Saturated C*-probability Spaces

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Abstract

The aim of this thesis is to develop the model theory of C^*-probability spaces, and investigate saturated models in this theory. We axiomatize several important classes of C^*-probability spaces and prove that many natural state-assigned subsets of a C^*-algebra are definable sets. As a first application, we resolve Robert's stable rank one/purely infinite dichotomy of selfless C^*-algebras. After this, we construct and classify the natural C^*-probability space analog of countable degree-n saturation of C^*-algebras. Most importantly, we investigate the analog of countable degree-1 saturation. This gives rise to the study of quantum P-points: states that inherit degree-1 saturation from their algebra. We prove the independence of the existence of quantum P-points. Finally, we prove an infinite transitivity theorem for the representation associated to a quantum P-point. Using this infinite transitivity theorem, we resolve the Farah-Hirshberg conjecture, proving that every countably degree-1 saturated C^*-algebra is either nonexact or subhomogeneous.

For Lesley.

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LLM Statement

Near the end of the writing of this thesis, ChatGPT 5.6 Sol was used in various results. Most notably, it provided a proof of proposition 3.2(3), a marginal yet important improvement on a prior result. It was also used for proofreading and grammatical checks.

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1 Introduction

1.1 Motivation

Continuous logic was first invented in 2008 [3], and has since had many applications in the theory of C^*-algebras. Simultaneously, in the 2010s, the theory of quantum filters was developed [4, 5, 13, 17], making tractable the application of combinatorial set theory to states on C^*-algebras. Most recently, the model theory of C^*-probability found an application in the classification of group C^*-algebras, with the advent of selfless C^*-probability spaces in 2025, defined in terms of an ultrapower. These breakthroughs motivate a thorough treatment of the model theory of C^*-probability spaces.

Alongside these discoveries was the advent of massive C^*-algebras, a generalization of ultraproducts in the category of C^*-algebras. It was demonstrated in [17], that coronas, $\mathscr{Q}(A) = \mathscr{M}(A)/A$ of nonunital, σ-unital A have a weak saturation called countable degree-1 saturation. This was remarkable, as in an arbitrary metric category, there is no reason to suspect the existence of (even weakly) saturated objects beyond ultraproducts, reduced products, and similar constructions. This motivates a study of analogous saturation on C^*-probability spaces.

1.2 Outline

In section 2, we will first define C^*-probability spaces and their metric language/theory. We then find several axiomatizable classes of C^*-probability spaces, most notably pure states. We next prove that, relative to some of these axiomatizable classes, the most important sets associated to a state φ, namely $\ker\varphi$, L_φ, $L_\varphi \cap L_\varphi^*$, and $\mathscr{F}_\varphi$, are definable sets, i.e. we can quantify over them. Finally, we use model theory to solve a problem in an area of research which has recently gained much attention: selfless C^*-probability spaces.

In section 3, we introduce the notion of saturated C^*-algebras and their classification by the degrees of the *-polynomials we are allowed to use inside their conditions. This motivates the analog for C^*-probability spaces: A bi-graded classification scheme for saturated C^*-probability spaces, graded by degree for the norm and state predicate, respectively, and denoted by degree-(m, n). We prove a reduction theorem which reduces this grading to at most 10 classes. We then provide proper examples inside 8 of these 10 classes. Finally, we note full saturation and the construction of ultrapowers of C^*-probability spaces and reduced powers of C^*-algebras, respectively.

In section 4, we focus on the classes which appear the most fruitful: degree-(n, n) for $n \geq 1$. To do this, we develop the noncommutative analog of P-points, which we call quantum P-points. The states corresponding to these quantum P-points, which we call P-states have a remarkable property. If φ is a P-state on a countably degree-n saturated C^*-algebra C, (C, φ) is countably degree-(n, n) saturated. That is, P-states inherit saturation from their algebra. Moreover, we show that this inheritance is equivalent to the state being a P-state. We prove, under the existence of a P-point, that certain corona algebras $\mathscr{Q}(A)$ admit a P-state. More generally, under the continuum hypothesis, sufficiently complex countably degree-1 saturated C admit many P-states. Finally, we find a model of ZFC in which many coronas fail to admit a P-state.

In section 5, we generalize a result of Farah, that ultraproducts of irreducible representations of C^*-algebras admit an infinite variant of the Kadison transitivity theorem. In particular, we show that the GNS representation associated to a countably degree-$(1,1)$ saturated C^*-probability space admits infinite transitivity, and we discuss several applications, including the resolution of a conjecture of Farah and Hirshberg. Finally, we prove that, in ZFC, there are states on the Calkin algebra whose GNS representations fail infinite transitivity.

2 Model Theory of C^*-probability Spaces

2.1 The Language of C^*-probability Spaces

In this section, we make explicit the theory of C^*-algebras, axiomatize various subclasses, and find definable sets in their theories. Before we begin this task, we should define C^*-probability spaces.

Definition 2.1. A *C^*-algebra* A is a complex Banach *-algebra satisfying the C^*-identity: $\|a^*a\| = \|a\|^2$. The *positive cone* A_+ of A is given by $A_+ = \{a^*a : a \in A\}$. A *state* φ on A is a positive, norm-1 linear functional. That is, a linear map $\varphi : A \to \mathbb{C}$ such that $\varphi[A_+] \subseteq [0, \infty)$ and $\|\varphi\| = \sup_{a \in A_1} |\varphi(a)| = 1$. A *$C^*$-probability space* is a pair (A, φ), where A is a unital C^*-algebra and φ is a state on A.

2.1.1 The Language of C^*-algebras

Definition 2.2. A *metric language* $\mathscr{L}$ is a tuple of data $\langle \mathcal{S}, \mathcal{F}, \mathcal{R}, \mathcal{C} \rangle$ consisting of:

1. The set $\mathcal{S}$ of sort symbols. For each $S \in \mathcal{S}$, we specify d_s and $M_s \geq 0$.
2. The set $\mathcal{F}$ of function symbols. For each $f \in F$, we specify $\mathrm{dom}(f) = (S_1, \ldots, S_n)$ and $\mathrm{rng}(f) = S$ for $S_j, S \in \mathcal{S}$. For each $j \leq n$, we specify $\delta_j^f : [0, \infty) \to [0, \infty)$.
3. The set $\mathcal{R}$ of relation (predicate) symbols. For $R \in \mathcal{R}$, we specify $\mathrm{dom}(R) = (S_1, \ldots, S_n)$ and $\mathrm{rng}(R) = K_r$, where $K_r \subseteq \mathbb{R}$ a compact interval. For $j \leq n$, we specify $\delta_j^R : [0, \infty) \to [0, \infty)$.
4. The set $\mathcal{C}$ of constant symbols. For $c \in \mathcal{C}$, we specify $c \in S_c \in \mathcal{S}$.

A *metric structure* $\mathfrak{A}$ is a tuple of data $\langle \mathfrak{S}, \mathfrak{F}, \mathfrak{R}, \mathfrak{C} \rangle$, where

5. For each $S \in \mathcal{S}$, we associate (S, d_S) a complete, bounded metric space with diameter $M_S \geq 0$.
6. For each $f \in \mathcal{F}$, we associate a continuous function $f : S_1 \times \cdots \times S_n \to S$ with
$$\delta_j^f(t) = \sup\{d(f(\bar{x}), f(\bar{y})) : d(x_j, y_j) \leq t \text{ and for } i \neq j, x_i = y_i\}.$$
7. For each $R \in \mathcal{R}$, we associate continuous $R : S_1 \times \cdots \times S_n \to K_R$ and
$$\delta_j^R(t) = \sup\{|R(\bar{x}) - R(\bar{y})| : d(x_j, y_j) \leq t \text{ and for } i \neq j, x_i = y_i\}.$$
8. For each $c \in \mathcal{C}$, we associate an element $c \in S_c$.

We also supply, for each $S \in \mathcal{S}$, infinitely many variables x_j^S, $j \in \mathbb{N}$.
The metric structure associated to a C^*-algebra A consists of:

9. Sorts $A_n = \{a \in A : \|a\| \leq n\}$.

10. Functions encoding addition $\cdot +_n \cdot : A_n^2 \to A_{2n}$, multiplication $\cdot \times_n \cdot : A_n^2 \to A_{n^2}$, involution $\cdot_n^* : A_n \to A_n$, and for $\lambda \in \mathbb{C}_1$, scalar multiplication $\lambda_n : A_n \to A_n$.

11. Predicates $\|\cdot\|_n : A_n \to [0, n]$.

12. Constants 0 and 1.

and the language $\mathscr{L}$ of C^*-algebras consists of corresponding symbols. See [17] for the theory of C^*-algebras.

2.1.2 The State Predicate

We expand the language $\mathscr{L}$ with two predicate symbols for each sort

$$\varphi_n^r, \varphi_n^i : A_n \to [-n, n],$$

with uniform continuity moduli $\delta_{\varphi_n^r} = \delta_{\varphi_n^i} = \mathrm{id}_{(0,n]}$, primed for interpretation as the real and imaginary parts of a sub-norm function $\varphi : A \to \mathbb{C}$ over the sorts (n-balls) of $\mathscr{L}$. Let $\mathscr{L}_{\mathrm{s}}$ denote this expanded language.

2.2 Terms and Formulas

Definition 2.3. We build terms and formulas in a metric language $\mathscr{L}$ recursively.

1. Variables x_j^S are terms, constants c are terms.

2. If $f \in \mathcal{F}$ and $\tau_1, \ldots, \tau_n$ are terms, then $f(\tau_1, \ldots, \tau_n)$ is a term.

3. If $R \in \mathcal{R}$ is a relation and $\tau_1, \ldots, \tau_n$ are terms with appropriate ranges, then $R(\tau_1, \ldots, \tau_n)$ is a formula.

4. If $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\psi_1, \ldots, \psi_n$ are formulas with appropriate ranges, then $g(\psi_1, \ldots, \psi_n)$ is a formula.

5. If $\varphi(x_j^S, \bar{y})$ is a formula, then both $\sup_{x \in S} \varphi(x, \bar{y})$ and $\inf_{x \in S} \varphi(x, \bar{y})$ are formulas.

An $\mathscr{L}$-formula with no free variables is called a sentence, and we denote the collection of such by $\mathrm{Sent}_{\mathscr{L}}$. An *$\mathscr{L}$-theory* is a subset of $\mathrm{Sent}_{\mathscr{L}}$.
In addition to the recursive construction for $\mathscr{L}$-formulas, we can also, relative to an $\mathscr{L}$-theory T, take uniform limits of such formulas [14]. We call these *definable predicates*, and we may treat them (relative to T) exactly the same as formulas.

Any formula derived by induction using the only the first 3 steps is called *atomic*. Since $\mathscr{L}_{\mathrm{s}}$ does not add new constant symbols or function symbols, atomic formulas look largely the same. Namely, an atomic formula will be of the form: $\|P(\bar{x})\|_n$, $\varphi_n^r(P(\bar{x}))$, or $\varphi_n^i(P(\bar{x}))$ for a *-polynomial $P(\bar{x})$ in noncommuting variables (of the appropriate sort).

Somewhat untraditionally, we abuse terminology by defining an atomic condition in $\mathscr{L}_{\mathrm{s}}$ to be of the form $\|P(\bar{x})\|_n = r$ or $\varphi_n(P(\bar{x})) = s$ for a *-polynomial $P(\bar{x})$, $r \geq 0$, and $s \in \mathbb{C}$. This causes no issues, and was implicitly adopted in [28], because φ_n^r, φ_n^i can be retrieved from φ_n using only quantifier-free (even degree-1) constructions, and clearly vice versa. It is important to note that, formally, metric formulas cannot be $\mathbb{C}$-valued.

2.3 Elementary Classes

Of course, this language is vague enough to encode a plethora of functions $\varphi : A \to \mathbb{C}$. For our purposes, the most general class we will define is φ a bounded linear functional. Several of the classes are described in terms of the GNS construction, which we outline now.

Definition 2.4. Let (A, φ) be a C^*-probability space. Define the *left kernel*

$$L_\varphi = \{a \in A : \varphi(a^*a) = 0\}.$$

Define a pre-inner product on the quotient vector space A/L_φ by $\langle a + L_\varphi, b + L_\varphi \rangle_\varphi = \varphi(b^*a)$. We denote by H_φ the *GNS Hilbert space* which is the completion of A/L_φ under $\langle \cdot, \cdot \rangle_\varphi$. We define the *GNS representation* $\pi_\varphi : A \to \mathscr{B}(H_\varphi)$ by the action $\pi_\varphi(a)(b + L_\varphi) = ab + L_\varphi$ extended continuously. We also specify a canonical cyclic vector $\xi_\varphi = 1 + L_\varphi$. We call this representation *faithful* if it is injective, equivalently if $\|a\| = \|\pi_\varphi(a)\|$. We call it *essential* if $\pi_\varphi[A] \cap \mathscr{K}(H_\varphi) = \{0\}$.

We will use this construction throughout.

Proposition 2.5. *The class of models (A, φ), where A is a C^*-algebra, and φ is a functional of the following kind, is $\forall\exists$-axiomatizable:*

1. *Unit functionals.*
2. *States.*
3. *Tracial states.*
4. *Characters.*
5. *Pure states.*
6. *Pure states inducing faithful GNS-representations.*
7. *Pure states inducing faithful, essential GNS-representations.*
8. *Weak*-limits of pure states.*

Proof. Since C^*-algebras themselves are $\forall\exists$-axiomatizable in $\mathscr{L}$, it suffices to show that each class is axiomatizable by sup-inf ($\forall\exists$) quantifications of nonnegative quantifier-free formulas.

For (1), in the following $\iota_n : A_1 \hookrightarrow A_n$ is the inclusion map, and λ are the scalar function symbols from $\mathscr{L}$, with real and imaginary parts λ_r, λ_i, resp.

$$\sup_{x\in A_n}\sup_{y\in A_m}\left|\varphi^r_{\lceil|\lambda|\rceil n+m}(\lambda x+y)-\lambda_r\varphi^r_n(x)+\lambda_i\varphi^i_n(x)-\varphi^r_m(y)\right|=0$$
$$\sup_{x\in A_n}\sup_{y\in A_m}\left|\varphi^i_{\lceil|\lambda|\rceil n+m}(\lambda x+y)-\lambda_r\varphi^i_n(x)-\lambda_i\varphi^r_n(x)-\varphi^i_m(y)\right|=0$$
$$\sup_{x\in A_1}\left(\left(\varphi^r_1(x)^2+\varphi^i_1(x)^2\right)\dot{-}\,1\right)=0$$
$$\inf_{x\in A_1}\left(1\dot{-}\left(\varphi^r_1(x)^2+\varphi^i_1(x)^2\right)\right)=0$$
$$\sup_{x\in A_1}\left|\varphi^r_n(\iota_n(x))-\varphi^r_1(x)\right|=0$$
$$\sup_{x\in A_1}\left|\varphi^i_n(\iota_n(x))-\varphi^i_1(x)\right|=0$$

The first two encode linearity, the third and fourth encode norm 1, and the last two ensure the predicates are glued together correctly. Since every constant in a sort can be decomposed as a linear combination of constants in the unit sort, this is sufficient.
For (2), we add to the axioms in (1) two more axioms encoding positivity.

$$\sup_{x\in A_1}\left(0\dot{-}\,\varphi^r_1(x^*x)\right)=0$$
$$\sup_{x\in A_1}\left|\varphi^i_1(x^*x)\right|=0$$

For (3), an expanded language is oftentimes overkill [6],[13], but the following axiom does the trick. From this point we will use $\varphi=\varphi^r+i\varphi^i$ and omit the subscripts, as is often done with the $\|\cdot\|$ predicate.

$$\sup_{x\in A_1}\sup_{y\in A_1}\left|\varphi(xy)-\varphi(yx)\right|=0$$

For (4), we use the following axiom.

$$\sup_{x\in A_1}\sup_{y\in A_1}\left|\varphi(xy)-\varphi(x)\varphi(y)\right|=0$$

For (5), some proof is required. Gao gave the following ingenious axiom [18].

$$\sup_{x\in A_1}\inf_{y\in A_1}\left(\left|\|y\|^2-\varphi(x^*x)\right|+\varphi((y-x)^*(y-x))\right)=0$$

The following argument that Gao's axiom axiomatizes pure states was given by Farah, though Gao provided their own argument.

