Fractions and Exponents

Why Do We Use Exponents?

- Mathematically, it's a simpler notation for expressing multiplication.
- Exponents appear frequently in economics and business!
- Examples include: compound interest, modeling, and forecasting.

1) Adding and Subtracting Exponents

$$(4)(4)(4)(4) = 4^4 \qquad (3)(3)(3)(3) = 3^4$$

Suppose a is unknown variable but we know it is a real number. ie $1/2$ or 3.14, $\sqrt{46}$

$$(a)(a)(a) = a^3$$

$$(2/3)(2/3) = (2/3)^2 = \frac{(2)(2)}{(3)(3)} = \frac{4}{9}$$

$$[(2)(2)(2)][(2)(2)] = [2^3][2^2]$$

$$= 2^{3+2} \Big\} \text{ adding the exponents}$$

$$= 2^5$$

Suppose m, n are positive integers (ie $3, 5, \ldots$) → not $1/2$, $\sqrt{5}$

$$a^m \cdot a^n = a^{m+n}$$

Rule 1

$a \to$ Base

$m, n \to$ exponents or powers

If you multiply two numbers together with the same Base, then add the exponents

ex. $\underline{12}^4 \cdot \underline{12}^2 = 12^6$

Division:
(same Base)

$$\frac{5^4}{5^2} = \frac{\cancel{(5)}\cancel{(5)}(5)(5)}{\cancel{(5)}\cancel{(5)}} = (5)(5)$$

$$= 5^2$$

Rule 1b: $\frac{5^4}{5^2} = 5^{4-2} = 5^2$

$$\frac{a^m}{a^n} = a^{m-n}$$

2) Multiplying Exponents

$(5^2)(5^2)(5^2) = (5^2)^3$

$(5)(5)(5)(5)(5)(5) = 5^6$

$(5^2)^3 = 5^{(2)(3)} = 5^6$

Rule 2: $(a^n)^m = a^{mn}$

ex) $(10^3)^{10} = 10^{(3)(10)} = 10^{30}$

3) Understanding Negative Exponents

$(5^6)\left((1/5)^3\right) = (5)(5)(5)\cancel{(5)}\cancel{(5)}\cancel{(5)}\cancel{\left(\frac{1}{5}\right)}\cancel{\left(\frac{1}{5}\right)}\cancel{\left(\frac{1}{5}\right)}$ (1)

$= 5^3$

Rule 1: $\frac{5^6}{5^3} = 5^{6-3} = 5^3$

$(5^6)(1/5)^3 = (5^6)(5^{-3})$

$\left\{ \text{so } \left(\frac{1}{5}\right)^3 = 5^{-3} \right\}$

$\underbrace{\left(\frac{1}{a}\right)^n = a^{-n}}_{\text{Rule 3}}$ ← a negative exponent represents a fraction.

$$= \left(2/3\right)^{-2} = \left(2/3\right)^{2(-1)} = \left(3/2\right)^{2} = \frac{9}{4}$$

example of rule 3

4) The Zero Exponent: a^0

Where a is any real number

Rule 4 $\quad a^0 = 1$

ex $(5^2)(5^{-2}) = 5^{2-2} = 5^0$ (Rule 1)

$\cancel{(5)}\cancel{(5)}\left(\cancel{\frac{1}{5}}\right)\left(\cancel{\frac{1}{5}}\right) = 1$

ex) $= (100x)^0$ where x is a real number.

$= 100^0 x^0 = (1)(1) = 1$

Solve the following

Assume a is a real number while b, c are positive integers.

$5^3\left(\frac{1}{5}\right)^3$

$(5)(5)(5)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)$

$= 1$

$(5^3)(5^{-3}) = 5^{3-3} = 5^0 = 1$

same

$a^{2b}/a^c = \dfrac{a^{2b}}{a^c} = a^{(2b-c)}$

No like-terms in the exponent

Exponents In Practice

$\nearrow$ annual compounding

If you save \$1,000 at an interest rate of 12%, how much would you have in one year?

