The binomial option pricing model: The trouble with dividends*

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Abstract

We identify a problem in the widely used binomial option pricing model when it is used to value options on an asset paying continuous dividends. It does not value pairs of European spot and futures options consistently even though they are theoretically equivalent. The inconsistency arises from the way dividend yield is incorporated into the jumps and probabilities. In addition, the model also has the tendency to undervalue American options due to suboptimal early exercise decisions. While the lingering effect of this problem diminishes asymptotically, it is nonetheless a concern for someone just beginning to learn the model or in applications where the use of a sufficiently large binomial tree is not practical or economical. We propose a simple modification to solve the problem and demonstrate the effectiveness of the solution.

JEL classifications: G13, G30, J33

Key words: Binomial option pricing model; Dividend yield; American options; Early exercise; Futures options.

1. Introduction

The binomial option pricing model is a powerful tool, not just for pricing more complex options that have no closed-form solution such as American, barrier or other path-dependent options but also for teaching risk-neutral valuation and the construction of riskless hedge portfolios. It is intuitive, easy to understand and simple to implement. The model is so commonly used that we almost take it for granted as much as we do the celebrated Black-Scholes-Merton (1973) model. The popularity of the model is also complemented by previous studies that confirm that option prices calculated using the binomial model converge to the correct limit as the number of periods increases to infinity. This is true for both European options (e.g., Cox, Ross and Rubinstein, 1979; Rendelman and Bartter, 1979; Hsia, 1983) and American options (e.g., Amin and Khanna, 1994; Leisen, 1998; Lamberton, 2002).

The most widely used form of the binomial model, referred to as the ***traditional binomial model*** hereafter, is the one first proposed by Cox, Ross and Rubinstein (1979). While the original version of the model is developed specifically for valuing options on a non-dividend paying stock, it is subsequently extended to allow for continuous dividends paid at a constant rate.[1] In this paper, we identify a problem in the traditional binomial model when it is used to value options on an asset paying continuous dividends. It is well known that a pair of European spot and futures (either call or put) options are theoretically equivalent if the two options have identical strike price and both options mature at the same time as the futures contract does.[2] If the traditional binomial model is

[1] This version of the model appeared as far back as Curran (1995) and McDonald and Schroder (1998) and some derivatives textbooks (e.g., Hull, 2018). For a survey and comparison of various versions of the binomial option pricing model, see Chance (2008).

[2] The theoretical equivalency of such pairs of options is explained in most derivatives textbooks (e.g., Section 18.3 of Hull, 2018, pp.384-385).

used to value such a pair of options, however, very different prices are usually calculated even if a fairly large binomial tree is used (e.g., with 50 periods). These differences cannot be explained by the usual discretization errors associated with numerical valuation methods such as binomial trees. In addition, the traditional binomial model also tends to provide a biased estimate of the gain from the early exercise for American options, leading to undervaluation due to suboptimal early exercises of these options.

We show that the culprit is the way dividend yield is incorporated into the jumps and probabilities of the traditional binomial model. When the underlying stock pays dividends continuously at a constant rate, future stock prices are expected to decline at the rate of the dividend yield in comparison with a comparable stock that is identical in every way except that it does not pay any dividends. In the traditional binomial model, the upward jump (u) and downward jump (d) in each period are determined as if the stock does not pay any dividends at all. Both jumps are only a function of stock return volatility and the length of the binomial period. Stock prices at all nodes in the binomial tree constructed this way are identical to those in a corresponding binomial tree for the matching non-dividend paying stock. As a result, future stock prices in the binomial tree remain the same whether or not the underlying stock pays any dividends. In order to maintain the no-arbitrage condition, the risk-neutral probabilities in each period is adjusted to reflect the dividend yield, making these probabilities different from those of the corresponding case without dividends. This adjustment to probabilities does ensure that the expected future stock price in the binomial tree declines at the rate of the dividend yield in comparison with the case for a non-dividend paying stock. It is thus technically correct and option prices do converge to the correct limit when the number of periods used is sufficiently large (e.g., Cox, Ross and Rubinstein, 1979; Amin and Khanna, 1994).

While this adjustment for dividend yield seems reasonable, it has some unintended consequences for option payoffs, valuation and early exercise, especially when the number of periods used is not sufficiently large. On the one hand, due to the lack of adjustment for dividends, the usual decline in stock price after a dividend payment does not happen in the traditional binomial tree, leading to higher projected stock prices in the binomial tree in all future periods. This upward bias in future stock prices makes the exercise value higher (lower) for call (put) options. On the other hand, dividend yield alters the binomial probabilities, making the upward (downward) jump less (more) likely to happen in comparison with the corresponding probabilities for the matching non-dividend paying stock. Holding everything else constant, this adjustment to probabilities may lead to a lower (higher) expected value (i.e., risk-neutral value) of holding the option for one more period for call (put) options. Together, these two effects may lead to incorrect assessment of the early exercise decision and possible undervaluation of American options. Extensive numerical analysis confirms this assessment of the traditional binomial model and the magnitude of the bias increases with the size of the dividend yield. This is particularly true when the binomial tree is built with a small or moderately sized number of periods.[3]

Another unintended consequence is that the traditional binomial model provides inconsistent pricing of pairs of European spot and futures options with identical strike price and maturity even though these options are theoretically equivalent. As long as the pair of (call or put) options mature at the same time as the futures contract does, their payoffs at maturity are always identical because the futures price converges to the spot price at maturity. However, when the traditional binomial model is used to value such a pair of options, different prices are always

[3] For further discussions on the convergence of the binomial model for American options, see Leisen and Reimer (1996), Leisen (1998), Chang and Palmer (2007), Joshi (2007), Chan et al. (2009) and Goldenberg (2010).

calculated for them throughout the entire tree as long as the current futures price is not equal to the current spot price (which is the norm rather than the exception). This is because the traditional binomial model does not adjust either the upward or downward jump for dividends. As a result, the sizes of upward and down jumps are identical for both the spot price tree and the futures price tree. A different initial price would always lead to different ending prices at maturity if the initial futures price is not equal to the initial spot price. This means that the European futures option would have different payoffs at maturity in comparison with the corresponding European spot option. These differences in payoffs lead to different prices for the matching pair of options even though they should have the same price. While the probabilities in the two binomial trees are also different, it is not enough to correct for the differences in payoffs as we demonstrate using numerical examples. While these differences do vanish in the limit as the number of periods is sufficiently large, it is a potential problem nevertheless, especially in some applications when it is not practical or economical to use a very large number of periods.[4]

Now that we have recognized the problem in the traditional binomial model, the next logical step is to find a solution for it. We propose a simple modification to the traditional binomial model by altering the size of the upward and downward jumps. The modification is similar in spirit to the dividend adjustment in the original Cox, Ross and Rubinstein (1979) binomial model except that they only consider the adjustment in the context of discrete dividends. The modified jumps adjust for the dividend payout in each period and reduce the stock prices in each period by the expected dividends. The risk-neutral probabilities are simultaneously adjusted in order to correctly

[4] For example, the Chicago Board Options Exchange provides real-time implied volatilities and Greeks for all options listed on the exchange via its Hanweck platform. With tens of thousands of options listed on the exchange, it is costly, both computationally and resource wise, to use a very large binomial tree to calculate implied volatilities and Greeks for these options. They need a binomial model that can calculate reasonably accurate prices with a moderately sized binomial tree (e.g., a 30-period binomial tree).

reflect the adjustment to the jumps such that the binomial model maintains the no-arbitrage condition that the stock price in the risk-neutral world grows at the rate of the risk-free rate in excess of the dividend yield. This requirement also ensures that the binomial distribution of stock prices converges to the correct lognormal distribution in the limit. This new binomial model will be referred to as the ***modified binomial model***.[5] Extensive numerical analysis demonstrates that the modified binomial model provides a good solution to the problem we have identified. It ensures that pairs of equivalent European spot and futures options are always priced exactly the same and the early exercise of American options is determined more accurately.

The rest of the paper proceeds as follows. Section 2 identifies the problem when the traditional binomial model is used to value American options and pairs of equivalent European spot and futures options on a dividend-paying asset. Section 3 describes the modified binomial model and how it compares with the traditional binomial model. The effectiveness of the solution is illustrated using numerical examples in Section 4. The final section concludes.

2. The trouble with dividends

The binomial model is constructed to approximate the geometric Brownian motion in the risk-neutral world:

$$\frac{dS}{S} = (r - q)dt + \sigma dz \qquad (1)$$

[5] The modified binomial model is different from the "forward tree" model in McDonald (2013) where future stock prices in the binomial tree increase at the rate of the risk-free rate net of the dividend yield. The modified binomial model can be regarded as an extension of the traditional binomial model because the two models are identical if the underlying asset does not pay any dividends. This is not the case with the "forward tree" binomial model. Users of the traditional binomial model are more likely to accept an extension to the original model instead of a brand new model.

where S is the stock price, r is the risk-free rate, q is the dividend yield, σ is the stock return volatility and z is a standard Wiener process. In an n-period traditional binomial model, the upward jump (u), downward jump (d) and the upward jump probability (p) are specified as follows:

$$u = \exp(\sigma\sqrt{\Delta t}) \tag{2a}$$

$$d = \exp(-\sigma\sqrt{\Delta t}) = 1/u \tag{2b}$$

$$p = \frac{\exp[(r-q)\Delta t] - d}{u - d} \tag{2c}$$

where $\Delta t = T/n$ is the length of one binomial period and T is option maturity. See Curran (1995), McDonald and Schroder (1998) or Hull (2018) for further details of the traditional binomial model. While there are various other versions of the binomial model, as discussed in Chance (2008) and McDonald (2013), we choose the version in Eqs. (2a - 2c) as it is the most widely used in practice.

Note that the upward and downward jumps in the traditional binomial tree are not adjusted for the dividend yield at all. After one period, the stock price would jump up by a factor of u or down by a factor of d as if there were no dividends at all. This is quite clear as both jump factors are only a function of stock return volatility (σ) and the length of the binomial period (Δt); dividend yield (q) does not factor into the jumps at all. While dividend yield is not reflected in the jumps, it is however incorporated into the upward and downward probabilities, p and $1-p$, respectively. A higher dividend yield would lead to a lower probability for the upward jump and a higher probability for the downward jump. This adjustment to probabilities maintains an important property of the binomial model that future stock prices are expected to grow at the risk-free rate in excess of the dividend yield in each period:

$$p \times uS + (1-p) \times dS = \exp[(r-q)\Delta t] \times S \tag{3}$$

It is a necessary condition for ruling out riskless arbitrage and ensures that the binomial distribution of stock prices converges to the correct lognormal distribution in the limit as $n \to \infty$.

While the traditional binomial model is widely used, both in practice and classrooms, it does have some problems due to the way dividend yield is treated in the model. For example, it is well known (e.g., McDonald, 2013, p.335) that the probability p (or $1-p$) may be negative if n or σ is too small. Such a problem, while theoretically true, is irrelevant in practice as it does not happen when a reasonably sized n is used. There are however other cases where the problem is more serious and may lead to troublesome outcomes even if a reasonably sized n (e.g., 50) is used. We describe two scenarios where the problem may manifest itself and potentially lead to the mispricing of options. The first one is the inconsistent pricing of European futures options relative the corresponding European spot options. The second scenario concerns the early exercise of American options.

2.1. The valuation of European spot and futures options

For many futures contracts traded on exchanges (e.g., the S&P 500 index futures), we often have options traded on both the underlying asset and the futures contract at the same time. In that case, a European option on the underlying asset (i.e., European spot option) is theoretically equivalent to a corresponding European option on the futures contract (i.e., European futures option) if the option and futures contracts mature at the same time, both options have the same strike price, and both options are of the same type (either both calls or puts). This is because the futures price converges to the spot price of the underlying asset at option maturity. As a result, the pair of European spot and futures options have identical payoffs at maturity and are thus theoretically equivalent. This theoretical equivalency result is well known and explained in most derivatives textbooks (e.g., Hull, 2018, pp. 384-385).