Suppose φ is pure. Then the map $A/L_\varphi \to H_\varphi$ is an isometric isomorphism, by [29, III.2.16], since $\|x + L_\varphi\| = \varphi(x^*x)^{\frac{1}{2}}$. Then for every $\epsilon > 0$, $x \in A_1$, there exists $y \in A_1$ s.t. $x - y \in L_\varphi$ and $|\|y\|^2 - \varphi(x^*x)| < \epsilon$. Thus the axiom holds.

Now suppose φ is not pure, and express $\varphi = r\psi_0 + (1-r)\psi_1$ for distinct states ψ_0, ψ_1, and $r \in (0,1)$. Since $\psi_0 \neq \psi_1$, there exists $a \in A_+$ such that $\|a\| = 1$ and $\psi_0(a) > \psi_1(a)$. Let $x = a^{\frac{1}{2}}$. Then

$$\psi_0(x^*x) = \psi_0(a) > \varphi(a) = \varphi(x^*x).$$

Suppose, towards contradiction, that Gao's axiom holds for this x. Then there are $y_n \in A_1$ such that $\varphi((y_n - x)^*(y_n - x)) \to 0$ and $\|y_n\|^2 \to \varphi(x^*x)$. Then

$$\psi_0((y_n - x)^*(y_n - x)) \leq \frac{1}{r}\varphi((y_n - x)^*(y_n - x)) \to 0.$$

Thus, as vectors in H_{ψ_0}, $y_n + L_{\psi_0} \to x + L_{\psi_0}$, so $\|y_n\|^2 \geq \psi_0(y_n^* y_n) \to \psi_0(x^*x)$, so

$$\liminf_{n\to\infty} \|y_n\|^2 \geq \psi_0(x^*x) > \varphi(x^*x),$$

a contradiction.

For (6), we claim the following axiom suffices in tandem with Gao's axiom.

$$\sup_{x\in A_1} \inf_{y\in A_1} (\|x\|^2 \dot{-} \varphi(y^*x^*xy)) = 0.$$

If this axiom holds, then

$$\|x\|^2 \leq \sup_{y\in A_1} \|xy\|_\varphi^2 \leq \sup_{\xi\in(H_\varphi)_1} \|\pi_\varphi(x)\xi\|^2 = \|\pi_\varphi(x)\|^2,$$

so π_φ is faithful. Conversely, if π_φ is faithful, for every $\epsilon > 0$, there is $\xi \in H_\varphi$ with $\|\xi\| < 1$ such that $\|\pi_\varphi(x)\xi\|^2 \geq \|x\|^2 - \epsilon$. By Gao's axiom, there exists $y \in A_1$ such that $\xi = y + L_\varphi$, thus

$$\|x\|^2 \leq \sup_{\xi\in H_\varphi, \|\xi\|<1} \|\pi_\varphi(x)\xi\|^2 \leq \sup_{y\in A_1} \|xy\|_\varphi^2,$$

so the axiom holds.

For (7), by Glimm's lemma, π_φ is essential if and only if φ is a weak*-limit of

$$\{\omega_\xi : \xi \in H_\varphi, \|\xi\| = 1, \langle \xi, \xi_\varphi \rangle = 0\},$$

if and only if it is a weak*-limit of

$$\{\omega_\xi : \xi \in H_\varphi, \|\xi\| < 1, \langle \xi, \xi_\varphi \rangle = 0\},$$

which is equivalent to the statement: for all $F \Subset A$, $\epsilon > 0$, there exists $\xi \in H_\varphi$ such that $\|\xi\| < 1$ and $\max_{a \in F} |\omega_\xi(a) - \varphi(a)| \leq \epsilon$. Any such ω_ξ can be expressed $\varphi(b^*ab)$, where $b \in A_1$, $\xi = b + L_\varphi$, so $\varphi(b) = 0$. Thus, we can axiomatize this by the axioms

$$\sup_{x_1 \in A_1} \cdots \sup_{x_n \in A_1} \inf_{y \in A_1} (|\varphi(y)| + \max_{j \leq n} |\varphi(y^* x_j y) - \varphi(x_j)|) = 0.$$

Finally, for (8), we use the fact [13, 5.7.4] that a state is a weak*-limit of pure states if and only if for every $F \Subset A$, $\epsilon > 0$, there exists $a \in A_{+,1}$ s.t. $\max_{b \in F} \|aba - \varphi(b)a^2\| < \epsilon$. The following axioms then encode this:

$$\sup_{x_1 \in A_1} \cdots \sup_{x_n \in A_1} \inf_{y \in A_{+,1}} \max_{j \leq n} \|yx_j y - \varphi(x_j)y^2\| = 0.$$

□

Proposition 2.6. *The class of models (A, φ), where A is a C^*-algebra, and φ is a functional of the following kind is not axiomatizable:*

1. *Faithful states.*
2. *Non-faithful states.*
3. *States inducing a faithful GNS representation.*
4. *States inducing a non-faithful GNS representation.*

Proof. Let M be a type II_1 factor with its unique tracial state φ, and $\mathscr{U}$ a free ultrafilter on $\mathbb{N}$. We denote by $(M, \tau)^{\mathscr{U}}$ its (C^*-)ultrapower. Necessarily, there exists $a_j \in M$ such that $\|a_j\| = 1$ and $\tau(a_j^* a_j) < \frac{1}{j}$. Let $a = (a_j) \in A^{\mathscr{U}}$, so $\|a\| = 1$ and $\tau^{\mathscr{U}}(a^*a) = \lim_{j \to \mathscr{U}} \tau(a_j^* a_j) = 0$. Then by traciality and Cauchy-Schwarz, $\|\pi_{\tau^{\mathscr{U}}}(a)\|^2 = 0$, so $\tau^{\mathscr{U}}$ is not GNS-faithful on $M^{\mathscr{U}}$. Since faithful states are GNS-faithful, we have a faithful state on a C^*-algebra whose ultrapower does not induce a faithful GNS-representation. Therefore, each of the classes is either not closed under ultraroots or ultrapowers. □

2.4 Definable Sets

Definable sets are a useful shorthand in continuous logic. They are precisely the sets in which we can quantify over, in the sense that definable predicates quantified over definable sets remain definable [14].

To assess the definability of sets in $\mathscr{L}_{\mathrm{s}}$-structures, we will use the indispensable fact that definable sets are exactly the zero-sets of weakly stable definable predicates. Let us define these terms.

Definition 2.7. For a definable predicate $\psi(\bar{x})$ relative to some $\mathscr{L}$-theory T, we define the *zero-set*

$$Z(\psi) = \{\bar{a} \in \mathrm{dom}(\psi) : \psi(\bar{a}) = 0\}.$$

We call a definable predicate $\psi(\bar{x})$ relative to some theory T *weakly stable* if there exists a homeomorphism $u : [0,\infty) \to [0,\infty)$ s.t.

$$\inf_{\bar{y}\in Z(\psi)} \|\bar{x} - \bar{y}\| \leq u(\psi(\bar{x})).$$

A set is definable relative to T if and only if it is of the form $Z(\psi)$ for some weakly stable predicate $\psi(\bar{x})$ relative to T, per [14, 3.2.5].

This is an instance where weak stability is exactly what we need. We will relay a proof given by Robert Israel [22].

Proposition 2.8. $\ker\varphi$ *is a definable set relative to unit functionals.*

Proof. We first show that $\inf_{b\in\ker\varphi} \|a-b\| = |\varphi(a)|$. First, for $b \in \ker\varphi$, $|\varphi(a)| = |\varphi(a-b)| \leq \|a-b\|$, so it suffices to show the other inequality. For every $\epsilon > 0$, there exists $c \in A$ s.t. $\varphi(c) = 1$, $\|c\| \leq 1+\epsilon$. Then $b = a - \varphi(a)c \in \ker\varphi$ has $\|a-b\| \leq |\varphi(a)c\| \leq (1+\epsilon)|\varphi(a)|$. Therefore, with $u = \mathrm{id}_{[0,\infty)}$, $|\varphi_n(x)|$ is a weakly stable predicate. By [14, 3.2.5], $\ker\varphi_n = Z(|\varphi_n(x)|)$ is a definable set. □

Proposition 2.9. L_φ *and* $L_\varphi \cap L_\varphi^*$ *are definable sets relative to pure states.*

Proof. By Gao's axiom for pure states, for $\varphi \in P(A)$, $n \in \mathbb{N}$, we have

$$\inf_{y\in(L_\varphi)_n} \|x-y\| = \varphi_n^r(x^*x)^{\frac{1}{2}}.$$

Since $(L_\varphi)_n = Z(\varphi_n^r(x^*x)^{\frac{1}{2}})$, $\varphi_n^r(x^*x)$ is a weakly stable predicate, so $(L_\varphi)_n$ is definable, i.e. L_φ is definable.
We claim $\psi(x) = \varphi_n^r(x^*x)^{\frac{1}{2}} + \varphi_n^r(xx^*)^{\frac{1}{2}}$ is weakly stable and clearly $Z(\psi(x)) = (L_\varphi \cap L_\varphi^*)_n$. Since $B = L_\varphi \cap L_\varphi^*$ is a C^*-subalgebra of A, it admits an approximate unit (e_λ). Let $x \in A_n$ and define $b_\lambda = e_\lambda x e_\lambda \in B$. Then $x - b_\lambda = x - xe_\lambda + xe_\lambda - e_\lambda x e_\lambda = x(1-e_\lambda) + (1-e_\lambda)xe_\lambda$, so

$$d(x, B_n) \leq \lim_\lambda (\|x(1-e_\lambda)\| + \|(1-e_\lambda)x\|) = \varphi_n^r(x^*x)^{\frac{1}{2}} + \varphi_n^r(xx^*)^{\frac{1}{2}} = \psi(x).$$

The middle equality is justified by excision (see [13, 5.2.1]). □

Definition 2.10. Let (A,φ) be a C^*-probability space. Define $A_{+,1} = \{a \in A_+ : \|a\| = 1\}$. We denote the *quantum filter of* φ by $\mathscr{F}_\varphi = \{a \in A_{+,1} : \varphi(a) = 1\}$. If there exists a projection $p \in A$ such that $\mathscr{F}_\varphi = \{a \in A_{+,1} : a \geq p\}$, we call $\mathscr{F}_\varphi$ (and φ) *principal*.

Corollary 2.11. *Principal pure states are axiomatizable.*

Proof. By 2.9, L_φ is definable when φ is pure. To the theory of pure states, we add the axiom

$$\inf_{x\in(L_\varphi\cap L_\varphi^*)_1} \sup_{y\in(L_\varphi\cap L_\varphi^*)_1} \|xy - y\| = 0$$

and claim this characterizes principal pure states.
First, assume φ is principal and pure. Then there is a minimal $p \in \mathscr{F}_\varphi$. Then $1-p \in L_\varphi \cap L_\varphi^*$ is the unit for $L_\varphi \cap L_\varphi^*$, so the axiom holds.
Now assume the axiom and let $B = L_\varphi \cap L_\varphi^*$. Then for each n, choose $e_n \in B_1$ such that $\sup_{y \in B_1} \|e_n y - y\| < \frac{1}{n}$. Then

$$\|e_n - e_m\| \leq \sup_{c \in B_1} \|(e_n - e_m)c\| \leq \sup_{c \in B_1} (\|e_n c - c\| + \|c - e_m c\|) < \frac{1}{n} + \frac{1}{m}$$

so (e_n) is Cauchy, so converges to $e \in B_1$ and clearly e is a unit for B. Then $p = 1 - e$ is minimal for $\mathscr{F}_\varphi$. □

Proposition 2.12. *The quantum filter $\mathscr{F}_\varphi$ is definable relative to characters.*

Proof. Now let φ be a character on arbitrary A. We claim the predicate

$$\psi(x) = |\varphi(x) - 1| + 2 \inf_{y \in A_1} \|x - y^* y\|$$

is weakly stable (and clearly $Z(\psi) = \mathscr{F}_\varphi$). Fix $y \in A_1$ arbitrary and set $a = y^* y$. Let $\lambda = \varphi(a)$ and by the continuous functional calculus let $b = f(a)$, where $f(t) = \min\{1, t + 1 - \lambda\}$. Then $0 \leq b \leq 1$ and since φ is a character, $\varphi(b) = f(\varphi(a)) = f(\lambda) = 1$, so $b \in \mathscr{F}_\varphi$. Moreover, $\|b - a\| \leq 1 - \lambda = |\varphi(a) - 1|$, so $d(a, \mathscr{F}_\varphi) \leq |\varphi(a) - 1|$. For arbitrary $x \in A_1$, we have

$$d(x, \mathscr{F}_\varphi) \leq \|x - a\| + d(a, F_\varphi) \leq \|x - a\| + |\varphi(a) - 1|$$

and since $|\varphi(x) - \varphi(a)| \leq \|x - a\|$, we have $d(x, \mathscr{F}_\varphi) \leq \|x - a\| + |\varphi(x) - 1| + \|x - a\|$. As $a = y^* y$ and y was arbitrary, we conclude

$$d(x, \mathscr{F}_\varphi) \leq |\varphi(x) - 1| + 2 \inf_{y \in A_1} \|x - y^* y\| = \psi(x).$$

□

2.5 Selfless C^*-probability Spaces

This section contains work from [20], wherein the author found multiple new applications of the model theory of C^*-probability spaces to the problem of selflessness.

Definition 2.13. We call an embedding $\theta : (B, \rho) \hookrightarrow (A, \varphi)$ *existential* if for every quantifier-free formula $\Phi(\bar{x}, \bar{y})$ over tuples $\bar{x}, \bar{y}$ in the unit ball,

$$\inf_{\bar{y} \in B_1} \Phi(\bar{a}, \bar{y})^{(B,\rho)} = \inf_{\bar{y} \in A_1} \Phi(\theta(\bar{a}), \bar{y})^{(A,\varphi)}.$$

It is worth mentioning that this is equivalent to the condition that there exists an embedding $\sigma : (A, \varphi) \hookrightarrow (B, \rho)^{\mathscr{U}}$ for some ultrafilter $\mathscr{U}$ such that $\sigma\theta$ is the diagonal embedding $\iota : B \hookrightarrow B^{\mathscr{U}}$,

though we never use this characterization.
A C^*-probability space (A, φ) is called *selfless* if $A \neq \mathbb{C}$, φ induces faithful GNS, and the first factor embedding $(A, \varphi) \hookrightarrow (A, \varphi) * (A, \varphi)$ is existential.

In this section, we will make frequent use of the Cuntz order. For $a, b \in A_+$, we write $a \precsim b$ if there exists $z_n \in A$ with $\|z_n^* b z_n - a\| \to 0$. The symmetrization of $\precsim$ is an equivalence relation, and we denote by $[a]$ the class of $a \in A_+$.

Definition 2.14. Let J be infinite. For each $j \in J$, let (A_j, φ_j) be a C^*-probability space with nonfaithful φ_j inducing a faithful GNS representation. Define $(C, \omega) = *_{j \in J}(A_j, \varphi_j)$. As φ_j is nonfaithful, fix $a_j \in (A_j)_+$ with $\|a_j\| = 1$, $\varphi_j(a_j) = 0$. For $F \subseteq J$ finite, define $a_F = \sum_{j \in F} a_j \in C$ after identifying $A_j \subseteq C$. Finally, let $I_j = \mathrm{Ideal}_{A_j}(\ker(\varphi_j)_+)$ and $\delta_j = \|\varphi_j|_{I_j}\|$.

