Porosity of Free Boundaries in A-Obstacle Problems

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Abstract

We establish the exact growth of the solution of the A-obstacle problem near the free boundary from which we deduce its porosity.

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Introduction

The solution of the obstacle problem in the degenerate and singular cases is known to be $C^{1,\alpha}$ but there is only little information regarding the free boundary. Recently, it has been proved, for an identically zero obstacle (see [4]), that the free boundary is porous and hence of Lebesgue's measure zero.

In this paper, we give the exact growth of the solution of the A-obstacle problem near the free boundary which agrees with the one for the p-Laplacian when $A(t) = t^p$. Then we deduce the porosity result.

1 Statement of the problem and preliminary results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$, $f \in L^\infty(\Omega)$ and $g \in W^{1,A}(\Omega) \cap L^\infty(\Omega)$, $g \geqslant 0$. We consider the obstacle problem for the A-Laplace equation

$$\begin{cases} \Delta_A u = div\Big(\dfrac{a(|\nabla u|)}{|\nabla u|}\nabla u\Big) = f(x) \qquad \text{in} \quad [u > 0], \\ u \geqslant 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

The weak formulation of this problem is given by the following variational inequality

$$(P)\begin{cases} \text{Find } u \in K_g \text{ such that :} \\ \displaystyle\int_\Omega \Big(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u.\nabla(v-u) + f(x)(v-u)\Big) dx \geqslant 0 \qquad \forall v \in K_g, \end{cases}$$

where $\quad K_g = \{v \in W^{1,A}(\Omega), \quad v - g \in W_0^{1,A}(\Omega), \quad v \geqslant 0 \quad \text{a.e. in } \Omega\}$,
$A(t) = \displaystyle\int_0^t a(\tau)d\tau$, a is a C^1 function from $[0,+\infty[$ into $[0,+\infty[$ such that $a(0) = 0$,

$$a_0 \leqslant \frac{ta'(t)}{a(t)} \leqslant a_1 \quad \forall t > 0, \quad a_0, a_1 \text{ are positive constants.} \tag{1.1}$$

As a consequence of (1.1), we can easily verify (see [2]) that

$$\Big(\frac{a(|\xi|)}{|\xi|}\xi - \frac{a(|\zeta|)}{|\zeta|}\zeta\Big)(\xi - \zeta) > 0 \qquad \forall \xi, \zeta \in \mathbb{R}^n \setminus \{0\}, \quad \xi \neq \zeta. \tag{1.2}$$

We recall the definition of the Orlicz class $\widetilde{L}^A(\Omega)$

$$\widetilde{L}^A(\Omega) = \{u : \Omega \to \mathbb{R} \cup \{-\infty, \infty\} \ / \ u \text{ is measurable and } \int_\Omega A(|u(x)|)dx < +\infty\}.$$

The Orlicz space $L^A(\Omega)$ is defined by

$$L^A(\Omega) = \{u : \Omega \to \mathbb{R}\cup\{-\infty, \infty\} \ / \ u \text{ is measurable and } \exists k > 0 : \int_\Omega A\Big(\frac{|u(x)|}{k}\Big)dx < +\infty\}.$$

We easily deduce from (1.1) (see [6]) that

$$A(st) \leqslant (1 + a_1)\max(s^{1+a_0}, s^{1+a_1})A(t) \qquad \forall s, t \geqslant 0$$

from which it is clear that $\widetilde{L}^A(\Omega) = L^A(\Omega)$. Moreover $L^A(\Omega)$ is a Banach space when equipped with the Luxemburg norm

$$\|u\|_A = \inf\Big\{k > 0, \quad \int_\Omega A\Big(\frac{|u(x)|}{k}\Big)dx \leqslant 1\Big\}.$$

The Orlicz-Sobolev space is defined by $W^{1,A}(\Omega) = \{u \in L^A(\Omega) \ / \ |\nabla u| \in L^A(\Omega)\}$. It is a Banach space when equipped with the norm $\|u\|_{1,A} = \|u\|_A + \|\nabla u\|_A$. The space $W_0^{1,A}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,A}(\Omega)$ with respect to the norm $\|.\|_{1,A}$. For definitions and properties of Orlicz and Orlicz-Sobolev spaces, we refer the reader to [1].