Since such a pair of European options are theoretically equivalent, we would expect the binomial model to provide an identical or nearly identical price for them as well. Unfortunately, this is not the case if we use the traditional binomial model to value these options. To see why this is the case, let S_0 be the current spot price of the underlying asset and F_0 be the corresponding price of a futures contract with maturity T. The underlying asset pays dividends continuously at a constant rate q. In this setup, we have the following initial futures price:

$$F_0 = S_0 \cdot \exp[(r - q)T]$$

under the assumption that interest rate is constant. When an n-period traditional binomial model is used to value the spot option, the ending spot prices at option maturity are:

$$S(n, j) = S_0 u^j d^{n-j} \tag{4}$$

for $j = 0, 1, \ldots, n$. In comparison, when an n-period traditional binomial model is used to value the corresponding futures option, the ending futures prices at option maturity are:

$$F(n, j) = F_0 u^j d^{n-j} \tag{5}$$

as the same jumps are used for the futures price tree as well (e.g., Hull, 2018, p.291). The ending futures prices in Eq. (5) would be different from the corresponding ending spot prices in Eq. (4) at every node (n, j) unless the initial futures price is the same as the initial spot price (i.e., $F_0 = S_0$). This means that unless the dividend yield (q) is equal to the risk-free rate (r), an extremely rare occurrence, the spot price tree would have different ending prices at maturity in comparison with the futures price tree. The pair of European spot and futures options would thus have different payoffs at maturity despite the fact that they are theoretically equivalent and should have identical payoffs at maturity.

Of course, the two binomial trees also have different upward and downward jump probabilities which may compensate for the differences in maturity payoffs. The upward probability in each period is:

$$p = \frac{\exp[(r - q)\Delta t] - d}{u - d}$$

in the spot price tree while the corresponding probability in the futures price tree is:

$$p^* = \frac{1-d}{u-d}$$

This is because the futures contract can be regarded as a hypothetical asset paying dividends continuously at the risk-free rate. These two probabilities are identical only if the dividend yield (q) is equal to the risk-free rate (r). The following numerical example shows that such differences in probabilities are not sufficient to compensate for the differences in maturity payoffs and the traditional binomial model indeed values the two equivalent options quite differently. Given put-call parity, we only use a pair of European call options to illustrate the inconsistencies in the valuation of spot and futures options when the traditional binomial model is used.

Consider a pair of theoretically equivalent European spot and futures call option with six months remaining to maturity. The futures contract also matures in six months. The current spot price of the underlying asset is normalized to be 100. The strike price is also 100 for both options. The risk-free rate is normalized to be 0. The dividend yield on the underlying asset is 4%. The volatility of the underlying asset is 20%. Given these parameter values, the current futures price is 98.02 for a six-month contract. We first use a four-period traditional binomial model to value both spot and futures options. Such a small binomial tree is chosen such that we can show details of every calculation in the binomial tree. The results are summarized in Panel A of Figure 1.

Figure 1 about here

As shown in Panel A of Figure 1, there are several differences in the valuation of the two European call options. The most important difference is that the two equivalent options are priced very differently by the traditional binomial model. While the spot call is priced at \$4.31, the futures call is valued at \$4.63 which is \$0.32 or 7.4% higher. The difference is thus quite large and difficult to ignore. Imagine such an example is used in the classroom or training programs to teach the binomial model. There is no easy way to explain the different prices calculated by the traditional binomial model for the two equivalent options.

Secondly, the ending spot prices at option maturity are clearly higher than the corresponding ending futures prices at option maturity because the initial spot price of $100 is higher than the initial futures price of $98.02. For example, the highest ending spot price is $132.69 which is $2.63 or 2.0% higher than the highest ending futures price of $130.06. Such differences are observed at all ending nodes, resulting in substantial differences in option payoffs. At the highest ending node, for example, the payoff of the spot call option is $32.69 which is $2.63 or 8.7% higher than the corresponding payoff of the futures option. Although the difference in dollars (i.e., $2.63) in the payoff between the two options is identical to the difference in dollars between the spot price and futures price, the percentage difference is much higher in the option payoffs (8.7%) than in the underlying asset prices (2.0%). In percentage terms, the differences between the spot price tree and the futures price tree may thus have an outsized impact on option value.

Finally, the difference in upward jump probabilities between the spot price tree and the futures price tree does not correct the differences in option payoffs at maturity. While the upward jump probability is 44.7% in the spot price tree, the corresponding probability is 48.2% in the futures price tree. The higher upward jump probability in the futures price tree should compensate or partially compensate for the lower payoffs of the futures call option. However, it does not work out that way as the values of these options are different at all nodes on the tree (with the exception where both options have a zero value). It appears that the higher upward jump probability in the futures price tree overcompensates for the lower payoffs of the futures option, leading to a higher initial price for the futures option in spite of the lower payoffs at maturity.

Of course, the problem exposed in this example could be an isolated incident and may not happen if a larger binomial tree with more periods is used. To make sure that is not the case, we use a series of binomial trees, with the number of periods (n) varying from 1 to 100, to value the same pair of spot and futures options. Their prices are then compared with the accurate price of \$4.647, calculated using the Black-Scholes-Merton model, and plotted in Panels B and C of Figure 1 for n ranging from 1 to 50 and from 50 to 100, respectively. It is clear from these plots that the binomial model calculates difference prices for these two equivalent options for all n. For example, when $n = 50$, the traditional binomial model calculates the prices of the equivalent European spot and futures call options to be \$4.619 and \$4.674, respectively. The difference of \$0.055 or about 1.2% of the option value is sizeable. Even when we increase n to 100, the difference is still \$0.023 or about 0.5% of the option value. The difference in prices appears to diminish quite slowly.

While the magnitude of the difference declines as the number of periods (n) increases, some interesting patterns emerge from these differences. The pricing error for the spot call option appears to oscillate symmetrically around the accurate price of \$4.647 and decreases quite slowly but consistently as n increases. The oscillation appears to be consistent with the usual discretization errors typically associated with the binomial model. In comparison, the pricing error for the futures call option has a very different pattern. The pricing error not only is always smaller than the corresponding error for the spot call option but also tends to have a positive bias. It also appears to decline faster over certain ranges of n (e.g., from 1 to 20) but slower for other ranges (e.g., from 40 to 60). These different patterns between spot and futures call prices provide further confirmation of the differences in the binomial prices of these call options.

2.2. Early exercise of American options

We next examine how the dividend adjustment in the traditional binomial model affects the early exercise decision of American options and its consequence for option valuation. Given the current stock price S, future stock prices in the binomial tree are not adjusted for any dividends paid out along the way in any subsequent period. This means that stock prices at all nodes in the binomial tree are positioned higher than they should be because dividend payout is expected to result in a corresponding decline in future stock prices. Instead, the binomial tree expands over time as if all future dividends are reinvested back in the stock immediately after they are paid out. A consequence of this treatment of dividends is that the exercise value at all future nodes in the binomial tree is higher than it should be for American call options and lower than it should be for American put options. If the present value (PV) of the option's expected future payoff (i.e., the risk-neutral value of the holding the option for one more period) remains the same, this upward bias of the stock price may lead to premature early exercise of American call options when it is not optimal to do so and delayed exercise of American put options when it is actually optimal to do so.

Of course, early exercise may not be affected at all if the dividend adjustment has an identical effect on the PV of the option's expected future payoff in comparison with the impact on the option's current exercise value. In that case, the difference between the exercise value and the PV of holding the option would remain unchanged. Unfortunately, that is not the case here. In fact, the opposite appears to be true for American call options. This is because the lack of adjustment to stock prices leads to a lower upward jump probability (p) and a higher downward jump probability (1-p), in comparison with the corresponding case of a non-dividend paying stock. These adjustments to probabilities reduce the risk-neutral value of holding the call option for one more period. At the same time, the dividend adjustment directly leads to upward-biased stock prices and hence higher exercise value for call options. Combining both effects, the dividend adjustment tends to bias towards more frequent and less optimal early exercises of American call options. For American put option, the effect on the early exercise decision is more nuanced. While the dividend adjustment leads to a downward bias to the put option's exercise value, its impact on the PV of expected future payoff is unclear. This is due to two confounding effects: while future exercise values of the put option is reduced by the dividend adjustment, the expected value of holding the option is higher due to the lower upward jump probability and higher downward probability. It is not clear which of the two opposing effects dominate. Using numerical examples, we subsequently show that the reduction in future exercise values appears to dominate and the traditional binomial model tends to bias towards more frequent and less optimal early exercises of American put options as well.

To illustrate the effect of the dividend adjustment on the early exercise decision of American options, we use the following numerical example: the stock price is \$100, the strike price is \$100, option maturity is six months, the risk-free rate is 4%, the dividend yield on the stock is 4%, stock return volatility is 20%. To keep things more manageable, we initially use a four-period binomial tree to illustrate the early exercise decision when the traditional binomial model is used to price the American option. The valuation and early exercise decision of the American options are shown in Panels A and B of Figure 2 for call and put options, respectively. At each node in the binomial tree, four pieces of information are listed in the order from top to bottom that summarize the calculations at that node: the stock price, the risk-neutral (RN) value of holding the option for one more period, the exercise value of the option and the decision to exercise the option (Yes or No).

Figure 2 about here

Consider the valuation of the American call option shown in Panel A. Early exercise is deemed optimal at three nodes (A, B and C) in the binomial tree. In each case, the exercise value is higher than the value of holding the option. For example, at the highest node in period 3 (Node B), the exercise value of the call is \$23.63 while the RN value of holding it for one more period is \$23.51. Exercising the option would thus gain an extra value of \$0.12 over the RN value. The optimal decision is thus to exercise the option at Node B. Of course, this optimal exercise decision is based on the valuation from the four-period traditional binomial model. Is it truly optimal to exercise the option at that node? In order to verify that, we need to know the location of the "true" optimal exercise boundary based on the continuous-time model of the stock price. Following Broadie and Detemple (1996), we use a 15,000-period traditional binomial model to calculate the "true" optimal early exercise boundary. Relevant to the early exercise decision for the American call option in Panel A, we list the three threshold prices from the optimal exercise boundary for periods 1, 2 and 3 at the bottom of the panel, respectively.

Comparing with the "true" optimal early exercise boundary, it is clear that early exercise decisions displayed in Panel A are not always optimal. In fact, two of the three early exercise decisions, those at Nodes A and C, are suboptimal. For example, in period 3 (i.e., $t = 0.375$ years) of the binomial tree in Panel A, the "true" optimal exercise boundary is \$119.60. This means that it is optimal to exercise the call option at Node B where the stock price is \$123.63 (above the "true" optimal exercise boundary) but not optimal at Node C where the stock price is \$107.33 (below the "true" optimal exercise boundary). Similarly, in period 2 (i.e., $t = 0.25$ years), the "true" optimal exercise boundary is \$125.88. It is thus not optimal to exercise the call option at any node in that period because the highest stock price is only \$115.19 (below the "true" optimal exercise boundary). Nevertheless, the traditional binomial model deems it optimal to exercise the call option at Node A in period 2. This example confirms that the use of the traditional binomial model may bias towards more frequent and less optimal early exercises of American call options, possibly leading to an undervaluation of the option.

In Panel B of Figure 2, we illustrate how early exercise decision for American put option is affected by the treatment of dividends in the traditional binomial model. Like the case of the corresponding American call option in Panel A, there are also three nodes where it is optimal to exercise the American put option, including Node D in period 2 and Nodes E and F in period 3. Checking with the "true" exercise boundary (shown at the bottom of the panel), it is clear that it is only optimal to exercise the put option at Node E but not at Nodes D and F. At Node E in period 3, the stock price of $80.89 is below the "true" early exercise boundary of $83.61; the early exercise at this node is thus correct. In contrast, at Node F in the same period, the stock price of $93.17 is above the "true" exercise boundary of $83.61; the early exercise decision at this node is thus suboptimal. Similarly, at Node D in period 2, the stock price of $86.81 is above the "true" exercise boundary of $79.50; the early exercise decision at this node is also suboptimal. This means that early exercise is premature in two of the three nodes that the put option is exercised when the four-period traditional binomial model is used. Such suboptimal exercise may lead to undervaluation of the American put option.