The following lemma is the main driver of theorem 2.16. We sacrifice the beauty of its statement to parse the steps involved.

Lemma 2.15. *Let J, (A_j, φ_j) be as in the preceding definition.*

1. *For all $i, j \in J$, $a_i \precsim a_j$. That is, $\alpha = [a_j]$ is independent of j.*
2. *For all nonzero $b \in C_+$, $m \in \mathbb{N}$, $m\alpha \leq [b]$.*
3. *$I = \mathrm{Ideal}_C(\alpha)$ is the minimal essential ideal of C.*
4. *I is purely infinite, simple.*

Proof. On the free product Hilbert space

$$H = \mathbb{C}\xi \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{j_1 \neq \cdots \neq j_k} \overset{\circ}{H}_{j_1} \otimes \cdots \otimes \overset{\circ}{H}_{j_k},$$

a_j annihilates the cyclic vector ξ and the subspace of centered words whose first letter does not come from A_j and outputs centered words with first letter from A_j. As the free product representation is faithful, $a_i a_j = 0$ whenever $i \neq j$.
Fix $i, j \in J$. If $i = j$, (i) is trivial, so assume $i \neq j$. Since φ_j induces faithful GNS, fix $v \in A_j$ such that $t = \omega(v^* a_j v) = \varphi_j(v^* a_j v) > 0$. We can express $v^* a_j v = t + c_0$, where c_0 is a centered element of A_j. Then $a_i v^* a_j v a_i = a_i(t + c_0)a_i = t a_i^2$, so $a_i \precsim a_i^2 \precsim a_j$.
For (2), fix nonzero $b \in C_+$ and $m \in \mathbb{N}$. Again, as the representation of ω is faithful, fix $v \in C$ such that $t = \omega(v^* b v) > 0$, let $c = v^* b v$, and note $c \precsim b$. Fix finite $G \subseteq J$ sufficiently large so that there is self-adjoint $d \in *_{i \in G}(A_i, \varphi_i)$ with $\|c - d\| < \frac{t}{3}$, so that $\omega(d) \geq \frac{2t}{3}$. We can express $d = \omega(d) + d_0$, where d_0 is a limit of a combination of words comprised of centered letters from the factors in G. Then for any $F \subseteq J \setminus G$ with $|F| = m$,

$$a_F d a_F = a_F(\omega(d) + d_0)a_F = \omega(d) a_F^2$$

by the same reasoning as in the preceding paragraph. Then

$$a_F c a_F \geq a_F d a_F + a_F(c-d)a_F \geq \omega(d)a_F^2 - \|c-d\|a_F^2 \geq \frac{t}{3}a_F^2,$$

so $a_F \precsim a_F^2 \precsim c \precsim b$. For any two distinct $i, j \in F$, $a_i a_j = 0$, so $[a_F] = m\alpha$. By (2), if K is a nontrivial ideal of C, $b \in K_+ \setminus \{0\}$ has $\alpha \leq [b]$, so $I \subseteq K$, so (3) follows.
Finally, for (4), since a_j is full in I, for all nonzero $b \in I_+$, $[b] \leq \infty\alpha$. Combining this with (2), some standard Cuntz semigroup analysis grants $2[b] \leq [b]$, so I is purely infinite, simple. □

This proof illuminates our mechanism for purely infinite simplicity. While nonfaithfulness tends to be a difficulty when proving purely infinite simplicity for finitely many factors, it actually becomes the vehicle when there are infinitely many factors, as we can use tails consisting of positive kernel elements to excise arbitrary positive elements, allowing for robust comparison under $\precsim$.

Theorem 2.16. *Let J be infinite. For each $j \in J$, let (A_j, φ_j) be a C^*-probability space with nonfaithful φ_j inducing faithful GNS.*

1. *Then $*_{j\in J}(A_j, \varphi_j)$ contains an essential and purely infinite, simple ideal.*
2. *If $\sum_{j\in J} \delta_j > 1$, then $*_{j\in J}(A_j, \varphi_j)$ is purely infinite, simple.*

Proof. Let φ_j be nonfaithful. It suffices to show that I is full in $C = *_{j\in J}(A_j, \varphi_j)$. First, note by 2.15(3) that I is the unique minimal essential ideal of C, so is independent of the choice of a_j in definition 2.14. Therefore, $\ker(\varphi_j)_+ \subseteq I$ and in particular, $I_j \subseteq I$ for all $j \in J$.
Choose finite $F \subseteq J$ such that $\sum_{j\in F} \delta_j > 1$. Then, for each $j \in F$, there exists $e_j \in (I_j)_+$ such that $\|e_j\| = 1$ and $\sum_{j\in F} \varphi_j(e_j) > 1$. Suppose, towards contradiction, that $c = \sum_{j\in F} e_j \not\geq \epsilon 1_C$ for any $\epsilon > 0$. Then there are unit vectors $\eta_n \in H$ such that $\langle c\eta_n, \eta_n\rangle \to 0$, so $\|e_j\eta_n\|^2 = \langle e_j^2\eta_n, \eta_n\rangle \leq \langle e_j\eta_n, \eta_n\rangle \to 0$ for all $j \in F$. Let $q_j = 1_C - e_j$.
Consider the subspace

$$H(j) = \mathbb{C}\xi \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{j\neq j_1\neq\cdots\neq j_k} \mathring{H}_{j_1} \otimes \cdots \otimes \mathring{H}_{j_k}.$$

There is a canonical isomorphism $H \cong H_j \otimes H(j)$ by $\xi_j \otimes \eta \mapsto \eta$ and $\zeta \otimes \eta \mapsto \zeta \otimes \eta$ for $\zeta \in \mathring{H}_j$, $\eta \in H(j)$, where ξ_j is the cyclic vector associated to the j-th factor. Let E_j be the projection of H onto $H(j)$, given by $E_j = P_{\mathbb{C}\xi_j} \otimes 1$. Hence $\|E_j q_j\| = \|P_{\mathbb{C}\xi_j} q_j\| = \|q_j\xi_j\|$, so

$$\|E_j\eta_n\| \leq \|E_j q_j \eta_n\| + \|E_j e_j \eta_n\| \leq \|q_j\xi_j\| + \|e_j\eta_n\|,$$

which grants

$$\limsup_{n\to\infty} \|E_j\eta_n\|^2 \leq \|q_j\xi_j\|^2 = \varphi_j((1-e_j)^2) \leq 1 - \varphi_j(e_j).$$

Fix an arbitrary unit vector $\eta \in H$ and express $\eta = \eta^0 + \sum_{j\in F} \eta^j + \eta^c$, where $\eta^0 \in \mathbb{C}\xi$, η_j begins with first letter from the j-th factor, and no component of η^c has first letter in the F-th factors. As

E_j only annihilates η^j, then $\|E_j\eta\|^2 = 1 - \|\eta^j\|^2$, so

$$\sum_{j\in F} \|E_j\eta\|^2 = |F| - \sum_{j\in F} \|\eta^j\|^2 \geq |F| - 1.$$

Then for $\eta = \eta_n$,

$$|F| - 1 \leq \limsup_{n\to\infty} \sum_{j\in F} \|E_j\eta_n\|^2 \leq \sum_{j\in F} \limsup_{n\to\infty} \|E_j\eta_n\|^2 \leq \sum_{j\in F}(1 - \varphi_j(e_j)),$$

so $\sum_{j\in F} \varphi_j(e_j) \leq 1$, a contradiction. Hence $c \geq \epsilon 1_C$ for some $\epsilon > 0$, so $1_C \precsim c$. Therefore $1_C \in I$, so by 2.15(4), $C = I$ is purely infinite, simple.

□

The following resolution of the selfless dichotomy involves our main model-theoretic trick.

Theorem 2.17. *Let (A, φ) be a selfless C^*-probability space with φ nonfaithful. Then A is purely infinite, simple.*

Proof. Assume (A, φ) is selfless with φ nonfaithful. As $\delta_j > 0$ is constant, $\sum_{j=1}^{\infty} \delta_j = \infty > 1$, so by theorem 2.16, $(C, \omega) = *_{j=1}^{\infty}(A, \varphi)$ is purely infinite, simple. As noted in [14, 3.13.7], purely infinite, simple C^*-algebras are axiomatized by

$$\sup_{x\in(C_+)_1} \sup_{y\in(C_+)_1} \min\{\|x\| \dot{-} \frac{1}{2}, \inf_{z\in C_1} \|2zxz^* - y\|\} = 0,$$

where $s \dot{-} t = \max\{s - t, 0\}$. By [28, 2.6(vi)], the first factor embedding $\theta : (A, \varphi) \hookrightarrow (C, \omega)$ is existential. Fix $a, b \in (A_+)_1$. Then

$$\inf_{z\in A_1} \min\{\|a\| \dot{-} \frac{1}{2}, \|2zaz^* - b\|\} = \inf_{z\in C_1} \min\{\|\theta(a)\| \dot{-} \frac{1}{2}, \|2z\theta(a)z^* - \theta(b)\|\} = 0.$$

As $a, b \in (A_+)_1$ were arbitrary, A is purely infinite, simple. □

A bit of model theory also allows us to remove a separability assumption in [28, 4.2].

Proposition 2.18. *Let (A, φ) be a selfless C^*-probability space, (B, ψ) inducing faithful GNS. Then $(A, \varphi) * (B, \psi)$ is selfless.*

Proof. If B is separable, then [28, 4.2] grants us the conclusion. Otherwise, express (B, ψ) as the direct limit of its separable elementary submodels $(B_\lambda, \psi_\lambda)$, where $\psi_\lambda = \psi|_{B_\lambda}$.
We claim that ψ_λ is GNS-faithful. Let $\theta : (B_\lambda, \psi_\lambda) \hookrightarrow (B, \psi)$ be the elementary embedding. Fix $a \in B_\lambda$ and suppose $\pi_{\psi_\lambda}(a) = 0$. Then

$$0 = \|\pi_{\psi_\lambda}(a)\| \geq \sup_{y\in(B_\lambda)_1} \|ay\|_\psi = \sup_{y\in B_1} \|\theta(a)y\|_\psi.$$

Thus, for all $y \in B$, $\|\theta(a)y\|_\psi = 0$, so $\|\pi_\psi(\theta(a))\| = 0$. As π_ψ is faithful, $\theta(a) = 0$, so $a = 0$. Since $a \in B_\lambda$ was arbitrary, π_{ψ_λ} is faithful. Hence, $(A, \varphi) * (B_\lambda, \psi_\lambda)$ is selfless. Then, as $(A, \varphi) * (B, \psi)$ is the direct limit of $(A, \varphi) * (B_\lambda, \psi_\lambda)$, $(A, \varphi) * (B, \psi)$ is selfless by [28, 4.1]. $\square$

3 Classification of Saturation

The notion of degree-n saturation [13, 15.2.1] was first introduced in the study of massive corona C^*-algebras. Most notably, it was demonstrated that coronas of nonunital, σ-unital C^*-algebras are countably degree-1 saturated, but not always countably degree-2 saturated. Using a power reduction trick, it was demonstrated that degree-2 saturation implies quantifier-free saturation, hence the classification only had 3 distinct classes: $n = 0, 1$, and quantifier-free. As we only added a predicate (scheme) to $\mathscr{L}$, there is a natural analog of this classification for C^*-probability spaces, which likewise has only finitely many distinct classes.[1]

3.1 Saturated C^*-algebras

Definition 3.1. Let $n, m \in \mathbb{N} \cup \{\infty\}$. We call a condition $\psi(\bar{x}) = r$ a *degree-n norm condition* if $\psi(\bar{x}) = \|P(\bar{x})\|$ for a degree-n *-polynomial, $r \geq 0$. For the sake of uniformity, we call such a condition degree-∞ when we do not want to distinguish the degree of n.
We call $\psi(\bar{x}) = s$ a *degree-n state condition* if $\psi(\bar{x}) = \varphi(P(\bar{x}))$ for a degree-n *-polynomial $P(\bar{x})$, $s \in \mathbb{C}$. We call a type $t(\bar{x})$ a *degree-(m, n) type* if each of its conditions are either degree-m norm conditions or degree-n state conditions.
Most generally, we express a countable degree-(m, n) type with

$$t(\bar{x}) = \{\rho_j(P_j(\bar{x})) = s_j : j \in \mathbb{N}\}$$

where ρ_j stands in for either $\|\cdot\|$ or φ and $P_j(\bar{x})$ is a *-polynomial of the appropriate degree, $s_j \in \mathbb{C}$. Such a type is *satisfiable* if for every $F \Subset \mathbb{N}$, $\epsilon > 0$, there exists $\bar{b}$ s.t.

$$\max_{j \in F} \|\rho_j(P_j(\bar{b})) - s_j\| < \epsilon.$$

It is *realized* if there exists $\bar{b}$ s.t. $\rho_j(P_j(\bar{b})) = s_j$ for all $j \in \mathbb{N}$. We call a C^*-probability space (C, φ) *countably degree-(m, n) saturated* if every satisfiable countable degree-(m, n) type is realized in C. We call it *properly countably degree-(m, n) saturated* if it is not countably degree-(m', n') saturated for any $m' \geq m$ and $n' \geq n$ if one inequality is strict.

As in the degree-n saturation of C^*-algebras, we have several redundancies in this classification scheme.

Proposition 3.2. *Let $m \in \mathbb{N} \cup \{\infty\}$, C a C^*-algebra, and $\varphi \in S(C)$.*

1. *If (C, φ) is countably degree-$(2, 0)$ saturated, then it is countably degree-$(\infty, 0)$ saturated.*
2. *If (C, φ) is countably degree-$(2, 1)$ saturated, then it is countably degree-(∞, ∞) (quantifier-free) saturated.*
3. *If (C, φ) is countably degree-$(m, 3)$ saturated, then it is countably degree-(m, ∞) saturated.*

[1] At the time of writing, there are at least 8 and at most 10.

