VOLUNTARY RETIREMENT AND OPTIMAL CONSUMPTION IN A STOCHASTIC MORTALITY ENVIRONMENT

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Abstract

My objective is to analyze a life cycle model in which an individual ages in a stochastic and non-linear fashion, in order to determine his optimal time to retire, and his plan for consuming optimally over the entire lifetime. Stochasticity is incorporated into the model using mortality rates that depend on biological age as well as calendar age. Biological age depends upon the status of the individual's health. Unlike calendar age, which always moves forward, biological age can occasionally move against the natural clock time, so individuals with the same birth dates may age at different rates. A rational retirement decision should be based on both ages. Huang, Milevsky and Salisbury [31] used biological age to analyze an individual's optimal retirement spending. This thesis is an extension of their work.

In my model there is labor income in the pre-retirement period. For post-retirement life there may be a pension to supplement retirement savings. I use a Cobb-Douglas utility function that provides a fixed multiplicative bonus to the utility of consumption in retirement. I derive and solve Hamilton Jacobi Bellman (HJB) equations in the above Cobb-Douglas utility framework.

I show among that consumers at same wealth level; the older consumer prefers to retire at a lower biological age while a younger consumer prefers to keep earning labor income until later biological ages. As per the common intuition, high wealth levels drive people to retire young and in better health status. The model predict some fascinating effects of the volatility of biological age on retirement curves. At lower wealth levels, high volatility drives less healthier people to retire later. The model with deterministic hazard rates also depicts an interesting non-uniqueness behavior in the wealth dynamics curves. Below a certain initial wealth level people slowly spend money over time but never retire while above this wealth they save for retirement and eventually retire. Then there is a wealth level beyond which they never enter the labor force. Also, over a fair range of age, health status and wealth, my results provide some reliable guidance for a to-be pensioner. For a moderately risk averse consumer with calendar age and biological age within a modest range around 65 years and who expects to have an exogenous pension of 40% of his annual labor income upon retirement, if he has savings equal to almost double the amount of his annual labor income he can consider retirement.

Dedication

To my wonderful and supportive husband

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Chapter 1

Introduction

The practice of retiring from the labor force dates back to 18th century. Retirement became a wide spread concept through many parts of the world during 19^{th} and 20^{th} centuries. In 1881, a German Chancellor, Otto von Bismarck, introduced the concept of pension vs retirement at a certain age. The German government created a pension system which provided retirement support to citizens aged 70 (life expectancy in Germany at that time) and above. Around 1920s, a large group of American industries had introduced retirement support for their workers with retirement age being 65 years. This age limit was mostly for economic reasons and had less to do with workers' health. In 1935, the Federal government of the USA officially declared the retirement age to be 65. It was beyond the life expectancy of American citizens (around age 58) back then. By 1960, economic and medical conditions in America improved resulting in longer living citizens. Life expectancy improved to 70 years. Companies were offering better and better retirement pension plans providing financial security to their workers post-retirement.This increased the population of retired American citizens who happily entered retirement life [38]. Between 1995 and 2000 early/voluntary retirement entered the picture. With the boom in stock markets, the funds available in retirement plans to employees increased drastically. This made retirement more and more accessible as well as more attractive enticing larger number of people to take a premature (before the standard retirement age of 65) exit from the labor market [28]. That was the beginning of early/voluntary retirement.

In the following paragraphs I will introduce some concepts at the core of my thesis along with a literature review.

1.0.1 Force of Mortality

The mortality rate is the number of deaths in a population per unit time. The force of mortality is simply the instantaneous mortality rate at a specific age. If a positive continuous random variable T denotes the lifetime of an individual with density function f and cumulative distribution function F, then, survival function $S(t) = P(T > t) = 1 - F(t)$ gives the probability that this individual will survive past a certain age. The hazard function or force of mortality of the survival function $h(t) = \frac{f(t)}{S(t)}$ is the probability that the person will die with in the next infinitesimal interval, given that he has survived until now.

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < T < t + \Delta t | T > t)}{\Delta t}$$

Until the early 19^{th} century, there were no formal laws to predict how many people might die in next few years. Benjamin Gompertz [27] discovered that the probability of dying for an average person doubled every 10 years [46]. He presented the famous Gompertz law of Mortality which predicts an exponential increase in human death rates with age for ages over 50 years. The hazard function

$$h(t) = \sigma^{-1} e^{\beta t/\sigma}; t > 0, \beta, \sigma > 0$$

Here, β is a dimensionless parameter and determines the growth of hazard rate with respect to the age. Makeham [43] introduced an age-independent component $\lambda > 0$ to the Gompertz hazard function and Gompertz-Makeham Model came into being:

$$h(t) = \lambda + \sigma^{-1} e^{\beta t/\sigma}, t > 0, \beta, \sigma > 0$$

This law is better at predicting mortality hazard rates at lower ages and t is allowed to grow to infinity, that is, there is no upper bound on the lifetime of an individual [11].

Perks [53] argued that "Makeham curves ran much too high at the older ages" and suggested a logistic type model (increases exponentially for small ages but presents a horizontal asymptote at advanced ages) which depicts a slower increase in mortality hazard rates at advanced ages; at least slower than an exponential increase. Now, this effect at later ages is known as late-life mortality deceleration. Beard et al. [8] also presented a logistic type model but with more variation in hazard rates at advanced ages. Thatcher et al. in [62] argued that the late life-mortality deceleration is majorly noticeable around the age of 80 years. Later, Thatcher [61] and Bebbington et al. [9] replaced the age-based exponential feature of Gompertz-Makeham law with a logistic function. The hazard function in its most general form [11] is given by:

$$h(t) = \lambda + \frac{A e^{\beta t/\sigma}}{1 + B e^{\beta t/\sigma}}, A > 0, B \geq 0$$

Beard emphasized in his research paper [7] that the heterogeneity between individual hazard rates can be captured via a gamma distribution. This was later extensively developed by Vaupel et al. [63] leading to the gamma-Gompertz-Makeham model.

1.0.2 Stochastic mortality

The force of mortality is deterministic over the lifetime of an individual if at time zero, the survival probabilities till the last possible time of death are known without any uncertainty.

While for a stochastic force of mortality the conditional survival probabilities for any future time, given that we already know the mortality rate at some current time, are not known. The probabilities are completely random. All we know at time 0 are the the expectations of the survival probabilities. Two important empirical studies about stochastic mortality can be found in a paper by Ronald D. Lee and Lawrence R. Carter [39] and the article by Cairns, Blake and Dowd [15]. Pitacco, Denuit, Haberman and Olivieri discussed stochastic mortality in their book [55]. Liu and Lin [42] and Zhou, Li and Tan [66] used stochastic mortality models for pricing mortality-linked securities.

1.0.3 Is there a limit to human life spans?

Determining a limit to human life spans is an interesting area of research from both a biological and economical point of view. In 20^{th} and 21^{st} century this research topic gained quite a lot of interest. Gampe [25] and Rootzen & Zholud [59] regard super-centenarian (people who reach the age of 110 years) data as exponentially distributed and Pearce & Raftery [52] argued that in this century it will be possible to observe a maximum recorded lifetime of 130 years , but it is highly unlikely to record any life above this limit. Barbi et al. [6] found that above age 105 human mortality appears constant over age. Belzile et. al [11] recently published their paper "Is there a cap on Longevity? A statistical review." In this article they discussed the methods of collecting data on extreme human lifetimes, correct ways of handling it, the models and strategies for analysis. They concluded that the remaining human lifetimes after age 109 are exponentially distributed. "Even if the exponential model imposes no upper bound on lifetimes, it does not imply that the highest human lifetime will be very large". They suggested that if there is an upper limit to the length of human life, it lies well above the highest lifetime we know so far. The most valid longest lifetime recorded is 122 years and 164 days, for the French woman Jeanne Calmet. The lower limit to a 95% confidence interval for human life span is around 130 years and there is no upper limit.

It is worth mentioning that for this thesis I explore a stochastic mortality model where mortality plateaus at advanced age. But also note that some researchers argue against that plateau, for example Gavrilova and Gavrilov [26] suggested that there is no conclusive evidence for a fixed biological limit to human life span.

1.0.4 Stochastic mortality in a life cycle model setting

The life cycle model (LCM) deals with the lifetime consumption and savings of an individual. It assumes a rational individual who is looking to maximize his discounted lifetime utility of consumption. It is based on smoothing out the consumption of an individual to have an almost consistent living standard throughout the life. Ramsey presented a rule of income saving for a nation in his very famous paper [56]. This formed the basis of all the further research done regarding savings, consumption and LCMs. Fisher [24], Modigliani and Brumberg [50] and Modigliani [49] published their articles about asset allocation models in a LCM setting which lie at the core of the life-cycle utility optimization literature. Menahem Yaari [65] presented a LCM where he used a stochastic time of death for an individual which introduced a factor of uncertainty in the model. Until that all LCM literature considered an end to lifetimes at some deterministic time. Stanley Fischer [23] used a discrete-time model with uncertain time of death to examine life-cycle patters of consumption, savings and insurance purchases.

In stochastic mortality LCM the randomness comes both from the hazard rate as well as the uncertain time of death. Most of the articles in the literature of LCM assume a deterministic force of mortality (mostly Gompertz law) where mortality simply adds up to the subjective discount rate. See for example, Levhari and Mirman [41], Davies [16], Deaton [17], Leung [40], Butler [14], Bodie et al. [13], Dybvig and Liu [20], Kingston and Thorp [34], Babbel and Merrill [5], Park [51], Wallmeier and Zainhofer [64], Feigenbaum [22], Huang and Milevsky [48], Lachance [37]. Huang et al. used the concept of the stochastic force of mortality in their paper [30]. This article introduced a model for a stochastic force of mortality. They used a lognormal mortality rate called the Dothan model [18] as a stochastic mortality force. Menoncin and Regis [44] used a stochastic force of mortality to solve for the pre-retirement consumption/investment problem. They considered a mean reverting stochastic extension of Gompertz law of mortality. Most of the above mentioned sources deal with the population hazard rates. Hunag et al. [31] expressed mortality hazard rates in terms of an individual's biological age. Their model used individual hazard rates where biological age is modeled using a stochastic generalized Brownian bridge process.

1.0.5 Voluntary Retirement as Optimal Stopping Time Problem

An optimal stopping time, in non-technical sense, is the time when a process/action stops to gain maximum profit or reward while minimizing the associated costs. Peskir and Shiryaev define stopping times in their book [54] as follows:

Consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$. $(\mathscr{F}_t)_{t\geq 0}$ is a filtration such that $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$ for all $s \leq t$. A random variable $\tau : \Omega \to [0, \infty]$ is called a *Markov time* if $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. A Markov time is called a *stopping time* if $\tau < \infty$ P-a.s. Typically the filtration $\mathscr{F}_t$ is the information available up to time t. The decision to stop is completely based on this information. Hence the stopping time τ "does not look into future" to decide whether to stop or not [60] .

The optimal stopping problem, in the control theory sense, is to determine the optimal stopping time guided by a number of controls (for example the consumption, portfolio, wages etc.). It is dealt with as a free-boundary problem. In the field of finance the American style options are widely studied using optimal stopping in order to find the optimal time to exercise the option and collect maximum gains. The early/voluntary retirement decision can be studied in a similar fashion when a decision has to be made about retiring at an optimal time, the retirement decision is irreversible, and the consumer is looking to collect maximum utility of consumption from this decision. The article by Bodie, Merton and Samuelson [12] considered the retirement decision problem as an optimal stopping problem. Kula [36] treated the option to retire as an investment process where we collect retirement wealth and once this accumulated wealth reaches a certain threshold, we retire. Koo, Koo, Shin [35] considered a voluntary retirement problem with a focus on optimal investment, consumption and leisure under a Cobb-Douglas utility framework. They used dynamic programming strategies to derive a closed form solution for the value function and optimal strategies for consumption, leisure, investment and retirement. Farhi and Panageas [21] used optimal stopping approach to solve for optimal consumption and portfolio choice problem to make the decision about retirement time while adjusting the labor supply. They wrote "The ability to time one's retirement introduces an option-type character to the optimal retirement decision. This option is most relevant for individuals with a high likelihood of early retirement, that is individual with high wealth levels". Francesco and Sergio [45] used martingale approach to solve for optimal consumption, labor supply and portfolio choice to find the optimal time to retire. They used a constant force of mortality and concluded that if a person enters labor market at age 25, then the optimal retirement age should be between 50 to 65 with an average of 55 years.

1.0.6 Biological Age

The biological age (b-age) is a measure of health status or the true age in contrast to the chronological age (c-age) which measures the number of calendar years passed since birth. With advancement in medical sciences, it may soon be possible to accurately know a person's biological age using telomeres (the protective ends of chromosomes) length. Research is also being conducted on DNA methylation and the epigenetic clock to be used as b-age markers. In a recent article [10] Belsky et al. discussed a blood-DNA methylation measure to measure variation in biological aging among individuals born in the same year. Human biological age can be 10-15 years more or less than the chronological age. Also, the variation of b-age around c-age is dependent upon c-age, that is, a middle aged (with respect to c-age) human will have a highest divergence of b-age about c-age as compared to a person at lower ages or old ages. There is some work done on the concept of biological age, for example, Ries & Pothing [57], Dubina et al. [19], Jackson et al. [32] and Jylhava et al. [33]. Milevsky gave a straightforward explanation of biological age in his book "Longevity Insurance for a Biological Age" [46]. Huang et al. [31] are the pioneers who used the concept of biological age (b-age) along with the chronological age (c-age) to determine retirement spending rates.

My thesis is an extension of the research initiated by Huang, Milevsky and Salisbury [31]. They used b-age and c-age to determine the optimal retirement spending rates. They came up with a mathematical stochastic model to measure the biological age at any chronological age. They used inverted Gompertz law of mortality to get the implied biological age at any chronological age. That is, the mortality rate of a person is known through his bio-markers. With these rates as an input, they used Gompertz law to solve for the age and called it biological age. For a pre-determined retirement age, a deterministic wealth variable, and no exogenous pension income they solved HJB equation to get optimal spending rates post-retirement. In their model the retiree is allowed have infinite (both positive and negative) b-age. I used the similar model and concepts for my thesis but my research aims at determining the optimal voluntary retirement time as a function of b-age as well as c-age along with the optimal consumption rates throughout the life of a consumer. Therefore, my model has HJB equations to be solved for pre-retirement and post-retirement while finding optimal retirement c-age, b-age and wealth. I also used a more practical finite domain for b-age. My research work began with a project for determining the voluntary retirement boundary for a rational consumer who wants to spend optimally throughout his life guided by a deterministic force of mortality. For the next project, I used biological age as the stochastic source of mortality and determined the optimal retirement boundary as a function of biological age, consumer's wealth and chronological age. For the next project I also incorporated an exogenous pension in the post-retirement phase of life and determined the optimal retirement boundary dependent upon b-age, c-age and wealth. The final chapter of my thesis is from the internship I did at CANNEX Financial Exchanges Limited, Toronto in 2019. For that project I determined the impact of market value adjustment on cash surrender values for a set a fixed income annuities.

