

1. $u \ll x$ implies $u \leq x$.

2. $u \ll x \leq z$ implies $u \ll z$.

3. $u \leq v \ll x$ implies $u \ll x$.

4. $x = \bigvee\{u \mid u \ll x\}$.

5. If $u \ll x$, then there exists v such that $u \ll v \ll x$.

Proof. (1) – (3) are trivial. For (4), $\bigvee\{u \mid u \ll x\} \leq x$ follows from (1). Putting $\Downarrow x$ in place of S in (2.1.6), one gets $x \leq \bigvee \Downarrow x = \bigvee\{u \mid u \ll x\}$. For (5), let $u \ll x$. Consider

$$S = \{w \mid \exists v : \ w \ll v \ll x\} = \bigcup\{\Downarrow v \mid v \ll x\}.$$

Observe that S is down-closed. Using (4), one gets

$$\bigvee S = \bigvee\{\bigvee \Downarrow v \mid v \ll x\} = \bigvee\{v \mid v \ll x\} = x.$$

Since $u \ll x$ and $x \leq \bigvee S$, $u \in S$. Hence there exists $v \ll x$ such that $u \ll v \ll x$. $\square$

Since $\mathrm{Dn}\, X$ is complete, X is ccd if and only if $\bigvee : \mathrm{Dn}\, X \to X$ is an inf-map. This means that for any family $\mathcal{X}$ of subsets of X, one has

$$\bigvee \Big(\bigcap_{S \in \mathcal{X}} \downarrow S \Big) = \bigwedge_{S \in \mathcal{X}} \bigvee S. \tag{2.1.9}$$

Let $\mathcal{X}^* = \{ T \subseteq \bigcup \mathcal{X} \mid \forall S \in \mathcal{X},\ T \cap S \neq \varnothing \}$. As $\bigcap_{S \in \mathcal{X}} \downarrow S = \downarrow \bigwedge_{T \in \mathcal{X}^*} T$, (2.1.9) implies

$$X \text{ is ccd} \iff \bigvee \bigwedge_{T \in \mathcal{X}^*} T = \bigwedge_{S \in \mathcal{X}} \bigvee S. \tag{2.1.10}$$

2.1.5 Complete distributivity

A complete lattice X is said to be *completely distributive* (cd) if for any family $\mathcal{X}$ of subsets of X

$$\bigvee_{f \in \prod \mathcal{X}} \bigwedge_{S \in \mathcal{X}} f(S) = \bigwedge_{S \in \mathcal{X}} \bigvee S. \tag{2.1.11}$$

Comparing the left-hand side of (2.1.9) and (2.1.11), one sees that if $f \in \prod \mathcal{X}$ then $\bigwedge_{S \in \mathcal{X}} f(S) \in \bigcap_{S \in \mathcal{X}} \downarrow S$. So

$$\bigvee_{f \in \prod \mathcal{X}} \bigwedge_{S \in \mathcal{X}} f(S) \leq \bigvee \Big(\bigcap_{S \in \mathcal{X}} \downarrow S \Big).$$

On the other hand, if $x \in \bigcap_{S \in \mathcal{X}} \downarrow S$, then for each $S \in \mathcal{X}$ there exists $s \in S$ such that $x \leq s$. Invoking the Axiom of Choice, one gets $f \in \prod_{S \in \mathcal{X}} S$ where $f(S) = s$ and $x \leq \bigwedge_{S \in \mathcal{X}} f(S)$. Hence

$$\bigvee_{f \in \prod \mathcal{X}} \bigwedge_{S \in \mathcal{X}} f(S) \geq \bigvee \Big(\bigcap_{S \in \mathcal{X}} \downarrow S \Big).$$

So complete distributivity and constructive complete distributivity are equivalent notions in the presence of the Axiom of Choice.

2.2 Filters

A *proper filter* $\mathfrak{x}$ on a set X is a collection of subsets of X such that

- $X \in \mathfrak{x}$ and $\varnothing \notin \mathfrak{x}$.
- If $A \in \mathfrak{x}$ and $B \in \mathfrak{x}$, then $A \cap B \in \mathfrak{x}$.
- If $A \in \mathfrak{x}$ and $A \subseteq B$, then $B \in \mathfrak{x}$.

Throughout this work we only deal with proper filters. So we drop the adjective proper and simply call them filters.

Given a map $f : X \to Y$ and a filter $\mathfrak{x}$ on X, the *image filter* $f(\mathfrak{x})$ on Y is the filter generated by the filter basis $\{f(A) \mid A \in \mathfrak{x}\}$, i.e.

$$f(\mathfrak{x}) = \uparrow \{f(A) \mid A \in \mathfrak{x}\} = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}. \tag{2.2.1}$$

If $\mathfrak{y}$ is a filter on Y and $f(X) \cap B \neq \varnothing$ for all $B \in \mathfrak{y}$, then one can define the *inverse image filter* $f^{-1}(\mathfrak{y})$ on X. This is the filter generated by the filter basis $\{f^{-1}(B) \mid B \in \mathfrak{y}\}$:

$$f^{-1}(\mathfrak{y}) = \uparrow \{f^{-1}(B) \mid B \in \mathfrak{y}\} = \{A \subseteq X \mid \exists B \in \mathfrak{y} : f^{-1}(B) \subseteq A\}.$$

Supposing that the inverse image filter is definable, one has

$$f^{-1}(f(\mathfrak{x})) \subseteq \mathfrak{x} \qquad \& \qquad f(f^{-1}(\mathfrak{y})) \supseteq \mathfrak{y}. \tag{2.2.2}$$

Let $\mathfrak{z}$ be a filter on X and $A \in \mathfrak{z}$. The *restriction of* $\mathfrak{z}$ *to* A, denoted by $\mathfrak{z}_{|A}$, is the inverse image of $\mathfrak{z}$ with respect to the embedding $i : A \hookrightarrow X$, i.e.

$$\mathfrak{z}_{|A} := i^{-1}(\mathfrak{y}) = \{A \cap M \mid M \in \mathfrak{z}\}.$$

If $\mathfrak{x}$ is a filter on $A \subseteq X$, then one can obtain the image filter $\uparrow \mathfrak{x} := i(\mathfrak{x})$ on X. In this case (2.2.2) becomes

$$(\uparrow \mathfrak{x})_{|A} = \mathfrak{x} \qquad \& \qquad \uparrow (\mathfrak{z}_{|A}) = \mathfrak{z}.$$

Hence there is a one-to-one correspondence between the filters on $A \subseteq X$ and the filters on X which contain A. Each collection is completely determined by the other.

A filter $\mathfrak{x}$ is called an *ultrafilter* if for any filter $\mathfrak{z}$, $\mathfrak{x} \subseteq \mathfrak{z}$ implies $\mathfrak{x} = \mathfrak{z}$. Given a map $f : X \to Y$ and an ultrafilter $\mathfrak{x}$ on X, $f(\mathfrak{x})$ is an ultrafilter on Y. If $A \in \mathfrak{x}$, then $\mathfrak{x}_{|A}$ is an ultrafilter as well. There is a one-to-one correspondence between the ultrafilters on $A \subseteq X$ and the ultrafilters on X which contain A.

The dual notion of a filter is an ideal. An *ideal* $\mathfrak{i}$ on a set X is a nonempty collection of subsets of X such that

- $A, B \in \mathfrak{i}$ implies $A \cup B \in \mathfrak{i}$,
- $A \subseteq B$ and $B \in \mathfrak{i}$ imply $A \in \mathfrak{i}$.

Lemma 2.2.1. Let $\mathfrak{a}$ be a filter basis and $\mathfrak{i}$ be an ideal on X such that $A \cap I = \varnothing$ for any $A \in \mathfrak{a}$ and $I \in \mathfrak{i}$. Then there exists an ultrafilter $\mathfrak{x}$ on X such that $\mathfrak{a} \subseteq \mathfrak{x}$ and $\mathfrak{x} \cap \mathfrak{i} = \varnothing$.

Proof. The assumption implies that $A \cap (X \smallsetminus I) \neq \varnothing$ for any $A \in \mathfrak{a}$ and $I \in \mathfrak{i}$. Consider $\mathfrak{b} = \{A \cap (X \smallsetminus I) \mid A \in \mathfrak{a},\ I \in \mathfrak{i}\}$ which is a filter basis that contains $\mathfrak{a}$. Let $\mathfrak{x}$ be an ultrafilter which contains $\mathfrak{b}$. Then $\mathfrak{a} \subseteq \mathfrak{x}$ and $\mathfrak{x} \cap \mathfrak{i} = \varnothing$ holds as $X \smallsetminus I \in \mathfrak{x}$ for any $I \in \mathfrak{i}$. □

2.3 V-relations

Classically a *relation* r from a set X to a set Y is a map $r : X \times Y \to 2$. Given $x \in X$, $y \in Y$, x is said to be r-related to y, written $x\,r\,y$, if $r(x, y) = 1$. It is desirable to generalize the definition of a relation by allowing it to assume a larger array of values, not just 0 and 1. This can be achieved by replacing 2 with an arbitrary quantale V.

A *V-relation* r from a set X to a set Y, denoted by $r : X \nrightarrow Y$, is a map $r : X \times Y \to V$. V-relations compose analogous to matrix multiplication. Given $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$, the composite $s.r : X \nrightarrow Z$ is defined by

$$s.r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

for $x \in X$, $z \in Z$. The composition is associative. For each set X one has the

V-relation $1_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ where $1_X(x,w)$ equals k if $x = w$ and $\perp$ otherwise. 1_X serves as the identity V-relation with respect to the composition. Hence sets and V-relations from a category denoted by

$$V\text{-}\mathbf{Rel}.$$

The hom-sets of V-**Rel** carry the pointwise order of V, i.e. for $q, r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$

$$q \le r \iff q(x,y) \le r(x,y) \;\; \forall x \in X, y \in Y.$$

Given $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $s_i : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ for $i \in I$, one has

$$(\bigvee_{i\in I} s_i).r = \bigvee_{i\in I}(s_i.r).$$

Since $(\_).r$ is a sup-map, it has a right adjoint $(\_) \multimapdotinv r$ defined by

$$s.r \le t \iff s \le t \multimapdotinv r$$

for any $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and $t : X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$. By (2.1.3), one gets

$$t \multimapdotinv r\ (y,z) = \bigvee\{s \mid s.r \le t\} = \bigwedge_{x\in X} t(x,z) \multimapinv r(x,y)\ . \tag{2.3.1}$$

$$\begin{tikzcd} Y \arrow[dr, dashed, "t \multimapdotinv r"] & \\ X \arrow[u, "r", "\shortmid"'] \arrow[r, "t"', "\shortmid"] & Z \end{tikzcd}$$

Similarly V-relational composition from the left has a right adjoint.

Given a V-relation $r : X \nrightarrow Y$, its *opposite relation* $r^\circ : Y \nrightarrow X$ is defined by

$$r^\circ(y,x) = r(x,y).$$

Trivially $(\_)^\circ : V\text{-}\mathbf{Rel}(X,Y) \to V\text{-}\mathbf{Rel}(Y,X)$ preserves order. Furthermore, one has $1_X{}^\circ = 1_X$ and $r^{\circ\circ} = r$. It is easy to see that $(s.r)^\circ = r^\circ.s^\circ$.

There is a functor $\mathbf{Set} \to V\text{-}\mathbf{Rel}$ that takes a map $f : X \to Y$ to the V-relation $f_\circ : X \nrightarrow Y$ where

$$f_\circ(x,y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$$

Since $k \neq \bot$, this functor is faithful. So it is safe to write f instead of $f_\circ$. V-relational composition becomes easier when maps are involved. For $s : Y \nrightarrow Z$ and $p : W \nrightarrow X$ one has

$$s.f(x,z) = s(f(x),z) \qquad \& \qquad f.p(w,y) = \bigvee_{x \in f^{-1}(y)} p(w,x).$$

Also $f^\circ.q(w,x) = q(w,f(x))$ for any $q : W \nrightarrow Y$.

Since V-$\mathbf{Rel}$ is an ordered category, one can consider adjunctions. Given V-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow X$, r is said to be a *left adjoint* to s, written $r \dashv s$, if

$$r.s \leq 1_Y \qquad \& \qquad s.r \geq 1_X.$$

In such a case s is said to be a *right adjoint* to r.

Lemma 2.3.1. Let $r', r : X \nrightarrow Y$, $s', s : Y \nrightarrow X$ be V-relations such that $r \vdash s$, $r' \vdash s'$. If $r \leq r'$ and $s \leq s'$, then $r = r'$.

Proof. Suppose the assumptions hold. Then $r' \leq r'.s.r \leq r'.s'.r \leq r$. Together with the hypothesis $r \leq r'$, one obtains $r = r'$.

$\square$

For a map $f : X \to Y$, $f \dashv f^\circ$ always holds as

$$f.f^\circ \leq 1_Y \qquad \& \qquad f^\circ.f \geq 1_X.$$

One has $f.f^\circ = 1_Y$ when f is surjective and $f^\circ.f = 1_X$ when f is injective.

Examples 2.3.2. 1. **2-Rel** = **Rel** is the category of sets and relations.

2. P_+-**Rel** is the category of fuzzy relations. The degree of relatedness of two elements is given by a nonnegative real number rather than just 0 and 1. The smaller the number the more the elements are related to each other.

2.4 Monads

One of the main components of the theory of $(\mathbb{T}, V)$-categories is the monad $\mathbb{T}$. We review the notions of a monad and an Eilenberg-Moore algebra of a monad in Subsection 2.4.1. We provide examples of some **Set**-monads that will be relevant in our context. The Beck-Chevalley condition, which will be one of the assumptions of our framework, is reviewed in Subsection 2.4.2. We recall what it means for a functor, a natural transformation and a monad to satisfy the Beck-Chevalley condition. Later we demonstrate that the ultrafilter monad satisfies this condition.

2.4.1 Definitions and examples

A *monad* $\mathbb{T} = (T, e, m)$ on a category $\mathscr{C}$ consists of a functor $T : \mathscr{C} \to \mathscr{C}$ and natural transformations $e : 1_{\mathscr{C}} \to T$ (unit), $m : T^2 \to T$ (multiplication) such that

$$m.mT = m.Tm \qquad \& \qquad m.eT = 1_T = Te.m,$$

i.e. the following diagrams commute for any object X in $\mathscr{C}$.

$$\begin{tikzcd} T^3X \arrow[r, "m_{TX}"] \arrow[d, "Tm_X"'] & T^2X \arrow[d, "m_X"] \\ T^2X \arrow[r, "m_X"'] & TX \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[r, "e_{TX}"] \arrow[dr, "1_{TX}"'] & T^2X \arrow[d, "m_X"] & TX \arrow[l, "Te_X"'] \arrow[dl, "1_{TX}"] \\ & TX & \end{tikzcd}$$

Now we list some **Set**-monads that will be relevant in our context.

Examples 2.4.1. 1. The identity functor $\mathrm{Id} : \mathbf{Set} \to \mathbf{Set}$ together with the identity natural transformation $1 : \mathrm{Id} \to \mathrm{Id}$ form the identity monad $\mathbb{1} : (\mathrm{Id}, 1, 1)$.

2. The list monad $\mathbb{L} = (L, e, m)$. Here $L : \mathbf{Set} \to \mathbf{Set}$ sends a set X to the set LX of all finite lists formed by the elements of X. One has $L\varnothing = \{()\}$ which

is the empty list. Given a map $f : X \to Y$ and $(x_1, x_2, \ldots, x_n) \in LX$,

$$Lf(x_1, x_2, \ldots, x_n) = (f(x_1), f(x_2), \ldots, f(x_n)).$$

For any set X, $e_X : X \to LX$ sends $x \in X$ to the list (x). Given a finite list of finite lists on X, $m_X : L^2X \to LX$ induces a finite list by removing the brakets in between, i.e.

$$m_X\big((x_1^1, \ldots, x_{n_1}^1), \ldots, (x_1^k, \ldots, x_{n_k}^k)\big) = \big(x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^k, \ldots, x_{n_k}^k\big).$$

3. Consider the contravariant powerset functor

$$P : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$$

which sends $f : X \to Y$ to $Pf : PY \to PX$ where $Pf(N) = f^{-1}(N)$ for any $N \subseteq Y$. One has the double powerset monad

$$\mathbb{P}^2 = (P^2 = P^{\mathrm{op}}.P, e, m)$$

where

$$P^2 f(\mathfrak{x}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}$$

for any $f : X \to Y$ and $\mathfrak{x} \in P^2X$. The natural transformations $e : 1_{\mathbf{Set}} \to P^2$ and $m : P^2.P^2 \to P^2$ are given by

$$e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\} \tag{2.4.1}$$

$$m_X(\mathfrak{X}) = \{A \subseteq X \mid A^\# \in \mathfrak{X}\} \tag{2.4.2}$$

for any set X, $\mathfrak{X} \in P^2.P^2X$ and $x \in X$ where $A^\# = \{\mathfrak{x} \in P^2X \mid A \in \mathfrak{x}\}$. One can also write

$$m_X(\mathfrak{X}) = \bigcup_{\mathcal{A} \in \mathfrak{X}} \bigcap_{\mathfrak{x} \in \mathcal{A}} \mathfrak{x}.$$

By restricting the double powerset functor and its monad structure, one obtains some interesting submonads.

- The filter monad $\mathbb{F} = (F, e, m)$. F sends a set X to the set of all filters on X. For any map $f : X \to Y$ and any filter $\mathfrak{x}$ on X,

  $$Ff(\mathfrak{x}) = \{N \subseteq Y \mid f^{-1}(N) \in \mathfrak{x}\}$$

  which is the image filter $f(\mathfrak{x})$ defined in (2.2.1). The natural transformations $e : 1_{\mathsf{Set}} \to F$ and $m : F^2 \to F$ are given by (2.4.1) and (2.4.2) where one replaces P^2 by F.

- The ultrafilter monad $\mathbb{U} = (U, e, m)$. U sends a set X to the set of all ultrafilters on X. All the definitions are the same as the filter monad except that one replaces "filter" by "ultrafilter".

Given a monad $\mathbb{T} = (T, e, m)$ on a category $\mathscr{C}$, a $\mathbb{T}$-*algebra* (or an *Eilenberg-Moore algebra* of $\mathbb{T}$) is a pair $(X, \alpha : TX \to X)$ such that the following diagrams commute.

$$
\begin{tikzcd}
T^2X \arrow[r, "m_X"] \arrow[d, "T\alpha"'] & TX \arrow[d, "\alpha"] \\
TX \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "\alpha"] \\
& X
\end{tikzcd}
$$

Given $\mathbb{T}$-algebras (X, α) and (Y, β), a morphism $f : X \to Y$ in $\mathscr{C}$ is called a $\mathbb{T}$-*homomorphism* if it makes the following diagram commute.

$$
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TY \arrow[d, "\beta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

The category of $\mathbb{T}$-algebras and $\mathbb{T}$-algebra homomorphisms is denoted by $\mathscr{C}^{\mathbb{T}}$.

2.4.2 The Beck-Chevalley condition

A commutative square

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
W \arrow[r, "k"'] & Z
\end{tikzcd}
\tag{2.4.3}
$$

in **Set** is said to satisfy the *Beck-Chevalley condition* (BC) if

$$h^\circ . k = f . g^\circ$$

in 2-**Rel**. This means that

$$1_Z(k(w), h(y)) = \bigvee_{x \in X} g(x, w) \wedge f(x, y)$$

for all $w \in W$ and $y \in Y$. Hence the square (2.4.3) satisfies (BC) if and only if for any $w \in W$ and $y \in Y$ such that $k(w) = h(y)$ there exists $x \in X$ with $f(x) = y$ and $g(x) = w$. This is equivalent to saying that the induced morphism $i : X \to W \times_Z Y$ going into the pullback of h and k is surjective.

$$\begin{tikzcd}
X \arrow[dr, "i"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& W \times_Z Y \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\
& W \arrow[r, "k"'] & Z
\end{tikzcd}$$

Hence the (BC) square (2.4.3) is also referred to as a *weak pullback*.

A functor $T : \mathbf{Set} \to \mathbf{Set}$ satisfies the *Beck-Chevalley condition* (BC) if T sends weak pullbacks to weak pullbacks. Assuming the axiom of choice, surjective maps are split epimorphisms in **Set**. Since functors preserve split epimorphisms, **Set** functors preserve surjections. So T satisfies (BC) if and only if it sends pullbacks to weak pullbacks.

If T belongs to a monad, then it preserves monomorphisms. The same is true if T satisfies (BC). To see this observe that a morphism $m : X \to Y$ is a monomorphism if

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "m"] \\ X \arrow[r, "m"'] & Y \end{tikzcd} \tag{2.4.4}$$

is a pullback. If m is a monomorphism and T satisfies (BC), then the image of (2.4.4) is a weak pullback, hence a pullback as $T1_X = 1_{TX}$. So Tm is a monomorphism.

If a functor T : **Set** $\to$ **Set** preserves monomorphisms, then $M \subseteq X$ implies $TM \subseteq TX$ in the sense that $TM \to TX$ is an isomorphism onto its image.

A natural transformation $\alpha : F \to G$ satisfies the *Beck-Chevalley condition* (BC) if every naturality square of α satisfies (BC).

A monad $\mathbb{T} = (T, e, m)$ is said to satisfy the *Beck-Chevalley condition* (BC) if both T and m satisfy (BC).

Except the double powerset monad, all the monads in Examples 2.4.1 satisfy the Beck-Chevalley condition. We demonstrate it for the ultrafilter monad $\mathbb{U} = (U, e, m)$ below.

Example 2.4.2. Firstly, we show that the ultrafilter functor U satisfies the Beck-

Chevalley condition. Suppose that the commutative square (2.4.3) satisfies (BC). Consider its image under U.

$$\begin{tikzcd} UX \arrow[r, "Uf"] \arrow[d, "Ug"'] & UY \arrow[d, "Uh"] \\ UW \arrow[r, "Uk"'] & UZ \end{tikzcd}$$

Let $Uk(\mathfrak{w}) = Uh(\mathfrak{y})$ for any $\mathfrak{w} \in UW$ and $\mathfrak{y} \in UY$. We need to show that there exists an ultrafilter $\mathfrak{x}$ on X with $Ug(\mathfrak{x}) = \mathfrak{w}$ and $Uf(\mathfrak{x}) = \mathfrak{y}$.

For any $W \in \mathfrak{w}$ and $Y \in \mathfrak{y}$, one has $k(W) \cap h(Y) \neq \emptyset$, since $Uk(\mathfrak{w}) = Uh(\mathfrak{y})$. Then $g^{-1}(W) \cap f^{-1}(Y) \neq \emptyset$, as the square (2.4.3) satisfes (BC). So sets of the form

$$g^{-1}(W) \cap f^{-1}(Y)$$

for $W \in \mathfrak{w}$ and $Y \in \mathfrak{y}$ constitute a filter basis. Consider an ultrafilter $\mathfrak{x}$ which contains this filter base. One has $A \in Ug(\mathfrak{x})$ if and only if $g^{-1}(A) \in \mathfrak{x}$. So $\mathfrak{w} \subseteq Ug(\mathfrak{x})$. Since $\mathfrak{w}$ is an ultrafilter, $Ug(\mathfrak{x}) = \mathfrak{w}$. Similarly $Uf(\mathfrak{x}) = \mathfrak{y}$. Therefore the ultrafilter functor U satisfies (BC).

Now we show that $m : U^2 \to U$ satisfies the Beck-Chevalley condition. Recall that $m_X(\mathfrak{X}) = \{A \subseteq X \mid A^\# \in \mathfrak{X}\}$ for any set X and $\mathfrak{X} \in U^2X$ where $A^\# = \{\mathfrak{x} \in UX \mid$

$A \in \mathfrak{x}\}$. Given a map $f : X \to Y$, one has the following naturality square.

$$\begin{tikzcd} U^2X \arrow[r, "m_X"] \arrow[d, "U^2f"'] & UX \arrow[d, "Uf"] \\ U^2Y \arrow[r, "m_Y"'] & UY \end{tikzcd}$$

Suppose that $\mathfrak{Y} \in U^2Y$ and $\mathfrak{x} \in UX$ with $m_Y(\mathfrak{Y}) = Uf(\mathfrak{x})$. Then $B^\# \in \mathfrak{Y}$ if and only if $f^{-1}(B) \in \mathfrak{x}$. Observe that if

$$A^\# \cap Uf^{-1}(\mathcal{B}) = A^\# \cap \{\mathfrak{z} \in UX \mid f(\mathfrak{z}) \in \mathcal{B}\} \neq \varnothing \tag{2.4.5}$$

for all $A \in \mathfrak{x}$, $\mathcal{B} \in \mathfrak{Y}$, then $\{A^\# \cap Uf^{-1}(\mathcal{B}) \mid A \in \mathfrak{x}, \mathcal{B} \in \mathfrak{Y}\}$ becomes a filter basis. For an ultrafilter $\mathfrak{X} \in U^2X$ containing this filter basis, one has $U^2f(\mathfrak{X}) = \mathfrak{Y}$ and $m_X(\mathfrak{X}) = \mathfrak{x}$. So it is enough to show that (2.4.5) holds for all $A \in \mathfrak{x}$, $\mathcal{B} \in \mathfrak{Y}$.

Take any $A \in \mathfrak{x}$, $\mathcal{B} \in \mathfrak{Y}$. Since $A \subseteq f^{-1}(f(A)) \in \mathfrak{x}$, $f(A)^\# \in \mathfrak{Y}$. Then $\mathcal{B} \cap f(A)^\# \neq \varnothing$. Let $\mathfrak{y} \in \mathcal{B} \cap f(A)^\#$. For any $B \in \mathfrak{y}$, one has $B \cap f(A) \neq \varnothing$ which implies $f^{-1}(B) \cap A \neq \varnothing$. Consider an ultrafilter $\mathfrak{z} \in UX$ which contains the filter basis $\{f^{-1}(B) \cap A \mid B \in \mathfrak{y}\}$. Then $A \in \mathfrak{z}$ and $f(\mathfrak{z}) = \mathfrak{y}$. Hence $\mathfrak{z} \in A^\# \cap Uf^{-1}(\mathcal{B}) \neq \varnothing$, (2.4.5) holds.

2.5 Topological theories

Let $\mathbb{T} = (T, e, m)$ be a **Set**-monad and V be a quantale. An Eilenberg-Moore algebra (X, α) of $\mathbb{T}$ consists of a set X and a map $\alpha : TX \to X$ such that

$$\alpha.T\alpha = \alpha.m_X \qquad \text{and} \qquad 1_X = \alpha.e_X. \tag{2.5.1}$$

An interesting direction to take is to replace the **Set**-morphism $\alpha : TX \to X$ with a V-relation $\alpha : TX \nrightarrow X$ and the composition in **Set** with the V-relational composition. But the term $T\alpha$ in (2.5.1) does not make sense unless T is defined on V-**Rel**. In order to obtain an extension of T to V-**Rel**, one has to make some investments in terms of the assumptions. Our choice will be to adopt the framework called "strict topological theory" introduced by Hofmann [33].

We give the definition of a topological theory in Subsection 2.5.1 and a strict topological theory in Subsection 2.5.2. We provide examples and show that the ultrafilter monad belongs to a strict topological theory. The extension of T to V-**Rel**, as defined in [33], is given in Subsection 2.5.3.

2.5.1 Definition and examples

A *topological theory* [33] is a triple $\mathcal{T} = (\mathbb{T}, V, \xi)$ where

- $\mathbb{T} = (T, e, m)$ is a **Set**-monad.

- V is a quantale

- $\xi : TV \to V$ is a map which is compatible with the monad $\mathbb{T}$ and the quantale V, which means:

(T1) $1_V \leq \xi.e_V$,

(T2) $\xi.T\xi \leq \xi.m_V$,

(T3) $k.!_1 \leq \xi.Tk$ where 1 is the singleton set $\{\star\}$ and $k : 1 \to V$ is the map that sends $\star$ to k,

$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!_1"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & V \end{tikzcd}$$

(T4) $\otimes.\langle \xi.T\pi_1, \xi.T\pi_2 \rangle \leq \xi.T(\otimes)$,

$$\begin{tikzcd} T(V \times V) \arrow[r, "T(\otimes)"] \arrow[d, "\langle \xi.T\pi_1, \xi.T\pi_2 \rangle"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ V \times V \arrow[r, "\otimes"'] & V \end{tikzcd}$$

(T5) $(\xi_X)_X : P_V \to P_V T$ is a natural transformation.

In the last condition $P_V : \mathbf{Set} \to \mathbf{Ord}$ is the V-powerset functor defined by $P_V X = V^X$ on objects. For $f : X \to Y$, $P_V f : V^X \to V^Y$ is given by

$$P_V f(\varphi)\,(y) = \bigvee_{x \in f^{-1}(y)} \varphi(x)$$

for any $\varphi \in V^X$. The map $\xi_X : P_V X \to P_V TX$ is defined by $\xi_X(\varphi) = \xi.T\varphi$. So one has the commutative square

$$\begin{tikzcd} V^X \arrow[r, "\xi_X"] \arrow[d, "P_V f"'] & V^{TX} \arrow[d, "P_V Tf"] \\ V^Y \arrow[r, "\xi_Y"'] & V^{TY} \end{tikzcd}$$

for any $f : X \to Y$ in **Set**.

Lemma 2.5.1 ([33]). Let $\mathcal{T} = (\mathbb{T}, V, \xi)$ be a topological theory. Then $\xi.T(\multimap) \leq \multimap .\langle \xi.T\pi_1, \xi.T\pi_2\rangle$.

$$\begin{tikzcd} T(V \times V) \arrow[r, "T(\multimap)"] \arrow[d, "\langle \xi.T\pi_1, \xi.T\pi_2\rangle"'] \arrow[dr, phantom, "\geq"] & TV \arrow[d, "\xi"] \\ V \times V \arrow[r, "\multimap"'] & V \end{tikzcd}$$

Proof. Take any $\mathfrak{v} \in T(V \times V)$. Let $\mathfrak{u} = T\langle \pi_1, \multimap\rangle(\mathfrak{v})$. One has

$$\otimes.\langle \pi_1, \multimap\rangle(u, v) = u \otimes (u \multimap v) \leq v \qquad \& \qquad \pi_2.\langle \pi_1, \multimap\rangle(u, v) = u \multimap v$$

for all $u, v \in V$. Hence

$$\otimes.\langle \pi_1, \multimap\rangle \leq \pi_2 \qquad \& \qquad \pi_2.\langle \pi_1, \multimap\rangle = \multimap .$$

$\xi_V : P_V V \to P_V TV$ is order preserving, as it is a morphism in **Ord**. This implies

$$\xi.T(\otimes.\langle \pi_1, \multimap\rangle) \leq \xi.T\pi_2 \qquad \& \qquad \xi.T(\pi_2\langle \pi_1, \multimap\rangle) = \xi.T(\multimap).$$

Then

$$\xi.T(\otimes)(\mathfrak{u}) = \xi.T(\otimes)(T\langle\pi_1, \multimap\rangle(\mathfrak{v})) = \xi.T(\otimes.\langle\pi_1, \multimap\rangle)(\mathfrak{v}) \leq \xi.T\pi_2(\mathfrak{v}). \qquad (\dagger)$$

Similarly

$$\xi.T\pi_1(\mathfrak{u}) = \xi.T\pi_1(\mathfrak{v}) \qquad \& \qquad \xi.T\pi_2(\mathfrak{u}) = \xi.T(\multimap)(\mathfrak{v}). \qquad (\ddagger)$$

$(T4)$ implies that

$$\xi.T\pi_1(\mathfrak{u}) \otimes \xi.T\pi_2(\mathfrak{u}) \leq \xi.T(\otimes)(\mathfrak{u}).$$

By $(\dagger)$ and $(\ddagger)$, one obtains

$$\xi.T\pi_1(\mathfrak{v}) \otimes \xi.T(\multimap)(\mathfrak{v}) \leq \xi.T\pi_2(\mathfrak{v}),$$

$$\xi.T(\multimap)(\mathfrak{v}) \leq \xi.T\pi_1(\mathfrak{v}) \multimap \xi.T\pi_2(\mathfrak{v}).$$

Since $\mathfrak{v} \in T(V \times V)$ is arbitrary, $\xi.T(\multimap) \leq \multimap .\langle\xi.T\pi_1, \xi.T\pi_2\rangle$. □

Example 2.5.2. Consider the ultrafilter monad $\mathbb{U} = (U, e, m)$. Let V be a constructively completely distributive (ccd) quantale and the map $\xi : UV \to V$ be given by

$$\xi(\mathfrak{x}) := \bigvee_{A\in\mathfrak{x}} \bigwedge A = \bigvee\{v \in V \mid \uparrow v \in \mathfrak{x}\}. \qquad (2.5.2)$$

Then $\mathcal{U} = (\mathbb{U}, V, \xi)$ is a topological theory. The details follow.

Firstly, we examine (2.5.2). Observe that

$$\bigvee_{A\in\mathfrak{x}} \bigwedge A \geq \bigvee\{v \in V \mid \uparrow v \in \mathfrak{x}\}$$

as $\bigwedge \uparrow v = v$. Conversely, for any $A \in \mathfrak{x}$, let $u = \bigwedge A$. Then $A \subseteq \uparrow u$ and $\uparrow u \in \mathfrak{x}$. Hence

$$\bigvee_{A \in \mathfrak{x}} \bigwedge A \leq \bigvee \{v \in V \mid \uparrow v \in \mathfrak{x}\}.$$

So the equality given in (2.5.2) is justified. Since V is ccd, one can write

$$\bigwedge_{A \in \mathfrak{x}} \bigvee A = \bigvee_{B \in \mathfrak{x}^*} \bigwedge B$$

where $\mathfrak{x}^* = \{B \subseteq V \mid \forall A \in \mathfrak{x},\ B \cap A \neq \varnothing\}$ by (2.1.10). Observe that $\mathfrak{x} = \mathfrak{x}^*$, as $\mathfrak{x}$ is an ultrafilter. Hence

$$\bigwedge_{A \in \mathfrak{x}} \bigvee A = \bigvee_{A \in \mathfrak{x}} \bigwedge A.$$

So $\xi : UV \to V$ is equivalently given by

$$\xi(\mathfrak{x}) = \bigwedge_{A \in \mathfrak{x}} \bigvee A = \bigwedge \{v \in V \mid \downarrow v \in \mathfrak{x}\}. \tag{2.5.3}$$

Following [33], we now show that conditions $(T1) - (T5)$ of Subsection 2.5.1 are satisifed. One actually has equalities for $(T1)$ and $(T2)$.

$(T1)$ For any $w \in V$,

$$\xi.e_V(w) = \xi(\dot{w}) = \bigvee \{v \in V \mid \uparrow v \in \dot{w}\} = \bigvee \{v \in V \mid v \leq w\} = w.$$

So $1_V = \xi.e_V$.

$(T2)$ Let $\mathfrak{X} \in U^2V$. By (2.5.2), one has

$$\xi.U\xi(\mathfrak{X}) = \bigvee \{v \in V \mid \uparrow v \in U\xi(\mathfrak{X})\},$$

$$\xi.m_V(\mathfrak{X}) = \bigvee\{v \in V \mid \uparrow v \in m_V(\mathfrak{X})\}.$$

Here $m_V(\mathfrak{X}) = \{A \subseteq V \mid A^{\#} \in \mathfrak{X}\}$ where $A^{\#} = \{\mathfrak{x} \in UV \mid A \in \mathfrak{x}\}$. Let $\uparrow v \in m_V(\mathfrak{X})$. Then $(\uparrow v)^{\#} \in \mathfrak{X}$. Given any $\mathfrak{x} \in (\uparrow v)^{\#}$, one has $\xi(\mathfrak{x}) \geq v$. Hence $\mathfrak{x} \in \xi^{-1}(\uparrow v)$, $(\uparrow v)^{\#} \subseteq \xi^{-1}(\uparrow v)$. This implies that $\xi^{-1}(\uparrow v) \in \mathfrak{X}$ and $\uparrow v \in U\xi(\mathfrak{X})$. Therefore

$$\xi.m_V \leq \xi.U\xi.$$

Similarly, using (2.5.3), one obtains

$$\xi.m_V \geq \xi.U\xi.$$

$(T3)$ Trivially $k.!_1 = k$. On the other hand, $U1 = 1$ and $Uk(\star) = \dot{k}$. Hence $k \leq \xi(\dot{k})$.

$(T4)$ Let $\mathfrak{w} \in U(V \times V)$ such that $U\pi_1(\mathfrak{w}) = \mathfrak{x}$ and $U\pi_2(\mathfrak{w}) = \mathfrak{y}$. One needs to show

$$\begin{aligned} \bigvee\{w \in V \mid \uparrow w \in U(\otimes)(\mathfrak{w})\} &\geq \bigvee\{u \in V \mid \uparrow u \in \mathfrak{x}\} \otimes \bigvee\{v \in V \mid \uparrow v \in \mathfrak{y}\} \\ &= \bigvee\{u \otimes v \in V \mid \uparrow u \in \mathfrak{x}, \uparrow v \in \mathfrak{y}\}. \end{aligned}$$

Assume that $\uparrow u \in \mathfrak{x}$ and $\uparrow v \in \mathfrak{y}$. Then $\uparrow u \times V \in \mathfrak{w}$ and $V \times \uparrow v \in \mathfrak{w}$, hence $\uparrow u \times \uparrow v \in \mathfrak{w}$. This implies that $\uparrow (u \otimes v) \in U(\otimes)(\mathfrak{w})$.

$(T5)$ Let $f : X \to Y$, $\varphi : X \to V$ be functions and $\mathfrak{y} \in UY$. Then

$$\begin{aligned} \xi_Y.P_V f(\varphi)\,(\mathfrak{y}) &= \xi.U(P_V f(\varphi))\,(\mathfrak{y}) \\ &= \bigvee\{v \in V \mid \uparrow v \in U(P_V f(\varphi))\,(\mathfrak{y})\} \\ &= \bigvee\{v \in V \mid Y_v \in \mathfrak{y}\} \end{aligned}$$

where $Y_v := \left(P_V f(\varphi)\right)^{-1}(\uparrow v) = \{y \in Y \mid v \le \bigvee_{x \in f^{-1}(y)} \varphi(x)\}$.

On the other hand,

$$\begin{aligned}
P_V Uf.\xi_X(\varphi)\,(\mathfrak{y}) &= \bigvee_{\mathfrak{x} \in Uf^{-1}(\mathfrak{y})} \xi.U\varphi(\mathfrak{x}) \\
&= \bigvee_{\mathfrak{x} \in Uf^{-1}(\mathfrak{y})} \bigvee\{v \in V \mid \uparrow v \in U\varphi(\mathfrak{x})\} \\
&= \bigvee_{\mathfrak{x} \in Uf^{-1}(\mathfrak{y})} \bigvee\{v \in V \mid X_v \in \mathfrak{x}\}
\end{aligned}$$

where $X_v := \varphi^{-1}(\uparrow v) = \{x \in X \mid v \le \varphi(x)\}$.

Suppose there exists $\mathfrak{x} \in Uf^{-1}(\mathfrak{y})$ such that $X_v \in \mathfrak{x}$. Since $f(X_v) \subseteq Y_v$, one gets $Y_v \in Uf(\mathfrak{x}) = \mathfrak{y}$. This shows $P_V Uf.\xi_X \le \xi_Y.P_V f$.

To obtain the reverse inequality, suppose that $Y_v \in \mathfrak{y}$. Let $u \ll v$. Then for any $y \in Y_v$ there exists $x \in f^{-1}(y)$ such that $u \le \varphi(x)$. Hence the restriction $f^{-1}(Y_v) \cap X_u \to Y_v$ of f is surjective.

$$\begin{tikzcd}
f^{-1}(Y_v) \cap X_u \arrow[r, two heads] \arrow[d, hook] & Y_v \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Consider $\mathfrak{y}_{|Y_v}$ which is the restriction of $\mathfrak{y}$ to Y_v. Since the ultrafilter functor preserves surjections, there exists an ultrafilter $\mathfrak{z}$ on $f^{-1}(Y_v) \cap X_u$ such that $Uf(\mathfrak{z}) = \mathfrak{y}_{|Y_v}$. Let $\mathfrak{x}_u$ be the image of $\mathfrak{z}$ under the inclusion $f^{-1}(Y_v) \cap X_u \hookrightarrow X$. Then $\mathfrak{x}_u \in UX$ with $X_u \in \mathfrak{x}_u$ and $Uf(\mathfrak{x}_u) = \mathfrak{y}$.

So for any $u \ll v$ there exists $\mathfrak{x}_u \in Uf^{-1}(\mathfrak{y})$ such that $X_u \in \mathfrak{x}_u$. Then

$$\begin{aligned} v = \bigvee_{u \ll v} u \quad &\leq \bigvee_{\mathfrak{x}_u \in Uf^{-1}(\mathfrak{y})} \bigvee \{ w \in V \mid X_w \in \mathfrak{x}_u \} \\ &\leq \bigvee_{\mathfrak{x} \in Uf^{-1}(\mathfrak{y})} \bigvee \{ w \in V \mid X_w \in \mathfrak{x} \}. \end{aligned}$$

This implies

$$\bigvee \{ v \in V \mid Y_v \in \mathfrak{y} \} \leq \bigvee_{\mathfrak{x} \in Uf^{-1}(\mathfrak{y})} \bigvee \{ v \in V \mid X_v \in \mathfrak{x} \}.$$

Therefore $\xi_Y . P_V f \leq P_V U f . \xi_X$.

2.5.2 Strict topological theories

Let $\mathcal{T} = (\mathbb{T}, V, \xi)$ be a topological theory. $\mathcal{T}$ is called *strict* if

- $\mathbb{T}$ satisfies the Beck-Chevalley condition;
- One has " $=$ " instead of " $\leq$ " in conditions $(T1)-(T4)$ of Subsection 2.5.1.

With equality in place, $(T1)$ and $(T2)$ imply that (V, ξ) is a $\mathbb{T}$-algebra. Likewise $(T3)$ and $(T4)$ imply that the maps $k : 1 \to V$ and $(\_) \otimes (\_) : V \times V \to V$ are $\mathbb{T}$-algebra homomorphisms.

Throughout this work we assume that $\mathcal{T}$ is a strict topological theory. Furthermore, we assume that T sends the singleton set to the singleton set, i.e. $T1 = 1$.

Examples 2.5.3. 1. $\mathcal{I}_V = (\mathbb{1}, V, 1_V)$ is a strict topological theory for any quantale V. Here $\mathbb{1}$ stands for the identity monad.

2. $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$ is a strict topological theory where $\mathbb{U}$ is the ultrafilter monad and $\xi_2 : U2 \to 2$ is induced by (2.5.2).

Since 2 is ccd, $\mathcal{U}_2$ is a topological theory by Example 2.5.2. We have also seen that $\mathbb{U}$ satisfies the Beck-Chevalley condition in Example 2.4.2. The map $\xi_2 : U2 \to 2$ sends the principle ultrafilter $\dot{x}$ to x for any $x \in \{0,1\}$. So ξ_2 is basically the identity map. One trivially has equality in conditions $(T1)$ - $(T4)$.

3. $\mathcal{U}_{\mathrm{P}_+} = (\mathbb{U}, \mathrm{P}_+, \xi_{\mathrm{P}_+})$ is a strict topological theory where $\xi_{\mathrm{P}_+} : U\mathrm{P}_+ \to \mathrm{P}_+$ is induced by (2.5.2), i.e.

$$\xi_{\mathrm{P}_+}(\mathfrak{x}) = \inf\{v \in V \mid [0,v] \in \mathfrak{x}\}.$$

Since P_+ is ccd, $\mathcal{U}_{\mathrm{P}_+}$ is a topological theory by Example 2.5.2. Furthermore, $(T1)$ and $(T2)$ hold with equality as shown in Example 2.5.2. So we only need to check $(T3)$ and $(T4)$. For $(T3)$, one has

$$\xi_{\mathrm{P}_+}.Uk(\star) = \xi_{\mathrm{P}_+}(\dot{k}) = k,$$

as $k = \top$. For $(T4)$, Example 2.5.2 implies

$$\xi.U(+) \leq +.\langle \xi.U\pi_1, \xi.U\pi_2 \rangle,$$

since the order on P_+ is reversed. To obtain $\xi.U(+) \geq +.\langle \xi.U\pi_1, \xi.U\pi_2 \rangle$, we use the equivalent formulation (2.5.3) for ξ_{P_+}:

$$\xi_{P_+}(\mathfrak{x}) = \sup\{v \in V \mid [v, \infty] \in \mathfrak{x}\}.$$

For any $\mathfrak{w} \in U(V \times V)$ with $U\pi_1(\mathfrak{w}) = \mathfrak{x}$ and $U\pi_2(\mathfrak{w}) = \mathfrak{y}$, one needs to show

$$\begin{aligned} \sup\{w \mid [w, \infty] \in U(+)(\mathfrak{w})\} &\geq \sup\{u \mid [u, \infty] \in \mathfrak{x}\} + \sup\{v \mid [v, \infty] \in \mathfrak{y}\} \\ &= \sup\{u + v \mid [u, \infty] \in \mathfrak{x}, [v, \infty] \in \mathfrak{y}\} \end{aligned}$$

where the equality holds since " + " preserves suprema in P_+. Assume that $[u, \infty] \in \mathfrak{x}$ and $[v, \infty] \in \mathfrak{y}$. Then $[u, \infty] \times [v, \infty] \in \mathfrak{w}$. This implies $[u + v, \infty] \in U(+)(\mathfrak{w})$.

4. $(\mathbb{L}, V, \xi)$ is a strict topological theory for any quantale V where $\mathbb{L} = (L, e, m)$ is the list monad (see Examples 2.4.1) and $\xi : LV \to V$ is given by

$$\xi(\,) = k,$$

$$\xi(v_1, v_2, \dots, v_n) = v_1 \otimes v_2 \otimes \dots \otimes v_n.$$

$\mathbb{L}$ satisfies the Beck-Chevalley condition. The conditions $(T1)$ and $(T3)$ trivially hold with equality. One has equality in $(T2)$ and $(T4)$ since " $\otimes$ " is associative and commutative respectively. For $(T5)$, let $f : X \to Y$ and $\varphi \in V^X$.

One needs to show $\xi_Y.P_V f(\varphi) = P_V Lf.\xi_X(\varphi)$. For any $(y_1, y_2, \ldots, y_n) \in LY$,

$$\begin{aligned}
\xi_Y.P_V f(\varphi)\,(y_1, y_2, \ldots, y_n) &= \xi.L(P_V f(\varphi))\,(y_1, y_2, \ldots, y_n) \\
&= \xi(P_V f(\varphi)(y_1), P_V f(\varphi)(y_2), \ldots, P_V f(\varphi)(y_n)) \\
&= \bigvee_{x_1 \in f^{-1}(y_1)} \varphi(x_1) \otimes \bigvee_{x_2 \in f^{-1}(y_2)} \varphi(x_2) \otimes \ldots \otimes \bigvee_{x_n \in f^{-1}(y_n)} \varphi(x_n) \\
&= \bigvee_{\substack{(x_1, x_2, \ldots, x_n) \\ \in Lf^{-1}(y_1, y_2, \ldots, y_n)}} \varphi(x_1) \otimes \varphi(x_2) \otimes \ldots \otimes \varphi(x_n) \\
&= \bigvee_{\substack{(x_1, x_2, \ldots, x_n) \\ \in Lf^{-1}(y_1, y_2, \ldots, y_n)}} \xi.L\varphi(x_1, x_2, \ldots, x_n) \\
&= P_V Lf.\xi_X(\varphi)\,(y_1, y_2, \ldots, y_n).
\end{aligned}$$

The assumption $T1 = 1$ implies the following result which will be helpful in the sequel.