Proof. To show the first two statements, we will demonstrate how basic degree-3 conditions can be unraveled into other conditions. The general proof is exactly the same, with (a lot of) extra bookkeeping.
For (1), consider a satisfiable condition $\|x^3\| = r$. We can split this up into the following 3 conditions

$$\begin{aligned} 0 &= \|y - x^2\|, \\ 0 &= \|z - xy\|, \\ r &= \|z\|. \end{aligned}$$

If $b \in A$ of the appropriate sort has $|\|(b^3)\| - r| < \epsilon$, then $(x, y, z) = (b, b^2, b^3)$ has

$$|\|(b^3)\| - r| < \epsilon, \|(b^3) - (b)(b^2)\| = 0, \|(b^2) - (b)^2\| = 0$$

so satisfies this new degree-$(2, 0)$ type. Therefore every satisfiable countable degree-$(\infty, 0)$ type is realized.
For (2), we also observe the condition $\varphi(x^3) = s$ and split it as

$$\begin{aligned} 0 &= \|y - x^2\|, \\ 0 &= \|z - xy\|, \\ s &= \varphi(z), \end{aligned}$$

and proceed exactly as in (1), leaving a degree-$(2, 1)$ type. This shows that all satisfiable types consisting of atomic formulas are realized. It is a basic compactness argument that this is sufficient to show that all satisfiable quantifier-free types are realized (regardless of metric language).
For (3), fix a satisfiable degree-(m, ∞) type

$$t(\bar{x}) = \{\rho_j(P_j(\bar{x})) = s_j : j \in \mathbb{N}\}.$$

Fix $\bar{b}_r$ such that $\max_{j \leq r} |\rho_j(P_j(\bar{b}_r)) - s_j| \leq \frac{1}{r}$. Fix a free ultrafilter $\mathscr{U}$ on $\mathbb{N}$ and set $(D, \psi) = (C, \varphi)^{\mathscr{U}}$ and $\bar{b} = (\bar{b}_r) \in D$. Let X be the set of coefficients of each P_j. Define $A = C^*(1, X)$, $B = C^*(1, \bar{b}, X)$, where by $\bar{b}$, we mean its entries. Then there exist countable subgroups H of $U(A)$, G of $U(B)$ such that $H \leq G$, $\mathbb{C}H$ is dense A, $\mathbb{C}G$ is dense in B and every entry of $(\bar{b})_i$ (the i-th entry of $\bar{b}$) is a finite linear combination of elements in G.
For each $g \in G$, let $y_g \in C_1$ with $y_g = g$ for $g \in H$ and y_g is a variable in C_1 otherwise. For $g, h, k \in G$, define $s_{ghk} = \psi(k^* gh)$ and consider the type

$$t'(\bar{y}) = \{\varphi(y_k^* y_g y_h) = \psi(k^* gh) : g, h, k \in G\}.$$

This is a satisfiable countable degree-$(0, 3)$ type, so is realized by $\bar{c}$. Let $\eta_g = \pi_\varphi(c_g)\xi_\varphi$, $K =$

$\overline{\text{span}}\{\eta_g : g \in G\}$. Since $\mathbb{C}G$ is dense in B, there is an isometry $V : H_{\psi\restriction B} \to K$ with

$$V\pi_\psi(g)\xi_\psi = \eta_g, \text{ for } g \in G.$$

Particularly, $V\xi_\psi = \xi_\varphi$ and

$$\begin{aligned}\langle \pi_\varphi(c_g)\eta_h, \eta_k\rangle &= \varphi(c_k^* c_g c_h)\\ &= \psi(k^* g h)\\ &= \langle V\pi_\psi(g)V^*\eta_h, \eta_k\rangle.\end{aligned}$$

Since the RHS is unitary, $P_K\pi_\varphi(c_g)P_K = V\pi_\psi(g)V^*$, so K reduces $\pi_\varphi(c_g)$, thus $\pi_\varphi(c_g)P_K = V\pi_\psi(g)V^*$. By linearity and continuity, K reduces $\pi_\varphi(a)$ and $\pi_\varphi(a)P_K = V\pi_\psi(a)V^*$ for every $a \in A$.
Express $(\bar{b})_i = \sum_{l=1}^{n_i} \lambda_{il} g_{il}$ and define $Q_i(\bar{y}) = \sum_{l=1}^{n_i} \lambda_{il} y_{g_{il}}$, $a_i = Q_i(\bar{c})$.
First, note that (2) resolves the case $m \geq 2$, so it suffices to show the claim for $m \in \{0, 1\}$. Case 1: $m = 0$. We must ensure a realization is in the prescribed r_i-ball of $(\bar{x})_i$. Define

$$f_i(t) = \begin{cases} 1, & t \leq r_i^2; \\ r_i t^{-\frac{1}{2}}, & t > r_i^2; \end{cases}$$

and define $d_i = a_i f_i(a_i^* a_i)$, so $\|d_i\| \leq r_i$. We claim $\bar{d} = (d_i)$ realizes $t(\bar{x})$. Since K reduces $\pi_\varphi(d_i)$ and $\pi_\varphi(a_i)P_K = V\pi_\psi(b_i)V^*$, we have $\pi_\varphi(d_i)P_K = V\pi_\psi(b_i)V^*$. Then, for every *-polynomial P with coefficients in X,

$$\pi_\varphi(P(\bar{d}))P_K = V\pi_\psi(P(\bar{b}))V^*.$$

Since, $V\xi_\psi = \xi_\varphi$, we obtain $\varphi(P_j(\bar{d})) = \psi(P_j(\bar{b})) = s_j$ for all j. In particular, $\bar{d}$ realizes $t(\bar{x})$.
Case 2: $m = 1$. Let $J = \{j \in \mathbb{N} : \rho_j = \|\cdot\|\}$. Define the degree-$(1, 3)$ type

$$t''(\bar{y}) = t'(\bar{y}) \cup \{\|Q_i(\bar{y})\| \leq r_i, \|P_j(\bar{Q}(\bar{y}))\| = s_j : i \in \mathbb{N}, j \in J\}.$$

Let $\bar{c}$ realize $t''(\bar{y})$. We claim $\bar{d} = \bar{Q}(\bar{c})$ realizes $t(\bar{x})$. Indeed, $\|d_i\| \leq r_i$, so it is in the correct sort. Also, for $j \in J$, $\|P_j(\bar{d})\| = s_j$. By the above reasoning, for every $j \in \mathbb{N} \setminus J$, $\varphi(P_j(\bar{d})) = s_j$, so $\bar{d}$ realizes $t(\bar{x})$. □

The proposition 3.2 greatly simplifies the variations. There are at most 10, as depicted in the following table.

	0	1	2	3
0	$(0,0)$	$(0,1)$	$(0,2)$	$(0,\infty)$
1	$(1,0)$	$(1,1)$	$(1,2)$	$(1,\infty)$
2	$(\infty,0)$	(∞,∞)		

Figure 1: All 10 degree-(m, n) saturation variations

3.2 Known Examples

Let us first demonstrate C^*-probability spaces with no saturation whatsoever, i.e. properly countable degree-$(0,0)$ saturation.

Example 3.3. *Let H be an infinite dimensional Hilbert space, $\xi \in H$ with $\|\xi\| = 1$ and $\varphi = \omega_\xi$ on $\widetilde{\mathscr{K}(H)}$. Then $(\widetilde{\mathscr{K}(H)}, \varphi)$ is properly countably degree-$(0,0)$ saturated. Indeed, since it is separable, by* [13, 15.6.4]*, it is not degree-$(1,0)$. To see it is not degree-$(0,1)$, fix an orthonormal basis ξ_j for H with $\xi_0 = \xi$. Let u_n be a unitary swapping ξ_0 with ξ_n. Then*

$$t(x) = \{\varphi(u_{2n+1}^* x u_{2n+1}) = 0, \varphi(u_{2n}^* x u_{2n}) = 1 : n \in \mathbb{N}\}$$

is satisfiable by $b_m = \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_{2j} : j \leq m\}}$ but is clearly not realized.

There are also simple examples with no norm saturation, but a great deal of state saturation, i.e. properly degree-$(0,\infty)$.

Example 3.4. *Let (M,τ) be an infinite dimensional tracial von Neumann algebra. Then the tracial ultrapower $(M,\tau)_{\mathscr{U}}$ (interpreted as a C^*-probability space) is properly countably degree-$(0,\infty)$ saturated. To see this, $(M,\tau)_{\mathscr{U}}$ is countably saturated in the language of tracial von Neumann algebras (see* [15, 2.3.2]*), so is saturated over countable types consisting of formulas $\tau(P(\bar{x})) = s$ for $P(\bar{x})$ a *-polynomial and $s \in \mathbb{C}$. However, it is known that an infinite dimensional, countably degree-1 saturated C^*-algebra cannot be a von Neumann algebra* [13, 15.6.4]*, so $(M,\tau)_{\mathscr{U}}$ is not countably degree-$(1,\infty)$ saturated.*

Example 3.5. *As we will see in the next section, there exist pure states φ on $\mathscr{Q}(H)$ for which $(\mathscr{Q}(H), \varphi)$ is countably degree-$(1,0)$ saturated and, under mild set-theoretic assumptions, states for which it is countably degree-$(1,1)$ saturated. Consequently, the canonical lift $\tilde{\varphi}$ to $\mathscr{B}(H)$ of the latter state will make $(\mathscr{B}(H), \tilde{\varphi})$ countably degree-$(0,1)$ saturated.*

We can also, under the continuum hypothesis, guarantee the existence of a properly countably degree-$(1,1)$ saturated example.

Example 3.6. *Let $\mathscr{U}_k$ be a sequence of RK-inequivalent P-points. Let $C = \ell^\infty/c_0$, $\varphi_k(a) = \lim_{j \to \mathscr{U}_k} a_j$ and $\varphi = \sum_{k=1}^\infty 2^{-k}\varphi_k$. Then, as we will see in corollary 5.2, (C,φ) is countably degree-$(1,1)$ saturated. If it were degree-$(2,1)$, it would be degree-(∞,∞), so it suffices to show it is not degree-$(0,2)$. Firstly, note that $H_\varphi = \bigoplus_{j\in\mathbb{N}} H_{\varphi_j} = \bigoplus_{j\in\mathbb{N}} \mathbb{C} = \ell^2$. Fix the obvious basis ξ_j for H_φ with lifts $\xi_j = a_j + L_\varphi$. Suppose, towards contradiction, that (C,φ) is degree-$(0,2)$. Fix the type*

$$t(x) = \{\varphi(x^*x) = 1, \varphi(a_j^* x) = 0 : j \in \mathbb{N}\}.$$

This is satisfiable by taking a_m for large m, so is realized. Thus, there is $\eta \in H_\varphi$ with $\|\eta\| = 1$ and $\langle \eta, \xi_j \rangle = 0$, a contradiction to the fact that (ξ_j) is a basis. This example has another curiosity: As C is a reduced product, (C,φ) is degree-$(\infty,0)$ and degree-$(1,1)$, but not degree-(∞,∞), so it is properly of both classes.

This also gives us a properly degree-$(0,1)$ example: $(\ell^\infty, \tilde{\varphi})$.

We can also, in ZFC, obtain a degree-$(1,\infty)$ example.

Example 3.7. *Let $C = \mathscr{Q}(H) \oplus \mathbb{C}$, $\varphi(a,\lambda) = \lambda$. Then C is countably degree-1 saturated, φ is a P-character, and C is not degree-2 saturated. Therefore, (C,φ) is properly degree-$(1,\infty)$.*

We note that it remains unknown whether there exists a model of ZFC in which there are properly countably degree-$(0,2)$ and $(1,2)$ saturated C^*-probability spaces.

3.3 Full Saturation

We would be remiss not to mention fully saturated C^*-probability spaces.

Definition 3.8. An $\mathscr{L}$-structure M is called countably saturated if every satisfiable type

$$t(\bar{x}) = \{\psi_j(\bar{x}) = r_j : j \in \mathbb{N}\}$$

is realized, where ψ_j is an $\mathscr{L}_M$-formula, and $r_j \in \mathbb{R}$.

The first and most natural examples of saturated objects are ultraproducts.

Theorem 3.9. *For $j \in \mathbb{N}$, let (A_j, φ_j) be a C^*-probability space, $\mathscr{U}$ a free ultrafilter on $\mathbb{N}$. Then $\prod_{\mathscr{U}}(A_j,\varphi_j)$ is countably saturated.*

Proof. See [13, 16.4.1]. □

As we will see in the next section, we can often build states on sufficiently saturated C^*-algebras which make the corresponding C^*-probability space saturated. Thus, while there is no analog of a reduced product state, it is worthwhile to know reduced product C^*-algebras are countably saturated.

Theorem 3.10. *For $j \in \mathbb{N}$, let A_j be a C^*-algebra. Then $\prod_j A_j / \bigoplus_j A_j$ is countably saturated.*

Proof. See [13, 16.5.1]. □

Since reduced product C^*-algebras C are countably saturated, for any state φ on C such that (C,φ) is degree-$(2,1)$ saturated, (C,φ) is countably degree-(∞,∞) saturated. Since this argument only needs quantifier-free saturation, the following fact is worth mentioning.

Proposition 3.11. *If C is a countably saturated C^*-algebra and A is a separable C^*-subalgebra, then $A' \cap C$ is countably quantifier-free saturated.*

Proof. See [13, 16.5.3]. □

4 Quantum P-points

In this section, we define quantum P-points and prove their possession of a powerful property, P-excision, which empowers them to inherit saturation from their parent algebra. Indeed, we show that this inheritance characterizes quantum P-points. Much of the set-theoretic content of this thesis lies at the end of this section, wherein we prove the existence and independence of P-states.

4.1 Quantum Filters

Before discussing quantum P-points, we must discuss quantum filters in general.

Definition 4.1. As before, given a state φ on a C^*-algebra A, the *quantum filter* associated to φ is the collection

$$\mathscr{F}_\varphi = \{a \in A_{+,1} : \varphi(a) = 1\},$$

where $A_{+,1}$ are the positive, norm 1 elements of A. The *projection filter* associated to φ is the collection

$$\mathscr{P}_\varphi = \{p \in \mathrm{Proj}(A) : \varphi(p) = 1\}.$$

We equip these collections with the order $\ll$ on $A_{+,1}$ by

$$a \ll b \text{ iff } ab = a \text{ (equiv. } ba = a).$$

Note that, if $p \in \mathrm{Proj}(A)$, $a \in A_{+,1}$, then $p \ll a$ iff $p \leq a$, the usual order on $A_{+,1}$.
There is also an abstract definition of a quantum filter, namely a *quantum filter in* A is a subset $\mathscr{F} \subseteq A_{+,1}$ with the property that for any $a_1, \ldots, a_n \in \mathscr{F}$, $\|a_1 \cdots a_n\| = 1$. A *maximal quantum filter* is a quantum filter not properly contained in any other.

Caution: Quantum filters are not always filters in the order-theoretic sense. More precisely, they are rarely downwards directed under $\ll$ or $\leq$, though interestingly they are always very close [13, 5.3.4] under both orderings. Thankfully, the projection filters $\mathscr{P}_\varphi$ studied in this note will be true P-filters under $\leq$.
The beauty of maximal quantum filters is their natural bijective correspondence with pure states, i.e. $\varphi \mapsto \mathscr{F}_\varphi$ is a bijection from pure states to maximal quantum filters [13, 5.3.8]. However, the map $\varphi \mapsto \mathscr{F}_\varphi$ from states to quantum filters is almost never injective or surjective. For example, $\{1\}$ is always a quantum filter, but is only of the form $\mathscr{F}_\varphi$ when φ is faithful and A is unital.

4.2 Excision, P-excision, and Quantum P-points

Quantum filters were first introduced in their relation to excision, a powerful property wherein the value of $|\varphi(b)|$ can be approximated by quantities $\|aba\|$ for $a \in A_{+,1}$.

Definition 4.2. For a state φ on a C^*-algebra A and subset $\mathscr{F} \subseteq A_{+,1}$, $X \subseteq A$, we say that $\mathscr{F}$ excises φ on X if for all $b \in X$,

$$\inf_{a \in \mathscr{F}} \|aba - \varphi(b)a^2\| = 0.$$

Most notably, when φ is pure, $\mathscr{F}_\varphi$ excises φ on the entirety of A [13, 5.2.1]. However, for small subsets $X \subseteq A$, we can usually excise φ using a small portion of $A_{+,1}$. In fact, if X is separable, then there is always some countable $\mathscr{F} \subseteq \mathscr{F}_\varphi$ excising φ on X. This fact was cleverly leveraged in [13, 15.3.8] to demonstrate that, when C is countably degree-1 saturated, for every separable C^*-subalgebra B of C, there exists a single $a \in C_{+,1}$ excising φ on B. This property is quite remarkable, though it has yet to find applications. This is likely due to the fact that a cannot be chosen inside $\mathscr{F}_\varphi$, so it is difficult to extract useful information from this single excising element. This is exactly the problem quantum P-points are made to solve, so let us define them now.

Definition 4.3. Let $(\mathscr{R}, <)$ be a set with a transitive, antisymmetric relation. A *P-filter* $\mathscr{F}$ is a subset of $\mathscr{R}$ which is upwards closed and σ-downwards directed. A *P-point* is an ultrafilter on $\mathbb{N}$ which is a P-filter under $\subseteq^*$, inclusion modulo finite sets. A *quantum P-point* is a maximal quantum filter which is a P-filter under $\ll$. That is, for any countable family $\mathscr{F} \subseteq \mathscr{F}_\varphi$, there is $a \in \mathscr{F}_\varphi$ such that $a \ll b$ for all $b \in \mathscr{F}$. We call φ a *P-state* if $\mathscr{F}_\varphi$ is a quantum P-point. Likewise, we call a representation π of C a *P-representation* if it is equivalent to the GNS-representation corresponding to a P-state.