Now we give some useful inequalities that will be needed afterwards and which can be deduced from (1.1) (see [6]).

$$\min(s^{1/a_0}, s^{1/a_1})a^{-1}(t) \leqslant a^{-1}(st) \leqslant \max(s^{1/a_0}, s^{1/a_1})a^{-1}(t) \qquad \forall s, t \geqslant 0, \tag{1.3}$$

$$\frac{a_0}{1+a_0} t a^{-1}(t) \leqslant \widetilde{A}(t) = \int_0^t a^{-1}(\tau)d\tau \leqslant t a^{-1}(t) \qquad \forall t \geqslant 0, \tag{1.4}$$

$$\widetilde{A}(st) \leqslant \frac{a_0}{1+a_0} \max(s^{1+1/a_0}, s^{1+1/a_1})\widetilde{A}(t) \qquad \forall s, t \geqslant 0. \tag{1.5}$$

a^{-1} is the inverse function of a which exists since a is continuous and increasing on $[0, \infty)$.

For the existence of a solution of (P), we refer to [3] or [5]. Moreover using the monotonicity property (1.2), it is easy to show that the solution is unique. Regarding regularity, we refer to [7] where it is proved that $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. In particular, $[u > 0]$ is an open set. As a matter of fact this regularity will not be used in the proofs of our results. Indeed only the continuity of the solution and the following properties will be used.

Proposition 1.1. *Let u be the solution of (P).*

$$\begin{array}{l} f \geqslant 0 \quad \text{in } \Omega \qquad \Longrightarrow \qquad 0 \leqslant u \leqslant \|g\|_\infty \quad \text{in } \Omega. \\ \Delta_A u = f \qquad \text{in } \mathcal{D}'([u > 0]). \\ f\chi([u > 0]) \leqslant \Delta_A u \leqslant f \qquad \text{a.e. in } \Omega. \end{array}$$

Proof. i) Note that $u \geqslant 0$ since $u \in K_g$. To get the upper bound of u, we take $\min(u, \|g\|_\infty) = u - (u - \|g\|_\infty)^+$ as a test function in (P). We obtain

$$\int_\Omega a(|\nabla(u - \|g\|_\infty)^+|)|\nabla(u - \|g\|_\infty)^+| \leqslant -\int_\Omega f(u - \|g\|_\infty)^+ \leqslant 0.$$

Then $\nabla(u - \|g\|_\infty)^+ = 0$ a.e. in Ω. We deduce that $(u - \|g\|_\infty)^+ = 0$ a.e. in Ω since $u = g$ on $\partial\Omega$.

ii) Let $\zeta \in \mathcal{D}([u > 0])$, $\zeta \geqslant 0$. Set $\delta = \min_{supp \zeta} u$ and take $u \pm \delta \frac{\zeta}{\|\zeta\|_\infty}$ as test functions for (P). We get

$$\int_{[u>0]} \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \nabla \zeta + f\zeta = 0 \quad \text{and then} \quad -\Delta_A u + f = 0 \quad \text{in } \mathcal{D}'([u > 0]).$$

iii) Let $\zeta \in \mathcal{D}(\Omega)$, $\zeta \geqslant 0$. Since $u + \zeta \in K_g$, we have

$$\int_\Omega \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \nabla \zeta + f\zeta \geqslant 0 \quad \text{and then} \quad -\Delta_A u + f \geqslant 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Now for $\epsilon > 0$, $u - H_\epsilon(u-\epsilon)\zeta \in K_g$ with $H_\epsilon(s) = \min\big(1, \dfrac{s^+}{\epsilon}\big)$. Then we have

$$\int_\Omega H_\epsilon(u-\epsilon)\frac{a(|\nabla u|)}{|\nabla u|}\nabla u \nabla\zeta + fH_\epsilon(u-\epsilon)\zeta \leqslant 0.$$