Of course, the examples in Figure 1 are not sufficient to confirm that the traditional binomial model is problematic in the valuation of American options. It is well known that the binomial model is only an approximation to its continuous-time counterpart. With only four periods used for the binomial tree, the suboptimal early exercise could be due to discretization errors. While such errors diminish in the limit, it can be large if the number of periods used is too small. The situation is even worse for the early exercise boundary calculated using the binomial model because the convergence to the "true" exercise boundary is much slower in comparison with the convergence of option prices in the limit (e.g., Lamberton, 2002; Goldenberg, 2010). The suboptimal exercise decisions illustrated in Figure 1 could be the result of the usual approximation errors of any binomial model instead of a serious problem.

To further explore the role played by dividend yield, we study the pricing errors and early exercise boundary of American put options when the number of periods (n) increases from 1 to 100. We compare approximation errors in American put option prices and early exercise boundaries between cases with and without dividends when the traditional binomial model is used to value American put options. If the bias illustrated in Figure 1 is indeed due to the treatment of dividend, we should observe a downward bias in the price of the American put option and an upward bias in its early exercise boundary if the underlying stock pays dividends in comparison with a matching stock that does not pay any dividends. This should be the case even if a relatively large n is used. We only consider American put options because it is never optimal to exercise American call options early if the underlying stock does not pay any dividends. Using the same American put option as in Panel B of Figure 2, we calculate the price of the American put option using the traditional binomial model as the number of periods increases from 1 to 100. To determine the pricing error in these prices, we need an accurate price of the American put option. Following Broadie and Detemple (1996), we use a 15,000-period traditional binomial model to determine the accurate price of the American put option. This accurate price allows us to determine the pricing error when an n-period traditional binomial model is used to value the American put option for any n ranging from 1 to 100. In Figure 3, these pricing errors are plotted and compared between the cases of a non-dividend stock and a stock paying dividends continuously at the 4% rate.

Figure 3 about here

In Panels A and B of Figure 3, we plot the pricing errors of the American put option as the number of periods (n) increases from 1 to 20 and from 20 to 100, respectively. We separately plot these ranges due to scale differences in the pricing errors. For the case without dividends ($q = 0$), we observe the usual oscillation of the pricing errors around zero and the magnitude of these errors declines as n increases. This is consistent with what is previously reported in the literature (e.g., Broadie and Detemple, 1996; Figlewski and Gao, 1999; Heston and Zhou, 2000). What is more relevant is how the traditional binomial model performs when the dividend yield is positive in comparison with the case of a non-dividend paying stock. From both panels of Figure 3, it is clear that the approximation errors are larger when the underlying stock pays dividends. The dividend adjustment in the traditional binomial model thus makes things worse for the case of a dividend-paying stock. More interestingly, the errors are downward biased in comparison with the corresponding errors when the stock does not pay any dividends, suggesting the traditional binomial model tends to underestimate the value of American put options when the underlying stock pays dividends. In contrast, the errors tend to oscillate evenly around zero, thus more likely to be due to discretization errors, when the underlying stock does not pay any dividends. It also appears that the downward bias does not disappear as more periods are used. This is clear from a comparison of the patterns of these errors in Panels A and B. While the pricing error generally declines as n increases, the relative difference between the two cases and the downward bias in the case of the dividend-paying stock remains.

The downward bias documented in Figure 3 is further confirmed by the early exercise boundary when the American put option is valued using the traditional binomial model. In Figure 4, we plot the early exercise boundary of the American put option using a 20-period traditional binomial model. The "true" early exercise boundary of the same option is approximated using a 15,000-period traditional binomial model. In Panel A, the case of the non-dividend paying stock is shown. It is clear that the early exercise boundary from a 20-period binomial model is a poor approximation of the "true" early exercise boundary, oscillating around the "true" boundary. The oscillation appears to be random, however, and equally likely on either side of the "true" early exercise boundary. In Panel B, we plot the case when the underlying stock pays dividends continuously at the 4% rate. There are at least two notable differences in comparison with the corresponding early exercise boundaries in Panel A. It is clear that the approximation errors are much larger in this case, especially closer to option maturity. More importantly, there is a substantial upward bias in the early exercise boundary in comparison with the case of the non-dividend paying stock (Panel A). The upward bias means that the early exercise of American put options is likely more frequent and less optimal in comparison with the case of the non-dividend paying stock, leading to a downward bias in the valuation of the American put option. The results in Figures 3 and 4 thus confirm that there is indeed a bias in the valuation and early exercise of American put options and it is unlikely due to the usual discretization errors commonly associated with the binomial model.

Figure 4 about here

3. A simple solution

We propose a simple solution to resole the problems identified in the previous section: modifying the upward and downward jumps such that future stock prices decline continuously at the rate of the dividend yield relative to an otherwise identical non-dividend paying stock. A binomial tree constructed this way is more consistent with how dividend payments actually impact future stock prices. Details of this solution are described below.[6]

Unlike the traditional binomial model, we propose the following specification of the upward and downward jumps in each period of the binomial tree:

$$u^* \equiv \exp(\sigma\sqrt{\Delta t} - q\Delta t) = u \cdot \exp(-q\Delta t) \tag{6a}$$

$$d^* \equiv \exp(-\sigma\sqrt{\Delta t} - q\Delta t) = d \cdot \exp(-q\Delta t) \tag{6b}$$

where u and d are the upward and downward jumps, respectively, in the traditional binomial model as defined in Eqs. (2a) and (2b). The modified jumps are similar in spirit to how Cox, Ross and Rubinstein (1979) incorporate discretely paid proportional dividends into their version of the binomial model.[7] This alternative specification is reasonable because it simply reduces the magnitude of both the upward and downward jumps in the traditional binomial model by a factor of $\exp(-q\Delta t)$. This reduction in stock price by the dividend yield is equivalent to having shareholders of a non-dividend paying stock giving away a fraction of their shares continuously at the constant rate q. If they own one share of the stock right now, their holdings will be slightly less than one share, or $\exp(-q\Delta t)$ shares to be exact, after one binomial period. This is more appropriate for option valuation because options do not have any claim on any dividends paid by the underlying stock until they are exercised after an exchange of cash payment and shares of the underlying stock.

[6] There are alternative modifications that might also solve the problem we have identified such as the "forward tree" model described in McDonald (2013). We choose this particular solution for both simplicity and as a true extension to the original binomial model in Cox, Ross and Rubinstein (1979) because the two models are identical if the underlying stock does not pay any dividends. It is thus more likely to gain acceptance by users of the binomial model in practice.

[7] See the discussion on dividends and put pricing on p. 255 of Cox, Ross and Rubinstein (1979).

With the jumps specified this way, the risk-neutral probabilities of the upward and downward jumps are determined such that the expected stock price in the next period grows from the current stock price at the risk-free rate net of the dividend yield (i.e., matching the mean return between the binomial tree and the underlying distribution):

$$p^* \times u^* S + (1 - p^*) \times d^* S = \exp[(r - q)\Delta t] \times S \tag{7}$$

Solving for the upward jump probability p^*, we have:

$$p^* = \frac{\exp[(r - q)\Delta t] - d^*}{u^* - d^*} \tag{8}$$

which simplifies further to a more compact form:

$$p^* = \frac{\exp(r\Delta t) - d}{u - d} \tag{8a}$$

Interestingly, this is exactly equal to the upward jump probability in the traditional binomial model for a non-dividend paying stock. By incorporating dividend yield into the jumps, we actually keep the binomial probabilities identical to the those in the traditional binomial model for a non-dividend paying stock. This alternative setup of the binomial tree thus has some clear differences in comparison with the traditional binomial tree. Instead of incorporating the dividend yield into the probabilities while leaving the jumps unchanged as in the traditional binomial model, we incorporate it into the jumps while leaving the probabilities unchanged.

Note that the modified binomial model is asymptotically equivalent to the traditional binomial model. This is because the differences between the jumps in the two models are in the order of $O(\Delta t)$. This is clear by a direct comparison between u and u^* in Eqs. (2a) and (6a) and between d and d^* in Eqs. (2b) and (6b). These differences are asymptotically dominated by the $\sigma\sqrt{\Delta t}$ term in the jumps and thus have a negligible impact as $n \to \infty$. However, differences between the two binomial models are magnified when the dividend yield (q) is high and the number of periods (n) is not sufficiently large. Numerical examples are used to illustrate such differences and whether or not the modified binomial model alleviates the problems we have identified in the traditional binomial model.

4. Numerical analysis

In this section, we use numerical examples to illustrate and compare the differences between the traditional and modified binomial models. In particular, we focus on how the modified binomial model evaluate pairs of equivalent European spot and futures options and how it handles early exercise decisions of American options.

4.1. Consistent pricing of European spot and futures options

Suppose that we use an n-period modified binomial model to value a European spot option. Given the initial spot price S_0, the ending spot prices at option maturity are:

$$S(n,j) = S_0(u^*)^j(d^*)^{n-j}$$

for all $j = 0, 1, \ldots, n$ where u^* and d^* are defined in Eqs. (6a) and (6b). By substitution, these ending spot prices are rewritten in terms of u and d:

$$S(n,j) = S_0 u^j d^{n-j} \exp(-qT) \tag{9}$$

To construct the n-period modified binomial tree for the futures price, we start with the initial futures price $F_0 = S_0 \exp[(r-q)T]$. Recall that futures price can be regarded as the price of an underlying asset that pays dividends continuously at the risk-free rate. This is quite clear as:

$$\frac{dF_t}{F_t} = \sigma dz \equiv (r-r)dt + \sigma dz$$

where F_t is the futures price at time t:

$$F_t = S_t \times e^{(r-q)(T-t)}$$

and S_t is the underlying stock price at time t.

Applying the modified binomial model to the futures contract, the ending futures prices at option maturity are:

$$F(n,j) = F_0 u^j d^{n-j} \exp(-rT) \tag{10}$$

for all j = 0, 1, …, n. By substituting out the initial futures price F_0, the ending futures prices are equal to:

$$F(n,j) = S_0 \exp[-(r-q)T] \times u^j d^{n-j} \exp(-rT) = S_0 u^j d^{n-j} \exp(-qT)$$

Comparing with the ending spot prices at option maturity in Eq. (9), it is clear that:

$$F(n,j) = S(n,j)$$

for all j. In other words, the ending futures prices are identical to the corresponding ending spot prices at option maturity. The pair of European options thus have identical payoffs if the modified binomial tree is used to evaluate them.

In addition, the binomial probabilities in the modified binomial model do not depend on the dividend yield, as shown in Eq. (8a). As a result, these probabilities are identical between the spot price tree and the futures price tree. With identical maturity payoffs and identical binomial probabilities, both binomial trees must calculate identical prices for the two options at all corresponding nodes including at the initial point in time. This is verified in the example provided in Figure 5 where we re-evaluate the same pair of European spot and futures call options shown in Figure 1, using the modified binomial model.

Figure 5 about here

As shown in Figure 5, both spot and futures options have identical payoffs at maturity. For example, the highest prices are \$130.06 in both trees while the lowest prices are \$73.87 in both trees. The upward jump probability in each period is 48.23% for both binomial trees. As a result, both binomial trees provide identical prices for the two options prior to maturity at all nodes. The identical price of $4.63 is calculated at the initial point in time. The modified binomial model provides a good solution to the problem identified in the traditional binomial model and values both European spot and futures options consistently. This is true regardless of the number of periods (n) used to build the binomial tree.

4.2. Early exercise and valuation of American options

We next evaluate whether or not the modified binomial model improves the early exercise decisions of American options and subsequently their valuation. As the stock prices in the modified binomial tree adjust for dividend payout along the way, we expect the early exercise decision to be more consistent and less likely to be suboptimal. We verify this in two ways: by first revisiting the example in Figure 2 using a 4-period modified binomial model and then demonstrating how the modified binomial model performs relative to the traditional binomial model as the number of periods n increases. As the results are similar for both American call and put options, we will perform the numerical analysis for American call options only.