As mentioned above, the most important quality of quantum P-points is their strong excision property. Namely that for every separable C^*-subalgebra B of C, φ is excised on B by a single $a \in \mathscr{F}_\varphi$. We call this property *P-excision*. On an intuitive level, it is not hard to see why this would make a serious difference, as $a \in \mathscr{F}_\varphi$ allows us to claim $|\varphi(b)| \leq \|aba\|$ for every $b \in C$, not only on the excised subalgebra. This will be crucial when realizing countable degree-$(1,1)$ types in theorem 4.7.

4.3 Equivalences

Let us give some order-theoretic terms relevant to our main equivalence theorem.

Definition 4.4. Let $(\mathscr{F}, \leq)$ be a set with a transitive, antisymmetric relation. A subset $\mathscr{P} \subseteq \mathscr{F}$ is *countably closed* if every decreasing sequence in $\mathscr{P}$ has a lower bound in $\mathscr{P}$. A subset $\mathscr{P}$ of $\mathscr{F}$ is *cofinal* if for every $a \in \mathscr{F}$, there exists $b \in \mathscr{P}$ such that $b \leq a$. Lastly, a totally ordered set $(\mathscr{P}, <)$ with no endpoints is an η_1-set if for all countable $X, Y \subseteq \mathscr{P}$ such that $a < b$ for all $a \in X$, $b \in Y$, there exists $c \in \mathscr{P}$ such that $a < c < b$ for all $a \in X$, $b \in Y$.

Note that our definition of cofinal is sometimes deemed *coinitial*, or, in [13], *downwards cofinal*. Before proceeding, let us define the *quantum filter type*

$$t_{\mathscr{F}}(x) = \{\|x\| = 1, x \geq 0, \varphi(x) = 1\},$$

a finite degree-1 type whose realizations are precisely $\mathscr{F}_\varphi$. Note that for $\|x\| \leq 1$, $x \geq 0$ is a degree-1 property, as it is equivalent to the following generalized degree-1 type: $\{x = x^*, \|x\| \leq 1, \|1 - x\| \leq 1\}$, per [13, 15.2.9].

Because $\ll$ is neither reflexive nor irreflexive, there are many equivalent characterizations of P-filters, as demonstrated below.

Lemma 4.5. *Let A be a unital C^*-algebra and φ a state on A. The following are equivalent:*

1. *$\mathscr{F}_\varphi$ is a P-filter.*
2. *$\mathscr{F}_\varphi$ is countably closed.*
3. *$\mathscr{F}_\varphi$ is cofinal in itself.*

If A has real rank zero, the following are equivalent to (1)-(3)*:*

4. *$\mathscr{P}_\varphi$ is a P-filter.*
5. *$\mathscr{P}_\varphi$ is countably closed.*
6. *$\mathscr{P}_\varphi$ is cofinal in $\mathscr{F}_\varphi$.*

If A is also purely infinite and simple, the following is equivalent to (1)-(6)*:*

7. *$\mathscr{P}_\varphi$ is a filter.*

Proof. (1) $\Rightarrow$ (2) is immediate. For (2) $\Rightarrow$ (3), suppose $\mathscr{F}_\varphi$ is countably closed and fix $a \in \mathscr{F}_\varphi$. Fix continuous $f_n : [0,1] \to [0,1]$ such that $f_n(t) = 0$ for $t < 1 - 2^{-n}$ and $f_n(t) = 1$ for $t > 1 - 2^{-n-1}$. Then $a_n = f_n(a)$ is decreasing, so has lower bound $b \in \mathscr{F}_\varphi$. Then

$$\|(1-a)b\| = \|(1-a)a_n b\| \leq \|(1-a)a_n\| \leq \sup_{t\in[0,1]} (1-t)f_n(t) \leq 2^{-n},$$

so $b \ll a$.
For (3) $\Rightarrow$ (1), suppose $\mathscr{F}_\varphi$ is cofinal in itself and fix (a_n) in $\mathscr{F}_\varphi$. Define $a = \sum_{n=1}^\infty 2^{-n} a_n$. Then $\varphi(a) = 1$, so $a \in \mathscr{F}_\varphi$. By cofinality, there is $b \in \mathscr{F}_\varphi$ with $b \ll a$. Then

$$0 = b(1-a)b = \sum_{n=1}^\infty 2^{-n} b(1-a_n)b$$

but each term of the RHS is positive, so $b(1-a_n)b = 0$ for all n. Thus

$$\|(1-a_n)b\|^2 \leq \|(1-a_n)^{\frac{1}{2}} b\|^2 = \|b(1-a_n)b\| = 0.$$

That is, $b \ll a_n$.
Suppose A has real rank zero. (4) $\Rightarrow$ (5) is trivial. For (5) $\Rightarrow$ (6), suppose $\mathscr{P}_\varphi$ is countably closed and fix $a \in \mathscr{F}_\varphi$. Define f_n as before and consider $a_n = f_n(a)$. By real rank zero [13, 2.7.5], for every n, there is $p_n \in \mathscr{P}_\varphi$ with $a_{2n+2} \ll p_n \ll a_{2n}$. Then (p_n) is decreasing, so has lower bound $p \in \mathscr{P}_\varphi$ and $p \ll a$.

(6) $\Rightarrow$ (3) is trivial. To complete this cycle, it is simplest to show (1) $\Rightarrow$ (4). Suppose $\mathscr{F}_\varphi$ is a P-filter and fix (p_n) in $\mathscr{P}_\varphi$. Then (p_n) has a lower bound $a \in \mathscr{F}_\varphi$. By recursion, we have $a, b, c \in \mathscr{F}_\varphi$ with $c \ll b \ll a \ll p_n$. Then by real rank zero, there is $p \in \mathscr{P}_\varphi$ with $c \ll p \ll a \ll p_n$.
Finally, suppose A is also purely infinite and simple. Apply both directions of [5, 4.1] to acquire (5) $\Leftrightarrow$ (7). $\square$

We state the above lemma separately from theorem 4.7, as it makes no assumptions on A or φ. Before proving our main equivalence theorem, it will be convenient to prove our P-excision claim now.

Lemma 4.6. *Let C be unital, countably degree-1 saturated and φ a P-state. Then φ has P-excision. If C has real rank zero, then the excising element can always be chosen to be a projection.*

Proof. Let B be a separable C^*-subalgebra of C. As noted in [13, 15.3.7], Löwenheim-Skolem grants us a countable $Y \subseteq \mathscr{F}_\varphi$ which excises φ on B. Since $\mathscr{F}_\varphi$ is a quantum P-point, this subset has a lower bound $a \in \mathscr{F}_\varphi$. Then as noted in [13, 15.3.8], for $b \in B$, $c \in Y$,

$$\|aba - \varphi(b)a^2\| = \|a(b - \varphi(b))a\| = \|ac(b - \varphi(b))ca\| \leq \|a\|^2 \|c(b - \varphi(b))c\|,$$

so

$$\|aba - \varphi(b)a^2\| \leq \|a\|^2 \inf_{c \in Y} \|cbc - \varphi(b)c^2\| = 0.$$

Since b was arbitrary, we are done. The second claim follows by lemma 4.5. $\square$

Theorem 4.7. *Let $n \in \{1, \infty\}$. Let C be unital, countably degree-n saturated and φ a pure state on C. The following are equivalent:*

1. *(C, φ) is countably degree-(n, n) saturated.*
2. *$\mathscr{F}_\varphi$ is a quantum P-point.*
3. *$\mathscr{F}_\varphi$ is countably closed.*
4. *$\mathscr{F}_\varphi$ is cofinal in itself.*

If in addition C has real rank zero, the following are equivalent to (1)-(4)*:*

5. *$\mathscr{P}_\varphi$ is a P-filter.*
6. *$\mathscr{P}_\varphi$ is countably closed.*
7. *$\mathscr{P}_\varphi$ is cofinal in $\mathscr{F}_\varphi$.*

If C also has no minimal projections, the following is equivalent to (1)-(7)*:*

8. *Maximal chains in $\mathscr{P}_\varphi \setminus \{1\}$ are η_1-sets.*

If C is also purely infinite and simple, the following is equivalent to (1)-(8)*:*

9. $\mathscr{P}_\varphi$ is a filter.

Proof. For (1) $\Rightarrow$ (2), assume (C,φ) is countably degree-(n,n) saturated and let $Y \subseteq \mathscr{F}_\varphi$ be countable and define the type

$$t(x) = t_{\mathscr{F}}(x) \cup \{ax = x : a \in Y\}.$$

Define an approximate unit (e_γ) for $L_\varphi \cap L_\varphi^*$. Notice that for each $a \in \mathscr{F}_\varphi$, $(1-e_\gamma)a - (1-e_\gamma) = (e_\gamma - 1)(1-a) = e_\gamma(1-a) - (1-a)$. Now fix $F \Subset Y$, $\epsilon > 0$. There exists γ such that for all $a \in F$, $\|e_\gamma(1-a) - (1-a)\| < \epsilon$. Therefore $\|(1-e_\gamma)a - (1-e_\gamma)\| = \|e_\gamma(1-a) - (1-a)\| < \epsilon$, so $(1-e_\gamma)$ satisfies $t(x)$, so it is realized by $b \in \mathscr{F}_\varphi$ such that $b \ll a$ for all $a \in Y$. Since Y was arbitrary, (2) follows.
For (2) $\Rightarrow$ (1), assume $\mathscr{F}_\varphi$ is a quantum P-point and fix a satisfiable degree-(n,n) type

$$t(\bar{x}) = \{\rho_j(P_j(\bar{x})) = s_j : j \in \mathbb{N}\}.$$

By the linearity of φ, for $\rho_j = \varphi$, we can replace $P_j(\bar{x})$ by $P_j(\bar{x}) - d_j$, where $d_j \in C$ such that $\varphi(d_j) = s_j$. Thus, without loss of generality, when $\rho_j = \varphi$, assume $s_j = 0$. Now fix, for each $m \in \mathbb{N}$, $\bar{b}_m$ such that $\max_{j \leq m} |\rho_j(P_j(\bar{b}_m)) - s_j| < \frac{1}{m}$. Let X be the collection of entries of each $\bar{b}_m$ and coefficients of each P_j. Then X is countable, so $B = C^*(X)$ is separable. By lemma 4.6, there exists a single $a \in \mathscr{F}_\varphi$ which excises φ on B. Now define the modified *-polynomials

$$P_j'(\bar{x}) = \begin{cases} aP_j(\bar{x})a, & \rho_j = \varphi; \\ P_j(\bar{x}), & \rho_j = \|\cdot\|; \end{cases}$$

and fix the countable degree-n norm type

$$t'(\bar{x}) = \{\|P_j'(\bar{x})\| = s_j : j \in \mathbb{N}\}$$

and notice $(\bar{b}_m)$ satisfies $t'(\bar{x})$, since for $\rho_j = \varphi$, we have $P_j(\bar{b}_m) \in B$, so P-excision yields

$$\|P_j'(\bar{b}_m)\| = \|aP_j(\bar{b}_m)a\| = |\varphi(P_j(\bar{b}_m))| < \frac{1}{m}.$$

As C is countably degree-n saturated, $t'(\bar{x})$ is realized. When $\rho_j = \varphi$, we have

$$|\varphi(P_j(\bar{b}))| = |\varphi(aP_j(\bar{b})a)| \leq \|aP_j(\bar{b})a\| = \|P_j'(\bar{b})\| = 0.$$

Therefore, $\bar{b}$ realizes $t(\bar{x})$. Since $t(\bar{x})$ was arbitrary, (1) follows.
The equivalences (2)-(4) follow by lemma 4.5. Suppose C has real rank zero. Again, lemma 4.5 grants the equivalences (2)-(7).
Now assume C has no minimal projections. For (6) $\Rightarrow$ (8), suppose $\mathscr{P}_\varphi$ is countably closed and fix a maximal chain $\mathscr{P}$ in $\mathscr{P}_\varphi \setminus \{1\}$. Since $\ll$ coincides with $\leq$ for projections, it will be more convenient

to work with the latter. If X, Y as in definition 4.4 are finite, then consider $p = \min(Y) - \max(X)$, where $\min(\varnothing) = 1, \max(\varnothing) = 0$. Since this cannot be minimal, choose a nonzero projection $q < p$. Then $a \ll \max(X) + q < b$ for all $a \in X, b \in Y$. If $\max(X) + q \notin \mathscr{P}$, then choose $c \in \mathscr{P}$ with $\max(X) + q \not\leq c$ and $c \not\leq \max(X) + q$. Then it must be the case that $a < c < b$ for all $a \in X, b \in Y$. Otherwise, $c = \max(X) + q$ suffices.
If X is infinite, use the type

$$t(x) = \{\|x\| = 1, x \geq 0, ax = a, bx = x, \|x - a\| = \|x - b\| = 1 : a \in X, b \in Y\},$$

and satisfy it by elements in X. This realizes $c_0 \in \mathscr{F}_\varphi$ such that $a \ll c_0 \ll b$ for all $a \in X, b \in Y$. Then use the type

$$t'(x) = \{\|x\| = 1, x \geq 0, ax = a, c_0 x = x, \|x - a\| = \|x - c_0\| = 1 : a \in X\}.$$

By iteration, we have $c_j \in \mathscr{F}_\varphi$ (since they have lower bounds in $X \subseteq \mathscr{F}_\varphi$) with $a \ll c_2 \ll c_1 \ll c_0 \ll b$ for all $a \in X, b \in Y$. By real rank zero, we have $c \in \mathscr{P}_\varphi$ with $a \ll c_2 \ll c \ll c_0 \ll b$. As before, either $c \in \mathscr{P}$ or there is a suitable replacement.
If Y is infinite, use the type

$$t(x) = \{\|x\| = 1, x \geq 0, \varphi(x) = 1, ax = a, bx = x, \|x - a\| = \|x - b\| = 1 : a \in X, b \in Y\},$$

where we have introduced $\varphi(x) = 1$ in case X is empty. By the mirrored iteration, we obtain $c \in \mathscr{P}_\varphi$ with $a \ll c \ll b$ for all $a \in X, b \in Y$, and if $c \notin \mathscr{P}$, we can find a replacement. Hence $\mathscr{P}$ is an η_1-set. Conversely, every decreasing sequence in $\mathscr{P}_\varphi \setminus \{1\}$ is contained in some maximal chain, and clearly adding 1 is benign, so (8) $\Rightarrow$ (6).
Finally, assume C is purely infinite, simple. As before, lemma 4.5 completes the picture. □

Compare condition (8) with [21, 3], which states that maximal chains of projections in $\mathscr{Q}(H)$ are η_1-sets. By the above reasoning, this theorem also holds for any countably degree-1 saturated C with real rank zero and no minimal projections. Now that we have these equivalences, we will simply refer to pure φ such that (C, φ) is countably degree-$(1, 1)$ saturated as a P-state.

4.4 Existence and Independence

The conclusion of theorem 4.7 can be leveraged to show it is relatively consistent with ZFC that there are no P-states on many corona algebras. However, in any model of ZFC which admits a P-point (an extremely weak assumption), we can produce a P-state on many coronas. To accomplish this, we want to mimic the construction of diagonal pure states on $\mathscr{Q}(H)$. As such, we need a generalization of an orthonormal sequence of vectors.

Definition 4.8. Let A be nonunital, σ-unital. We call a sequence (q_j) of projections a *sub-σ-sequence of A* if there is a σ-unit (e_j) of A such that $q_j \leq e_j - e_{j-1}$, $e_{-1} = 0$. For a sub-σ-sequence,

define the map $p : \mathscr{P}(\mathbb{N}) \to \mathrm{Proj}(\mathscr{M}(A))$ by p_X equal to the strict limit of the sequence

$$p_{X,n} = \sum_{\substack{j \in X \\ j \leq n}} q_j,$$

which is strictly Cauchy by the sub-σ property and [13, 13.1.11]. We abuse the notation $p_X = \sum_{j \in X} q_j$. For each $\varphi \in S(\mathscr{Q}(A))$, define

$$\mathscr{X}_\varphi = \{X \subseteq \mathbb{N} : \varphi(\pi(p_X)) = 1\}.$$

We call a sub-σ-sequence *scalar* if q_j is a scalar projection (i.e. $q_j A q_j = \mathbb{C} q_j$) for all j and there is a faithful, nondegenerate representation $\rho : A \to \mathscr{B}(H)$ such that $\rho(q_j)$ has rank one for all j. We call a sub-σ-sequence a *σ-sequence* if $p_{\mathbb{N}} = 1_{\mathscr{M}(A)}$, equivalently if $q_j = e_j - e_{j-1}$ for some σ-unit (e_j) of A.