Letting $\epsilon \longrightarrow 0$, we obtain

$$\int_\Omega \frac{a(|\nabla u|)}{|\nabla u|}\nabla u\nabla\zeta + f\chi([u>0])\zeta \leqslant 0 \quad \text{ and then } \quad -\Delta_A u + f\chi([u>0]) \leqslant 0 \quad \text{ in } \mathcal{D}'(\Omega).$$

□

In all what follows, we assume that there exists two positive constants λ_0 and Λ_0 such that

$$0 < \lambda_0 \leqslant f \leqslant \Lambda_0 \qquad \text{a.e. in } \Omega.$$

2 A class of functions on the unit ball

In this section, we study a family $\mathcal{F}_A$ of problems defined on the unit ball $B_1 = B_1(0)$. More precisely, $u \in \mathcal{F}_A$ if it satisfies :

$$u \in W^{1,A}(B_1), \qquad \|\Delta_A u\|_{L^\infty(B_1)} \leqslant 1$$
$$0 \leqslant u \leqslant 1 \quad \text{ in } B_1, \qquad u(0) = 0.$$

We know (see [7]) that $u \in C^{1,\alpha}(B_1)$ for some $\alpha \in (0,1)$. The following theorem gives a growth of the elements of $\mathcal{F}_A$. This extends a result proved in [4] for the p-Laplace operator where the authors have proved a growth of order $p' = \frac{p}{p-1}$.

Theorem 2.1. *There exists a positive constant $C = C(a,n)$ such that for every $u \in \mathcal{F}_A$, we have*

$$0 \leqslant u(x) \leqslant C\widetilde{A}(|x|) \qquad \forall x \in B_1$$

where $\widetilde{A}$ is the function defined in (1.4).

In order to prove Theorem 2.1, we need to introduce some notations inspired from [4]. For a nonnegative bounded function u, we define the quantities

$$S(r,u,y) = \sup_{x \in B_r(y)} u(x), \qquad S(r,u) = S(r,u,0).$$

We also define for $u \in \mathcal{F}_A$ the set

$$\mathbb{M}(u) = \{j \in \mathbb{N} / \quad mS(2^{-j-1}, u) \geqslant S(2^{-j}, u)\}$$

where $m = 2^{1+\frac{1}{a_0}} \dfrac{a_0}{1+a_0}$ is the number that appears in the inequality (1.5) for $s = 2$, i.e.

$$\widetilde{A}(2t) \leqslant m\widetilde{A}(t) \qquad \forall t \geqslant 0.$$

Then we have

Lemma 2.1. *Assume that $\mathbb{M}(u) \neq \emptyset$. Then, there exists a constant $C_1 = C_1(a, n)$ such that*

$$S(2^{-j-1}, u) \leqslant C_1\widetilde{A}(2^{-j}) \qquad \forall u \in \mathcal{F}_A, \quad \forall j \in \mathbb{M}(u).$$

Proof. We argue by contradiction. Then,

$$\forall k \in \mathbb{N}, \quad \exists u_k \in \mathcal{F}_A, \quad \exists j_k \in \mathbb{M}(u_k) \quad \text{ such that } \quad S(2^{-j_k-1}, u_k) \geqslant k\widetilde{A}(2^{-j_k}).$$

Consider $\omega_k(x) = \dfrac{u_k(2^{-j_k}x)}{2^{-j_k}}$ defined in B_1. We have

$$\Delta_A\omega_k(x) = 2^{-j_k}(\Delta_A u_k)(2^{-j_k}x), \qquad \|\Delta_A\omega_k\|_\infty \leqslant 2^{-j_k}$$

$$0 \leqslant \omega_k \leqslant \frac{1}{2^{-j_k}} \quad \text{ in } B_1, \qquad \inf_{B_{1/2}} \omega_k = 0.$$