Chapter 2

Deterministic Mortality Rate

2.1 Problem

In order to keep the focus on determining the optimal voluntary retirement times, I am beginning with a relatively simpler model and using a minimum number of the factors affecting wealth and retirement.

Consider a consumer of age x with a stochastic (random) time of death or remaining lifetime ς. He is interested in determining the optimal time to retire while maximizing his expected utility during the course of remaining life. Currently, the consumer is employed and earning a labor income of \$1 annually until retirement. He does not have bequest motives and there is no pension income stream for his post-retirement life. He is willing to invest only in risk-free assets earning an instantaneous return r. All this work is done in real inflation adjusted terms.

2.2 Mathematical Setup

If the time scale, in the unit of years, begins on the day when this consumer turned x years old, without any loss of generality, I can say that time $t = 0$ denotes age x. I am considering an infinite time horizon problem, that is, the maximum time a person lives for is infinite. But for the sake of practicality and numerics of the problem, I am assuming that the maximum possible time of death is given by some large enough time T representing infinity (that is, the maximum possible age at the time of death is $x + T$). Even though the time of death is stochastic but the the consumer's mortality rate or aging is determined by a deterministic mortality hazard rate λ_t at any time t of a cohort of population. I assume that this deterministic mortality law is given by the famous Gompertz law of mortality, that is, $\lambda_t = \frac{1}{b} e^{(\frac{x_t - m}{b})}$. The parameters defining this law are, x_t is the age at time t, m is the modal value of life in years and b is the dispersion parameter in years [47]. The probability of the consumer surviving beyond time t is given by

$$Pr[\varsigma > t] = p(t, \lambda_0) = {}_t p_x = e^{-\int_0^t \lambda_q dq} \tag{2.1}$$

Let c_t be the consumption process at time t and l_t be the leisure rate process at time t. Call t^* the time of voluntary retirement from labor and w_t is the customer's wealth at time t. Let $V(w,l)$ be the value function defining the objective of the consumer. With current wealth level $w_0 = w$, the consumer wants to maximize the discounted utility of leisure and consumption while determining an optimal time for voluntary retirement from labor force. That is:

$$V(w,l) = \max_{c_t, t^*} E\left[\int_0^T e^{-\rho s} u(l_s, c_s) 1_{\{s \leq \varsigma\}} ds \middle| w_0 = w, l_0 = l\right] \tag{2.2}$$

where ρ is subjective discount rate. The per period leisure process l_t [21] is assumed to be a piece-wise constant function. A consumer is endowed with a constant $\bar{l}$ units of leisure for retirement.

$$l_t = \begin{cases} l_1, \text{ pre-retirement time period} \\ \bar{l}, \text{ post-retirement time period} \end{cases} \tag{2.3}$$

where, $\bar{l} > 1$ and without any loss of generality l_1 is normalized to be equal to 1. In my thesis, the leisure is not denominated in time (as in usual economics literature), it is a parameter that says how much enjoyment is there in consuming post-retirement as compared to pre-retirement. The utility of consumer is described by a *Cobb-Douglas utility function* [21]:

$$u(c,l) = \frac{1}{\alpha} \frac{\left(c^\alpha l^{1-\alpha}\right)^{1-\gamma^*}}{1-\gamma^*} \tag{2.4}$$

Here $\alpha(0 < \alpha < 1)$ measures the weight of consumption contribution to the consumers utility per period. $\gamma^*(\gamma^* \neq 1$and $\gamma^* > 0)$ is relative risk aversion coefficient of the consumer for two goods; consumption c and leisure l. For $\gamma^* > 1$, c and l are substitutes for each other while for $\gamma^* < 1$ they are complements. Let $\gamma = 1 - \alpha(1-\gamma^*)$, then utility takes the form

$$u(c,l) = l^{(1-\alpha)(1-\gamma^*)} \frac{c^{1-\gamma}}{1-\gamma} \tag{2.5}$$

and for any time $0 \leq t \leq T$,

$$u(c_t, l_t) = \begin{cases} u_1 = \frac{c_t^{1-\gamma}}{1-\gamma} & \text{, pre-retirement} \\ \overline{u} = \bar{l}^{(1-\alpha)(1-\gamma^*)} \frac{c_t^{1-\gamma}}{1-\gamma} & \text{, post- retirement} \end{cases} \tag{2.6}$$

Define

$$V(t,w,l) = \max_{c_t, t^*} E\left[\int_t^T e^{-\rho(s-t)} u(l_s, c_s) 1_{\{s \leq \varsigma\}} ds \middle| w_t = w, \lambda_t = \lambda, l_t = l\right] \tag{2.7}$$

Hence, using Fubini's theorem

$$V(t,w,l) = \max_{c_t, t^*} \left[\int_t^T e^{-\rho(s-t)} u(l_s, c_s) E\left\{1_{\{s \leq \varsigma\}}\right\} ds\right]$$

$$V(t,w,l) = \max_{c_t, t^*} \left[\int_t^T e^{-\rho(s-t)} u(l_s, c_s) {}_s p_x ds\right] \tag{2.8}$$

where, $E\left[1_{\{s\leq\varsigma\}}\right] = {}_sp_x$. Then,

$$V(t,w,l) = \max_{c,t^*}\left[\int_t^T e^{-\rho(s-t)}u(l_s,c_s)_sp_x ds\middle| w_t = w, \lambda_t = \lambda, l_t = l\right] \tag{2.9}$$

and

$$V(t,w,l) \equiv \left\{\begin{array}{ll} \overline{V}(t,w); l = \bar{l} & \text{, post-retirement} \\ V_1(t,w); l = 1 & \text{, pre-retirement} \end{array}\right\}. \tag{2.10}$$

The utility from using consumption stream c_s for $0 \leq s \leq t$ and then using optimal behavior after t is given by

$$Z_t = \int_0^t e^{-\rho s}u(l_s,c_s)_sp_x ds + e^{-\rho t}{}_tp_x V(t,w_t,l_t) \tag{2.11}$$

Therefore, Z_t is constant with optimal behavior and decreasing in general. I'll break this optimization problem into two inter-linked problems: 1. Pre-retirement case ($t < t^*$) 2. Post-retirement case ($t \geq t^*$)

2.2.1 PRE-RETIREMENT CASE ($t < t^*$)

Lets consider the pre-retirement case first. For pre-retirement utility and value, from equation 2.11, I get

$$Z_t = \int_0^t e^{-\rho s}u_1(1,c_s)_sp_x ds + e^{-\rho t}{}_tp_x V_1(t,w_t) \tag{2.12}$$

Differentiating gives

$$\begin{aligned} dZ_t = u_1(c_t)e^{-\rho t}{}_tp_x dt - (\rho+\lambda_t)e^{-\rho t}{}_tp_x V_1(t,w_t) \\ + e^{-\rho t}{}_tp_x\{V_{1t}(t,w_t)dt + V_{1w}(t,w_t)dw_t\} \leq 0 \end{aligned} \tag{2.13}$$

Since Z_t is decreasing for a consumption process c_t and retirement time t^*, $dZ_t \leq 0$ and this equals zero for optimal choice of c_t and t^*. The consumer is working and earning an annual income of 1 unit, therefore his wealth is increasing by $(1 + rw_t)dt$ while he is consuming $c_t dt$ in an infinitesimal time period dt. Then wealth dynamics for pre-retirement time period are

$$dw_t = (1 + rw_t - c_t)dt \tag{2.14}$$

For optimal consumption c_t^* and optimal voluntary retirement time t^*, equations 2.13 and 2.14 give

$$\begin{aligned} \sup_{c_t,t^*}[u_1(c_t)e^{-\rho t}{}_tp_x dt - (\rho+\lambda_t)e^{-\rho t}{}_tp_x V_1(t,w_t) \\ + e^{-\rho t}{}_tp_x\{V_{1t}(t,w_t)dt + V_{1w}(t,w_t)(1 + rw_t - c_t)dt\}] = 0 \end{aligned} \tag{2.15}$$

and for an optimal choice of t^* (imagining there is an optimal t^* and it's incorporated into the following equation) , factoring out $e^{-\rho t}{}_tp_x(\neq 0$ being a product of positive quantities)

$$\sup_{c_t}\{u_1(c_t) - c_tV_{1w}\} - (\rho+\lambda_t)V_1 + V_{1t} + V_{1w}(1 + rw_t) = 0 \tag{2.16}$$

My aim is to find the optimal t^* numerically and use the convex dual of the utility to work out the optimization of 2.16 over c_t.

Optimization over c_t

Consider the convex dual of $u_1(x)$ [58]

$$\tilde{u_1}(y) = \sup_x \{u_1(x) - xy\} \tag{2.17}$$

for $u_1(x) = \frac{x^{1-\gamma}}{1-\gamma}$. Call, $q \equiv u_1(x) - xy$. Differentiating q w.r.t x, setting the derivative equal to zero yields $x = y^{-\tilde{\gamma}}, \tilde{\gamma} = \frac{1}{\gamma}$. Hence, the supremum is

$$\tilde{u_1}(y) = -\frac{y^{1-\tilde{\gamma}}}{1-\tilde{\gamma}}$$

The optimal pre-retirement consumption $c_t^* = (V_{1w})^{-\tilde{\gamma}}$ and optimization over c_t develops as

$$\sup_{c_t}\{u_1(c_t) - c_t V_{1w}\} = \tilde{u_1}(V_{1w}) = -\frac{{V_{1w}}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} \tag{2.18}$$

Equations 2.16 and 2.18 give the PDE (Hamilton Jacobi equation) for $V_1(t,w); t < t^*$

$$V_{1t} - (\rho + \lambda_t)V_1 + (1 + rw)V_{1w} - \frac{{V_{1w}}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} = 0 \tag{2.19}$$

I am considering the consumer's wealth to stay within the domain $0 \leq w \leq W$, where W is some large enough wealth adjusted using numerical observations of the results.

Boundary Conditions

$t = t^*$ is the moving boundary which is the (deterministic) optimal time of retirement to be determined numerically. t^* being optimal stopping time, the value matching property holds along the boundary, that is, $V_1(t^*, w) = \overline{V}(t^*, w, \bar{l})$. I am assuming that at very large wealth W the individual will prefer retirement from work at all ages, hence, $V_1(t, W) = \overline{V}(t, W, \bar{l})$, where $\overline{V}(t, w, \bar{l})\forall t$ is the post-retirement value of the consumer. Also, at maximum possible time I am forcing the consumer to retire, that is, $V_1(T, w) = \overline{V}(T, w, \bar{l})$,

2.2.2 POST-RETIREMENT CASE $(t \geq t^*)$

For the post-retirement case utility $u(c_t, l_t) = \overline{u}(c_t, \bar{l})$ and the value is $V(t, w_t, l_t) = \overline{V}(t, w_t, \bar{l})$. Therefore the lifetime utility from 2.11 takes the form

$$\int_0^t e^{-\rho s}\overline{u}(c_s, l_s)_s p_x ds + e^{-\rho t}{}_t p_x \overline{V}(t, w_t, \bar{l}) \tag{2.20}$$

Differentiating gives

$$\overline{u}(c_t, \bar{l})e^{-\rho t}{}_t p_x dt - (\rho + \lambda_t)e^{-\rho t}{}_t p_x \overline{V}(t, w_t, \bar{l})$$

$$+ e^{-\rho t}{}_t p_x\{\overline{V}_t(t, w_t, \bar{l})dt + \overline{V}_w(t, w_t, \bar{l})dw_t\} \leq 0 \tag{2.21}$$

and this is equal to zero for the optimal choice of c_t and t^*, where $\overline{u}(c_t, \bar{l}) = \bar{l}^{(1-\alpha)(1-\gamma*)}\frac{c^{1-\gamma}}{1-\gamma}$. I am assuming there is no pension income, hence the retiree can only consume from his pre-retirement savings which is earning an annual interest rate r. Therefore the wealth dynamics for post-retirement are

$$dw_t = (rw_t - c_t)dt \tag{2.22}$$

Substituting for the wealth dynamics give

$$\begin{aligned}\sup_{c_t, t^*}[&\overline{u}(c_t, \bar{l})e^{-\rho t}{}_t p_x dt - (\rho + \lambda_t)e^{-\rho t}{}_t p_x \overline{V}(t, w_t, \bar{l}) \\ &+ e^{-\rho t}{}_t p_x\{\overline{V}_t(t, w_t, \bar{l})dt + \overline{V}_w(t, w_t, \bar{l})(rw_t - c_t)dt\}] = 0\end{aligned} \tag{2.23}$$

As in the pre-retirement case, imagine the optimal choice of t^* is incorporated into the following equation. Factoring out $e^{-\rho t}{}_t p_x$ gives

$$\sup_{c_t}\{\overline{u}(c_t, \bar{l}) - c_t\overline{V}_w(t, w_t, \bar{l})\} - (\rho + \lambda_t)\overline{V}(t, w_t, \bar{l}) + \overline{V}_t + \rho w_t \overline{V}_w = 0 \tag{2.24}$$

Once again, I'll work out the above equation's optimization over c_t via a convex dual and later find the optimal t^* numerically.