Lemma 2.5.4 ([36]). Let $\mathbb{T} = (T, e, m)$ be a **Set**-monad where $T1 = 1$ and m satisfy (BC). Then $m^\circ.e = eT.e$.

Proof. Since $\mathbb{T}$ is a monad, $1_{TX} = m_X.e_{TX}$. Composing both sides by m_X° on the left, one gets $m_X^\circ \geq e_{TX}$ which in turn implies $m_X^\circ.e_X \geq e_{TX}.e_X$.

Now take any $x \in X$ and $\mathfrak{X} \in T^2X$. One has $m_X^\circ.e_X(x, \mathfrak{X}) \leq e_{TX}.e_X(x, \mathfrak{X})$ if $m_X(\mathfrak{X}) = e_X(x)$ implies $e_{TX}.e_X(x) = \mathfrak{X}$. So suppose that $m_X(\mathfrak{X}) = e_X(x)$. Consider the following naturality squares of e and m where $x : 1 \to X$ is the map that picks

$x \in X$.

$$
\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "x"'] & T1 \arrow[r, equal] \arrow[d, "Tx"'] & T^2 1 \arrow[d, "T^2 x"] \\
X \arrow[r, "e_X"'] & TX \arrow[r, shift right, "e_{TX}"'] & T^2 X \arrow[l, shift right, "m_X"']
\end{tikzcd}
$$

One has $m_X(\mathfrak{X}) = e_X(x) = Tx(\star)$. Since m satisfies (BC), $T^2x(\star) = \mathfrak{X}$. But $e_{TX}.e_X(x) = T^2x(\star)$, as the outer diagram commutes. Hence $e_{TX}.e_X(x) = \mathfrak{X}$. □

One has $eT.e = Te.e$ by the naturality of e. Hence the lemma implies $m^\circ.e = Te.e$ as well.

2.5.3 Extension of T to V-Rel

Given a topological theory $\mathcal{T} = (\mathbb{T}, V, \xi)$, one can extend the **Set** functor T to V-**Rel** as given in [33]. For any $r : X \nrightarrow Y$, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$, the extension $\overline{T}$ of T is defined by

$$\overline{T}r(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\xi.Tr(\mathfrak{w}) \mid \mathfrak{w} \in T(X \times Y) : T\pi_1(\mathfrak{w}) = \mathfrak{x},\, T\pi_2(\mathfrak{w}) = \mathfrak{y}\} \qquad (2.5.4)$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection maps.

So $\overline{T}r$ is the smallest map $q : TX \times TY \to V$ such that $\xi.Tr \le q.\langle T\pi_1, T\pi_2\rangle$.

$$\begin{tikzcd} T(X\times Y) \arrow[rr, "\langle T\pi_1, T\pi_2\rangle"] \arrow[dr, "\xi.Tr"'] & & TX\times TY \arrow[dl, dashed, "\overline{T}r"] \\ & V & \end{tikzcd}$$

The reader shall keep in mind that the extension of T to V-**Rel** is dependent on the choice of ξ although this is not reflected in the notation for the sake of simplicity.

Remark 2.5.5. By Example 2.5.2 we know that $\mathcal{U} = (\mathbb{U}, V, \xi)$ is a topological theory when V is ccd and $\xi : UV \to V$ is given by $\xi(\mathfrak{x}) = \bigvee\{v \in V \mid \uparrow v \in \mathfrak{x}\}$.

Now we formulate the extension of the ultrafilter functor U to V-**Rel** following (2.5.4). For $r : X \nrightarrow Y$,

$$\begin{aligned} \overline{U}r(\mathfrak{x},\mathfrak{y}) &= \bigvee\{\xi.Ur(\mathfrak{w}) \mid \mathfrak{w} \in U(X\times Y) : U\pi_1(\mathfrak{w}) = \mathfrak{x},\ U\pi_2(\mathfrak{w}) = \mathfrak{y}\} \\ &= \bigvee\{\bigvee\{v \in V \mid \uparrow v \in Ur(\mathfrak{w})\} \mid \mathfrak{w} \in U(X\times Y) : U\pi_1(\mathfrak{w}) = \mathfrak{x},\ U\pi_2(\mathfrak{w}) = \mathfrak{y}\} \\ &= \bigvee\{v \in V \mid \mathfrak{w} \in U(X\times Y) : r^{-1}(\uparrow v) \in \mathfrak{w},\ U\pi_1(\mathfrak{w}) = \mathfrak{x},\ U\pi_2(\mathfrak{w}) = \mathfrak{y}\} \end{aligned}$$

There exists $\mathfrak{w} \in U(X\times Y)$ such that $r^{-1}(\uparrow v) \in \mathfrak{w}$, $U\pi_1(\mathfrak{w}) = \mathfrak{x}$ and $U\pi_2(\mathfrak{w}) = \mathfrak{y}$ if and only if $r^{-1}(\uparrow v) \cap \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \neq \varnothing$ for any $A \in \mathfrak{x}$, $B \in \mathfrak{y}$. This is equivalent to saying that for any $A \in \mathfrak{x}$, $B \in \mathfrak{y}$ there exists $x \in A$, $y \in B$ such that $v \le r(x,y)$. Hence

$$\overline{U}r(\mathfrak{x},\mathfrak{y}) = \bigvee\{v \in V \mid \forall A \in \mathfrak{x},\ \forall B \in \mathfrak{y},\ \exists x \in A,\ \exists y \in B :\ v \le r(x,y)\}.$$

If v belongs to the set over which this supremum is formed, then $v \leq \bigwedge_{\substack{A\in\mathfrak{x}\\B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\y\in B}} r(x,y)$, which implies

$$\overline{U}r(\mathfrak{x},\mathfrak{y}) \leq \bigwedge_{\substack{A\in\mathfrak{x}\\B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\y\in B}} r(x,y).$$

On the other hand, let $u \ll \bigwedge_{\substack{A\in\mathfrak{x}\\B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\y\in B}} r(x,y)$. Then for any $A \in \mathfrak{x}$, $B \in \mathfrak{y}$ there exists $x \in A$, $y \in B$ such that $u \leq r(x,y)$. Hence $u \leq \overline{U}r(\mathfrak{x},\mathfrak{y})$. Taking suprema over all $u \ll \bigwedge_{\substack{A\in\mathfrak{x}\\B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\y\in B}} r(x,y)$, one gets

$$\bigwedge_{\substack{A\in\mathfrak{x}\\B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\y\in B}} r(x,y) \leq \overline{U}r(\mathfrak{x},\mathfrak{y}).$$

Therefore the extension of U to V-**Rel** is

$$\overline{U}r(\mathfrak{x},\mathfrak{y}) = \bigwedge_{\substack{A\in\mathfrak{x}\\B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\y\in B}} r(x,y). \tag{2.5.5}$$

Examples 2.5.6. 1. Let $r : X \nrightarrow Y$ be a 2-relation. In accordance with (2.5.5), the extension of U to 2-**Rel** is given by

$$\mathfrak{x}\,\overline{U}r\,\mathfrak{y} \iff \forall A \in \mathfrak{x},\ \forall B \in \mathfrak{y}, \exists x \in A,\ \exists y \in B :\ x\,r\,y$$

for any $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$.

Equivalently,

$$\begin{aligned} \mathfrak{x}\,\overline{U}r\,\mathfrak{y} &\iff \forall A \in \mathfrak{x},\ r(A) \in \mathfrak{y} \\ &\iff \forall B \in \mathfrak{y},\ r^\circ(B) \in \mathfrak{x} \end{aligned} \tag{2.5.6}$$

where $r(A) = \{y \in Y \mid \exists x \in A :\ x\,r\,y\}$ and $r^\circ(B) = \{x \in X \mid \exists y \in B :\ x\,r\,y\}$.

2. Let $r : X \nrightarrow Y$ be a P_+-relation. Following (2.5.5), the extension of U to P_+-**Rel** is given by

$$\overline{U}r(\mathfrak{x},\mathfrak{y}) = \sup_{\substack{A\in\mathfrak{x}\\ B\in\mathfrak{y}}} \inf_{\substack{x\in A\\ y\in B}} r(x,y) \tag{2.5.7}$$

for any $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$.

3. For any quantale V, the list functor L is extended to V-**Rel** by (2.5.4) as follows:

Let $r : X \nrightarrow Y$ be a V-relation, $\mathfrak{x} = (x_1, x_2, \ldots, x_n)$ and $\mathfrak{y} = (y_1, y_2, \ldots, y_m)$. Then

$$\overline{L}r(\mathfrak{x},\mathfrak{y}) = \begin{cases} r(x_1,y_1) \otimes r(x_2,y_2) \otimes \ldots \otimes r(x_n,y_n) & \text{if } n = m, \\ \bot & \textit{else}. \end{cases}$$

If $\mathcal{T} = (\mathbb{T}, V, \xi)$ is a strict topological theory, then the extension $\overline{T}$ of T to V-**Rel** becomes a functor. $m : \overline{T}^2 \to \overline{T}$ becomes a natural transformation and $e : 1_{V\text{-}\mathbf{Rel}} \to \overline{T}$ becomes an op-lax natural transformation.

Proposition 2.5.7 ([33]). Let $\mathcal{T} = (\mathbb{T}, V, \xi)$ be a strict topological theory. Suppose that the extension $\overline{T}$ of the functor $T : \mathbf{Set} \to \mathbf{Set}$ to V-**Rel** is defined as in (2.5.4). Then $\overline{T} : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ becomes a functor. Furthermore, given any map $f : X \to Y$ and V-relations $r, s : X \nrightarrow Y$ the following assertions hold:

1. $\overline{T}f = Tf$.

2. $(\overline{T}r)^\circ = \overline{T}r^\circ$.

3. $r \leq s$ implies $\overline{T}r \leq \overline{T}s$.

4. $e_Y.r \leq \overline{T}r.e_X$.

5. $m_Y.\overline{T}^2 r = \overline{T}r.m_X$.

2.6 $(\mathbb{T}, V)$-relations

A $(\mathbb{T}, V)$-*relation* r from X to Y, denoted by $r : X \rightharpoondown Y$, is a V-relation $r : TX \nrightarrow Y$. Composition of two $(\mathbb{T}, V)$-relations $r : X \rightharpoondown Y$ and $s : Y \rightharpoondown Z$ is given by the *Kleisli convolution*,

$$s \circ r := s.\overline{T}r.m_X^\circ.$$

$(\mathbb{T}, V)$-relations inherit the order on V-relations. Kleisli convolution is an associative operation that respects the order on $(\mathbb{T}, V)$-relations. For any $r : X \rightharpoondown Y$, one has $r \circ e_X^\circ = r$ and $e_Y^\circ \circ r \geq r$.

Like V-relational composition, Kleisli convolution from the right is a sup-map. Hence for $r : X \rightharpoondown Y$, $(\_) \circ r$ has a right adjoint $(\_) \multimapdotinv r$ defined by

$$s \circ r \leq t \iff s \leq t \multimapdotinv r \tag{2.6.1}$$

for any $s : Y \rightharpoondown Z$ and $t : X \rightharpoondown Z$. Similar to (2.3.1), one gets

$$t \multimapdotinv r\,(\mathfrak{y}, z) = \bigvee\{s \mid s \circ r \leq t\} = \bigwedge_{\mathfrak{x} \in TX} \left(t(\mathfrak{x}, z) \multimapinv \overline{T}r.m_X^\circ(\mathfrak{x}, \mathfrak{y})\right). \tag{2.6.2}$$

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "t \multimapdotinv r"] & \\
X \arrow[u, "r", "\shortmid"'] \arrow[r, "t"', "\shortmid"] & Z
\end{tikzcd}
\tag{2.6.3}
$$

Kleisli convolution from the left does not have a right adjoint in general [35].

2.7 $(\mathbb{T}, V)$-categories

The notion of a $(\mathbb{T}, V)$-category is the main ingredient of the theory developed in the subsequent chapters. In Subsection 2.7.1 we provide the definition of a $(\mathbb{T}, V)$-category and its main examples as given in [14], [17], [16]. Subsection 2.7.2 is devoted to the basic facts about $(\mathbb{T}, V)$-categories. Firstly, we consider some important functors between the category of $(\mathbb{T}, V)$-categories and the category of V-categories. Following [15], we provide the definitions of a free Eilenberg-Moore algebra and a dual $(\mathbb{T}, V)$-category and show that the quantale V itself can be considered as a $(\mathbb{T}, V)$-category. Furthermore, we review $\otimes$-exponentiability [33] and some basic limits in the category of $(\mathbb{T}, V)$-categories.

2.7.1 Definitions and examples

From this point on we assume that $\mathcal{T} = (\mathbb{T}, V, \xi)$ is a strict topological theory where $T1 = 1$.

A $(\mathbb{T},V)$-*category* (X,a) is a set X together with a $(\mathbb{T},V)$-relation $a: X \rightharpoonup X$ which satisfies

$$e_X^\circ \leq a \qquad \& \qquad a \circ a \leq a. \tag{2.7.1}$$

Note that composing both sides of $e_X^\circ \leq a$ with a yields $a \leq a \circ a$. So one actually has $a \circ a = a$. Expressed elementwise, (2.7.1) means

$$k \leq a(e_X(x), x) \qquad \& \qquad \overline{T}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x) \leq a(m_X(\mathfrak{X}),x) \tag{2.7.2}$$

for all $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$.

A $(\mathbb{T},V)$-*functor* $f:(X,a) \to (Y,b)$ is a map from X to Y which satisfies

$$f.a \leq b.Tf$$

or, equivalently,

$$a(\mathfrak{x},x) \leq b(Tf(\mathfrak{x}), f(x))$$

for all $\mathfrak{x} \in TX, x \in X$.

$(\mathbb{T},V)$-categories together with $(\mathbb{T},V)$-functors form the category

$$(\mathbb{T},V)\text{-}\mathbf{Cat}.$$

If $\mathbb{T}$ is the identity monad $\mathbb{1}$, one calls an $(\mathbb{1},V)$-category simply a V-*category.* Similarly an $(\mathbb{1},V)$-functor is called a V-*functor.* The category they form is called

$$V\text{-}\mathbf{Cat}.$$

Remark 2.7.1. Given a $(\mathbb{T}, V)$-category (X, a), (2.7.1) can be equivalently written as

$$1_X \leq a.e_X \qquad \& \qquad a.\overline{T}a \leq a.m_X.$$

So (X, a) can be seen as a lax $\mathbb{T}$-algebra (or a lax Eilenberg-Moore algebra) where the composition is replaced with the V-relational composition. In this respect a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ is a lax $\mathbb{T}$-algebra homomorphism (see Subsection 2.4.1).

Now we look at the main examples of $(\mathbb{T}, V)$-categories.

Examples 2.7.2. 1. 2-**Cat** $\cong$ **Ord**.

Given a 2-category (X, a) and $x, y, z \in X$ one has

$$1 \leq a(x, x) \qquad \& \qquad a(x, y) \wedge a(y, z) \leq a(x, z).$$

This means that a is a reflexive and transitive relation on X. Hence X is an ordered set. On the other hand, a 2-functor $f : (X, a) \to (Y, b)$ is a monotone map, as

$$a(x, z) \leq b(f(x), f(z))$$

for all $x, z \in X$. Therefore 2-**Cat** is isomorphic to **Ord**.

2. $\mathbb{P}_{+}$-**Cat** $\cong$ **Met**.

Given a $\mathbb{P}_+$-category (X, a) and $x, y, z \in X$,

$$0 \geq a(x,x) \qquad \& \qquad a(x,y) + a(y,z) \geq a(x,z). \tag{2.7.3}$$

One can consider the value $a(x,y)$ as the distance from the point x to the point y. Then (2.7.3) implies that every point has zero distance to itself and the distance function satisfies the triangle inequality. So (X, a) is a (pre)metric space. Following the convention in category theory, we will simply call (X, a) a metric space. A $\mathbb{P}_+$-functor $f : (X, a) \to (Y, b)$ satisfies

$$a(x,z) \geq b(f(x), f(z))$$

for all $x, z \in X$. This means that f is a nonexpansive map.

Therefore $\mathbb{P}_+$-**Cat** is isomorphic to **Met** which is the category of metric spaces and nonexpansive maps.

A metric space (X, a) which satisfies $a(x,y) = a(y,x)$ for all $x, y \in X$ is called a *symmetric metric space.* If $a(x,y) = 0$ implies $x = y$, then X is callled a *separated metric space.*

3. $(\mathbb{U}, 2)$-**Cat** $\cong$ **Top** [4].

We will demonstrate the correspondance between $(\mathbb{U}, 2)$-**Cat** and **Top** following [56]. Let (X, τ) be a topological space. Consider the ultrafilter convergence relation $a : UX \times X \to 2$ given by

$$\mathfrak{x}\, a\, x \iff \forall O \subseteq X \text{ open } (x \in O \Rightarrow O \in \mathfrak{x}).$$

for $\mathfrak{x} \in UX$, $x \in X$. Then (X, a) is a $(\mathbb{U}, 2)$-category as follows:

The conditions in (2.7.2) translates to

$$\dot{x}\, a\, x \tag{2.7.4}$$

$$\mathfrak{X}\, \overline{U}a\, \mathfrak{x} \quad \& \quad \mathfrak{x}\, a\, x \implies m_X(\mathfrak{X})\, a\, x \tag{2.7.5}$$

for all $\mathfrak{X} \in U^2X$, $\mathfrak{x} \in UX$ and $x \in X$.

Trivially, the principle ultrafilter $\dot{x}$ converges to x for any $x \in X$. To see that the second condition is satisfied, suppose that $\mathfrak{X}\, \overline{U}a\, \mathfrak{x}$ and $\mathfrak{x}\, a\, x$. Let O be any open neighbourhood of x. Since $\mathfrak{x}$ converges to x, $O \in \mathfrak{x}$. As $\mathfrak{X}\, \overline{U}a\, \mathfrak{x}$, this implies $a^\circ(O) = \{\mathfrak{z} \in UX \mid \exists z \in O : \mathfrak{z}\, a\, z\} \in \mathfrak{X}$ (see (2.5.6)). Then

$$a^\circ(O) \subseteq \{\mathfrak{z} \in UX \mid O \in \mathfrak{z}\} = O^\# \in \mathfrak{X}.$$

Hence $O \in m_X(\mathfrak{X})$, $m_X(\mathfrak{X})$ converges to x. So (X, a) is a $(\mathbb{U}, 2)$-category.

Conversely, let (X, a) be a $(\mathbb{U}, 2)$-category. Then it satisfies (2.7.4) and (2.7.5). One can define open sets of X by

$$O \text{ a-open in X} \iff a^\circ(O) \subseteq O^\#.$$

Trivially, $a^\circ(X) \subseteq X^\#$. For any a-open $O_1, O_2 \subseteq X$,

$$a^\circ(O_1 \cap O_2) \subseteq a^\circ(O_1) \cap a^\circ(O_2) \subseteq O_1^\# \cap O_2^\# = (O_1 \cap O_2)^\#.$$

Furthermore,

$$a^\circ(\bigcup_{i\in I} O_i) = \bigcup_{i\in I} a^\circ(O_i) \subseteq \bigcup_{i\in I} O_i^\# \subseteq (\bigcup_{i\in I} O_i)^\#$$

for any collection $\{O_i \subseteq X \mid O_i \text{ a-open}, i \in I\}$. Therefore a-open sets form a topology on X.

The correspondence between topological spaces and $(\mathbb{U}, 2)$-categories is bijective [4], [38].

A map $f : (X, \tau) \to (Y, \tau')$ is continuous if and only if for any $\mathfrak{x} \in UX$, $x \in X$ if $\mathfrak{x}$ converges to x, then $f(\mathfrak{x})$ converges to $f(x)$. This precisely means that $f : (X, a) \to (Y, b)$ is a $(\mathbb{U}, 2)$-functor where τ and τ' correspond to a and b respectively.

4. $(\mathbb{U}, \mathrm{P}_+)$-**Cat** $\cong$ **App** [14].

An approach space, introduced by Lowen [43], is a simultaneous generalization of a topological space and a metric space. There are several equivalent characterizations of an approach space. We will mention the characterizations by "distances", "towers" and "regular function frames".

An *approach space* (X, δ) consists of a set X and a distance function $\delta : PX \times X \to [0, \infty]$ which satisfies the following conditions:

- $\delta(\{x\}, x) = 0$,
- $\delta(\varnothing, x) = \infty$,
- $\delta(A \cup B, x) = \min\{\delta(A, x),\ \delta(B, x)\}$,
- $\delta(A, x) \le \delta(A^{(\varepsilon)}, x) + \varepsilon$ where $A^{(\varepsilon)} = \{z \in X \mid \delta(A, z) \le \varepsilon\}$,

for all $x \in X$ and $A, B \subseteq X$. Let (X, δ) and (Y, δ') be approach spaces. A map $f : X \to Y$ is called a *contraction* if

$$\delta(A, x) \ge \delta'(f(A), f(x))$$

for all $A \subseteq X$, $x \in X$. Approach spaces and contractions form the category **App**.

The bijective correspondence between approach spaces and $(\mathbb{U}, \mathrm{P}_+)$-categories is established as follows [14]:

Given an approach space (X, δ), one has the corresponding $(\mathbb{U}, \mathrm{P}_+)$-category (X, a) where $a : UX \times X \to [0, \infty]$ is defined by

$$a(\mathfrak{x}, x) = \sup\{\delta(A, x) \mid A \in \mathfrak{x}\} \qquad (2.7.6)$$

for $\mathfrak{x} \in UX$, $x \in X$. Conversely, given a $(\mathbb{U}, \mathrm{P}_+)$-category (X, a), the distance function $\delta : PX \times X \to [0, \infty]$ of the corresponding approach space is given by

$$\delta(A, x) = \inf\{a(\mathfrak{x}, x) \mid A \in \mathfrak{x}\} \qquad (2.7.7)$$

for $A \subseteq X$, $x \in X$. Futhermore, $f : (X, \delta) \to (Y, \delta')$ is a contraction if and only if $f : (X, a) \to (Y, b)$ is a $(\mathbb{U}, \mathrm{P}_+)$-functor where a and b correspond to δ and δ' respectively via (2.7.6) and (2.7.7).

Proposition 2.7.3 ([43]). Let (X, δ) be an approach space. For any $x \in X$, $A, B \subseteq X$, the following assertions hold:

1. $x \in A$ implies $\delta(A, x) = 0$.
2. $A \subseteq B$ implies $\delta(B, x) \leq \delta(A, x)$.
3. $\delta(A, x) \leq \sup_{y \in B} \delta(A, y) + \delta(B, x)$.

An approach structure on a set X is equivalently given by a tower. A *tower* is a family of functions

$$t_\varepsilon : PX \to PX, \quad \varepsilon \in [0, \infty]$$

that satisfies the following conditions for all $A, B \subseteq X$ and $\varepsilon, \sigma \in [0, \infty]$:

- $A \subseteq t_\varepsilon(A)$,
- $t_\varepsilon(\varnothing) = \varnothing$,
- $t_\varepsilon(A \cup B) = t_\varepsilon(A) \cup t_\varepsilon(B)$,
- $t_\varepsilon(t_\sigma(A)) \subseteq t_{\varepsilon+\sigma}(A)$,
- $t_\varepsilon(A) = \bigcap_{\varepsilon < \sigma} t_\sigma(A)$.

Transition from a distance function δ to a tower $\{t_\varepsilon \mid \varepsilon \in [0,\infty]\}$ is done by setting

$$t_\varepsilon(A) = \{x \in X \mid \delta(A,x) \leq \varepsilon\}.$$

We will denote the set $\{x \in X \mid \delta(A,x) \leq \varepsilon\}$ by $A^{(\varepsilon)}$.

One can also put an approach structure on a set X by a regular function frame. A *regular function frame* is a collection $\mathcal{R}$ of functions from X to $[0,\infty]$ which satisfy

- $\forall \mathcal{S} \subseteq \mathcal{R}$, $\bigvee \mathcal{S} \in \mathcal{R}$,
- $\forall \mu, \nu \in \mathcal{R}$, $\mu \wedge \nu \in \mathcal{R}$,
- $\forall \mu \in \mathcal{R}$, $\forall \alpha \in [0,\infty]$, $\mu + \alpha \in \mathcal{R}$,
- $\forall \mu \in \mathcal{R}$, $\forall \alpha \in [0,\infty]$, $\max\{\mu - \alpha, 0\} \in \mathcal{R}$.

Given an approach space (X,δ), its regular function frame $\mathcal{R}$ is the set of contractions from (X,δ) to (P_+,δ') where

$$\delta'(A,x) = \max\{x - \sup A, 0\}.$$

Remark 2.7.4. Consider the following commutative diagram which will be useful in the sequel.

$$\begin{tikzcd}
\mathbf{Ord} \arrow[r, hook] \arrow[d, hook] & \mathbf{Top} \arrow[d, hook] \\
\mathbf{Met} \arrow[r, hook] & \mathbf{App}
\end{tikzcd} \tag{2.7.8}$$

The embedding $2 \hookrightarrow \mathrm{P}_+$ which takes $\perp$ to ∞ and $\top$ to 0 induces the vertical embeddings which are both reflective and coreflective. An ordered set $(X, \leq)$ becomes a metric space (X, d) with the distance

$$d(y, x) = \begin{cases} 0 & \text{if } y \leq x, \\ \infty & \text{otherwise}. \end{cases}$$

Similarly any topological space (X, τ) can be construed as an approach space (X, δ) where

$$\delta(A, x) = \begin{cases} 0 & \text{if } x \in \overline{A}, \\ \infty & \text{otherwise}. \end{cases}$$

The coreflection of an approach space (X, δ) in **Top** has the closure operator defined by

$$\overline{A} = \{x \in X \mid \delta(A, x) = 0\}.$$

The horizontal embeddings of the diagram are coreflective. The embedding **Ord** $\hookrightarrow$ **Top** is the Alexandroff topology functor which takes $(X, \leq)$ to the topological space whose open sets are the down-closed sets. The collection $\{\downarrow x \mid x \in X\}$ of principal down-closures form a basis for this topology. The embedding $\mathbf{Met} \hookrightarrow \mathbf{App}$ takes a metric space (X, d) to the approach space (X, δ) where

$$\delta(A, x) = \inf_{y \in A} d(y, x)$$

for $A \subseteq X$, $x \in X$.

2.7.2 Basic facts

There are some important functors between $(\mathbb{T}, V)$-$\mathbf{Cat}$ and V-$\mathbf{Cat}$. One defines the functors M and A° as follows:

$$\begin{aligned} \mathsf{M} : (\mathbb{T}, V)\text{-}\mathbf{Cat} &\longrightarrow V\text{-}\mathbf{Cat} \\ (X, a) &\longmapsto (TX, \overline{T}a.m_X^\circ) \\ f &\longmapsto Tf \end{aligned}$$

$$\begin{aligned} \mathsf{A}^\circ : V\text{-}\mathbf{Cat} &\longrightarrow (\mathbb{T}, V)\text{-}\mathbf{Cat} \\ (X, b) &\longmapsto (X, e_X^\circ.\overline{T}b) \\ f &\longmapsto f \end{aligned}$$

To see that M is a functor, take a $(\mathbb{T}, V)$-category (X, a). One has $e_X^\circ \leq a$ which implies $Te_X^\circ \le \overline{T}a$. Then $1_{TX} = Te_X^\circ . m_X^\circ \le \overline{T}a.m_X^\circ$. Secondly,

$$\begin{aligned}
\overline{T}a.m_X^\circ.\overline{T}a.m_X^\circ &= \overline{T}a.\overline{T}^2 a.m_{TX}^\circ.m_X^\circ \\
&= \overline{T}a.\overline{T}^2 a.Tm_X^\circ.m_X^\circ \\
&= \overline{T}(a.\overline{T}a.m_X^\circ).m_X^\circ \\
&\le \overline{T}a.m_X^\circ
\end{aligned}$$

Hence $\mathsf{M}X = (TX, \overline{T}a.m_X^\circ)$ is a V-category. Since T is a functor, $\mathsf{M}(g.f) = \mathsf{M}g.\mathsf{M}f$. Therefore M is a functor.

To see that A° is a functor, take a V-category (X, b). One has $1_X \le b$ which implies $1_{TX} \le \overline{T}b$. Then $e_X^\circ \le e_X^\circ.\overline{T}b$. Secondly,

$$\begin{aligned}
(e_X^\circ.\overline{T}b) \circ (e_X^\circ.\overline{T}b) &= e_X^\circ.\overline{T}b.\overline{T}(e_X^\circ.\overline{T}b).m_X^\circ \\
&= e_X^\circ.\overline{T}b.(Te_X)^\circ.\overline{T}^2 b.m_X^\circ \\
&= e_X^\circ.\overline{T}b.(Te_X)^\circ.m_X^\circ.\overline{T}b \\
&= e_X^\circ.\overline{T}b.\overline{T}b \\
&= e_X^\circ.\overline{T}(b.b) \\
&\le e_X^\circ.\overline{T}b
\end{aligned}$$

Hence $\mathsf{A}^\circ X = (X, e_X^\circ.\overline{T}b)$ is a $(\mathbb{T}, V)$-category and A° is a functor.

For each $(\mathbb{T}, V)$-category (X, a), there are two important $(\mathbb{T}, V)$-categories to consider. These are the *free Eilenberg-Moore algebra*

$$|X| = (TX, m_X)$$

and the *dual* $(\mathbb{T}, V)$*-category*

$$X^{\mathrm{op}} = \mathsf{A}^{\circ}(\mathsf{M}(X)^{\mathrm{op}}).$$

Lemma 2.7.5. The functor A° has a right adjoint defined by

$$\begin{aligned} \mathsf{A} : (\mathbb{T}, V)\text{-}\mathbf{Cat} &\longrightarrow V\text{-}\mathbf{Cat} \\ (X, a) &\longmapsto (X, a.e_X) \\ f &\longmapsto f \end{aligned}$$

Proof. Firstly, we show that A is a functor. Let (X, a) be a $(\mathbb{T}, V)$-category. Then $e_X^{\circ} \leq a$ implies $1_X \leq a.e_X$. On the other hand,

$$a.e_X.a.e_X \leq a.\overline{T}a.e_{TX}.e_X \leq a.\overline{T}a.m_X^{\circ}.e_X \leq a.e_X.$$

Hence $\mathsf{A}X = (X, a.e_X)$ is a V-category and A is a functor.

Now we show that $\mathsf{A}^{\circ} \dashv \mathsf{A}$. Observe that

$$b \leq e_X^{\circ}.\overline{T}b.e_X$$

for any V-category $X = (X, b)$ as e is an op-lax natural transformation. This means $1_X : X \to \mathsf{A}(\mathsf{A}^{\circ}(X))$ is a V-functor. On the other hand, given any $(\mathbb{T}, V)$-category

(X, a),

$$e_X^\circ.\overline{T}(a.e_X) = e_X^\circ.\overline{T}a.Te_X \le e_X^\circ.\overline{T}a.m_X^\circ \le a.\overline{T}a.m_X^\circ \le a$$

which implies that $1_X : \mathsf{A}^\circ(\mathsf{A}(X)) \to X$ is a $(\mathbb{T}, V)$-functor. Therefore $\mathsf{A}^\circ \dashv \mathsf{A}$ where the unit and the counit of the adjunction are the identity natural transformations.

□

Remarks 2.7.6. 1. The embeddings **Ord** $\hookrightarrow$ **Top** and **Met** $\hookrightarrow$ **App** in diagram (2.7.8) are instances of the functor $\mathsf{A}^\circ : V\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Cat}$. The functor A sends a topological space X to the ordered set $(X, \le)$ with the dual specialization order, i.e. $x \le y$ if and only if $\overline{\{y\}} \subseteq \overline{\{x\}}$. For an approach space $X = (X, \delta)$, $\mathsf{A}X$ is the metric space (X, d) where $d(x, y) = \delta(\{x\}, y)$ for $x, y \in X$.

2. Consider the functor M. Let $X = (X, a)$ be a topological space with the convergence relation $a : UX \nrightarrow X$ and $\mathsf{M}(X) = (UX, \le)$. As shown in [15], given ultrafilters $\mathfrak{x}, \mathfrak{y} \in UX$, one has

$$\begin{aligned}
\mathfrak{x} \le \mathfrak{y} &\iff \exists \mathfrak{X} \in U^2X : m_X(\mathfrak{X}) = \mathfrak{x} \quad \& \quad \mathfrak{X}\,\overline{U}a\,\mathfrak{y} \\
&\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y}, \ \exists \mathfrak{w} \in UA, y \in B : \mathfrak{w}\ a\ y \\
&\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y}, \ \overline{A} \cap B \ne \varnothing \\
&\iff \forall A \in \mathfrak{x}, \ \overline{A} \in \mathfrak{y}.
\end{aligned}$$

Let $X = (X, a)$ be an approach space with the convergence P_+-relation $a : UX \nrightarrow X$. Suppose that "a" corresponds to the approach distance $\delta : PX \times X \to [0, \infty]$. Then $\mathsf{M}X$ is the metric space (UX, d) where

$$d(\mathfrak{x}, \mathfrak{y}) = \inf\{v \in [0, \infty] \mid \forall A \in \mathfrak{x},\ A^{(v)} \in \mathfrak{y}\} \tag{2.7.9}$$

for all ultrafilters $\mathfrak{x}, \mathfrak{y}$ on X [15]. The details follow.

One has $\mathsf{M}X = (UX, \overline{U}a.m_X^\circ)$ where

$$\overline{U}a.m_X^\circ(\mathfrak{x}, \mathfrak{y}) = \inf\{\overline{U}a(\mathfrak{X}, \mathfrak{y}) \mid \mathfrak{X} \in U^2X :\ m_X(\mathfrak{X}) = \mathfrak{x}\}.$$

Furthermore, $\overline{U}a(\mathfrak{X}, \mathfrak{y}) = \sup\limits_{\substack{\mathcal{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \inf\limits_{\substack{\mathfrak{z} \in \mathcal{A} \\ y \in B}} a(\mathfrak{z}, y)$ by (2.5.7).

Let

$$u = \inf\{\sup_{\substack{\mathcal{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \inf_{\substack{\mathfrak{z} \in \mathcal{A} \\ y \in B}} a(\mathfrak{z}, y) \mid \mathfrak{X} \in U^2X :\ m_X(\mathfrak{X}) = \mathfrak{x}\}$$

and

$$w = \inf\{v \in [0, \infty] \mid \forall A \in \mathfrak{x},\ A^{(v)} \in \mathfrak{y}\}.$$

We will show that $u = w$. For $\mathfrak{X} \in U^2X$ with $m_X(\mathfrak{X}) = \mathfrak{x}$, $A \in \mathfrak{x}$ implies $A^\# = \{\mathfrak{z} \in UX \mid A \in \mathfrak{z}\} \in \mathfrak{X}$. Hence

$$\sup_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{y}}} \inf_{\substack{\mathfrak{z} \in A^\# \\ y \in B}} a(\mathfrak{z}, y) \le u.$$

So for any $\varepsilon > 0$, $A \in \mathfrak{x}$, $B \in \mathfrak{y}$, there exists $\mathfrak{w} \in A^\#, y \in B$ such that $a(\mathfrak{w}, y) \le u + \varepsilon$. Then

$$\delta(A, y) = \inf\{a(\mathfrak{z}, y) \mid \mathfrak{z} \in A^\#\} \le a(\mathfrak{w}, y) \le u + \varepsilon.$$

So $y \in A^{(u+\varepsilon)} \cap B \neq \varnothing$. Since $\mathfrak{y}$ is an ultrafilter, $A^{(u+\varepsilon)} \in \mathfrak{y}$. As this holds for all $A \in \mathfrak{x}$ and $\varepsilon > 0$, one gets $w \leq u$.

For the reverse inequality, observe that $A^{(w+\varepsilon)} \cap B \neq \varnothing$ for all $A \in \mathfrak{x}$, $B \in \mathfrak{y}$ and $\varepsilon > 0$ by the definition of w. For any $y \in A^{(w+\varepsilon)} \cap B$, one has

$$\inf\{a(\mathfrak{z}, y) \mid \mathfrak{z} \in A^{\#}\} = \delta(A, y) \leq w + \varepsilon.$$

Hence $\inf\limits_{\substack{\mathfrak{z} \in A^{\#} \\ y \in B}} a(\mathfrak{z}, y) \leq w + \varepsilon$. Since this holds for all $A \in \mathfrak{x}$ and $B \in \mathfrak{y}$,

$$\sup_{\substack{A \in \mathfrak{x} \\ B \in \mathfrak{y}}} \inf_{\substack{\mathfrak{z} \in A^{\#} \\ y \in B}} a(\mathfrak{z}, y) \leq w + \varepsilon.$$

Let $\mathfrak{A} = \{A^{\#} \subseteq UX \mid A \in \mathfrak{x}\}$ and $\mathfrak{J} = \{\mathcal{I} \subseteq UX \mid \sup\limits_{B \in \mathfrak{y}} \inf\limits_{\substack{\mathfrak{z} \in \mathcal{I} \\ y \in B}} a(\mathfrak{z}, y) > w + \varepsilon\}$. Then $\mathfrak{A}$ is a filter basis and $\mathfrak{J}$ is an ideal such that $A^{\#} \cap \mathcal{I} = \varnothing$ for any $A^{\#} \in \mathfrak{A}$, $\mathcal{I} \in \mathfrak{J}$. By Lemma 2.2.1, there exist $\mathfrak{X} \in U^2X$ with $\mathfrak{A} \subseteq \mathfrak{X}$ and $\mathfrak{X} \cap \mathfrak{J} = \varnothing$. This means that $m_X(\mathfrak{X}) = \mathfrak{x}$ and $\sup\limits_{\substack{\mathcal{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \inf\limits_{\substack{\mathfrak{z} \in \mathcal{A} \\ y \in B}} a(\mathfrak{z}, y) \leq w + \varepsilon$. Hence $u \leq w$.

One can put a V-category structure on the quantale V. Consider the map $\multimap : V \times V \to V$ given in (2.1.5). Since k is the unit element with respect to $\otimes$, $k \leq v \multimap v$ for any $v \in V$. One also has

$$u \otimes (u \multimap v) \otimes (v \multimap w) \leq v \otimes (v \multimap w) \leq w$$

which implies that

$$(u \multimap v) \otimes (v \multimap w) \leq (u \multimap w)$$

for any $u, v, w \in V$. So $1_V \leq \multimap$ and $\multimap . \multimap \leq \multimap$. In other words $(V, \multimap)$ is a V-category.

One can also put a $(\mathbb{T}, V)$-category structure on the quantale V.

Proposition 2.7.7 ([15], [33]). (V, hom_ξ) is a $(\mathbb{T}, V)$-category with $hom_\xi = TV \xrightarrow{\xi} V \overset{\multimap}{\nrightarrow} V$.

Proof. Firstly, $1_V \leq \multimap$ and $1_V = \xi.e_V$ imply $1_V \leq \multimap .\xi.e_V$. Hence

$$e_V^\circ \leq \multimap .\xi.$$

Recall from Lemma 2.5.1 that

$$\xi.T(\multimap) \leq \multimap .\langle \xi.T\pi_1, \xi.T\pi_2 \rangle = \multimap .(\xi \times \xi).\langle T\pi_1, T\pi_2 \rangle.$$

Then $\overline{T}(\multimap) \leq \multimap .(\xi \times \xi)$ by the definition of the extension of T to V-**Rel** (2.5.4). Using V-relational composition, one can write it as $\overline{T}(\multimap) \leq \xi^\circ . \multimap .\xi$. Hence

$$\xi.\overline{T}(\multimap) \leq \multimap .\xi.$$

Also $\multimap . \multimap \leq \multimap$, since V is a V-category. $\xi.T\xi \leq \xi.m_V$ holds, as $\mathcal{T}$ is a topological theory. Then

$$(\multimap .\xi).\overline{T}(\multimap .\xi) = \multimap .\xi.\overline{T}(\multimap).T\xi \leq \multimap . \multimap .\xi.T\xi \leq (\multimap .\xi).m_V.$$

So $(\multimap .\xi).\overline{T}(\multimap .\xi).m_V^\circ \leq (\multimap .\xi)$ which means that

$$(\multimap .\xi) \circ (\multimap .\xi) \leq (\multimap .\xi).$$

Hence (V, hom_ξ) is a $(\mathbb{T}, V)$-category. $\square$

Examples 2.7.8. The $(\mathbb{U}, 2)$-category 2 is the Sierpinski space with $\{0\}$ open. The $(\mathbb{U}, \mathrm{P}_+)$-category P_+ is the approach space (P_+, δ) where

$$\delta(A, x) = \max\{x - \sup A, 0\}$$

for $A \subseteq \mathrm{P}_+$, $x \in \mathrm{P}_+$.

Given $(\mathbb{T}, V)$-categories (X, a) and (Y, b), one can form their tensor product

$$(X, a) \otimes (Y, b) = (X \times Y, a \otimes b)$$

where

$$a \otimes b(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) \otimes b(T\pi_2(\mathfrak{w}), y)$$

for $\mathfrak{w} \in T(X \times Y), (x, y) \in X \times Y$. The singleton set together with the constant relation k, denoted by (E, k), is the $\otimes$-neutral object.

In general $(\mathbb{T}, V)$-**Cat** is not a closed category. But one has the following result.

Proposition 2.7.9 ([33]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. $X \otimes (\_)$ has a right adjoint $(\_)^X$ if $a.\overline{T}a = a.m_X$

A $(\mathbb{T}, V)$-category (X, a) which satisfies the conditon of Prop. 2.7.9 is called $\otimes$-*exponentiable*. Given another $(\mathbb{T}, V)$-category (Y, b), the underlying set of the tensor exponential object $(Y^X, [\![a, b]\!])$ is the set of all $(\mathbb{T}, V)$-functors from X to Y.

The structure $[\![a,b]\!]$ is defined by

$$[\![a,b]\!](\mathfrak{p},h) = \bigvee \left\{ v \in V \mid \forall x \in X, \mathfrak{q} \in T\pi_2^{-1}(\mathfrak{p});\ a(T\pi_1(\mathfrak{q}),x) \otimes v \leq b(Tev(\mathfrak{q}),h(x)) \right\} \tag{2.7.10}$$

where $\mathfrak{p} \in T(Y^X), h \in Y^X$, $ev : Y^X \times X \to Y$ is the evaluation map, $\pi_1 : X \times Y^{|X|} \to X$ and $\pi_2 : X \times Y^{|X|} \to Y^{|X|}$ are the projection maps.

Lemma 2.7.10 ([36]). Let $X = (X,a)$ and $Y = (Y,b)$ be $(\mathbb{T},V)$-categories where X is $\otimes$-exponentiable and $f,g \in Y^{|X|}$. Then

$$[\![a,b]\!](e_{Y^X}(f),g) = \bigwedge_{x \in X} b(e_Y(f(x)),g(x))$$

Given a $(\mathbb{T},V)$-category $X = (X,a)$ and $i : M \hookrightarrow X$, M is subcategory of X with the structure $i^\circ.a.Ti$. Since $T : \mathbf{Set} \to \mathbf{Set}$ preserves monomorphisms, one has $Ti : TM \hookrightarrow TX$. So we will simply denote this subcategory by (M,a).

Since the forgetful functor from $(\mathbb{T},V)$-**Cat** to **Set** is topological, the limits in $(\mathbb{T},V)$-**Cat** are formed in **Set** with the initial structure on them. In particular we will denote the cartesian product of (X,a) and (Y,b) by

$$(X \times Y, a \times b)$$

where $a \times b = (\pi_1{}^\circ.a.T\pi_1) \wedge (\pi_2{}^\circ.b.T\pi_2)$, i.e.

$$a \times b(\mathfrak{w},(x,y)) = a(T\pi_1(\mathfrak{w}),x) \wedge b(T\pi_2(\mathfrak{w}),y)$$

for all $\mathfrak{w} \in T(X \times Y), (x,y) \in X \times Y$. The terminal object is the singleton set with the constant relation $\top$, which will be denoted by $1 = (1,\top)$.

2.8 $(\mathbb{T}, V)$-modules

$(\mathbb{T}, V)$-modules play an important role in developing the notions of separation and completeness for $(\mathbb{T}, V)$-categories. Firstly, we define $(\mathbb{T}, V)$-modules and review the notions of full faithfulness, L-density and L-equivalence for $(\mathbb{T}, V)$-functors as given in [36]. Secondly, we explore the relationship between $(\mathbb{T}, V)$-functors and $(\mathbb{T}, V)$-modules following [15]. We provide the Yoneda functor and the Yoneda lemma for $(\mathbb{T}, V)$-categories [15].

Let (X, a), (Y, b) be $(\mathbb{T}, V)$-categories and $\varphi : X \nrightarrow Y$ be a $(\mathbb{T}, V)$-relation. φ is called a $(\mathbb{T}, V)$-*module* if

$$\varphi \circ a \leq \varphi \quad \& \quad b \circ \varphi \leq \varphi.$$

In such a case we write $\varphi : X \rightsquigarrow Y$. $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-modules with the Kleisli convolution form the category

$$(\mathbb{T}, V)\text{-}\mathbf{Mod}.$$

Since $a \geq e_X^\circ$ and the Kleisli convolution preserves order, $\varphi \circ a \geq \varphi \circ e_X^\circ = \varphi$. Similarly $b \circ \varphi \geq \varphi$. So one actually has $\varphi \circ a = \varphi$ and $b \circ \varphi = \varphi$. As a result $a : X \rightsquigarrow X$ functions as the identity morphism of (X, a) in $(\mathbb{T}, V)$-$\mathbf{Mod}$. $(\mathbb{T}, V)$-$\mathbf{Mod}$ is a 2-category, as $(\mathbb{T}, V)$-modules inherit the order on $(\mathbb{T}, V)$-relations. This allows one to consider adjunctions in $(\mathbb{T}, V)$-$\mathbf{Mod}$.