By Harnack's inequality (see Corollary 1.4 [6]), we have for some constant $C = C(n, a_0, a_1)$,

$$\sup_{B_{1/2}} \omega_k \leqslant C\big(\inf_{B_{1/2}} \omega_k + a^{-1}(2^{-j_k})\big) = Ca^{-1}(2^{-j_k}).$$

Now let $v_k(x) = \dfrac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}\omega_k(x) = \dfrac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)}$ defined in B_1. We have

$$1 = \sup_{B_{1/2}} v_k \leqslant C\frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}a^{-1}(2^{-j_k}).$$

By assumption, we have $\dfrac{1}{S(2^{-j_k-1}, u_k)} \leqslant \dfrac{1}{k\widetilde{A}(2^{-j_k})}$. Then

$$1 \leqslant \frac{C}{k} \cdot \frac{2^{-j_k}a^{-1}(2^{-j_k})}{\widetilde{A}(2^{-j_k})}.$$

By (1.4), we have $\dfrac{ta^{-1}(t)}{\widetilde{A}(t)} \leqslant \dfrac{1+a_0}{a_0} \qquad \forall t > 0$. Hence, we get

$$1 \leqslant \frac{C}{k} \cdot \frac{1+a_0}{a_0}.$$

Letting $k \longrightarrow +\infty$, we get a contradiction. $\square$

Proof of Theorem 2.1. First, we prove that for $C_2 = \max(mC_1, 1/\widetilde{A}(1))$ we have

$$S(2^{-j}, u) \leqslant C_2\widetilde{A}(2^{-j}) \qquad \forall j \in \mathbb{N}.$$

We proceed by induction.
For $j = 0$, we have $S(2^{-0}, u) = S(1, u) \leqslant 1 \leqslant C_2\widetilde{A}(2^{-0})$ since $C_2 \geqslant 1/\widetilde{A}(1)$.
Let $j \geqslant 1$. Assume that $S(2^{-j}, u) \leqslant C_2\widetilde{A}(2^{-j})$. We distinguish two cases :

– If $j \in \mathbb{M}(u)$, we have by Lemma 2.1 and (1.5),

$$S(2^{-j-1}, u) \leqslant C_1\widetilde{A}(2^{-j}) = C_1\widetilde{A}(2.2^{-j-1}) \leqslant C_1 m\widetilde{A}(2^{-j-1}) \leqslant C_2\widetilde{A}(2^{-j-1}).$$

– If $j \notin \mathbb{M}(u)$, we have $S(2^{-j-1}, u) < S(2^{-j}, u)/m$. Using the induction assumption and (1.5), we get

$$S(2^{-j-1}, u) \leqslant \frac{1}{m}C_2\widetilde{A}(2^{-j}) \leqslant C_2\widetilde{A}(2^{-j-1}).$$

Now, let $x \in B_1$ and set $r = |x|$. Then there exists $j \in \mathbb{N}$ such that $2^{-j-1} \leqslant r \leqslant 2^{-j}$ and we have

$$u(x) \leqslant \sup_{\overline{B_r}} u = S(r, u) \leqslant S(2^{-j}, u) \leqslant C_2\widetilde{A}(2^{-j}) \leqslant C_2\widetilde{A}(2r) \leqslant C_2.m\widetilde{A}(r) = C\widetilde{A}(|x|).$$

□

3 Porosity of the free boundary

The following lemma gives the growth of the solution of (P) near the free boundary $(\partial[u > 0]) \cap \Omega$.