Recall that when the 4-period traditional binomial model is used to evaluate the American call option in Figure 2 (Panel A), early exercise is deemed optimal at three nodes in the tree. Referencing the "true" early exercise boundary, however, early exercise is optimal in only one (Node B) of the three cases. At Nodes A and C, early exercise is suboptimal because the stock price is actually below the "true" early exercise boundary. In Figure 6, we show the parallel results when the 4-period modified binomial model is used to evaluate the same American call option. It is clear that the modified binomial model does a better job handling early exercise decisions. Early exercise is only optimal at Node B but not at Node A or C, which is consistent with the correct exercise decision based on the "true" early exercise boundary shown at the bottom of the figure.

Figure 6 about here

Of course, the example in Figure 6 is not sufficient to confirm that the modified binomial model consistently does a better job in making early exercise decisions. With a 4-period binomial model, the discretization errors are likely very large and may swamp any improvement in the estimation of the early exercise boundary. To minimize the impact of discretization errors, we plot the early exercise boundary using a 20-period binomial tree of both the traditional and modified binomial models in Figure 7. The "true" early exercise boundary calculated using a 15,000-period binomial tree is also included for comparison. It is clear from Figure 7 that the early exercise boundary from the 20-period traditional binomial model is severely downward biased when it is compared to the "true" early exercise boundary, especially closer to option maturity. It provides further support that the early exercise of American call option is more frequent and less optimal if the traditional binomial model is used to evaluate the option. In contrast, the early exercise boundary from the 20-period modified binomial model is a much closer approximation to the "true" early exercise boundary, especially closer to maturity. There is also no obvious bias, either upward or downward, as it appears to oscillate around the "true" boundary more closely and symmetrically. The same comparison is also performed when a larger n (e.g., 50) is used and the differences between the two models are quite similar to those shown in Figure 7, both in pattern and magnitude. We thus confirm that the modified binomial model consistently does a better job making early exercise decisions in comparison with the traditional binomial model.

Figure 7 about here

Finally, we want to find out whether or not the modified binomial model improves the accuracy of the estimated price of the American option in comparison with the traditional binomial model. While improving the early exercise decision is certainly important, does it also make the modified binomial model more accurate? For the same American call option in Figure 6, we directly compare the traditional and modified binomial models and use both of them to value the American call option for the same number of binomial periods (n) and then plot these prices for n from 1 to 150. To assess the accuracy of either binomial model, we again estimate the accurate price of the American call option using a 15,000-period traditional binomial model. We also verify this price by using a 15,000-period modified binomial model to estimate the option price. Both models calculate an identical accurate price of \$5.546. The results are summarized in Figure 8.

Figure 8 about here

As shown in Figure 8, it is clear that the modified binomial model is more accurate than the traditional binomial model for pricing the American call option for relatively small or moderately sized n. The difference appears quite pronounced over a wide range of n. Consider first the example shown Panel A where the number of periods is 50 or less. The difference in accuracy between the two models is quite striking. For example, when $n = 4$ which is the case illustrated in Figure 2a and Figure 6, the traditional binomial model calculates the call price to be \$5.237. Compared to the accurate price of \$5.546, the pricing error is –\$0.309. The traditional binomial model undervalues the call by about \$0.31 or 5.6%. In comparison, the corresponding call price calculated by the modified binomial model is \$5.520. The modified binomial model undervalues the same call by only \$0.025 or 0.5%. While not all differences in accuracy are this large, the size of these differences is clearly visible in the plot and appears to be significant for most cases regardless of how large or small n is. Although it appears that the difference in accuracy between the two models diminishes when more than 40 periods are used to evaluate the American call option, this is actually not the case. In Panel B of Figure 8, we plot the prices of the same American call option calculated from the two binomial models as the number of periods (n) increases from 50 to 150. It is clear that the same pattern of relative differences is observed for larger n as well, with the modified binomial model more accurate than the traditional binomial model. Granted, such differences diminish and eventually disappear as a sufficiently large n (e.g., 1,000) is used. It is nevertheless important to point out these differences when it is not practical or economical to use a very large n in some applications.

4.3. More accurate pricing of European options

Interestingly, the modified binomial model is also more accurate than the traditional binomial model when it is used to value European options, especially when a relatively small or moderately sized n is used. This is illustrated in Figure 9, using the same call option as in Figure 8 except that it has European instead of American exercise style. A direct comparison with Panel A of Figure 8 shows the improvement in accuracy for European options by the modified binomial model over the traditional model is quite similar to the corresponding improvement for American options. For example, when $n = 4$, the European call price calculated by the traditional binomial model is \$5.194. Compared to the accurate price of \$5.526, the pricing error is –\$0.332, representing a 6.0% undervaluation. In comparison, the corresponding call price calculated by the modified binomial model is \$5.505. The modified binomial model undervalues the same call by merely \$0.021 or 0.4%. The level of improvement in accuracy for the European call option by the modified binomial model is thus nearly identical to the corresponding improvement for the corresponding American call option. Even the pattern of pricing errors for the European call option are quite similar between the two binomial models, in comparison with the corresponding pattern observed for the matching American call option. These similarities suggest that correctly modeling dividend payment in the binomial model is not just important for early exercise decisions; it also helps improve the accuracy of the binomial model even when early exercise is not allowed. Adjusting probabilities alone, as is done in the traditional binomial model, is perhaps not the best approach to building binomial trees.

Figure 9 about here

5. Conclusions

In this paper, we point out a potential problem in the widely used binomial option pricing model. Rather than letting the stock price fall after dividend payment, it leaves future stock prices unaffected by dividends and makes adjustment to probabilities in order to match the expected return between the binomial tree and the continuous distribution of the underlying asset. This particular choice of dividend adjustment has the unintended consequence of the inconsistent pricing of European spot and futures options and suboptimal early exercise of American options. We propose a simple solution to alleviate the problem by modifying the binomial tree such that future stock prices decline in tandem with the dividend payout. Stock price paths along the modified binomial tree are more natural as they are more consistent with how dividend payments actually impact stock prices in the marketplace. More importantly, we use both theoretical argument and numerical analysis to show that the modified binomial model values pairs of European spot and futures options consistently and improves the early exercise decisions of American options in comparison with the traditional binomial model. Interestingly, the modified binomial model is also more accurate than the traditional binomial model for pricing both

European and American options. The benefits of using the modified binomial tree are thus multifaceted and goes beyond improving early exercise decisions.

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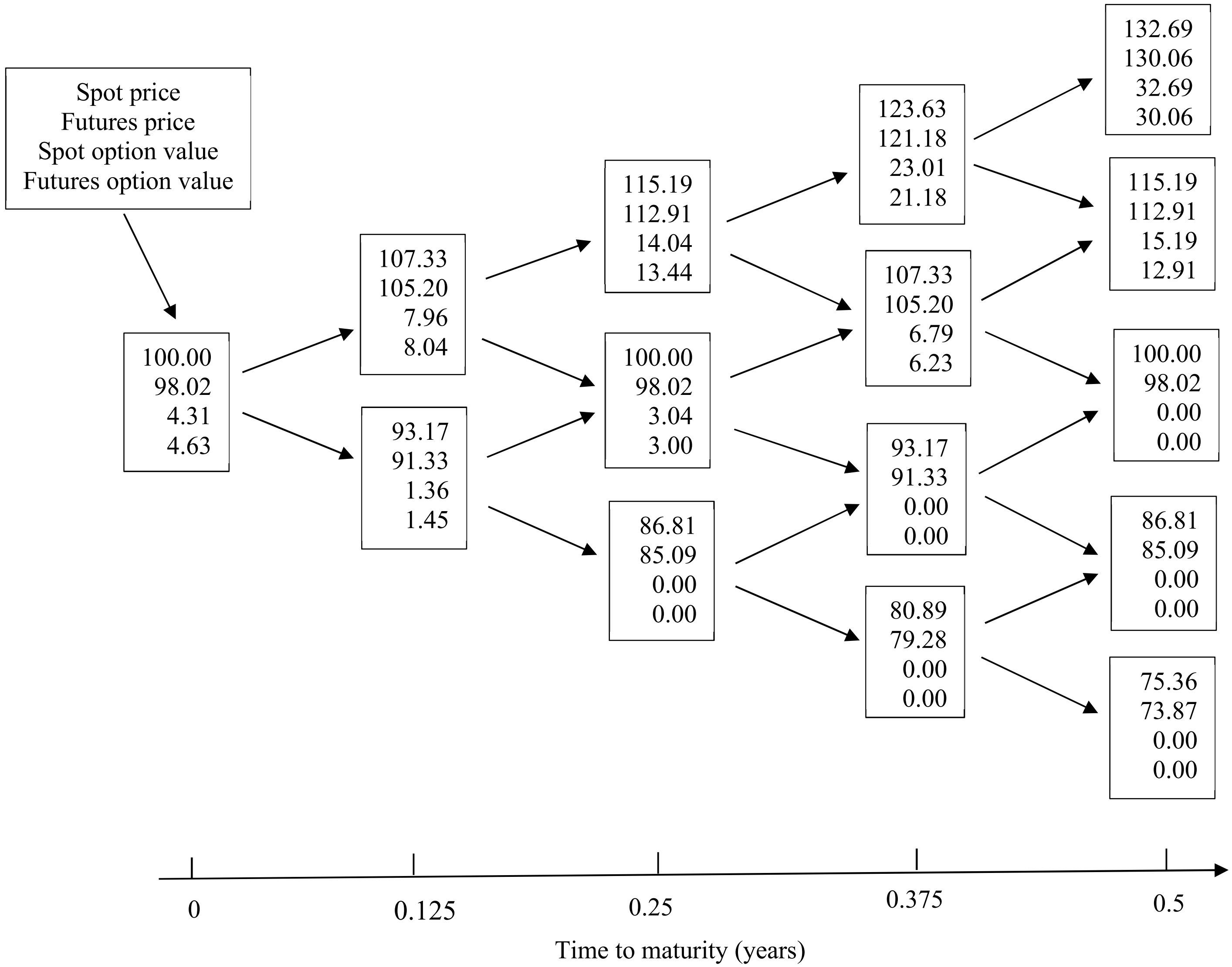
Figure 1

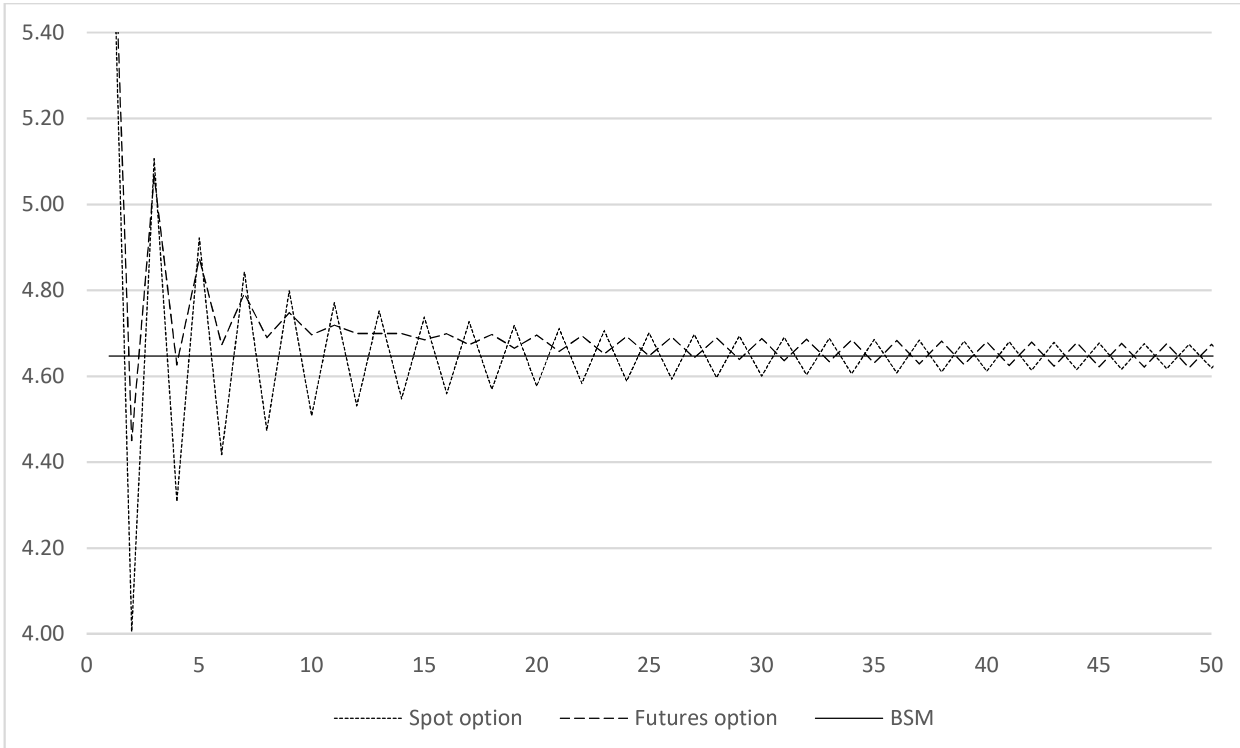
The valuation of European spot and futures call options using the traditional binomial model

The traditional binomial model is used to value a pair of European spot and futures call options that have identical strike price and maturity. The two options theoretically equivalent as the futures contract matures at the same time as the options. The following parameter values are used: the spot price of the underlying asset is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 0, the dividend yield on the stock is 4% and stock return volatility is 20%. In Panel A, a 4-period binomial model is used to illustrate the details of the calculations. At each node of the binomial tree, the calculations are summarized inside a box and listed in the order of (from top to bottom): the spot price of the underlying asset, the futures price, the value of the spot option and the value of the futures option. In Panel B, the binomial prices of spot and futures options are plotted against the number of periods (n) used in the binomial tree as n increases from 1 to 50. For reference, the accurate price is \$4.647 for both options, calculated using the Black-Scholes-Merton model.