A $(\mathbb{T},V)$-functor $f:(X,a)\to(Y,b)$ induces two important $(\mathbb{T},V)$-modules: $f_*:(X,a)\rightsquigarrow(Y,b)$ and $f^*:(Y,b)\rightsquigarrow(X,a)$ given by

$$f_* = b.Tf \qquad \& \qquad f^* = f^\circ.b.$$

For $(\mathbb{T},V)$-modules $\varphi:(Y,b)\rightsquigarrow(Z,c)$ and $\psi:(Z,c)\rightsquigarrow(Y,b)$, one has

$$\varphi\circ f_* = \varphi.Tf \qquad \& \qquad f^*\circ\psi = f^\circ.\psi.$$

These identities follow from

$$\varphi\circ f_* = \varphi.\overline{T}b.T^2f.m_X^\circ = \varphi.\overline{T}b.m_Y^\circ.Tf = \varphi.Tf,$$

$$f^*\circ\psi = f^\circ.b.\overline{T}\psi.m_Z^\circ = f^\circ.\psi.$$

Given $(\mathbb{T},V)$-functors $f:(X,a)\to(Y,b)$, $g:(Y,b)\to(Z,c)$ one has

$$(g.f)_* = g_*\circ f_* \qquad \& \qquad (g.f)^* = f^*\circ g^*.$$

Observe that for any $(\mathbb{T},V)$-category (X,a), $a=(1_X)^*=(1_X)_*$. So one has the lower star functor and the upper star functor

$$(\_)_*:(\mathbb{T},V)\textbf{-Cat}\to(\mathbb{T},V)\textbf{-Mod} \qquad \& \qquad (\_)^*:((\mathbb{T},V)\textbf{-Cat})^{\mathrm{op}}\to(\mathbb{T},V)\textbf{-Mod}$$

which are identical on objects and which take a $(\mathbb{T},V)$-functor f to f_* and f^* respectively.

One sees that $f_*\dashv f^*$ as

$$f^*\circ f_* = f^\circ.b.Tf \geq a = (1_X)^*,$$

$$f_* \circ f^* = b.Tf.Tf^\circ.\overline{T}b.m_Y^\circ \leq b.\overline{T}b.m_Y^\circ = b \circ b \leq b = (1_Y)^*.$$

Let $f, g : (X, a) \to (Y, b)$ be $(\mathbb{T}, V)$-functors. By taking advantage of the adjunctions $f_* \dashv f^*$ and $g_* \dashv g^*$,

$$f^* \geq g^* \iff f_* \leq g_*$$

One defines $f \geq g$ if $f^* \geq g^*$ or, equivalently, $f_* \leq g_*$. f and g are called *equivalent*, written $f \simeq g$, if $f \leq g$ and $f \geq g$.

Lemma 2.8.1 ([36]). Let $f, g : (X, a) \to (Y, b)$ be $(\mathbb{T}, V)$-functors. Then $f \leq g$ if and only if $k \leq b(e_Y(f(x)), g(x))$ for all $x \in X$.

Proof. Suppose that $f \leq g$. Then $g_* \leq f_*$. For all $x \in X$, one has

$$\begin{aligned} k \leq a(e_X(x), x) \leq g^* \circ g_*(e_X(x), x) \leq g^* \circ f_*(e_X(x), x) &= b(Tf.e_X(x), g(x)) \\ &= b(e_Y(f(x)), g(x)). \end{aligned}$$

Conversely, suppose that $k \leq b(e_Y(f(x)), g(x))$ for all $x \in X$. Since $e : \overline{T} \to \mathrm{Id}$ is an op-lax natural transformation and (Y, b) is a $(\mathbb{T}, V)$-category, one gets

$$\begin{aligned} f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x)) &\leq \overline{T}b(e_{TY}(\mathfrak{y}), e_Y(f(x))) \otimes b(e_Y(f(x)), g(x)) \\ &\leq b(m_Y.e_{TY}(\mathfrak{y}), g(x)) \\ &= b(\mathfrak{y}, g(x)) \\ &= g^*(\mathfrak{y}, x). \end{aligned}$$

for all $x \in X$, $\mathfrak{y} \in TY$. Hence $f^* \leq g^*$, $f \leq g$.

$\square$

One calls a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ *fully faithful* if

$$f^* \circ f_* = (1_X)^*$$

or, equivalently,

$$b(Tf(\mathfrak{x}), f(x)) = a(\mathfrak{x}, x) \tag{2.8.1}$$

for all $x \in X$, $\mathfrak{x} \in TX$. Since $f^* \circ f_* \geq (1_X)^*$ always holds, f is fully faithful if and only if $f^* \circ f_* \leq (1_X)^*$. A fully faithful $(\mathbb{T}, V)$-functor that is also injective is called a *full emdedding*. f is called *L-dense* if

$$f_* \circ f^* = (1_Y)^*.$$

Since $f_* \circ f^* \leq (1_Y)^*$ always holds, f is L-dense if and only if $f_* \circ f^* \geq (1_Y)^*$. Composition of fully faithful (L-dense) $(\mathbb{T}, V)$-functors are fully faithful (L-dense). A $(\mathbb{T}, V)$-functor which is both fully faithful and L-dense is called an *L-equivalence*. L-equivalences are isomorphisms in $(\mathbb{T}, V)$-**Mod**.

The following proposition will be useful in the sequel.

Proposition 2.8.2 ([36]). Let $f : (X, a) \to (Y, b)$, $g : (Y, b) \to (Z, c)$ be $(\mathbb{T}, V)$-functors.

1. If $g.f$ is fully faithful, then f is fully faithful.

2. If $g.f$ is L-dense, then g is L-dense.

3. If $g.f$ is fully faithful and f is L-dense, then g is fully faithful.

4. If $g.f$ is L-dense and g is fully faithful, then f is L-dense.

5. If f is surjective, then it is L-dense.

Proof. We only show 1,3 and 5. Proofs of 2 and 4 are similar.

1. Suppose that $g.f$ is fully faithful. Then

$$(1_X)^* = (g.f)^* \circ (g.f)_* = f^* \circ g^* \circ g_* \circ f_* \geq f^* \circ f_*.$$

3. Suppose that $g.f$ is fully faithful and f is L-dense. Then

$$(1_Y)^* = f_* \circ f^* = f_* \circ (1_X)^* \circ f^* = f_* \circ (f^* \circ g^* \circ g_* \circ f_*) \circ f^* = g^* \circ g_*.$$

5. Suppose that f is surjective. Then $f.f^\circ = 1_Y$. Hence

$$f_* \circ f^* = b.Tf.\overline{T}(f^\circ.b).m_Y^\circ = b.T(f.f^\circ).\overline{T}b.m_Y^\circ = b.\overline{T}b.m_Y^\circ = b = (1_Y)^*.$$

□

Proposition 2.8.3. Fully faithful $(\mathbb{T}, V)$-functors are pullback stable.

Proof. Let $g : (Y, b) \to (Z, c)$ be a fully faithful $(\mathbb{T}, V)$-functor. Consider its pullback along a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Z, c)$.

$$\begin{tikzcd}
(X \times_Z Y, a \times b) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, b) \arrow[d, "g"] \\
(X, a) \arrow[r, "f"'] & (Z, c)
\end{tikzcd}$$

Take any $\mathfrak{w} \in T(X \times_Z Y)$ and $(x, y) \in T(X \times_Z Y)$. Then

$$a(T\pi_1(\mathfrak{w}), x) \le c(Tf.T\pi_1(\mathfrak{w}), f(x)) = c(Tg.T\pi_2(\mathfrak{w}), g(y)) = b(T\pi_2(\mathfrak{w}), y).$$

This means that $a(T\pi_1(\mathfrak{w}), x) = a(T\pi_1(\mathfrak{w}), x) \wedge b(T\pi_2(\mathfrak{w}), y)$. So

$$(1_{X \times_Z Y})^* = a \times b = \pi_1^\circ.a.T\pi_1 = (\pi_1)^* \circ (\pi_1)_*.$$

Hence π_1 is fully faithful. $\square$

Examples 2.8.4.

1. In **Met**, a nonexpansive map $f : (X, d) \to (Y, d')$ is fully faithful if and only if $d(x, z) = d'(f(x), f(z))$ for all $x, z \in X$. So fully faithful maps are precisely isometries.

2. Given a continuous map $f : X \to Y$ in **Top**, let $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ denote the corresponding frame homomorphism between the lattice of open sets of Y and the lattice of open sets of X.

Claim: $f : X \to Y$ is fully faithful if and only if $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is surjective.

To see this, let f be fully faithful and $U \subseteq X$ be an open set. Take any point $x \in U$. The collection $\mathscr{F} = \{V \subseteq X \mid \exists O \subseteq Y \text{ open} : f(x) \in O, f^{-1}(O) \subseteq V\}$ is a filter on X. Let $\mathfrak{x}$ be any ultrafilter that contains $\mathscr{F}$. Then $f(\mathfrak{x})$ converges to $f(x)$, since for any open neighbourhood O of $f(x)$, $f^{-1}(O) \in \mathscr{F} \subseteq \mathfrak{x}$. As f is fully faithful, $\mathfrak{x}$ converges to x (see (2.8.1)). Given that $\mathfrak{x}$ is arbitrary, any ultrafilter refining $\mathscr{F}$ converges to x. This implies that $\mathscr{F}$ converges to x. Hence $U \in \mathscr{F}$. Then there exists an open set $O_x \subseteq Y$ such that $x \in f^{-1}(O_x) \subseteq U$. Repeating this for each $x \in U$, one gets $U = \bigcup\limits_{x \in U} f^{-1}(O_x) = f^{-1}(\bigcup\limits_{x \in U} O_x)$. Hence f^{-1} is surjective.

Conversely, assume that f^{-1} is surjective. Let $\mathfrak{x}$ be any ultrafilter on X such that $f(\mathfrak{x})$ converges to $f(x)$. Take any open neighbourhood $U \subseteq X$ of x. Since f^{-1} is surjective, there exists an open neighbourhood $O \subseteq Y$ of $f(x)$ such that $f^{-1}(O) = U$. As $f(\mathfrak{x})$ converges to $f(x)$, $O \in f(\mathfrak{x})$. Then $f^{-1}(O) = U \in \mathfrak{x}$ and $\mathfrak{x}$ converges to x. Therefore f is fully faithful.

In light of this characterization, f is fully faithful if and only if $\mathcal{O}X = \{f^{-1}(O) \mid O \text{ open in } Y\}$. If f is also injective, i.e. a full embedding, then X is homeomorphic with $f(X)$. Hence f is a subspace embedding. On the other hand, every subspace embedding is a full emdedding. So full embeddings in **Top** are precisely subspace embeddings.

There is a close relationship between $(\mathbb{T}, V)$-modules and $(\mathbb{T}, V)$-functors.

Proposition 2.8.5 ([15]). Let $(X, a), (Y, b)$ be $(\mathbb{T}, V)$-categories and $\psi : (X, a) \nrightarrow (Y, b)$ be a $(\mathbb{T}, V)$-relation. The following are equivalent:

1. $\psi : (X, a) \rightsquigarrow (Y, b)$ is a $(\mathbb{T}, V)$-module.

2. Both $\psi : |X| \otimes Y \to V$ and $\psi : X^{\mathrm{op}} \otimes Y \to V$ are $(\mathbb{T}, V)$-functors.

Proof. Suppose that $\psi : (X, a) \rightsquigarrow (Y, b)$ is a $(\mathbb{T}, V)$-module. $\psi : |X| \otimes Y \to V$ is a $(\mathbb{T}, V)$-functor if and only if

$$m_X(T\pi_1(\mathfrak{W}), \mathfrak{x}) \otimes b(T\pi_2(\mathfrak{W}), y) \leq \xi.T\psi(\mathfrak{W}) \multimap \psi(\mathfrak{x}, y)$$

for any $\mathfrak{W} \in T(TX \times Y)$ and $(\mathfrak{x}, y) \in TX \times Y$. This inequality holds trivially if $m_X.T\pi_1(\mathfrak{W}) \neq \mathfrak{x}$. So assume $m_X.T\pi_1(\mathfrak{W}) = \mathfrak{x}$. In that case one needs to show

$$\xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y).$$

Observe that

$$\xi.T\psi(\mathfrak{W}) \leq \overline{T}\psi(T\pi_1(\mathfrak{W}), T\pi_2(\mathfrak{W})) \qquad (\dagger)$$

by the definition of the extension of T to V-**Rel** (2.5.4). As ψ is a T-module, one has $b.\overline{T}\psi.m_X^\circ \leq \psi$ or, equivalently, $b.\overline{T}\psi \leq \psi.m_X$. Then

$$\begin{aligned}
\xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) &\leq \overline{T}\psi(T\pi_1(\mathfrak{W}), T\pi_2(\mathfrak{W})) \otimes b(T\pi_2(\mathfrak{W}), y) \\
&\leq \psi.m_X(T\pi_1(\mathfrak{W}), y) \\
&= \psi(\mathfrak{x}, y)
\end{aligned}$$

Hence $\psi : |X| \otimes Y \to V$ is a $(\mathbb{T}, V)$-functor.

Now we look at $\psi : X^{\mathrm{op}} \otimes Y \to V$ where $X^{\mathrm{op}} = \mathsf{A}^\circ(\mathsf{M}(X)^{\mathrm{op}}) = (TX, e_{TX}^\circ.Tm_X.\overline{T}^2 a^\circ)$. $\psi : X^{\mathrm{op}} \otimes Y \to V$ is a $(\mathbb{T}, V)$-functor if and only if

$$e_{TX}^\circ.Tm_X.\overline{T}^2 a^\circ(T\pi_1(\mathfrak{W}), \mathfrak{x}) \otimes b(T\pi_2(\mathfrak{W}), y) \leq \xi.T\psi(\mathfrak{W}) \multimap \psi(\mathfrak{x}, y)$$

or, equivalently,

$$\overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, T\pi_1(\mathfrak{W})) \otimes \xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y)$$

for any $\mathfrak{W} \in T(TX \times Y)$ and $(\mathfrak{x}, y) \in TX \times Y$. This inequality is obtained as follows:

$$\overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, T\pi_1(\mathfrak{W})) \otimes \xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y)$$

$$\leq \overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, T\pi_1(\mathfrak{W})) \otimes \overline{T}\psi(T\pi_1(\mathfrak{W}), T\pi_2(\mathfrak{W})) \otimes b(T\pi_2(\mathfrak{W}), y)$$

$$\leq b.\overline{T}\psi.\overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, y)$$

$$\leq b.\overline{T}\psi.\overline{T}^2 a.Tm_X^\circ.m_X^\circ(\mathfrak{x}, y)$$

$$= b.\overline{T}\psi.\overline{T}^2 a.m_{TX}^\circ.m_X^\circ(\mathfrak{x}, y)$$

$$= b.\overline{T}\psi.m_X^\circ.\overline{T}a.m_X^\circ(\mathfrak{x}, y)$$

$$\leq \psi.\overline{T}a.m_X^\circ(\mathfrak{x}, y)$$

$$\leq \psi(\mathfrak{x}, y)$$

The steps depend on the inequality (†), the facts that (T, e, m) is a monad which satisfies (BC) and ψ is a $(\mathbb{T}, V)$-module.

Conversely, suppose that both $\psi : |X| \otimes Y \to V$ and $\psi : X^{\mathrm{op}} \otimes Y \to V$ are $(\mathbb{T}, V)$-functors. One needs to show $b.\overline{T}\psi.m_X^\circ \leq \psi$ and $\psi.\overline{T}a.m_X^\circ \leq \psi$. Pick any $\mathfrak{x} \in TX$ and $y \in Y$. Then

$$\begin{aligned} b.\overline{T}\psi.m_X^\circ(\mathfrak{x}, y) &= \bigvee_{\substack{\mathfrak{X}\in T^2X \\ m_X(\mathfrak{X})=\mathfrak{x} \\ \mathfrak{y}\in TY}} \overline{T}\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \\ &= \bigvee_{\substack{\mathfrak{X}\in T^2X \\ m_X(\mathfrak{X})=\mathfrak{x} \\ \mathfrak{y}\in TY}} \bigvee_{\substack{\mathfrak{W}\in T(TX\times Y) \\ T\pi_1(\mathfrak{W})=\mathfrak{X} \\ T\pi_2(\mathfrak{W})=\mathfrak{y}}} \xi.T\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y). \end{aligned}$$

Since T satisfies (BC), for any $\mathfrak{y} \in TY$ and $\mathfrak{X} \in T^2X$ with $m_X(\mathfrak{X}) = \mathfrak{x}$ there exists

$\mathfrak{W} \in T(TX \times Y)$ such that $T\pi_1(\mathfrak{W}) = \mathfrak{X}$ and $T\pi_2(\mathfrak{W}) = \mathfrak{y}$. So

$$b.\overline{T}\psi.m_X^\circ(\mathfrak{x}, y) = \bigvee_{\substack{\mathfrak{W} \in T(TX \times Y) \\ m_X.T\pi_1(\mathfrak{W}) = \mathfrak{x}}} \xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y).$$

As $\psi : |X| \otimes Y \to V$ is a $(\mathbb{T}, V)$-functor,

$$\xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y)$$

for any $\mathfrak{W} \in T(TX \times Y)$ with $m_X.T\pi_1(\mathfrak{W}) = \mathfrak{x}$. Hence

$$\bigvee_{\substack{\mathfrak{W} \in T(TX \times Y) \\ m_X.T\pi_1(\mathfrak{W}) = \mathfrak{x}}} \xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y).$$

Therefore $b.\overline{T}\psi.m_X^\circ \leq \psi$.

For the second inequality, observe that

$$\psi.\overline{T}a.m_X^\circ \leq b.e_Y.(\psi.\overline{T}a.m_X^\circ) \leq b.\overline{T}(\psi.\overline{T}a.m_X^\circ).e_{TX} = b.\overline{T}\psi.\overline{T}^2 a.Tm_X^\circ.e_{TX},$$

as (Y, b) is a $(\mathbb{T}, V)$-category and e is an op-lax natural transformation. Hence

$$\begin{aligned}
\psi.\overline{T}a.m_X^\circ(\mathfrak{x}, y) &\leq b.\overline{T}\psi.\overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, y) \\
&= \bigvee_{\substack{\mathfrak{X} \in T^2X \\ \mathfrak{y} \in TY}} \overline{T}\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \otimes \overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, \mathfrak{X}) \\
&= \bigvee_{\substack{\mathfrak{X} \in T^2X \\ \mathfrak{y} \in TY}} \bigvee_{\substack{\mathfrak{W} \in T(TX \times Y) \\ T\pi_1(\mathfrak{W}) = \mathfrak{X} \\ T\pi_2(\mathfrak{W}) = \mathfrak{y}}} \xi.T\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \otimes \overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, \mathfrak{X}) \\
&= \bigvee_{\mathfrak{W} \in T(TX \times Y)} \xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \otimes \overline{T}^2 a.Tm_X^\circ.e_{TX}(\mathfrak{x}, T\pi_1(\mathfrak{W}))
\end{aligned}$$

where the last equality follows as T satisfies (BC).

Since $\psi : X^{\mathrm{op}} \otimes Y \to V$ is a $(\mathbb{T}, V)$-functor, one has

$$\xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \otimes \overline{T}^2 a.Tm_X^{\circ}.e_{TX}(\mathfrak{x}, T\pi_1(\mathfrak{W})) \leq \psi(\mathfrak{x}, y)$$

for any $\mathfrak{W} \in T(TX \times Y)$. Hence

$$\bigvee_{\mathfrak{W} \in T(TX \times Y)} \xi.T\psi(\mathfrak{W}) \otimes b(T\pi_2(\mathfrak{W}), y) \otimes \overline{T}^2 a.\overline{T}m_X^{\circ}.e_{TX}(\mathfrak{x}, T\pi_1(\mathfrak{W})) \leq \psi(\mathfrak{x}, y).$$

Therefore $\psi.\overline{T}a.m_X^{\circ} \leq \psi$. □

In particular for any $(\mathbb{T}, V)$-category (X, a), $a : X \rightsquigarrow X$ can be seen as a $(\mathbb{T}, V)$-functor $a : |X| \otimes X \to V$. Since $|X|$ is $\otimes$-exponentiable, one consider its mate

$$y = \ulcorner a \urcorner : X \to V^{|X|}$$

called the *Yoneda functor of* X. It is given by $y(x) = a(\_, x)$. The following result corresponds to the Yoneda lemma for $(\mathbb{T}, V)$-categories.

Proposition 2.8.6 ([15]). Let (X, a) be a $(\mathbb{T}, V)$-category and $\psi : |X| \to V$ be a $(\mathbb{T}, V)$-functor. Then

1. $[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}), \psi) \leq \psi(\mathfrak{x})$ for all $\mathfrak{x} \in TX$,

2. $[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}), \psi) \geq \psi(\mathfrak{x})$ for all $\mathfrak{x} \in TX$ if and only if $\psi : X^{\mathrm{op}} \to V$ is a $(\mathbb{T}, V)$-functor.

Proof. 1. One has the following commutative diagrams where the square on the left is a pullback.

$$
\begin{tikzcd}
TX \times X \arrow[r, "1_{TX} \times y"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & TX \times V^{|X|} \arrow[d, "\pi_2"] \\
X \arrow[r, "y"'] & V^{|X|}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \\
TX \times X \arrow[ur, "a"] \arrow[r, "1_{TX} \times y"'] & TX \times V^{|X|} \arrow[u, "ev"']
\end{tikzcd}
$$

Let $\psi \in V^{|X|}$ and $\mathfrak{x} \in TX$. By (2.7.10),

$$
\begin{aligned}
[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}), \psi) &= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{z} \in T\pi_2{}^{-1}(Ty(\mathfrak{x}));\\
&\qquad\qquad m_X(T\pi_1(\mathfrak{z}), \mathfrak{y}) \otimes v \leq hom_\xi(Tev(\mathfrak{z}), \psi(\mathfrak{y}))\}\\
&= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{z} \in T(TX \times V^{|X|}):\\
&\qquad\qquad T\pi_1(\mathfrak{z}) = \mathfrak{Y}, T\pi_2(\mathfrak{z}) = Ty(\mathfrak{x});\ v \leq \xi.Tev(\mathfrak{z}) \multimap \psi(\mathfrak{y})\}.
\end{aligned}
$$

Observe the following commutative diagram. The square below is a weak pullback as T satisfies (BC).

$$\begin{tikzcd}
 & & TV \\
T(TX \times X) \arrow[urr, "Ta"] \arrow[rr, "T(1_{TX} \times y)"'] \arrow[d, "T\pi_2"'] & & T(TX \times V^{|X|}) \arrow[u, "Tev"'] \arrow[d, "T\pi_2"] \\
TX \arrow[rr, "Ty"'] & & TV^{|X|}
\end{tikzcd}$$

So one has

$$\begin{aligned}
[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}), \psi) &= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times X) : \\
&\qquad T\pi_1(\mathfrak{W}) = \mathfrak{Y}, T\pi_2(\mathfrak{W}) = \mathfrak{x};\ \ v \le \xi.Ta(\mathfrak{W}) \multimap \psi(\mathfrak{y})\} \\
&= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y});\ \ v \le \bigwedge_{\substack{\mathfrak{W} \in T(TX \times X) \\ T\pi_1(\mathfrak{W}) = \mathfrak{Y} \\ T\pi_2(\mathfrak{W}) = \mathfrak{x}}} \big(\xi.Ta(\mathfrak{W}) \multimap \psi(\mathfrak{y})\big)\} \\
&= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y});\ \ v \le \big(\bigvee_{\substack{\mathfrak{W} \in T(TX \times X) \\ T\pi_1(\mathfrak{W}) = \mathfrak{Y} \\ T\pi_2(\mathfrak{W}) = \mathfrak{x}}} \xi.Ta(\mathfrak{W})\big) \multimap \psi(\mathfrak{y})\} \\
&= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y});\ \ v \le \overline{T}a(\mathfrak{Y}, \mathfrak{x}) \multimap \psi(\mathfrak{y})\} \\
&= \bigvee\{v \in V \mid \forall \mathfrak{y} \in TX;\ \ \overline{T}a.m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes v \le \psi(\mathfrak{y})\}. \qquad (2.8.2)
\end{aligned}$$

Suppose that $v \in V$ belongs to the set over which the supremum in (2.8.2) is taken. Then for $\mathfrak{y} = \mathfrak{x}$, one gets

$$v = k \otimes v \le \overline{T}a.m_X^\circ(\mathfrak{x}, \mathfrak{x}) \otimes v \le \psi(\mathfrak{x}).$$

Hence $[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}),\psi) \leq \psi(\mathfrak{x})$.

2. Let $\psi \in V^{|X|}$. As $X^{\mathrm{op}} = \mathsf{A}^\circ(\mathsf{M}(X)^{\mathrm{op}})$, $\psi : X^{\mathrm{op}} \to V$ is a $(\mathbb{T}, V)$-functor if and only if $\psi : \mathsf{M}X \to V$ is a V-functor. The latter means

$$\overline{T}a.m_X^\circ(\mathfrak{y},\mathfrak{x}) \otimes \psi(\mathfrak{x}) \leq \psi(\mathfrak{y})$$

for any $\mathfrak{x},\mathfrak{y} \in TX$. That is equivalent to $[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}),\psi) \geq \psi(\mathfrak{x})$ for all $\mathfrak{x} \in TX$ by (2.8.2).

□

One defines the $(\mathbb{T}, V)$-category $\widehat{X} = (\widehat{X}, \widehat{a})$ by

$$\widehat{X} = \{\psi \in V^{|X|} \mid \psi : X^{\mathrm{op}} \to V \text{ is a } (\mathbb{T}, V)\text{-functor}\}.$$

$\widehat{X}$ is considered as a subcategory of $V^{|X|}$ and $\widehat{a}$ is the restriction of $[\![m_X, hom_\xi]\!]$ to $\widehat{X}$. The following is an important property of the Yoneda functor $y : X \to V^{|X|}$.

Corollary 2.8.7 ([15]). The Yoneda functor $y : (X,a) \to (\widehat{X},\widehat{a})$ is fully faithful.

Proof. By Prop. 2.8.5, $a : X^{\mathrm{op}} \otimes X \to V$ is a $(\mathbb{T}, V)$-functor. Then $y(x) = a(\_, x) : X^{\mathrm{op}} \to V$ is a $(\mathbb{T}, V)$-functor and $y(x) \in \widehat{X}$ for all $x \in X$. So one has $y : (X,a) \to (\widehat{X},\widehat{a})$.

Letting $\psi = y(x)$ in Prop. 2.8.6, one gets

$$[\![m_X, hom_\xi]\!](Ty(\mathfrak{x}), y(x)) = y(x)(\mathfrak{x}) = a(\mathfrak{x}, x)$$

for all $\mathfrak{x} \in TX$, $x \in X$. Hence the Yoneda functor is functor is fully faithful. □

3 L-completeness, L-separation, L-closure

This chapter reviews some important notions for $(\mathbb{T}, V)$-categories like L-completeness [15], L-separation, L-closure and L-injectivity [36], as well their interactions. In addition to these known concepts, we introduce two new closure operators in $(\mathbb{T}, V)$-**Cat**, namely the natural closure and the dual closure, in Section 3.4. By investigating L-closure in detail, we show that L-closure for approach spaces is equal to its Zariski closure [26]. Furthermore, we provide a concrete characterization of this closure for approach spaces.

3.1 L-separation

Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. Recall that two $(\mathbb{T}, V)$-functors $f, g : Z \to X$ are called equivalent, written $f \simeq g$, if $f_* = g_*$ or, equivalently, $f^* = g^*$. X is called *L-separated* if given any $(\mathbb{T}, V)$-functors $f, g : Z \to X$, $f \simeq g$ implies $f = g$.

In this context it is enough to consider the $\otimes$-neutral object E in the place of Z as the next proposition shows.

Proposition 3.1.1 ([36]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. Then the following are equivalent:

1. X is L-separated.

2. For any $x, y \in X$, $x \simeq y$ implies $x = y$.

3. For any $x, y \in X$, $k \leq a(e_X(x), y)$ and $k \leq a(e_X(y), x)$ implies $x = y$.

4. The Yoneda functor $y : X \to \widehat{X}$ is injective.

Proof. ($1 \Leftrightarrow 2$) Suppose that X is L-separated. One can consider an element $x \in X$ as a map $x : E \to X$. Then (2) means, given any $x, y : E \to X$, $x \simeq y$ implies $x = y$. Letting $Z = E$, one gets the result.

For the reverse implication suppose that $x \simeq y$ implies $x = y$ for all $x, y \in X$. Let $f \simeq g$. One has

$$(f(z))_* = (f.z)_* = f_* \circ z_* = g_* \circ z_* = (g.z)_* = (g(z))_*,$$

i.e. $f(z) \simeq g(z)$ for all $z \in Z$. By the hypothesis, $f(z) = g(z)$ for all $z \in Z$. Hence $f = g$, X is L-separated.

($2 \Leftrightarrow 3 \Leftrightarrow 4$) One has

$$\begin{aligned} y(x) = y(y) &\iff x^* = y^* \\ &\iff x \simeq y \\ &\iff x \leq y \quad \& \quad y \leq x \\ &\iff k \leq a(e_X(x), y) \quad \& \quad k \leq a(e_X(x), y) \end{aligned}$$

where the last equivalence is due to Lemma 2.8.1. □

Corollary 3.1.2. 1. The $(\mathbb{T}, V)$-category $V = (V, hom_\xi)$ is L-separated.

2. Let $X = (X, a)$, $Y = (Y, b)$ be $(\mathbb{T}, V)$-categories where X is $\otimes$-exponentiable

and Y is L-separated. Then Y^X is L-separated. In particular $V^{|X|}$ is L-separated.

3. Any subcategory of an L-separated $(\mathbb{T}, V)$-category is L-separated. In particular $\widehat{X}$ is L-separated for any $(\mathbb{T}, V)$-category X.

Proof. 1. Let $u, w \in V$ such that $k \leq \xi(e_V(u)) \multimap w$ and $k \leq \xi(e_V(w)) \multimap u$. Since $\xi.e_V = 1_V$, one has $k \leq u \multimap w$ and $k \leq w \multimap u$. This implies $u \leq w$ and $w \leq u$. Hence $u = w$.

2. Let $f, g \in Y^{|X|}$ such that $k \leq [\![a,b]\!](e_{Y^X}(f), g)$ and $k \leq [\![a,b]\!](e_{Y^X}(g), f)$. By Lemma 2.7.10, one gets $k \leq b(e_Y(f(x)), g(x))$ and $k \leq b(e_Y(g(x)), f(x))$ for all $x \in X$. Since Y is L-separated, $f(x) = g(x)$ for all $x \in X$ by Prop. 3.1.2. Hence $f = g$, Y^X is L-separated.

Since V is L-separated and $|X|$ is $\otimes$-exponentiable, $V^{|X|}$ is L-separated.

3. Trivial.

□

Examples 3.1.3. 1. An ordered set $(X, \leq)$ is L-separated if and only if the order $\leq$ is antisymmetric. Hence an L-separated ordered set is precisely what is usually called a partially ordered set.

2. A metric space (X, d) is L-separated if and only if for any $x, y \in X$, $d(x, y) =$

$d(y,x) = 0$ implies $x = y$.

3. A topological space (X,τ) is L-separated if and only if for any $x, z \in X$, $\dot{x} \to z$ and $\dot{z} \to x$ implies $x = z$ or, equivalently, $z \in \overline{\{x\}}$ and $x \in \overline{\{z\}}$ implies $x = z$. So for any two distinct points $x, z \in X$ there exists an open set which contains one but not the other. This means that X is T_0.

4. An approach space (X,δ) is L-separated if and only if for any $x, z \in X$, $\delta(\{x\}, z) = \delta(\{z\}, x) = 0$ implies that $x = z$. The coreflection of (X,δ) in **Top** has the closure operator defined by $\overline{A} = \{x \in X \mid \delta(A,x) = 0\}$. So (X,δ) is L-separated if and only if $z \in \overline{\{x\}}$ and $x \in \overline{\{z\}}$ implies $x = z$. This means that the coreflection of (X,δ) is T_0.

3.2 L-completeness

Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. X is called *L-complete* [15] if for any adjunction $\varphi \dashv \psi$ with $\varphi : Z \rightsquigarrow X$, $\psi : X \rightsquigarrow Z$ there exists a $(\mathbb{T}, V)$-functor $f : Z \to X$ such that $\varphi = f_*$ or, equivalently, $\psi = f^*$.

Assuming the axiom of choice, Z can be replaced by the $\otimes$-neutral object E.

Proposition 3.2.1 ([15]). For a $(\mathbb{T}, V)$-category $X = (X, a)$ the following are equivalent:

1. X is L-complete.

2. For any left adjoint $(\mathbb{T}, V)$-module $\varphi : E \rightsquigarrow X$, there exists $x \in X$ such that $\varphi = x_*$.

3. For any right adjoint $(\mathbb{T}, V)$-module $\psi : X \rightsquigarrow E$, there exists $x \in X$ such that $\psi = x^*$.

Proof. One has $(2 \Leftrightarrow 3)$ by uniqueness of adjoints. $(1 \Rightarrow 3)$ is trivial. We show $(3 \Rightarrow 1)$.

Let $\varphi \dashv \psi : (X, a) \rightsquigarrow (Z, c)$. Given $z \in Z$, one has $\varphi \circ z_* \dashv z^* \circ \psi$. By hypothesis, there exists $x \in X$ such that $z^* \circ \psi = x^*$. Repeating this for all $z \in Z$, one obtains a function $f : Z \to X$ where $f(z) = x$.

Then

$$\psi(\mathfrak{x}, z) = z^* \circ \psi(\mathfrak{x}, \star) = x^*(\mathfrak{x}, \star) = (f(z))^*(\mathfrak{x}, \star) = z^* \circ f^*(\mathfrak{x}, \star) = f^*(\mathfrak{x}, z)$$

for any $z \in Z$, $\mathfrak{x} \in UZ$. Hence $\psi = f^* = f^\circ.a$.

To see that f is a $(\mathbb{T}, V)$-functor, one needs to show $f.c \le a.Tf$ or, equivalently, $c.Tf^\circ \le f^\circ.a$. This holds, as

$$c.Tf^\circ \le c.Tf^\circ.\overline{T}a.Te_X \le c.Tf^\circ.\overline{T}a.m_X^\circ = c.\overline{T}\psi.m_X^\circ \le \psi = f^\circ.a.$$

□

Let $X = (X, a)$, $Y = (Y, b)$ be a $(\mathbb{T}, V)$-categories. Recall from Prop. 2.8.5 that a $(\mathbb{T}, V)$-relation $\varphi : (X, a) \nrightarrow (Y, b)$ is a $(\mathbb{T}, V)$-module if and only if both

$\varphi : |X| \otimes Y \to V$ and $\varphi : X^{\mathrm{op}} \otimes Y \to V$ are $(\mathbb{T}, V)$-functors. A $(\mathbb{T}, V)$-functor $\psi : |X| \to V$ is called *tight* [36] if $\psi : X^{\mathrm{op}} \to V$ is a $(\mathbb{T}, V)$-functor and as a $(\mathbb{T}, V)$-module $\psi : X \rightsquigarrow E$ is a right adjoint. We will denote the collection of tight $(\mathbb{T}, V)$-functors by $\tilde{X}$ and consider it as a subcategory of $\widehat{X}$. Observe that the codomain of the Yoneda functor y can be taken as $(\tilde{X}, \widehat{a})$ since $x_* \dashv x^*$ for all $x \in X$.

Corollary 3.2.2. Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. X is L-complete if and only if the Yoneda functor $y : X \to \tilde{X}$ is surjective.

Now we investigate the conditions under which a $(\mathbb{T}, V)$-module $\psi : (X, a) \rightsquigarrow (E, k)$ is a right adjoint. Suppose that ψ has a left adjoint $\varphi : (E, k) \rightsquigarrow (X, a)$. Then $\varphi \circ \psi \leq (1_X)^*$ implies $\varphi \leq (1_X)^* \multimapinv \psi$ (see (2.6.1)). On the other hand, one has $((1_X)^* \multimapinv \psi) \circ \psi \leq (1_X)^*$ and $(1_E)^* \leq \psi \circ \varphi$ which implies $(1_X)^* \multimapinv \psi \leq \varphi$. Hence if ψ is a right adjoint, then its left adjoint is necessarily $(1_X)^* \multimapinv \psi$. Since $((1_X)^* \multimapinv \psi) \circ \psi \leq (1_X)^*$ always holds, ψ is a right adjoint if and only if

$$(1_E)^* \leq \psi \circ ((1_X)^* \multimapinv \psi).$$

Given that $\psi : (X, a) \rightsquigarrow (E, k)$ is a $(\mathbb{T}, V)$-module, one has

$$\psi = k \circ \psi = k.\overline{T}\psi.m_X^\circ = \overline{T}\psi.m_X^\circ. \tag{3.2.1}$$

Also $\psi : |X| \to V$, $\psi : X^{\mathrm{op}} \to V$ are $(\mathbb{T}, V)$-functors by Prop. 2.8.5, hence $\psi \in \widehat{X}$. Using (2.6.2), (3.2.1) and Lemma 2.7.10, one finds

$$
\begin{aligned}
((1_X)^* \multimapinv \psi)(\star, x) &= \bigwedge_{\mathfrak{x}\in TX} (1_X)^*(\mathfrak{x}, x) \multimapinv \overline{T}\psi.m_X^\circ(\mathfrak{x}, \star) \\
&= \bigwedge_{\mathfrak{x}\in TX} a(\mathfrak{x}, x) \multimapinv \psi(\mathfrak{x}) \\
&= \bigwedge_{\mathfrak{x}\in TX} \xi.e_V(\psi(\mathfrak{x})) \multimap a(\mathfrak{x}, x) \\
&= \bigwedge_{\mathfrak{x}\in TX} \xi.e_V(\psi(\mathfrak{x})) \multimap y(x)(\mathfrak{x}) \\
&= [\![m_X, hom_\xi]\!](e_{\widehat{X}}(\psi), y(x)) \\
&= \widehat{a}(e_{\widehat{X}}(\psi), y(x)).
\end{aligned}
\tag{3.2.2}
$$

Lemma 3.2.3 ([36]). Let $\psi : (X, a) \rightsquigarrow (E, k)$ be a $(\mathbb{T}, V)$-module and $\varphi = (1_X)^* \multimapinv \psi$. Then

$$\overline{T}\varphi(\mathfrak{x}) = \overline{T}\widehat{a}(e_{T\widehat{X}}.e_{\widehat{X}}(\psi), Ty(\mathfrak{x}))$$

for all $\mathfrak{x} \in TX$.

Proof. Consider $\psi \in \widehat{X}$ as a map $\psi : 1 \to \widehat{X}$. One has

$$\varphi(\star, x) = \widehat{a}(e_{\widehat{X}}(\psi), y(x)) = y^\circ.\widehat{a}.e_{\widehat{X}}.\psi(\star, x)$$

for all $x \in X$. So $\varphi = y^\circ.\widehat{a}.e_{\widehat{X}}.\psi$. This implies $\overline{T}\varphi = \overline{T}(y^\circ.\widehat{a}.e_{\widehat{X}}.\psi) = Ty^\circ.\overline{T}\widehat{a}.Te_{\widehat{X}}.T\psi$.

Since $T1 = 1$ and $e : \mathrm{Id} \to T$ is a natural transformation, one gets

$$\begin{aligned}\overline{T}\varphi(\star,\mathfrak{x}) &= Ty^\circ.\overline{T}\widehat{a}.Te_{\widetilde{X}}.T\psi(\star,\mathfrak{x}) \\ &= \overline{T}\widehat{a}(Te_{\widetilde{X}}.T\psi(\star),Ty(\mathfrak{x})) \\ &= \overline{T}\widehat{a}(Te_{\widetilde{X}}.e_{\widetilde{X}}(\psi),Ty(\mathfrak{x})) \\ &= \overline{T}\widehat{a}(e_{T\widetilde{X}}.e_{\widetilde{X}}(\psi),Ty(\mathfrak{x}))\end{aligned}$$

□

Proposition 3.2.4 ([36]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. A $(\mathbb{T}, V)$-module $\psi : X \rightsquigarrow E$ is a right adjoint if and only if

$$k \leq \bigvee_{\mathfrak{x}\in TX} \psi(\mathfrak{x}) \otimes \overline{T}\widehat{a}(e_{T\widetilde{X}}.e_{\widetilde{X}}(\psi),Ty(\mathfrak{x}))$$

In such a case ψ has a left adjoint $\varphi : E \rightsquigarrow X$ where $\varphi(x) = \widehat{a}(e_{\widetilde{X}}(\psi), y(x))$.

Proof. Following the discussion above, $\psi : (X, a) \rightsquigarrow (E, k)$ is a right adjoint if and only if its left adjoint is $\varphi = (1_X)^* \multimapinv \psi$. This is equivalent to the condition $(1_E)^* \leq \psi \circ \varphi$, i.e. $(1_E)^* \leq \psi.\overline{T}\varphi.m_E^\circ$. One obtains the desired inequality by Lemma 3.2.3. Also $\varphi(x) = ((1_X)^* \multimapinv \psi)(x) = \widehat{a}(e_{\widetilde{X}}(\psi), y(x))$ as given in (3.2.2). □

In particular a V-module $\psi : (X, a) \rightsquigarrow (E, k)$ is a right adjoint if and only if

$$k \leq \bigvee_{z\in X} \psi(z) \otimes \Big(\bigwedge_{x\in X} \psi(x) \multimap a(x,z)\Big). \tag{3.2.3}$$

Examples 3.2.5. 1. Every ordered set $(X, \leq)$ is L-complete as follows:

A 2-module $\psi : X \rightsquigarrow E$ corresponds to the monotone map $\psi : X^{\mathrm{op}} \to 2$ which is the characteristic function of a set $A \subseteq X$. If $x \leq z$ and $z \in A$, then $1 = \psi(z) \leq \psi(x)$ which means that $x \in A$. Hence ψ corresponds to a down-closed set $A \subseteq X$.

By (3.2.3), ψ is a right adjoint if and only if there exists $z \in A$ such that for all $x \in A$, $x \leq z$. This means $A = \downarrow z$ or, equivalently, $\psi = z^*$. So every right adjoint 2-module $\psi : X \rightsquigarrow E$ is representable. Therefore X is L-complete.

2. A metric space $X = (X, d)$ is L-complete if and only if X is Cauchy complete [42]. Here a sequence (x_n) converges to a point $x \in X$ if $\lim_{n \to \infty} \big(d(x_n, x) + d(x, x_n)\big) = 0$.

Firstly, we show that adjoint P_+-modules $\varphi \dashv \psi : X \rightsquigarrow E$ correspond to equivalence classes of Cauchy sequences in X and vice versa.

Given $\varphi \dashv \psi : X \rightsquigarrow E$, one has

$$\psi(x) + \varphi(z) \geq d(x, z) \qquad \& \qquad \inf_{x \in X} \{\varphi(x) + \psi(x)\} = 0 \tag{3.2.4}$$

for $x, z \in X$. For any $n \in \mathbb{N}$, pick $x_n \in X$ such that $\varphi(x_n) + \psi(x_n) \leq \frac{1}{n}$. Then

$$d(x_n, x_m) + d(x_m, x_n) \leq \psi(x_n) + \varphi(x_m) + \psi(x_m) + \varphi(x_n) \leq \frac{1}{n} + \frac{1}{m}$$

for $n, m \in \mathbb{N}$. So (x_n) is a Cauchy sequence. If one obtains another Cauchy sequence (y_n) during this process, then

$$d(x_n, y_n) + d(y_n, x_n) \leq \psi(x_n) + \varphi(y_n) + \psi(y_n) + \varphi(x_n) \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.$$

Hence (x_n) and (y_n) are equivalent. So the adjunction $\varphi \dashv \psi : X \rightsquigarrow E$ corresponds to an equivalence class of Cauchy sequences in X.

Conversely, let $\{(x_n)\}$ be an equivalence class Cauchy sequences in X. Define $\varphi : X \to \mathrm{P}_+$ and $\psi : X^{\mathrm{op}} \to \mathrm{P}_+$ by

$$\varphi(x) = \lim_{n\to\infty} d(x_n, x) \qquad \& \qquad \psi(x) = \lim_{n\to\infty} d(x, x_n)$$

for all $x \in X$. For any $n \in \mathbb{N}$, $x, z \in X$, one has $d(x,z) \leq d(x, x_n) + d(x_n, z)$. Then

$$\begin{aligned} d(x,z) &\leq \lim_{n\to\infty} \big(d(x,x_n) + d(x_n,z)\big) \\ &\leq \lim_{n\to\infty} d(x,x_n) + \lim_{n\to\infty} d(x_n,z) \\ &\leq \psi(x) + \varphi(z). \end{aligned}$$

Furthermore, $\inf_{x\in X}\{\lim_{n\to\infty} \big(d(x_n,x) + d(x,x_n)\big)\} = 0$, since (x_n) is Cauchy. Hence

$$\inf_{x\in X}\{\varphi(x) + \psi(x)\} = \inf_{x\in X}\{\lim_{n\to\infty} d(x_n,x) + \lim_{n\to\infty} d(x,x_n)\} = 0$$

So $\varphi \dashv \psi : X \rightsquigarrow E$. Therefore there is a one-to-one correspondence between adjoint P_+-modules $\varphi \dashv \psi : X \rightsquigarrow E$ and equivalence classes of Cauchy sequences in X.

Suppose that $\varphi \dashv \psi : X \rightsquigarrow E$ corresponds to the equivalence class $\{(x_n)\}$ of Cauchy sequences. Now we show that

$$\varphi = x_* \quad \Longleftrightarrow \quad \lim_{n\to\infty} x_n = x.$$

If $\varphi = x_*$ for some $x \in X$, then $\varphi(z) = d(x,z)$ and $\psi(z) = d(z,x)$ for all $z \in X$. (3.2.4) implies

$$d(x,x_n) + d(x_n,x) = \varphi(x_n) + \psi(x_n) \le \frac{1}{n}.$$

Letting $n \to \infty$, one gets $\lim_{n\to\infty} x_n = x$. Conversely, if (x_n) converges to $x \in X$, then $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x,x_n) = 0$. Take any $z \in X$. $d(x_n,z) \le d(x_n,x) + d(x,z)$ implies

$$\begin{aligned}\lim_{n\to\infty} d(x_n,z) &\le \lim_{n\to\infty} d(x_n,x) + d(x,z)\\ \varphi(z) &\le 0 + d(x,z).\end{aligned}$$

Similarly $d(x,z) \le d(x,x_n) + d(x_n,z)$ implies $d(x,z) \le \varphi(z)$. Hence $\varphi(z) = d(x,z)$ for all $z \in X$. This means $\varphi = x_*$.

So a left adjoint $\varphi : E \rightsquigarrow X$ is representable by a point $x \in X$ precisely when the corresponding Cauchy sequence converges to x. Therefore a metric space is L-complete if and only if it is Cauchy complete.

3. As shown in [15], a topological space is L-complete if and only if it is quasi sober, i.e. every irreducible closed set can be written as the closure of a

point. Here an irreducible closed set is a closed set which cannot be written as a union of two proper closed subsets.

To see this assume that $\varphi \dashv \psi : (X, a) \rightsquigarrow (E, k)$ is a pair of adjoint $(\mathbb{U}, 2)$-modules. The $(\mathbb{U}, 2)$-module φ corresponds to the continuous map $\varphi : X \to 2$ where 2 is the Sierpinski space. So φ can be identified with the closed subset $A := \varphi^{-1}(\{1\})$ of X. Similarly ψ corresponds to the continuous maps $\psi : |X| \to 2$ and $\psi : X^{\mathrm{op}} \to 2$. So it can be identified with $\mathcal{A} \subseteq UX$ where $\mathcal{A} := \psi^{-1}(\{1\})$. $\mathcal{A}$ is closed both in $|X|$ and X^{op}. The topology on $|X|$ is given by the Zariski closure where for an ultrafilter $\mathfrak{z}$, $\mathfrak{z} \in \overline{\mathcal{A}}$ if and only if $\mathfrak{z} \subseteq \bigcup \mathcal{A}$. $X^{\mathrm{op}} = \mathsf{A}^{\circ}(\mathsf{M}(X)^{\mathrm{op}})$ has the Alexandroff topology of the dual order on $\mathsf{M}X$. The order on $\mathsf{M}X$ is given by

$$\mathfrak{y} \leq \mathfrak{z} \iff \forall A \in \mathfrak{y},\ \overline{A} \in \mathfrak{z}$$

for ultrafilters $\mathfrak{y}, \mathfrak{z}$ (see Remarks 2.7.6). The closed subsets of X^{op} are precisely the down-closed subsets of $\mathsf{M}X$.