Optimization over c_t

Consider the convex dual [58] of $\overline{u}(x, \bar{l})$

$$\tilde{\overline{u}}(y, \bar{l}) = \sup_x\{\overline{u}(x, \bar{l}) - xy\} \tag{2.25}$$

Consider $q = \overline{u}(x, \bar{l}) - xy = \bar{l}^{(1-\alpha)(1-\gamma*)}\frac{x^{1-\gamma}}{1-\gamma} - xy$. Differentiating q with respect to x and setting the derivative equal to zero yields

$$\begin{aligned}\bar{l}^{(1-\alpha)(1-\gamma*)}x^{-\gamma} - y &= 0 \\ y &= \bar{l}^{(1-\alpha)(1-\gamma*)}x^{-\gamma} \\ x^{\gamma} &= \bar{l}^{(1-\alpha)(1-\gamma*)}y^{-1} \\ x^* &= \bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}y^{-\tilde{\gamma}}\end{aligned} \tag{2.26}$$

where, $\tilde{\gamma} = \frac{1}{\gamma}$. Using this optimal x^* in $\tilde{\overline{u}}(y, \bar{l})$

$$\tilde{\overline{u}}(y, \bar{l}) = \frac{\bar{l}^{(1-\alpha)(1-\gamma*)}}{1-\gamma}[\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}y^{-\tilde{\gamma}}]^{1-\gamma} - [\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}y^{-\tilde{\gamma}}]y$$

$$= \frac{\bar{l}^{(1-\alpha)(1-\gamma*)}}{1-\gamma}[\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}y^{-\tilde{\gamma}}][y\bar{l}^{-(1-\alpha)(1-\gamma*)}] - \bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}y^{-\tilde{\gamma}}y$$
$$= \bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}y^{(1-\tilde{\gamma})}[\frac{1}{1-\gamma} - 1]$$
$$= -\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}\frac{y^{(1-\tilde{\gamma})}}{1-\tilde{\gamma}} \tag{2.27}$$

Thus, optimization over c_t develops as

$$\sup_{c_t}\{\overline{u}(c_t,\bar{l}) - c_t\overline{V}_w\} = \tilde{\overline{u}}(\overline{V}_w,\bar{l}) \tag{2.28}$$

and the optimal choice of c_t is

$${c_t}^* = \bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}{\overline{V}_w}^{-\tilde{\gamma}} \tag{2.29}$$

Equations 2.24 and 2.28 gives the PDE (HJ equation) for $\overline{V}(t,w,\bar{l}); t \geq t^*$

$$\overline{V}_t - (\rho+\lambda_t)\overline{V} + rw\overline{V}_w - \frac{\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}}{1-\tilde{\gamma}}{\overline{V}_w}^{(1-\tilde{\gamma})} = 0 \tag{2.30}$$

Due to the absence of labor income, a natural scaling relation for $\overline{V}(t,w,\bar{l})$ exists for some scalar k

$$\overline{V}(t,kw,\bar{l}) = k^{1-\gamma}\overline{V}(t,w,\bar{l})$$
$$=> \overline{V}(t,w,\bar{l}) = F(t,\bar{l})\frac{w^{1-\gamma}}{1-\gamma} \tag{2.31}$$

where $F(t,\bar{l})$ is the scaled post-retirement value function with a dimension less than original $\overline{V}(t,w,\bar{l})$. This gives

$$\overline{V}_w = F(t,\bar{l})w^{-\gamma}$$
$$(\overline{V}_w)^{1-\tilde{\gamma}} = F(t,\bar{l})^{1-\tilde{\gamma}}w^{1-\gamma}$$

Since $\bar{l}$ is a constant, call $F(t,\bar{l}) = F(t)$ and $B \equiv \bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}$. So, 2.30 takes the form

$$rwFw^{-\gamma} - \frac{B}{1-\tilde{\gamma}}F^{1-\tilde{\gamma}}w^{1-\gamma} + F'\frac{w^{1-\gamma}}{1-\gamma} - (\rho+\lambda_t)F\frac{w^{1-\gamma}}{1-\gamma} = 0$$

Factoring out $w^{1-\gamma}$

$$rF + \frac{\gamma B}{1-\gamma}F^{1-\tilde{\gamma}} + \frac{F'}{1-\gamma} - (\rho+\lambda_t)\frac{F}{1-\gamma} = 0$$

$$F' + r(1-\gamma)F + B\gamma F^{1-\tilde{\gamma}} - \rho F - \lambda_t F = 0$$

Call $F(t)^{\tilde{\gamma}} = f(t)$ and its derivative is $\frac{dF}{dt} = \gamma F^{1-\tilde{\gamma}}\frac{df}{dt}$. This reduces the above equation to

$$\gamma F^{1-\tilde{\gamma}}\frac{df}{dt} + [r(1-\gamma) - (\rho + \lambda_t)]F + \gamma B F^{1-\tilde{\gamma}} = 0$$

If $\gamma > 1$ and $F(t, \bar{l}) \neq 0$, I can factor out $\gamma F^{1-\tilde{\gamma}}$ and the ODE

$$\frac{df}{dt} + \tilde{\gamma}[r(1-\gamma) - (\rho + \lambda_t)]f + B = 0$$

Boundary Conditions

At infinite time the hazard rate is so high the consumer might die tomorrow. So he'll consume at a very large rate. For $\gamma > 1$ the utility $u(c) \to 0$ which brings down the value function to zero at infinite time: $\overline{V}(T, w, \bar{l}) = 0, 0 \leq w \leq W$. This boundary condition results in $F(T) = 0$ and eventually $f(t) = 0$.

Hence the dimension for post-retirement value function is reduced to give a first order non-linear ODE

$$\begin{aligned} &\frac{df}{dt} + \tilde{\gamma}[r(1-\gamma) - (\rho + \lambda_t)]f + B = 0 \\ &f(T) = 0 \end{aligned} \tag{2.32}$$

with value $\overline{V}(t, w) = f(t)^{\gamma}\frac{w^{1-\gamma}}{1-\gamma}$ and consumption $c^*(t) = B(f^{\gamma}w^{-\gamma})^{-1/\gamma} = Bf^{-1}w; t \geq \tau$

2.3 Numerical Scheme

Remember that the labor income of a consumer is 1 unit annually. I am using a wealth scale with units of labor income. That means $w = 5$ is basically 5 times the annual labor income.

I used a Matlab routine *ODE 45* [1] to solve the ordinary differential equation 2.32 over the time domain $0 \leq t \leq T$ and got the post-retirement value function 2.31.

$$\overline{V}(t, w, \bar{l}) = [f(t)]^{\gamma}\frac{w^{(1-\gamma)}}{1-\gamma}; 0 \leq w \leq W \tag{2.33}$$

For the pre-retirement partial differential equation 2.19 I used an upwind explicit finite-difference scheme. In order to capture the behavior of the value function for small values of wealth $0 \leq w_t < 1$, a log-transform of wealth variable is used. Call $y = lnw$ which gives $w = e^y; 0 \leq w < 1$ and $V_{1w} = e^{-y}V_{1y}; -\infty < y < 0$. Hence the PDE in the actual wealth domain and the transformed domain is given by:

$$\left[e^{-y} + r - \left(\frac{e^{-y(1-\tilde{\gamma})}V_1{}^{-\tilde{\gamma}}}{1-\tilde{\gamma}}\right)\right]V_{1y} + V_{1t} - (\rho + \lambda_t)V_1 = 0; -\infty < y_t < 0$$

$$V_{1t} - (\rho + \lambda_t)V_1 + (1 + rw)V_{1w} - \frac{{V_{1w}}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} = 0; 1 \leq w \leq W \tag{2.34}$$

Unless mentioned, I assumed $\rho = r$, a large wealth $W = 30$ and a maximum calendar age 120 giving $T = 120 - x$. For the sake of numerical stability of the scheme, I also assumed that the consumer is forced to retire when he reaches the age of $\tau = 110$, provided he is alive and not retired by that time. This restriction is applied in order to avoid stability issues arising for extremely small wealth levels when a consumer wishes to work until the last possible time of death. Also, it is assumed that once a person retires, he can not go back to work and has to spend his retirement life solely on retirement savings. Consider the numerical grids $(t_i, y_j), 0 \leq t_i \leq T, Y1 \leq y_j < 0$ and $(t_i, w_k), 0 \leq t_i \leq T, 1 \leq w_k < W$. Here $Y1$ represents a small wealth level close to $w = 0$. Let Δt, Δy and Δw be the time step-size, log-wealth step-size and the wealth step-size, respectively. N, M and M_w be the total number of time nodes, total log-wealth nodes and total wealth nodes, respectively. Then $t_i = i\Delta t,\ i = N, N-1, ..., 0$, $y_j = Y1 + j\Delta y,\ j = 0, 1, ..., M$ and $w_k = 1 + k\Delta w,\ k = 0, 1, ..., M_w$ where $t_0 = 0,\ t_n = T,\ y_0 = Y1, y_M = 1, w_0 = 1$ and $w_{M_w} = W$. Let $i = N_{ret}$ denotes the time grid point for the forced retirement time $t = \tau - x$. Denote $\overline{V}(t_i, y_j) = \overline{V}^i_j$, $\overline{V}(t_i, w_k) = \overline{V}^i_k$, $V_1(t_i, y_j) = v^i_j$ and $V_1(t_i, w_k) = v^i_k$.

Let,

$$\beta^{i+1}_j = e^{-y_j} + r - \frac{(e^{-y_j})^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} {V_{1y}}^{1-\tilde{\gamma}}(t_{i+1}, y_j)$$

and

$$\beta^{i+1}_k = 1 + rw_k - \frac{{V_{1w}}^{1-\tilde{\gamma}}(t_{i+1}, w_k)}{1-\tilde{\gamma}}$$

The backward finite difference for the time derivative and the upwind finite difference for the log-wealth derivative at (t_{i+1}, y_i) is:

$$V_{1t}(t_{i+1}, y_j) = \frac{v^{i+1}_j - v^i_j}{dt}$$

$$V_{1y}(t_{i+1}, y_j) = \begin{cases} \frac{v^{i+1}_{j+1} - v^{i+1}_j}{dy} & , \beta^{i+1}_j \geqslant 0 \\ \frac{v^{i+1}_j - v^{i+1}_{j-1}}{dy} & , \beta^{i+1}_j \leq 0 \end{cases}$$

Also, the backward finite difference for the time derivative and the upwind finite difference for the wealth derivative at (t_{i+1}, w_k) is:

$$V_{1t}(t_{i+1}, w_k) = \frac{v^{i+1}_k - v^i_k}{dt}$$

$$V_{1w}(t_{i+1}, w_k) = \begin{cases} \frac{v^{i+1}_{k+1} - v^{i+1}_k}{dw} & , \beta^{i+1}_k \geqslant 0 \\ \frac{v^{i+1}_k - v^{i+1}_{k-1}}{dw} & , \beta^{i+1}_k \leq 0 \end{cases}$$

Then for $i = N_{ret} - 1, ..., 0, j = 0, 1, ..., M$ and $k = 0, 1, ..., M_w - 1$

$$v^i_j = \frac{1}{1 + dt(r + \lambda_i)} \Bigg[v^{i+1}_j$$

$$
\begin{aligned}
&+ dt\Big\{\frac{\beta_j^{i+1}+|\beta_j^{i+1}|}{2}\frac{v_{j+1}^{i+1}-v_j^{i+1}}{dy}+\frac{\beta_j^{i+1}-|\beta_j^{i+1}|}{2}\frac{v_j^{i+1}-v_{j-1}^{i+1}}{dy}\Big\}\Big]\\
v_k^i =&\frac{1}{1+dt(r+\lambda_i)}\Big[v_k^{i+1}\\
&+ dt\Big\{\frac{\beta_k^{i+1}+|\beta_k^{i+1}|}{2}\frac{v_{k+1}^{i+1}-v_k^{i+1}}{dw}+\frac{\beta_k^{i+1}-|\beta_k^{i+1}|}{2}\frac{v_k^{i+1}-v_{k-1}^{i+1}}{dw}\Big\}\Big]
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
&v_{M_w}^i = \overline{V}^i_{M_w}; \forall i, \text{retire if the consumer has large wealth W.}\\
&v_j^i = \overline{V}^i_j, \forall j, \text{Value matching holds at } t = t^*\\
&v_k^i = \overline{V}^i_k, \forall k, \text{Value matching holds at } t = t^*\\
&v_j^i = \overline{V}^i_j, \forall j, i \geq N_{ret}, \text{consumer is forced to retire at } \tau\\
&v_k^i = \overline{V}^i_k, \forall k, i \geq N_{ret} \text{consumer is forced to retire at } \tau
\end{aligned}
$$

2.35 along with the boundary conditions solve PDE 2.34.

The lifetime optimal consumption for all wealth levels $Y1 \leq y_j < 0$, $1 \leq w_k \leq W$ at all time nodes $0 \leq t_i \leq T$ is calculated as follows:

$$
\begin{aligned}
c(t_i, y_j) &= \begin{cases} B(e^{-y_j}\frac{v_{j+1}^{i+1}-v_j^{i+1}}{dy})^{-\tilde{\gamma}} & , t_i \geq t^* \\ (e^{-y_j}\frac{v_{j+1}^{i+1}-v_j^{i+1}}{dy})^{-\tilde{\gamma}} & , t_i < t^* \end{cases}\\
c(t_i, w_k) &= \begin{cases} B(\frac{v_{k+1}^{i+1}-v_k^{i+1}}{dw})^{-\tilde{\gamma}} & , t_i \geq t^* \\ (\frac{v_{k+1}^{i+1}-v_k^{i+1}}{dw})^{-\tilde{\gamma}} & , t_i < t^* \end{cases}
\end{aligned}
\tag{2.36}
$$

Model vs Numerics (done in Matlab) $c(t_i, w_j), 0 \leq t_i \leq T, 0 \leq w_j \leq W$ is one matrix combined for log-wealth grid as well as the wealth grid.

2.4 Wealth Dynamics

Now I am trying to see how does the wealth of a 30 years old consumer evolve over the course of his lifetime given he has w_0 wealth in hand at $t = 0$ and his optimal consumption is c_t^* and retirement time is t^* as found above. Once again the assumptions are intact that he is not allowed to borrow any money and he can only invest in risk-free assets earning an instantaneous interest rate r. There is no pension stream and he doesn't have any bequest motives. Consider the post-retirement and pre-retirement wealth dynamics:

$$
\frac{dw}{dt} = 1 + rw(t) - c(t); t < t^*
$$

$$\frac{dw}{dt} = rw(t) - c(t); t \geq t^*$$

Or,

$$\frac{dw}{dt} = 1 + rw(t) - c(t); t < t^*$$
$$\frac{dw}{dt} = rw(t) - Bf^{-1}(t)w(t); t \geq t^*$$

Using the same time grid and wealth grid, discretizing the wealth ODEs using forward difference in wealth give

$$\frac{w_{i+1} - w_i}{dt} = 1 + rw_i - c(t_i, w_i); t_i < t^*$$
$$\frac{w_{i+1} - w_i}{dt} = rw(t_i) - Bf^{-1}(t_i)w_i; t_i \geq t^*$$

where $0 \leq t_i \leq T$ and $w_i = w(t_i)$. Simplification gives

$$\begin{aligned} w_{i+1} &= dt\left[1 + \left(r + \frac{1}{dt}\right)w_i - c(t_i, w_i)\right]; t_i < t^* \\ w_{i+1} &= w_i dt\left[r + \frac{1}{dt} - Bf^{-1}(t_i)\right]; t_i \geq t^* \end{aligned} \tag{2.37}$$

with $w(t_0) = w_0$. Beginning at t_0 and solving forwards in time gives the wealth dynamics of the consumer.

2.5 Algorithm

The algorithm to solve the problem with deterministic mortality is as follows:

1. Form the numerical grids $(t_i, y_j), 0 \leq t_i \leq T, Y1 \leq y_j < 0$ and $(t_i, w_k), 0 \leq t_i \leq T, 1 \leq w_k < W$.

2. Solve Post-retirement ODE given by 2.32 on the formed numerical grid using Matlab routine ODE45 [1] and save the post-retirement value function $\overline{V}(t_i, y_j) = \overline{V}^i_j$, $\overline{V}(t_i, w_k) = \overline{V}^i_k$.