Panel A: Valuation using a 4-period traditional binomial model



Panel B: Comparison of option prices as *n* increases from 1 to 50



Panel C: Comparison of option prices as *n* increases from 50 to 100

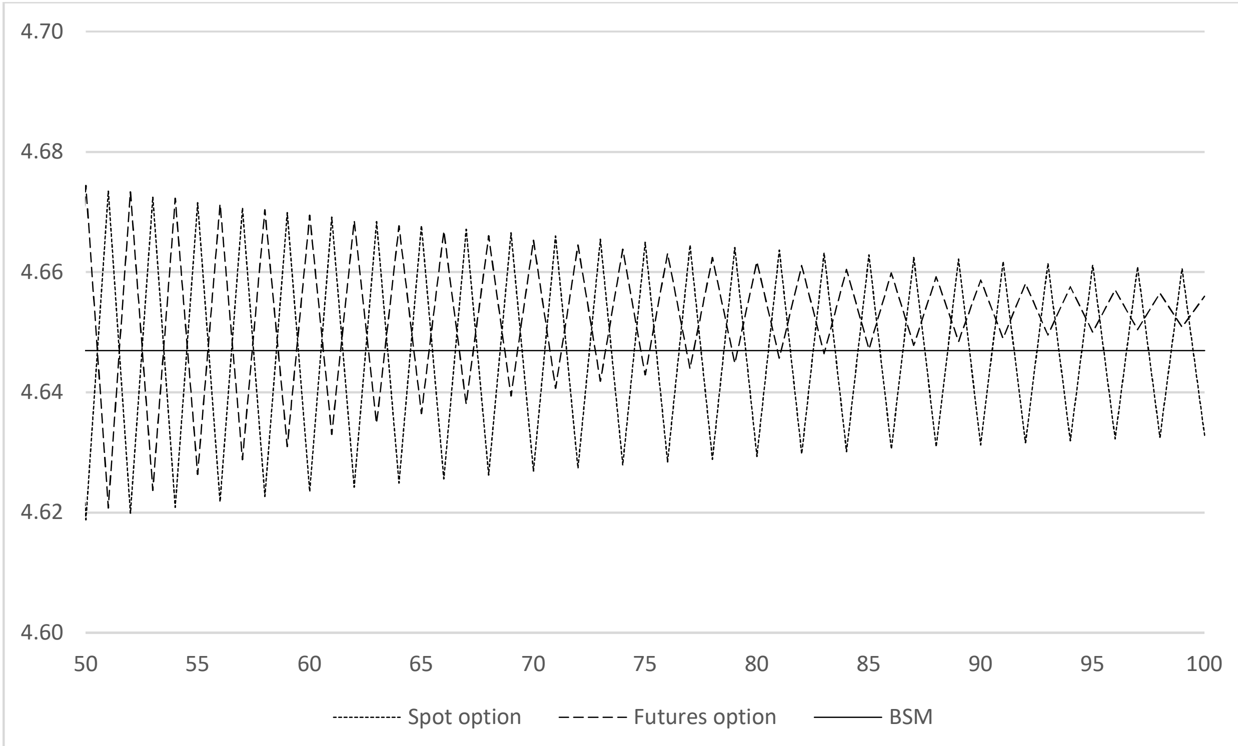
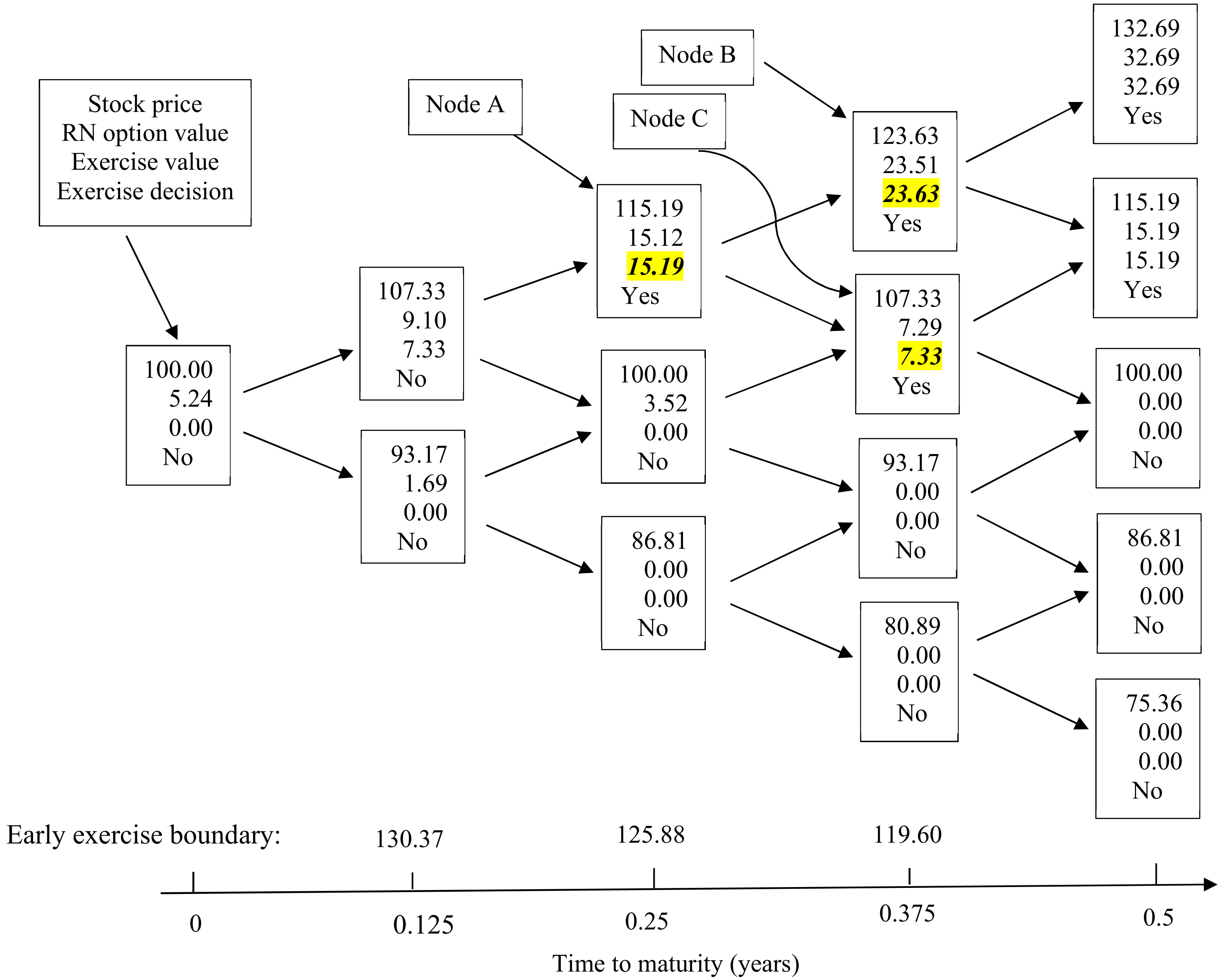


Figure 2
The value and early exercise of American options using the traditional binomial model

A 4-period binomial tree is used to value American options with the following parameter values: the stock price is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield on the stock is 4% and stock return volatility is 20%. The figure illustrates the valuation and early exercise decision at each node in the binomial tree for the American call and put options in Panels A and B, respectively. Inside the box at each node, four pieces of information are shown for the calculation at that node (from the top to the bottom): the stock price, the risk-neutral (RN) value of holding the option for one more period, the exercise value and the decision to exercise (Yes or No). At the bottom of the figure, the early exercise boundary is shown at a given point in time: early exercise is "truly" optimal if the stock price at the node is above (below) the early exercise boundary for the American call (put) option.