As a result of the adjunction inequalities, one finds that $\varphi \dashv \psi$ if and only if there exists a closed set $A \subseteq X$, a down-closed and Zariski closed $\mathcal{A} \subseteq UX$ such that

$$\exists \mathfrak{x} \in UX :\ A \in \mathfrak{x}, \mathfrak{x} \in \mathcal{A} \quad \& \quad \forall \mathfrak{z} \in \mathcal{A}, x \in A, \mathfrak{z} \to x. \qquad (3.2.5)$$

This implies that $\mathcal{A}$ is the down-closure of $\mathfrak{x}$ in $\mathsf{M}X$,

$$\mathcal{A} = \Downarrow \mathfrak{x} = \{\mathfrak{z} \in UX \mid \forall x \in A, \mathfrak{z} \to x\}.$$

Having an ultrafilter $\mathfrak{x}$ which contains the closed set A and which converges to every point of A is equivalent to saying that A is an irreducible closed set. So $\varphi \dashv \psi$ corresponds to the irreducible closed set A. There exists $x \in X$ such that $\varphi = x_*$ if and only if $\varphi(z) = x_*(z) = a(\dot{x}, z)$ for all $z \in X$. One obtains

$$z \in A \iff \varphi(z) = 1 \iff \dot{x} \to z \iff z \in \overline{\{x\}}.$$

So X is L-complete if and only if for any irreducible closed set $A \subseteq X$ there exists $x \in X$ such that $A = \overline{\{x\}}$, i.e. X is quasi sober.

Now we provide an alternative perspective on L-completeness in **Top** which will be useful in the sequel. For a filter $\mathfrak{z}$ on X, let $conv\,\mathfrak{z} := \{x \in X \mid \mathfrak{z} \to x\}$. $\mathfrak{z}$ is called *irreducible* [32] if $conv\,\mathfrak{z} \in \mathfrak{z}$. Given an irreducible filter $\mathfrak{z}$, $conv\,\mathfrak{z}$ is an irreducible closed set. With this terminology a topological space X is quasi sober if and only if for any irreducible ultrafilter $\mathfrak{z} \in UX$ there exists $x \in X$ such that $conv\,\mathfrak{z} = conv\,\dot{x}$ [32].

We establish a bijective correspondence between irreducible ultrafilters on X and pairs of adjoint $(\mathbb{U}, 2)$-modules between X and E as follows:

Given any $\varphi \dashv \psi : X \rightsquigarrow E$, consider the corresponding ultrafilter $\mathfrak{x}$ mentioned in (3.2.5). Firstly, $\mathfrak{x}$ converges to all the points of A. Secondly, $A \in \mathfrak{x}$. So if $\mathfrak{x}$ converges to a point $x \in X$, then $x \in \overline{A} = A$. Hence *conv* $\mathfrak{x} = A$, $\mathfrak{x}$ is an irreducible ultrafilter. We also know that $\mathcal{A} = \Downarrow \mathfrak{x}$. Conversely, given any irreducible ultrafilter $\mathfrak{x}$ on X, let $A := conv\ \mathfrak{x}$. Then A is a closed set such that $A \in \mathfrak{x}$ and $\mathfrak{x}$ converges to every point in A. This implies that $\Downarrow \mathfrak{x} = \{\mathfrak{z} \in UX \mid \forall x \in A, \mathfrak{z} \to x\}$. Setting $\mathcal{A} := \Downarrow \mathfrak{x}$, $\mathcal{A}$ becomes a down-closed and Zariski closed subset of UX that satisfies (3.2.5). This implies the existence of an adjunction $\varphi \dashv \psi : X \rightsquigarrow E$. Hence there is a bijective correspondence between the adjoint pairs $\varphi \dashv \psi : X \rightsquigarrow E$ and irreducible ultrafilters $\mathfrak{x} \in UX$ with *conv* $\mathfrak{x} = A$ and $\Downarrow \mathfrak{x} = \mathcal{A}$.

Suppose that $\varphi \dashv \psi : X \rightsquigarrow E$ corresponds to an irreducible ultrafilter $\mathfrak{x}$. Then the left adjoint φ is representable by $x \in X$ if and only if $A = \overline{\{x\}} = conv\ \dot{x}$ or, equivalently, *conv* $\mathfrak{x} = conv\ \dot{x}$. Therefore a topological space X is L-complete if and only if for any irreducible ultrafilter $\mathfrak{x} \in UX$, there exists $x \in X$ such that *conv* $\mathfrak{x} = conv\ \dot{x}$. This exactly means that X is quasi sober [32].

4. An approach space is L-complete if and only if every irreducible variable closed set is representable by a point [15]. The details follow.

   There is a bijective correspondence between contractions $\rho : X \to \mathrm{P}_+$ and families $(A_v)_{v \in \mathrm{P}_+}$ of subsets $A_v \subseteq X$ satisfying

$$A_v = \bigcap_{u > v} A_u, \tag{3.2.6}$$

$$(A_u)^{(v)} \subseteq A_{u+v} \quad \forall u, v \in \mathrm{P}_+, \tag{3.2.7}$$

where $A^{(v)} = \{x \in X \mid \delta(A, x) \leq v\}$ and $\delta(A, x) = \inf\{a(\mathfrak{x}, x) \mid A \in \mathfrak{x}\}$ for $v \in \mathrm{P}_+$, $A \subseteq X$. Given $\rho : X \to \mathrm{P}_+$, one defines $A_v := \rho^{-1}([0, v])$ for all $v \in \mathrm{P}_+$. Conversely, having a family $(A_v)_{v \in \mathrm{P}_+}$, one gets the contraction ρ by defining $\rho(x) := \inf\{v \in \mathrm{P}_+ \mid x \in A_v\}$. A family $A = (A_v)_{v \in \mathrm{P}_+}$ of subsets $A_v \subseteq X$ satisfying (3.2.6) is called a *variable set.* If A satisfies (3.2.7), then it is called *closed.*

Assume that $\varphi \dashv \psi : (X, a) \rightsquigarrow (E, k)$ is a pair of adjoint $(\mathbb{U}, \mathrm{P}_+)$-modules. The $(\mathbb{U}, \mathrm{P}_+)$-module φ is essentially a contraction from X to P_+. Hence it corresponds to a closed variable set $A = (A_v)_{v \in \mathrm{P}_+}$. Similarly, the right adjoint $\psi : X \rightsquigarrow E$ determines a variable set $\mathcal{A} = (\mathcal{A}_v)_{v \in \mathrm{P}_+}$ of subsets $\mathcal{A}_v \subseteq UX$ where $\mathcal{A}_v := \psi^{-1}([0, v])$.

The adjunction condition $(1_X)^* \geq \varphi \circ \psi$ translates to $a \leq \varphi.\overline{U}\psi.m_X^\circ = \varphi.\psi$. So

$$a(\mathfrak{x}, x) \leq \psi(\mathfrak{x}) + \varphi(x)$$

for all $x \in X, \mathfrak{x} \in UX$. On the other hand, $(1_E)^* \leq \psi \circ \varphi$ translates to

$$\begin{aligned} 0 = \inf_{\mathfrak{x} \in UX}\{\psi(\mathfrak{x}) + \xi.\varphi(\mathfrak{x})\} &= \inf_{\mathfrak{x} \in UX}\{\psi(\mathfrak{x}) + \inf\{v \in V \mid [0, v] \in U\varphi(\mathfrak{x})\}\} \\ &= \inf_{\mathfrak{x} \in UX}\{\psi(\mathfrak{x}) + \inf\{v \in V \mid A_v \in \mathfrak{x}\}\}. \end{aligned}$$

Upon further examination, one finds that

$$\mathcal{A}_v = \{\mathfrak{x} \in UX \mid \forall u \in \mathrm{P}_+, \forall x \in A_u,\ a(\mathfrak{x}, x) \leq u + v\}$$

for all $v \in \mathrm{P}_+$. Furthermore, $\varphi \dashv \psi$ if and only if

$$\forall v \in \mathrm{P}_+ \ (v > 0 \ \Rightarrow \ UA_v \cap \mathcal{A}_v \neq \varnothing) \tag{3.2.8}$$

A variable set A is called irreducible if it satisfies (3.2.8).

If $\varphi = x_*$ for some $x \in X$, then $\varphi = a(\dot{x}, \_)$. Hence

$$\varphi(z) = a(\dot{x}, z) = \inf_{\mathfrak{x} \in UX} \{a(\mathfrak{x}, z) \mid \{x\} \in \mathfrak{x}\} = \delta(\{x\}, z)$$

for all $z \in X$. This means that

$$A_v = \{z \in X \mid \delta(\{x\}, z) \leq v\} = \{x\}^{(v)} \tag{3.2.9}$$

for all $v \in \mathrm{P}_+$. So X is L-complete if and only if every irreducible variable closed set $A = (A_v)_{v \in \mathrm{P}_+}$ is representable by a point $x \in X$ as in (3.2.9).

L-completeness turns out to be sobriety for approach spaces [2], [15]. An approach space is called *sober* [2] if it is a fixed point of the dual adjunction between **App** and the category **AFrm** of approach frames and homomorphisms.

3.3 L-injectivity

A $(\mathbb{T}, V)$-category X is called *L-injective* if given any $(\mathbb{T}, V)$-functor $h : Y \to X$ and any L-equivalence $f : Y \to Z$, there exists a $(\mathbb{T}, V)$-functor $g : Z \to X$ such that

$g.f \simeq h$.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "h"'] & Z \arrow[d, dashed, "g"] \\ & X \end{tikzcd}$$

The $(\mathbb{T}, V)$-category V is L-injective. To see this, let $h : Y \to V$ be a $(\mathbb{T}, V)$-functor and $f : Y \to Z$ be an L-equivalence. Since $|E| = E^{\mathrm{op}} = E$, one can consider h as a $(\mathbb{T}, V)$-module $h : E \rightsquigarrow Y$ by Prop. 2.8.5. Then the $(\mathbb{T}, V)$-module $f_* \circ h : E \rightsquigarrow Z$ corresponds to a $(\mathbb{T}, V)$-functor $g : Z \to V$. One has

$$g.f(y) = g(f(y)) = f_* \circ h(\star, f(y)) = f^* \circ f_* \circ h(\star, y) = h(y)$$

for all $y \in Y$. Hence $g.f = h$, V is L-injective.

Proposition 3.3.1 ([36])**.** Let X, Y be a $(\mathbb{T}, V)$-categories where X is $\otimes$-exponentiable and Y is L-injective. Then Y^X is L-injective.

Proof. Let $h : A \to Y^X$ be a $(\mathbb{T}, V)$-functor and $f : A \to B$ be an L-equivalence. Consider the mate $\llcorner h \lrcorner : A \otimes X \to Y$ of h. Since f is an L-equivalence, so is $f \otimes 1_X : A \otimes X \to B \otimes X$. Then there exists a $(\mathbb{T}, V)$-functor $\llcorner g \lrcorner : B \otimes X \to Y$ with $\llcorner g \lrcorner.(f \otimes 1_X) \simeq \llcorner h \lrcorner$, as Y is L-injective. The corresponding $(\mathbb{T}, V)$-functor $g : B \to Y^X$ satisfies $g.f \simeq h$. $\square$

Corollary 3.3.2. $V^{|X|}$ is L-injective for any $(\mathbb{T}, V)$-category X.

3.4 Closure operators in $(\mathbb{T}, V)$-Cat

3.4.1 Natural closure and dual closure

Definition 3.4.1. Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category, $M \subseteq X$ and $x \in X$. We say that x is in the *natural closure* of M, denoted by $x \in \overrightarrow{M}$, if

$$k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x).$$

M is called *n-closed* if $\overrightarrow{M} = M$.

In the definition above one actually has $a(Ti(\mathfrak{x}), x)$ where $i : M \hookrightarrow X$ is the inclusion map. We omit this style of writing for the sake of simplicity.

Proposition 3.4.2. Let $f : (X, a) \to (Y, b)$ be a $(\mathbb{T}, V)$-functor, $M, N \subseteq X$, $O \subseteq Y$ and $x \in X$. Then one has:

1. $M \subseteq \overrightarrow{M}$; $N \subseteq M$ implies $\overrightarrow{N} \subseteq \overrightarrow{M}$.
2. If $T\varnothing = \varnothing$, then $\overrightarrow{\varnothing} = \varnothing$.
3. $\overrightarrow{\overrightarrow{M}} = \overrightarrow{M}$.
4. $f(\overrightarrow{M}) \subseteq \overrightarrow{f(M)}$ and $f^{-1}(\overrightarrow{O}) \supseteq \overrightarrow{f^{-1}(O)}$.
5. If $N \subseteq M$, then $\overrightarrow{N}^{M} = \overrightarrow{N}^{X} \cap M$.
6. If k is $\vee$-irreducible and T preserves finite sums, then $\overrightarrow{M \cup N} = \overrightarrow{M} \cup \overrightarrow{N}$.

Proof. 1. Let $x \in M$. As $k \le a(e_X(x), x) \le \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x)$, $x \in \overrightarrow{M}$. Hence $M \subseteq \overrightarrow{M}$.

$N \subseteq M$ implies $TN \subseteq TM$ since $T : \mathbf{Set} \to \mathbf{Set}$ preserves monomorphisms. Then

$$k \le \bigvee_{\mathfrak{x} \in TN} a(\mathfrak{x}, x) \le \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x).$$

So $\overrightarrow{N} \subseteq \overrightarrow{M}$.

2. Since $T\varnothing = \varnothing$, $k \nleq \bigvee_{\mathfrak{x} \in \varnothing} a(\mathfrak{x}, x) = \bot$. Hence $\overrightarrow{\varnothing} = \varnothing$.

3. $\overrightarrow{M} \subseteq \overrightarrow{\overrightarrow{M}}$ follows from (1). Observe that $\overrightarrow{M}$ is precisely the set which makes the following diagram lax commutative.

$$\begin{tikzcd}
\overrightarrow{M} \arrow[rr, "a^\circ", "|" marking] \arrow[dr, "!"'] & \arrow[d, phantom, "\le"] & TM \arrow[dl, "!"] \\
& 1 &
\end{tikzcd}$$

Applying the functor T to this diagram, one finds that $T\overrightarrow{M}$ is precisely the set such that $(T\overrightarrow{M} \xrightarrow{!} 1 \le T\overrightarrow{M} \overset{\overline{T}a^\circ}{\nrightarrow} T^2M \xrightarrow{!} 1)$. So for any $\mathfrak{y} \in T\overrightarrow{M}$, one has $k \le \bigvee_{\mathfrak{X} \in T^2M} \overline{T}a(\mathfrak{X}, \mathfrak{y})$. To show $\overrightarrow{\overrightarrow{M}} \subseteq \overrightarrow{M}$, let $x \in \overrightarrow{\overrightarrow{M}}$.

Then

$$\begin{aligned}
k &\le \bigvee_{\mathfrak{y}\in T\overrightarrow{M}} a(\mathfrak{y},x) \\
&\le \bigvee_{\mathfrak{y}\in T\overrightarrow{M}} \Big(\bigvee_{\mathfrak{X}\in T^2M} \overline{T}a(\mathfrak{X},\mathfrak{y})\Big) \otimes a(\mathfrak{y},x) \\
&\le \bigvee_{\mathfrak{y}\in T\overrightarrow{M}} \Big(\bigvee_{\mathfrak{X}\in T^2M} \overline{T}a(\mathfrak{X},\mathfrak{y}) \otimes a(\mathfrak{y},x)\Big) \\
&\le \bigvee_{\mathfrak{X}\in T^2M} a(m_X(\mathfrak{X}),x) \\
&= \bigvee_{\mathfrak{x}\in TM} a(\mathfrak{x},x).
\end{aligned}$$

Hence $x \in \overrightarrow{M}$, $\overrightarrow{\overrightarrow{M}} \subseteq \overrightarrow{M}$.

4. Let $y \in f(\overrightarrow{M})$. Then there exists $x \in M$ such that $k \le \bigvee_{\mathfrak{x}\in TM} a(\mathfrak{x},x)$ and $f(x) = y$. One has

$$k \le \bigvee_{\mathfrak{x}\in TM} a(\mathfrak{x},x) \le \bigvee_{\mathfrak{x}\in TM} b(Tf(\mathfrak{x}),f(x)) \le \bigvee_{\mathfrak{y}\in T(f(M))} b(\mathfrak{y},y).$$

Hence $y \in \overrightarrow{f(M)}$, $f(\overrightarrow{M}) \subseteq \overrightarrow{f(M)}$. Letting $M = f^{-1}(O)$ gives $f^{-1}(\overrightarrow{O}) \supseteq \overrightarrow{f^{-1}(O)}$.

5. Let $N \overset{i'}{\hookrightarrow} M \overset{i}{\hookrightarrow} X$ with $i.i' = j : N \hookrightarrow X$. To distinguish between the natural closure of N in X and in M, we will write $\overrightarrow{N}^X$ and $\overrightarrow{N}^M$ respectively.

Then

$$\begin{aligned}
x \in \overrightarrow{N}^{X} \cap M &\iff k \leq \bigvee_{\mathfrak{x} \in TN} a(Tj(\mathfrak{x}), x) \\
&\iff k \leq \bigvee_{\mathfrak{x} \in TN} a(Ti.Ti'(\mathfrak{x}), i(x)) \\
&\iff k \leq \bigvee_{\mathfrak{x} \in TN} i^{\circ}.a.Ti(Ti'(\mathfrak{x}), x) \\
&\iff x \in \overrightarrow{N}^{M}.
\end{aligned}$$

6. Suppose that k is $\vee$-irreducible and T preserves finite sums. Recall from Subsection 2.1.3 that k is $\vee$-irreducible if $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$. One has $\overrightarrow{M \cup N} \supseteq \overrightarrow{M} \cup \overrightarrow{N}$ by monotonicity of the natural closure. $\overrightarrow{M \cup N} \subseteq \overrightarrow{M} \cup \overrightarrow{N}$ follows, as

$$k \leq \bigvee_{\mathfrak{x} \in T(M \cup N)} a(\mathfrak{x}, x) = \bigvee_{\mathfrak{x} \in TM \cup TN} a(\mathfrak{x}, x) = \Big(\bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \Big) \vee \Big(\bigvee_{\mathfrak{x} \in TN} a(\mathfrak{x}, x) \Big).$$

$\square$

Corollary 3.4.3. If k is $\vee$-irreducible and T preserves finite sums, then the natural closure induces a functor $\mathsf{N} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$.

Examples 3.4.4. 1. For an ordered set $(X, \leq)$, $x \in \overrightarrow{M}$ if and only if there exists $y \in M$ such that $y \leq x$.

2. For a metric space (X, d), $x \in \overrightarrow{M}$ if and only if $\inf_{y \in M} d(y, x) = 0$.

3. For a topological space (X, τ), $x \in \overrightarrow{M}$ if and only if there exists an ultrafilter $\mathfrak{x}$ on M that converges to x. This is equivalent to saying that $M \cap O \neq \varnothing$ for

any open neighbourhood O of x. Hence the natural closure of X is equal to the closure induced by τ.

4. For an approach space (X,δ), $x \in \overrightarrow{M}$ if and only if

$$\delta(M,x) = \inf\{a(\mathfrak{x},x) \mid M \in \mathfrak{x}\} = 0.$$

So $\overrightarrow{M} = M^{(0)}$.

Given $(\mathbb{T},V)$-category $X = (X,a)$, consider $\mathsf{A}^{\circ}(\mathsf{A}(X)^{\mathrm{op}}) = (X, (\overline{T}a.Te_X.e_X)^{\circ})$ (see Section 2.7.2). We will define the dual closure of X as the natural closure of $\mathsf{A}^{\circ}(\mathsf{A}(X)^{\mathrm{op}})$.

Definition 3.4.5. Let $X = (X,a)$ be a $(\mathbb{T},V)$-category, $M \subseteq X$ and $x \in X$. We say that x is in the *dual closure* of M, denoted by $x \in \overleftarrow{M}$, if

$$k \leq \bigvee_{\mathfrak{x}\in TM} \overline{T}a(Te_X.e_X(x),\mathfrak{x}).$$

M is called *d-closed* if $\overleftarrow{M} = M$.

Proposition 3.4.6. Let $f:(X,a)\to(Y,b)$ be a $(\mathbb{T},V)$-functor, $M,N\subseteq X$, $O\subseteq Y$ and $x\in X$. Then one has:

1. $M \subseteq \overleftarrow{M}$; $N\subseteq M$ implies $\overleftarrow{N}\subseteq\overleftarrow{M}$.

2. If $T\varnothing = \varnothing$, then $\overleftarrow{\varnothing} = \varnothing$.

3. $\overleftarrow{\overleftarrow{M}} = \overleftarrow{M}$.

4. $f(\overleftarrow{M}) \subseteq \overleftarrow{f(M)}$ and $f^{-1}(\overleftarrow{O}) \supseteq \overleftarrow{f^{-1}(O)}$.

5. If $N \subseteq M$, then $\overleftarrow{N}^M = \overleftarrow{N}^X \cap M$.

6. If k is $\vee$-irreducible and T preserves finite sums, then $\overleftarrow{M \cup N} = \overleftarrow{M} \cup \overleftarrow{N}$.

Proof. Follows from Prop. 3.4.2 and the fact that the dual closure of $X = (X, a)$ is the natural closure of $\mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}})$. □

Corollary 3.4.7. If k is $\vee$-irreducible and T preserves finite sums, then the dual closure induces a functor $\mathsf{D} : (\mathbb{T}, V)$-**Cat** $\to$ **Top**.

Examples 3.4.8. 1. When $\mathbb{T}$ is the identity monad, $(X, (\overline{T}a.Te_X.e_X)^\circ) = (X, a^\circ)$. So for an ordered set $(X, \leq)$, $x \in \overleftarrow{M}$ if and only if there exists $y \in M$ such that $x \leq y$.

2. For a metric space (X, d), $x \in \overleftarrow{M}$ if and only if $\inf_{y \in M} d(x, y) = 0$.

3. Let (X, τ) be a topological space which corresponds to the $(\mathbb{U}, 2)$-category (X, a). One has $m_X^\circ.e_X = Ue_X.e_X$ by Prop. 2.5.4. So $\overline{U}a(Ue_X.e_X(x), \mathfrak{x}) = \overline{U}a.m_X^\circ(e_X(x), \mathfrak{x})$. Recall from Remarks 2.7.6 that $\overline{U}a.m_X^\circ$ is the structure on $\mathsf{M}X = (UX, \leq)$ where

$$\mathfrak{x} \leq \mathfrak{y} \iff \forall A \in \mathfrak{x},\ \overline{A} \in \mathfrak{y}. \tag{3.4.1}$$

So $x \in \overleftarrow{M}$ if and only if there exists an ultrafilter $\mathfrak{x}$ on M such that $e_X(x) \leq \mathfrak{x}$. By (3.4.1), this means that for any set N containing x, $\overline{N} \in \mathfrak{x}$. That is equivalent to saying that $\overline{\{x\}} \in \mathfrak{x}$. Therefore

$$x \in \overleftarrow{M} \iff \overline{\{x\}} \bigcap M \neq \varnothing.$$

4. Let (X, δ) be an approach space which corresponds to the $(\mathbb{U}, \mathrm{P}_+)$-category (X, a). As above one has $\overline{U}a(Ue_X.e_X(x), \mathfrak{x}) = \overline{U}a.m_X^\circ(e_X(x), \mathfrak{x})$ where $\overline{U}a.m_X^\circ$ is the structure of the metric space $\mathsf{M}X = (UX, d)$. By Remarks 2.7.6,

$$d(\mathfrak{x}, \mathfrak{y}) = \inf\{\varepsilon \in [0, \infty] \mid \forall A \in \mathfrak{x},\ A^{(\varepsilon)} \in \mathfrak{y}\}.$$

Then $x \in \overleftarrow{M}$ if and only if

$$\begin{aligned}
0 &= \inf_{\mathfrak{x} \in UM} \overline{U}a.(Ue_X.e_X(x), \mathfrak{x}) \\
&= \inf_{\mathfrak{x} \in UM} \overline{U}a.m_X^\circ(e_X(x), \mathfrak{x}) \\
&= \inf_{\mathfrak{x} \in UM} \inf\{\varepsilon \in [0, \infty] \mid \forall A \in \dot{x},\ A^{(\varepsilon)} \in \mathfrak{x}\} \\
&= \inf_{\mathfrak{x} \in UM} \inf\{\varepsilon \in [0, \infty] \mid \{x\}^{(\varepsilon)} \in \mathfrak{x}\} \\
&= \inf_{\mathfrak{x} \in UX} \inf\{\varepsilon \in [0, \infty] \mid \{x\}^{(\varepsilon)} \bigcap M \in \mathfrak{x}\} \\
&= \inf\{\varepsilon \in [0, \infty] \mid \exists \mathfrak{x} \in UX :\ \{x\}^{(\varepsilon)} \bigcap M \in \mathfrak{x}\} \\
&= \inf\{\varepsilon \in [0, \infty] \mid \{x\}^{(\varepsilon)} \bigcap M \neq \varnothing\}
\end{aligned}$$

Hence

$$x \in \overleftarrow{M} \iff \forall \varepsilon > 0,\ \{x\}^{(\varepsilon)} \bigcap M \neq \varnothing.$$

3.4.2 L-closure

L-closure [36] is a hybrid of the natural closure and the dual closure.

Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category, $M \subseteq X$ and $x \in X$. One says that x is in the *L-closure* of M, denoted by $x \in \overleftrightarrow{M}$, if

$$k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}).$$

M is called *L-closed* if $\overleftrightarrow{M} = M$.

Remark 3.4.9. Given a $(\mathbb{T}, V)$-category (X, a), one has the $(\mathbb{T}, V)$-category $X \otimes \mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}}) = (X \times X, a \otimes (\overline{T}a.Te_X.e_X)^\circ)$. Consider X as a subcategory of $X \otimes \mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}})$ via the map $\delta_X : X \hookrightarrow X \times X$. Then the structure a^{sym} on X is given by

$$\begin{aligned} a^{\mathrm{sym}}(\mathfrak{x}, x) &= \delta^\circ.a \otimes (\overline{T}a.Te_X.e_X)^\circ.T\delta(\mathfrak{x}, x) \\ &= a(T\pi_1.T\delta(\mathfrak{x}), \pi_1.\delta(x)) \otimes (\overline{T}a.Te_X.e_X)^\circ(T\pi_2.T\delta(\mathfrak{x}), \pi_2.\delta(x)) \\ &= a(\mathfrak{x}, x) \otimes \overline{T}a.Te_X.e_X(x, \mathfrak{x}) \end{aligned}$$

for any $x \in X$, $\mathfrak{x} \in TX$.

Define the functor

$$\mathsf{S} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Cat}$$

which is identical on morphisms and sends a $(\mathbb{T}, V)$-category (X, a) to (X, a^{sym}).

The functoriality of S follows from the commutative diagram below.

$$\begin{tikzcd}
X \otimes \mathsf{A}^{\circ}(\mathsf{A}(X)^{\mathrm{op}}) \arrow[r, "f \otimes f"] & Y \otimes \mathsf{A}^{\circ}(\mathsf{A}(Y)^{\mathrm{op}}) \arrow[r, "g \otimes g"] & Z \otimes \mathsf{A}^{\circ}(\mathsf{A}(Z)^{\mathrm{op}}) \\
X \arrow[u, hook, "\delta_X"] \arrow[r, "f"'] & Y \arrow[u, hook, "\delta_Y"'] \arrow[r, "g"'] & Z \arrow[u, hook, "\delta_Z"']
\end{tikzcd}$$

Given a $(\mathbb{T}, V)$-category $X = (X, a)$, we call $\mathsf{S}X$ the symmetrization of X. So L-closure of X is the natural closure of its symmetrization.

Proposition 3.4.10 ([36]). Let (X, a) be a $(\mathbb{T}, V)$-category, $M \subseteq X$ and $x \in X$. Suppose that $i : M \hookrightarrow X$ is the inclusion map. Then the following are equivalent:

1. $x \in \overleftrightarrow{M}$.

2. $k \leq i_* \circ i^*(e_X(x), x)$.

3. $(1_E)^* \leq x^* \circ i_* \circ i^* \circ x_*$.

4. $i^* \circ x_* \dashv x^* \circ i_*$.

5. $x_* : E \rightsquigarrow X$ factors through $i_* : M \rightsquigarrow X$ by a morphism $\varphi : E \rightsquigarrow M$ in $(\mathbb{T}, V)$-**Mod**.

6. For all $(\mathbb{T}, V)$-functors $g, h : X \to Y$ with $g.i = h.i$, one has $g(x) \simeq h(x)$.

7. For all $(\mathbb{T}, V)$-functors $g, h : X \to V$ with $g.i = h.i$, one has $g(x) = h(x)$.

Proof. ($1 \Leftrightarrow 2 \Leftrightarrow 3$) Observe that

$$k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}) \qquad (\dagger)$$

is equivalent to $k \leq (a.Ti).(Ti^\circ.\overline{T}a.Te_X.e_X(x,x))$. By Lemma 2.5.4, $m_X^\circ.e_X = Te_X.e_X$. Hence

$$\begin{aligned} k &\leq (a.Ti).(Ti^\circ.\overline{T}a.m_X^\circ.e_X)(x,x) \\ &= (a.Ti).\overline{T}(i^\circ.a).m_X^\circ.(e_X(x),x) \\ &= i_* \circ i^*(e_X(x),x) \\ &= i_* \circ i^*(Tx(\star),x(\star)) \\ &= x^* \circ i_* \circ i^* \circ x_*(\star,\star) \end{aligned}$$

So ($\dagger$) is equivalent to $k \leq i_* \circ i^*(e_X(x),x)$ and $(1_E)^* \leq x^* \circ i_* \circ i^* \circ x_*$.

($3 \Leftrightarrow 4$) One always has

$$i^* \circ x_* \circ x^* \circ i_* \leq i^* \circ i_* = (1_M)^*,$$

since $i : M \hookrightarrow X$ is fully faithful. Therefore $(1_E)^* \leq x^* \circ i_* \circ i^* \circ x_*$ if and only if $i^* \circ x_* \dashv x^* \circ i_*$.

($3 \Rightarrow 5$) $(1_E)^* \leq x^* \circ i_* \circ i^* \circ x_*$ implies

$$x_* \leq x_* \circ x^* \circ i_* \circ i^* \circ x_* \leq i_* \circ i^* \circ x_* \leq x_*.$$

Hence $x_* = i_* \circ i^* \circ x_*$ where $i^* \circ x_* : E \rightsquigarrow M$.

$(5 \Rightarrow 6)$ Let $x_* = i_* \circ \varphi$ for $\varphi : E \rightsquigarrow M$. If $g.i = h.i$, then

$$(g(x))_* = g_* \circ x_* = g_* \circ i_* \circ \varphi = (g.i)_* \circ \varphi = (h.i)_* \circ \varphi = h_* \circ i_* \circ \varphi = h_* \circ x_* = (h(x))_*.$$

$(6 \Rightarrow 7)$ Follows as V is L-separated.

$(7 \Rightarrow 3)$ Consider $x_* : E \rightsquigarrow X$ and $i_* \circ i^* \circ x_* : E \rightsquigarrow X$. By Prop. 2.8.5, these $(\mathbb{T}, V)$-modules correspond to $(\mathbb{T}, V)$-functors $g : X \to V$ and $h : X \to V$ respectively. For $z \in M$, one has

$$\begin{aligned} g.i(z) = g(i(z)) = x_*(\star, i(z)) = i^* \circ x_*(\star, z) &= i^* \circ i_* \circ i^* \circ x_*(\star, z) \\ &= i_* \circ i^* \circ x_*(\star, i(z)) \\ &= h(i(z)) \\ &= h.i(z). \end{aligned}$$

By hypothesis, $g(x) = h(x)$. This means that $x_*(\star, x) = i_* \circ i^* \circ x_*(\star, x)$. Then

$$(1_E)^* \leq x^* \circ x_* = x^* \circ i_* \circ i^* \circ x_*.$$

$\square$

Let $\psi : (X, a) \rightsquigarrow (X, a)$ be a $(\mathbb{T}, V)$-module. Observe that $a = (1_X)^* \leq \psi$ implies $e_X^\circ \leq a \leq \psi$. Conversely, if $e_X^\circ \leq \psi$, then $a = a \circ e_X^\circ \leq a \circ \psi = \psi$. Hence $(1_X)^* \leq \psi$ if and only if $e_X^\circ \leq \psi$.

Proposition 3.4.11 ([36]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category and $i : M \hookrightarrow X$ be the inclusion map. Then i is L-dense if and only if $x \in \overleftrightarrow{M}$ for all $x \in X$.

Proof. i is L-dense if and only if $(1_X)^* \leq i_* \circ i^*$ which is equivalent to $e_X^\circ \leq i_* \circ i^*$. That is equivalent to $k \leq i_* \circ i^*(e_X(x), x)$ for all $x \in X$. Hence i is L-dense if and only if $x \in \overleftrightarrow{M}$ for all $x \in X$ by Prop. 3.4.10. $\square$

Prop. 3.4.10 implies that $M \subseteq \overleftrightarrow{M}$. So $\overleftrightarrow{M}$ is the largest subset D of X such that $M \hookrightarrow D$ is L-dense by Prop. 3.4.11.

Remark 3.4.12. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. Similar to the proof of Prop. 3.4.11, one gets

$$f \text{ is L-dense} \quad \Longleftrightarrow \quad \forall y \in Y,\ \ k \leq f_* \circ f^*(e_Y(y), y).$$

Consider the canonical factorization $X \overset{f'}{\twoheadrightarrow} f(X) \overset{i}{\hookrightarrow} Y$ of f. Since surjective $(\mathbb{T}, V)$-functors are L-dense, f is L-dense if and only if i is L-dense by Prop. 2.8.2. Then Prop. 3.4.11 gives

$$f \text{ is L-dense} \quad \Longleftrightarrow \quad \forall y \in Y,\ \ y \in \overleftrightarrow{f(X)}.$$

Surjectivity of f' implies that $g.f = h.f$ if and only if $g.i = h.i$ for all $(\mathbb{T}, V)$-functors $g, h : Y \to Z$. Since f being L-dense is equivalent to i being L-dense,

$$f \text{ is L-dense} \quad \Longleftrightarrow \quad \forall g, h : Y \to Z,\ \ (g.f = h.f \Rightarrow g \simeq h)$$

by Prop. 3.4.10 and Prop. 3.4.11. So L-dense $(\mathbb{T}, V)$-functors are "epimorphisms up to $\simeq$" in $(\mathbb{T}, V)$-**Cat**. Similarly, one gets

$$f \text{ is L-dense} \quad \Longleftrightarrow \quad \forall g, h : Y \rightharpoonup V,\ \ (g.f = h.f \Rightarrow g = h).$$

Proposition 3.4.13 ([36]). Let $f : (X, a) \to (Y, b)$ be a $(\mathbb{T}, V)$-functor, $M, N \subseteq X$, $O \subseteq Y$ and $x \in X$. Then one has:

1. $M \subseteq \overleftrightarrow{M}$; $N \subseteq M$ implies $\overleftrightarrow{N} \subseteq \overleftrightarrow{M}$.
2. If $T\varnothing = \varnothing$, then $\overleftrightarrow{\varnothing} = \varnothing$.
3. $\overleftrightarrow{\overleftrightarrow{M}} = \overleftrightarrow{M}$.
4. $f(\overleftrightarrow{M}) \subseteq \overleftrightarrow{f(M)}$ and $f^{-1}(\overleftrightarrow{O}) \supseteq \overleftrightarrow{f^{-1}(O)}$.
5. If $N \subseteq M$, then $\overleftrightarrow{N}^{M} = \overleftrightarrow{N}^{X} \cap M$.
6. If k is $\vee$-irreducible and T preserves finite sums, then $\overleftrightarrow{M \cup N} = \overleftrightarrow{M} \cup \overleftrightarrow{N}$.

Proof. We only show idempotency of the L-closure. The other assertions hold by Prop. 3.4.2 and the fact that L-closure of $X = (X, a)$ is the natural closure of its symmetrization $\mathsf{S}X$.

By Prop. 3.4.11, both $M \hookrightarrow \overleftrightarrow{M}$ and $\overleftrightarrow{M} \hookrightarrow \overleftrightarrow{\overleftrightarrow{M}}$ are L-dense. Then the composite $M \hookrightarrow \overleftrightarrow{\overleftrightarrow{M}}$ is L-dense. Since $\overleftrightarrow{M}$ is the largest subset of X that contains M as an L-dense subset, one has $\overleftrightarrow{\overleftrightarrow{M}} \subseteq \overleftrightarrow{M}$. □

Corollary 3.4.14 ([36]). If k is $\vee$-irreducible and T preserves finite sums, then the L-closure induces a functor $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$.

Corollary 3.4.15. Let $N \subseteq M \subseteq X$. Then N is L-closed in M if and only if there exists an L-closed set $A \subseteq X$ such that $N = A \cap M$.

Proof. Assume that N is L-closed in M. Then by Prop. 3.4.13, $N = \overleftrightarrow{N}^M = \overleftrightarrow{N}^X \cap M$ where $\overleftrightarrow{N}^X$ is L-closed in X. Conversely, assume that there exists an L-closed set $A \subseteq X$ such that $N = A \cap M$. Then

$$\overleftrightarrow{N}^M = \overleftrightarrow{(A \cap M)}^M \subseteq \overleftrightarrow{A}^M \cap \overleftrightarrow{M}^M = (\overleftrightarrow{A}^X \cap M) \cap M = A \cap M = N.$$

Hence N is L-closed in M. □

Examples 3.4.16. 1. For an ordered set $(X, \leq)$, $x \in \overleftrightarrow{M}$ if and only if there exists $y \in M$ such that $x \leq y$ and $y \leq x$.

2. For a metric space (X, d), $x \in \overleftrightarrow{M}$ if and only if $\inf_{y \in M}\{d(x,y) + d(y,x)\} = 0$.

3. For a topological space (X, τ), $x \in \overleftrightarrow{M}$ if and only if there exists an ultrafilter $\mathfrak{x}$ which converges to x and contains both $\overline{\{x\}}$ and M. This implies

$$x \in \overleftrightarrow{M} \quad \Longleftrightarrow \quad \forall O \text{ open nbhd of x, } M \cap \overline{\{x\}} \cap O \neq \varnothing.$$

Hence L-closure of X is equal to its b-closure or Skula closure [3], [52].

4. Let (X, δ) be an approach space which corresponds to the $(\mathbb{U}, \mathrm{P}_+)$-category

(X,a). Then $x \in \overleftrightarrow{M}$ if and only if

$$\begin{aligned}
0 &= \inf_{\mathfrak{x} \in UM} \{a(\mathfrak{x},x) + \overline{U}a.(Ue_X.e_X(x),\mathfrak{x})\} \\
&= \inf_{\mathfrak{x} \in UM} \left\{a(\mathfrak{x},x) + \overline{U}a.m_X^{\circ}(e_X(x),\mathfrak{x})\right\} \\
&= \inf_{\mathfrak{x} \in UM} \left\{a(\mathfrak{x},x) + \inf\{\varepsilon \in \mathrm{P}_+ \mid \{x\}^{(\varepsilon)} \in \mathfrak{x}\}\right\} \\
&= \inf_{\varepsilon \in \mathrm{P}_+} \{\varepsilon + \inf_{\substack{\mathfrak{x} \in UM \\ \{x\}^{(\varepsilon)} \in \mathfrak{x}}} a(\mathfrak{x},x)\} \\
&= \inf_{\varepsilon \in \mathrm{P}_+} \{\varepsilon + \inf_{\substack{\mathfrak{x} \in UX \\ M \cap \{x\}^{(\varepsilon)} \in \mathfrak{x}}} a(\mathfrak{x},x)\} \\
&= \inf_{\varepsilon \in \mathrm{P}_+} \{\varepsilon + \delta(M \cap \{x\}^{(\varepsilon)}, x)\}
\end{aligned}$$

Hence

$$x \in \overleftrightarrow{M} \iff \forall \varepsilon > 0,\ \delta(M \cap \{x\}^{(\varepsilon)}, x) = 0. \tag{3.4.2}$$

Remark 3.4.17. Let $X = (X,\delta)$ be an approach space. The *Zariski closure* of $M \subseteq X$ [26], [19] is given by

$$\overline{M}^{\mathcal{Z}} := \{x \in X \mid \forall g,h \in \mathcal{R},\ (g_{|M} = h_{|M} \Rightarrow g(x) = h(x)\}.$$

Here $\mathcal{R}$ is the regular function frame of (X,δ) which is actually the set of contractions from X to P_+. The concrete characterization of Zariski closure for approach spaces has not been known for some time. However, by taking advantage of Prop. 3.4.10, one can write L-closure of $M \subseteq X$ as

$$\overleftrightarrow{M} = \{x \in X \mid \forall g,h : X \to \mathrm{P}_+,\ (g_{|M} = h_{|M} \Rightarrow g(x) = h(x)\}.$$

So Zariski closure of an approach space is equal to its L-closure. The concrete characterization of the Zariski closure is given by (3.4.2).

3.5 Connections between L-separation, L-completeness and L-closure

One can formulate L-separatedness via L-closure.

Proposition 3.5.1 ([36]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category and $\Delta \subseteq X \times X$ be the diagonal. Then $\overleftrightarrow{\Delta} = \{(x, y) \in X \times X \mid x \simeq y\}$.

Proof. Assume that $(x, y) \in \overleftrightarrow{\Delta}$. Let $\pi_1, \pi_2 : X \times X \to X$ be the projection maps and $i : \Delta \hookrightarrow X \times X$ be the inclusion map . Since $\pi_1.i = \pi_2.i$, one gets $x = \pi_1(x, y) \simeq \pi_2(x, y) = y$ by Prop. 3.4.10.

Conversely assume that $x \simeq y$. For $(z, w) \in X \times X$, one has

$$\begin{aligned}
(x,y)_*(z,w) &= a \times a(e_{X\times X}(x,y),(z,w)) \\
&= a(T\pi_1.e_{X\times X}(x,y),z) \wedge a(T\pi_2.e_{X\times X}(x,y),w) \\
&= a(e_X.\pi_1(x,y),z) \wedge a(e_X.\pi_2(x,y),w) \\
&= a(e_X(x),z) \wedge a(e_X(y),w) \\
&= a(e_X(x),z) \wedge a(e_X(x),w) \\
&= (x,x)_*(z,w).
\end{aligned}$$

So $(x,y) \simeq (x,x)$. Let $g,h : X \times X \to Y$ be $(\mathbb{T},V)$-functors such that $g.i = h.i$. Then $g(x,y) \simeq g(x,x) = h(x,x) \simeq h(x,y)$. Hence $(x,y) \in \overleftrightarrow{\Delta}$ by Prop. 3.4.10.

$\square$

Corollary 3.5.2 ([36]). A $(\mathbb{T},V)$-category $X = (X,a)$ is L-separated if and only if the diagonal Δ is L-closed in $X \times X$.

Exploring the relationship between L-closure and L-completeness leads to interesting results.

Proposition 3.5.3 ([36]). Let $X = (X,a)$ be a $(\mathbb{T},V)$-category and $M \subseteq X$.

1. If X is L-complete and M is L-closed then M is L-complete.

2. If X is L-separated and M is L-complete then M is L-closed.

Proof. 1. Let $\varphi \dashv \psi : M \nrightarrow E$ and $i : M \hookrightarrow X$ be the inclusion map. Then $i_* \circ \varphi \dashv \psi \circ i^* : X \nrightarrow E$. Since X is L-complete, there exists $x \in X$ such that $i_* \circ \varphi = x_*$. This means $x \in \overleftrightarrow{M} = M$, as M is L-closed. $i_* \circ \varphi = x_*$ implies $\varphi = i^* \circ x_*$. Hence

$$\varphi(\star, y) = i^* \circ x_*(\star, y) = x_*(\star, i(y)) = x_*(\star, y)$$

for $y \in M$. Since $x \in M$, $\varphi = x_*$ and M is L-complete.

2. Let $x \in \overleftrightarrow{M}$. Then $i^* \circ x_* \dashv x^* \circ i_* : M \nrightarrow E$ by Prop. 3.4.10. Since M is complete, there exists $y \in M$ such that $y^* = x^* \circ i_*$. Then $i(y)^* = x^*$, as i

is fully faithful. Since X is L-separated, one gets $i(y) = x$. So $x \in M$, M is L-closed.

□

Proposition 3.5.4 ([36]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. The Yoneda functor $y : X \to \tilde{X}$ is L-dense.

Proof. Take any $\psi \in \tilde{X}$. Then

$$k \leq \bigvee_{\mathfrak{x} \in TX} \widehat{a}(Ty(\mathfrak{x}), \psi) \otimes \overline{T}\widehat{a}(e_{T\tilde{X}}.e_{\tilde{X}}(\psi), Ty(\mathfrak{x}))$$

by Prop. 3.2.4 and Prop. 2.8.6. As $e_{T\tilde{X}}.e_{\tilde{X}} = m^{\circ}_{\tilde{X}}.e_{\tilde{X}}$, that is equivalent to

$$k \leq y_* \circ y^*(e_{\tilde{X}}(\psi), \psi).$$

Then $y : X \to \tilde{X}$ is L-dense by Remark 3.4.12. □

Corollary 3.5.5. Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. Then $\tilde{X} = \overleftrightarrow{y(X)}$ where $y : X \to \tilde{X}$ is the Yoneda functor.

Proof. $\tilde{X} = \overleftrightarrow{y(X)}$ if and only if $y : X \to \tilde{X}$ is L-dense by Remark 3.4.12. □

Theorem 3.5.6 ([36]). Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category. X is L-complete if and only if X is L-injective.

Proof. Suppose that X is L-complete. Let $f : Y \to Z$ be an L-equivalence and $h : Y \to X$ be a $(\mathbb{T}, V)$-functor. Since f is an L-equivalence, one has $h_* \circ f^* \dashv f_* \circ h^*$.

As X is L-complete, there exists a $(\mathbb{T}, V)$-functor $g : Z \to X$ such that $g_* = h_* \circ f^*$. This implies $g_* \circ f_* = h_*$, $g.f \simeq h$, i.e. $g.f \simeq h$. Hence X is L-injective.