3. Beginning at $i = N_{ret} - 1$ and $j = 1$, solve the pre-retirement value function $V_1(t_i, y_j) = v^i_j$ at each grid point (t_i, y_j) using numerical scheme 2.35. Check for $\max(v^i_j, \overline{V}^j_i)$ and replace existing v^i_j by the maximum. Go through all the log-wealth steps $j = 2, 3, ..., M - 1$

4. For the same time step, beginning at $k = 1$ solve the pre-retirement value function $V_1(t_i, w_k) = v_k^i$ at each grid point (t_i, w_k) using numerical scheme 2.35. Check for $\max(v_k^i, \overline{V}_i^k)$ and replace existing v_k^i with the maximum. Go through all the wealth steps $k = 1, 2, ..., M_w - 1$

5. Walk through the grid backwards in time while solving for each log-wealth grid points and then for wealth grid points for every time step.

6. For each time step, tag the log-wealth steps and the wealth steps where $\max(v_j^i, \overline{V}_i^j)$ and $\max(v_k^i, \overline{V}_i^k)$ changes, for the first time, from $\overline{V}_i^j$ to v_j^i and $\overline{V}_i^k$ to v_k^i, respectively. These tagged time and wealth grid points give a plot for t^* the optimal time to retire as a function of optimal wealth levels.

7. Once, v_i^j and v_k^i have been evaluated for all gird points $0 \leq t_i \leq T, Y1 \leq y_j < 0$ and $0 \leq t_i \leq T, 1 \leq w_k < W$ they give the value function for the entire life time of the consumer.

8. Calculate optimal consumption for all wealth levels $Y1 \leq y_j < 0$, $1 \leq w_k \leq W$ at each time step $0 \leq t_i \leq T$ according to equations 2.36.

9. Now calculate wealth dynamics of the consumer with wealth w_0 at time $t = 0$ using forward difference scheme given in 3.40. Begin at $t_0 = 0$, solve forwards in time. For each set of $(t_i, w(t_i))$ find the consumption from $c(t_i, w_i)$ matrix or $f(t_i)$ depending upon if t_i belongs to the pre-retirement phase or post-retirement phase of life.

2.6 Numerical Results

For all numerical results the parameter values used are $r = 2.5\%, \rho = r, \alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82$ and $\gamma = 2$, unless mentioned. I am using the grid sizes of $dt = 0.0005, dw = 0.01, dy = 0.01$. The computations took around 48 hours at this fine grid size. Qualitative as well as quantitative results stayed the same if I changed the grid sizes to $dt = 0.001, dw = 0.02, dy = 0.02$. Except for 2.1, all figures are plotted at the $dt = 0.001, dw = 0.02, dy = 0.02$ grid sizes.

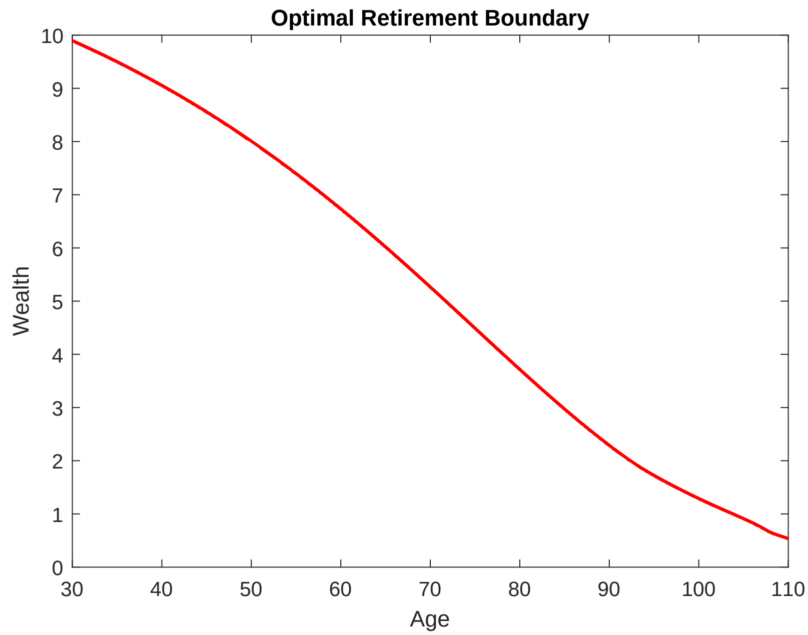


Figure 2.1: Optimal Retirement Boundary.
Assuming $\gamma = 2$, $r = 2.5\%, \rho = r$, $\alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

Figure 2.1 plots the optimal retirement boundary, as a function of age and wealth level. For any age level it gives the optimal wealth required to retire from the labor force. The portion above this boundary is the post-retirement region and below is the pre-retirement region. A consumer in the pre-retirement region, will retire as soon as he hits the boundary. Notice that, the wealth to be accumulated for retirement decreases as the consumer pushes retirement forward in time. A consumer age x can retire right away if he has a wealth level of \$9.89 or more. People with wealth levels below \$0.53 are willing to work till the maximum possible age. They give more utility to the pre-retirement stage of life and are forced to retire at age 110.

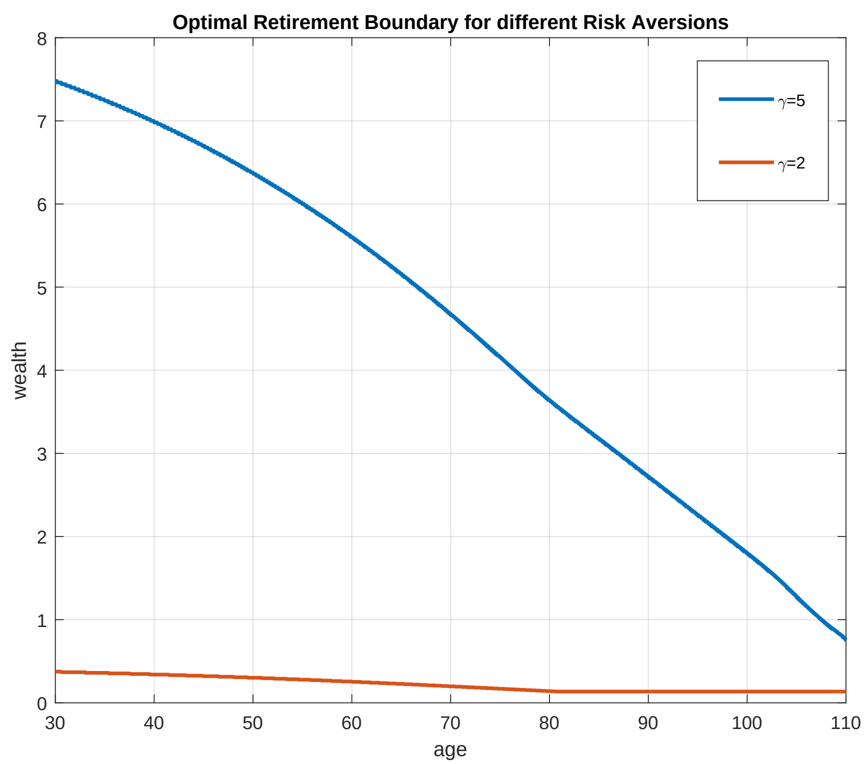


Figure 2.2: Optimal retirement age and wealth for varying risk aversion.
$\gamma = 2$ assumes $\alpha = 0.2, \gamma^* = 6$
$\gamma = 5$ assumes $\alpha = 0.5, \gamma^* = 9$
$B^\gamma = 5.6 \times 10^{-4}, r = 2.5\%, \rho = r, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

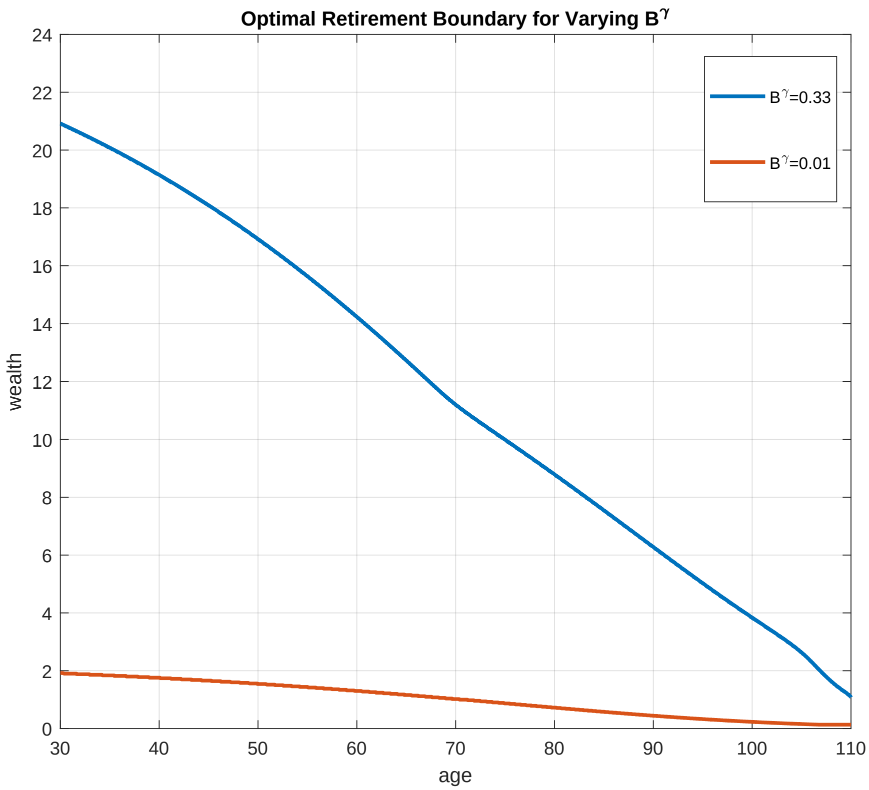


Figure 2.3: Optimal retirement age and wealth for varying B^γ.
$B^\gamma = 0.33$ assumes $\alpha = 0.5, \gamma^* = 3$
$B^\gamma = 0.01$ assumes $\alpha = 0.2, \gamma^* = 6$
$\gamma = 2, r = 2.5\%, \rho = r, \bar{l} = 3, b = 9.44, m = 88.82, \tau = 110$

In Figure 2.2 I did a comparison of the attitude of two consumers with different risk aversions γ while holding B^γ constant at 5.6×10^{-4}. The parameter B^γ controls the shift in marginal utility of consumption as the consumer enters retirement. As age increases, individuals require less wealth to be able to retire, the decrease is smaller (more gradual) for less risk-averse individuals as they are more averse to the risk associated with a longer life-span. Higher risk aversion shifts the retirement wealth threshold upwards. More risk-averse individuals, having greater fear of living longer, require more wealth to optimally retire. Figure 2.3 does a comparison of retirement wealth thresholds as a function of age for different levels of the B^γ parameter while keeping $\gamma = 2$. From equation 2.6 B^γ is the ratio of pre-retirement to post-retirement utility. As B^γ reduces, indicating more dis-utility of labor, the retirement wealth decreases. A consumer with lower B^γ cares more about the post-retirement leisure and wants to retire earlier with lower wealth levels.

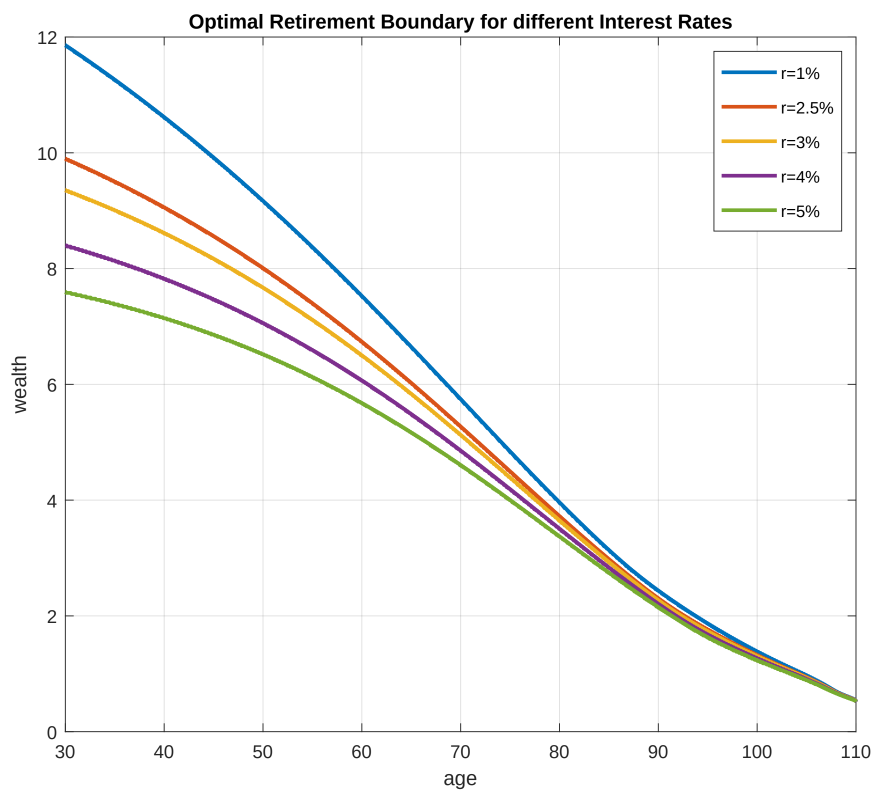


Figure 2.4: Optimal retirement age and wealth for different values of annual risk free rate. Assuming $\gamma = 2, \rho = 2.5\%,\ \alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

Figure 2.4 shows the effect of the interest rate r on the behavior of optimal retirement. With increasing interest rate the retirement boundary shift downwards. With higher interest rates, the need to accumulate more wealth to sustain post-retirement consumption decreases, because in the post-retirement phase the retirement savings will earn higher interest rates.