To justify the definition of a scalar sub-σ-sequence, let $A = \bigoplus_{i \in I} \mathscr{K}(H_i)$ such that $\bigoplus_{i \in I} H_i$ is infinite dimensional, separable. Then A has a scalar sub-σ-sequence. Namely, we identify $A \subseteq \mathscr{B}(\bigoplus_i H_i)$, fix an orthonormal basis (ξ_j) of $\bigoplus_i H_i$ compatible with the direct sum, making $q_j = \mathrm{proj}_{\mathbb{C}\xi_j}$ scalar sub-σ. Any subsequence of a scalar sub-σ-sequence is still scalar sub-σ.
The following lemma was originally proven for the Calkin algebra by Zamora-Avilés [30, 2.5.10]. Here we provide a version generalized to coronas of algebras with a sub-σ-sequence.

Lemma 4.9. *Let A be a nonunital, σ-unital C^*-algebra. For every sub-σ-sequence in A, decreasing sequence $X_n \subseteq \mathbb{N}$, and $a \in \mathscr{Q}(A)_{+,1}$, we have $a \ll \pi(p_{X_n})$ for all n if and only if there exists a pseudointersection $X \subseteq \mathbb{N}$ of (X_n) such that $a \ll \pi(p_X)$.*

Proof. Without loss of generality, suppose $X_1 = \mathbb{N}$ and $X_n \setminus X_{n+1}$ infinite. Enumerate $X_n \setminus X_{n+1} = \{j_i^n : i \in \mathbb{N}\}$. Now define $p^n : \mathscr{P}(\mathbb{N}) \to \mathrm{Proj}(\mathscr{M}(A))$ by $p^n_Y = p_{\{j_i^n : i \in Y\}}$. Fix a lift $\tilde{a} \in \mathscr{M}(A)_{+,1}$ of a, so $a = \pi(\tilde{a}) \ll \pi(p_{X_n})$, and we have

$$\pi(p_{X_n \setminus X_{n+1}} \tilde{a}) = (\pi(p_{X_n}) - \pi(p_{X_{n+1}}))a = a - a = 0.$$

By the sub-σ property, for each m, there is k such that $(1 - e_k)p^n_{\mathbb{N}} = p^n_{\mathbb{N} \setminus m}$. This yields

$$\lim_{k \to \infty} \|p^n_{\mathbb{N} \setminus k} \tilde{a}\| = \lim_{k \to \infty} \|(1 - e_k) p_{X_n \setminus X_{n+1}} \tilde{a}\| = 0,$$

so the function $f_\epsilon \in \mathbb{N}^{\mathbb{N}}$ by

$$f_\epsilon(n) = \min \left\{ k : \|p^n_{\mathbb{N} \setminus k} \tilde{a}\| < \frac{\epsilon}{2^{n+1}} \right\}$$

is well-defined. Fix $f \in \mathbb{N}^{\mathbb{N}}$ such that $f_{\frac{1}{m}} \leq^* f$ for all m. That is, $f_{\frac{1}{m}}(n) \leq f(n)$ for eventually all n. Now we claim that $X = \bigcup_{n \in \mathbb{N}} \{j_i^n : i \leq f(n)\}$ is our desired pseudointersection. Since X_n

decreases from $\mathbb{N}$, $X_n \setminus X_{n+1}$ partitions $\mathbb{N}$, so

$$X \setminus X_n = \bigcup_{k=1}^{n-1} X \cap (X_k \setminus X_{k+1}) = \bigcup_{k=1}^{n-1} \{j_i^k : i \leq f(k)\},$$

so $|X \setminus X_n| = \sum_{j=1}^{n-1} f(j) < \infty$, thus X is a pseudointersection. Now we claim $a \ll \pi(p_X)$, and it suffices to show that $\|(1-e_k)p_{\mathbb{N}\setminus X}\tilde{a}\| \to 0$. By $f_{\frac{1}{m}} \leq^* f$, we have $N \in \mathbb{N}$ such that

$$\begin{aligned}
\|(1-e_k)p_{\mathbb{N}\setminus X}\tilde{a}\| &\leq \sum_{n=1}^{\infty} \|(1-e_k)p^n_{\mathbb{N}\setminus f(n)}\tilde{a}\| \\
&\leq \sum_{n=1}^{N-1} \|(1-e_k)p^n_{\mathbb{N}\setminus f(n)}\tilde{a}\| + \sum_{n=N}^{\infty} \frac{1}{m2^{n+1}} \\
&\leq \sum_{n=1}^{N-1} \|(1-e_k)p_{X_n\setminus X_{n+1}}\tilde{a}\| + \frac{1}{m} \\
&\leq \frac{1}{m} + \frac{1}{m} = \frac{2}{m}
\end{aligned}$$

by taking k so that $\|(1-e_k)p_{X_n\setminus X_{n+1}}\tilde{a}\| \leq \frac{1}{(N-1)m}$ for all $n < N$. □

The following lemma concretely justifies quantum P-points as the noncommutative analog of P-points.

Theorem 4.10. *Let A be nonunital, σ-unital.*

1. *For every sub-σ-sequence, there exists a natural $\varphi : \beta\mathbb{N} \setminus \mathbb{N} \hookrightarrow P(\mathscr{Q}(A))$ such that $\mathscr{X}_{\varphi_{\mathscr{U}}} = \mathscr{U}$ and $\varphi_{\mathscr{U}}$ is a P-state only if $\mathscr{U}$ is a P-point.*

2. *For every scalar sub-σ-sequence, there exists a canonical $\varphi : \beta\mathbb{N} \setminus \mathbb{N} \hookrightarrow P(\mathscr{Q}(A))$ such that $\mathscr{X}_{\varphi_{\mathscr{U}}} = \mathscr{U}$ and $\varphi_{\mathscr{U}}$ is a P-state if and only if $\mathscr{U}$ is a P-point.*

Proof. Let (q_j) be a sub-σ-sequence. Fix $\varphi_j \in P(A)$ such that $\varphi_j(q_j) = \|q_j\| = 1$. By [27, 3.3.9], these each have unique extensions $\tilde{\varphi_j} \in S(\mathscr{M}(A))$ and $\tilde{\varphi_j}$ is pure by [13, 5.4.3]. Now define $\varphi : \beta\mathbb{N} \setminus \mathbb{N} \hookrightarrow S(\mathscr{Q}(A))$ by $\varphi_{\mathscr{U}}(a) = \lim_{j\to\mathscr{U}} \tilde{\varphi_j}(\tilde{a})$, for any lift $\tilde{a} \in \mathscr{M}(A)$ of a. We know these are states because they lift to weak*-limits of pure states on $\mathscr{M}(A)$ and they are pure by [2, 1].
Suppose $\mathscr{U}$ is not a P-point, and fix decreasing $X_n \in \mathscr{U}$ with no pseudointersection in $\mathscr{U}$. Our claim is that $\pi(p_{X_n})$ has no lower bound in $\mathscr{F}_{\varphi_{\mathscr{U}}}$. By 4.9, any lower bound $a \in \mathscr{F}_{\varphi_{\mathscr{U}}}$ of $(\pi(p_{X_n}))$ must in fact have $a \ll \pi(p_X)$, where X is a pseudointersection of (X_n), therefore no such a exists.
If (q_j) is also scalar, there is a unique $\varphi_j \in P(A)$ such that

$$\mathscr{F}_{\varphi_j} = \{a \in A_{+,1} : q_j \leq a\}.$$

This is the sense in which the construction is canonical. We define $\varphi : \beta\mathbb{N}\setminus\mathbb{N} \hookrightarrow P(\mathscr{Q}(A))$ as before.

Suppose $\mathscr{U}$ is a P-point. Fix faithful nondegenerate $\rho : A \to \mathscr{B}(H)$ such that $\rho(q_j)$ rank one. There is a unique extension $\tilde{\rho} : \mathscr{M}(A) \to \mathscr{B}(H)$, per [13, 13.2.1]. Fix $a \in \mathscr{F}_{\varphi_{\mathscr{U}}}$ and a lift $\tilde{a} \in \mathscr{M}(A)_{+,1}$ of a.

Now, let

$$X_n = \{k \in \mathbb{N} : \tilde{\varphi_k}(\tilde{a}) > 1 - \frac{1}{n}\}.$$

Then $X_n \in \mathscr{U}$ and $q_k(1-\tilde{a})q_k = (1 - \tilde{\varphi_k}(\tilde{a}))q_k$, so $\|q_k(1-\tilde{a})q_k\| \leq \frac{1}{n}$. We give a multiplier version of Anderson paving. Since q_j is scalar, write $q_j(1-\tilde{a})q_j = \lambda_j q_j$ and $c = \sum_{j \in X_n} \lambda_j q_j$, $d = p_{X_n}(1-\tilde{a})p_{X_n} - c$. Then $q_j d q_j = 0$ for all $j \in X_n$. The separable Anderson paving noted in [26] gives a finite partition $X_n = \bigsqcup_{j=1}^{r_n} X_{nj}$ such that for all $j \leq r_n$, $\|p_{X_{nj}} d p_{X_{nj}}\| \leq \frac{1}{n}$. Thus, $\|p_{X_{nj}}(1-\tilde{a})p_{X_{nj}}\| \leq \frac{2}{n}$. Since $\mathscr{U}$ is an ultrafilter, for some $j \leq r_n$, $Y_n = X_{nj} \in \mathscr{U}$. Moreover, as $\mathscr{U}$ is a P-point, choose $X \in \mathscr{U}$ such that $X \subseteq^* Y_n$ for all n. Then

$$\|\pi(p_X(1-\tilde{a})p_X)\| \leq \|\pi(p_{Y_n}(1-\tilde{a})p_{Y_n})\| \leq \|p_{Y_n}(1-\tilde{a})p_{Y_n}\| \leq \frac{2}{n}.$$

Finally,

$$\|(1-a)\pi(p_X)\|^2 \leq \|(1-a)^{\frac{1}{2}}\pi(p_X)\|^2 = \|\pi(p_X(1-\tilde{a})p_X)\| \leq \frac{2}{n},$$

so $\pi(p_X) \ll a$. As a was arbitrary, $\mathscr{F}_{\varphi_{\mathscr{U}}}$ is a quantum P-point. □

We write $\mathfrak{c} = 2^{\aleph_0}$. The continuum hypothesis is the axiom $\mathfrak{c} = \aleph_1$. The existence of a P-point appears extremely common in models of ZFC. In fact, until recently [8], there were very few known models of ZFC which have no P-points. Their existence is implied by the continuum hypothesis, is consistent with various standard forcing axioms, and even OCA_T. For the operator algebraist reader who does not fancy the continuum hypothesis, rest assured that P-states are likely relatively consistent with whichever axioms one fancies most. If $\mathscr{Q}(A)$ as in theorem 4.10 is simple, the existence of just one P-point implies that P-states are weakly*-dense in $S(\mathscr{Q}(A))$. More generally, we have the following result.

Theorem 4.11. *If there exists a P-state φ on C such that π_φ is faithful and essential, then the set of P-states is weakly*-dense in $S(C)$.*

Proof. Since π_φ is faithful and essential, by Glimm's lemma [13, 5.2.6], the set of vector states

$$\{\omega_\xi \circ \pi_\varphi : \xi \in H_\varphi, \|\xi\| = 1\}$$

is weakly*-dense in $S(C)$. By the Kadison transitivity theorem [13, 3.5.4], there exists a unitary $u \in C$ such that $\pi_\varphi(u)\xi_\varphi = \xi$, so $\omega_\xi \circ \pi_\varphi = \varphi \circ \mathrm{Ad}(u)$, so each state in this set is a P-state. □

The continuum hypothesis allows us to directly construct many P-states on a huge class.

Theorem 4.12. *Assume the continuum hypothesis. Let C be countably degree-1 saturated with no minimal projections. If $|C| = \mathfrak{c}$, there exist $2^\mathfrak{c}$-many unitarily inequivalent P-states on C.*

Proof. By the continuum hypothesis, enumerate $C_{+,1} = \{b_\alpha : \alpha < \aleph_1\}$. We will recursively construct a tree $(\mathscr{F}_s : s \in 2^{<\aleph_1})$ of countable quantum filters such that, for $\beta < \aleph_1, s \in 2^\beta$, $\mathscr{F}_s \cup \{b_\beta\}$ is a quantum filter implies $b_\beta \in \mathscr{F}_{s^\frown j}$ and b_β has lower bound in $\mathscr{F}_{s^\frown j}$ for $j = 0, 1$.
Fix $a_0 \in C_{+,1}$ and $\mathscr{F}_\varnothing = \{a_0\}$. If $\beta < \aleph_1$ is a limit, $s \in 2^\beta$, let $\mathscr{F}_s = \bigcup_{\alpha<\beta} \mathscr{F}_{s\restriction\alpha}$. Let $\beta < \aleph_1$, $s \in 2^\beta$. Define $\mathscr{G}_s = \mathscr{F}_s \cup \{b_\beta\}$ if the RHS is a quantum filter and $\mathscr{G}_s = \mathscr{F}_s$ if not.
We claim there exist orthogonal $a_{s0}, a_{s1} \in C_{+,1}$ with $a_{sj} \ll a$ for all $a \in \mathscr{G}_s$. Fix the type

$$t(x) = \{\|x\| = 1, x \geq 0, ax = x : a \in \mathscr{G}_s\}.$$

As $\mathscr{G}_s$ is a quantum filter, extend it to a maximal quantum filter $\mathscr{F}_\psi$. As in the proof of theorem 4.7 (1) $\Rightarrow$ (2), $(1 - e_\gamma)$ satisfies $t(x)$, where (e_γ) is an approximate unit of $L_\psi \cap L_\psi^*$. Let $b \in C_{+,1}$ realize $t(x)$.
If b is a projection, it is not minimal, so choose $c \in (bCb)_{+,1}$ which is not a projection. If b is not a projection, choose $c = b$. Now choose $f_0, f_1 \in C(\sigma(c))_{+,1}$ with disjoint supports such that $f_j(0) = 0$. By approximating $f_j(c)$ with polynomials over c without constant terms, $f_j(c)a = f_j(c)$ for all $a \in \mathscr{G}_s$, so $a_{sj} = f_j(c)$ grants the desired pair.
Let $\mathscr{F}_{s^\frown j} = \mathscr{G}_s \cup \{a_{sj}\}$. As a_{sj} is a lower bound for $\mathscr{G}_s$, $\mathscr{F}_{s^\frown j}$ is a countable quantum filter with the required property. This completes the recursive construction. For $s \in 2^{\aleph_1}$, we claim $\mathscr{F}_s = \bigcup_{\beta<\aleph_1} \mathscr{F}_{s\restriction\beta}$ is a quantum P-point. Fix $b_1, \ldots, b_n \in \mathscr{F}_s$. Then there exists $\beta < \aleph_1$ such that $a_{s\restriction\beta} \ll b_j$ for $j \leq n$, so

$$\|b_1 \cdots b_n\| \geq \|b_1 \cdots b_n a_{s\restriction\beta}\| = \|a_{s\restriction\beta}\| = 1.$$

Thus, $\mathscr{F}_s$ is a quantum filter which is cofinal in itself. To see it is maximal, suppose $\mathscr{F}_s \cup \{b_\beta\}$ is a quantum filter. Then $\mathscr{F}_{s\restriction\beta} \cup \{b_\beta\}$ is as well, so $b_\beta \in \mathscr{F}_{s\restriction\beta+1} \subseteq \mathscr{F}_s$. Therefore $\mathscr{F}_s = \mathscr{F}_{\varphi_s}$ for some pure state φ_s. Moreover, by theorem 4.7, φ_s is a P-state. Since our recursion was chosen to split, the map $2^{\aleph_1} \to P(C)$, $s \mapsto \varphi_s$ is an injection. Since $|C| = \mathfrak{c}$, this family must yield $2^\mathfrak{c}$-many distinct unitary orbits, as $|U(C)| \leq \mathfrak{c}$. □

It should be noted that the assumption of no minimal projections in the last proof was only used to build the tree. The same recursive construction builds at least one P-state on an arbitrary countably degree-1 saturated C^*-algebra C with $|C| = \mathfrak{c}$, under CH. If we weaken the claim of 4.12 to weak*-density in the pure state space, we can achieve full generality.