Panel A: American call option



Panel B: American put option

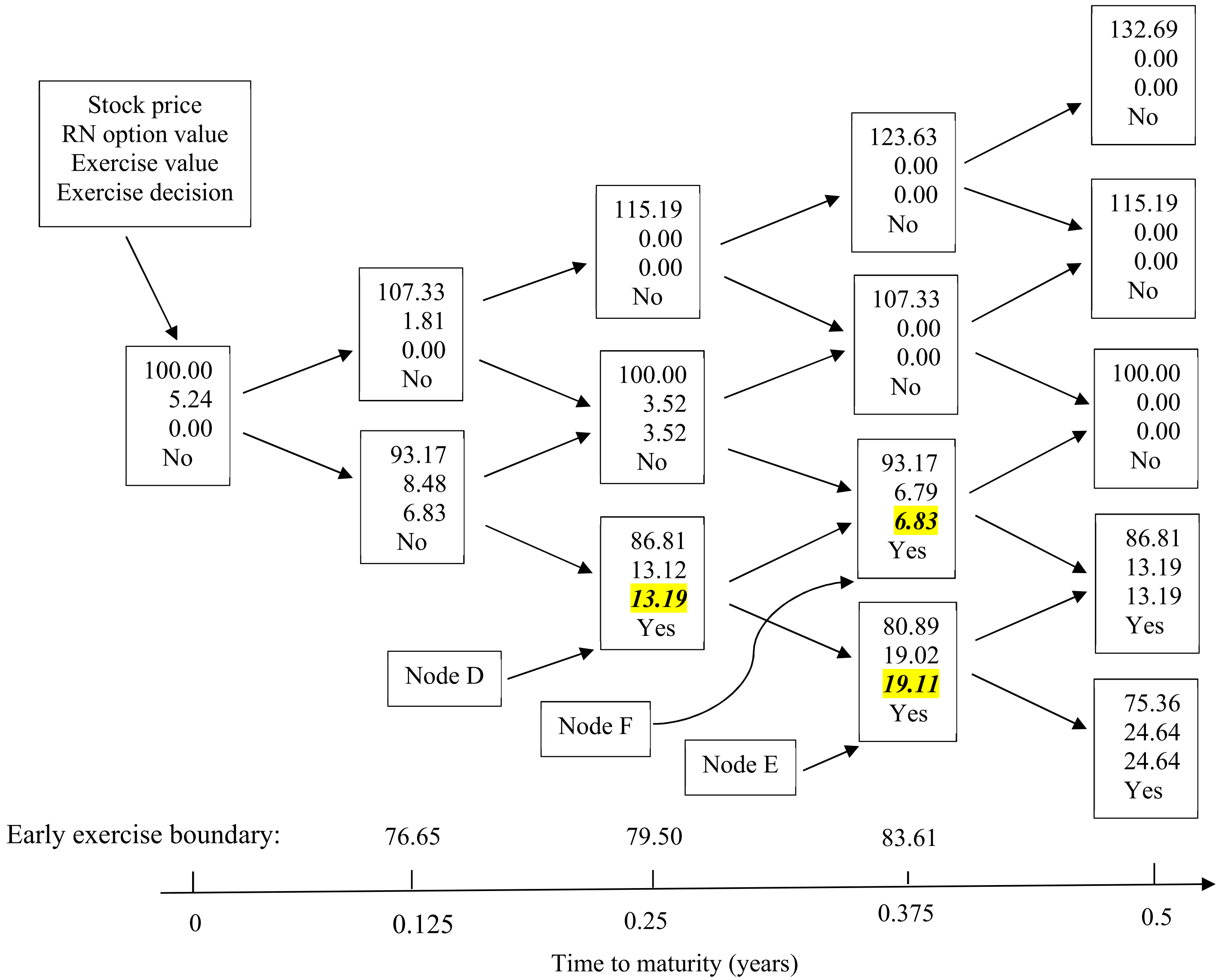
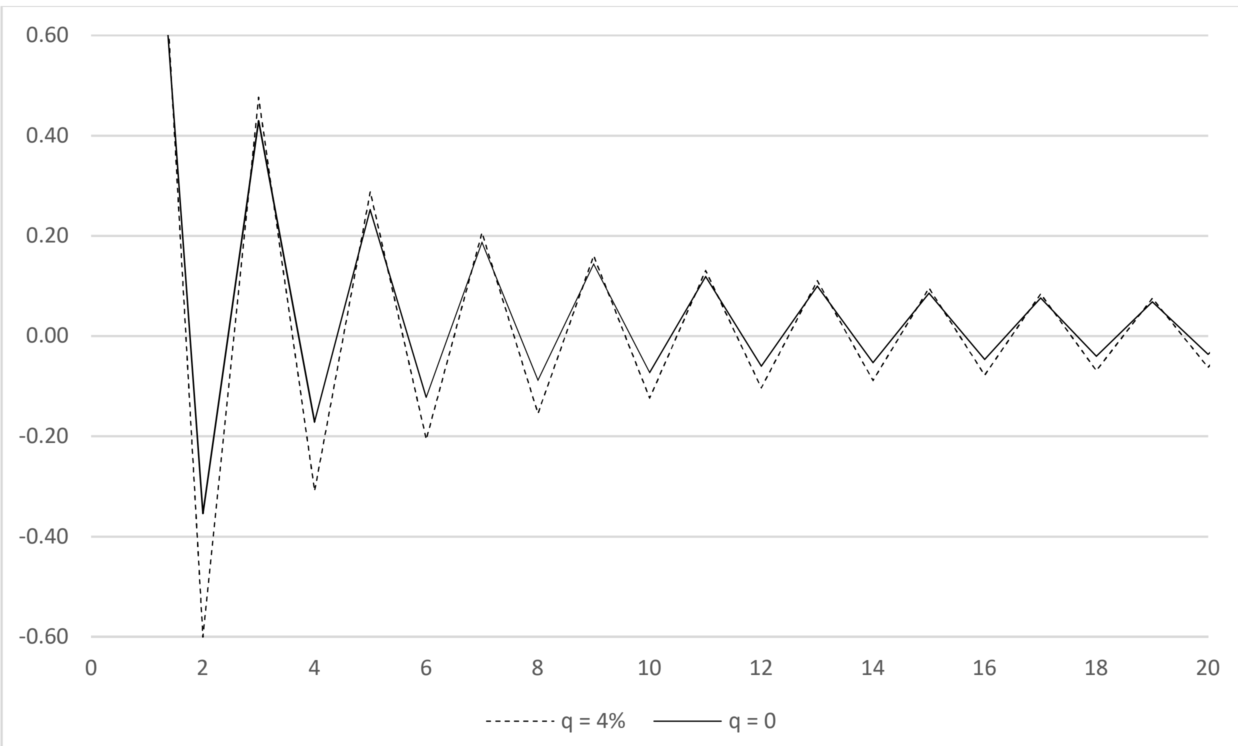


Figure 3

Pricing errors of American put options on a stock with and without dividends

In this figure, we compare the pricing errors of American put options in cases where the underlying stock pay and do not pay dividends. The traditional binomial model with number of periods (n) ranging from 1 to 100 is used to value the American put option. The following parameter values are used: the stock price is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield (q) on the stock is either 4% or 0% and stock return volatility is 20%. The accurate price of the American put option is calculated using a 15,000-period binomial tree. Comparing with this accurate price, the pricing error of the American put option is plotted for the two cases with and without dividends. Panels A and B show the plot for n ranging from 1 to 20 and from 20 to 100, respectively.

Panel A: The number of periods ranging from 1 to 20



Panel B: The number of periods ranging from 20 to 100

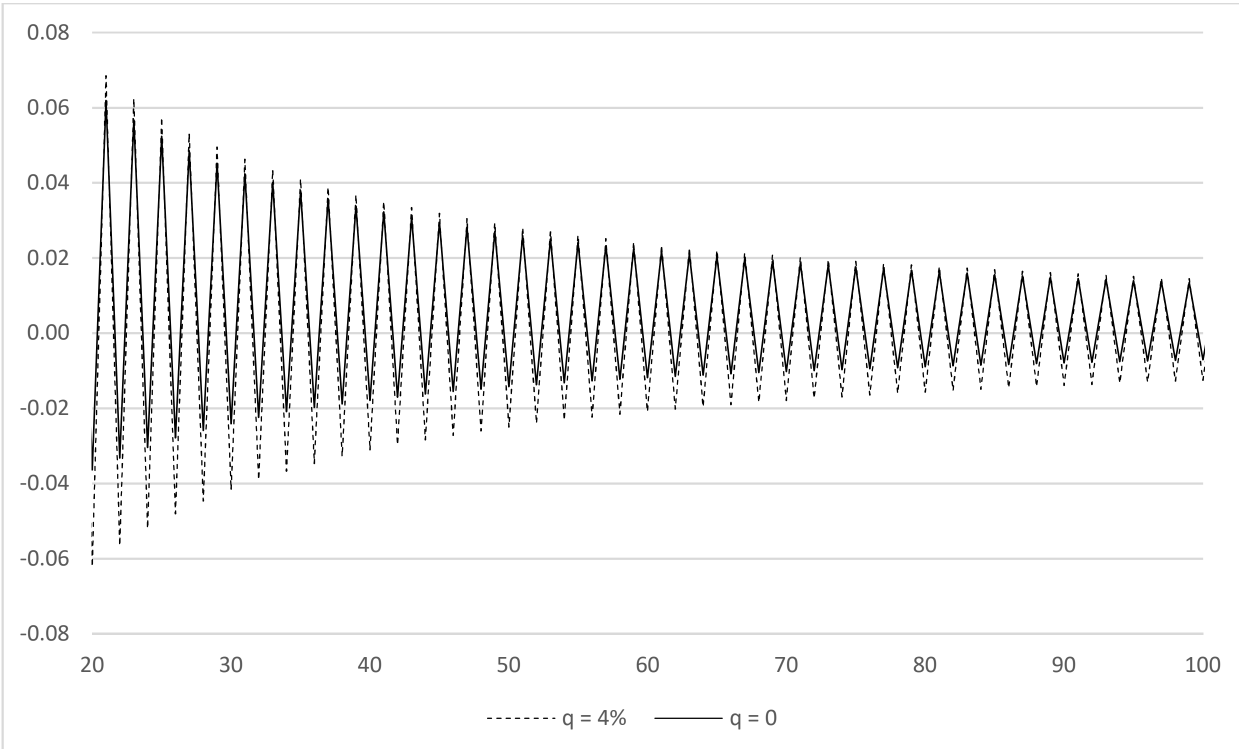
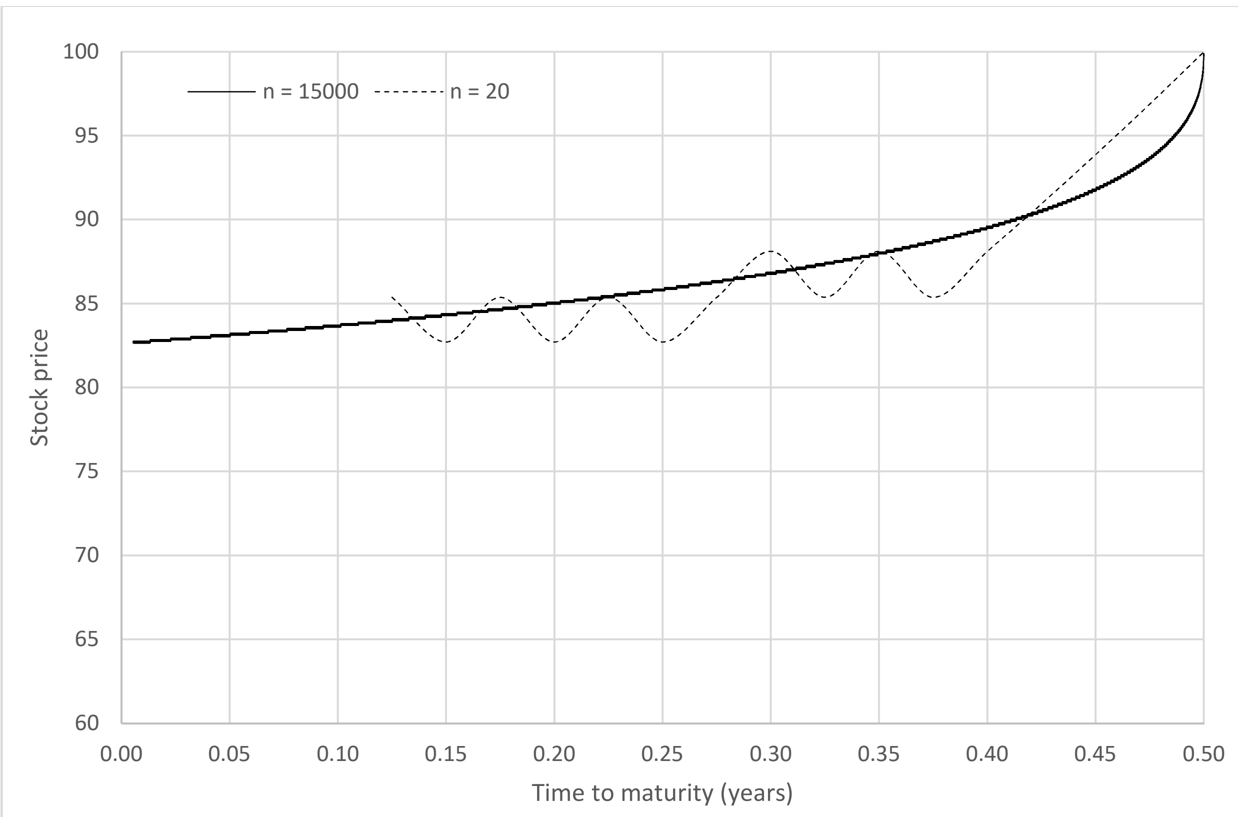


Figure 4

Early exercise boundary of American put options on a stock with and without dividends

In this figure, we compare the early exercise boundary of American put options in cases where the underlying stock pay and do not pay dividends. A 20-period traditional binomial model is used to illustrate the error in the estimated early exercise boundary. The following parameter values are used: the stock price is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield (q) on the stock is either 4% or 0% and stock return volatility is 20%. The "true" early exercise boundary of the American put option is approximated using the one calculated from a 15,000-period traditional binomial model. Panels A and B show the early exercise boundary for the case of a non-dividend paying stock and a case of a stock paying dividends continuously at the 4% rate, respectively.