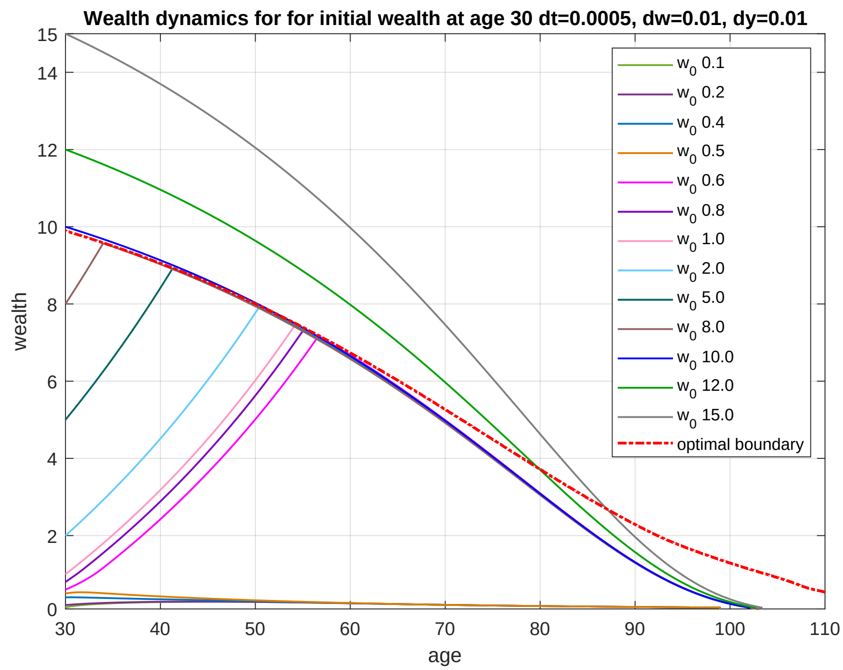


Figure 2.5: Wealth dynamics for different wealth levels at age 30.
Assuming $r = 2.5\%, \rho = r,\ \gamma = 2, \alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

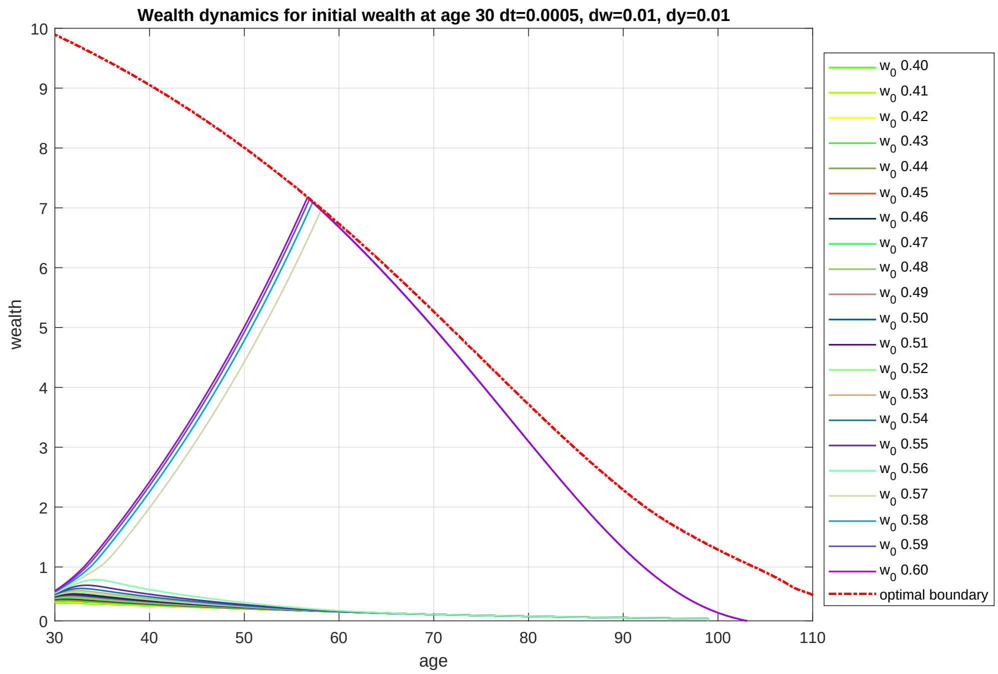


Figure 2.6: Wealth dynamics for smaller wealth levels at age 30.
Assuming $r = 2.5\%, \rho = r,\ \gamma = 2, \alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

Figures 2.5 and 2.6 plot the wealth dynamics of consumers aged 30 who are sitting at different levels of their current savings. As long as the consumer is not retired and his wealth-age combination is below the optimal retirement boundary, their wealth follows the pre-retirement dynamics which takes into consideration the labor income. Once they retire, their wealth evolves as per the post-retirement dynamics. If they are above the optimal boundary at age 30, they will not enter the labor force ever. They have enough money saved to spend all their life enjoying the post-retirement life. Note that between age 95 and 105, all consumers have completely exhausted their savings and need some external money source (for example pension income) for survival. I also noticed an interesting non-uniqueness behavior in wealth curves. Figure 2.6 shows this behavior more clearly where the wealth dynamics are plotted for initial wealth between $w = 0.4$ and $w = 0.6$. Starting with wealth at age 30 above a certain wealth level (approximately around wealth =0.5, I call it $\tilde{w}$), there is a unique optimal strategy. It involves saving till the consumer reaches the optimal retirement curve, and then retiring. For age-30 wealth below $\tilde{w}$, there is a unique optimal strategy that involves slowly spending money over time, but never retiring. At age-30 wealth level $\tilde{w}$ the wealth curve is a non-uniqueness curve shown and explained in Figure 2.7. For this wealth level, there are multiple optimal strategies, all of which give the same utility.

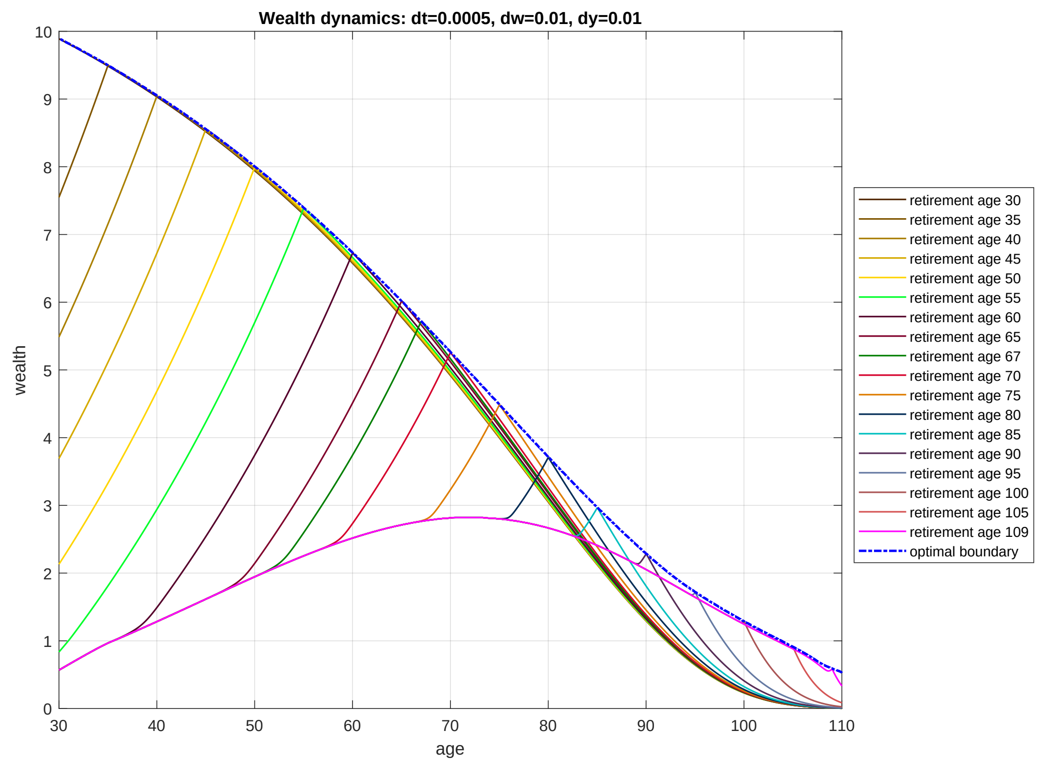


Figure 2.7: Wealth dynamics for different retirement ages.
Assuming $r = 2.5\%, \rho = r, \gamma = 2, \alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

For figure 2.7 I picked a range of retirement ages between age 30 to age 110. For each retirement age I marked the optimal retirement wealth level. Then using that age and wealth level as an initial condition I solved the forward wealth dynamics and using the same age and wealth pair as a final condition I solved the backwards wealth dynamics. This exercise was done to study the non-uniqueness behavior shown in figures 2.5 and 2.6. All forward curves are a scalar multiple of each other because the post-retirement value function involves wealth scaling. On the other hand, all the backward wealth curves coalesced to form the a special curve (call it the uncommitted curve, shown in magenta color) for wealth, along which consumers aren't committed to retire or not retire. But if wealth moves above it, optimal behavior will be to eventually retire. And if wealth moves below it, optimal behavior will be to never retire. At every point on the uncommitted curve the consumer has a choice of whether to stay on the curve, to move up, or to move down, with all three choices producing the same utility. A consumer beginning with wealth $\tilde{w}$ at age 30 has multiple optimal strategies. Each one consists of following the uncommitted curve (the magenta curve) for some period of time and then either moving upwards towards the retirement curve or downwards on the path

where retirement is not an option (for example, all the curves for age 30 wealth 0.5 and less, shown in figures 2.5 and 2.6) .
In other words, I have found three regions. Above the blue optimal boundary, optimal behavior is to retire right away. Between the blue curve and the magenta curve, optimal behavior is to save with the intention of retiring. Below the magenta uncommitted curve, optimal behavior is to plan consumption around never retiring. For each point on the magenta curve I can decide to go up, to go down, or to stay on the magenta curve till later. All three choices give the same utility.

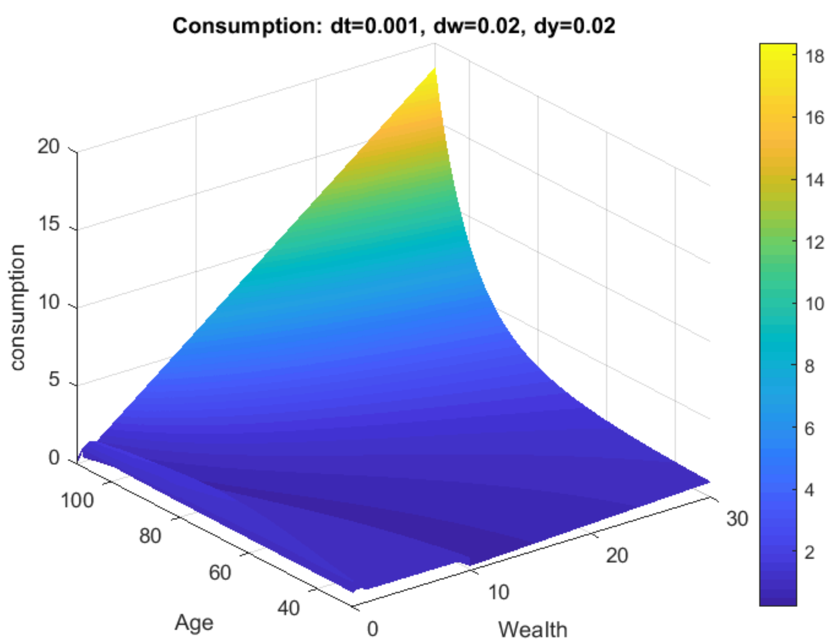


Figure 2.8: Optimal Consumption Surface.
Assuming $r = 2.5\%, \rho = r,\ \gamma = 2, \alpha = 0.5, \bar{l} = 6.49, b = 9.44, m = 88.82, \tau = 110$

Figure 2.8 shows the life time optimal consumption surface of a consumer as a function of age and wealth.

2.6.1 Efforts to Calibrate $\bar{l}$

There are some interesting numbers for retirement savings and age in an article *How much do I need to save for retirement?* [29] at Fidelity.com. It shows approximately how much a person needs to save (in terms of his starting salary), if he is 30 years old now, wants to retire at age 67 and aims to maintain his current lifestyle. Table 2.1 exhibit these values.

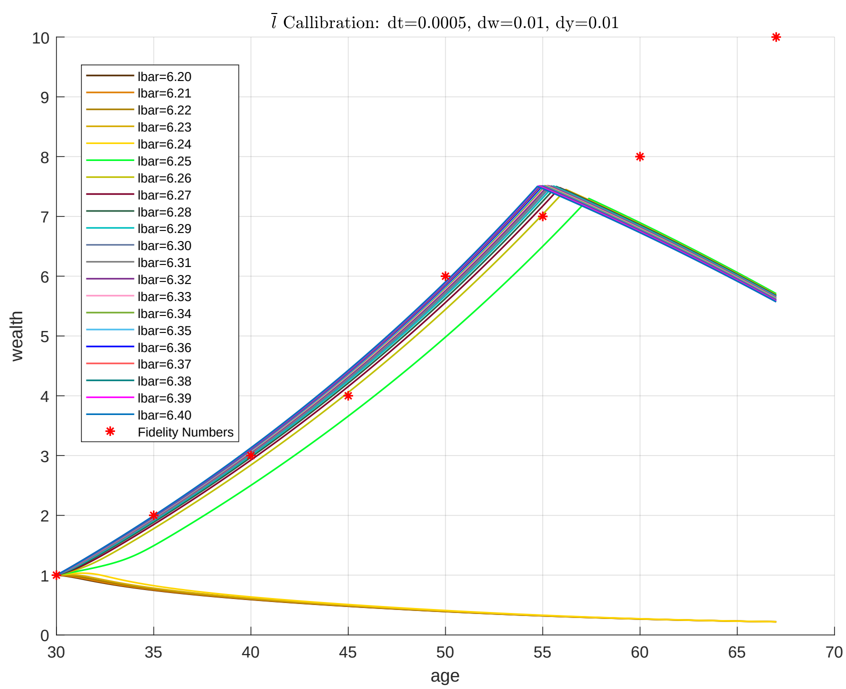


Figure 2.9: Comparison between Fidelity numbers and my Optimal Wealth Dynamics. Assuming $r = 2.5\%, \rho = r,\ \gamma = 2, \alpha = 0.5, b = 9.44, m = 88.82, \tau = 110$

I tried using these numbers to find the value of $\bar{l}$ in my model by calibrating Fidelity's retirement savings curve to the optimal wealth dynamics. Using the Fidelity number $w = 1$

at age 30 ($t = 0$), I solved the optimal wealth dynamics forward in time until age 67. From figure 2.9 I can see that there is a slight agreement between Fidelity numbers and my numbers between ages 30 and 55 for values of $\bar{l}$ between 6.26 and 6.40 but this agreement is not quite encouraging. After age 55, clearly the wealth dynamics changed to post-retirement but as per Fidelity number the consumer retires at age 67. I also tried working on Mean squared Error (MSE) between the Fidelity numbers and my wealth dynamics numbers. There are 8 sample points from the Fidelity article, see Table 2.1. Then $MSE = \frac{1}{8}\sum_{i=1}^{8}(Y_i - \hat{Y}_i)^2$, where

Age	Wealth
30	1
35	2
40	3
45	4
50	6
55	7
60	8
67	10

Table 2.1: Fidelity numbers [29] for retirement age and savings to retire at age 67 with wealth 10x the current annual income.

Y_i are the wealth numbers from table 2.1 and $\hat{Y}_i$ are wealth numbers from my optimal model for corresponding ages. Table 2.2 depicts the MSE calculated for varying values of $\bar{l}$. The sudden jump in MSE for $\bar{l} = 6.24$ and $\bar{l} = 6.25$ is due to the non-uniqueness behavior of wealth dynamics discussed in Figure 2.7. I can notice that for all values of $\bar{l}$ MSE is quite high being away from 0. The values of $\bar{l}$ between 6.25 and 6.5 seems a better fit with lower MSE numbers as compared to rest of the $\bar{l}$ values tested. That means the Fidelity model is not a good fit for my model to calibrate for $\bar{l}$. Possible reasons could be the simplicity of my model, while the Fidelity model takes into consideration a lot of factors related to retirement. I'll get back to calibration for $\bar{l}$ in my model with pension involved.