Interest: $\$1000(0.12) = \$120$

Principal: $\$1000(1.00) = \$1{,}000$

$\$1{,}120$

$$\$1{,}000(1) + \$1{,}000(0.12)$$

$$\underbrace{\$1{,}000}_{\text{initial amount}}(1 + \underbrace{0.12}_{\text{interest rate}}) = \$1{,}000(1.12) = \underbrace{\$1{,}120}_{\text{Future amount}}$$

Exponents In Practice

annual compounding

If you save \$1,000 at an interest rate of 12%,
How much would you have in three years?

year 1) $\$1,000(1.12) = \$1,120$

year 2) $\$1,120(1.12) = \$1,254.40$

year 3) $\$1,254.40(1.12) = \$1,404.93$ ✓

year 3) $\$1,000(1.12)(1.12)(1.12)$

$= \underbrace{\$1,000(1.12)^3}$ $= \$1,404.93$

3 = # of years

Exponents In Practice

annual compounding

You saved some money 8 years ago at an interest rate of 4 percent. Today, it is worth \$50,000. How much savings did you start with?

let x = the money saved 8 years ago.

$$x(1.04)^8 = \$50{,}000$$

$$x = \frac{\$50{,}000}{(1.04)^8} = \frac{\$50{,}000}{1.369}$$

$$x = \$36{,}535$$

Exponents In Practice

A blouse is regularly priced at 89.99. The store is offering it a discount of 20%. It is also a long weekend where all store items receive an additional 33% off. The sales tax is 13%. How much does the blouse actually cost?

$$\$(89.99)(100\% - 20\%)(100\% - 33\%)(100\% + 13\%)$$

$$\$89.99(1 - .2)(1 - .33)(1 + .13)$$

$$\underbrace{\$(89.99)(.8)(.67)(1.13)}$$

$$= \$\underline{54.50} \qquad = (89.99)(1.13)(.67)(.8)$$

Some Important Algebra Rules

Let a, b, c be real numbers and let d be a positive integer, then $\rightarrow \sqrt{2}, 4/7, 2.5$

1. The order of addition does not matter:
   $a + b + c = c + b + a$
2. The order of multiplication does not matter:
   $ab = ba$
3. Any real number multiplied by zero is zero:
   $0^d = 0$

Let a, b, c be real numbers and let d be a positive integer, then

4. Any real number divided by zero is *undefined*:

$2/0 = ?$ undefined

$0^{-2} = \left(\frac{1}{0}\right)\left(\frac{1}{0}\right)$ = undefined

5. $(a)(b + c) = ab + ac$ Rule 5 has imporant implications:

BEDMAS

↳ Bracket, exponent, division and multiplication, addition and subtraction

$2(3+4) = 2(7) = 14$

$2(3) + 2(4)$

$= 6 + 8$

$= 14$

$\underbrace{(2+3)}_{a}(4+5) = (5)(9) = 45$

$a(4+5)$

$= 4a + 5a$

$= 4(2+3) + 5(2+3)$

$= 8 + 12 + 10 + 15 = 45$

Rule 5 is consistent with BEDMAS

Rule 5 Acronym: FOIL: First Outside Inside Last

$(2+3)(4+5)$

F F

O O

I I

L L

$(2)(4) + (2)(5) + (3)(4) + (3)(5)$

$= 8 + 10 + 12 + 15$

$= 45$

A bedsheet has a width of $a + 3$ and a length of $b + 3$. What is the area of the bedsheet?

$$\text{Area} = (\text{width})(\text{length})$$
$$= (a + 3)(b + 3)$$
$$= ab + 3a + 3b + 9$$

Notation

Algebraic Expressions

$\underline{2x^2} + \underline{3y^2} - \underline{x^2}$

- 3 terms
- a term is separated by a +/−
- multiplication with a term is allowed.

$$2x^2 \leftarrow \text{power}$$

coefficient → 2, base → x

- if two terms have the same base and power, then they are "like terms" and we can add or subtract their coefficients.

$= 2x^2 + 4x - x^2 + 4$

$= x^2 + 4x + 4$

Quadratic Identities

A quadratic expression has no variable with a power higher than 2.