Lemma 3.1. *Suppose that $u \in W^{1,A}(\Omega)$ is a nonnegative continuous function satisfying*

$$\Delta_A u = f \quad in \quad \mathcal{D}'([u > 0]).$$

Then for each $y \in \overline{[u > 0]}$ and $r > 0$ such that $B_r(y) \subset \Omega$, we have

$$S(r, u, y) \geqslant \sup_{\partial B_r(y)} u \geqslant \frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}r\Big) + u(y).$$

Proof. It is enough to prove the result for $y \in [u > 0]$. We consider for $\epsilon > 0$ the following functions

$$u_\epsilon(x) = u(x) - (1 - \epsilon)u(y), \qquad v(x) = \frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}|x - y|\Big).$$

Then we have

$$\nabla v = \widetilde{A}'\Big(\frac{\lambda_0}{n}|x-y|\Big)\frac{x-y}{|x-y|} = a^{-1}\Big(\frac{\lambda_0}{n}|x-y|\Big)\frac{x-y}{|x-y|}, \qquad |\nabla v| = a^{-1}\Big(\frac{\lambda_0}{n}|x-y|\Big)$$

and then

$$\Delta_A v = div\Big(\frac{a(|\nabla v|)}{|\nabla v|}\nabla v\Big) = div\Big(\frac{\lambda_0}{n}|x-y|\frac{1}{a^{-1}\Big(\frac{\lambda_0}{n}|x-y|\Big)}.a^{-1}\Big(\frac{\lambda_0}{n}|x-y|\Big).\frac{x-y}{|x-y|}\Big)$$

$$= div\Big(\frac{\lambda_0}{n}(x-y)\Big) = \lambda_0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Now we have

$$\Delta_A u_\epsilon = \Delta_A u = f \geqslant \lambda_0 = \Delta_A v \quad \text{in} \quad B_r(y) \cap [u>0].$$

Moreover

$$u_\epsilon = -(1-\epsilon)u(y) \leqslant 0 \leqslant v \quad \text{on} \quad (\partial[u>0]) \cap B_r(y).$$

If we assume that

$$u_\epsilon \leqslant v \quad \text{on} \quad (\partial B_r(y)) \cap [u>0],$$

then we get by the weak maximum principle

$$u_\epsilon \leqslant v \quad \text{in} \quad B_r(y) \cap [u>0].$$

But $u_\epsilon(y) = \epsilon u(y) > 0 = v(y)$ which is a contradiction.

So we have

$$\sup_{\partial B_r(y)} (u-(1-\epsilon)u(y)) = \sup_{\partial B_r(y)} u_\epsilon \geqslant \sup_{\partial B_r(y)\cap[u>0]} u_\epsilon \geqslant \sup_{\partial B_r(y)\cap[u>0]} v = \frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}r\Big).$$

Letting $\epsilon \longrightarrow 0$, we get

$$S(r,u,y) = \sup_{\overline{B_r(y)}} u \geqslant \sup_{\partial B_r(y)} u \geqslant \frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}r\Big) + u(y).$$

□

The main result of this paper is the following theorem establishing the porosity of the free boundary $(\partial[u>0]) \cap \Omega$.

Theorem 3.1. *Let u be a solution of (P). Let $R>0$ and $x_0 \in \Omega$. Then for every closed ball $\overline{B_R(x_0)} \subset \Omega$, the intersection $(\partial[u>0]) \cap \overline{B_R(x_0)}$ is porous with porosity constant depending only on $\|g\|_\infty$, λ_0, Λ_0, $dist(\overline{B_R(x_0)}, \partial\Omega)$, a and n.*

We recall that a set $E \subset \mathbb{R}^n$ is called porous with porosity δ, if there is an $r_0 > 0$ such that

$$\forall x \in E, \quad \forall r \in (0, r_0), \quad \exists y \in \mathbb{R}^n \quad \text{such that} \quad B_{\delta r}(y) \subset B_r(x) \setminus E.$$

A porous set has Hausdorff dimension not exceeding $n - c\delta^n$, where $c = c(n) > 0$ is a constant depending only on n. In particular a porous set has Lebesgue measure zero.