$\bar{l}$	MSE	$\bar{l}$	MSE
3	31.4649	6.31	2.5926
4	31.4649	6.32	2.6024
5	31.4649	6.33	2.6164
6	31.4643	6.34	2.6333
6.1	31.4597	6.35	2.6524
6.2	31.4170	6.36	2.6638
6.21	31.4036	6.37	2.6723
6.22	31.3811	6.38	2.6807
6.23	31.3551	6.39	2.6888
6.24	31.2878	6.4	2.7077
6.25	2.6944	6.5	2.8150
6.26	2.5315	6.6	2.9414
6.27	2.5348	6.7	3.0736
6.28	2.5391	6.8	3.2009
6.29	2.5541	6.9	3.3437
6.3	2.5733	7	3.4903

Table 2.2: MSE for varying $\bar{l}$.
Assuming $r = 2.5\%, \rho = r,\ \gamma = 2, \alpha = 0.5, b = 9.44, m = 88.82, \tau = 110$

Chapter 3

Stochastic Mortality Rate

3.1 Problem

Unlike deterministic mortality where the survival probabilities are known in advance at all times, in stochastic mortality I do not know in advance how the survival probability will evolve over time. At time 0 I only have an expectation of the survival probability at any future time. My assumption is that, even though the force of mortality will grow stochastically, one can accurately measure an individual's current mortality rate. That is, the stochastic mortality rates are random but observable. Now, these rates are tied to probabilities and people find it easier to understand mortality in terms of time or age than probability. Hence, phrasing these mortality rates in terms of an age make things easier to understand. Huang et al. [31] used the concept of biological age for the first time in LCM setting. This paper has a very detailed explanation about the b-age process. For this project I am using two types of ages associated to a consumer.
C-age: the calendar age measured from the time of birth.
B-age: the biological age. It is the true age based on the health status of the consumer. It can be measured using different bio-markers and can be 10-15 years more or less than the calendar age.

Now, consider a consumer with chronological age (C-age) κ_t and biological age (B-age) A_t at time t. Time $t; 0 \leq t \leq \infty$ is measured from current C-age x. It's an infinite time horizon problem. Assume that the consumer invested all his wealth at a risk-free rate r with no bequest motives. He is interested in determining the optimal B-age a^* and wealth w^* to retire at an optimal calendar age κ_{t^*} (where, t^* is the time when the consumer reaches c-age κ_{t^*}) while having an optimal consumption plan for his entire lifetime.

3.2 Mathematical Setup

I assume the classical Gompertz law of mortality based on biological age A_t (following Huang et al. [31]) hence making the mortality hazard rates $\lambda_t = \lambda(A_t)$ stochastic in nature. Here

$$\lambda(a) = \frac{1}{b}e^{\frac{a-m}{b}} \tag{3.1}$$

Note that $\lambda(a)$ is a real function but λ_t is a stochastic process. (m, b) are found using pinned down hazard rates $\overline{\lambda}$ and λ_T at b-ages 60 and κ_T. T is a time beyond which aging ceases and the mortality plateaus [6]. Huang et al. [31] used $\overline{\lambda} = 0.005$ and $\lambda_T = 1$. The assumption $\lambda_T = 1$ implies that after age κ_T the consumer stops aging and from now onwards his life expectancy is one year. From the hazard rate equation 3.1

$$0.005 = \frac{1}{b}e^{\frac{60-m}{b}}$$

$$1 = \frac{1}{b}e^{\frac{\kappa_T-m}{b}}$$

Solving these equations simultaneously results in

$$b = \frac{\kappa_T - 60}{log(1/0.005)}, m = 60 - b\, log(0.005b)$$

Huang et al. [31] assumed that at any time t, C-age κ_t can be written as $\kappa_t = \kappa_0 + t$ for κ_0 typically equal to x and the biological age is $A_t = \kappa_t + Y_t$ for $t \leq T$. Y_t or A_t is a stochastic process called *generalized Brownian Bridge*. For $\xi = \sigma = 1$, Y_t is the *Brownian Bridge* process with the property that end points Y_0 and Y_T both are 0. That is, the process begins at 0 at $t = 0$ and reverts back to 0 at time $t = T$. I have,

$$dY_t = -\xi\frac{Y_t}{T-t}dt + \sigma dB_t \tag{3.2}$$

where, σ is the volatility of the process and ξ is a mean reversion parameter and B_t is the standard Brownian motion. Since, c-age κ_t is deterministic $d\kappa_t = dt$, therefore,

$$dA_t = \left(1 + \xi\frac{\kappa_t - A_t}{T-t}\right)dt + \sigma dB_t \tag{3.3}$$

The most important concept of my thesis is the filtration. $\mathscr{G}_t = \sigma\{\lambda_q | q \leq t\}$ is a filtration which contains all information about an individual's hazard rate until time t except for the stochastic time of death ς. Consumption and leisure are adapted to this filtration. The point is that I want to base the filtration on something which I can observe, that is, the individual's hazard rate and then adjust the leisure and consumption according to those hazard rates. For deterministic mortality an individual's mortality rate is still stochastic, but the Gompertz law is giving population hazard rates which are known and used for the individual. For stochastic mortality an individual's hazard rates are observable being based on biological age. Hence, it makes sense to consider the consumption c_t and leisure processes l_t adapted to this filtration. By the definition of hazard rate, the survival probability is

$$E\left[1_{\{s\leq\varsigma\}}\middle|\,\mathscr{G}_s\right] = P\left(\varsigma > s|\mathscr{G}_s\right) = e^{-\int_0^s \lambda_q dq} \tag{3.4}$$

Recall from chapter 2 and section 2.2., the deterministic leisure process l_t is given by 2.3.

$$l_t = \begin{cases} l_1, \text{ pre-retirement time period} \\ \bar{l}, \text{ post-retirement time period} \end{cases}$$

It is adapted to the filtration $\mathscr{G}_t$. Once again I am working in Cobb-Douglas utility framework given by 2.5 and 2.6.

$$u(c_t, l_t) = \begin{cases} u_1 = \frac{c_t^{1-\gamma}}{1-\gamma} & \text{, pre-retirement} \\ \overline{u} = \bar{l}^{(1-\alpha)(1-\gamma^*)} \frac{c_t^{1-\gamma}}{1-\gamma} & \text{, post- retirement} \end{cases}$$

A consumer with current wealth $w_0 = w$ and current b-age $A_0 = a$ (at $t = 0$), wants to maximize his expected life time utility of leisure and consumption while determining the optimal stopping time t^* to retire from the labor force. Then,

$$V(w, l, a) = \max_{c_s, t^*} E\left[\int_0^\infty e^{-\rho s} u(l_s, c_s) 1_{\{s\leq\varsigma\}} ds \middle| w_0 = w, A_0 = a\right] \tag{3.5}$$

Using Fubini's theorem gives

$$\begin{aligned} V(w, l, a) &= \max_{c_s, t^*} \int_0^\infty E\left[e^{-\rho s} u(l_s, c_s) 1_{\{s\leq\varsigma\}}\right] ds \\ &= \max_{c_s, t^*} \int_0^\infty E\left[E\left\{e^{-\rho s} u(l_s, c_s) 1_{\{s\leq\varsigma\}}\middle|\, \mathscr{G}_s\right\}\right] ds \\ &= \max_{c_s, t^*} \int_0^\infty E\left[e^{-\rho s} u(l_s, c_s) E\left\{1_{\{s\leq\varsigma\}}\middle|\, \mathscr{G}_s\right\}\right] ds \end{aligned} \tag{3.6}$$

Hence,

$$V(w, l, a) = \max_{c_s, t^*} E\left[\int_0^\infty e^{-\int_0^s (\rho + \lambda_q) dq} u(l_s, c_s) ds\right] \tag{3.7}$$

Define

$$V(t, w, l, a) = \max_{c_s, t^*} E\left[\int_t^\infty e^{-\int_t^s (\rho + \lambda_q) dq} u(l_s, c_s) ds \middle| A_t = a, w_t = w, l_t = l\right] \tag{3.8}$$

Consider the expected value of maximal total benefit given all information up-till time 't'

$$Z_t = E\left[\int_0^\infty e^{-\int_0^s (\rho + \lambda_q) dq} u(l_s, c_s) ds \middle| \mathscr{G}_t\right]$$

$$= \int_0^t e^{-\int_0^s(\rho+\lambda_q)dq} u(l_s, c_s)ds + E\left[\int_t^\infty e^{-\int_0^s(\rho+\lambda_q)dq} u(l_s, c_s)ds \middle| \mathscr{G}_t\right]$$

where the integral from 0 to t is $\mathscr{G}_t$-measurable. Thus, using 3.8, Z_t takes the form:

$$Z_t = \int_0^t e^{-\int_0^s(\rho+\lambda_q)dq} u(l_s, c_s)ds + e^{-\int_0^t(\rho+\lambda_q)dq} V(t, w_t, l, A_t) \tag{3.9}$$

It is a martingale for optimal choice of controls c_s and t^*. A similar argument shows that it is a super martingale for sub-optimal choice of control variables. As discussed in the deterministic mortality chapter 2 and section 2.2, I'll divide the problem into pre-retirement case and post-retirement case.

3.2.1 PRE-RETIREMENT $(t < t^*)$

For the pre-retirement period I am considering leisure $l_t = l_1 = 1$ (normalized), utility of consumption $u(c_t, l_t) = u_1(c_t) = \frac{{c_t}^{1-\gamma}}{1-\gamma}, \gamma \neq 1$ and the pre-retirement value $V(t, w, l, a) = V_1(t, w, l, a)$. Then for an optimal choice of consumption c_t and time to retire t^*

$$Z_t = \int_0^t e^{-\int_t^s(\rho+\lambda_q)dq} u_1(c_s)ds + e^{-\int_0^t(\rho+\lambda_q)dq} V_1(t, w_t, A_t) \tag{3.10}$$

Applying Ito's lemma to Z_t

$$\begin{aligned} dZ_t =& e^{-\int_0^t(\rho+\lambda_q)dq}\bigg(-(\rho+\lambda_t)+0\bigg)V_1(t, w_t, A_t)dt + \\ & e^{-\int_0^t(\rho+\lambda_q)dq} V_{1t}(t, w_t, A_t)dt + e^{-\int_0^t(\rho+\lambda_q)dq} u_1(c_t)dt + \\ & e^{-\int_0^t(\rho+\lambda_q)dq} V_{1w}dw_t + e^{-\int_0^t(\rho+\lambda_q)dq}\bigg(V_{1a}dA_t + \frac{1}{2}V_{1aa}dA_t.dA_t\bigg) \end{aligned}$$

and factoring out $e^{-\int_t^s(\rho+\lambda_q)dq}$ gives

$$dZ_t = e^{-\int_t^s(\rho+\lambda_q)dq}\bigg[u_1(c_t)dt - (\rho+\lambda(A_t))V_1dt + V_{1t}dt + V_{1w}dw_t + V_{1a}dA_t + \frac{1}{2}V_{1aa}dA_t.dA_t\bigg]$$

Since assumption for the consumer's investment and consumption stays the same as in deterministic mortality case; chapter 2 and section 2.2, the wealth dynamics for pre-retirement are given by 2.14.

$$dw_t = (1 + rw_t - c_t)dt$$

Using dynamics of B-age from 3.3 gives

$$dZ_t = e^{-\int_t^s(\rho+\lambda_q)dq}\bigg[u_1(c_t)dt - (\rho+\lambda(A_t))V_1dt + V_{1t}dt + V_{1w}(1 + rw_t - c_t)dt +$$

$$V_{1a}[(1+\xi\frac{\kappa_t - A_t}{T-t})dt + \sigma dB_t] + \frac{1}{2}V_{1aa}\sigma^2 dt\Big]$$

for $t < T$. Z_t is a martingale for optimal choice of c_t and t^*, therefore, the Z_t process shouldn't have any drift for this choice of controls. For sub-optimal choice of controls $dZ_t \leq 0$, that is, Z_t is a super martingale. In the continuation region $t < t^*$, I have

$$\sup_c \Big[u_1(c) - (\rho + \lambda(a))V_1 + V_{1t} + (1 + rw - c)V_{1w}\Big] + \\ + \left[\left(1 + \xi\frac{\kappa_t - a}{T-t}\right)\right]V_{1a} + \frac{\sigma^2}{2}V_{1aa} = 0 \tag{3.11}$$

when $t < T$. For optimization over c, consider the convex dual of $u_1(c)$

$$\tilde{u_1}(V_{1w}) = \sup_c\{u_1(c) - cV_{1w}\} = -\frac{{V_{1w}}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} \tag{3.12}$$

The details about finding this convex dual has been discussed in chapter 2 and section 2.2.1, equations 2.17 and 2.18.

Using equation 3.12 in equation 3.11 results in the Hamilton Jacobi Bellman equation for $V_1(t,w,a); 0 < t < t^*$ and $t < T$:

$$V_{1t} + (1 + rw)V_{1w} - \frac{{V_{1w}}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} - (\rho + \lambda(a))V_1 + \left[\left(1 + \xi\frac{\kappa_t - a}{T-t}\right)\right]V_{1a} + \frac{\sigma^2}{2}V_{1aa} = 0 \tag{3.13}$$

Free Boundary $\tau(t^*, w^*, a^*)$

Variables we don't control usually require smooth pasting. At the free boundary smooth pasting holds for the stochastic variable a. $V_{1a}(t,w,a^*) = \overline{V}_a(t,w,a^*)$. Also, value matching must hold, that is, post-retirement value is equal to the pre-retirement value. $V_1(t^*,w^*,a^*) = \overline{V}(t^*,w^*,a^*)$. This boundary will be determined numerically.

Boundary Condition at large Wealth W

Consider the wealth domain $0 < w < W$ for a large enough maximum wealth W. I am assuming that at a very large wealth W, the consumer will have enough money to live a happy retired life so he will prefer to retire $V_1(t,W,a) = \overline{V}(t,W,a)$.

Boundary Conditions for b-age

Let's consider the domain discussed in the article by Huang et al. [31] for b-age $-\infty < a < \infty$. It is an infinite domain. $a = -\infty$ depicts that that a person can be in an infinite healthy state and $a = +\infty$ simply means a human can live to an infinite b-age. A very large hazard rate at $a = \infty$ would kill the consumer, so an obvious choice of boundary condition is $V_1(t,w,\infty) = 0$.

For $a = -\infty$, consumer is in extreme good health, so hazard rate $\lambda(a) = 0$ and V_1 does not depend upon a. Hence, a simpler PDE has to be solved

$$V_{1t} - \rho V_1 + (1 + rw)V_{1w} - \frac{{V_{1w}}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} = 0$$

with the boundary conditions

$V_1(t^*, w) = \overline{V}(t^*, w, -\infty)$ that is, the consumer should retire when $t = t^*$

$V_1(t, W) = \overline{V}(t^*, W, -\infty)$ that is, the consumer will prefer to retire at large wealth W.

where, $\overline{V}(t, w, a)$ is post-retirement value function.
New set of Boundary Conditions for B-Age:
In reality a person cannot live for an infinite time and by no-means can be younger than when he was born, that is age 0. Keeping this in mind I changed the domain of A_t from $-\infty < a < +\infty$ to $0 \leq a \leq \kappa_T$.