1. $(a+b)^2 =$ $(a+b)(a+b)$
   $a^2 + 2ab + b^2$

2. $(a-b)^2 =$ $(a-b)(a-b)$
   $= a^2 - 2ab + b^2$

3. $(a+b)(a-b) =$ $a^2 - b^2$

Negative Parentheses

$$a(b+c) = ab + ac$$

$$-(4-3+5-2) =$$ if $a = -1$

$$= a(4-3+5-2)$$

$$= 4a - 3a + 5a - 2a$$

$$= -4 + 3 - 5 + 2$$

$$= -4$$

More than 2 Parentheses

BEDMAS

$$(2+3)(4+5)(6+7) = (5)(9)(13) = (45)(13) = 585$$

FOIL

$$= \big((2)(4) + (2)(5) + (3)(4) + (3)(5)\big)(6+7)$$

$$= (8 + 10 + 12 + 15)(6+7)$$

$$= (45)(6+7)$$

$$= 585$$

consistent

Rule 5

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

$$= (a^2 + 2ab + b^2)(a+b)$$

$$= a^2(a+b) + 2ab(a+b) + b^2(a+b)$$

$$= a^3 + \underline{a^2b} + \underline{2a^2b} + 2ab^2 + ab^2 + b^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

Factoring

$$\overbrace{(4)(6)}^{\text{term}} = 24$$

factor factor

$\Downarrow$

Express an algebraic expression in its simplest factors:

$$49 = (7)(7)$$

$$4x^3y^2 = (2)(2)(x)(x)(x)(y)(y)$$

Factor the Following Expressions

$$\frac{108}{12} = \frac{(2)(54)}{(3)(4)} = \frac{(2)(2)(27)}{(3)(2)(2)} = \frac{\cancel{(2)}\cancel{(2)}\cancel{(3)}(3)(3)}{\cancel{(2)}\cancel{(2)}\cancel{(3)}}$$

$$\frac{3^2}{81} = \frac{(3)(3)}{(9)(9)} = \frac{(3)(3)}{(2)(3)(3)(3)} = \frac{1}{(2)(3)}$$

Factor the Following Expressions

$(1+d)Z+(1+d)Zd$ $= (1+d)(z+zd)$

$z(1+d)(1+d)$

$z(1+d)^2$

$2x^{-2}+4dx^{-3}$ $= 2(x^{-2}+2dx^{-3})$

$= 2x^{-2}(1+2dx^{-1})$

$= 2x^{-3}(x+2x)$

Factor Using the Identities

$$(a-b)(a+b) = a^2 - b^2$$

$36a^2 - 1$

$$(6a-1)(6a+1) = 36a^2 + 6a - 6a - 1$$

$$= 36a^2 - 1$$

$$4x^2 + 8xy + 4y^2 = 4\underbrace{(x^2 + 2xy + y^2)}$$

$$(a+b)^2 = a^2 + 2ab + y^2$$

$$= 4(x+y)^2$$

Factoring Harder Cases

$x^2 - 3x - 40$ $= (x-8)(x+5)$

$= x^2 + 5x - 8x - 40$

$= x^2 - 3x - 40$

$2x^2 + 2x - 24$ $= 2(x+4)(x-3)$

expand using FOIL

$2(x^2 - 3x + 4x - 12)$

$= 2(x+4)(x-3)$

Fractions

Let a, b, c, d be positive integers (whole numbers).

$$\frac{a}{b} \qquad \frac{\leftarrow \text{ numerator}}{\leftarrow \text{ denominator}}$$

$$2(1/4) = 1/4 + 1/4 = 2/4 = 1/2$$

$$2(a/b) = a/b + a/b = \frac{a+a}{b} = \frac{2a}{b}$$

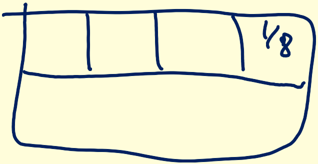
$$2(a/b) = \frac{2a}{b}$$

$$cd\left(\frac{a}{b}\right) = \frac{acd}{b}$$

$$-\frac{a}{b} = \frac{(-1)(a)}{b} = (-1)\left(\frac{a}{b}\right) = -\frac{a}{b}$$