Conversely, suppose that X is L-injective. X is L-complete if $y : X \to \tilde{X}$ is surjective by Cor. 3.2.2. So it is enough to show that y is a retraction. One has $1_X : X \to X$ and $y : X \to \tilde{X}$ which is an L-equivalence by Cor. 2.8.7 and Prop. 3.5.4. Since X is L-injective, there exists $m : \tilde{X} \to X$ such that $m.y \simeq 1_X$. This implies $y.m.y \simeq y.1_X = 1_{\tilde{X}}.y$. Then $y.m.y = 1_{\tilde{X}}.y$, as $\tilde{X}$ is L-separated by Cor. 3.1.2. Since y is L-dense, one gets $y.m \simeq 1_{\tilde{X}}$ by Remark 3.4.12. Using again the fact that $\tilde{X}$ is L-separated, $y.m = 1_{\tilde{X}}$. □

Corollary 3.5.7. $V^{|X|}$ is L-complete for any $(\mathbb{T}, V)$-category X.

Proof. $V^{|X|}$ is L-injective by Cor. 3.3.2. □

Corollary 3.5.8. $\tilde{X}$ is L-complete for any $(\mathbb{T}, V)$-category X.

Proof. By Cor. 3.5.5 and Cor. 3.5.7, $\tilde{X}$ is an L-closed subset of $V^{|X|}$ which is L-complete. Then $\tilde{X}$ is L-complete by Prop. 3.5.3. □

We will denote the full subcategory of L-complete and L-separated $(\mathbb{T}, V)$-categories by $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{cpl\ \&\ sep}}$.

Theorem 3.5.9 ([36]). $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{cpl\ \&\ sep}}$ is a reflective subcategory of $(\mathbb{T}, V)$-$\mathbf{Cat}$ with reflection maps $y : X \to \tilde{X}$ for each $(\mathbb{T}, V)$-category X.

Proof. Let X be a $(\mathbb{T}, V)$-category and $f : X \to Y$ be a $(\mathbb{T}, V)$-functor where Y is L-complete and L-separated. As $y : X \to \tilde{X}$ is an L-equivalence and Y is L-injective, there exists a $(\mathbb{T}, V)$-functor $g : \tilde{X} \to Y$ such that $g.y \simeq f$. This implies $g.y = f$, since Y is L-separated. If there exists another $(\mathbb{T}, V)$-functor $h : \tilde{X} \to Y$ with $h.y \simeq f$, then $h.y \simeq g.y$. As Y is L-separated and y is L-dense, one gets $h = g$.

□

4 L-completeness, L-separation, L-injectivity for morphisms

The previous chapters summarized the framework for $(\mathbb{T}, V)$-categories as presented in [36], [33] and [15]. The parts that should be highlighted include the concepts of L-completeness, L-separation and L-injectivity as well as the results showing their interactions. In this section we introduce morphism counterparts of these notions [53]. For our purposes L-complete morphisms will be the most important among the others. Later in Chapter 8 we will develop a topological theory based on L-complete morphisms in the style of [13].

4.1 L-complete morphisms

Definition 4.1.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. We say that f is *L-complete* if for any left adjoint $(\mathbb{T}, V)$-module $\varphi : Z \rightsquigarrow X$ and any $(\mathbb{T}, V)$-functor $h : Z \to Y$ such that $f_* \circ \varphi = h_*$, there exists a $(\mathbb{T}, V)$-functor $g : Z \to X$ with $\varphi = g_*$ and $f.g = h$.

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\exists g"] \arrow[dr, "h"'] & & X \arrow[dl, "f"] & {} \arrow[r, mapsto, "(_)_*"] & {} & Z \arrow[rr, squiggly, "\varphi"] \arrow[dr, squiggly, "h_*"'] & & X \arrow[dl, squiggly, "f_*"] \\
& Y & & & & & Y &
\end{tikzcd}$$

Recall the lower star functor of Section 2.8. Since $f_* \dashv f^*$ for any $(\mathbb{T}, V)$-functor f, one has $(\_)_* : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Mod}_l$ where $(\mathbb{T}, V)\text{-}\mathbf{Mod}_l$ is the subcategory of $(\mathbb{T}, V)$-**Mod** whose morphisms are the left adjoint $(\mathbb{T}, V)$-modules. Taking the functor $(\_)_*$ into account, we see that a $(\mathbb{T}, V)$-functor is L-complete if and only if it is a $(\_)_*$-quasi cartesian morphism in the sense of fibrational category theory [27]. Here quasi refers to that fact the morphism g in the definition is only unique up to $\simeq$.

The above definition can be equivalently expressed with the upper star notation, i.e. f is L-complete if given any right adjoint $(\mathbb{T}, V)$-module $\psi : X \rightsquigarrow Z$ and any $(\mathbb{T}, V)$-functor $h : Z \to Y$ such that $\psi \circ f^* = h^*$, there exists a $(\mathbb{T}, V)$-functor $g : Z \to X$ with $\psi = g^*$ and $f.g = h$.

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\exists g"] \arrow[dr, "h"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}
\xmapsto{\quad (_)^* \quad}
\begin{tikzcd}
Z & & X \arrow[ll, squiggly, "\psi"'] \\
& Y \arrow[ul, squiggly, "h^*"] \arrow[ur, squiggly, "f^*"'] &
\end{tikzcd}$$

Now considering $(\_)^* : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Mod}_r$ where $(\mathbb{T}, V)\text{-}\mathbf{Mod}_r$ is the subcategory of $(\mathbb{T}, V)$-**Mod** whose morphisms are the right adjoint $(\mathbb{T}, V)$-modules, we conclude that a $(\mathbb{T}, V)$-functor is L-complete if and only if it is a $(\_)^*$-quasi cocartesian morphism.

Similar to the case of L-complete objects, one can replace the $(\mathbb{T}, V)$-category Z by the $\otimes$-neutral object E assuming that the axiom of choice holds.

Proposition 4.1.2. For a $(\mathbb{T}, V)$-functor $f : X \to Y$, the following are equivalent:

1. $f : X \to Y$ is L-complete.

2. For any left adjoint $(\mathbb{T}, V)$-module $\varphi : E \nrightarrow X$ and $y \in Y$ such that $f_* \circ \varphi = y_*$, there exists $x \in X$ with $\varphi = x_*$ and $f(x) = y$.

3. For any right adjoint $(\mathbb{T}, V)$-module $\psi : X \nrightarrow E$ and $y \in Y$ such that $\psi \circ f^* = y^*$, there exists $x \in X$ with $\psi = x^*$ and $f(x) = y$.

Examples 4.1.3. 1. In **Ord**, every ordered set X is L-complete as shown in Examples 3.2.5. So there is a one-to-one correspondence between the left adjoint 2-modules $\varphi : E \nrightarrow X$ and the elements of X. Hence a monotone map $f : (X, \leq) \to (Y, \leq)$ is L-complete if and only if given any $x \in X$ with $f(x) \simeq y$ for some $y \in Y$, there exists $w \in f^{-1}(\{y\})$ such that $x \simeq w$.

$$\begin{tikzcd}
x \arrow[r, dashed, dash, "\sim"] \arrow[d, "f"'] & w \arrow[d, dashed, "f"] \\
f(x) \arrow[r, dash, "\sim"'] & y
\end{tikzcd}$$

2. In **Met**, there is a one-to-one correspondence between the adjoint P_+-modules $\varphi \dashv \psi : X \nrightarrow E$ and the equivalence classes of Cauchy sequences in X as shown in Examples 3.2.5. The left adjoint $\varphi : E \nrightarrow X$ is representable by a point

$x \in X$ precisely when the elements of the equivalence class of Cauchy sequences corresponding to $\varphi \dashv \psi$ converges to x.

Let $f : (X, d) \to (Y, d')$ be a nonexpansive map and $\varphi \dashv \psi : X \rightsquigarrow E$ be a pair of adjoint P_+-modules. Suppose that the pair $\varphi \dashv \psi$ corresponds to the equivalence class of Cauchy sequences $\{(x_n)\}$ in X.

<u>Claim</u>: The pair $f_* \circ \varphi \dashv \psi \circ f^*$ corresponds to the equivalence class of Cauchy sequences $\{(f(x_n))\}$ in Y.

Since (x_n) is Cauchy and f is nonexpansive, $(f(x_n))$ is Cauchy. Its equivalence class corresponds to the left adjoint P_+-module $\lim\limits_{n\to\infty} d'(f(x_n), \_)$. So showing $f_* \circ \varphi = \lim\limits_{n\to\infty} d'(f(x_n), \_)$ will suffice. Pick any $y \in Y$, then

$$\begin{aligned} f_* \circ \varphi(y) &= \inf_{x \in X}\{\varphi(x) + f_*(x, y)\} \\ &= \inf_{x \in X}\{\varphi(x) + d'(f(x), y)\}. \end{aligned}$$

Since $\varphi \dashv \psi$ corresponds to the equivalence class of Cauchy sequences $\{(x_n)\}$, we have $\varphi(x) = \lim\limits_{n\to\infty} d(x_n, x)$. So

$$f_* \circ \varphi(y) = \inf_{x \in X}\{\lim_{n\to\infty} d(x_n, x) + d'(f(x), y)\}.$$

Now,

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x) + d'(f(x), y) &\geq \lim_{n\to\infty} d'(f(x_n), f(x)) + d'(f(x), y) \\
&= \lim_{n\to\infty} \Big(d'(f(x_n), f(x)) + d'(f(x), y) \Big) \\
&\geq \lim_{n\to\infty} d'(f(x_n), y).
\end{aligned}$$

This holds for all $x \in X$. So

$$\inf_{x\in X}\{\lim_{n\to\infty} d(x_n, x) + d'(f(x), y)\} \geq \lim_{n\to\infty} d'(f(x_n), y).$$

Hence, $f_* \circ \varphi \geq \lim_{n\to\infty} d'(f(x_n), \_)$.

To obtain the reverse inequality, observe that we have

$$\inf_{x\in X}\{\lim_{k\to\infty} d(x_k, x) + d'(f(x), y)\} \leq \lim_{k\to\infty} d(x_k, x_n) + d'(f(x_n), y)$$

for any $n \in \mathbb{N}$. Taking the limit of both sides,

$$\begin{aligned}
\inf_{x\in X}\{\lim_{k\to\infty} d(x_k, x) + d'(f(x), y)\} &\leq \lim_{n\to\infty} \Big(\lim_{k\to\infty} d(x_k, x_n) + d'(f(x_n), y) \Big) \\
&= 0 + \lim_{n\to\infty} d'(f(x_n), y).
\end{aligned}$$

Hence $f_* \circ \varphi \leq \lim_{n\to\infty} d'(f(x_n), \_)$. Therefore our claim is justified.

So a nonexpansive map $f : X \to Y$ is L-complete if and only if given any Cauchy sequence (x_n) in X with $\lim_{n\to\infty} f(x_n) = y$, there exists $x \in f^{-1}(\{y\})$

such that $\lim_{n\to\infty} x_n = x$

$$\begin{tikzcd}
(x_n) \arrow[r, dashed] \arrow[d, "f"'] & x \arrow[d, dashed, "f"] \\
f(x_n) \arrow[r] & y
\end{tikzcd}$$

3. Now we characterize L-complete morphisms in **Top**. Let $f : X \to Y$ be a continuous map and $\varphi \dashv \psi : X \rightsquigarrow E$ be a pair of adjoint $(\mathbb{U}, 2)$-modules. In Example 3.2.5 we have seen that $\varphi \dashv \psi : X \rightsquigarrow E$ corresponds bijectively to a closed set $A \subseteq X$ and a down-closed and Zariski closed $\mathcal{A} \subseteq UX$ such that

$$\exists \mathfrak{x} \in UX : \; A \in \mathfrak{x}, \, \mathfrak{x} \in \mathcal{A} \quad \text{and} \quad \forall \mathfrak{z} \in \mathcal{A}, \, x \in A, \, \mathfrak{z} \to x. \tag{4.1.1}$$

Here $A := \varphi^{-1}(\{1\})$ and $\mathcal{A} := \psi^{-1}(\{1\})$. Upon further examination, one finds that

$$\mathcal{A} = \Downarrow \mathfrak{x} = \{\mathfrak{z} \in UX \mid \forall x \in A, \, \mathfrak{z} \to x\}$$

and A is an irreducible closed set. The left adjoint φ is representable by a point $x \in X$ if and only if $A = \overline{\{x\}}$.

The first step towards the characterization of L-complete morphisms in **Top** is finding the counterparts of the adjoint pair $f_* \circ \varphi \dashv \psi \circ f^*$. Suppose that $f_* \circ \varphi \dashv \psi \circ f^*$ corresponds to an irreducible closed set $B \subseteq Y$ and a Zariski closed and down-closed set $\mathcal{B} \subseteq UY$.

<u>Claim</u>: $B = \overline{f(A)}$.

One has

$$f_* \circ \varphi(y) = f_*.U\varphi(y) = \bigvee_{\mathfrak{z} \in UX} U\varphi(\mathfrak{z}) \otimes b(Uf(\mathfrak{z}), y)$$

for all $y \in Y$. Since $B = (f_* \circ \varphi)^{-1}(\{1\})$, we get

$$B = \{y \in Y \mid \exists\, \mathfrak{z} \in UX : A \in \mathfrak{z} \ \&\ f(\mathfrak{z}) \to y\}.$$

Let $y \in f(A)$ where $y = f(x)$ for some $x \in A$. Then there is $\mathfrak{x} \in UX$, given in (4.1.1), with $\mathfrak{x} \to x$ and $A \in \mathfrak{x}$. This implies that $f(\mathfrak{x}) \to y$. Hence $y \in B$, $f(A) \subseteq B$. Since B is closed, we get $\overline{f(A)} \subseteq B$. Conversely, let $y \in B$. Then there exists $\mathfrak{z} \in UX$ such that $A \in \mathfrak{z}$ and $f(\mathfrak{z}) \to y$. This implies that $f(A) \in f(\mathfrak{z})$. As $f(\mathfrak{z}) \to y$, $f(A) \cap O \neq \varnothing$ for any open neighbourhood O of y. Hence $y \in \overline{f(A)}$, $B \subseteq \overline{f(A)}$.

So if $\varphi \dashv \psi : X \rightsquigarrow E$ corresponds to an irreducible closed set $A \subseteq X$, then $f_* \circ \varphi \dashv \psi \circ f^* : Y \rightsquigarrow E$ corresponds to the irreducible closed set $\overline{f(A)} \subseteq Y$.

Hence a continuous map $f : X \to Y$ is L-complete if and only if for any irreducible closed set $A \subseteq X$ with $\overline{f(A)} = \overline{\{y\}}$ for some $y \in Y$, there exists $x \in f^{-1}(\{y\})$ such that $A = \overline{\{x\}}$. We call such maps *quasi fibrewise sober.* In case the point x is unique, f is called fibrewise sober [48].

Now we identify L-complete maps from an alternative perspective. Recall that Example 3.2.5 also provides a bijective correspondence between the adjoint pairs $\varphi \dashv \psi : X \rightsquigarrow E$ and irreducible ultrafilters $\mathfrak{x} \in UX$ with *conv* $\mathfrak{x} = A$

and $\Downarrow \mathfrak{x} = \mathcal{A}$. The left adjoint φ is representable by $x \in X$ if and only if $conv\, \mathfrak{x} = conv\, \dot{x}$.

Consider the adjoint pair $f_* \circ \varphi \dashv \psi \circ f^* : Y \rightsquigarrow E$. It corresponds to an irreducible ultrafilter $\mathfrak{w} \in UY$ such that

$$conv\, \mathfrak{w} = B \qquad \& \qquad \Downarrow \mathfrak{w} = \mathcal{B}. \tag{4.1.2}$$

<u>Claim:</u> $f(\mathfrak{x})$ satisfies the conditions in (4.1.2).

We know that $conv\, \mathfrak{x} = A$. Hence $f(A) \subseteq conv\, f(\mathfrak{x})$, as f is continuous. On the other hand, $f(A) \in f(\mathfrak{x})$ implies $conv\, f(\mathfrak{x}) \subseteq \overline{f(A)}$. Since the limit points of a filter is a closed set, $conv\, f(\mathfrak{x}) = \overline{f(A)}$. $f(\mathfrak{x})$ is irreducible, as $\overline{f(A)} \in f(\mathfrak{x})$. This also shows that $conv\, f(\mathfrak{x}) = B$.

Secondly, $\Downarrow f(\mathfrak{x}) = \mathcal{B}$ as follows: We have $\mathcal{B} = \{\mathfrak{y} \in UY \mid \forall y \in \overline{f(A)}, \mathfrak{y} \to y\}$, since $B = \overline{f(A)}$. Take any $\mathfrak{y} \in \mathcal{B}$. For $\mathcal{B} \subseteq \Downarrow f(\mathfrak{x})$, one needs to show $\mathfrak{y} \leq f(\mathfrak{x})$ in $\mathsf{M}Y$. Let $C \in \mathfrak{y}$. Since $\mathfrak{y}$ converges to all points of $f(A)$, $f(A) \subseteq \overline{C}$. Then we have $\overline{C} \in f(\mathfrak{x})$, as $A \in \mathfrak{x}$. So $\mathfrak{y} \leq f(\mathfrak{x})$. Conversely, $f(\mathfrak{x}) \in \mathcal{B}$, as $\overline{f(A)} = conv\, f(\mathfrak{x})$. Since $\mathcal{B}$ is down-closed, $\Downarrow f(\mathfrak{x}) \subseteq \mathcal{B}$.

So if $\varphi \dashv \psi : X \rightsquigarrow E$ corresponds to an irreducible ultrafilter $\mathfrak{x} \in UX$, then $f_* \circ \varphi \dashv \psi \circ f^* : Y \rightsquigarrow E$ corresponds to the irreducible ultrafilter $f(\mathfrak{x}) \in UY$.

Hence a continuous map $f : X \to Y$ is L-complete if and only if for any irreducible ultrafilter $\mathfrak{x} \in UX$ with $conv\, f(\mathfrak{x}) = conv\, \dot{y}$, there exists $x \in f^{-1}(\{y\})$

such that $conv\,\mathfrak{x} = conv\,\dot{x}$.

$$\begin{tikzcd}
\mathfrak{x} \arrow[r, dash, dashed, "\sim"] \arrow[d, "f"'] & \dot{x} \arrow[d, dashed, "f"] \\
f(\mathfrak{x}) \arrow[r, dash, "\sim"'] & \dot{y}
\end{tikzcd}$$

Now we investigate properties of L-complete $(\mathbb{T}, V)$-functors.

Proposition 4.1.4. 1. Fully faithful and surjective $(\mathbb{T}, V)$-functors are L-complete. In particular isomorphisms in $(\mathbb{T}, V)$-**Cat** are L-complete.

2. L-complete $(\mathbb{T}, V)$-functors are closed under composition.

3. If $g.f$ is L-complete and g is monic then f is L-complete.

4. If $g.f$ is L-complete and f is an L-equivalence then g is L-complete.

Proof. 1. Let $f : X \to Y$ be a fully faithful and surjective $(\mathbb{T}, V)$-functor. Suppose that $\varphi : E \rightsquigarrow X$ is a left adjoint $(\mathbb{T}, V)$-module such that $f_* \circ \varphi = y_*$ for some $y \in Y$. Since f is fully faithful, one gets $\varphi = f^* \circ y_*$. As f is surjective, there exists $x \in X$ such that $f(x) = y$. Then

$$\varphi = f^* \circ (f(x))_* = f^* \circ f_* \circ x_* = x_*.$$

Hence f is L-complete. Isomorphisms in $(\mathbb{T}, V)$-**Cat** are fully faithful bijective $(\mathbb{T}, V)$-functors. So isomorphisms are L-complete.

2. Let $f : X \to Y$, $g : Y \to Z$ be L-complete $(\mathbb{T}, V)$-functors. Assume that $\varphi : E \rightsquigarrow X$ is a left adjoint $(\mathbb{T}, V)$-module such that $(g.f)_* \circ \varphi = z_*$ for some $z \in Z$. Then $g_* \circ (f_* \circ \varphi) = z_*$. Since $f_* \circ \varphi : E \rightsquigarrow Y$ is a left adjoint $(\mathbb{T}, V)$-module and g is L-complete, there exists $y \in Y$ such that $f_* \circ \varphi = y_*$ and $g(y) = z$. Using the fact that f is L-complete, there exists $x \in X$ such that $\varphi = x_*$ and $f(x) = y$. So $\varphi = x_*$ and $g.f(x) = z$. Hence $g.f$ is L-complete.

3. Let $f : X \to Y$, $g : Y \to Z$ be $(\mathbb{T}, V)$-functors where $g.f$ is L-complete and g is monic. Assume that $\varphi : E \rightsquigarrow X$ is a left adjoint $(\mathbb{T}, V)$-module such that $f_* \circ \varphi = y_*$ for some $y \in Y$. Then

$$(g.f)_* \circ \varphi = g_* \circ f_* \circ \varphi = g_* \circ y_* = (g(y))_*$$

Since $g.f$ is L-complete, there exists $x \in X$ such that $\varphi = x_*$ and $g.f(x) = g(y)$. Then $f(x) = y$, as g is monic. Hence f is L-complete.

4. Let $f : X \to Y$, $g : Y \to Z$ be $(\mathbb{T}, V)$-functors where $g.f$ is L-complete and f is an L-equivalence. Assume that $\varphi : E \rightsquigarrow Y$ is a left adjoint $(\mathbb{T}, V)$-module such that $g_* \circ \varphi = z_*$ for some $z \in Z$. Since f is an L-equivalence, $f_* \circ f^* \circ \varphi = \varphi$

and $f^* \circ \varphi$ is left adjoint $(\mathbb{T}, V)$-module.

$$\begin{tikzcd} & X \arrow[dr, squiggly, "f_*"] & \\ E \arrow[ur, squiggly, "f^* \circ \varphi"] \arrow[rr, squiggly, "\varphi"] \arrow[dr, squiggly, "z_*"'] & & Y \arrow[dl, squiggly, "g_*"] \\ & Z & \end{tikzcd}$$

As $g.f$ is L-complete, there exists $x \in X$ such that $f^* \circ \varphi = x_*$ and $g.f(x) = z$. Then

$$\varphi = f_* \circ f^* \circ \varphi = f_* \circ x_* = (f(x))_*.$$

Hence g is L-complete.

□

Theorem 4.1.5. L-complete $(\mathbb{T}, V)$-functors are stable under pullback.

Proof. Let $g : (Y, b) \to (Z, c)$ be an L-complete $(\mathbb{T}, V)$-functor. Consider its pullback along a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Z, c)$.

$$\begin{tikzcd} (X \times_Z Y, a \times b) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, b) \arrow[d, "g"] \\ (X, a) \arrow[r, "f"'] & (Z, c) \end{tikzcd}$$

We need to show that π_1 is L-complete. So assume $(\pi_1)_* \circ \varphi = (x_0)_*$ for some $x_0 \in X$ and $\varphi \dashv \psi : (X \times_Z Y, a \times b) \rightsquigarrow (E, k)$. Then we have the following commutative diagram in $(\mathbb{T}, V)$-**Mod**.

$$\begin{tikzcd}
(E,k) \arrow[r, rightsquigarrow, "\varphi"] \arrow[dr, rightsquigarrow, "(x_0)_*"'] & (X \times_Z Y, a \times b) \arrow[r, rightsquigarrow, "(\pi_2)_*"] \arrow[d, rightsquigarrow, "(\pi_1)_*"] & (Y,b) \arrow[d, rightsquigarrow, "g_*"] \\
& (X,a) \arrow[r, rightsquigarrow, "f_*"'] & (Z,c)
\end{tikzcd}$$

So $g_* \circ (\pi_2)_* \circ \varphi = f_* \circ (x_0)_* = (f(x_0))_*$ where $(\pi_2)_* \circ \varphi \dashv \psi \circ (\pi_2)^*$. Since g is L-complete, there exists $y_0 \in Y$ such that $(\pi_2)_* \circ \varphi = (y_0)_*$ and $g(y_0) = f(x_0)$. Hence $(x_0, y_0) \in X \times_Z Y$.

<u>Claim</u>: $\varphi = (x_0, y_0)_*$.

It suffices to show that $\varphi \leq (x_0, y_0)_*$ and $\psi \leq (x_0, y_0)^*$ by Lemma 2.3.1.

Since φ is a $(\mathbb{T}, V)$-module, one has $\varphi = (a \times b) \circ \varphi = (a \times b).\overline{T}\varphi.m_E^\circ$. So for any $(x, y) \in X \times_Z Y$,

$$\begin{aligned}
\varphi(x,y) &= \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} \overline{T}\varphi(\mathfrak{w}) \otimes (a \times b)(\mathfrak{w}, (x,y)) \\
&= \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} \overline{T}\varphi(\mathfrak{w}) \otimes (a(T\pi_1(\mathfrak{w}), x) \wedge b(T\pi_2(\mathfrak{w}), y)) \\
&\leq \bigvee_{\mathfrak{w} \in T(X \times_Z Y)} \overline{T}\varphi(\mathfrak{w}) \otimes a(T\pi_1(\mathfrak{w}), x) \\
&= (\pi_1)_* \circ \varphi(x) = (x_0)_*(x) = a(e_X(x_0), x).
\end{aligned}$$

Similarly $\varphi(x,y) \le b(e_Y(y_0),y)$. Hence

$$\varphi(x,y) \le a(e_X(x_0),x) \wedge b(e_Y(y_0),y) = (x_0,y_0)_*(x,y).$$

So $\varphi \le (x_0,y_0)_*$.

Now we show $\psi \le (x_0,y_0)^*$. By hypothesis, $(x_0)^* = \psi \circ (\pi_1)^*$. So for any $\mathfrak{x} \in TX$,

$$a(\mathfrak{x},x_0) = \bigvee_{\mathfrak{w}\in T(X\times_Z Y)} \overline{T}a(m_X^\circ(\mathfrak{x}), T\pi_1(\mathfrak{w})) \otimes \psi(\mathfrak{w}).$$

Considering $\mathfrak{x} = T\pi_1(\mathfrak{w})$, we get

$$\overline{T}a(m_X^\circ.T\pi_1(\mathfrak{w}), T\pi_1(\mathfrak{w})) \otimes \psi(\mathfrak{w}) \le a(T\pi_1(\mathfrak{w}),x_0).$$

As T is order preserving, $1_{TX} = Te_X^\circ.m_X^\circ \le \overline{T}a.m_X^\circ$. This means that

$$k \le \overline{T}a(m_X^\circ.T\pi_1(\mathfrak{w}), T\pi_1(\mathfrak{w})).$$

Hence $\psi(\mathfrak{w}) \le a(T\pi_1(\mathfrak{w}),x_0)$. Similarly, $(y_0)^* = \psi\circ(\pi_2)^*$ implies $\psi(\mathfrak{w}) \le b(T\pi_2(\mathfrak{w}),y_0)$. Then

$$\psi(\mathfrak{w}) \le a(T\pi_1(\mathfrak{w}),x_0) \wedge b(T\pi_2(\mathfrak{w}),y_0) = a\times b(\mathfrak{w},(x_0,y_0)) = (x_0,y_0)^*(\mathfrak{w})$$

for all $\mathfrak{w} \in T(X\times_Z Y)$. Hence $\psi \le (x_0,y_0)^*$.

$\square$

A pair (f,S) consisting of a morphism $f: A \to B$ and a source $S = \{\pi_i : A \to A_i\}_I$, is called a *multiple pullback* of a sink $\{f_i : A_i \to B\}_I$ provided that

- $f = f_i.\pi_i$ for all $i \in I$,
- For each pair (f', S') where $f' : A' \to B$ is a morphism and $S' = \{\pi'_i : A' \to A_i\}_I$ is a source such that $f' = f_i.\pi'_i$ for all $i \in I$, there exists a unique morphism $g : A' \to A$ with $f' = f.g$ and $\pi'_i = \pi_i.g$ for all $i \in I$.

Proposition 4.1.6. L-complete $(\mathbb{T}, V)$-functors are stable under multiple pullback.

Proof. Consider a sink $\{f_i : (X_i, a_i) \to (Z, c)\}_I$ of L-complete $(\mathbb{T}, V)$-functors. Taking the limit of the diagram, one gets

$$\begin{tikzcd}
(P, \prod a_i) \arrow[d, "\pi_i"'] \arrow[dr, "f"] & \\
(X_i, a_i) \arrow[r, "f_i"'] & (Z, c)
\end{tikzcd}$$

where $(P, \prod a_i)$ is the fibred product of (X_i, a_i)'s over Z and π_i's are the projection maps. To prove the claim, one needs to show that f is L-complete. So it is enough to show that there exists $j \in I$ for which π_j is L-complete. This is done similarly to the proof of Theorem 4.1.5. $\square$

Remark 4.1.7. In Section 3.5, we have seen that every $(\mathbb{T}, V)$-category (X, a) has the L-completion $(\tilde{X}, \widehat{a})$ consisting of tight $(\mathbb{T}, V)$-functors. Let $\mathcal{Y}$ denote the L-completion functor.

$$\mathcal{Y} : \quad (\mathbb{T}, V)\textbf{-Cat} \quad \to \quad (\mathbb{T}, V)\textbf{-Cat}_{\text{cpl \& sep}}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & & \tilde{X} \arrow[d, "\tilde{f}"] \\ Y & \arrow[r, mapsto] & \tilde{Y} \end{tikzcd}$$

To see the action of $\mathcal{Y}$ on morphisms, recall that $\tilde{X}$ can also be seen as the collection of right adjoint $(\mathbb{T}, V)$-modules from X to E. For a $(\mathbb{T}, V)$-functor $f : X \to Y$, one has $\mathcal{Y}(f) = \tilde{f}$ where

$$\tilde{f}(\psi) = \psi \circ f^*$$

for $\psi \in \tilde{X}$. The family of the Yoneda functors $y_X : X \to \tilde{X}$ form a natural transformation

$$y : 1_{(\mathbb{T},V)\text{-}\mathbf{Cat}} \to \mathcal{Y}.$$

This gives us another way to characterize L-complete morphisms.

Proposition 4.1.8. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. Then f is L-complete if and only if the naturality square

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ Y \arrow[r, "y_Y"'] & \tilde{Y} \end{tikzcd}$$

is a weak pullback.

Proof. The naturality square is a weak pullback if and only if for any $\psi \in \tilde{X}$ and $y \in Y$ such that $\tilde{f}(\psi) = y_Y(y)$, there exists $x \in X$ satisfying $\psi = y_X(x)$ and $f(x) = y$. This is equivalent to saying that for any right adjoint $(\mathbb{T}, V)$-module $\psi : X \rightsquigarrow E$ and $y \in Y$ such that $\psi \circ f^* = y^*$ there exists $x \in X$ satisfying $\psi = x^*$ and $f(x) = y$. That is equivalent to f being L-complete. □

Corollary 4.1.9. A $(\mathbb{T}, V)$-category X is L-complete if and only if $!_X : X \to 1$ is L-complete.

Proof. By Prop. 4.1.8, $!_X : X \to 1$ is L-complete if and only if

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "!_X"'] & \tilde{X} \arrow[d, "\tilde{!_X}"] \\ 1 \arrow[r, "y_1"'] & \tilde{1} \end{tikzcd}$$

is a weak pullback. Since $\tilde{1} = 1$, this is equivalent to saying that y_X is surjective. y_X is surjective if and only if X is L-complete by Cor. 3.2.2. □

This result is another confirmation that the morphism notion of L-completeness is the natural extension of the corresponding object notion.

4.2 L-separated morphisms

Let Y be a $(\mathbb{T}, V)$-category. Consider the comma category $(\mathbb{T}, V)$-**Cat**$/Y$ whose objects are the $(\mathbb{T}, V)$-functors with the codomain Y. A morphism from $k : Z \to Y$ to $f : X \to Y$ in this category is a $(\mathbb{T}, V)$-functor $g : Z \to X$ such that $f.g = k$.

Definition 4.2.1. We call a $(\mathbb{T}, V)$-functor $f : X \to Y$ L-separated if it is an L-separated object in the ordered category $(\mathbb{T}, V)$-**Cat**$/Y$. This means, given any morphisms $g, h : k \to f$ such that $g \simeq h$, one has $g = h$.

So a $(\mathbb{T}, V)$-functor $f : X \to Y$ is L-separated if and only if given any $(\mathbb{T}, V)$-functors $g, h : Z \to X$ such that $g \simeq h$ and $f.g = f.h$, one has $f = g$. This means that f L-separated if and only if it is jointly monic with the lower star (upper star) operation.

As in the case of L-separated objects, it is sufficient to consider the $\otimes$-neutral object E instead of Z. Hence $f : X \to Y$ is L-separated if and only if $x \simeq w$ and $f(x) = f(w)$ implies $x = w$ for all $x, w \in X$.

Examples 4.2.2.
1. In **Ord**, a monotone map $f : X \to Y$ is L-separated if and only if its fibres are partially ordered sets.

2. In **Met**, a nonexpansive map $f : (X, d) \to (Y, d')$ is L-separated if and only if for all $x, w \in X$ in the same fibre of f, $d(x, w) = d(w, x) = 0$ implies $x = w$.

3. In **Top**, a continous map $f : X \to Y$ is L-separated if and only if its fibres are T_0.

4. In **App**, a contraction $f : X \to Y$ is L-separated if and only if its coreflection in **Top** has T_0 fibres.

Proposition 4.2.3. 1. If X is L-separated then any $(\mathbb{T}, V)$-functor $f : X \to Y$ is L-separated.

2. L-separated $(\mathbb{T}, V)$-functors are stable under pullback.

Proof. (1) is trivial. For (2), consider the following pullback diagram where $g : Y \to Z$ is L-separated.

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Let $(x, y), (x', y') \in X \times_Z Y$ such that $(x, y)_* = (x', y')_*$ and $\pi_1(x, y) = \pi_1(x', y')$. Then $x = x'$ and we have $g(y) = f(x) = f(x') = g(y')$. On the other hand, $(x, y)_* = (x', y')_*$ implies that $y_* = y'_*$. Since g is L-separated, one gets $y = y'$. Hence $(x, y) = (x', y')$, π_1 is L-separated. □

Proposition 4.2.4. Let $f : X \to Y$ be $(\mathbb{T}, V)$-functor such that X is L-complete and Y is L-separated. Then f is L-complete.

Proof. Suppose the assumptions hold. Let $\varphi : E \rightsquigarrow X$ be a left adjoint $(\mathbb{T}, V)$-module such that $f_* \circ \varphi = y_*$ for some $y \in Y$. Since X is L-complete, there exists $x \in X$ such that $\varphi = x_*$. Then $f_* \circ x_* = (f(x))_* = y_*$. As Y is L-separated, $f(x) = y$. Hence f is L-complete. $\square$

4.3 L-injective morphisms

Definition 4.3.1. We call a $(\mathbb{T}, V)$-functor $f : X \to Y$ L-injective if it is an L-injective object in $(\mathbb{T}, V)$-**Cat**$/Y$. This means, given any morphism $j : k \to f$ and any L-equivalence $i : k \to h$, there exists a morphism $g : h \to f$ such that $g.i \simeq j$.

Observe that $f : X \to Y$ is L-injective if and only if given any solid arrow commutative square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd} \tag{4.3.1}$$

in $(\mathbb{T}, V)$-**Cat** where i is an L-equivalence, there exists $g : B \to X$ such that $f.g = h$ and $g.i \simeq j$. Indeed, having $j : k \to f$ in $(\mathbb{T}, V)$-**Cat**$/Y$ means that $k = f.j$. So the commutative square (4.3.1) corresponds to the morphisms $j : k \to f$ and $i : k \to h$. In $(\mathbb{T}, V)$-**Cat**$/Y$ there exists $g : B \to X$ with the desired properties if and only if there exists $g : h \to f$ such that $g.i \simeq j$.

So, in the language of general homotopy theory, L-injective $(\mathbb{T}, V)$-functors are the morphisms in $(\mathbb{T}, V)$-**Cat** which have weak right lifting property with respect to L-equivalences.

In Section 3.5 we have seen that L-completeness and L-injectivity are equivalent notions at the level of objects. These notions are also equivalent at the level of morphisms as the next theorem shows.

Theorem 4.3.2. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. f is L-complete if and only if f is L-injective.

Proof. Let f be L-complete. Suppose that we have the commutative square (4.3.1) where i is an L-equivalence. Then $f_* \circ j_* = h_* \circ i_*$. As i is an L-equivalence, we get $f_* \circ j_* \circ i^* = h_*$ and $j_* \circ i^* \dashv i_* \circ j^*$. Since f is L-complete, there exists $g : B \to X$ such that $j_* \circ i^* = g_*$ and $f.g = h$. Then $j \simeq g.i$ and f is L-injective.

Conversely, let f be L-injective. By Prop. 4.1.8, f is L-complete if the following diagram is a weak pullback or, equivalently, the induced map i is surjective. So it is enough to show that i is a retraction.

$$\begin{tikzcd}
X \arrow[dr, "i"] \arrow[drr, bend left, "y_X"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_{\tilde{Y}} \tilde{X} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X} \arrow[d, "\tilde{f}"] \\
& Y \arrow[r, "y_Y"'] & \tilde{Y}
\end{tikzcd}$$

Firstly, $\tilde{f}$ is L-complete and L-separated by Prop. 4.2.3 and 4.2.4. So π_1 is L-complete and L-separated as a pullback of $\tilde{f}$. On the other hand, π_2 is fully faithful as a pullback of y_Y which is fully faithful. Since $y_X : X \to \tilde{X}$ is an L-equivalence and π_2 is fully faithful, i is an L-equivalence by Prop. 2.8.2. We have the following commutative square:

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "\pi_1.i = f"] \\
Y \times_{\tilde{Y}} \tilde{X} \arrow[r, "\pi_1"'] \arrow[ur, dashed, "m"'] & Y
\end{tikzcd}$$

Since f is L-injective, there exists $m : Y \times_{\tilde{Y}} \tilde{X} \to X$ such that $m.i \simeq 1_X$ and $\pi_1.i.m = \pi_1$. Then we have $\pi_1.i.m.i = \pi_1.i$ with $i.m.i \simeq i$. As π_1 is L-separated, $i.m.i = i$. That implies $i.m \simeq 1_{Y \times_{\tilde{Y}} \tilde{X}}$, as i is L-dense. Now we have $\pi_1.i.m = \pi_1.1_{Y \times_{\tilde{Y}} \tilde{X}}$ with

$i.m \simeq 1_{Y \times_{\tilde{Y}} \tilde{X}}$. Using again the fact that π_1 is L-separated, we obtain $i.m = 1_{Y \times_{\tilde{Y}} \tilde{X}}$. Hence i is a retraction.

$\square$

Since L-injectivity and L-completeness are equivalent notions, L-complete mophisms have the weak right lifting property with respect to L-equivalences. This fact becomes very helpful for showing that L-complete and L-separated morphisms belong to a factorization system. We will consider this factorizations system in detail in Chapter 8.

5 The functor L

In this chapter we investigate the functor $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ that is induced by the L-closure (see Subsection 3.4.2). Our results complement Chapter 7 where we explore the functional topology on $(\mathbb{T}, V)\text{-}\mathbf{Cat}$ with respect to "L-closed maps".

A theme that we will frequently encounter in Chapter 7 is the preservation of finite products by the functor L. We investigate the necessary conditions for L to have this property in Section 5.1. In Section 5.2, we examine the functor $\mathsf{L}_{(\mathbb{U},\mathsf{P}_+)}$: $\mathbf{App} \to \mathbf{Top}$ which will be important in studying compactness with respect to "L-closed maps" for approach spaces. This concept is known as Zariski compactness [26] and has not been characterized yet. We will provide such a characterization in Section 7.2.

5.1 Preservation of finite products

Suppose that the quantale V is constructively completely distributive (ccd). In this setting the L-closure can be equivalently formulized as follows.

Proposition 5.1.1. Let V be a ccd quantale, (X, a) be a $(\mathbb{T}, V)$-category, $M \subseteq X$ and $x \in X$. Then the following assertions are equivalent:

1. $x \in \overleftrightarrow{M}$.

2. For all $\varepsilon \ll k$ there exists $\mathfrak{x} \in TM$ such that $\varepsilon \leq a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x})$.

3. For all $\varepsilon \ll k$ there exists $\mathfrak{x} \in TM$ such that $\varepsilon \ll a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x})$.

Proof. ($1 \Rightarrow 2$) Let $x \in \overleftrightarrow{M}$. Then $k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x})$. If $\varepsilon \ll k$, then there exists $\mathfrak{x} \in TM$ such that $\varepsilon \leq a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x})$ by (2.1.8).

($2 \Rightarrow 3$) Let $\varepsilon \ll k$. By Lemma 2.1.3, there exists $\varepsilon_1 \in V$ such that $\varepsilon \ll \varepsilon_1 \ll k$. For $\varepsilon_1 \ll k$, there exists $\mathfrak{x} \in TM$ with

$$\varepsilon_1 \leq a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x})$$

by hypothesis. Since $\varepsilon \ll \varepsilon_1$, we get

$$\varepsilon \ll a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}).$$

($3 \Rightarrow 1$) Suppose that for all $\varepsilon \ll k$ there exists $\mathfrak{x}_\varepsilon \in TM$ such that

$$\varepsilon \ll a(\mathfrak{x}_\varepsilon, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}_\varepsilon).$$

Since V is ccd, $k = \bigvee_{\varepsilon \ll k} \varepsilon$ by Lemma 2.1.3. Then

$$k = \bigvee_{\varepsilon \ll k} \varepsilon \leq \bigvee_{\mathfrak{x}_\varepsilon \in TM} a(\mathfrak{x}_\varepsilon, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}_\varepsilon) \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}).$$

Hence $x \in \overleftrightarrow{M}$. □

Recall from Prop. 3.4.13 that the L-closure is additive when T preserves finite sums and k is $\vee$-irreducible. Assuming that V is ccd, we replace $\vee$-irreducibility by the condition (5.1.1) below.

Proposition 5.1.2. Suppose that T preserves finite sums, V is ccd and satisfies the following condition:

$$u \ll k \quad \& \quad w \ll k \quad \Longrightarrow \quad u \vee w \ll k. \tag{5.1.1}$$

Then the L-closure is additive.

Proof. Let (X,a) be a $(\mathbb{T},V)$-category and $M,N \subseteq X$. Since L-closure is monotone, it is enough to show $\overleftrightarrow{M \cup N} \subseteq \overleftrightarrow{M} \cup \overleftrightarrow{N}$. Suppose that $x \notin \overleftrightarrow{M} \cup \overleftrightarrow{N}$. Then there exists $\varepsilon_1 \ll k$ such that for all $\mathfrak{x} \in TM$,

$$\varepsilon_1 \not\ll a(\mathfrak{x},x) \otimes \overline{T}a(Te_X.e_X(x),\mathfrak{x})$$

and there exists $\varepsilon_2 \ll k$ such that for all $\mathfrak{x} \in TN$,

$$\varepsilon_2 \not\ll a(\mathfrak{x},x) \otimes \overline{T}a(Te_X.e_X(x),\mathfrak{x}).$$

By (5.1.1), one has $\varepsilon_1 \vee \varepsilon_2 \ll k$. Since T preserves finite sums, for all $\mathfrak{x} \in TM \cup TN = T(M \cup N)$,

$$\varepsilon_1 \vee \varepsilon_2 \not\ll a(\mathfrak{x},x) \otimes \overline{T}a(Te_X.e_X(x),\mathfrak{x}).$$

This exactly means that $x \notin \overleftrightarrow{M \cup N}$. $\square$

So the L-closure induces the functor $\mathsf{L} : (\mathbb{T},V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ given that the conditions of Prop. 5.1.2 are satisfied.

Remark 5.1.3. Condition (5.1.1) implies $\vee$-irreducibility of k as shown in [34]. To see this, let $k \leq u \vee v$, $A = \{x \in V \mid x \ll k\}$, $A_u = \{x \in A \mid x \leq u\}$ and $A_v = \{x \in A \mid x \leq v\}$. For any $x \ll k$, one has $x \leq u$ or $x \leq v$. Hence $A \subseteq A_u \cup A_v$. Trivially $A \supseteq A_u \cup A_v$. So $A = A_u \cup A_v$.

Furthermore, $k = \bigvee A = \bigvee\{x \in A \mid x \geq y\}$ for any $y \in A$ as follows. Given any $x \in A$, one has $x \vee y \in A$ by condition (5.1.1) where $x \vee y \geq y$ and $x \vee y \geq x$. So $\bigvee A \leq \bigvee\{x \in A \mid x \geq y\}$. On the other hand, one trivially has $\bigvee A \geq \bigvee\{x \in A \mid x \geq y\}$.

Consider the following cases. If $k = \bigvee A_u$, then $k \leq u$. If $k \neq \bigvee A_u$, then there exists $y \in A$ such that $y \notin A_u$. Take any $x \in A$ with $x \geq y$. Then $x \in A_u$ or $x \in A_v$. One gets $x \in A_v$, as $x \in A_u$ will imply $y \in A_u$ which is a contradiction. So $\{x \in A \mid x \geq y\} \subseteq A_v$. Taking supremum of both sides, $k = \bigvee\{x \in A \mid x \geq y\} \subseteq \bigvee A_v \leq v$. Hence $k \leq u$ or $k \leq v$ which means that k is $\vee$-irreducible.

Lemma 5.1.4 ([24]). Suppose that V is a ccd quantale and (5.1.1) holds. Then for every $w \ll k$ there exists $u \ll k$ such that $w \ll u \otimes u$.

Proof. For any $u_1, u_2 \ll k$, $u_1 \vee u_2 \ll k$ by (5.1.1). Then

$$u_1 \otimes u_2 \leq (u_1 \vee u_2) \otimes (u_1 \vee u_2)$$

and

$$\bigvee_{u_1, u_2 \ll k} u_1 \otimes u_2 \leq \bigvee_{u \ll k} u \otimes u.$$

One gets

$$k = k \otimes k = \bigvee_{u_1 \ll k} u_1 \otimes \bigvee_{u_2 \ll k} u_2 = \bigvee_{u_1, u_2 \ll k} u_1 \otimes u_2 \leq \bigvee_{u \ll k} u \otimes u.$$

Then for any $w \ll k$ there exists $u \ll k$ such that $w \ll u \otimes u$. □

Remark 5.1.5. Let $X = (X, a)$, $Y = (Y, b)$ be $(\mathbb{T}, V)$-categories. Consider the $(\mathbb{T}, V)$-category $\mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}})$ whose structure is $(\overline{T}a.Te_X.e_X)^\circ$ (see Subsection 2.7.2).

The functor A preserves products, since $\mathsf{A}^\circ \dashv \mathsf{A}$. Furthermore, the operation $(\_)^{\mathrm{op}}$ is compatible with products. If A° preserves finite products, then

$\mathsf{A}^\circ(\mathsf{A}(X \times Y)^{\mathrm{op}}) = \mathsf{A}^\circ((\mathsf{A}X \times \mathsf{A}Y)^{\mathrm{op}}) = \mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}} \times \mathsf{A}(Y)^{\mathrm{op}}) = \mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}}) \times \mathsf{A}^\circ(\mathsf{A}(Y)^{\mathrm{op}})$.