Proof of Theorem 3.1. Let $R > 0$ such that $\overline{B_{4R}(x_0)} \subset \Omega$ and let $x \in \partial[u > 0] \cap \overline{B_R(x_0)}$. By Lemma 3.1, we have for each $0 < r < R$ such that $B_r(x) \subset B_{2R}(x_0) \subset \Omega$

$$\sup_{\partial B_r(x)} u \geqslant \frac{n}{\lambda_0} \widetilde{A}\Big(\frac{\lambda_0}{n} r\Big) + u(x) = \frac{n}{\lambda_0} \widetilde{A}\Big(\frac{\lambda_0}{n} r\Big).$$

There exists then $y \in \partial B_r(x)$ such that $u(y) = \sup_{\partial B_r(x)} u$. Then we have

$$u(y) \geqslant \frac{n}{\lambda_0} \widetilde{A}\Big(\frac{\lambda_0}{n} r\Big) > 0.$$

Hence $y \in B_{2R}(x_0) \cap [u > 0]$. Now we define $d(y) = dist(y, \overline{B_{2R}(x_0)} \cap [u = 0])$ the distance from y to the set $\overline{B_{2R}(x_0)} \cap [u = 0]$. By continuity of d, there exists $y_0 \in \overline{B_{2R}(x_0)} \cap [u = 0]$ such that $d(y) = |y - y_0|$.

We define a function v in B_1 by $v(z) = \dfrac{u(y_0 + Rz)}{R}$. Then we have

$$\Delta_A v = div\Big(\Big(\frac{a(|\nabla u|)}{|\nabla u|} \nabla u\Big)(y_0 + Rz)\Big) = R(\Delta_A u)(y_0 + Rz).$$

It follows that v satisfies

$$\begin{aligned} &||\Delta_A v||_\infty \leqslant R\Lambda_0 \\ &0 \leqslant v \leqslant \frac{||g||_\infty}{R} \\ &v(0) = 0. \end{aligned}$$

Now in order to apply Theorem 2.1, we introduce the functions

$$b(t) = \frac{a(tM)}{a(M)}, \qquad w(z) = \frac{v(z)}{M} \qquad \text{with} \qquad M = \max\Big(a^{-1}(R\Lambda_0), \frac{||g||_\infty}{R}\Big).$$

Note that b satisfies (1.1) with the same constants a_0 and a_1.
If we set $B(t) = \displaystyle\int_0^t b(\tau) d\tau = \frac{A(Mt)}{Ma(M)}$, we see that w satisfies

$$\|\Delta_B w\|_\infty = \|\frac{1}{a(M)}\Delta_A v\|_\infty \leqslant \frac{R\Lambda_0}{a(M)} \leqslant 1$$
$$0 \leqslant w \leqslant \frac{\|g\|_\infty}{MR} \leqslant 1$$
$$w(0) = 0.$$

Hence $w \in \mathcal{F}_B$ and we get by Theorem 2.1 for a positive constant C depending only on n, a_0 and a_1

$$w(z) \leqslant C\widetilde{B}(|z|) \qquad \forall z \in B_1 \qquad \text{with} \qquad \widetilde{B}(t) = \int_0^t b^{-1}(\tau)d\tau = \frac{\widetilde{A}(ta(M))}{Ma(M)}.$$

In particular for $z = \frac{y-y_0}{R} \in B_1$, we obtain

$$w\Big(\frac{y-y_0}{R}\Big) \leqslant C\widetilde{B}\Big(\frac{|y-y_0|}{R}\Big) \qquad \Longleftrightarrow \qquad u(y) \leqslant CMR\widetilde{B}\Big(\frac{d(y)}{R}\Big).$$

We deduce that

$$\frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}r\Big) \leqslant u(y) \leqslant CMR.\frac{\widetilde{A}(\frac{d(y)}{R}a(M))}{Ma(M)} = C.\frac{\widetilde{A}(d(y).\frac{a(M)}{R})}{\frac{a(M)}{R}}$$

By (1.5), we have

$$\frac{\widetilde{A}(d(y).\frac{a(M)}{R})}{\frac{a(M)}{R}} \leqslant \frac{a_0}{1+a_0}\max((\frac{a(M)}{R})^{1/a_0},(\frac{a(M)}{R})^{1/a_1})\widetilde{A}(d(y)) = \theta(\frac{a(M)}{R})\widetilde{A}(d(y)).$$