$$\frac{-a}{-b} = \frac{-a}{-b} = \frac{a}{b}$$

$(1/4)(1/2) \to 1/4$ of $1/2$



$$(1/4)(1/2) = \frac{1}{(2)(4)}$$

$$= 1/8$$

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$$

$$\frac{\frac{6}{10}}{1/10} = \left(\frac{6}{10}\right)\left(\frac{10}{1}\right) = \frac{60}{10} = 6$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

$$\frac{1}{2} + \frac{1}{4} = 3/4 = \frac{1(2)}{2(2)} + 1/4 = \frac{2}{4} + \frac{1}{4} = 3/4$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

To cancel out terms, they must be common to both the numerator and the denominator:

$$\frac{ab + ac}{ad} = \frac{\cancel{a}(b+c)}{\cancel{a}(d)} = \frac{b+c}{d}$$

$$\frac{ab + ac}{ad + a^2} = \frac{a(b+c)}{a(d+a)} = \frac{b+c}{a+d}$$

Fractions: Common Denominators

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = \left(\frac{1}{2}\right)\left(\frac{3}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{2}\right) + \frac{1}{6}$$

$$= \frac{3+2+1}{6} = 1$$

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{1(4)(5)}{3(4)(5)} + \frac{1(3)(5)}{4(3)(5)} + \frac{1(3)(4)}{5(3)(4)}$$

$$= \frac{20}{60} + \frac{15}{60} + \frac{12}{60} = \frac{47}{60}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{bc}{abc} + \frac{ac}{abc} + \frac{ab}{abc}$$

$$= \frac{ab+ac+bc}{abc}$$

Simplifying Fractions

$$\frac{5a^2b^3}{15ab^2c} = \left(\frac{5}{15}\right)\left(\frac{a^2}{a}\right)\left(\frac{b^3}{b^2}\right)\left(\frac{1}{c}\right)$$

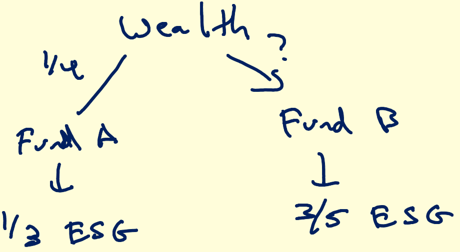
$$= \left(\frac{1}{3}\right) a^{2-1} b^{3-2} c^{0-1}$$

$$= \frac{ab}{3c}$$

$$\frac{2a^2+2ab}{a^2-b^2} = \frac{2a\cancel{(a+b)}}{(a-b)\cancel{(a+b)}} = \frac{2a}{a-b}$$

Word Problem with Fractions

You invest $\frac{1}{4}$ of your wealth in Fund A which contains $\frac{1}{3}$ ESG (Environmental, Social, and Governance) approved assets. The remaining wealth you invest in Fund B which contains $\frac{3}{5}$ ESG approved assets. How much of your wealth is invested in ESG approved assets?



weight on Fund B $= 1 - 1/4 = \frac{4}{4} - \frac{1}{4} = 3/4$

How much of your wealth is in ESG approved assets?

$$\left(\tfrac{1}{4}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{3}{5}\right)$$

$$\frac{1}{12} + \frac{9}{20} = \frac{1}{(2)(2)(3)} + \frac{(3)(3)}{(2)(2)(5)}$$

$$= \frac{1(5)}{(2)(2)(3)(5)} + \frac{(3)(3)(3)}{(2)(2)(5)(3)}$$

$$= \frac{5}{60} + \frac{27}{60} = \frac{32}{60}$$

$$= \frac{(2)(16)}{(6)(10)} = \frac{\cancel{(2)}\cancel{(2)}(2)(2)(2)}{\cancel{(2)}(3)\cancel{(2)}(5)} = \frac{8}{15}$$

Investing 1/4 in Fund A and 3/4 in Fund B means 8/15 of your wealth is in ESG funds.