That means

$$(\overline{T}(a \times b).Te_{X\times Y}.e_{X\times Y})^\circ = (\overline{T}a.Te_X.e_X)^\circ \times (\overline{T}b.Te_Y.e_Y)^\circ.$$

Proposition 5.1.6. Assume that T and V satisfies the conditions of Prop. 5.1.2, $k = \top$ and $\mathsf{A}^\circ : V\text{-}\mathbf{Cat} \to (\mathbb{T}, V)\text{-}\mathbf{Cat}$ preserves finite products. Then $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ preserves finite products.

Proof. Let $X = (X, a)$, $Y = (Y, b)$ be $(\mathbb{T}, V)$-categories. We need to show that $\mathsf{L}(X \times Y) \cong \mathsf{L}X \times \mathsf{L}Y$, i.e. $\overleftrightarrow{M} = \overline{M}$ for all $M \subseteq X \times Y$ where $\overline{M}$ is the closure of M in $\mathsf{L}X \times \mathsf{L}Y$.

Take any $M \subseteq X \times Y$. Since L is a functor, $id : \mathsf{L}(X \times Y) \to \mathsf{L}X \times \mathsf{L}Y$ is continuous. Hence $\overleftrightarrow{M} \subseteq \overline{M}$. To show that $\overline{M} \subseteq \overleftrightarrow{M}$, let $(x, y) \in \overline{M}$ and fix $\varepsilon_0 \ll k$. Let π_1, π_2 be the projection maps. Then $(x, y) \in \overline{M}$ implies $x \in \pi_1(\overline{M}) \subseteq \overleftrightarrow{\pi_1(M)}$, since π_1 is continuous. Similarly, one has $y \in \overleftrightarrow{\pi_2(M)}$. There exists $\varepsilon \ll k$ such that $\varepsilon_0 \leq \varepsilon \otimes \varepsilon$ by Lemma 5.1.4. As $x \in \overleftrightarrow{\pi_1(M)}$, there exists $\mathfrak{x} \in T(\pi_1(M))$ such that

$$\varepsilon \ll a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}).$$

Similarly, there exists $\mathfrak{y} \in T(\pi_2(M))$ such that

$$\varepsilon \ll b(\mathfrak{y}, y) \otimes \overline{T}b(Te_Y.e_Y(y), \mathfrak{y}),$$

as $y \in \overleftrightarrow{\pi_2(M)}$. Since $k = \top$, we have

$$\varepsilon \ll a(\mathfrak{x}, x) \quad \& \quad \varepsilon \ll \overline{T}a(Te_X.e_X(x), \mathfrak{x}),$$

$$\varepsilon \ll b(\mathfrak{y}, y) \quad \& \quad \varepsilon \ll \overline{T}b(Te_Y.e_Y(y), \mathfrak{y}).$$

Then $\varepsilon = \varepsilon \wedge \varepsilon \le a(\mathfrak{x}, x) \wedge b(\mathfrak{y}, y)$. Similarly $\varepsilon \le \overline{T}a(Te_X.e_X(x), \mathfrak{x}) \wedge \overline{T}b(Te_Y.e_Y(y), \mathfrak{y})$. Hence

$$\varepsilon_0 \ll \varepsilon \otimes \varepsilon \le \big(a(\mathfrak{x}, x) \wedge b(\mathfrak{y}, y)\big) \otimes \big(\overline{T}a(Te_X.e_X(x), \mathfrak{x}) \wedge \overline{T}b(Te_Y.e_Y(y), \mathfrak{y})\big).$$

Since T satisfies the Beck-Chevalley property, the map $\langle T\pi_1, T\pi_2\rangle : TM \to T(\pi_1(M)) \times T(\pi_2(M))$ is surjective. So there exists $\mathfrak{w} \in TM$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{x}$ and $T\pi_2(\mathfrak{w}) = \mathfrak{y}$. Hence

$$\begin{aligned} \varepsilon_0 &\ll a(T\pi_1(\mathfrak{w}), x) \wedge b(T\pi_2(\mathfrak{w}), y) \\ &\qquad \otimes \overline{T}a(Te_X.e_X(x), T\pi_1(\mathfrak{w})) \wedge \overline{T}b(Te_Y.e_Y(y), T\pi_2(\mathfrak{w})) \\ &= (a \times b)(\mathfrak{w}, (x, y)) \otimes ((\overline{T}a.Te_X.e_X)^\circ \times (\overline{T}b.Te_Y.e_Y)^\circ)(\mathfrak{w}, (x, y)). \end{aligned}$$

Since A° preserves finite products, Remark 5.1.5 implies

$$\begin{aligned} \varepsilon_0 &\ll (a \times b)(\mathfrak{w}, (x, y)) \otimes (\overline{T}(a \times b).Te_{X\times Y}.e_{X\times Y})^\circ(\mathfrak{w}, (x, y)) \\ &= (a \times b)(\mathfrak{w}, (x, y)) \otimes \overline{T}(a \times b)(Te_{X\times Y}.e_{X\times Y}(x, y), \mathfrak{w}). \end{aligned}$$

Hence $(x, y) \in \overleftrightarrow{M}$. Therefore $\overline{M} \subseteq \overleftrightarrow{M}$. $\square$

Now we look at the implications of this result for our main examples.

Corollary 5.1.7. $\mathsf{L}_2 : \mathbf{Ord} \to \mathbf{Top}$ and $\mathsf{L}_{\mathsf{P}_+} : \mathbf{Met} \to \mathbf{Top}$ preserve finite products.

Proof. In these cases T and A° are identity functors. The quantales 2 and P_+ are ccd where $k = \top$ and (5.1.1) holds. Hence Prop. 5.1.6 applies. □

Proposition 5.1.8. Let V be a completely distributive (cd) quantale. Then $\mathsf{A}^\circ : V\text{-}\mathbf{Cat} \to (\mathbb{U}, V)\text{-}\mathbf{Cat}$ preserves finite products.

Proof. Let $X = (X, a)$, $Y = (Y, b)$ be V-categories. One has $\mathsf{A}^\circ(X) = (X, e_X^\circ.\overline{U}a)$, $\mathsf{A}^\circ(Y) = (Y, e_Y^\circ.\overline{U}b)$ and $\mathsf{A}^\circ(X \times Y) = (X \times Y, e_{X\times Y}^\circ.\overline{U}(\pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2))$. If

$$\overline{U}(\pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2) = U\pi_1^\circ.\overline{U}a.U\pi_1 \wedge U\pi_2^\circ.\overline{U}b.U\pi_2, \quad (\dagger)$$

then

$$\begin{aligned} e_{X\times Y}^\circ.\overline{U}(\pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2) &= e_{X\times Y}^\circ.(U\pi_1^\circ.\overline{U}a.U\pi_1 \wedge U\pi_2^\circ.\overline{U}b.U\pi_2) \\ &= e_{X\times Y}^\circ.U\pi_1^\circ.\overline{U}a.U\pi_1 \wedge e_{X\times Y}^\circ.U\pi_2^\circ.\overline{U}b.U\pi_2 \\ &= \pi_1^\circ.e_X^\circ.\overline{U}a.U\pi_1 \wedge \pi_2^\circ.e_Y^\circ.\overline{U}b.U\pi_2. \end{aligned}$$

Since $\pi_1^\circ.e_X^\circ.\overline{U}a.U\pi_1 \wedge \pi_2^\circ.e_Y^\circ.\overline{U}b.U\pi_2$ is the structure on $\mathsf{A}^\circ(X) \times \mathsf{A}^\circ(Y)$, the result follows. In the remaining part of the proof we will show that $(\dagger)$ holds.

As $\overline{U}$ preserves order, we have $\overline{U}(\pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2) \le U\pi_1^\circ.\overline{U}a.U\pi_1 \wedge U\pi_2^\circ.\overline{U}b.U\pi_2$. To show the reverse inequality, recall from (2.5.5) that for $r : X \nrightarrow Y$ in V-**Rel**,

$$\overline{U}r(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{\substack{A\in\mathfrak{x} \\ B\in\mathfrak{y}}} \bigvee_{\substack{x\in A \\ y\in B}} r(x, y)$$

for all $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$.

Let $\mathfrak{w}, \mathfrak{z}$ be two ultrafilters on $X \times Y$, then

$$\begin{aligned}
\big(U\pi_1^\circ.\overline{U}a.U\pi_1 \wedge U\pi_2^\circ.\overline{U}b.U\pi_2\big)(\mathfrak{w},\mathfrak{z}) &= \overline{U}a(U\pi_1(\mathfrak{w}), U\pi_1(\mathfrak{z})) \wedge \overline{U}b(U\pi_2(\mathfrak{w}), U\pi_2(\mathfrak{z})) \\
&= \Big(\bigwedge_{\substack{A \in U\pi_1(\mathfrak{w}) \\ C \in U\pi_1(\mathfrak{z})}} \bigvee_{\substack{x \in A \\ z \in C}} a(x,z) \Big) \wedge \Big(\bigwedge_{\substack{B \in U\pi_2(\mathfrak{w}) \\ D \in U\pi_2(\mathfrak{z})}} \bigvee_{\substack{y \in B \\ w \in D}} b(y,w) \Big) \\
&= \bigwedge_{\substack{A \in U\pi_1(\mathfrak{w}) \\ C \in U\pi_1(\mathfrak{z})}} \bigvee_{\substack{x \in A \\ z \in C}} \bigwedge_{\substack{B \in U\pi_2(\mathfrak{w}) \\ D \in U\pi_2(\mathfrak{z})}} \bigvee_{\substack{y \in B \\ w \in D}} \Big(a(x,z) \wedge b(y,w) \Big) \\
&\leq \bigwedge_{\substack{A \in U\pi_1(\mathfrak{w}) \\ C \in U\pi_1(\mathfrak{z})}} \bigwedge_{\substack{B \in U\pi_2(\mathfrak{w}) \\ D \in U\pi_2(\mathfrak{z})}} \bigvee_{\substack{x \in A \\ z \in C}} \bigvee_{\substack{y \in B \\ w \in D}} \Big(a(x,z) \wedge b(y,w) \Big) \quad (CD) \\
&= \bigwedge_{\substack{A \times B \in \mathfrak{w} \\ C \times D \in \mathfrak{z}}} \bigvee_{\substack{(x,y) \in A \times B \\ (z,w) \in C \times D}} \Big(a(x,z) \wedge b(y,w) \Big) \\
&\leq \bigwedge_{\substack{F \in \mathfrak{w} \\ G \in \mathfrak{z}}} \bigvee_{\substack{(x,y) \in F \\ (z,w) \in G}} \Big(a(x,z) \wedge b(y,w) \Big) \\
&= \bigwedge_{\substack{F \in \mathfrak{w} \\ G \in \mathfrak{z}}} \bigvee_{\substack{(x,y) \in F \\ (z,w) \in G}} \pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2((x,y),(z,w)) \\
&= \overline{U}(\pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2)(\mathfrak{w},\mathfrak{z}).
\end{aligned}$$

So $U\pi_1^\circ.\overline{U}a.U\pi_1 \wedge U\pi_2^\circ.\overline{U}b.U\pi_2 \leq \overline{U}(\pi_1^\circ.a.\pi_1 \wedge \pi_2^\circ.b.\pi_2)$. □

Corollary 5.1.9. $\mathsf{L}_{(\mathbb{U},2)} : \mathbf{Top} \to \mathbf{Top}$ **and** $\mathsf{L}_{(\mathbb{U},\mathsf{P}_+)} : \mathbf{App} \to \mathbf{Top}$ **preserve finite products.**

Proof. The ultrafilter functor U and the quantales 2 and P_+ satisfy the conditions of Prop. 5.1.2. Furthermore, 2 and P_+ are completely distributive quantales where

$k = \top$. Hence Prop. 5.1.8 and Prop. 5.1.6 applies. $\square$

5.2 $\mathsf{L}_{(\mathbb{U},\mathsf{P}_+)}$ from a bitopological viewpoint

In this section we investigate the functor $\mathsf{L}_{(\mathbb{U},\mathsf{P}_+)} : \mathbf{App} \to \mathbf{Top}$ via the framework of bitopological spaces.

A *bitopological space* (X, τ, σ) is a set X equipped with two topologies τ, σ. Let (Y, τ', σ') be another bitopological space. A map $f : X \to Y$ is called *bicontinuous* if it is continuous with respect to both topologies, i.e. both $f : (X, \tau) \to (Y, \tau')$ and $f : (X, \sigma) \to (Y, \sigma')$ are continuous maps in **Top**. Bitopological spaces and bicontinuous maps form the category **BiTop**.

Recall the natural closure, the dual closure and the L-closure as defined in Section 3.4. If k is $\vee$-irreducible and T preserves finite sums, then these closures induce the functors $\mathsf{N} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$, $\mathsf{D} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ and $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$. Given a $(\mathbb{T}, V)$-category $X = (X, a)$, let

$$\mathsf{N}X = (X, \overrightarrow{\tau}), \qquad \mathsf{D}X = (X, \overleftarrow{\tau}), \qquad \mathsf{L}X = (X, \overleftrightarrow{\tau}).$$

Then (X, a) naturally corresponds to the bitopological space $(X, \overrightarrow{\tau}, \overleftarrow{\tau})$. Let

$$\mathsf{B} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{BiTop}$$

be the functor which sends (X, a) to $(X, \overrightarrow{\tau}, \overleftarrow{\tau})$. Also consider the functor

$$\mathsf{J} : \mathbf{BiTop} \to \mathbf{Top}$$

which joins the topologies of a bitopological space, i.e. $\mathsf{J}(X,\tau,\sigma) = (X,\tau\vee\sigma)$ where $\tau\vee\sigma$ is the coarsest topology which contains both τ and σ. At each point $x \in X$, $\tau\vee\sigma$ has the local basis $\{O\cap U \mid x \in O \in \tau, x \in U \in \sigma\}$. The final aim of this section is to show that for an approach space, the topology induced by its L-closure is the join of the topologies induced by its natural closure and dual closure, i.e. $\overleftrightarrow{\tau} = \overrightarrow{\tau} \vee \overleftarrow{\tau}$ or, $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)} = \mathsf{J}.\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}$.

We start by examining the topologies induced by the natural closure and the dual closure in our examples.

Let $X = (X,a)$ be a $(\mathbb{T},V)$-category, $M \subseteq X$ and $x \in X$. Recall from Subsection 3.4.1 that x is in the natural closure of M, denoted by $x \in \overrightarrow{M}$, if

$$k \leq \bigvee_{\mathfrak{x}\in TM} a(\mathfrak{x},x).$$

Examples 5.2.1. For an object $X = (X,a)$ in **Ord**, **Met**, **Top** or **App**, we have $\mathsf{N}X = (X,\overrightarrow{\tau})$.

1. For an ordered set $(X,\leq)$, $x \in \overrightarrow{M}$ if and only if there exists $y \in M$ such that $y \leq x$. So M is closed in $\mathsf{N}X$ if it is up-closed and open if it is down-closed. The collection $\{\downarrow x \mid x \in X\}$ of principal down-closures forms a basis for $\overrightarrow{\tau}$. So $\mathsf{N}_2 : \mathbf{Ord} \to \mathbf{Top}$ is the Alexandroff topology functor.

2. For a metric space (X,d), $x \in \overrightarrow{M}$ if and only if $\inf\limits_{y\in M} d(y,x) = 0$. Therefore $\mathsf{N}_{\mathbb{P}_+} : \mathbf{Met} \to \mathbf{Top}$ is the usual forgetful functor. The collection $\{B^r_\varepsilon(x) \mid x \in$

$X, \varepsilon > 0\}$ forms a basis for $\overrightarrow{\tau}$ where $B^r_\varepsilon(x) := \{y \in X \mid d(y,x) < \varepsilon\}$. We call $B^r_\varepsilon(x)$ the right open ball of radius ε at x.

3. The natural closure of a topological space (X, τ) is the closure induced by τ. So $\tau = \overrightarrow{\tau}$, $\mathsf{N}_{(\mathbb{U},2)} : \mathbf{Top} \to \mathbf{Top}$ is the identity functor.

4. For an approach space (X, δ), $x \in \overrightarrow{M}$ if and only if $\delta(M, x) = 0$. So M is closed in $\mathsf{N}X$ if $M^{(0)} = M$ and open if $\delta(X \setminus M, x) > 0$ for all $x \in M$. The functor $\mathsf{N}_{(\mathbb{U},\mathbb{P}_+)} : \mathbf{App} \to \mathbf{Top}$ is the coreflector.

By taking the functor N into consideration, we can augment diagram (2.7.8) and obtain the following commutative diagram.

$$\begin{tikzcd}
\mathbf{Ord} \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, hook, "\mathsf{N}_2"] & & \mathbf{Top} \arrow[dd, hook] \arrow[dl, equal, "\mathsf{N}_{(\mathbb{U},2)}"'] \\
& \mathbf{Top} & \\
\mathbf{Met} \arrow[rr, hook] \arrow[ur, "\mathsf{N}_{\mathbb{P}_+}"] & & \mathbf{App} \arrow[ul, "\mathsf{N}_{(\mathbb{U},\mathbb{P}_+)}"']
\end{tikzcd} \tag{5.2.1}$$

Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category, $M \subseteq X$ and $x \in X$. Recall from Subsection 3.4.1 that x is in the dual closure of M, denoted by $x \in \overleftarrow{M}$, if

$$k \leq \bigvee_{\mathfrak{x} \in TM} \overline{T}a(Te_X.e_X(x), \mathfrak{x}).$$

The dual closure of X is the natural closure of $\mathsf{A}^\circ(\mathsf{A}(X)^{\mathrm{op}})$. This implies that $\mathsf{D} = \mathsf{N} . \mathsf{A}^\circ(\mathsf{A}(\_)^{\mathrm{op}})$.

Examples 5.2.2. For an object $X = (X, a)$ in **Ord**, **Met**, **Top** or **App**, we have $\mathsf{D}X = (X, \overleftarrow{\mathcal{T}})$.

1. For an ordered set $(X, \leq)$, $x \in \overleftarrow{M}$ if and only if there exists $y \in M$ such that $x \leq y$. So $M \subseteq X$ is closed in $\mathsf{D}X$ if it is down-closed and open if it is up-closed. The collection $\{\uparrow x \mid x \in X\}$ of the principal up-closures forms a basis for $\overleftarrow{\mathcal{T}}$.

2. For a metric space (X, d), $x \in \overleftarrow{M}$ if and only if $\inf\limits_{y \in M} d(x, y) = 0$. The collection $\{B^l_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$ forms a basis for $\overleftarrow{\mathcal{T}}$ where $B^l_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$. We call $B^l_\varepsilon(x)$ the left open ball of radius ε at x.

3. Let (X, τ) be a topological space. Recall from Examples 3.4.8 that $x \in \overleftarrow{M}$ if and only if $\overline{\{x\}} \cap M \neq \varnothing$.

   Let $O \subseteq X$. Then

$$
\begin{aligned}
O \in \overleftarrow{\mathcal{T}} &\iff X \setminus O \text{ closed in } \mathsf{D}X \\
&\iff \overleftarrow{(X \setminus O)} \subseteq X \setminus O \\
&\iff \forall x \in X \left(\overline{\{x\}} \bigcap (X \setminus O) \neq \varnothing \Rightarrow x \in X \setminus O\right) \\
&\iff \forall x \in O,\ \overline{\{x\}} \subseteq O.
\end{aligned}
$$

Furthermore, $\overline{\{x\}} \in \overleftarrow{\mathcal{T}}$, since for any $y \in \overline{\{x\}}$, $\overline{\{y\}} \subseteq \overline{\{x\}}$. So the collection of point closures $\{\overline{\{x\}} \mid x \in X\}$ forms a basis for $\overleftarrow{\mathcal{T}}$.

4. Let (X, δ) be an approach space. Recall from Examples 3.4.8 that $x \in \overleftarrow{M}$ if and only if $\{x\}^{(\varepsilon)} \bigcap M \neq \varnothing$ for all $\varepsilon > 0$. Let $O \subseteq X$. Then

$$\begin{aligned} O \in \overleftarrow{\mathcal{T}} &\iff X \setminus O \text{ closed in } \mathrm{D}X \\ &\iff \overleftarrow{(X \setminus O)} \subseteq X \setminus O \\ &\iff \forall x \in X \; (\forall \varepsilon > 0,\ \{x\}^{(\varepsilon)} \textstyle\bigcap (X \setminus O) \neq \varnothing \implies x \in X \setminus O) \\ &\iff \forall x \in O,\ \exists \varepsilon > 0 :\ \{x\}^{(\varepsilon)} \subseteq O. \end{aligned}$$

As in the case of topological spaces, one may try to show that the collection $\{\{x\}^{(\varepsilon)} \mid x \in X,\ \varepsilon > 0\}$ is a basis for $\overleftarrow{\mathcal{T}}$. Unfortunately that does not hold. But we can reach a comparable result by a slight modification.

The dual closure can be equivalently characterized by

$$x \in \overleftarrow{M} \iff \forall \varepsilon > 0,\ \{x\}^{((\varepsilon))} \textstyle\bigcap M \neq \varnothing$$

where $\{x\}^{((\varepsilon))} := \{z \in X \mid \delta(\{x\}, z) < \varepsilon\}$. Observe that if $x \in \overleftarrow{M}$, then $\varnothing \neq \{x\}^{(\varepsilon/2)} \bigcap M \subseteq \{x\}^{((\varepsilon))} \bigcap M$ for any $\varepsilon > 0$. Conversely, $\{x\}^{((\varepsilon))} \bigcap M \neq \varnothing$ trivially implies $\{x\}^{(\varepsilon)} \bigcap M \neq \varnothing$.

So

$$O \in \overleftarrow{\mathcal{T}} \iff \forall x \in O,\ \exists \varepsilon > 0 :\ \{x\}^{((\varepsilon))} \subseteq O. \tag{5.2.2}$$

Lemma 5.2.3. The collection $\mathcal{B} = \{\{x\}^{((\varepsilon))} \mid x \in X,\ \varepsilon > 0\}$ is a basis for $\overleftarrow{\mathcal{T}}$.

Proof. Clearly any element of $\overleftarrow{\mathcal{T}}$ can be written as a union of the elements of $\mathcal{B}$ by (5.2.2). So it is enough to show that every element of $\mathcal{B}$ is in $\overleftarrow{\mathcal{T}}$.

Take any $\{x\}^{((\varepsilon))} \in \mathcal{B}$ and $z \in \{x\}^{((\varepsilon))}$. Let $\delta(\{x\}, z) = \varepsilon_1$. Pick ε_2 such that $0 < \varepsilon_2 < \varepsilon - \varepsilon_1$. Then by Prop. 2.7.3,

$$\delta(\{x\}, y) \le \sup_{w \in \{z\}} \delta(\{x\}, w) + \delta(\{z\}, y) = \delta(\{x\}, z) + \delta(\{z\}, y) < \varepsilon_1 + \varepsilon_2 < \varepsilon$$

for any $y \in \{z\}^{((\varepsilon_2))}$. Hence $\{z\}^{((\varepsilon_2))} \subseteq \{x\}^{((\varepsilon))}$.

Therefore for arbitrary $z \in \{x\}^{((\varepsilon))}$, there exists $\varepsilon_2 > 0$ such that $\{z\}^{((\varepsilon_2))} \subseteq \{x\}^{((\varepsilon))}$. So $\{x\}^{((\varepsilon))} \in \overleftarrow{\mathcal{T}}$. □

The composite $\mathsf{A}^\circ(\mathsf{A}(\_)^{\mathrm{op}})$ commutes with the embeddings of diagram (2.7.8). Having diagram (5.2.1) and the fact that $\mathsf{D} = \mathsf{N} \,.\, \mathsf{A}^\circ(\mathsf{A}(\_)^{\mathrm{op}})$, we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathbf{Ord} \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, "\mathsf{D}_2"] & & \mathbf{Top} \arrow[dd, hook] \arrow[dl, "\mathsf{D}_{(\mathbb{U},2)}"'] \\
& \mathbf{Top} & \\
\mathbf{Met} \arrow[rr, hook] \arrow[ur, "\mathsf{D}_{\mathbb{P}_+}"] & & \mathbf{App} \arrow[ul, "\mathsf{D}_{(\mathbb{U},\mathbb{P}_+)}"']
\end{tikzcd} \tag{5.2.3}$$

The commutative diagrams (5.2.1) and (5.2.3) induce the following commutative diagram for the functor $\mathsf{B} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{BiTop}$.

$$\begin{tikzcd}
\mathbf{Ord} \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, "\mathsf{B}_2"] & & \mathbf{Top} \arrow[dd, hook] \arrow[dl, "\mathsf{B}_{(\mathbb{U},2)}"'] \\
& \mathbf{BiTop} & \\
\mathbf{Met} \arrow[rr, hook] \arrow[ur, "\mathsf{B}_{\mathbb{P}_+}"] & & \mathbf{App} \arrow[ul, "\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}"']
\end{tikzcd} \tag{5.2.4}$$

Now we examine the topologies induced by the L-closure in our examples.

Let $X = (X, a)$ be a $(\mathbb{T}, V)$-category, $M \subseteq X$ and $x \in X$. Recall from Subsection 3.4.2 that x is in the L-closure of M, denoted by $x \in \overleftrightarrow{M}$, if

$$k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}).$$

The L-closure of X is the natural closure of its symmetrization $\mathsf{S}X$. This implies that $\mathsf{L} = \mathsf{N}.\mathsf{S}$. We denote $\mathsf{L}X$ by $(X, \overleftrightarrow{\tau})$.

Examples 5.2.4. 1. For an ordered set $(X, \leq)$, $x \in \overleftrightarrow{M}$ if and only if there exists $y \in M$ such that $x \simeq y$, i.e. $x \leq y$ and $y \leq x$. Let $\tilde{x}$ denote equivalence class of x with respect to $\simeq$. M is closed in $\mathsf{L}X$ if and only if $\tilde{x} \subseteq M$ for all $x \in M$. Observe that this also the characterization of M being open in $\mathsf{L}X$. The collection $\{\tilde{x} \mid x \in X\}$ forms a basis for $\overleftrightarrow{\tau}$.

2. For a metric space (X,d), $x \in \overleftrightarrow{M}$ if and only if $\inf\limits_{y \in M}\{d(x,y)+d(y,x)\} = 0$. The collection $\{B^s_\varepsilon(x) \mid x \in X, \varepsilon > 0\}$ forms a basis for $\overleftrightarrow{\mathcal{T}}$ where $B^s_\varepsilon(x) := \{y \in X \mid d(y,x)+d(x,y) < \varepsilon\}$. We call $B^s_\varepsilon(x)$ the symmetrized open ball of radius ε at x.

3. Let (X,τ) be a topological space. Recall from Examples 3.4.16 that the L-closure of X is equal to its b-closure, i.e. $x \in \overleftrightarrow{M}$ if and only if $M \cap \overline{\{x\}} \cap O \neq \varnothing$ for any open neighbourhood O of x. $\mathsf{L}_{(\mathbb{U},2)} : \mathbf{Top} \to \mathbf{Top}$ is the b-topology functor.

4. Let (X,δ) be an approach space. Recall from Examples 3.4.16 that $x \in \overleftrightarrow{M}$ if and only if $\delta(M \cap \{x\}^{(\varepsilon)}, x) = 0$ for all $\varepsilon > 0$.

   Let $O \subseteq X$. Then

$$\begin{aligned}
O \in \overleftrightarrow{\mathcal{T}} &\iff X \smallsetminus O \text{ closed in } \mathsf{L}X \\
&\iff \overleftrightarrow{(X \smallsetminus O)} \subseteq X \smallsetminus O \\
&\iff \forall x \in X \left(\forall \varepsilon > 0,\ \delta(x, (X \smallsetminus O) \bigcap \{x\}^{(\varepsilon)}) = 0 \implies x \in X \smallsetminus O\right) \\
&\iff \forall x \in O, \exists \varepsilon > 0 :\ \delta(x, (X \smallsetminus O) \bigcap \{x\}^{(\varepsilon)}) > 0.
\end{aligned}$$

   Similar to dual closure, $\{x\}^{(\varepsilon)}$ can be equivalently replaced by $\{x\}^{((\varepsilon))}$ in the formulation of the L-closure, i.e.

$$x \in \overleftrightarrow{M} \iff \forall \varepsilon > 0,\ \delta(M \cap \{x\}^{((\varepsilon))}, x) = 0.$$

The symmetrization functor S commutes with the embeddings of diagram (2.7.8). Having diagram (5.2.1) and the fact that $\mathsf{L} = \mathsf{N}.\mathsf{S}$, we obtain the following commutative diagram.

$$
\begin{tikzcd}
\mathbf{Ord} \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, "\mathsf{L}_2"] & & \mathbf{Top} \arrow[dd, hook] \arrow[dl, "\mathsf{L}_{(\mathbb{U},2)}"'] \\
& \mathbf{Top} & \\
\mathbf{Met} \arrow[rr, hook] \arrow[ur, "\mathsf{L}_{\mathbb{P}_+}"] & & \mathbf{App} \arrow[ul, "\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)}"']
\end{tikzcd}
\tag{5.2.5}
$$

Proposition 5.2.5. $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)} = \mathsf{J}.\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}$.

Proof. Let $X = (X,\delta)$ be an approach space. $\mathsf{J}.\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}(X)$ is the topological space $(X, \overrightarrow{\mathcal{T}} \vee \overleftarrow{\mathcal{T}})$ with the local basis $\mathcal{B}_x = \{\{x\}^{((\varepsilon))} \cap O \mid \varepsilon > 0,\ O \in \overrightarrow{\mathcal{T}} : x \in O\}$ for each $x \in X$ by Lemma 5.2.3. Let $M \subseteq X$. The closure operator associated with $\mathsf{J}.\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}(X)$ is given by

$$x \in \overline{M} \quad \Longleftrightarrow \quad \forall \big(\{x\}^{((\varepsilon))} \cap O\big) \in \mathcal{B}_x,\ M \cap \{x\}^{((\varepsilon))} \cap O \neq \varnothing.$$

On the other hand, the topology $\overleftrightarrow{\mathcal{T}}$ on $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)}X$ is induced by the L-closure. One has

$$\begin{aligned} x \in \overleftrightarrow{M} &\iff \forall \varepsilon > 0,\ \delta(M \cap \{x\}^{((\varepsilon))}, x) = 0 \\ &\iff \forall \varepsilon > 0,\ x \in \overrightarrow{M \cap \{x\}^{((\varepsilon))}}. \end{aligned}$$

Since the natural closure for approach spaces is idempotent, $x \in \overrightarrow{M \cap \{x\}^{((\varepsilon))}}$ if and only if $M \cap \{x\}^{((\varepsilon))} \cap O \neq \varnothing$ for any $O \in \overrightarrow{\mathcal{T}}$ that contains x. Hence

$$\begin{aligned} x \in \overleftrightarrow{M} &\iff \forall \varepsilon > 0,\ \forall O \in \overrightarrow{\mathcal{T}} : x \in O,\ M \cap \{x\}^{((\varepsilon))} \cap O \neq \varnothing \\ &\iff \forall \big(\{x\}^{((\varepsilon))} \cap O\big) \in \mathcal{B}_x,\ M \cap \{x\}^{((\varepsilon))} \cap O \neq \varnothing. \end{aligned}$$

So $\overline{M} = \overleftrightarrow{M}$. Therefore $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)}X = \mathsf{J}.\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}(X)$. □

Remark 5.2.6. Observe that commutativity of the diagrams (5.2.4) and (5.2.5) makes the factorization $\mathsf{L} = \mathsf{J}.\mathsf{B}$ also work for **Ord**, **Met** and **Top**. So for any object X in these categories, one has $\overleftrightarrow{\mathcal{T}} = \overrightarrow{\mathcal{T}} \vee \overleftarrow{\mathcal{T}}$.

In **Ord**, this means that $\overset{\leftrightarrow}{\tau}$ has the local basis $\downarrow x \bigcap \uparrow x$ for each $x \in X$ which is exactly the set $\tilde{x}$. In **Met**, one sees that the topology induced by the symmetrized open balls is the same as the topology induced by the left open balls and the right open balls. In **Top**, this implies that $O \cap \overline{\{x\}}$ where O is an open neighbourhood of x is a local basis of the b-topology for $x \in X$.

6 Functional topology

Given a finitely complete category $\mathscr{C}$ equipped with a proper factorization system, one can pursue topological notions by using a distinguished class $\mathcal{F}$ of "closed morphisms" [13]. This chapter will provide the basics of the framework of [13] together with some of its results. We will use this setting to develop topological notions in $(\mathbb{T}, V)$-**Cat** in the following chapters. It is important to emphasize that the results in this chapter originally appeared in [13] unless stated otherwise.

6.1 The setting

6.1.1 Factorization system

A *factorization system* for a category $\mathscr{C}$ consists of two classes of morphisms $\mathcal{E}$ and $\mathcal{M}$ such that

(FS1) $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms and are closed under composition.

(FS2) For every morphism f in $\mathscr{C}$ there exists $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $f = m.e$.

(FS3) $\mathcal{E}$ is orthogonal to $\mathcal{M}$, i.e. given any commutative square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "e"'] & Y \arrow[d, "m"] \\ Z \arrow[r, "v"'] \arrow[ur, dashed, "f"] & W \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique morphism f that makes the whole diagram commutative.

$(\mathcal{E}, \mathcal{M})$ is called a *proper factorization system* if every morphism in $\mathcal{E}$ is an epimorphism and every morphism in $\mathcal{M}$ is a monomorphism.

Proposition 6.1.1. Let $(\mathcal{E}, \mathcal{M})$ be a proper factorization system. Then,

1. $\mathcal{E} \cap \mathcal{M} = \mathrm{Iso}$
2. $g.f \in \mathcal{M}$ imples $f \in \mathcal{M}$.
3. $g.f \in \mathcal{E}$ imples $g \in \mathcal{E}$.
4. $\mathcal{M}$ is stable under pullback.

6.1.2 Subobjects

Let X be an object in $\mathscr{C}$. Morphisms in $\mathcal{M}$ with codomain X are called *subobjects* of X. There is a preorder "$\leq$" on the subobjects of X defined by $i \leq j$ if there exists a morphism k such that $i = j.k$. Having $\leq$, one defines the equivalence relation $\simeq$ as $i \simeq j$ if and only if $i \leq j$ and $j \leq i$. In a concrete category, equivalence classes of the subobjects of X are in one-to-one correspondence with the subsets of X.

Given a morphism $f : X \to Y$ in $\mathscr{C}$ and a subobject $i : M \to X$ of X, the *image of i under f*, denoted by $f[i]$, is given by the $(\mathcal{E}, \mathcal{M})$ factorization of $f.i$ as shown in the following diagram:

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d, "i"'] & f[M] \arrow[d, "f[i]"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

For a subobject $j : N \to Y$ of Y, the *preimage of j under f*, denoted by $f^{-1}[j]$, is the pullback of j along f.

$$\begin{tikzcd} f^{-1}[N] \arrow[r, "f''"] \arrow[d, "f^{-1}[j]"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

A pullback of f along a morphism in $\mathcal{M}$ is a called a *restriction* of f.

6.1.3 Closed maps

A topology is characterized by its open sets or, equivalently its closed sets. In the category theoretical setting of [13], one uses a collection $\mathcal{F}$ of morphisms which are thought of as being "closed". The collection $\mathcal{F}$ has to satisfy the following conditions:

(C1) $\mathcal{F}$ is closed under composition and it contains all isomorphisms.

(C2) $\mathcal{F} \cap \mathcal{M}$ is stable under pullback.

(C3) $\mathcal{F}$ has the right cancellation property with respect to $\mathcal{E}$, i.e. if $g.f \in \mathcal{F}$ and $f \in \mathcal{E}$, then $g \in \mathcal{F}$.

A morphism in $\mathcal{F}$ is called an $\mathcal{F}$-closed map. $\mathcal{F} \cap \mathcal{M}$ constitute the collection of $\mathcal{F}$-closed subobjects.

Example 6.1.2. Consider **Top** with the proper factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the collection of surjective continuous maps and $\mathcal{M}$ is the collection of subspace embeddings. The collection $\mathcal{F} = \{\text{closed maps}\}$ satisfies $(C1) - (C3)$. Another possible candidate for $\mathcal{F}$ is the collection of open continuous maps.

From now on we assume that $\mathscr{C}$ is a finitely complete category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ and a collection $\mathcal{F}$ of morphisms satisfying (C1)-(C3).

Remark 6.1.3. Let $\mathscr{C}/Y$ denote the comma category of objects (A, s) over Y where $s : A \to Y$ is a morphism in $\mathscr{C}$. A morphism $f : (A, s) \to (B, t)$ of $\mathscr{C}/Y$ is just a morphism $f : A \to B$ of $\mathscr{C}$ for which $t.f = s$. Let $U_Y : \mathscr{C}/Y \to \mathscr{C}$ be the forgetful functor that sends $f : (A, s) \to (B, t)$ to $f : A \to B$. Then $(\mathcal{E}_Y, \mathcal{M}_Y) = (U_Y^{-1}(\mathcal{E}), U_Y^{-1}(\mathcal{M}))$ becomes a proper factorization system for $\mathscr{C}/Y$ with $\mathcal{F}_Y = U_Y^{-1}(\mathcal{F})$ satisfying (C1)-(C3). We will drop the subscript Y when there is no danger of confusion.

6.2 Dense maps

A morphism $f : X \to Y$ is called $\mathcal{F}$*-dense* if in any factorization $f = i.g$, if $i \in \mathcal{F} \cap \mathcal{M}$, then i is an isomorphism.

Proposition 6.2.1. 1. If $f \in \mathcal{E}$, then f is $\mathcal{F}$-dense.

2. Any $\mathcal{F}$-dense $\mathcal{F}$-closed subobject is an isomorphism.

Proof. 1. Let $f \in \mathcal{E}$. Then in any factorization $f = i.g$, $i \in \mathcal{E}$ by Prop. 6.1.1. As $i \in \mathcal{M}$, i is an isomorphism.

2. Let f be an $\mathcal{F}$-dense $\mathcal{F}$-closed subobject. Consider an $(\mathcal{E}, \mathcal{M})$ factorization $f = i.g$ for f. Since $f \in \mathcal{F}$ and $g \in \mathcal{E}$, $i \in \mathcal{F}$ by condition (C3). As f is $\mathcal{F}$-dense and i is an $\mathcal{F}$-closed subobject, i becomes an isomorphism. On the other hand, $g \in \mathcal{M}$, since $f \in \mathcal{M}$. So $g \in \mathcal{E} \cap \mathcal{M}$ and it is an isomorphism as well. This implies that f is an isomorphism.

□

6.3 Compactness

6.3.1 Proper maps

Morphisms which are stably $\mathcal{F}$-closed deserve special attention. A morphism f is called $\mathcal{F}$*-proper* if every pullback of f is in $\mathcal{F}$. The class of $\mathcal{F}$-proper maps will be denoted by $\mathcal{F}^*$. The following characterization of $\mathcal{F}$-proper maps will useful in the sequel.

Proposition 6.3.1. A map $f : X \to Y$ is $\mathcal{F}$-proper if and only if every restriction of $f \times 1_Z : X \times Z \to Y \times Z$ is $\mathcal{F}$-closed for any object Z.

Proof. Let f be $\mathcal{F}$-proper and Z be an object in $\mathscr{C}$. Since $f \times 1_Z$ is a pullback of f, any restriction f' of $f \times 1_Z$ is a pullback of f as well. Hence f' is $\mathcal{F}$-closed. Conversely, suppose that every restriction of $f \times 1_Z$ is $\mathcal{F}$-closed for any object Z. Consider the pullback of f along a morphism $g : B \to Y$.

$$\begin{tikzcd}
A \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
B \arrow[r, "g"'] & Y
\end{tikzcd}$$

This diagram can be extended as follows:

$$\begin{tikzcd}
A \arrow[r, "\langle g', f' \rangle"] \arrow[d, "f'"'] & X \times B \arrow[r, "\pi_1"] \arrow[d, "f \times 1_B"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
B \arrow[r, "\langle g, 1_B \rangle"'] & Y \times B \arrow[r, "\pi_1"'] & Y
\end{tikzcd}$$

Since the outer square and the right-hand side square are pullbacks, the left-hand side square is a pullback. One has $\langle g, 1_B \rangle \in \mathcal{M}$, since $\pi_2.\langle g, 1_B \rangle = 1_B \in \mathcal{M}$. So f' is a restriction of $f \times 1_B$ and by hypothesis it is $\mathcal{F}$-closed.

□

If $\mathcal{F}$ is stable under restrictions, then Prop. 6.3.1 can be simplified further.

Corollary 6.3.2. Suppose that $\mathcal{F}$ stable under restrictions. Then f is $\mathcal{F}$-proper if and only if $f \times 1_Z : X \times Z \to Y \times Z$ is $\mathcal{F}$-closed for any object Z.

$\mathcal{F}^*$ is closed under composition and it is the largest pullback stable subclass of $\mathcal{F}$. As a consequence, one has $\mathcal{F} \cap \mathcal{M} \subseteq \mathcal{F}^*$. $\mathcal{F}$-proper maps have some nice cancellation properties.

Proposition 6.3.3. 1. If $g.f \in \mathcal{F}^*$ and g is a monomorphism, then $f \in \mathcal{F}^*$.

2. If $g.f \in \mathcal{F}^*$ and $f \in \mathcal{E}^*$, where $\mathcal{E}^*$ denotes the morphisms that are stably in $\mathcal{E}$, then $g \in \mathcal{F}^*$

Proof. 1. Let $f : X \to Y$, $g : Y \to Z$. Assume that $g.f \in \mathcal{F}^*$ and g is monomorphism. Take the pullback of f along an arbitrary morphism $m : B \to Y$. Since g is monic, the right-hand side square of the diagram below is a pullback. Then $1_B.s = s \in \mathcal{F}$ as a pullback of $g.f \in \mathcal{F}^*$. Hence $f \in \mathcal{F}^*$.

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "1_B"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g.m"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

2. This statement follows from the composablity of adjacent pullback diagrams and condition (C3).

□

6.3.2 Compact objects

A topological space X is compact if and only if the unique map to the singleton set is proper. Taking this fact as a reference, an object X in $\mathscr{C}$ is called *$\mathcal{F}$-compact* if and only if $!_X : X \to 1$ is $\mathcal{F}$-proper where 1 is the terminal object of $\mathscr{C}$.

The Kuratowski-Mrowka theorem states that a topological space X is compact if and only if the projection map $\pi_2 : X \times Y \to Y$ is closed for any topological space Y. One can easily prove the Kuratowski-Mrowka theorem in our categorical setting.

Proposition 6.3.4. The following are equivalent:

1. X is $\mathcal{F}$-compact.

2. For any object Y, the projection map $\pi_2 : X \times Y \to Y$ is $\mathcal{F}$-closed.

3. For any object Y, the projection map $\pi_2 : X \times Y \to Y$ is $\mathcal{F}$-proper.

4. For any $\mathcal{F}$-compact object Y, $X \times Y$ is $\mathcal{F}$-compact.

Proof. $(1 \Leftrightarrow 2)$, $(1 \Leftrightarrow 3)$ Pullbacks of $!_X : X \to 1$ are precisely the projection maps $\pi_2 : X \times Y \to Y$ for Y in $\mathscr{C}$.

$(3 \Rightarrow 4)$ $(X \times Y \to 1) = (X \times Y \to Y \to 1)$. $(4 \Rightarrow 1)$ Take $Y = 1$. □

$\mathcal{F}$-compactness is carried backward by $\mathcal{F}$-proper maps and forward by maps that are stably in $\mathcal{E}$.

Proposition 6.3.5. 1. If $f : X \to Y$ is $\mathcal{F}$-proper with Y $\mathcal{F}$-compact, then X is $\mathcal{F}$-compact.

2. If $f : X \to Y$ is in $\mathcal{E}^*$ with X $\mathcal{F}$-compact, then Y is $\mathcal{F}$-compact.

Proof. (1) $(X \to 1) = (X \xrightarrow{f} Y \to 1)$. (2) Follows from Prop. 6.3.3. □

A morphism $f : X \to Y$ is $\mathcal{F}$-proper if and only f if it is an $\mathcal{F}$-compact object in the comma category $\mathscr{C}/Y$. This follows from the fact that the unique map $!_f : (X, f) \to (Y, 1_Y)$ going to the terminal object is f itself.

6.4 Separation

6.4.1 Separated morphisms, separated objects

A morphism $f : X \to Y$ is called *$\mathcal{F}$-separated* if the induced morphism $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is $\mathcal{F}$-proper.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\delta_f"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{6.4.1}$$

Since $\delta_f \in \mathcal{M}$, f is $\mathcal{F}$-separated if and only if δ_f is $\mathcal{F}$-closed.

Proposition 6.4.1. 1. $\mathcal{F}$-separated maps are closed under composition and contain all monomorphisms.

2. $\mathcal{F}$-separated maps are stable under pullback.

3. If $g.f$ is $\mathcal{F}$-separated, then f is $\mathcal{F}$-separated.

4. If $g.f$ is $\mathcal{F}$-separated and $f \in \mathcal{E} \cap \mathcal{F}^*$, then g is $\mathcal{F}$-separated.

Proof. See [13]. □

Like $\mathcal{F}$-compactness, one calls a $(\mathbb{T}, V)$-category X *$\mathcal{F}$-separated* (or *$\mathcal{F}$-Hausdorff*) if and only if $!_X : X \to 1$ is $\mathcal{F}$-separated. This means that $\delta_X = \langle 1_X, 1_X \rangle : X \to X \times X$ is $\mathcal{F}$-proper. Since $\delta_X \in \mathcal{M}$ this is equivalent to δ_X being $\mathcal{F}$-closed.

Example 6.4.2. Consider **Top** with $\mathcal{F} = \{\text{closed maps}\}$. A continuous map f is $\mathcal{F}$-separated if and only if two distinct points in the same fibre of f is separated by disjoint open sets. A topological space X is $\mathcal{F}$-separated if and only if it is Hausdorff. In the case of $\mathcal{F} = \{\text{open maps}\}$, $\mathcal{F}$-separation coincide with local injectivity for maps and discreteness for objects.

Proposition 6.4.3. The following are equivalent:

1. X is $\mathcal{F}$-separated.

2. Any morphism $f : X \to Y$ is $\mathcal{F}$-separated.

3. There exists an $\mathcal{F}$-separated morphism $f : X \to Y$ with Y $\mathcal{F}$-separated.

4. For any Y, the projection $X \times Y \to Y$ is $\mathcal{F}$-separated.

5. For any $\mathcal{F}$-separated Y, $X \times Y$ is $\mathcal{F}$-separated.

6. For any $f : X \to Y$ such that $f \in \mathcal{E} \cap \mathcal{F}^*$, Y is $\mathcal{F}$-separated.

7. In any equalizer diagram

$$\begin{tikzcd} A \arrow[r, "s"] & B \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$

s is $\mathcal{F}$-proper.

Proof. (1) $\Rightarrow$ (2) $(X \to 1) = (X \xrightarrow{f} Y \to 1)$, apply Prop. 6.4.1.

(2) $\Rightarrow$ (3) Take $Y = 1$. (3) $\Rightarrow$ (1) By Prop. 6.4.1.

(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) Similar to Prop. 6.3.4.

(1) $\Rightarrow$ (6) By $(X \to 1) = (X \xrightarrow{f} Y \to 1)$ and Prop. 6.4.1. (6) $\Rightarrow$ (1) Take $f = 1_X$.