Corollary 4.13. *Assume the continuum hypothesis. For every countably degree-1 saturated C with $|C| = \mathfrak{c}$, P-states are weakly*-dense in $P(C)$.*

Proof. Fix a pure state ψ, $F \Subset C$, $\epsilon > 0$. Since $\mathscr{F}_\psi$ excises ψ, choose $a \in \mathscr{F}_\psi$ such that $\|aba - \psi(b)a^2\| < \epsilon$ for $b \in F$. Repeat a single branch of the recursion argument in the proof of 4.12,

starting with $\mathscr{F}_\varnothing = \{a\}$. Then we obtain a P-state φ on C with, for $b \in F$,

$$\begin{aligned} |\varphi(b) - \psi(b)| &= |\varphi(aba) - \psi(b)\varphi(a^2)| \\ &= |\varphi(aba - \psi(b)a^2)| \\ &\leq \|aba - \psi(b)a^2\| < \epsilon. \end{aligned}$$

Thus every weak*-open subset of $P(C)$ contains a P-state. □

The problem of finding nondiagonalizable pure states on $\mathscr{Q}(H)$ stood for decades. The first progress was made by Akemann and Weaver, who proved that such states exist under the continuum hypothesis [1] in 2006. In 2020, Koszmider proved the existence of such a state [24]. A modification of the above argument also allows for the construction of nondiagonalizable P-states on $\mathscr{Q}(H)$. We first define the quantum cardinal invariants

$$\mathfrak{p}^* = \min\{|\mathscr{P}| : \mathscr{P} \subseteq \mathrm{Proj}(\mathscr{Q}(H)), \mathscr{P} \text{ is a quantum filter with no lower bound}\}.$$

We build nondiagonalizable P-states using the weakening $\mathfrak{p}^* = \mathfrak{c}$ of the continuum hypothesis, following the argument and notation in [13, 12.5.10].

Theorem 4.14. *Assume $\mathfrak{p}^* = \mathfrak{c}$. There exist $2^\mathfrak{c}$-many unitarily inequivalent nondiagonalizable P-states on $\mathscr{Q}(H)$.*

Proof. By $\mathfrak{p}^* = \mathfrak{c}$, enumerate $\mathrm{Proj}(\mathscr{Q}(H)) = \{q_\alpha : \alpha < \mathfrak{p}^*\}$ and a $\leq^*$-cofinal subset $\{L_\alpha : \alpha < \mathfrak{p}^*\}$ of Part_{ℓ^2} (potentially with repetitions). We will recursively construct a tree $(\mathscr{P}_s : s \in 2^{<\mathfrak{p}^*})$ of quantum filters in $\mathrm{Proj}(\mathscr{Q}(H))$ such that, for $\beta < \mathfrak{p}^*, s \in 2^\beta$,

1. $\mathscr{P}_{s^\frown j}$ has a minimal element p_{sj},
2. $p_{sj} \in \pi[\mathscr{E}(L_\beta^{\mathrm{even}}) \cap \mathscr{E}(L_\beta^{\mathrm{odd}})]$,
3. $\mathscr{P}_s \cup \{q_\beta\}$ is a quantum filter implies $p_{sj} \ll q_\beta$.

Let $\mathscr{P}_\varnothing = \{1\}$. If $\beta < \mathfrak{p}^*$ is a limit, $s \in 2^\beta$, let $\mathscr{P}_s = \bigcup_{\alpha<\beta} \mathscr{P}_{s\restriction\alpha}$. Let $\beta < \mathfrak{p}^*$, $s \in 2^\beta$. Define $\mathscr{G}_s = \mathscr{P}_s \cup \{q_\beta\}$ if the RHS is a quantum filter and $\mathscr{G}_s = \mathscr{P}_s$ if not.
Since $|\mathscr{G}_s| < \mathfrak{p}^*$ and $\mathscr{G}_s$ is a quantum filter, there exists a lower bound $q \in \mathrm{Proj}(\mathscr{Q}(H))$ for $\mathscr{G}_s$. As before, we can find orthogonal $p_{sj} \in \mathrm{Proj}(\mathscr{Q}(H))$ with $p_{sj} \ll p$ for all $p \in \mathscr{G}_s$. By [13, 12.5.13], we can choose $p_{sj} \in \pi[\mathscr{E}(L_\beta^{\mathrm{even}}) \cap \mathscr{E}(L_\beta^{\mathrm{odd}})]$. Let $\mathscr{P}_{s^\frown j} = \mathscr{G}_s \cup \{p_{sj}\}$. This completes the recursive construction. For $s \in 2^{\mathfrak{p}^*}$, we claim

$$\mathscr{F}_s = \{a \in \mathscr{Q}(H)_{+,1} : a \text{ has a lower bound in some } \mathscr{P}_{s\restriction\beta}\}$$

is a quantum P-point. This follows as in 4.12, as does the claim of unitary inequivalence. □

Lemma 4.15. *Let A be nonunital with a σ-sequence (q_j), and $\varphi \in P(\mathscr{Q}(A))$. If φ is a P-state, then $\mathscr{X}_\varphi$ is a non-meagre P-filter. Also, for any downwards directed $X_n \subseteq \mathbb{N}$, there exists a pseudointersection X of (X_n) such that $\varphi(\pi(p_X)) = \inf_n \varphi(\pi(p_{X_n}))$. Moreover, for arbitrary (X_n), if $\varphi(\pi(p_{X_n})) \not\to 0$, there exists (k_n) in $\mathbb{N}$ increasing and $X \subseteq \mathbb{N}$ such that $X \subseteq^* \bigcup_{j\in[k_n,k_{n+1})} X_j$ and $\varphi(\pi(p_X)) > 0$.*

Proof. We mimic the proof of [5, 6.2]. To show that $\mathscr{X}_\varphi$ is non-meagre, take an arbitrary interval partition (I_n) of $\mathbb{N}$ and let B be the A-strict completion of the C^*-subalgebra generated by $\{p_{I_n} : n \in \mathbb{N}\}$. This is isomorphic to ℓ^∞ and its quotient $\pi[B]$ is isomorphic to ℓ^∞/c_0. Then $\mathrm{Proj}(\pi[B])$ is countably closed and atomless. Since (q_j) is a σ-sequence, $\varphi(p_{\mathbb{N}}) = 1$, so φ restricts to a state on $\pi[B]$. By [5, 6.1], $\{1\} \neq \mathscr{P}^B_\varphi \hookrightarrow \mathscr{X}_\varphi$. As (I_n) was arbitrary, $\mathscr{X}_\varphi$ is non-meagre by [13, 9.9.1].
Let (X_n) be downwards directed. If some X_n is finite, then $X = \varnothing$ is the desired pseudointersection. Otherwise, define the type

$$t(x) = \{\|x\| = 1, x \geq 0, \pi(p_{X_n})x = x, \varphi(x) = \inf_n \varphi(\pi(p_{X_n})) : n \in \mathbb{N}\}$$

and satisfy it by $\pi(p_{X_n})$ for sufficiently large n. Its realization is $a \in \mathscr{Q}(A)_{+,1}$ such that $a \ll \pi(p_{X_n})$ and $\varphi(a) = \inf_n \varphi(\pi(p_{X_n}))$. Then by 4.9, there is a pseudointersection X of (X_n) such that $a \ll \pi(p_X)$, so $\inf_n \varphi(\pi(p_{X_n})) = \varphi(a) \leq \varphi(\pi(p_X))$. The other inequality is trivial. It then follows that $\mathscr{X}_\varphi$ is a P-filter.
For the last part of the claim, let $\epsilon = \limsup_{n\to\infty} \frac{\varphi(\pi(p_{X_n}))}{2} > 0$. If

$$\varphi(\pi(p_{X_{m_{i+1}} \setminus \bigcup_{j\in[k,m_i)} X_j})) \geq \frac{\epsilon}{3^{n+1}}$$

for $i < l$, where $k < m_0 < \cdots < m_l$, then $l\epsilon < 3^{n+1}$, i.e. there are only finitely many l for which this may hold. Thus, we may choose k_n such that $\varphi(\pi(p_{X_j})) > \epsilon$ for some $k_n \leq j < k_{n+1}$ and

$$\varphi(\pi(p_{X_m \setminus \bigcup_{j\in[k_n,k_{n+1})} X_j})) < \frac{\epsilon}{3^{n+1}}$$

for all $m \geq k_{n+1}$. Then

$$\varphi(\pi(p_{\bigcap_{n\leq m} \bigcup_{j\in[k_n,k_{n+1})} X_j})) > \epsilon - \sum_{n=1}^{m} \frac{\epsilon}{3^{n+1}} \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

By the pseudointersection ensured above, we have $X \subseteq^* \bigcup_{j\in[k_n,k_{n+1})} X_j$ so that $\varphi(\pi(p_X)) \geq \frac{\epsilon}{2} > 0$. □

Theorem 4.16. *Both of the following statements are relatively consistent with ZFC.*

1. *For every nonunital, σ-unital A with $|\mathscr{Q}(A)| = \mathfrak{c}$, there exist $2^\mathfrak{c}$-many unitarily inequivalent P-states on $\mathscr{Q}(A)$. Also, for every nonunital, σ-unital A admitting a scalar sub-σ sequence, there exist $2^\mathfrak{c}$-many P-states on $\mathscr{Q}(A)$.*

2. *For every nonunital A with a σ-unit consisting of projections, there exist no P-states on $\mathscr{Q}(A)$.*

Proof. For (1), assume the continuum hypothesis. For nonunital, σ-unital A, $\mathscr{Q}(A)$ has no minimal projections [9, 1.5], so the first claim of (1) follows by theorem 4.12.

For the second claim of (1), the continuum hypothesis also grants $2^{\mathfrak{c}}$-many selective ultrafilters (which are P-points) on $\mathbb{N}$, so the map $\varphi : \beta\mathbb{N} \setminus \mathbb{N} \hookrightarrow P(\mathscr{Q}(A))$ in 4.10 produces $2^{\mathfrak{c}}$-many distinct P-states.

For (2), note that A has a σ-sequence if and only if it has a σ-unit consisting of projections. In Bice's model [5, 6.5], there are no $\varphi \in P(\mathscr{Q}(A))$ such that $\mathscr{X}_\varphi$ satisfies the conclusion of lemma 4.15, because this reproduces the conclusion of [5, 6.2]. Therefore, there are no P-states on $\mathscr{Q}(A)$. $\square$

5 Infinite Transitivity

We say a representation $\pi : C \to \mathscr{B}(H)$ of a C^*-algebra C admits *infinite transitivity* if for every separable subspace K of H, the map $C \to \mathscr{B}(K)$ by $c \mapsto P_K \pi(c) P_K$ is onto, where P_K is the projection onto K. In [12], Farah showed that ultraproducts $(A, H, \pi)^{\mathscr{U}}$ admit infinite transitivity, where $\pi : A \to \mathscr{B}(H)$ is an irreducible representation. In this section, we will use quantum P-points on countably degree-1 saturated C^*-algebras to produce new examples of infinite transitivity.

5.1 Vector Equation Conditions

The reader may have noticed a concerning gap between the countable degree-$(1,1)$ saturation of (C, φ) and an infinite Kadison transitivity theorem: Vector equations in H_φ are degree-2 conditions of the form

$$0 = \|xa - b\|_\varphi = \varphi((xa-b)^*(xa-b))^{\frac{1}{2}}.$$

The following lemma shows the true potential of P-states and addresses these concerns.

Lemma 5.1. *Let C be countably degree-1 saturated. Then C is saturated over countable types consisting of the following conditions*

1. $\|P(\bar{x})\| \in K$,
2. $\varphi(P(\bar{x})) \in \Lambda$,
3. $\|P(\bar{x})\|_\varphi \leq r$,

where $P(\bar{x})$ is a degree-1 $*$-polynomial, $K \subseteq [0, \infty)$ and $\Lambda \subseteq \mathbb{C}$ are compact, $r \geq 0$, and φ may be any P-state, varying by condition.

Proof. Let $t(\bar{x}) = \{\rho_j(P_j(\bar{x})) \in K_j : j \in \mathbb{N}\}$ be a satisfiable countable type consisting of conditions above. To prove that it is realized, we will perform two conversions of $t(\bar{x})$.
Since $t(\bar{x})$ is satisfiable, fix $\bar{b}_m$ such that $\max_{j \leq m} d(\rho_j(P_j(\bar{b}_m)), K_j) \leq \frac{1}{m}$. Then fix an ultrafilter $\mathscr{U}$ on $\mathbb{N}$. Let $s_j = \lim_{m \to \mathscr{U}} \rho_j(P_j(\bar{b}_m))$. Then

$$t'(\bar{x}) = \{\rho_j(P_j(\bar{x})) = s_j : j \in \mathbb{N}\}$$

is satisfiable type whose realizations realize $t(\bar{x})$. Now for each φ appearing as some ρ_j, choose $a_\varphi \in \mathscr{F}_\varphi$ excising the C^*-subalgebra B generated by coefficients of $P_j(\bar{x})$ and entries of $\bar{b}_m$. Then, for each j such that $\rho_j = \varphi$, choose $d_j \in B$ such that $\varphi(d_j) = s_j$. Define

$$P'_j(\bar{x}) = \begin{cases} a_\varphi(P_j(\bar{x}) - d_j)a_\varphi, & \rho_j = \varphi; \\ P_j(\bar{x})a_\varphi, & \rho_j = \|\cdot\|_\varphi; \\ P_j(\bar{x}), & \rho_j = \|\cdot\|; \end{cases}$$

$$r_j = \begin{cases} 0, & \rho_j = \varphi; \\ s_j, & \text{otherwise.} \end{cases}$$

The type

$$t''(\bar{x}) = \{\|P'_j(\bar{x})\| = r_j : j \in \mathbb{N}\}$$

is a countable degree-1 norm type. It is satisfiable by P-excision, as in the proof of theorem 4.7 (2) $\Rightarrow$ (1). To see that realizations of $t''(\bar{x})$ realize $t(\bar{x})$, we only need to check the new case where $\rho_j = \|\cdot\|_\varphi$,

$$\|P_j(\bar{b})\|_\varphi = \varphi(P_j(\bar{b})^* P_j(\bar{b}))^{\frac{1}{2}} \leq \|a_\varphi^* P_j(\bar{b})^* P_j(\bar{b}) a_\varphi\|^{\frac{1}{2}} = \|P_j(\bar{b}) a_\varphi\| = r_j \in K_j.$$

In this case, K_j is of the form $[0, r]$, so $\|P_j(\bar{b})\|_\varphi \in K_j$. Therefore, since $t''(\bar{x})$ is realized, so is $t(\bar{x})$. □

The prior lemma also allows us to justify the claim in example 3.6.