Panel A: Non-dividend paying stock (q = 0)



Panel B: Dividend paying stock (q = 4%)

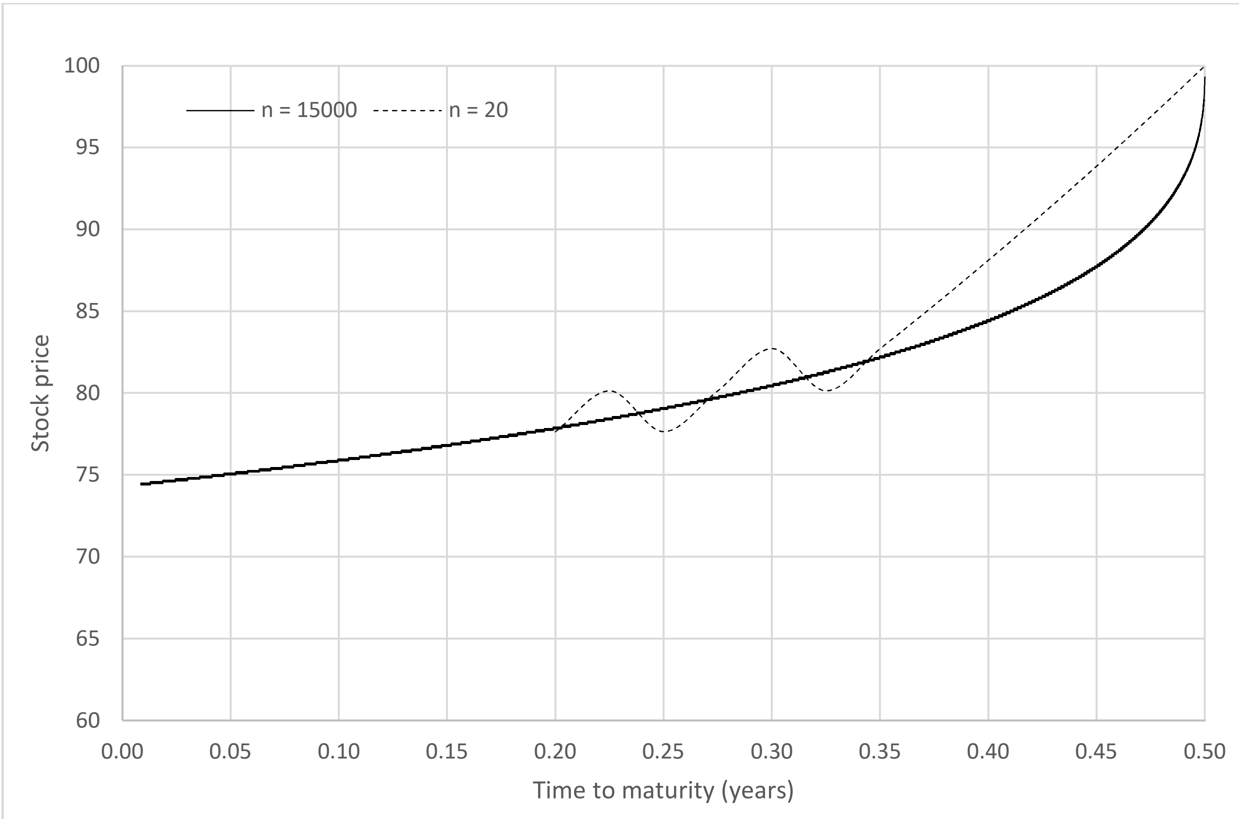


Figure 5

The valuation of European spot and futures call options using the modified binomial model

A 4-period modified binomial model is used to value a pair of European spot and futures call options that have identical strike price and maturity. The two options theoretically equivalent as the futures contract matures at the same time as the options. The following parameter values are used: the spot price of the underlying asset is $100, the strike price is $100, option maturity is 6 months, the risk-free rate is 0, the dividend yield on the stock is 4% and stock return volatility is 20%. In the figure, the calculations at each node are summarized inside a box and listed in the order of (from top to bottom): the spot price of the underlying asset, the futures price, the value of the spot option and the value of the futures option. For reference, the accurate price is $4.647 for both spot and futures options, calculated using the Black-Scholes-Merton model.

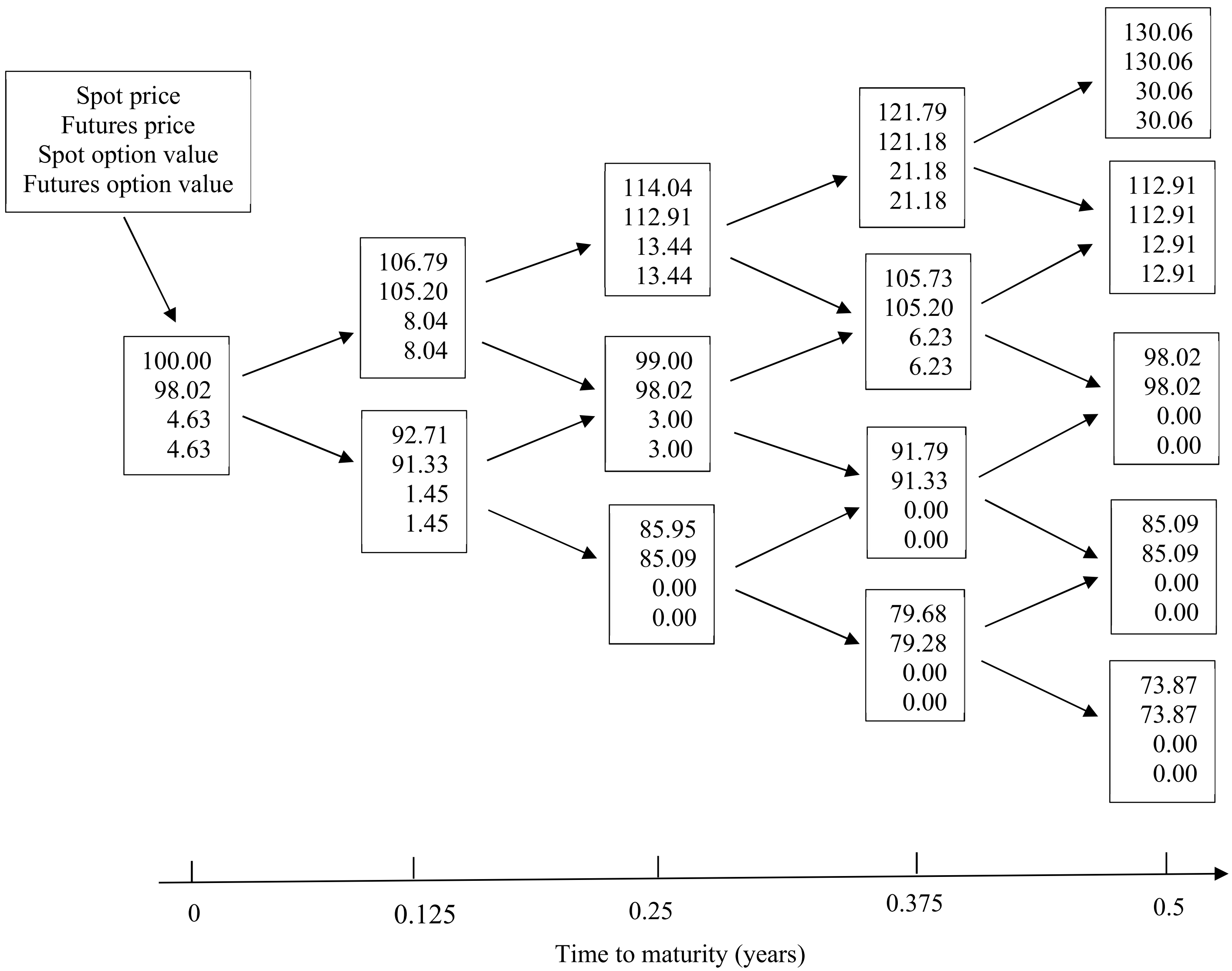

Spot price
Futures price
Spot option value
Futures option value
100.00
98.02
4.63
4.63
106.79
105.20
8.04
8.04
92.71
91.33
1.45
1.45
114.04
112.91
13.44
13.44
99.00
98.02
3.00
3.00
85.95
85.09
0.00
0.00
121.79
121.18
21.18
21.18
105.73
105.20
6.23
6.23
91.79
91.33
0.00
0.00
79.68
79.28
0.00
0.00
130.06
130.06
30.06
30.06
112.91
112.91
12.91
12.91
98.02
98.02
0.00
0.00
85.09
85.09
0.00
0.00
73.87
73.87
0.00
0.00
0
0.125
0.25
0.375
0.5
Time to maturity (years)

Figure 6

The value and early exercise of American call option using the modified binomial model

A 4-period modified binomial tree is used to value an American call option with the following parameter values: the stock price is $100, the strike price is $100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield on the stock is 4% and stock return volatility is 20%. The figure illustrates the valuation and early exercise decision at each node in the binomial tree for the American call option. Inside the box at each node, four pieces of information are shown for the calculation at that node (from the top to the bottom): the stock price, the risk-neutral (RN) value of holding the option for one more period, the exercise value and the decision to exercise (Yes or No). At the bottom of the figure, the early exercise boundary is shown at a given point in time: early exercise is optimal if the stock price at the node is above the early exercise boundary for the American call option.

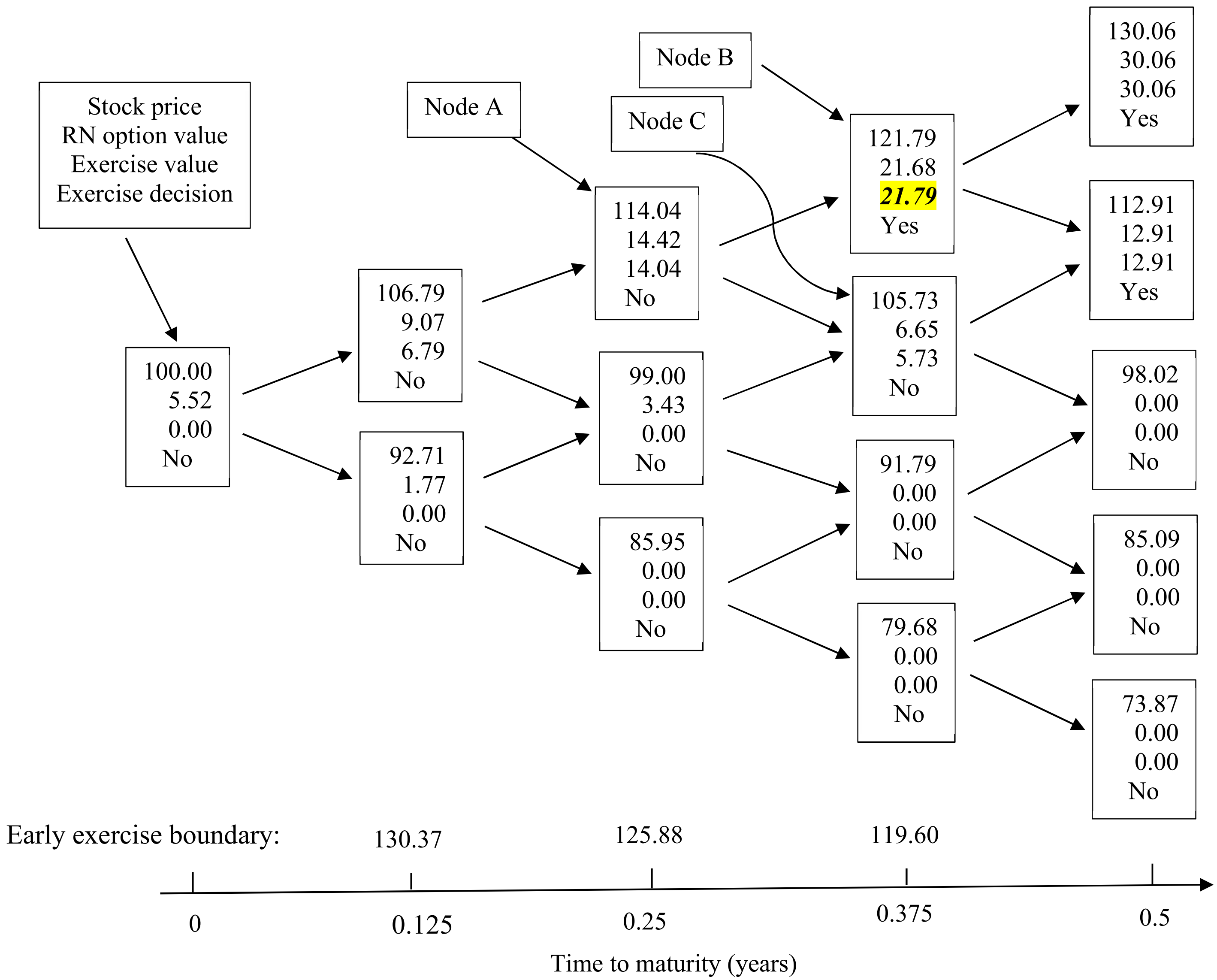

Stock price
RN option value
Exercise value
Exercise decision
Node A
Node B
Node C
100.00
5.52
0.00
No
106.79
9.07
6.79
No
92.71
1.77
0.00
No
114.04
14.42
14.04
No
99.00
3.43
0.00
No
85.95
0.00
0.00
No
121.79
21.68
21.79
Yes
105.73
6.65
5.73
No
91.79
0.00
0.00
No
79.68
0.00
0.00
No
130.06
30.06
30.06
Yes
112.91
12.91
12.91
Yes
98.02
0.00
0.00
No
85.09
0.00
0.00
No
73.87
0.00
0.00
No
Early exercise boundary:
130.37
125.88
119.60
0
0.125
0.25
0.375
0.5
Time to maturity (years)

Figure 7
Comparison of early exercise boundary of American call options between the traditional and modified binomial models

In this figure, we compare the early exercise boundary of American call options between the traditional and modified binomial models. A 20-period binomial model is used in both cases. The following parameter values are used: the stock price is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield on the stock is 4% and stock return volatility is 20%. The "true" early exercise boundary of the American call option is approximated by the one calculated from a 15,000-period traditional binomial model.