(1) $\Leftrightarrow$ (7) The pullbacks of δ_X are exactly such equalizers s. $\square$

Corollary 6.4.4. The full subcategory of $\mathcal{F}$-compact objects and the full subcategory of $\mathcal{F}$-separated objects is closed under finite limits in $\mathscr{C}$.

Proof. The full subcategory of $\mathcal{F}$-compact objects is closed under finite limits by Prop. 6.3.4, Prop. 6.4.3-(7) and Prop. 6.3.5. The full subcategory of $\mathcal{F}$-separated objects is closed under finite limits by Prop. 6.4.3-(3),(5) and the fact that monomorphisms are $\mathcal{F}$-separated. $\square$

Observing diagram (6.4.1), it is easy to see as $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is the morphism $\delta_{(X,f)} = \langle 1_X, 1_X \rangle : (X, f) \to (X, f) \times (X, f)$ in $\mathscr{C}/Y$. Hence a morphism $f : X \to Y$ in $\mathscr{C}$ is $\mathcal{F}$-separated if and only if (X, f) is an $\mathcal{F}$-separated object in the comma category $\mathscr{C}/Y$.

6.4.2 Relationship between compactness and separation

In **Top**, a continuous map between a compact domain and a Hausdorff codomain is proper. In our categorical framework one has the following result.

Proposition 6.4.5. Any morphism $f : X \to Y$ with X $\mathcal{F}$-compact and Y $\mathcal{F}$-separated is $\mathcal{F}$-proper.

Proof. Let $f : X \to Y$ be a morphism where X is $\mathcal{F}$-compact and Y is $\mathcal{F}$-separated. f can be written as $\pi_2.\langle 1_X, f\rangle$. Since $\langle 1_X, f\rangle$ is a pullback of δ_Y and Y is $\mathcal{F}$-separated, $\langle 1_X, f\rangle$ is $\mathcal{F}$-proper. On the other hand π_2 is $\mathcal{F}$-proper as it is a pullback of $!_X : X \to 1$ and X is $\mathcal{F}$-compact. Hence $f = \pi_2.\langle 1_X, f\rangle$ is $\mathcal{F}$-proper. □

Corollary 6.4.6. Let Y be an $\mathcal{F}$-compact and $\mathcal{F}$-separated object. Then a morphism $f : X \to Y$ is $\mathcal{F}$-proper if and only if X is $\mathcal{F}$-compact.

Proof. Follows by Prop. 6.4.5 and Prop. 6.3.5. □

Corollary 6.4.7. If $g.f$ is $\mathcal{F}$-proper and g is $\mathcal{F}$-separated then f is $\mathcal{F}$-proper.

Proof. Let $f : X \to Y$ and $g : Y \to Z$. Consider the comma category $\mathscr{C}/Z$. One has $f : g.f \to g$ where $g.f$ is an $\mathcal{F}$-compact object and g is an $\mathcal{F}$-separated object in $\mathscr{C}/Z$. By Prop. 6.4.5, f is an $\mathcal{F}$-proper map in $\mathscr{C}/Z$ and hence in $\mathscr{C}$. □

An $\mathcal{F}$-proper morphism cannot be extended along an $\mathcal{F}$-dense subobject with an $\mathcal{F}$-separated codomain.

Corollary 6.4.8. Let $f : X \to Y$ be an $\mathcal{F}$-proper morphism. In any factorization $f = (X \xrightarrow{g} Z \xrightarrow{h} Y)$ where g is an $\mathcal{F}$-dense subobject and Z is $\mathcal{F}$-separated, g is an isomorphism.

Proof. Since Z is $\mathcal{F}$-separated, h is $\mathcal{F}$-separated. Then g is $\mathcal{F}$-proper by Cor. 6.4.7. So g is an $\mathcal{F}$-dense $\mathcal{F}$-closed subobject, which is indeed an isomorphism by Prop. 6.2.1. □

6.5 Perfect maps

A morphism f is called *$\mathcal{F}$-perfect* if it is both $\mathcal{F}$-proper and $\mathcal{F}$-separated. So an $\mathcal{F}$-perfect morphism $f : X \to Y$ is an $\mathcal{F}$-compact Hausdorff object in $\mathscr{C}/Y$.

Proposition 6.5.1.

1. $\mathcal{F}$-perfect morphisms are closed under composition and stable under pullback.
2. If $g.f$ is $\mathcal{F}$-perfect and g is $\mathcal{F}$-separated then f is $\mathcal{F}$-perfect.
3. If $g.f$ is $\mathcal{F}$-perfect, f is $\mathcal{F}$-perfect and stably in $\mathcal{E}$, then g is $\mathcal{F}$-perfect.

Proof. Follows from Prop. 6.3.3, Prop. 6.4.1 and Cor. 6.4.7. □

Example 6.5.2. Consider **Top** with $\mathcal{F} = \{\text{open maps}\}$. In this case, $\mathcal{F}$-perfect maps correspond precisely to open and locally injective maps. These maps are local homeomorphisms in **Top**.

6.6 Compactifications

An $\mathcal{F}$*-compactification* of an object X is an $\mathcal{F}$-dense embedding $i : X \to K$ where K is $\mathcal{F}$-compact and $\mathcal{F}$-separated. An object X is called $\mathcal{F}$*-Tychonoff* if X admits an $\mathcal{F}$-compactification. Let $\mathcal{F}$-Tych denote the full subcategory of $\mathcal{F}$-Tychonoff objects and $\mathcal{F}$-CompHaus denote the full subcategory of $\mathcal{F}$-compact and $\mathcal{F}$-separated objects. Consider a functor $\beta : \mathcal{F}\text{-Tych} \to \mathcal{F}\text{-CompHaus}$ which comes with a natural transformation $\{\beta_X : X \to \beta X\}$. One calls β a *functorial* $\mathcal{F}$*-compactification* if each β_X is an $\mathcal{F}$-dense embedding. (By abuse of notation the functor and the natural transformation will be denoted by the same letter).

Example 6.6.1. The main example of an $\mathcal{F}$-compactification is the Stone-Cech compactification in **Top** with $\mathcal{F}$ = closed maps.

By considering $\mathcal{F}$-compactifications in the comma category, one can also develop an $\mathcal{F}$-compactification of morphisms. Let $f : X \to Y$. Consider the following diagram:

$$
\begin{tikzcd}
X \arrow[dr, "\beta_f"] \arrow[drr, bend left, "\beta_X"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r, "\mu_f"'] \arrow[d, "\kappa_f"] \arrow[dr, phantom, very near start, "\lrcorner"] & \beta X \arrow[d, "\beta f"] \\
& Y \arrow[r, "\beta_Y"'] & \beta Y
\end{tikzcd}
\tag{6.6.1}
$$

β_f is an $\mathcal{F}$-compactification of f as follows. Firstly, $\mu_f \in \mathcal{M}$ since it is a pullback of $\beta_Y \in \mathcal{M}$. One has $\beta_f \in \mathcal{M}$ as $\beta_X \in \mathcal{M}$. Then $\beta_X = \mu_f.\beta_f$ where β_X is $\mathcal{F}$-dense and $\mu_f, \beta_f \in \mathcal{M}$. This implies that β_f is $\mathcal{F}$-dense, assuming that morphisms in $\mathcal{M}$ are $\mathcal{F}$-initial (a morphism $g : Y \to Z$ is called *$\mathcal{F}$-initial* if any $\mathcal{F}$-closed subobject of Y is a pullback of an $\mathcal{F}$-closed subobject of Z along g). On the other hand, κ_f is $\mathcal{F}$-perfect as a pullback of βf. Hence we have an $\mathcal{F}$-dense embedding $\beta_f : f \to \kappa_f$ in the comma category $\mathscr{C}/Y$ where κ_f is $\mathcal{F}$-compact and $\mathcal{F}$-separated. So one can think $\beta_f : f \to \kappa_f$ as an $\mathcal{F}$-compactification of f in $\mathscr{C}/Y$.

The following proposition provides a characterization of $\mathcal{F}$-perfect morphisms. The analogous result in **Top** with respect to the Stone-Čech compactification belongs to Isbell and Henriksen [29]. It states that β_f sends $\beta X \setminus X$ into $\beta Y \setminus Y$ which is equivalent to saying that the naturality square (6.6.2) is a pullback.

Proposition 6.6.2. Suppose that β is a functorial $\mathcal{F}$-compactification and morphisms in $\mathcal{M}$ are $\mathcal{F}$-initial. For a morphism $f : X \to Y$, the following are equivalent:

1. f is $\mathcal{F}$-perfect.

2. f cannot be extended along an $\mathcal{F}$-dense subobject with an $\mathcal{F}$-separated codomain.

3. The naturality diagram below is a pullback.

$$\begin{tikzcd} X \arrow[r, "\beta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \beta X \arrow[d, "\beta f"] \\ Y \arrow[r, "\beta_Y"'] & \beta Y \end{tikzcd} \tag{6.6.2}$$

Proof. $(1 \Rightarrow 2)$ By Cor. 6.4.8.

$(2 \Rightarrow 3)$ Under the assumptions the morphism β_f in diagram (6.6.1) is an $\mathcal{F}$-dense subobject, hence an isomorphism. This means that the naturality square is a pullback.

$(3 \Rightarrow 1)$ f is $\mathcal{F}$-perfect, since it is a pullback of βf which is $\mathcal{F}$-perfect. □

If $\mathcal{F}$-Comp Haus is reflective in $\mathcal{F}$-Tych with the reflection morphisms $\beta_X : X \to \beta X$, then β is a functorial compactification. Furthermore, the reflexivity induces ($\mathcal{F}$-antiperfect, $\mathcal{F}$-perfect) factorization system on $\mathcal{F}$-Tych where a morphism is called *$\mathcal{F}$-antiperfect* if its image under β is an isomorphism. Given a morphism $f : X \to Y$, one has the factorization $f = \kappa_f.\beta_f$ as given in diagram (6.6.1) where

κ_f is an $\mathcal{F}$-perfect morphism and β_f is an $\mathcal{F}$-antiperfect morphism. Let $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{A}_{\mathcal{F}}$ denote the collection of $\mathcal{F}$-perfect morphisms and $\mathcal{F}$-antiperfect morphisms in $\mathcal{F}$-Tych respectively.

Proposition 6.6.3. Suppose that $\mathcal{F}$-Comp Haus is reflective in $\mathcal{F}$-Tych and morphisms in $\mathcal{M}$ are $\mathcal{F}$-initial. Then $(\mathcal{A}_{\mathcal{F}}, \mathcal{P}_{\mathcal{F}})$ is a (generally non-proper) factorization system for $\mathcal{F}$-Tych.

Proof. See [13]. □

6.7 Open maps

Now we consider another important topological concept, openness for morphisms. A continuous open map $f : X \to Y$ in **Top** reflects dense subsets in the sense that given a dense subset $M \subseteq Y$, $f^{-1}(M)$ is dense in X. Secondly, in **Top** open maps are stable under pullback. With the help of these observations one can define open morphisms in our category theoretical setting.

A morphism $f : X \to Y$ in $\mathscr{C}$ is said to *reflect $\mathcal{F}$-density* if for any $\mathcal{F}$-dense subobject m of Y, $f^{-1}[m]$ is an $\mathcal{F}$-dense subobject of X. One calls f *$\mathcal{F}$-open* if it stably reflects $\mathcal{F}$-density, i.e. every pullback of f reflects $\mathcal{F}$-density. The collection $\mathcal{F}$-open morphisms is denoted by $\mathcal{F}^{+}$.

Proposition 6.7.1. 1. $\mathcal{F}^{+}$ is closed under composition and contains isomorphisms.

2. $\mathcal{F}^+$ is stable under pullback.

3. If $g.f \in \mathcal{F}^+$ and g is monic then $f \in \mathcal{F}^+$.

4. If $g.f \in \mathcal{F}^+$ and $f \in \mathcal{E}^*$ then $g \in \mathcal{F}^+$.

Proof. (1) and (2) are trivial. (3) can be proven similar to Prop. 6.3.3. To show (4), let $g.f \in \mathcal{F}^+$ and $f \in \mathcal{E}^*$. A pullback of $g.f$ will be $g'.f'$ where g' is a pullback of g and $f' \in \mathcal{E}$ is a pullback of f. So it is enough to show that g reflects $\mathcal{F}$-density whenever $g.f$ does and $f \in \mathcal{E}^*$. Let m be an $\mathcal{F}$-dense subobject of the codomain of g. Since $f \in \mathcal{E}$, one has $g^{-1}[m] = f[f^{-1}[g^{-1}[m]]] = f[(g.f)^{-1}[m]]$. $(g.f)^{-1}[m]$ is $\mathcal{F}$-dense, as $g.f$ reflects $\mathcal{F}$-density. Then $f[(g.f)^{-1}[m]]$ is $\mathcal{F}$-dense, since $f \in \mathcal{E}$. □

As a consequence, $\mathcal{F}^+$ satisfies the conditions $(C1)$ – $(C3)$ and induces a topological structure on $\mathscr{C}$ in case $\mathcal{E}$ is stable under pullback.

A topological space is discrete if and only if $\delta_X : X \to X \times X$ is an open map. A continuous map is a local homeomorphism if and only if it is open and locally injective. Taking these as a reference, one calls an object X in $\mathscr{C}$ *$\mathcal{F}$-discrete* if X is $\mathcal{F}^+$-separated. A morphism $f : X \to Y$ is called an *$\mathcal{F}$-local homeomorphism* if f is $\mathcal{F}^+$-perfect, i.e. both f and δ_f are $\mathcal{F}$-open.

7 Functional topology with respect to L-closed morphisms

In this chapter we develop topological notions for $(\mathbb{T}, V)$-categories using the categorical framework outlined in Chapter 6. Our main parameter $\mathcal{F}$ will be the collection of "L-closed" $(\mathbb{T}, V)$-functors.

7.1 Factorization system, L-closed morphisms

Let

$$\mathcal{E} = \{\text{surjective } (\mathbb{T}, V)\text{-functors}\} \qquad \& \qquad \mathcal{M} = \{\text{full embeddings}\}.$$

Then $(\mathcal{E}, \mathcal{M})$ is a proper factorization system, as $(\mathbb{T}, V)$-**Cat** is topological over **Set**. This will be the default factorization system we will consider for $(\mathbb{T}, V)$-**Cat**.

Definition 7.1.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. We say that f is *L-closed* if it preserves L-closure, i.e. $f(\overleftrightarrow{M}) = \overleftrightarrow{f(M)}$ for all $M \subseteq X$.

For a $(\mathbb{T}, V)$-functor $f : X \to Y$, one has $f(\overleftrightarrow{M}) \subseteq \overleftrightarrow{f(M)}$ by Prop. 3.4.13. So

$$f \text{ is L-closed} \iff \overleftrightarrow{f(M)} \subseteq f(\overleftrightarrow{M}) \quad \forall M \subseteq X.$$

In particular, an embedding $i : M \hookrightarrow X$ is L-closed if and only if $\overleftrightarrow{N}^X \subseteq \overleftrightarrow{N}^M$ for all $N \subseteq M$. Since L-closure is hereditary,

$$i : M \hookrightarrow X \text{ is L-closed} \iff M \text{ is an L-closed subset of } X.$$

One can equivalently define an L-closed $(\mathbb{T}, V)$-functor as a morphism which preserves L-closed subobjects. We will denote the collection of L-closed $(\mathbb{T}, V)$-functors by $\mathcal{C}$.

Proposition 7.1.2. $\mathcal{C}$ satisfies conditions $(C1)$ – $(C3)$ of Subsection 6.1.3.

Proof. (C1) L-closed maps are trivially closed under composition. Let $f : X \to Y$ be an isomorphism in $(\mathbb{T}, V)$-**Cat** with the inverse $g : Y \to X$. By Prop. 3.4.13,

$$\overleftrightarrow{f(M)} = \overleftrightarrow{g^{-1}(M)} \subseteq g^{-1}(\overleftrightarrow{M}) = f(\overleftrightarrow{M})$$

for all $M \subseteq X$. So f is L-closed.

(C2) Consider the following pullback square where $N \hookrightarrow Y$ is an L-closed embedding.

$$\begin{tikzcd} f^{-1}(N) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Since $N \hookrightarrow Y$ is L-closed, N is L-closed, i.e. $\overleftrightarrow{N} = N$. By Prop. 3.4.13, we have

$$\overleftrightarrow{f^{-1}(N)} \subseteq f^{-1}(\overleftrightarrow{N}) = f^{-1}(N).$$

This means that $f^{-1}(N)$ an is L-closed subset of X. Hence $f^{-1}(N) \hookrightarrow X$ is L-closed.

(C3) Let $f : X \to Y$, $g : Y \to Z$ be $(\mathbb{T}, V)$-functors such that $g.f$ is L-closed and $f \in \mathcal{E}$. Take any $N \subseteq Y$. Since f is surjective, there exists $M \subseteq X$ such that

$f(M) = N$. Then

$$\overleftrightarrow{g(N)} = \overleftrightarrow{g(f(M))} = g.f(\overleftrightarrow{M}) \subseteq g(\overleftrightarrow{f(M)}) = g(\overleftrightarrow{N}).$$

Hence g is L-closed.

$\square$

Proposition 7.1.3. $\mathcal{C}$ is stable under restrictions.

Proof. Let $f : X \to Y$ be L-closed. It is enough to consider the pullback of f along an embedding $N \hookrightarrow Y$. Consider the following pullback diagram:

$$\begin{tikzcd}
f^{-1}(N) \arrow[r, "f'"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Take any L-closed set $C \subseteq f^{-1}(N)$. By Prop. 3.4.15, there exists an L-closed set $A \subseteq X$ such that $C = f^{-1}(N) \cap A$. We have

$$f(C) = f(f^{-1}(N) \cap A) = N \cap f(A).$$

Since f is L-closed, $f(A)$ is L-closed. This implies that $f(C)$ is L-closed in N. Therefore f' preserves L-closed subobjects, f' is L-closed. $\square$

The following lemma will be helpful in the sequel.

Lemma 7.1.4. 1. Fully faithful and surjective $(\mathbb{T}, V)$-functors are L-closed.

2. Let $f : X \to Y$ be a fully faithful $(\mathbb{T}, V)$-functor. Then f is L-closed if and only if $f(X)$ is L-closed in Y.

Proof. 1. Let $f : (X, a) \to (Y, b)$ be a fully faithful and surjective $(\mathbb{T}, V)$-functor. Take $M \subseteq X$ and $y \in \overleftrightarrow{f(M)}$. Then

$$k \leq \bigvee_{\mathfrak{y} \in T(f(M))} b(\mathfrak{y}, y) \otimes \overline{T}b(Te_Y.e_Y(y), \mathfrak{y}).$$

Since f is surjective, $y = f(x)$ for some $x \in X$. Tf is surjective, as T preserves surjections. So

$$\begin{aligned} k &\leq \bigvee_{\mathfrak{x} \in TM} b(Tf(\mathfrak{x}), f(x)) \otimes \overline{T}b(Te_Y.e_Y(f(x)), Tf(\mathfrak{x})) \\ &= \bigvee_{\mathfrak{x} \in TM} b(Tf(\mathfrak{x}), f(x)) \otimes \overline{T}b(e_{TY}.e_Y.f(x), Tf(\mathfrak{x})) \\ &= \bigvee_{\mathfrak{x} \in TM} b(Tf(\mathfrak{x}), f(x)) \otimes \overline{T}b(Tf^2.e_{TX}.e_X(x), Tf(\mathfrak{x})) \\ &= \bigvee_{\mathfrak{x} \in TM} (f^\circ.b.Tf)(\mathfrak{x}, x) \otimes (Tf^\circ.\overline{T}b.Tf^2)(Te_X.e_X(x), \mathfrak{x}) \\ &= \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes \overline{T}a(Te_X.e_X(x), \mathfrak{x}). \end{aligned}$$

This means that $x \in \overleftrightarrow{M}$ and $y \in f(\overleftrightarrow{M})$. Therefore $\overleftrightarrow{f(M)} \subseteq f(\overleftrightarrow{M})$, f is L-closed.

2. Let $f : X \to Y$ be fully faithful. Consider the canonical $(\mathcal{E}, \mathcal{M})$ factorization $X \overset{f'}{\twoheadrightarrow} f(X) \overset{i}{\hookrightarrow} Y$ of f. Since f is fully faithful, so is f'. Hence f' is fully faithful and surjective which implies that it is L-closed.

Suppose that f is L-closed, then i is L-closed by condition $(C3)$. Conversely, if i is L-closed, then $f = i.f'$ is L-closed. Hence f is L-closed if and only if i is L-closed. The latter is equivalent to $f(X)$ being L-closed in Y.

□

7.2 $\mathcal{C}$-compactness

Let X be a $(\mathbb{T}, V)$-category. Following Subsection 6.3.2, X is called *$\mathcal{C}$-compact* if and only if $!_X : X \to 1$ is $\mathcal{C}$-proper, i.e. stably L-closed. Equivalently, X is $\mathcal{C}$-compact if and only if for any $(\mathbb{T}, V)$-category Y the projection map $\pi_2 : X \times Y \to Y$ is L-closed.

Proposition 7.2.1. A $(\mathbb{T}, V)$-functor $f : X \to Y$ is $\mathcal{C}$-proper if and only if $f \times 1_Z : X \times Z \to Y \times Z$ is L-closed for any $(\mathbb{T}, V)$-category Z.

Proof. Follows by Cor. 6.3.2 and Prop. 7.1.3. □

Examples 7.2.2. 1. In **Ord**, every object is $\mathcal{C}$-compact. To show this, we first characterize $\mathcal{C}$-proper morphisms in **Ord**.

Claim: A monotone map $f : (X, \leq) \to (Y, \leq)$ is $\mathcal{C}$-proper if and only if for any $x \in X$ with $f(x) \simeq y$ there exists $w \in f^{-1}(\{y\})$ with $x \simeq w$.

We prove the claim using Prop. 7.2.1. Suppose that f is $\mathcal{C}$-proper, then $f \times 1_Y$ is L-closed. Take $x \in X$ with $f(x) \simeq y$ and let $K = \{(x, y)\}$. Then

$(y,y) \in \overleftrightarrow{f \times 1_Y(K)} \subseteq f \times 1_Y(\overleftrightarrow{K})$, as $f \times 1_Y$ is L-closed. So there exists $(w,y) \in \overleftrightarrow{K}$ such that $f \times 1_Y(w,y) = (y,y)$. This implies that $x \simeq w$ and $f(w) = y$. Conversely, suppose that for any $x \in X$ with $f(x) \simeq y$ there exists $w \in f^{-1}(\{y\})$ with $x \simeq w$. Take an ordered set $(Z, \leq)$ and $M \subseteq X \times Z$. Let $(y,z) \in \overleftrightarrow{f \times 1_Z(M)}$. Then there exists $(x,z) \in M$ such that $f(x) \simeq y$. By hypothesis, there exists $w \in f^{-1}(\{y\})$ with $x \simeq w$. Hence $(w,z) \in \overleftrightarrow{M}$ and $f \times 1_Z(w,z) = (y,z)$, which implies $(y,z) \in f \times 1_Z(\overleftrightarrow{M})$. Therefore $\overleftrightarrow{f \times 1_Z(M)} \subseteq f \times 1_Z(\overleftrightarrow{M})$ and $f \times 1_Z$ is L-closed.

Following this characterization, one sees that $!_X : X \to 1$ is $\mathcal{C}$-proper for any ordered set X. Therefore every ordered set is $\mathcal{C}$-compact.

Observe that $\mathcal{C}$-proper maps coincide with L-complete maps in **Ord** (see Examples 4.1.3).

2. In **Met**, a metric space (X,d) is $\mathcal{C}$-compact if and only if $\mathsf{L}X$ is compact where $\mathsf{L} : \mathbf{Met} \to \mathbf{Top}$ is the functor induced by the L-closure. To show this, we first characterize $\mathcal{C}$-proper morphisms in **Met**.

   Claim: A nonexpansive map $f : (X,d) \to (Y,d')$ is $\mathcal{C}$-proper if and only if for any sequence (x_n) in X with $\lim\limits_{n\to\infty} f(x_n) = y \in Y$, there exists a subsequence (x_{n_k}) of (x_n) and $x \in f^{-1}(\{y\})$ such that $\lim\limits_{n\to\infty} x_{n_k} = x$.

   We prove the claim using Prop. 7.2.1. Firstly, assume that f is $\mathcal{C}$-proper.

Then $f \times 1_Y : (X \times Y, d \times d') \to (Y \times Y, d' \times d')$ is L-closed. Take any sequence (x_n) in X such that $f(x_n) \to y$ for some $y \in Y$. Let $K \subseteq X \times Y$ be the elements of the sequence (x_n, y). Since $f(x_n)$ converges to y, for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$d'(y, f(x_m)) + d'(f(x_m), y) < \varepsilon.$$

Then

$$\max\{d'(y, f(x_m)), d'(y, y)\} + \max\{d'(f(x_m), y), d'(y, y)\} < \varepsilon,$$

which means

$$d' \times d'((y, y), (f(x_m), y)) + d' \times d'((f(x_m), y), (y, y)) \leq \varepsilon.$$

Hence

$$\inf_{(x_n, y) \in K} \{d' \times d'((y, y), (f(x_n), y)) + d' \times d'((f(x_n), y), (y, y))\} = 0.$$

So $(y, y) \in \overleftrightarrow{f \times 1_Y(K)}$. Since $f \times 1_Y$ is L-closed, $(y, y) \in f \times 1_Y(\overleftrightarrow{K})$. Hence there exists $(x, y) \in \overleftrightarrow{K}$ such that $f \times 1_Y(x, y) = (y, y)$, i.e. $f(x) = y$. As f is nonexpansive, we have

$$\inf_{(x_n, y) \in K} \{d \times d'((x, y), (x_n, y)) + d \times d'((x_n, y), (x, y))\} = 0.$$

So

$$\inf_{(x_n, y) \in K} \{\max\{d(x, x_n), d'(y, y)\} + \max\{d(x_n, x), d'(y, y)\}\} = 0,$$

$$\inf_{(x_n,y)\in K}\{d(x,x_n)+d(x_n,x)\}=0.$$

For any $k\in\mathbb{N}$, pick x_{n_k} such that

$$d(x,x_{n_k})+d(x_{n_k},x)<\frac{1}{k}.$$

Then one obtains a subsequence (x_{n_k}) of (x_n) with $\lim_{n\to\infty} x_{n_k}=x\in f^{-1}(\{y\})$.

Conversely, assume that any sequence (x_n) such that $f(x_n)\to y\in Y$, has a convergent subsequence (x_{n_k}) whose limit is in $f^{-1}(\{y\})$. Take any metric space (Z,d''). We need to show that $f\times 1_Z:(X\times Z,d\times d'')\to(Y\times Z,d'\times d'')$ is L-closed. Let $M\subseteq X\times Z$ and $(y,z)\in\overleftrightarrow{f\times 1_Z(M)}$. This means

$$\inf_{(x_\circ,z_\circ)\in M}\{\max\{d'(y,f(x_\circ)),d''(z,z_\circ)\}+\max\{d'(f(x_\circ),y),d''(z_\circ,z)\}\}=0.$$

Then for any $n\in\mathbb{N}$ there exists $(x_n,z_n)\in M$ such that

$$\max\{d'(y,f(x_n)),d''(z,z_n)\}+\max\{d'(f(x_n),y),d''(z_n,z)\}<\frac{1}{2n}.$$

The sequence $f(x_n)$ converges to y, as

$$d'(y,f(x_n))+d'(f(x_n),y)<\frac{1}{2n}.$$

Similarly (z_n) converges to z. Then (x_n) has a subsequence (x_{n_k}) such that $(x_{n_k})\to x$ and $f(x)=y$ by hypothesis. Adjusting the indices if necessary, for any $n_k\in\mathbb{N}$ one can find $(x_{n_k},z_{n_k})\in M$ such that

$$\max\{d(x,x_{n_k}),d''(z,z_{n_k})\}+\max\{d(x_{n_k},x),d''(z_{n_k},z)\}<\frac{1}{2n_k}+\frac{1}{2n_k}<\frac{1}{n_k}.$$

Hence $(x,z) \in \overleftrightarrow{M}$ and $(y,z) \in f \times 1_Z(\overleftrightarrow{M})$. Therefore $\overleftrightarrow{f \times 1_Z(M)} \subseteq f \times 1_Z(\overleftrightarrow{M})$, $f \times 1_Z$ is L-closed.

Following this characterization, $!_X : X \to 1$ is $\mathcal{C}$-proper if and only if every sequence in X has a convergent subsequence. So a metric space (X,d) is $\mathcal{C}$-compact if and only if (X,d) is sequentially compact. The latter is equivalent to $\mathsf{L}X$ being compact, where the topology of $\mathsf{L}X$ is induced by the L-closure (see Examples 5.2.4).

In the remaining part of this section we will try to formulate $\mathcal{C}$-compactness in **Top** and **App**. For topological spaces this notion is known as b-compactness [51]. In **App**, $\mathcal{C}$-compactness coincides with Zariski compactness [26]. This concept has not been characterized in concrete terms yet. To do that we will require some intermediate results. As a principle we will try to obtain these results in the most general terms.

Recall that a $(\mathbb{T},V)$-category X is $\mathcal{C}$-compact if and only if for any $(\mathbb{T},V)$-category Y the projection map $\pi_2 : X \times Y \to Y$ is L-closed. Consider the functor $\mathsf{L} : (\mathbb{T},V)$-**Cat** $\to$ **Top** induced by the L-closure. One sees that $\pi_2 : X \times Y \to Y$ being L-closed is equivalent to $\mathsf{L}\pi_2 : \mathsf{L}(X \times Y) \to \mathsf{L}Y$ being closure preserving. So X is $\mathcal{C}$-compact if and only if $\mathsf{L}\pi_2 : \mathsf{L}(X \times Y) \to \mathsf{L}Y$ is closure preserving for any $(\mathbb{T},V)$-category Y.

Our first aim is to characterize $\mathcal{C}$-compactness of a $(\mathbb{U},V)$-category X, for $V = 2$

or P_+, in terms of compactness of the topological space $\mathsf{L}X$. Observe that, different from the characterization of $\mathcal{C}$-compactness, this requires the projection map $\pi_2 : \mathsf{L}X \times Y \to Y$ to be closure preserving for any topological space Y.

Now we look at a characterization of compactness for topological spaces. Given an ultrafilter $\mathscr{F}$ on a set X, define the test space $X_{\mathscr{F}} = X \cup \{\infty\}$ with the following topology [5]:

- For each $x \neq \infty$, the neighbourhoods of x are all subsets of $X_{\mathscr{F}}$ that contain x.
- The neighbourhoods of the point ∞ are $\{F \cup \{\infty\} \mid F \in \mathscr{F}\}$.

Proposition 7.2.3. [21] For a topological space X, the following are equivalent:

1. X is compact, i.e. every ultrafilter in X converges.
2. For each topological space Y, the projection $\pi_2 : X \times Y \to Y$ is closure preserving.
3. For each nonconvergent ultrafilter $\mathscr{F}$ on X, the projection $\pi_{X_{\mathscr{F}}} : X \times X_{\mathscr{F}} \to X_{\mathscr{F}}$ is closure preserving.

By generalizing a result in [21], we obtain the following characterization of $\mathcal{C}$-compactness.

Theorem 7.2.4. Suppose that $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ satisfies the following condition:

($\dagger$) For each $(\mathbb{T}, V)$-category X and each nonconvergent ultrafilter $\mathscr{F}$ on X, $X_{\mathscr{F}} \in \mathsf{L}((\mathbb{T}, V)\text{-}\mathbf{Cat})$.

Then $\mathsf{L}X$ is compact if X is $\mathcal{C}$-compact. Furthermore, if L preserves finite products, then X is $\mathcal{C}$-compact if and only if $\mathsf{L}X$ is compact.

Proof. Assume that X is a $(\mathbb{T}, V)$-category and ($\dagger$) holds.

Let X be $\mathcal{C}$-compact and $\mathscr{F}$ be a nonconvergent ultrafilter on X. In light of the Prop. 7.2.3, we need to show that $\pi_{X_{\mathscr{F}}} : \mathsf{L}X \times X_{\mathscr{F}} \to X_{\mathscr{F}}$ is closure preserving. By condition ($\dagger$), there exists a $(\mathbb{T}, V)$-category Y such that $\mathsf{L}Y = X_{\mathscr{F}}$. So it is enough to show that $\pi_{\mathsf{L}Y} : \mathsf{L}X \times \mathsf{L}Y \to \mathsf{L}Y$ is closure preserving. One has the following commutative diagram in **Top**:

$$\begin{tikzcd}
\mathsf{L}(X \times Y) \arrow[d, "id"'] \arrow[dr, "\mathsf{L}\pi_2"] & \\
\mathsf{L}X \times \mathsf{L}Y \arrow[r, "\pi_{\mathsf{L}Y}"'] & \mathsf{L}Y
\end{tikzcd}$$

Since X is $\mathcal{C}$-compact, $\mathsf{L}\pi_2$ is closure preserving. One has $\mathsf{L}\pi_2 = \pi_{\mathsf{L}Y}.id$ where $\mathsf{L}\pi_2$ is closure preserving and id is surjective. This implies that $\pi_{\mathsf{L}Y}$ is closure preserving.

Conversely, assume that $\mathsf{L}X$ is compact. Then $\pi_{\mathsf{L}Y} : \mathsf{L}X \times \mathsf{L}Y \to \mathsf{L}Y$ is closure preserving for any $(\mathbb{T}, V)$-category Y. If L preserves finite products, then $id : \mathsf{L}(X \times$

$Y) \to \mathsf{L}X \times \mathsf{L}Y$ is a homeomorphism, hence closure preserving. Then $\mathsf{L}\pi_2 = \pi_{\mathsf{L}Y}.id$ is closure preserving for any $(\mathbb{T}, V)$-category Y. Therefore X is $\mathcal{C}$-compact.

$\square$

Corollary 7.2.5. Let X be a $(\mathbb{U}, V)$-category where V is 2 or P_+. Then X is $\mathcal{C}$-compact if and only if $\mathsf{L}X$ is compact.

Proof. We check that the conditions of Theorem 7.2.4 are satisfied. Firstly, $\mathsf{L}_{(\mathbb{U},2)}$ and $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)}$ preserve finite products by Prop. 5.1.9. Condition (†) is satisfied as follows:

Given a topological space X and any nonconvergent ultrafilter $\mathscr{F}$ on X, consider the topological space $X'_{\mathscr{F}} = X \cup \{\infty\}$ with the following topology [21]:

- For each $x \neq \infty$, the neighbourhoods of x are the cofinite subsets of X which contain x and ∞.
- The neighbourhoods of the point ∞ are $\{F \cup \{\infty\} \mid F \in \mathscr{F}\}$.

One has $\mathsf{L}_{(\mathbb{U},2)}X'_{\mathscr{F}} = X_{\mathscr{F}}$ [21]. To see this, take $x \in X$. For any $y \in X$, y has a neighbourhood that does not contain x. So $y \notin \overline{\{x\}}$. Also $\infty \notin \overline{\{x\}}$, as $\mathscr{F}$ is nonconvergent. Then $\overline{\{x\}} = \{x\}$. So $\{x\}$ is open in $\mathsf{L}_{(\mathbb{U},2)}X'_{\mathscr{F}}$ and any subset of $X \cup \{\infty\}$ that contains x is a neighbourhood of x. On the other hand, $\overline{\{\infty\}} = X'_{\mathscr{F}}$, since any neighbourhood of any $x \in X$ contains ∞. Then the neighbourhoods of ∞ in $\mathsf{L}_{(\mathbb{U},2)}X'_{\mathscr{F}}$ are $(F \cup \{\infty\}) \cap X'_{\mathscr{F}} = F \cup \{\infty\}$ for $F \in \mathscr{F}$. Therefore $\mathsf{L}_{(\mathbb{U},2)}X'_{\mathscr{F}} = X_{\mathscr{F}}$.

Given an approach space X and any nonconvergent ultrafilter $\mathscr{F}$ on X, consider the appoach space $i(X'_{\mathscr{F}})$ where $i : \mathbf{Top} \hookrightarrow \mathbf{App}$ is the subcategory embedding. Then $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)}(i(X'_{\mathscr{F}})) = \mathsf{L}_{(\mathbb{U},2)}X'_{\mathscr{F}} = X_{\mathscr{F}}$ by commutativity of diagram (5.2.5). So condition (†) of Theorem 7.2.4 is satisfied for **Top** and **App**.

□

Remark 7.2.6. Examples 7.2.2 and Cor. 7.2.5 show that $\mathcal{C}$-compactness can be characterized in terms of the functor L for metric, topological and approach spaces with the exception of ordered sets where every ordered set is $\mathcal{C}$-compact.

We will characterize $\mathcal{C}$-compactness (Zariski compactness) for approach spaces using bitopological spaces and Salbany's notion of 2-compactness. We present some useful terminology from [51] below.

Let $X = (X, \tau, \sigma)$ be a bitopological space. X is called *2-compact* if $\tau \vee \sigma$ is compact. At each point $x \in X$, $\tau \vee \sigma$ has the local basis $\{O \cap U \mid x \in O \in \tau, x \in U \in \sigma\}$. One calls X *2-separated* if $\tau \vee \sigma$ is Hausdorff. X is called *2-regular* if both of the following conditions are satisfied:

- For any point $x \in X$ and any τ-closed set C, there exist a disjoint τ-open set U and a σ-open set O such that $x \in U$, $C \subseteq O$.
- For any point $x \in X$ and any σ-closed set K, there exist a disjoint τ-open set U and a σ-open set O such that $x \in O$, $K \subseteq U$.

Recall from Section 5.2 that an approach space $X = (X, \delta)$ corresponds to the bitopological space $\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}X = (X, \overrightarrow{\tau}, \overleftarrow{\tau})$ where $\overrightarrow{\tau}$ and $\overleftarrow{\tau}$ are the topologies induced by the natural closure and the dual closure respectively. Furthermore, one has $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)} = \mathsf{J}.\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}$ by Prop. 5.2.5, i.e. $\mathsf{L}_{(\mathbb{U},\mathbb{P}_+)}X = \mathsf{J}(X, \overrightarrow{\tau}, \overleftarrow{\tau}) = (X, \overrightarrow{\tau} \vee \overleftarrow{\tau})$. By Cor. 7.2.5, the approach space X is $\mathcal{C}$-compact if and only if $(X, \overrightarrow{\tau} \vee \overleftarrow{\tau})$ is compact or, equivalently, $(X, \overrightarrow{\tau}, \overleftarrow{\tau})$ is 2-compact. One has the following characterization of 2-compactness for 2-regular bitopological spaces.

Proposition 7.2.7. [51] Let $X = (X, \tau, \sigma)$ be a 2-regular bitopological space. X is 2-compact if and only if every τ-closed set is σ-compact and every σ-closed set is τ-compact.

In the above proposition, τ-compact means compact with respect to the topology τ, i.e. every τ-open cover has a finite subcover. Likewise for σ-compactness.

Lemma 7.2.8. Let $X = (X, \delta)$ be an approach space. Then $\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}X = (X, \overrightarrow{\tau}, \overleftarrow{\tau})$ is 2-regular.

Proof. Let $K \subseteq X$ be $\overleftarrow{\tau}$-closed and $x \in X \smallsetminus K$. Since $X \smallsetminus K$ is $\overleftarrow{\tau}$-open, there exists $\varepsilon > 0$ such that $\{x\}^{(\varepsilon)} \subseteq X \smallsetminus K$ by Examples 5.2.2. $\{x\}^{(\varepsilon)}$ is $\overrightarrow{\tau}$-closed, as $(\{x\}^{(\varepsilon)})^{(0)} \subseteq \{x\}^{(\varepsilon)}$. So there exist disjoint $\overleftarrow{\tau}$-open set $\{x\}^{((\varepsilon))}$ and $\overrightarrow{\tau}$-open set $X \smallsetminus \{x\}^{(\varepsilon)}$ such that $x \in \{x\}^{((\varepsilon))}$ and $K \subseteq X \smallsetminus \{x\}^{(\varepsilon)}$.

Now suppose that $C \subseteq X$ is $\overrightarrow{\tau}$-closed and $x \in X \smallsetminus C$. Since $X \smallsetminus C$ is $\overrightarrow{\tau}$-open,

$\delta(C,x) > 0$ by Examples 5.2.1. Pick ε such that $\delta(C,x) > \varepsilon > 0$. Consider $\bigcup_{y\in C}\{y\}^{((\varepsilon))}$. Since each $\{y\}^{((\varepsilon))}$ is $\overleftarrow{\mathcal{T}}$-open, so is $\bigcup_{y\in C}\{y\}^{((\varepsilon))}$. Observe that $\bigcup_{y\in C}\{y\}^{(\varepsilon)} \subseteq C^{(\varepsilon)}$. So $(\bigcup_{y\in C}\{y\}^{(\varepsilon)})^{(0)} \subseteq (C^{(\varepsilon)})^{(0)} \subseteq C^{(\varepsilon)}$. Hence we have

$$C \subseteq \bigcup_{y\in C}\{y\}^{((\varepsilon))} \subseteq (\bigcup_{y\in C}\{y\}^{(\varepsilon)})^{(0)} \subseteq C^{(\varepsilon)}.$$

Since $\delta(C,x) > \varepsilon$, $x \in \left(X \setminus C^{(\varepsilon)}\right) \subseteq \left(X \setminus (\bigcup_{y\in C}\{y\}^{(\varepsilon)})^{(0)}\right)$ which is $\overrightarrow{\mathcal{T}}$-open. On the other hand, C is a subset of the $\overleftarrow{\mathcal{T}}$-open set $\bigcup_{y\in C}\{y\}^{((\varepsilon))}$ where $\left(\bigcup_{y\in C}\{y\}^{((\varepsilon))}\right)\bigcap\left(X \setminus (\bigcup_{y\in C}\{y\}^{(\varepsilon)})^{(0)}\right) = \varnothing$. So we are done

□

Corollary 7.2.9. Let $X = (X,\delta)$ be an approach space with $\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}X = (X, \overrightarrow{\mathcal{T}}, \overleftarrow{\mathcal{T}})$. X is $\mathcal{C}$-compact (Zariski compact) if and only if the following conditions hold:

1. Every $\overrightarrow{\mathcal{T}}$-closed subset of X is $\overleftarrow{\mathcal{T}}$-compact.

2. Every $\overleftarrow{\mathcal{T}}$-closed subset of X is $\overrightarrow{\mathcal{T}}$-compact.

Proof. Follows from Cor. 7.2.5, Prop. 5.2.5, Prop. 7.2.7 and Prop. 7.2.8. □

This result also applies to **Top**. Furthermore, the above characterization can be carried one step further.

Corollary 7.2.10. [51] Let X be a topological space. X is $\mathcal{C}$-compact (b-compact) if and only if the following conditions hold.

1. Every subset of X is compact.

2. Every closed subset of X can be written as a finite union of point closures.

Proof. Let X be a topological space, $\mathsf{L}X = (X, \overleftrightarrow{\mathcal{T}})$. Suppose that X is $\mathcal{C}$-compact, i.e. $\mathsf{L}X$ is compact. Take any $M \subseteq X$ and let $\{O_i \mid i \in I\}$ be an arbitrary open cover of M. Then $\{O_i \mid i \in I\} \bigcup \{\overline{\{x\}} \mid x \notin \bigcup_{i \in I} O_i\}$ is an $\overleftrightarrow{\mathcal{T}}$-open cover of X. Since $\mathsf{L}X$ is compact, there exists $n, m \in \mathbb{N}$ such that $X = (\bigcup_{i=1}^{n} O_i) \bigcup (\bigcup_{j=1}^{m} \overline{\{x_j\}})$. But $\overline{\{x_j\}} \subseteq X \setminus \bigcup_{i \in I} O_i$ for $1 \leq j \leq m$, since $X \setminus \bigcup_{i \in I} O_i$ is closed. Hence $M \subseteq \bigcup_{i=1}^{n} O_i$, M is compact. Now let $C \subseteq X$ be closed. $\{\overline{\{x\}} \mid x \in C\}$ is a $\overleftarrow{\mathcal{T}}$-open cover of C. Since X is $\mathcal{C}$-compact, every closed subset of X is $\overleftarrow{\mathcal{T}}$-compact. Hence there exists $x_1, \ldots, x_n \in C$ such that $C = \bigcup_{i=1}^{n} \overline{\{x_i\}}$.

Conversely, assume that every subset of X is compact and every closed subset of X can be written as a finite union of point closures. Then trivially every $\overleftarrow{\mathcal{T}}$-closed subset of X is compact. Let $C \subseteq X$ be closed. Consider an open cover $\{\overline{\{y_i\}} \mid i \in I\}$ of C by $\overleftarrow{\mathcal{T}}$-basic open sets. One has $C = \bigcup_{j=1}^{n} \overline{\{x_j\}}$ by hypothesis. Then for any $1 \leq j \leq n$, there exists y_{i_j} such that $x_j \in \overline{\{y_{i_j}\}}$. Hence $C = \bigcup_{j=1}^{n} \overline{\{x_j\}} \subseteq \bigcup_{j=1}^{n} \overline{\{y_{i_j}\}}$. So C is $\overleftarrow{\mathcal{T}}$-compact.

□

7.3 $\mathcal{C}$-separation, $\mathcal{C}$- compact $\mathcal{C}$-separated objects

Following Section 6.4, a $(\mathbb{T}, V)$-functor $f : X \to Y$ is called *$\mathcal{C}$-separated* if and only if $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is L-closed.

Proposition 7.3.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. Then f is $\mathcal{C}$-separated if and only if it is L-separated.

Proof. f is $\mathcal{C}$-separated if and only if δ_f is L-closed. Since δ_f is fully faithful, it is enough to consider whether $\delta_f(X)$ is L-closed in $X \times_Y X$ by Lemma 7.1.4. Observe that

$$\begin{aligned}
\overleftrightarrow{\delta_f(X)}^{X \times_Y X} &= \overleftrightarrow{\delta_f(X)}^{X \times X} \cap (X \times_Y X) \\
&= \overleftrightarrow{\Delta}^{X \times X} \cap (X \times_Y X) \\
&= \{(x, z) \mid x \simeq z,\ f(x) = f(z)\}
\end{aligned}$$

by Prop. 3.4.13 and Prop. 3.5.1. $\overleftrightarrow{\delta_f(X)}^{X \times_Y X} = \delta_f(X) = \Delta$ if and only if $x \simeq z$ and $f(x) = f(z)$ implies $x = z$ for any $x, z \in X$. This is equivalent to f being L-separated. $\square$

A $(\mathbb{T}, V)$-category X is called *$\mathcal{C}$-separated* if and only if $!_X : X \to 1$ is $\mathcal{C}$-separated.

Corollary 7.3.2. Let X be a $(\mathbb{T}, V)$-category. Then X is $\mathcal{C}$-separated if and only if it is L-separated.