Corollary 5.2. *Let C be countably degree-1 saturated and (φ_k) a sequence of P-states on C. Then for any infinite convex combination $\varphi = \sum_{k=1}^{\infty} t_k \varphi_k$, (C, φ) is countably degree-$(1,1)$ saturated.*

Proof. Let $t(\bar{x}) = \{\rho_j(P_j(\bar{x})) = s_j : j \in \mathbb{N}\}$ be a satisfiable degree-$(1,1)$ type. Since $t(\bar{x})$ is satisfiable, fix $\bar{b}_m$ such that $\max_{j \leq m} |\rho_j(P_j(\bar{b}_m)) - s_j| \leq \frac{1}{m}$. Then fix an ultrafilter $\mathscr{U}$ on $\mathbb{N}$. Let

$$X = \{j \in \mathbb{N} : \rho_j = \varphi\}.$$

For $\rho_j = \varphi$, let $s_{jk} = \lim_{m \to \mathscr{U}} \varphi_k(P_j(\bar{b}_m))$. Then define

$$t'(\bar{x}) = \{\|P_i(\bar{x})\| = s_i, \varphi_k(P_j(\bar{x})) = s_{jk} : i \in \mathbb{N} \setminus X, j \in X, k \in \mathbb{N}\}$$

Then $t'(\bar{x})$ is satisfiable by $\bar{b}_m$ and, since φ is a uniform limit of finite combinations of φ_k, realizations of $t'(\bar{x})$ realize $t(\bar{x})$. □

5.2 Infinite Glimm-Kadison Theorem

Theorem 5.3. *Let C be a countably degree-1 saturated C^*-algebra, and $\pi : C \to \mathscr{B}(H)$ a direct sum of countably many inequivalent P-representations $\pi_j : C \to \mathscr{B}(H_j)$ and K_j a separable subspace of H_j. Then the map $C \to \prod_j \mathscr{B}(K_j)$ by $c \mapsto P_K \pi(c) P_K$ is onto, where $K = \bigoplus_j K_j$. In particular, for every $T \in \prod_j \mathscr{B}(K_j)$ self-adjoint (positive, unitary, resp.), there exists $c \in C$ such that $\|c\| = \|T\|$, $P_K \pi(c) P_K = T$, and c is self-adjoint (positive, unitary, resp.).*

Proof. Let K_j be a separable subspace of H_j and enumerate an orthonormal basis (ξ_i) of $K = \bigoplus_j K_j$ compatible with each K_j. Every $T \in \prod_j \mathscr{B}(K_j)$ is determined by $(T\xi_i)$. For each $i, j \in \mathbb{N}$, lift

$a_{ij} + L_{\varphi_j} = (\xi_i)_j$ and $b_{ij} + L_{\varphi_j} = (T\xi_i)_j$. Define the type

$$t(x) = \{\|x\| \leq \|T\|, \|xa_{ij} - b_{ij}\|_{\varphi_j} = 0 : i, j \in \mathbb{N}\}.$$

Since $t(x)$ consists of conditions from 5.1, it suffices to show that it is satisfiable. Fix $m \in \mathbb{N}$. Because $\{\xi_j : j \leq m\}$ is orthogonal, by the Glimm-Kadison theorem, for each $\epsilon > 0$ there exists $d \in C$ such that $\pi_j(d)\xi_i = T\xi_i$ for all $i, j \leq m$ and $\|d\| < \|T\| + \epsilon$. That is, $\|da_{ij} - b_{ij}\|_{\varphi_j} = 0$, for all $j \leq m$ and $\|d\| < \|T\| + \epsilon$. Therefore $t(x)$ is satisfiable and is realized by $c \in C$. Finally, $\|c\| \geq \|\pi(c)\| \geq \|T\|$, so $\|c\| = \|T\|$ and $P_K\pi(c)P_K = T$.

For self-adjoint or positive, add the necessary generalized degree-1 norm conditions to $t(x)$ [13, 15.2.9], yielding a satisfiable type which is realized by countably degree-1 saturated C, thanks to 5.1.

For unitary, notice that by the Borel functional calculus, T has self-adjoint complex logarithm $S \in \prod_j \mathscr{B}(K_j)$. Then, by the self-adjoint case, there exists self-adjoint $s \in C$ so that $P_K\pi(s)P_K = S$, so $P_K\pi(e^{is})P_K = T$, so $c = e^{is}$ is our desired unitary. □

5.3 Applications of Infinite Transitivity

Corollary 5.4. *Suppose C is countably degree-1 saturated, $\pi : C \to \mathscr{B}(H)$ a direct sum of inequivalent P-representations, and E a separable projection in $\pi[C]''$. Let $U = \{E\}' \cap U_0(C)$. Then $u \mapsto uE$ defines a surjective group homomorphism from U onto $U(E[H])$.*

Proof. By 5.3, for every $u \in U(E[H])$, there exists $v \in U_0(C)$ such that $vE = u$ and the fact that the map is a homomorphism is plain. Now,

$$vv^* = vEv^* + v(1-E)v^* = uu^* + v(1-E)v^* = E + v(1-E)v^*.$$

Then $v = Ev + v(1-E)$, so $0 = Ev - vE$. □

Corollary 5.5. *Let C be countably degree-1 saturated. Let $(\varphi_j), (\psi_j)$ be sequences of inequivalent P-states on C such that for all j, $\|\varphi_j - \psi_j\| < 2$. Then there exists $a \in C_{+,1}$ such that $u = e^{i2\pi a}$ has $\varphi_j \circ Ad(u) = \psi_j$ for all j.*

Proof. By lemma [13, 3.8.1], $\varphi_j \sim \psi_j$ for all j. By [13, 3.8.2], for each j, there exists $0 \leq c_j \leq 2\pi$ such that $u_j = e^{ic_j}$ has $\varphi_j \circ \mathrm{Ad}(u_j) = \psi_j$. By theorem 5.3, there exists $0 \leq c \leq 2\pi$ such that $u = e^{ic}$ has $\pi(u)\xi_{\psi_j} = \xi_{\varphi_j}$ for all j, so $\varphi_j \circ \mathrm{Ad}(u) = \psi_j$ for all j. □

The next application does not directly follow from infinite transitivity, but is in the same spirit, as its finite analog was used in tandem with transitivity to prove homogeneity of the pure state space [23].

Proposition 5.6. *Let C be countably degree-1 saturated and $\pi : C \to \mathscr{B}(H)$ a direct sum of inequivalent P-representations. If $E \in \pi[C]''$ is a separable projection, $\epsilon > 0$, and $a_j \in C$ a sequence s.t. $\|\pi(a_j)E\| \leq \epsilon$ for all j, then there is $e \in C_{+,1}$ s.t. $E \leq \pi(e)$ and $\|a_j e\| \leq \epsilon$ for all j.*

Proof. Fix an orthonormal basis (ξ_j) for E which factors through the direct sum and lifts $b_{ij} \in C$ of ξ_j, so that $(\xi_j)_i = b_{ij} + L_{\varphi_i}$. Then fix the type

$$t(x) = \{\|x\| = 1, x \geq 0, \|a_j x\| \leq \epsilon, \|xb_{ij} - b_{ij}\|_{\varphi_j} = 0 : i, j \in \mathbb{N}\}.$$

Any finite subset of this type is included in a type of the form

$$t'(x) = \{\|x\| = 1, x \geq 0, \|a_j x\| \leq \epsilon, \|xb_i - b_i\|_{\varphi_j} = 0 : j < n, i < m\}$$

which is realized by [13, 3.9.6]. Therefore, by 5.1, $t(x)$ is realized by $e \in C_{+,1}$ s.t. $\|a_j e\| \leq \epsilon$ and $E \leq \pi(e)$, the element we were looking for. □

Now for our most exciting application, we resolve a conjecture of Farah-Hirshberg [16].

Lemma 5.7. *Assume the continuum hypothesis. Every countably degree-1 saturated C with $|C| = \mathfrak{c}$ is either subhomogeneous or nonexact.*

Proof. First, we claim that, if C admitted an infinite-dimensional P-representation, C would be nonexact. Indeed, fix some separable, infinite dimensional subspace K of H. Set

$$D = \{c \in C : K \text{ reduces } \pi_\varphi(c)\}.$$

For every symmetry $T \in \mathscr{B}(K)$, there exists $c \in C_{\text{sa}}$ such that $P_K \pi_\varphi(c) P_K = T$ by 5.3. Since T is unitary and $\pi(c)$ is self-adjoint, this forces that K reduces $\pi(c)$, so $c \in D$. Since every symmetry gives rise to a projection and vice versa, $\mathscr{B}(K) \subseteq \overline{\text{span}}(D)$, so there is a surjective *-homomorphism $D \to \mathscr{B}(K)$. If C were exact, so would D and $\mathscr{B}(K)$ be, but the latter is not.
Second, we claim that if C admitted P-representations of arbitrarily large finite dimension, C would also be nonexact. Fix a sequence π_j of P-representations on H_j of dimension $n(j) \to \infty$. Let $K = \bigoplus_j H_j$. Let D be as before. Again, by theorem 5.3, there is a surjective *-homomorphism $D \mapsto \prod_j M_{n(j)}$, so C is nonexact. Therefore, if C is exact, there exists $n \in \mathbb{N}$ such that for every P-state φ, $\dim(H_\varphi) \leq n$.
Now, the set $S_n(C) = \{\psi \in S(C) : \dim(H_\psi) \leq n\}$ is weak*-closed. By 4.13, P-states are weak*-dense in $P(C)$, yet are all contained in $S_n(C)$. Therefore, $P(C) \subseteq S_n(C)$, so C is subhomogeneous. □

We now remove the assumption of CH and $|C| = \mathfrak{c}$, by using the standard collapse forcing outlined in [25].

Corollary 5.8. *Every countably degree-1 saturated C^*-algebra is either nonexact or subhomogeneous.*

Proof. Work in a model M and let C be an exact countably degree-1 saturated C^*-algebra, Assume, without loss of generality, $C \neq \{0\}$, so that $|C|^M \geq \mathfrak{c}^M$. Let $\kappa = \max\{|C|^M, \mathfrak{c}^M\}$,

$$\mathbb{P} = \{p : X \to \kappa : X \in [\aleph_1]^{\leq \aleph_0}\}$$

ordered by reverse inclusion. It is plain that $\mathbb{P}$ is countably closed. If $G \subseteq \mathbb{P}$ is a generic filter, then $g = \bigcup G$ is a surjection from $\aleph_1$ onto κ. Thus, $|\kappa|^{M[G]} = \aleph_1$. Since $\mathbb{P}$ is countably closed, it adds no new countable sequences of ordinals, so adds no new reals. In particular, $|C|^{M[G]} = \mathfrak{c}^{M[G]} = \aleph_1$.
Since the value of an $\mathscr{L}$-sentence stays the same under forcing, to see that C is still a C^*-algebra in $M[G]$, it suffices to show that $\mathbb{C}$ is unchanged and C remains complete. The first fact follows since $M[G]$ contains no new reals. For the second fact, every sequence of C in $M[G]$ exists in M, and the metric is described by $\mathscr{L}$-sentences, so every Cauchy sequence in $C^{M[G]}$ is also Cauchy in C^M, so converges in C^M. Therefore, every Cauchy sequence of C in $M[G]$ converges.
By [10, 5.19], C is countably degree-1 saturated in $M[G]$. Also, C remains exact. Since C is exact in M, there is a faithful nuclear representation $\pi : C \to \mathscr{B}(H)$. Since $C^M = C^{M[G]}$, composing π with the canonical inclusion $\mathscr{B}(H)^M \hookrightarrow \mathscr{B}(H)^{M[G]}$, shows that $\pi^{M[G]}$ is nuclear, so C is exact in $M[G]$. Because $|C|^{M[G]} = \mathfrak{c}^{M[G]} = \aleph_1$, by lemma 5.7, C is subhomogeneous in $M[G]$, so it is n-subhomogeneous for some n. By [7, IV.1.4.6], n-subhomogeneity is axiomatizable, so C is (n-)subhomogeneous in M. □

5.4 Counterexample to Infinite Transitivity

The question of whether infinite transitivity of $(\mathscr{Q}(H), \pi_\varphi)$ implies degree-1 saturation of $(\mathscr{Q}(H), \varphi)$ is fairly opaque. In this section, we at least prove that in ZFC, the Calkin algebra has an irreducible representation which does not admit infinite transitivity.

Definition 5.9. For $X \subseteq \mathbb{N}$, $j \in \mathbb{N}$, we define $X + j = \{m + j : m \in X\}$ and for an ultrafilter $\mathscr{U} \in \beta\mathbb{N} \setminus \mathbb{N}$, $\mathscr{U} + j = \{X + j : X \in \mathscr{U}\}$. Since $\beta\mathbb{N} \setminus \mathbb{N}$ is compact, we can define $\mathscr{U} + \mathscr{V} = \lim_{j \to \mathscr{V}} (\mathscr{U} + j)$. We call a nonprincipal ultrafilter $\mathscr{U}$ *idempotent* if $\mathscr{U} + \mathscr{U} = \mathscr{U}$. Following definitions, idempotency is equivalent to the statement: For all $X \in \mathscr{U}$, $\{j \in \mathbb{N} : X - j \in \mathscr{U}\} \in \mathscr{U}$, where $X - j = \{m - j : m \in X \setminus j\}$.

By [11, 1], every compact topological space with a semigroup structure whose left operation is continuous admits an idempotent element. The addition defined in 5.9 plainly grants this semigroup structure on $\beta\mathbb{N} \setminus \mathbb{N}$, so there is $\mathscr{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ idempotent.

Theorem 5.10. *There exists an irreducible representation of $\mathscr{Q}(H)$ which does not admit infinite transitivity.*

Proof. Let $\mathscr{U}$ be an idempotent ultrafilter on $\mathbb{N}$. We claim that $\pi = \pi_{\tilde{\varphi}_\mathscr{U}}$ (as a representation of $\mathscr{B}(H)$ factoring through $\mathscr{Q}(H)$) fails infinite transitivity. Let $s \in \mathscr{B}(H)$ be the right shift along an orthonormal basis (η_n) of H and $\varphi_\mathscr{U}$ as in 4.10 along (η_n), with lift $\tilde{\varphi}_\mathscr{U}$. Let $\xi_j = s^j + L_{\tilde{\varphi}_\mathscr{U}}$. This family is orthonormal, since $\varphi_\mathscr{U}((s^j)^* s^i) = \delta_{ij}$. Suppose there is $a \in \mathscr{Q}(H)$ satisfying

$$\langle \pi(\tilde{a})\xi_i, \xi_j \rangle = \delta_{i0}\delta_{ij}.$$

This implies

$$\tilde{\varphi}((s^j)^* \tilde{a}(s^j)) = \lim_{n \to \mathscr{U}} \tilde{a}_{n+j} = \delta_{j0},$$

where $\tilde{a}_n = \langle \tilde{a}\eta_n, \eta_n \rangle$. Thus $X_0 = \{n \in \mathbb{N} : |\tilde{a}_n - 1| < \frac{1}{2}\} \in \mathscr{U}$ and for all $j > 0$, $X_j = \{n \geq j : |\tilde{a}_{n+j}| < \frac{1}{2}\} \in \mathscr{U}$. By idempotency, $\{j \in \mathbb{N} : X_0 - j \in \mathscr{U}\} \in \mathscr{U}$, so for some $j > 0$, $X_0 - j \in \mathscr{U}$, so

$$\varnothing = \left\{n \geq j : |\tilde{a}_{n+j}| < \frac{1}{2}, |\tilde{a}_{n+j} - 1| < \frac{1}{2}\right\} = X_j \cap (X_0 - j) \in \mathscr{U},$$

a contradiction. Therefore, there is no $a \in \mathscr{Q}(H)$ such that $\pi(\tilde{a})\xi_j = \delta_{j0}\xi_j$, so infinite transitivity fails. □

This tells us that infinite transitivity is not a property intrinsic to all irreducible representations of countably degree-1 saturated C^*-algebras, so the infinite transitivity of P-representations comes from the nontrivial interplay between the state and the algebra. This also gives us an example of a countably degree-$(1,0)$ saturated C^*-probability space whose failure to be degree-$(1,1)$ is visible at the representation level.

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