At $a = 0$, I applied homogeneous Neumann condition, that is, whenever $V_1(t, w, a)$ hits this boundary it reflects back; $V_{1a}(t, w, 0) = 0$.

I make $a = \kappa_T$ an absorbing boundary. So, once a person reaches maximum b-age, he'll have $V_{1a} = 0 = V_{1aa}, V_{1t} = 0$. Hence, the value at this boundary is the solution to ODE 3.14 discussed below under *Boundary Condition at* $t = T$. Therefore, at maximum b-age κ_T, $V_1(t, w, \kappa_T) = V_T(w)$.

Boundary Condition at $t = T$

At time T I am assuming that person hits maximum b-age. Hence, the pinned down mortality $\lambda_T = 1$ makes the person stop aging with a life expectancy of exactly one year. Thus, $V_{1a} = 0 = V_{1aa}$. Also, at time T, if we look at two different times $t_1 > T$ and $t_2 > T$, their maximization problems are the same, that is, both the problem and the dynamics become time homogeneous. So they have the same solution, which will not depend on t. Therefore, the problem is time-invariant, so $V_{1t} = 0$. Let's call $V_1(T, w, a) \equiv V_T(w)$. Therefore a suitable boundary condition at T is to solve a simpler ODE for $V_T(w)$ obtained from 3.13 by setting $V_{1a} = 0 = V_{1aa}, V_{1t} = 0$ and $\lambda(a) = \lambda_T$.

$$\begin{aligned} &(1 + rw)\frac{dV_T}{dw} - \frac{1}{1-\tilde{\gamma}}\left(\frac{dV_T}{dw}\right)^{1-\tilde{\gamma}} - (\rho + \lambda_T)V_T = 0, 0 \leq w \leq W \\ &V_T(W) = \overline{V}_T(W) \end{aligned} \tag{3.14}$$

$\overline{V}_T(W)$ is the post-retirement value at large wealth $w = W$ and time $t = T$.

The pre-retirement boundary condition at T is exactly the same as in chapter 4 section 4.2.2: *Boundary condition at* $t = T$. I will discuss this problem in great detail there. The only difference is notation. In section 4.2.2 the post-retirement value is $\overline{V_T^\pi}$ and the pre-retirement value is V_T^π, while over here the post-retirement value is denoted by $\overline{V_T}$ and the pre-retirement value is V_T. For the results and discussion, with a chance of repetition, I'll only mention the main results over here.

At $t = T$, if the consumer retires at some some wealth level $\overline{w}$ then the optimal value function for the consumer agrees with the post -retirement value $\overline{V_T}(w)$ which I will calculate in equation 3.25 for some interval $[\overline{w}, W]$, and the pre-retirement ODE 3.14 on the interval $[0, \overline{w}]$. For the pre-retirement phase $w < \overline{w}$ if $S(\nu) = (1 + rw)\nu - \frac{1}{1-\tilde{\gamma}}\nu^{1-\tilde{\gamma}}$ then ODE 3.14 is reduced to an algebraic equation for the variable $\nu \equiv \frac{dV_T}{dw}$

$$S(\nu) = (\rho + \lambda_T)V_T \tag{3.15}$$

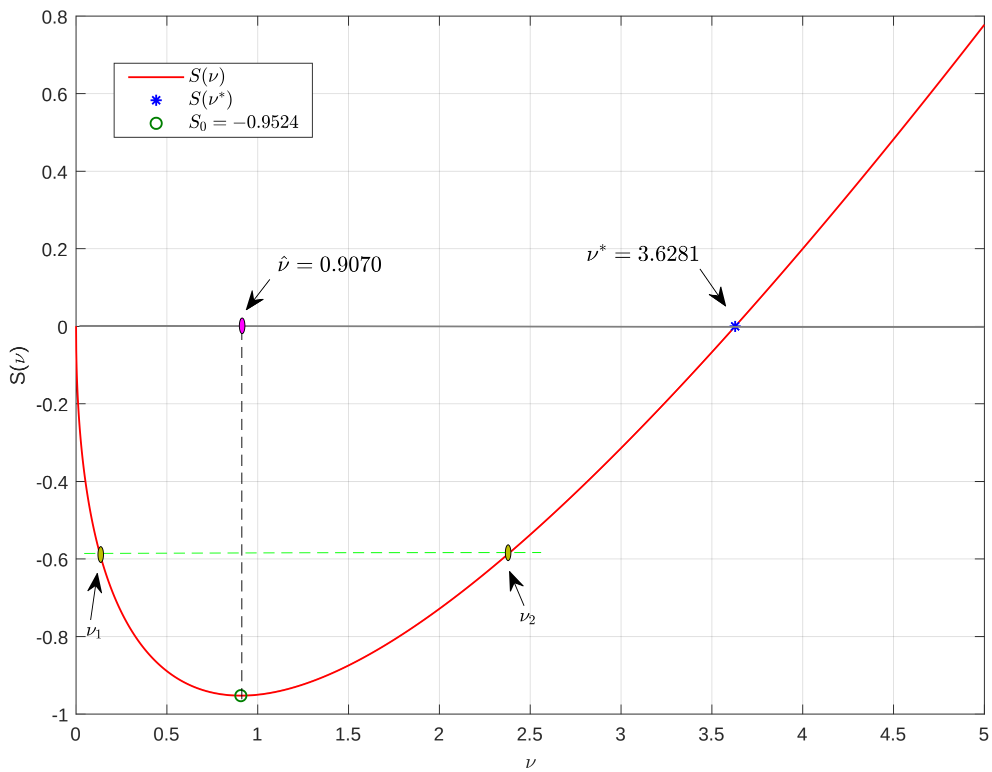


Figure 3.1: Example for the function $S(\nu)$ at $w = 2$, $\gamma = 2, r = 2.5\%$

Then, S has a critical point at $\hat{\nu} = (1 + rw)^{-\gamma}$, a minimum of $S_0 = \frac{(1+rw)^{1-\gamma}}{1-\gamma}$ and the subsistence consumption level $\tilde{c} = 1 + rw$. The zeros are $\nu = 0$ and $\nu^* = (1 + rw)(1 - \tilde{\gamma})^{-\gamma}$. $\nu^* > 0$ is the right hand side zero. The 2 possible solutions to the algebraic equation (3.15) in an interval $(S_0, 0]$; are ν_1 and ν_2 with $\nu_1 < \hat{\nu} < \nu_2$. The pre-retirement values V_{T1} and

V_{T2} are the solutions for ODE (3.14) corresponding to ν_1 and ν_2, respectively. I will in fact consider both solutions (for details of the results mentioned in this paragraph, please see chapter 4 section 4.2.2: *Boundary condition at* $t = T$). See figure 3.1 for an example of the function $S(\nu)$. We can clearly see the left hand zero , the minima and the critical point. The dotted green line is simply the horizontal line test showing an example of existence of two possible solutions for the algebraic equation 3.15.

Therefore, consider the situation when both solutions $V_{T1}(w)$ and $V_{T2}(w)$ cross at some wealth $w^* \in [0, W]$. The maximum of two solutions; $max(V_{T1}, V_{T2})$ will have a lower $(V_T)_w$ (higher consumption) to the left of w^* and higher $(V_T)_w$ (lower consumption) to the right of w^*. Thus, the curve for the maximum of two solutions, with increasing wealth allows for switching transversely from high consumption to low consumption but not vice versa. That is, there can be at most one such wealth w^*. This shows the potential for there to be a wealth $w^* < \overline{w}$ at which the $(V_T)_w$ is not continuous and smooth pasting doesn't hold. See for example see figure 3.3. If there was noise in this problem then it could've forced wealth to move in both directions across w^* forcing the smooth pasting to hold (for details of the results mentioned in this paragraph, please see chapter 4 section 4.2.2: *Boundary condition at* $t = T$).

Now, consider the case when smooth pasting holds at $\overline{w}$. Call that $\overline{w} \equiv w^\circ$. Then, $(\overline{V}_T)_w\Big|_{w=w^\circ} = (V_T)_w\Big|_{w=w^\circ}$. At $w = w^\circ$, the post-retirement and pre-retirement values match; $(\overline{V}_T)\Big|_{w=w^\circ} = (V_T)\Big|_{w=w^\circ}$. Also, under these conditions I am in a position to equate the $t = T$ post-retirement ODE 3.24 and pre-retirement ODE 3.14 at $w = w^\circ$.

$$rw^\circ(\overline{V}_T)_w\Big|_{w=w^\circ} - B\frac{{(\overline{V}_T)_w}^{1-\tilde{\gamma}}}{1-\tilde{\gamma}}\Big|_{w=w^\circ} - (\rho + \lambda_t)\overline{V}_T\Big|_{w=w^\circ} = 0$$
$$(1 + rw^\circ)\frac{dV_T}{dw}\Big|_{w=w^\circ} - \frac{1}{1-\tilde{\gamma}}{(\frac{dV_T}{dw})}^{1-\tilde{\gamma}}\Big|_{w=w^\circ} - (\rho + \lambda_T)V_T\Big|_{w=w^\circ} = 0$$

Subtracting both ODEs result in:

$$(V_T)_w\Big|_{w=w^\circ} - \frac{1}{1-\tilde{\gamma}}(1-B){(V_T)_w}^{1-\tilde{\gamma}}\Big|_{w=w^\circ} = 0 \tag{3.16}$$
$$\implies (V_T)_w\Big|_{w=w^\circ} = \left(\frac{1-\tilde{\gamma}}{1-B}\right)^{-\gamma}$$

I have a closed form solution for post-retirement ODE 3.24 due to the presence of scaling. That is, $\overline{V}_T(w) = f_T\frac{w^{1-\gamma}}{1-\gamma}$, where, f_T is a constant. Then, $(\overline{V}_T)_w\Big|_{w=w^\circ} = (V_T)_w\Big|_{w=w^\circ} = f_T(w^\circ)^{-\gamma}$. Plugging this in equation 3.17 results in

$$f_T(w^\circ)^{-\gamma} = \left(\frac{1-\tilde{\gamma}}{1-B}\right)^{-\gamma} \tag{3.17}$$
$$\implies w^\circ = \left(\frac{1}{f_T}\right)^{-\tilde{\gamma}}\left(\frac{1-\tilde{\gamma}}{1-B}\right)$$

(For details of the results mentioned above, please see chapter 4 section 4.2.2: *Boundary condition at $t = T$*).

Now, the following possibilities exist for the solution to the pre-retirement utility V_T. I will use examples to show that any of these possibilities can occur. Consider the notation $\tilde{V_T}$ (blue) for the pre-retirement solution corresponding to ν_1 and solved over the whole wealth domain $[0, W]$ (practically it is the $\tilde{V}_T^{\pi}$ from Pension income problem in chapter 4 section 4.2.2). V_{T2} (green) is the pre-retirement solution corresponding to larger ν_2. $\overline{V}_T$ (black) is the post-retirement solution. Call $\tilde{w}$ for the wealth level where $\overline{V_T}$ and $\tilde{V_T}$ cross and $\tilde{w} = 0$ if $\overline{V_T} > \tilde{V_T}$ everywhere. w^* is the wealth where $\tilde{V_T}$ and V_{T2} meet. At w°, smooth pasting holds and $\overline{V}_T$ and V_{T2} join smoothly into one curve. The pre-retirement solution V_T is the maximum of $\tilde{V_T}, V_{T2}$ and $\overline{V}_T$ over the wealth domain $[0, W]$. The retirement boundary wealth $\overline{w}$ is maximum of $\tilde{w}, w^\circ$.

- CASE I $w^\circ > \tilde{w}$. In this case, $\overline{w} = w^\circ$ and smooth pasting holds, with V_T solving the pre-retirement HJB using the larger solution V_{T2} to the left of $\overline{w}$. Two things may occur.

  1. That solution may extend all the way to $w = 0$ without crossing $\tilde{V_T}$. In that case we always either retire or save for retirement. See figure 3.2.
  2. That solution crosses $\tilde{V_T}$ at a point $w^* > 0$. In that case, $V_T = \tilde{V_T}$ to the left of w^*. To the left of w^* we give up on retirement and wealth declines, but between w^* and $\overline{w}$ we save for retirement and wealth increases. See figure 3.3.

- CASE II $w^\circ \leq \tilde{w}$ and $\tilde{w} > 0$. In this case $\overline{w} = \tilde{w}$ and $V_T = \tilde{V_T}$ to the left of $\overline{w}$. We retire immediately if $w > \overline{w}$, or we give up on retirement if $w < \overline{w}$. See figure 3.4.