Proof. By Prop. 7.3.1, X is $\mathcal{C}$-separated if and only if $!_X : X \to 1$ is L-separated. This is equivalent to X being L-separated. $\square$

$\mathcal{C}$-compact $\mathcal{C}$-separated objects in **Top** are the b-compact T_0 spaces. As stated in [51], a topological space X is T_0 if and only if its b-topology is Hausdorff. We also know that X is b-compact if and only if its b-topology is compact by Cor. 7.2.5. Hence a topological space X is $\mathcal{C}$-compact $\mathcal{C}$-separated if and only if its b-topology is compact Hausdorff.

In **Top**, separated objects with respect to the closed maps are precisely Hausdorff spaces. So a topological space X is $\mathcal{C}$-separated if and only if $\mathsf{L}X$ is separated. This result can also be extended to approach spaces.

Proposition 7.3.3. Let $X = (X, \delta)$ be an approach space. X is $\mathcal{C}$-separated if and only if $\mathsf{L}X$ is separated (Hausdorff).

Proof. Let $X = (X, \delta)$ be $\mathcal{C}$-separated and $x, z \in X$ be two distinct points. Since X is $\mathcal{C}$-separated, i.e. L-separated, $\mathsf{N}X = (X, \overrightarrow{\tau})$ is T_0 by Examples 3.1.3. Without loss of generality, assume that there exists $O \in \overrightarrow{\tau}$ that contains z but not x. Then $x \in X \setminus O$ which is $\overrightarrow{\tau}$-closed. Since $\mathsf{B}_{(\mathbb{U},\mathbb{P}_+)}X = (X, \overrightarrow{\tau}, \overleftarrow{\tau})$ is 2-regular, there exists disjoint $V \in \overrightarrow{\tau}$, $U \in \overleftarrow{\tau}$ such that $x \in X \setminus O \subseteq U$ and $z \in V$. Then $\mathsf{L}X$ is Hausdorff, since its topology is the join of $\overrightarrow{\tau}$ and $\overleftarrow{\tau}$ by Prop. 5.2.5.

Conversely, let $\mathsf{L}X$ be Hausdorff. We need to show that $\mathsf{N}X = (X, \overrightarrow{\tau})$ is T_0. Let

$x, z \in X$ such that $\overrightarrow{\{x\}} = \overrightarrow{\{z\}}$, i.e. $\{x\}^{(0)} = \{z\}^{(0)}$. By way of contradiction, assume $x \neq z$. Since $\mathsf{L}X$ is Hausdorff, there exists $\varepsilon, \lambda > 0$, $U, V \in \overrightarrow{\mathcal{T}}$ such that $x \in U \cap \{x\}^{((\varepsilon))}$, $z \in V \cap \{z\}^{((\lambda))}$ with $U \cap \{x\}^{((\varepsilon))} \cap V \cap \{z\}^{((\lambda))} = \varnothing$. Then $U \cap \{x\}^{(0)} \cap V \cap \{z\}^{(0)} = \varnothing$. But this is a contradiction, since $x \in U \cap \{x\}^{(0)} \cap V \cap \{z\}^{(0)}$. Hence $x = z$, $\mathsf{N}X$ is T_0. $\square$

So an approach space $X = (X, \delta)$ is $\mathcal{C}$-compact $\mathcal{C}$-separated if and only if $\mathsf{L}X$ is compact Hausdorff. Hence one has the following is a pullback diagram in **CAT** where $\mathcal{C}$-cpct $\mathcal{C}$-sep denotes the full subcategory of $\mathcal{C}$-compact $\mathcal{C}$-separated approach spaces.

$$\begin{tikzcd}
\mathcal{C}\text{-cpct } \mathcal{C}\text{-sep} \arrow[r, "\mathsf{L}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \text{Cpct Haus} \arrow[d, hook] \\
\mathbf{App} \arrow[r, "\mathsf{L}"'] & \mathbf{Top}
\end{tikzcd}$$

In fact we have the following commutative diagram where the right hand side square is a pullback by the definitions of 2-compactness and 2-separation. Since $\mathsf{L} = \mathsf{J}.\mathsf{B}$, the left hand side square is a pullback as well.

$$\begin{tikzcd}
\mathcal{C}\text{-cpct } \mathcal{C}\text{-sep} \arrow[r, "\mathsf{B}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \text{2-cpct 2-sep} \arrow[r, "\mathsf{J}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \text{Cpct Haus} \arrow[d, hook] \\
\mathbf{App} \arrow[r, "\mathsf{B}"'] & \mathbf{BiTop} \arrow[r, "\mathsf{J}"'] & \mathbf{Top}
\end{tikzcd}$$

7.4 $\mathcal{C}$-density

In accordance with Section 6.2, a $(\mathbb{T}, V)$-functor f is called *$\mathcal{C}$-dense* if and only if in any factorization $f = i.g$, if $i \in \mathcal{C} \cap \mathcal{M}$, then i is an isomorphism.

Proposition 7.4.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. f is $\mathcal{C}$-dense if and only if f is L-dense.

Proof. Let f be $\mathcal{C}$-dense. Consider the $X \xrightarrow{f'} \overleftrightarrow{f(X)} \overset{i}{\hookrightarrow} Y$ factorization for f. Since $\overleftrightarrow{f(X)}$ is L-closed in Y, i is L-closed. Then $i \in \mathcal{C} \cap \mathcal{M}$. As f is $\mathcal{C}$-dense, i becomes an isomorphism. Hence $\overleftrightarrow{f(X)} = Y$ and f is L-dense by Remark 3.4.12.

Conversely, suppose that f is L-dense. Consider any factorization $X \xrightarrow{g} Z \xrightarrow{i} Y$ of f where $i \in \mathcal{C} \cap \mathcal{M}$. By Prop. 2.8.2, i is L-dense. So $\overleftrightarrow{i(Z)} = Y$. On the other hand, $i(Z)$ is L-closed, since i is L-closed. Hence $i(Z) = Y$, $i \in \mathcal{E}$. Therefore $i \in \mathcal{E} \cap \mathcal{M}$, i is an isomorphism. $\square$

This means that a $(\mathbb{T}, V)$-functor $f : X \to Y$ is $\mathcal{C}$-dense if and only if $\overleftrightarrow{f(X)} = Y$.

Corollary 7.4.2. Any L-dense L-closed subobject is an isomorphism.

Proof. Follows from Prop. 6.2.1 and Prop. 7.4.1. □

Example 7.4.3. Suppose that $f : X \to Y$ is a continuous map between the topological spaces. Let $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ denote the corresponding frame homomorphism between the lattice of open sets of Y and the lattice of open sets of X.

Claim: $f : X \to Y$ is $\mathcal{C}$-dense (L-dense) if and only if $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is injective.

To see this, let $f : X \to Y$ be $\mathcal{C}$-dense. Take open sets $O_1, O_2 \subseteq Y$ such that $f^{-1}(O_1) = f^{-1}(O_2)$, i.e. $f(X) \cap O_1 = f(X) \cap O_2$. We show that $O_1 = O_2$. Let $y \in O_1$. Since f is $\mathcal{C}$-dense, $\overleftrightarrow{f(X)} = Y$ which means that $f(X)$ is dense in LY. So $f(X) \cap O_1 \cap \overline{\{y\}} \neq \varnothing$. This implies $f(X) \cap O_2 \cap \overline{\{y\}} \neq \varnothing$. Hence there exists $z \in O_2 \cap \overline{\{y\}}$. Since $z \in \overline{\{y\}}$, any open set containing z also contains y. In particular, $y \in O_2$. Hence $O_1 \subseteq O_2$. Similarly, we get $O_2 \subseteq O_1$. Therefore f^{-1} is injective.

Conversely, assume that f^{-1} is injective. Let $g, h : Y \to 2$ be continuous maps where 2 is the Sierpinski space. Suppose that $g \neq h$. Then $g^{-1}(\{0\}) \neq h^{-1}(\{0\})$. Since f^{-1} is injective, $f^{-1}(g^{-1}(\{0\})) \neq f^{-1}(h^{-1}(\{0\}))$. Hence $g.f \neq h.f$. So given any $g, h : Y \to 2$, $g.f = h.f$ implies $g = h$. Then f is L-dense by Remark 3.4.12.

7.5 $\mathcal{C}$-openness

Following Section 6.7, a $(\mathbb{T}, V)$-functor is called *$\mathcal{C}$-open* if and only if it stably reflects $\mathcal{C}$-density, i.e. L-density. We will denote the collection of $\mathcal{C}$-open maps by $\mathcal{C}^{+}$.

Definition 7.5.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. We say that f is *L-open* if $f^{-1}(\overleftrightarrow{N}) = \overleftrightarrow{f^{-1}(N)}$ for all $N \subseteq Y$.

Recall from Prop. 3.4.13 that $f^{-1}(\overleftrightarrow{N}) \supseteq \overleftrightarrow{f^{-1}(N)}$ always holds. So f is L-open if and only if $f^{-1}(\overleftrightarrow{N}) \subseteq \overleftrightarrow{f^{-1}(N)}$ for all $N \subseteq Y$.

Proposition 7.5.2. For a $(\mathbb{T}, V)$-functor $f : X \to Y$, the following are equivalent:

1. f is L-open.
2. $\mathsf{L}f : \mathsf{L}X \to \mathsf{L}Y$ is an open map in **Top**, i.e. for any $O \subseteq X$ open in $\mathsf{L}X$, $f(O)$ is open in $\mathsf{L}Y$.

Proof. Let f be L-open, $O \subseteq X$ be an open set of $\mathsf{L}X$. Then $X \setminus O$ is closed in $\mathsf{L}X$, i.e. $\overleftrightarrow{X \setminus O} \subseteq X \setminus O$. Since f is L-open,

$$f^{-1}(\overleftrightarrow{Y \setminus f(O)}) = \overleftrightarrow{f^{-1}(Y \setminus f(O))} \subseteq \overleftrightarrow{X \setminus O} \subseteq X \setminus O.$$

This implies that $\overleftrightarrow{Y \setminus f(O)} \subseteq f(X \setminus O) \subseteq Y \setminus f(O)$. Hence $Y \setminus f(O)$ is closed in $\mathsf{L}Y$, $f(O)$ is open in $\mathsf{L}Y$. Therefore $\mathsf{L}f$ is an open map.

Conversely, assume that $\mathsf{L}f : \mathsf{L}X \to \mathsf{L}Y$ is an open map in **Top**. Then $f(M^\circ) \subseteq f(M)^\circ$ for all $M \subseteq X$ where $(\_)^\circ$ represents the interior operators of $\mathsf{L}X$ and $\mathsf{L}Y$, i.e. $M^\circ = X \setminus \overleftrightarrow{(X \setminus M)}$, $f(M)^\circ = Y \setminus \overleftrightarrow{(Y \setminus f(M))}$. In particular, $f(f^{-1}(K)^\circ) \subseteq f(f^{-1}(K))^\circ \subseteq K^\circ$ for any $K \subseteq Y$. This implies $f^{-1}(K)^\circ \subseteq f^{-1}(K^\circ)$. Rewriting this inclusion by closures and putting $K = Y \setminus N$, one obtains $X \setminus \overleftrightarrow{f^{-1}(N)} \subseteq f^{-1}(Y \setminus \overleftrightarrow{N})$. Since $f^{-1}(Y \setminus \overleftrightarrow{N}) = X \setminus f^{-1}(\overleftrightarrow{N})$, $f^{-1}(\overleftrightarrow{N}) \subseteq \overleftrightarrow{f^{-1}(N)}$ for any $N \subseteq Y$. Therefore f is L-open.

□

Proposition 7.5.3. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. Then we have $(1 \Rightarrow 2 \Rightarrow 3)$.

1. f is $\mathcal{C}$-open.

2. f is L-open

3. f reflects L-density.

Proof. $(1 \Rightarrow 2)$ Let f be $\mathcal{C}$-open and $N \subseteq Y$. Consider the following pullback diagram.

$$\begin{tikzcd}
f^{-1}(N) \arrow[r, "f''"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, hook] \\
f^{-1}(\overleftrightarrow{N}) \arrow[r, "f'"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \overleftrightarrow{N} \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

As f is $\mathcal{C}$-open, f' reflects L-density. Then $f^{-1}(N) \hookrightarrow f^{-1}(\overleftrightarrow{N})$ is L-dense as a pullback of $N \hookrightarrow \overleftrightarrow{N}$ along f'. Hence $\overleftrightarrow{f^{-1}(N)} = f^{-1}(\overleftrightarrow{N})$ by Prop. 3.4.11, f is L-open.

($2 \Rightarrow 3$) Assume that f is L-open and $N \hookrightarrow Y$ is L-dense, i.e. $\overleftrightarrow{N} = Y$. Taking the pullback of f along $N \hookrightarrow Y$, one gets

$$\begin{tikzcd}
f^{-1}(N) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Since f is L-open, $f^{-1}(\overleftrightarrow{N}) = \overleftrightarrow{f^{-1}(N)}$. So $X = f^{-1}(Y) = f^{-1}(\overleftrightarrow{N}) = \overleftrightarrow{f^{-1}(N)}$. Hence $f^{-1}(N) \hookrightarrow X$ is L-dense, f reflects L-density. $\square$

The proposition implies that the collection of $\mathcal{C}$-open maps are included in the collection of L-open maps. Furthermore, both type of maps reflect L-density. But $\mathcal{C}$-open maps is the largest pullback stable collection which has this property. So if L-open maps are pullback stable, then these collections will be equal. Hence we will have the equivalence of $\mathcal{C}$-openness and L-openness.

Proposition 7.5.4. Suppose that $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ preserves finite products. Then L-open maps are pullback stable.

Proof. Let $f : X \to Z$, $g : Y \to Z$ be $(\mathbb{T}, V)$-functors. Suppose that f is L-open. Consider its pullback along g.

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

By Prop. 7.5.2, it is enough to show that $\mathsf{L}\pi_2 : \mathsf{L}(X \times_Z Y) \to \mathsf{L}Y$ is open in **Top**. Since L preserves finite products, $\mathsf{L}(X \times Y) \cong \mathsf{L}X \times \mathsf{L}Y$. So $\{M \times N \mid M$ open in $\mathsf{L}X$, N open in $\mathsf{L}Y\}$ is a basis for $\mathsf{L}(X \times Y)$.

Take a basic open set in $\mathsf{L}(X \times_Z Y)$, say $(M \times N) \cap (X \times_Z Y)$ where M is open in $\mathsf{L}X$ and N is open in $\mathsf{L}Y$. Then $\pi_2(M \times N \cap (X \times_Z Y)) = N \cap g^{-1}(f(M))$. Since f is L-open and g is continuous, $g^{-1}(f(M))$ is open in $\mathsf{L}Y$. Then $N \cap g^{-1}f(M)$ is open in $\mathsf{L}Y$. Hence π_2 is L-open.

$\square$

Corollary 7.5.5. Suppose that $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ preserves finite products. Then a $(\mathbb{T}, V)$-functor is $\mathcal{C}$-open if and only if it is L-open.

Proof. Follows by Prop. 7.5.3 and Prop. 7.5.4. $\square$

As we have seen in Section 5.1, the functor L preserves finite products in the case of **Ord**, **Met**, **Top** and **App**. So Cor. 7.5.5 applies in these examples.

$\mathcal{C}$-open maps also satisfy the conditions $(C1)-(C3)$ of Subsection 6.1.3. So one can develop topological notions by considering $\mathcal{C}$-open maps. Following Section 6.7, a $(\mathbb{T}, V)$-category X is called *$\mathcal{C}$-discrete* if X is $\mathcal{C}^{+}$-separated. i.e. $\delta_X : X \times X \to X$ is $\mathcal{C}$-open.

Proposition 7.5.6. Let X be a $(\mathbb{T}, V)$-category and $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ preserve finite products. Then X is *$\mathcal{C}$-discrete* if and only if $\mathsf{L}X$ is a discrete topological space.

Proof. We get the characterization of $\mathcal{C}$-discreteness as follows:

$$\begin{aligned}
X\ \mathcal{C}\text{-discrete} &\iff \delta_X : X \to X \times X\ \mathcal{C}\text{-open} \\
&\iff \delta_X : X \to X \times X\ \text{L-open} \\
&\iff \delta_X(X) = \Delta \text{ open in } \mathsf{L}(X \times X) \cong \mathsf{L}X \times \mathsf{L}X \\
&\iff \forall x \in X,\ \exists U, V \ni x \text{ open in } \mathsf{LX} : (x,x) \subseteq U \times V \subseteq \Delta \\
&\iff \forall x \in X,\ \exists W \ni x \text{ open in } \mathsf{LX} : (x,x) \subseteq W \times W \subseteq \Delta \\
&\iff \forall x \in X,\ \exists W \ni x \text{ open in } \mathsf{LX} : W = \{x\} \\
&\iff \forall x \in X,\ \{x\} \text{ open in } \mathsf{LX} \\
&\iff \mathsf{L}X \text{ discrete.}
\end{aligned}$$

□

Example 7.5.7. Let X be a topological space. $M \subseteq X$ is called *locally closed* if there exist an open set O and a closed set C such that $M = O \cap C$. X is called a *T_d space* [1] if each singleton is locally closed. T_d is a separation axiom between T_0 and T_1. By Prop.7.5.6, a topological space X is $\mathcal{C}$-discrete if and only if its $\mathsf{L}X$ is discrete. This is equivalent to X being a T_d space as follows:

Take any $x \in X$. Suppose that X is a T_d space. Then there exists $U \subseteq X$ open and $C \subseteq X$ closed such that $\{x\} = U \cap C$. As $U \cap C$ is open $\mathsf{L}X$, $\mathsf{L}X$ is discrete. Conversely, let $\mathsf{L}X$ be discrete. Then $\{x\}$ is open in $\mathsf{L}X$. Recall from Remark 5.2.6 that the collection $\{O \cap \overline{\{x\}} \mid x \in O \text{ open}\}$ is a local basis at x. So there exists an open neighbourhood U of x such that $U \cap \overline{\{x\}} = \{x\}$. Hence $\{x\}$ is locally closed, X is T_d.

A $(\mathbb{T}, V)$-functor $f : X \to Y$ is called *$\mathcal{C}$-local homeomorphism* if f is $\mathcal{C}^+$-perfect.

Proposition 7.5.8. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor and $\mathsf{L} : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to \mathbf{Top}$ preserve finite products. Then f is $\mathcal{C}$-local homeomorphism if and only if Lf is open and locally injective, i.e. each $x \in X$ has a neighbourhood W in $\mathsf{L}X$ such that $f_{|W}$ is injective.

Proof. Since $\mathcal{C}^+$ is pullback stable, f is $\mathcal{C}^+$-perfect if and only if both f and $\delta_f : X \times_Y X \to X$ are $\mathcal{C}$-open. By Prop. 7.5.2 and Cor. 7.5.5, $\mathsf{L}f$ is an open map. The equivalence of $\delta_f : X \times_Y X \to X$ being $\mathcal{C}$-open and $\mathsf{L}f$ being locally injective can be shown as follows:

$$\begin{aligned}
\delta_f \ \mathcal{C}\text{-open} &\iff \delta_f \ \text{L-open} \\
&\iff \delta_f(X) = \Delta \text{ open in } \mathsf{L}(X \times_Y X) \\
&\iff \forall x \in X,\ \exists U, V \ni x \text{ open in } \mathsf{L}X : U \times V \cap X \times_Y X \subseteq \Delta \\
&\iff \forall x \in X,\ \exists W \ni x \text{ open in } \mathsf{L}X : W \times W \cap X \times_Y X \subseteq \Delta \\
&\iff \forall x \in X,\ \exists W \ni x \text{ open in } \mathsf{L}X : \forall y, z \in W \ \big(f(y) = f(z) \Rightarrow y = z\big) \\
&\iff \forall x \in X,\ \exists W \ni x \text{ open in } \mathsf{L}X : f_{|W} \text{ is injective.}
\end{aligned}$$

□

8 Functional topology with respect to L-complete morphisms

In this chapter we develop topological notions for $(\mathbb{T}, V)$-categories using the categorical framework outlined in Chapter 6. Our main parameter $\mathcal{F}$ will be the collection of L-complete $(\mathbb{T}, V)$-functors.

8.1 L-complete morphisms vs. L-closed morphisms

Firstly, we compare L-complete maps, L-closed maps and $\mathcal{C}$-proper maps in our main examples. This will give us a first idea about how the topological theory based on L-complete maps will be similar to or different from the one based on L-closed maps.

Examples 8.1.1. 1. In **Ord**, a monotone map $f : (X, \leq) \to (Y, \leq)$ is L-complete if and only if given any $x \in X$ with $f(x) \simeq y$, there exists $w \in f^{-1}(\{y\})$ with $x \simeq w$ (see Examples 4.1.3). Recall from Examples 7.2.2 that this is also the characterization of $\mathcal{C}$-proper maps. So L-complete maps and $\mathcal{C}$-proper maps coincide in **Ord**.

Furthermore, L-completeness is equivalent to L-closedness. To see this, suppose that $f : (X, \leq) \to (Y, \leq)$ is L-closed. Let $f(x) \simeq y$ for some $y \in Y$. Then $y \in \overleftrightarrow{\{f(x)\}}$. As f is L-closed, $\overleftrightarrow{\{f(x)\}} = f(\overleftrightarrow{\{x\}})$. So there exists $w \in \overleftrightarrow{\{x\}}$ such that $f(w) = y$. Since $w \in \overleftrightarrow{\{x\}}$ means $x \simeq w$, f is L-complete. Conversely, every L-complete map is $\mathcal{C}$-proper, hence L-closed.

So the notions of L-completeness, L-closedness and $\mathcal{C}$-properness coincide in **Ord**.

2. In **Met**, a nonexpansive map $f:(X,d)\to(Y,d')$ is L-complete if and only if given any Cauchy sequence (x_n) in X with $\lim\limits_{n\to\infty} f(x_n)=y$, there exists $x\in f^{-1}(\{y\})$ such that $\lim\limits_{n\to\infty} x_n=x$ (see Examples 4.1.3).

   Recall from Examples 7.2.2 that a nonexpansive map f is $\mathcal{C}$-proper if for any sequence (x_n) in X with $\lim\limits_{n\to\infty} f(x_n)=y\in Y$, there exists a subsequence (x_{n_k}) of (x_n) and $x\in f^{-1}(\{y\})$ such that $\lim\limits_{n\to\infty} x_{n_k}=x$. Since a Cauchy sequence which has a convergent subsequence is itself convergent, $\mathcal{C}$-proper maps are L-complete. One also has the following:

   Claim: L-closed nonexpansive maps are L-complete.

   Suppose that $f:(X,d)\to(Y,d')$ is an L-closed nonexpansive map and (x_n) is a Cauchy sequence in X with $\lim\limits_{n\to\infty} f(x_n)=y$. Let M be the set consisting of the elements of the sequence (x_n). Then $y\in\overleftrightarrow{f(M)}=f(\overleftrightarrow{M})$. So there exists $x\in\overleftrightarrow{M}$ such that $f(x)=y$. $x\in\overleftrightarrow{M}$ means that $\inf\limits_{x_n\in M}\big(d(x,x_n)+d(x_n,x)\big)=0$. Picking x_{n_k} for each $k\in\mathbb{N}$, one obtains a subsequence (x_{n_k}) of (x_n) such that $\lim\limits_{n\to\infty} x_{n_k}=x$. Since (x_n) is Cauchy, it converges to x as well. Hence f is L-complete.

   Unlike the case in **Ord**, L-completeness, L-closedness and $\mathcal{C}$-properness are distinct from each other in **Met**. Consider the following pullback diagram in **Met**.

$$\begin{tikzcd} \mathbb{R} \times \mathbb{R} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "!_{\mathbb{R}}"] \\ \mathbb{R} \arrow[r, "!_{\mathbb{R}}"'] & 1 \end{tikzcd}$$

$!_{\mathbb{R}} : \mathbb{R} \to 1$ is L-complete, since $\mathbb{R}$ is a complete metric space. Furthermore, $!_{\mathbb{R}}$ is L-closed, since its codomain is the one point set. As L-complete maps are stable under pullback, $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is L-complete. But π_1 is not L-closed. To see this, let G be the graph of the function $y = \frac{1}{x}$. G is an L-closed set in $\mathbb{R} \times \mathbb{R}$, but $\pi_1(G) = \mathbb{R} \setminus \{0\}$ is not. So π_1 is not L-closed. Hence π_1 is an L-complete map that is not L-closed. This also shows that $!_{\mathbb{R}}$ is an L-closed map which is not $\mathcal{C}$-proper. Therefore in **Met**, we have

$$\mathcal{C}\text{-proper maps} \subsetneq \text{L-closed maps} \subsetneq \text{L-complete maps}.$$

3. In **Top**, a continuous map $f : X \to Y$ is L-complete if and only if for any irreducible closed set $A \subseteq X$ with $\overline{f(A)} = \overline{\{y\}}$ for some $y \in Y$, there exists $x \in f^{-1}(\{y\})$ such that $A = \overline{\{x\}}$ (see Examples 4.1.3).

L-closed maps are not generally L-complete in **Top**. Consider an infinite set

X with the cofinite topology. The closed subsets are all finite subsets of X and the set X itself. So X cannot be written as a union of two proper closed subsets. Hence X is an irreducible closed set. Consider $!_X : X \to 1$. We have $\overline{!_X(X)} = \overline{\{*\}}$. But there is no $x \in X$ such that $\overline{\{x\}} = X$. So $!_X$ is not L-complete. Hence $!_X : X \to 1$ is L-closed but not L-complete.

Therefore in general we have

$\mathcal{C}$-proper $(\mathbb{T}, V)$-functors $\subsetneqq$ L-closed $(\mathbb{T}, V)$-functors $\neq$ L-complete $(\mathbb{T}, V)$-functors.

L-complete $(\mathbb{T}, V)$-functors are different from L-closed $(\mathbb{T}, V)$-functors. But there is an important subcollection of $(\mathbb{T}, V)$-functors in which L-completeness and L-closedness coincide.

Lemma 8.1.2. Let $f : X \to Y$ be a fully faithful $(\mathbb{T}, V)$-functor. Then f is L-complete if and only if f is L-closed.

Proof. Assume that f is L-closed. Let $\varphi : E \rightsquigarrow X$ be a left adjoint $(\mathbb{T}, V)$-module such that $f_* \circ \varphi = y_*$. Considering the canonical $(\mathcal{E}, \mathcal{M})$ factorization $X \overset{f'}{\twoheadrightarrow} f(X) \overset{i}{\hookrightarrow} Y$ of f, we can write $i_* \circ (f'_* \circ \varphi) = y_*$. Then we have $y \in \overleftrightarrow{f(X)}$ by Prop. 3.4.10. Since f is L-closed, $f(X)$ is L-closed by Lemma 7.1.4. Hence $y \in f(X)$, i.e. there exists $x \in X$ such that $f(x) = y$. Then $f_* \circ \varphi = y_* = f_* \circ x_*$. Since f is fully faithful, $\varphi = x_*$. Therefore f is L-complete.

Conversely, assume that f is L-complete. Write $f = i.f'$ as above. As f' is surjective, it is L-dense. It is also fully faithful, since f is fully faithful. Hence f' is an L-equivalence. Then i is L-complete by Prop. 4.1.4. Since f is fully faithful, it is enough to show that $f(X)$ is L-closed by Lemma 7.1.4. Take $y \in \overleftrightarrow{f(X)}$. By Prop. 3.4.10, $(1_E)^* \le y^* \circ i_* \circ i^* \circ y_*$. Composing both sides with y_* and taking advantage of the adjunctions, we see that

$$y_* \le y_* \circ y^* \circ i_* \circ i^* \circ y_* \le i_* \circ i^* \circ y_* \le y_*.$$

Hence $i_* \circ i^* \circ y_* = y_*$. Since $i^* \circ y_* : E \rightsquigarrow f(X)$ is a left adjoint $(\mathbb{T}, V)$-module and i is L-complete, there exists $f(x) \in f(X)$ such that $i(f(x)) = f(x) = y$. Hence $y \in f(X)$, $\overleftrightarrow{f(X)} \subseteq f(X)$. $\square$

8.2 The setting

We keep $(\mathcal{E}, \mathcal{M})$ to denote the proper factorization system of surjective $(\mathbb{T}, V)$-functors and full embeddings. We denote the collection of L-complete $(\mathbb{T}, V)$-functors by $\mathcal{L}$.

$\mathcal{L}$ is closed under composition and contains isomorphisms by Prop. 4.1.4. Since $\mathcal{L}$ is stable under pullback, so is $\mathcal{L} \cap \mathcal{M}$. Hence $\mathcal{L}$ satisfies conditions $(C1)$ and $(C2)$ of Subsection 6.1.3. However $\mathcal{L}$ does not satisfy condition $(C3)$, i.e. $g.f \in \mathcal{L}$ and $f \in \mathcal{E}$ does not necessarily imply that $g \in \mathcal{L}$.

Example 8.2.1. Let G be the graph of the function $y = \frac{1}{x}$. Consider $\pi_1' : G \to \mathbb{R}$ which is the restriction of the projection map $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to G. We know from Examples 8.1.1 that π_1 is L-complete. On the other hand, $i : G \hookrightarrow \mathbb{R} \times \mathbb{R}$ is L-closed, since G is L-closed in $\mathbb{R} \times \mathbb{R}$. Then i is L-complete by Lemma 8.1.2. This implies that $\pi_1' = \pi_1.i$ is L-complete. Consider the canonical $(\mathcal{E}, \mathcal{M})$ factorization $G \overset{p}{\twoheadrightarrow} \mathbb{R} \smallsetminus \{0\} \overset{j}{\hookrightarrow} \mathbb{R}$ for π_1'. Since $\mathbb{R} \smallsetminus \{0\}$ is not L-closed in $\mathbb{R}$, $j : \mathbb{R} \smallsetminus \{0\} \hookrightarrow \mathbb{R}$ is not L-closed. Hence it is not L-complete by Lemma 8.1.2. So one has $\pi_1' = j.p$ where π_1' is L-complete, p is surjective but j is not L-complete.

Instead of condition $(C3)$, we have an analogous result by replacing surjective $(\mathbb{T}, V)$-functors with L-equivalences, i.e. $g.f \in \mathcal{L}$ and f is an L-equivalence imply that $g \in \mathcal{L}$. This is given by Prop. 4.1.4.

Most of the results in this chapter will be corollaries of the general results presented in Chapter 6. Proofs will be omitted unless they require condition $(C3)$ to hold.

8.3 $\mathcal{L}$-density

In accordance with Section 6.2, a $(\mathbb{T}, V)$-functor f is called *$\mathcal{L}$-dense* if and only if in any factorization $f = i.g$, if $i \in \mathcal{L} \cap \mathcal{M}$, then i is an isomorphism.

Proposition 8.3.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. f is $\mathcal{L}$-dense if and only if it is L-dense.

Proof. A full embedding is L-complete if and only if it is L-closed by Lemma 8.1.2. So $\mathcal{L}$-density coincides with $\mathcal{C}$-density. $\mathcal{C}$-dense maps are precisely L-dense maps by Prop. 7.4.1. □

Corollary 8.3.2. Any L-dense L-complete subobject is an isomorphism.

Proof. A subobject is L-complete if and only if it is L-closed by Lemma 8.1.2. An L-dense L-closed subobject is an isomorphism by Cor. 7.4.2. □

8.4 $\mathcal{L}$-compactness

Following Subsection 6.3.2, a $(\mathbb{T}, V)$-category X is called *$\mathcal{L}$-compact* if and only if the unique map $!_X : X \to 1$ is in $\mathcal{L}^* = \mathcal{L}$.

Proposition 8.4.1. X is $\mathcal{L}$-compact if and only if X is L-complete.

Proof. X is $\mathcal{L}$-compact if and only if $!_X : X \to 1$ is L-complete. This is equivalent to X being L-complete by Cor. 4.1.9. □

The following is an analogue of the Kuratowski-Mrowka theorem for L-completeness.

Corollary 8.4.2. For a $(\mathbb{T}, V)$-category X, the following are equivalent.

1. X is L-complete

2. For any $(\mathbb{T}, V)$-category Y, the projection $X \times Y \to Y$ is L-complete.

3. For any L-complete $(\mathbb{T}, V)$-category Y, $X \times Y$ is L-complete

Proof. Follows from Prop. 6.3.4. □

Moreover, L-completeness is carried forwards by L-equivalences and backwards by L-complete morphisms.

Corollary 8.4.3. 1. If $f : X \to Y$ is L-complete with Y L-complete, then X is L-complete.

2. If $f : X \to Y$ is an L-equivalence with X L-complete, then Y is L-complete.

Proof. (1) follows from Prop. 6.3.5. (2) follows from Prop. 6.3.5 and Prop. 4.1.4. □

Consider a $(\mathbb{T}, V)$-functor $f : X \to Y$ as an object of the comma category $(\mathbb{T}, V)$-**Cat**$/Y$. One sees that $f : X \to Y$ is L-complete if and only if f is an $\mathcal{L}$-compact object in $(\mathbb{T}, V)$-**Cat**$/Y$.

8.5 $\mathcal{L}$-separation

Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. In accordance with Section 6.4, f is called *$\mathcal{L}$-separated* if and only if $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is in $\mathcal{L}^* = \mathcal{L}$.

Proposition 8.5.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. f is $\mathcal{L}$-separated if and only if it is L-separated.

Proof. Observe that $\delta_f = \langle 1_X, 1_X \rangle : X \to X \times_Y X$ is fully faithful. Then δ_f is L-complete if and only if it is L-closed by Lemma 8.1.2. Hence f is $\mathcal{L}$-separated if and only if it is $\mathcal{C}$-separated. We also know that $\mathcal{C}$-separated maps are precisely L-seperated maps by Prop. 7.3.1. ☐

Corollary 8.5.2.

1. L-separated maps are closed under composition and contain all monomorphisms.
2. L-separated maps are stable under pullback.
3. If $g.f$ is L-separated then f is L-separated.
4. Suppose that $g.f$ is L-separated and f is an L-equivalence which is L-complete, then g is L-separated.

Proof. (1)–(3) follow from Prop. 6.4.1. (4) follows from Prop. 6.4.1 and Prop. 4.1.4. ☐

One calls a $(\mathbb{T}, V)$-category X $\mathcal{L}$-separated if and only if $!_X : X \to 1$ is $\mathcal{L}$-separated.

Corollary 8.5.3. Let X be a $(\mathbb{T}, V)$-category. Then X is $\mathcal{L}$-separated if and only if it is L-separated.

Proof. X is $\mathcal{L}$-separated if and only if $!_X : X \to 1$ is L-separated by Prop. 8.5.1. This is equivalent to X being L-separated. □

As in the case of $\mathcal{L}$-compactness, one sees that a $(\mathbb{T}, V)$-functor $f : X \to Y$ is L-separated if and only if it is an $\mathcal{L}$-separated object in $(\mathbb{T}, V)$-**Cat**$/Y$.

Corollary 8.5.4. For a $(\mathbb{T}, V)$-category X, the following are equivalent:

1. X is L-separated.

2. Any morphism $f : X \to Y$ is L-separated.

3. There exists an L-separated morphism $f : X \to Y$ with Y L-separated.

4. For any $(\mathbb{T}, V)$-category Y, the projection $X \times Y \to Y$ is L-separated.

5. For any L-separated $(\mathbb{T}, V)$-category Y, $X \times Y$ is L-separated.

6. For any L-complete L-equivalence $f : X \to Y$, Y is L-separated.

7. In any equalizer diagram

$$A \xrightarrow{s} B \rightrightarrows X$$

   s is L-complete.

Proof. (1) is equivalent to (6) by Prop. 6.4.3 and Cor. 8.5.2. The rest follows from Prop. 6.4.3.

□

In **Top**, a continuous map between a compact domain and a Hausdorff codomain is proper. The analogous statement about the maps with $\mathcal{L}$-compact domain and $\mathcal{L}$-separated codomain gives us the following result.

Corollary 8.5.5. Any $(\mathbb{T}, V)$-functor $f : X \to Y$ with X L-complete and Y L-separated is L-complete.

Proof. Follows from Prop. 6.4.5. □

Corollary 8.5.6. Let Y be an L-complete and L-separated object. Then a $(\mathbb{T}, V)$-functor $f : X \to Y$ is L-complete if and only if X is L-complete.

Proof. Follows from Prop. 6.4.6. □

Corollary 8.5.7. If $g.f$ is L-complete with g L-separated, then f is L-complete.

Proof. Follows from Cor. 6.4.7. □

An L-complete morphism cannot be extended along an L-dense subobject with an L-separated codomain.

Corollary 8.5.8. Let $f : X \to Y$ be an L-complete $(\mathbb{T}, V)$-functor. In any factorization $f = X \xrightarrow{g} Z \xrightarrow{h} Y$ where g is an L-dense subobject and Z is L-separated, g is an isomorphism.

Proof. Follows from Cor. 6.4.8. □

8.6 $\mathcal{L}$-perfect maps

Following Section 6.5, a $(\mathbb{T}, V)$-functor f is called *$\mathcal{L}$-perfect* if it is both $\mathcal{L}$-proper and $\mathcal{L}$-separated. So a $(\mathbb{T}, V)$-functor is $\mathcal{L}$-perfect if and only if it is L-complete and L-separated. Hence an L-complete L-separated morphism $f : X \to Y$ is an $\mathcal{L}$-compact $\mathcal{L}$-separated object in $(\mathbb{T}, V)$-$\mathbf{Cat}/Y$.

Corollary 8.6.1.

1. If $g.f$ is L-complete L-separated and g is L-separated then f is L-complete L-separated.

2. If $g.f$ is L-complete L-separated, f is an L-complete L-equivalence, then g is L-complete L-separated.

Proof. (1) follows from Prop. 6.5.1. (2) follows from Cor. 8.5.2 and Prop. 4.1.4. □

8.7 $\mathcal{L}$-compactifications

Recall from Section 6.6 that an $\mathcal{F}$-compactification of an object X is an $\mathcal{F}$-dense embedding $i : X \to K$ where K is $\mathcal{F}$-compact and $\mathcal{F}$-separated. Replacing $\mathcal{F}$ by $\mathcal{L}$, one gets an L-dense embedding $i : X \hookrightarrow K$ where K is L-complete and L-separated. But such an embedding exists only for L-separated objects by Cor. 8.5.4. To extend the notion to a larger collection, we ask i to be only fully faithful.

Definition 8.7.1. Let X be a $(\mathbb{T}, V)$-category. An *$\mathcal{L}$-compactification* of X is given by an L-equivalence $i : X \to K$ where K is $\mathcal{L}$-compact and $\mathcal{L}$-separated.

We are particularly interested in a *functorial $\mathcal{L}$-compactification* of $(\mathbb{T},V)$-categories. By that we mean a functor $\Gamma : (\mathbb{T},V)\textbf{-Cat} \to (\mathbb{T},V)\textbf{-Cat}_{\text{cpl \& sep}}$ which comes with a natural transformation $\{\gamma_X : X \to \Gamma X\}_{X\in(\mathbb{T},V)\textbf{-Cat}}$ where each γ_X is an L-equivalence and each ΓX is L-complete and L-separated.

Proposition 8.7.2. The L-completion functor $\mathcal{Y} : (\mathbb{T},V)\textbf{-Cat} \to (\mathbb{T},V)\textbf{-Cat}_{\text{cpl \& sep}}$ together with the natural transformation $\{y_X : X \to \mathcal{Y}(X)\}_{X\in(\mathbb{T},V)\textbf{-Cat}}$ is a functorial $\mathcal{L}$-compactification.

Proof. See Theorem 3.5.9 and Remark 4.1.7. □

Examples 8.7.3. In **Met**, $\mathcal{L}$-compactification of a metric space is its Cauchy completion. In **Top**, $\mathcal{L}$-compactification takes the form of soberification where $\mathcal{Y}$ is the soberification functor [40].

Working in the comma category, one can extend the $\mathcal{L}$-compactification notion to morphisms.

Definition 8.7.4. Let $f : X \to Y$ be a $(\mathbb{T},V)$-functor. *$\mathcal{L}$-compactification of f* is given by an L-equivalence $i : f \to g$ where g is $\mathcal{L}$-compact and $\mathcal{L}$-separated.

An $\mathcal{L}$-compactification of a morphism corresponds to its L-completion. The functorial L-completion $\mathcal{Y}$ for objects provides such an L-completion for morphisms.

To see this, let $f : X \to Y$ be any $(\mathbb{T}, V)$-functor. Consider the following diagram:

$$
\begin{tikzcd}
X \arrow[dr, "i"] \arrow[drr, bend left, "y_X"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_{\tilde{Y}} \tilde{X} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{X} \arrow[d, "\tilde{f}"] \\
& Y \arrow[r, "y_Y"'] & \tilde{Y}
\end{tikzcd}
\tag{8.7.1}
$$

By Cor. 8.5.4 and Cor. 8.5.5, $\tilde{f}$ is L-complete and L-separated. Then π_1 is L-complete and L-separated as a pullback of $\tilde{f}$. On the other hand, π_2 is fully faithful as a pullback of y_Y which is fully faithful. Since y_X is an L-equivalence and π_2 is fully faithful, i is an L-equivalence by Prop. 2.8.2. Hence $i : f \to \pi_1$ is an L-equivalence where π_1 is L-complete and L-separated. This means that $i : f \to \pi_1$ is an L-completion of f.

The Isbell-Henriksen theorem [29] which describes the perfect maps in topology translates into a characterization of $\mathcal{L}$-perfect maps.

Proposition 8.7.5. For a $(\mathbb{T}, V)$-functor $f : X \to Y$, the following are equivalent:

1. f is L-complete and L-separated.

2. The naturality square

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "\tilde{f}"] \\ Y \arrow[r, "y_Y"'] & \tilde{Y} \end{tikzcd} \tag{8.7.2}$$

is a pullback.

Proof. Assume that f is L-complete and L-separated. Then the naturality square is a weak pullback by Prop. 4.1.8. Given $y \in Y$, $\psi \in \tilde{X}$ with $\tilde{f}(\psi) = y_Y(y)$, assume that there exists $x, z \in X$ such that $f(x) = y$, $\psi = x^*$ and $f(z) = y$, $\psi = z^*$. This means that $f(x) = f(z)$ and $x \simeq z$. Since f is L-separated, $x = z$. So, the naturality square is a pullback. Conversely, suppose that naturality square is a pullback. Then f is L-complete and L-separated as a pullback of $\tilde{f}$ which is L-complete and L-separated.

□

Considering the functors $(\_)_* : (\mathbb{T}, V)$-**Cat** $\to (\mathbb{T}, V)$-**Mod**$_l$ and $(\_)^* : (\mathbb{T}, V)$-**Cat** $\to$ $(\mathbb{T}, V)$-**Mod**$_r$ of Section 2.8, we also see that an L-complete L-separated $(\mathbb{T}, V)$-functor is a $(\_)_*$-cartesian morphism or equivalently a $(\_)^*$-cocartesian morphism.

8.8 ($\mathcal{L}$-antiperfect, $\mathcal{L}$-perfect) factorization system for $(\mathbb{T}, V)$-Cat

The (Antiperfect, Perfect) factorization of the continuous maps of Tychonoff spaces [30], [54], [12], [55] is obtained with the help of the left adjoint Stone-Čech compactification functor. Here an "antiperfect map" stands for a map which is sent to an isomorphism by the compactification functor. Analogously in our context the reflector $\mathcal{Y}$ induces ($\mathcal{L}$-antiperfect, $\mathcal{L}$-perfect) factorization system for $(\mathbb{T}, V)$-**Cat**. Here $\mathcal{L}$-*antiperfect* morphisms are precisely the ones that are mapped to isomorphisms by $\mathcal{Y}$. These types of factorization systems are studied in [9], [39], [49].

Lemma 8.8.1. Let $f : X \to Y$ be a $(\mathbb{T}, V)$-functor. f is $\mathcal{L}$-antiperfect if and only if f is an L-equivalence.

Proof. Suppose that f is $\mathcal{L}$-antiperfect, i.e. $\mathcal{Y}(f) = \tilde{f}$ is an isomorphism. Then $\tilde{f}$ is an L-equivalence. The naturality square (8.7.2) gives $\tilde{f}.y_X = y_Y.f$ where $y_X, y_Y, \tilde{f}$ are L-equivalences. Then f is an L-equivalence by Prop 2.8.2.

Conversely, suppose that f is an L-equivalence. Define $\breve{f} : \tilde{Y} \to \tilde{X}$ by $\breve{f}(\psi) = \psi \circ f_*$ for any right adjoint $(\mathbb{T}, V)$-module $\psi : Y \rightsquigarrow E$. Then $\breve{f} = (\tilde{f})^{-1}$. Hence $\mathcal{Y}(f) = \tilde{f}$ is an isomorphism. $\square$

Theorem 8.8.2. Let $\mathscr{E}$ be the collection of L-equivalences and $\mathscr{M}$ be the collection of L-complete and L-separated morphisms. Then $(\mathscr{E}, \mathscr{M})$ is a factorization system for $(\mathbb{T}, V)$-**Cat**.

Proof. We check conditions (FS1)-(FS3) of Subsection 6.1.1. Firstly, both $\mathscr{E}$ and $\mathscr{M}$ contain all isomorphisms and they are closed under composition. Secondly, given any $(\mathbb{T}, V)$-functor $f : X \to Y$, one has an $(\mathscr{E}, \mathscr{M})$ factorization $f = i.\pi_1$ as shown in diagram (8.7.1). To show that $\mathscr{E} \perp \mathscr{M}$, consider the following commutative square where i is an L-equivalence and f is L-complete and L-separated.

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Since f is L-complete, it is L-injective by Theorem 4.3.2. So there exists $g : B \to X$ such that $g.i \simeq j$ and $f.g = h$. Then we have $f.g.i = f.j$. As f is L-separated, $g.i = j$. For the uniqueness part, suppose that there exists $m : B \to X$ such that $f.m = h$ and $m.i = j$. Then $f.g = f.m$ and $g.i = m.i$. Since i is L-dense, $g \simeq m$. Invoking L-separatedness of f again, we get $g = m$. □

Examples 8.8.3. 1. Let $f : X \to Y$ be a nonexpansive map in **Met**. f is an L-equivalence if and only if it is a dense isometry. f is L-complete and L-separated if and only if for any Cauchy sequence (x_n) in X with $\lim_{n\to\infty} f(x_n) = y$, there exists a unique $x \in f^{-1}(\{y\})$ such that $\lim_{n\to\infty} x_n = x$. So (Dense isometry, L-complete L-separated) is a factorization system in **Met** [53].

2. Let $f : X \to Y$ be a continuous map in **Top**. f is L-complete and L-separated if and only if it is fibrewise sober [48]. Let $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ denote the corresponding frame homomorphism between the lattice of open sets of Y and the lattice of open sets of X. We know that f is fully faithful if and only if $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is surjective by Examples 2.8.4. Furthermore, f is L-dense if and only if $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is injective by Example 7.4.3. Therefore, f is an L-equivalence if and only if $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is a frame isomorphism or, equivalently, f is an isomorphism as a continuous map in the category of locales, i.e. the opposite category of the category of frames. Let's denote the canonical functor from the category of topological spaces to the category of locales by $\mathscr{L} : \mathbf{Top} \to \mathbf{Loc}$. Then $(\mathscr{L}^{-1}(\text{Iso}), \text{Fibrewise sober})$ is a factorization system in **Top** [53].

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