Then
$$\frac{1}{C}.\frac{1}{\theta\big(\frac{a(M)}{R}\big)}.\frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}r\Big) \leqslant \widetilde{A}(d(y)).$$

Since $a(M)/R \geqslant \Lambda_0$ and $\theta\big(\frac{a(M)}{R}\big) \geqslant \theta(\Lambda_0)$, then if we choose C such that

$$C > \frac{1}{\theta(\Lambda_0)}.\frac{n}{\lambda_0} = \frac{a_0+1}{a_0}.\Big(\max(\Lambda_0^{1/a_0},\Lambda_0^{1/a_1})\Big)^{-1}.\frac{n}{\lambda_0}$$

we get $\frac{1}{C}.\frac{1}{\theta\big(\frac{a(M)}{R}\big)}.\frac{n}{\lambda_0} < 1$ and by the convexity of $\widetilde{A}$, we obtain

$$\widetilde{A}(\frac{1}{C}.\frac{1}{\theta\big(\frac{a(M)}{R}\big)}r) \leqslant \frac{1}{C}.\frac{1}{\theta\big(\frac{a(M)}{R}\big)}.\frac{n}{\lambda_0}\widetilde{A}\Big(\frac{\lambda_0}{n}r\Big) \leqslant \widetilde{A}(d(y))$$

from which we deduce that
$$\frac{1}{C}.\frac{1}{\theta\big(\frac{a(M)}{R}\big)}r \leqslant d(y).$$

Let $\delta = \min(\frac{1}{2}, \frac{1}{C}.\frac{1}{\theta\big(\frac{a(M)}{R}\big)}\Big) < 1$ and $y^* \in [x, y]$ such that $|y - y^*| = \delta r/2$. Then we have

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x).$$

Indeed, let $m \in B_{\delta r/2}(y^*)$. We have

$$|m - y| \leqslant |m - y^*| + |y^* - y| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r$$
$$|m - x| \leqslant |m - y^*| + (|x - y| - |y^* - y|) \leqslant \frac{\delta r}{2} + (r - \frac{\delta r}{2}) = r.$$

Moreover, we have

$$B_{\delta r}(y) \cap B_r(x) \subset [u > 0]$$

since $d(y) \geqslant \delta r > 0$ and $B_{\delta r}(y) \subset B_{d(y)}(y) \subset [u > 0]$.
Hence, if we set $E = \partial[u > 0] \cap \overline{B_R(x_0)}$, we have

$$B_{\frac{\delta}{2}r}(y^*) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial[u > 0] \subset B_r(x) \setminus E.$$

□

References

[1] R. A. Adams : Sobolev spaces. Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] S. Challal, A. Lyaghfouri : Hölder Continuity of Solutions to the A-Laplace Equation Involving Measures. To appear in Communications in Pure and Applied Analysis.

[3] J. P. Gossez, V. Mustonen : Variational inequalities in Orlicz-Sobolev spaces. Nonlinear Anal. Vol. 11, No. 3, 379-392 (1987).

[4] L. Karp, T. Kilpeläinen, A. Petrosyan and H. Shahgholian : On the Porosity of Free Boundaries in Degenerate Variational Inequalities. J. Differential Equations Vol. 164, 110-117 (2000).

[5] V. K. Le, K. Schmitt : Quasilinear elliptic equations and inequalities with rapidly growing coefficients. J. London Math. Soc. (2) 62, No. 3, 852-872 (2000).

[6] G. M. Lieberman : The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations. Comm. Partial Differential Equations Vol. 16, No. 2-3, 311-361 (1991).

[7] G. M. Lieberman : Regularity of Solutions to Some degenerate Double Obstacle Problems. Indiana University Mathematics Journal, Vol. 40, No. 3, 1009-1028 (1991).