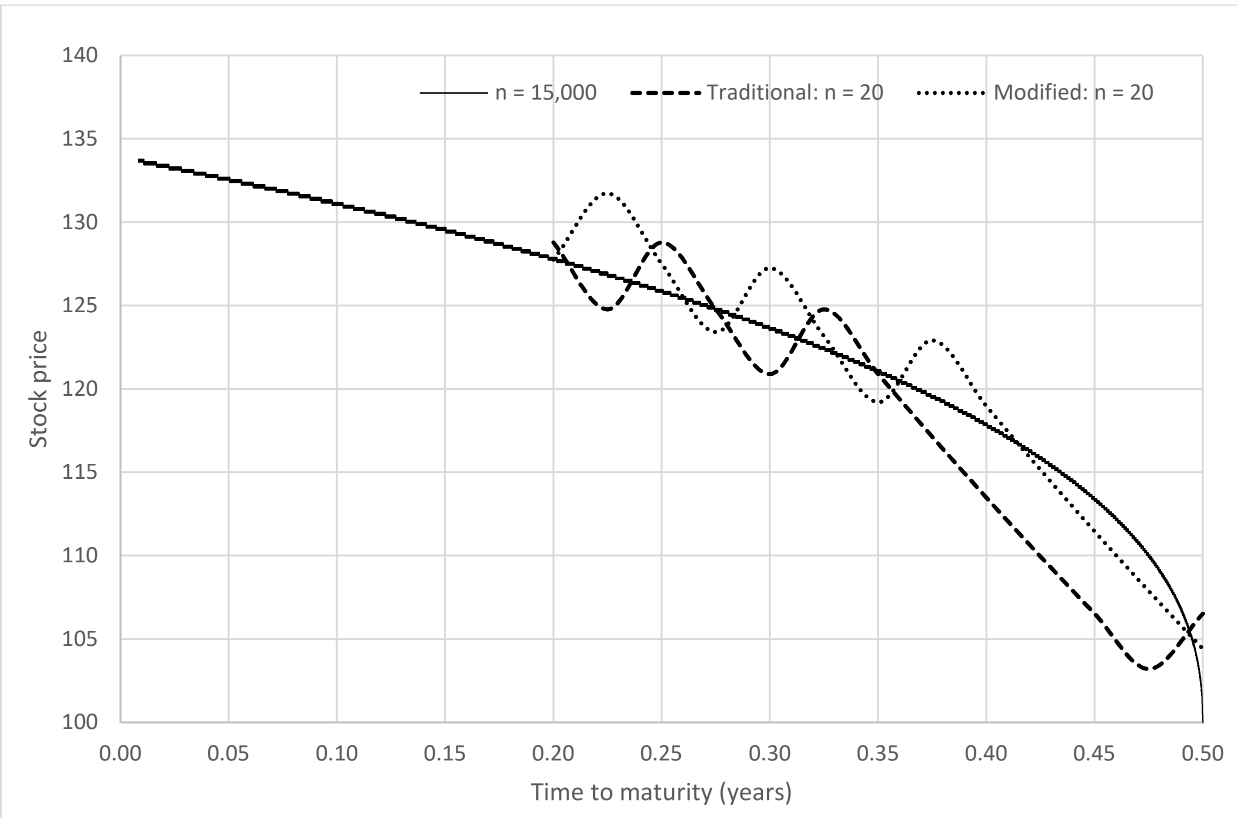
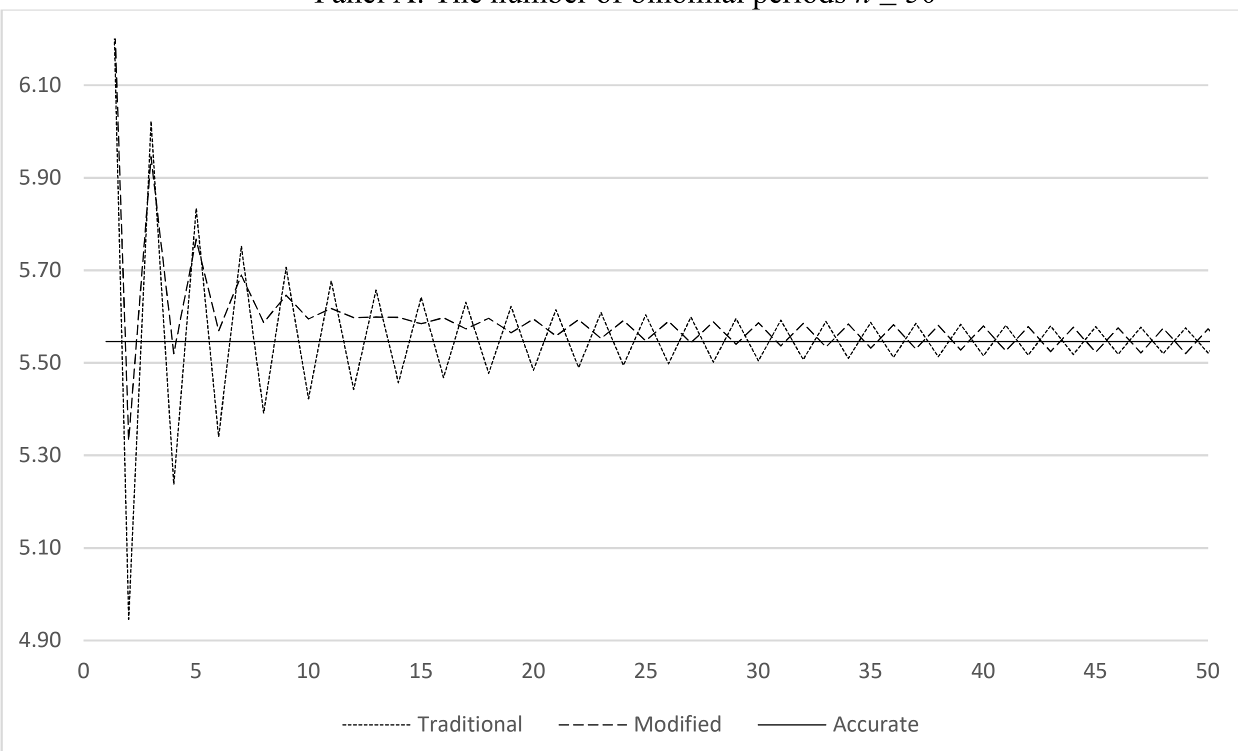


Figure 8

Comparison of accuracy between the traditional and modified binomial models: American call option

Both the traditional and modified binomial models are used to evaluate an American call option with the following parameter values: the stock price is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield on the stock is 4% and stock return volatility is 20%. The figure plots and compare the call prices from the two binomial models as the number of binomial periods (n) increases from 1 to 50 (Panel A) and from 50 to 150 (Panel B). The accurate price of the American call option is estimated using a 15,000-period traditional binomial tree.

Panel A: The number of binomial periods $n \leq 50$



Panel B: The number of binomial periods (n) between 50 and 150

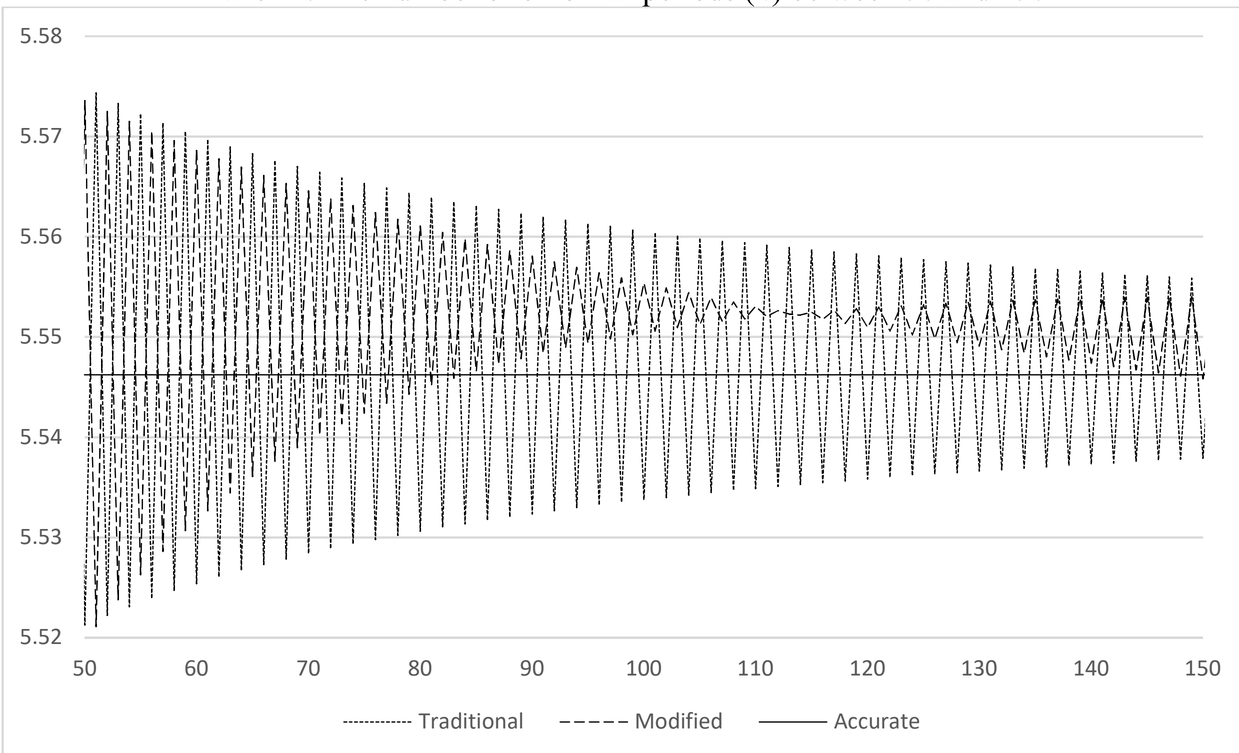


Figure 9

Comparison of accuracy between the traditional and modified binomial models: European call option

Both the traditional and modified binomial models are used to evaluate a European call option with the following parameter values: the stock price is \$100, the strike price is \$100, option maturity is 6 months, the risk-free rate is 4%, the dividend yield on the stock is 4% and stock return volatility is 20%. The figure plots and compare the call prices from the two binomial models as the number of binomial periods (n) increases from 1 to 50. The closed-form solution of the Black-Scholes-Merton model is used to calculate the accurate price of the European call option.