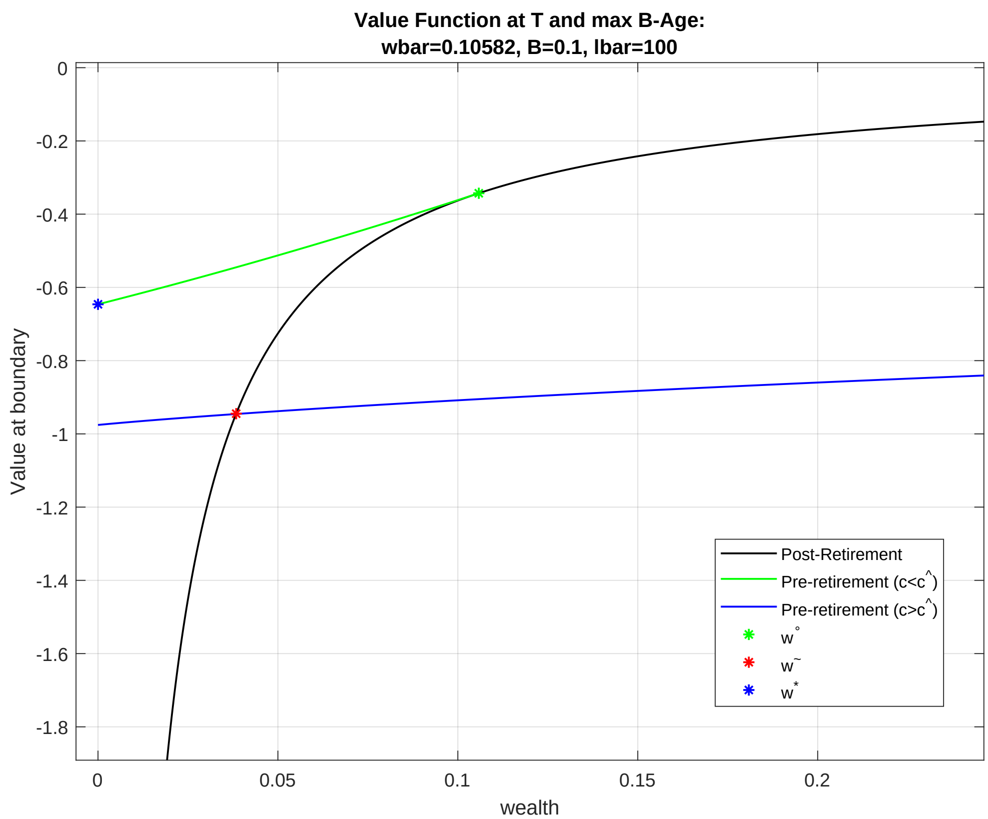


Figure 3.2: Example for Case I-1 for Value function at time T and maximum b-age κ_T. $\gamma = 2, \sigma = 1, r = 2.5\%, \bar{l} = 100$

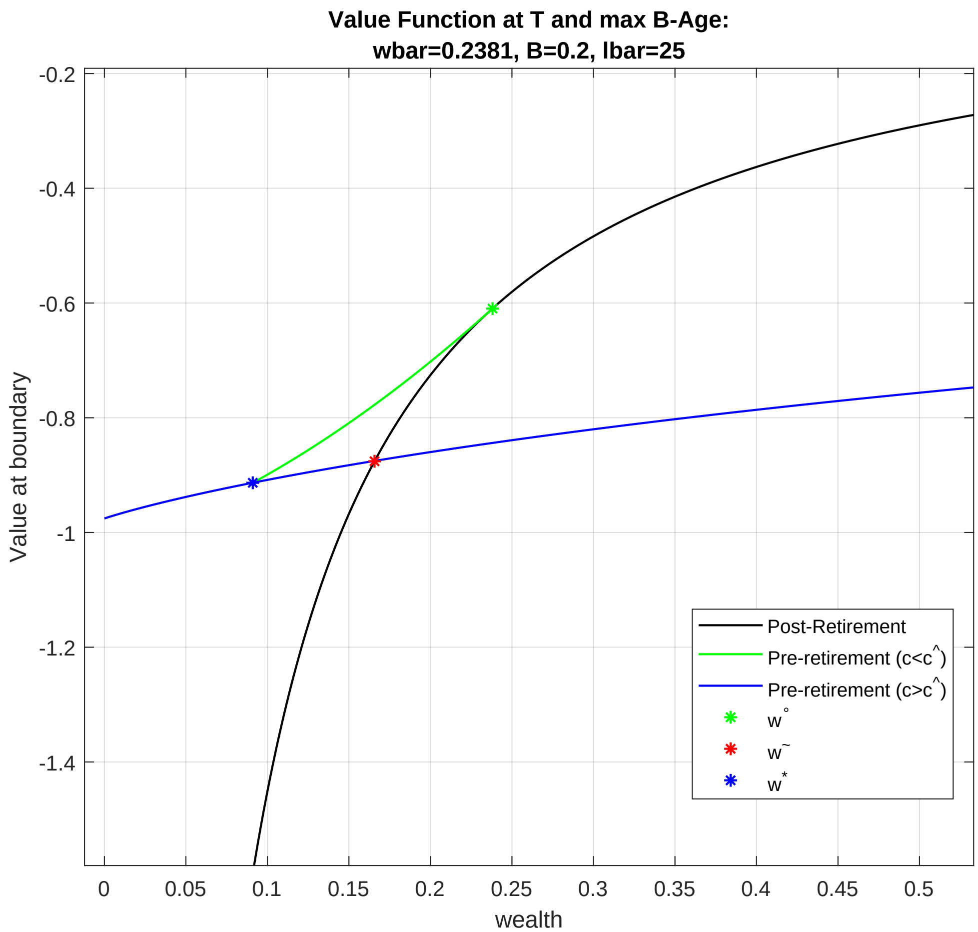


Figure 3.3: Example for Case I-2 for Value function at time T and maximum b-age κ_T. $\gamma = 2, \sigma = 1, r = 2.5\%, \bar{l} = 25$

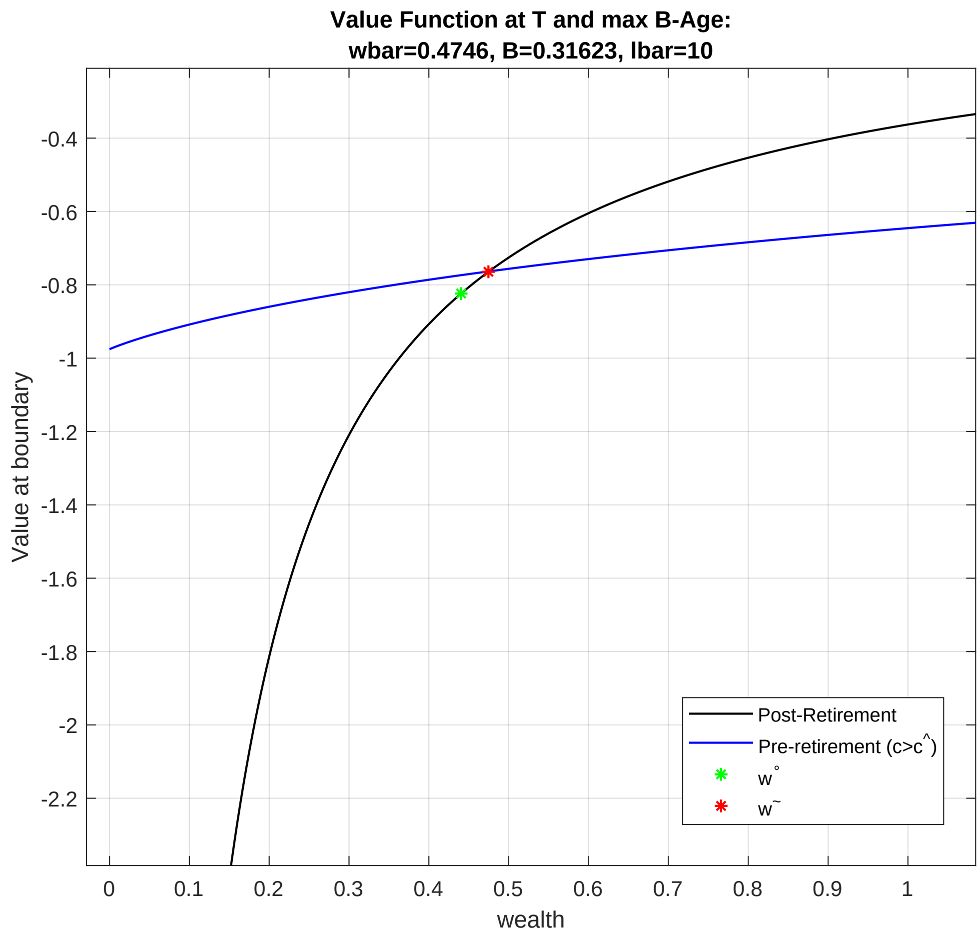


Figure 3.4: Example for Case II for Value function at time T and maximum b-age κ_T. $\gamma = 2, \sigma = 1, r = 2.5\%, \bar{l} = 10$

Therefore, all the boundary conditions used for PDE 3.13 are:

- $V_1(T, w, a) = V_T(w)$, where, $V_T(w)$ is the solution of following ODE for all w.

$$(1 + rw)\frac{dV_T}{dw} - \frac{1}{1-\tilde{\gamma}}(\frac{dV_T}{dw})^{1-\tilde{\gamma}} - (\rho + \lambda_T)V_T = 0, 0 \leq w \leq W$$
$$V_T(W) = \overline{V}(a = \kappa_T, w = W)$$

- $V_1(t, W, a) = \overline{V}(t, W, a), \forall t, a$
- $V_{1a}(t, w, 0) = 0, \forall t, w$
- $V_1(t, w, \kappa_T) = V_T(w), \forall w$

At the free retirement boundary $\tau(t^*, w^*, a^*)$ the following conditions hold which are used to find the boundary:

- Smooth pasting holds for the stochastic variable a. $V_{1a}(t, w, a^*) = \overline{V}_a(t, w, a^*)$.
- Value matching holds. $V_1(t, w, a^*) = \overline{V}(t, w, a^*)$.

3.2.2 POST-RETIREMENT $(t \geq t^*)$

As discussed in chapter 2 section 2.2, for post-retirement, the leisure process $l_t = \bar{l}$, the utility of consumption and leisure is $u(c_t, l_t) = \overline{u}(c_t, \bar{l}) = \bar{l}^{(1-\alpha)(1-\gamma*)}\frac{c^{1-\gamma}}{1-\gamma}, \gamma \neq 1$ and the value $V(t, w, l, a) = \overline{V}(t, w, l, a)$. Z_t the maximum total benefit given all information up till time t, from 3.9, for an optimal choice of consumption c_t and t^*, gives

$$Z_t = \int_0^t e^{-\int_0^s(\rho+\lambda_q)dq}\overline{u}(c_s, l_s)ds + e^{-\int_0^t(\rho+\lambda_q)dq}\overline{V}(t, w_t, l, A_t) \tag{3.18}$$

Recall that in the absence of labor income for post-retirement life, the wealth dynamics are

$$dw_t = (rw_t - c_t)dt \tag{3.19}$$

Applying Ito's lemma to Z_t and using the wealth dynamics and the dynamics of B-age from 3.3 gives

$$dZ_t = e^{-\int_t^s(\rho+\lambda_q)dq}\Big[\overline{u}(c_t, \bar{l})dt - (\rho + \lambda(A_t))\overline{V}dt + \overline{V}_t dt + \overline{V}_w(rw_t - c_t)dt +$$
$$\overline{V}_a[(1 + \xi\frac{\kappa_t - A_t}{T - t})dt + \sigma dB_t] + \frac{1}{2}\overline{V}_{aa}\sigma^2 dt\Big]$$

for $t < T$. Z_t is a martingale for optimal controls c_t. Hence, setting the drift term equal to 0 gives

$$\sup_c \Big[\overline{u}(c, \bar{l}) - (\rho + \lambda(a))\overline{V} + \overline{V}_t + \overline{V}_w(rw - c) +$$

$$\overline{V}_a\left[\left(1+\xi\frac{\kappa_t-a}{T-t}\right)\right]+\frac{1}{2}\overline{V}_{aa}\sigma^2\right]=0 \tag{3.20}$$

for $t<T$. For optimization of c I refer to the convex dual calculated in chapter 2 section 2.2.2, 2.28 and 2.29. The, optimization over c develops as

$$\sup_c\{\overline{u}(c,\bar{l})-c\overline{V}_w\}=\tilde{\overline{u}}(\overline{V}_w,\bar{l})$$

and the optimal choice of c is

$$c^*=\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}\overline{V}_w^{-\tilde{\gamma}}$$

Using the convex dual of c in 3.20 gives the HJB equation for the post-retirement value function:

$$\overline{V}_t+rw\overline{V}_w-\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}\frac{\overline{V}_w^{1-\tilde{\gamma}}}{1-\tilde{\gamma}}-(\rho+\lambda(a))\overline{V}+\left[\left(1+\xi\frac{\kappa_t-a}{T-t}\right)\right]\overline{V}_a+\frac{\sigma^2}{2}\overline{V}_{aa}=0 \tag{3.21}$$

Due to the absence of labor income, there exists a scaling relation for the post-retirement value function $\overline{V}$.

$$\overline{V}(t,kw,\bar{l},a)=k^{1-\gamma}\overline{V}(t,w,\bar{l},a)$$
$$\implies \overline{V}(t,w,\bar{l},a)=f(t,a)\frac{w^{1-\gamma}}{1-\gamma} \tag{3.22}$$

This reduces the dimension of PDE 3.21 and gives a PDE for scaled value function for $t<T$

$$f_t+\left[\left(1+\xi\frac{\kappa_t-a}{T-t}\right)\right]f_a+\frac{\sigma^2}{2}f_{aa}+r(1-\gamma)f-(\rho+\lambda(a))f+\gamma Bf^{1-\tilde{\gamma}}=0 \tag{3.23}$$

where, $B=\bar{l}^{\tilde{\gamma}(1-\alpha)(1-\gamma*)}$. The consumption becomes $c^*(t,w,a)=B(f(t,a)w^{-\gamma})^{-1/\gamma}$. For $B=1$, the post-retirement problem reduces to the Huang et al. paper's model [31].

Boundary Condition at $t=T$

For $t\geqslant T$, there is no aging after time T, hence $\lambda(a)=\lambda_T$ which is the pinned down hazard rate at time T. Since, the consumer is not aging in any sense, the problem is independent of b-age; $\overline{V}_a=0=\overline{V}_{aa}$. Call $\overline{V}(T,w,a)\equiv\overline{V}_T(w)$. Hence HJB equation 3.21 becomes

$$(\overline{V}_T)_t+rw(\overline{V}_T)_w-B\frac{(\overline{V}_T)_w^{1-\tilde{\gamma}}}{1-\tilde{\gamma}}-(\rho+\lambda_T)\overline{V}_T=0 \tag{3.24}$$

Using the scaling relations mentioned above in 3.22 I get a scaled version of HJB equation 3.24

$$f_t+r(1-\gamma)f-(\rho+\lambda_T)f+\gamma Bf^{1-\tilde{\gamma}}=0$$

Sine the hazard rate is staying constant at λ_T for $t \geqslant T$, the time doesn't have any impact on person's age hence, the problem is time invariant, $f_t = 0$ and call $f(t,a) \equiv f_T; t \geq T$. This assumption gives

$$r(1-\gamma)f - (\rho + \lambda_T)f + \gamma B f^{1-\tilde{\gamma}} = 0$$

That means there exists an explicit solution to the ODE 3.24

$$\begin{aligned} f_T &= \left(\frac{\rho + \lambda_T - r(1-\gamma)}{\gamma B}\right)^{-\gamma} \\ \overline{V}_T(w) &= f_T \frac{w^{1-\gamma}}{1-\gamma}; \forall w \end{aligned} \tag{3.25}$$

Here f_T is a constant. Therefore at $t = T$ I am using the boundary condition $f(T,a) = f_T$ and $\overline{V}(T,w,a) \equiv \overline{V}_T(w)$.

Boundary Conditions for b-age

The Huang et al. paper [31] considered an infinite domain for the b-age $-\infty < a < \infty$. A very large hazard rate at infinite b-age will kill the retiree hence dropping the value to 0; $f(t,\infty) = 0$. Similar to the pre-retirement problem, for extremely small a (extremely good health), the hazard rate is 0; $\lambda(a) = 0$ and the problem becomes independent of the person's b-age, that is, $f_a = 0 = f_{aa}$. Then from 3.23 the boundary condition $f(t,-\infty)$ is the solution of following ordinary differential equation:

$$\begin{aligned} &f_t + r(1-\gamma)f - (\rho + \lambda_T)f + \gamma B f^{1-\tilde{\gamma}} = 0 \\ &f(T) = f_T \end{aligned} \tag{3.26}$$

New set of Boundary Conditions for b-age:
Changing the domain of A_t from $-\infty < a < +\infty$ to more realistic numbers $0 \leq a \leq \kappa_T$. At $a = 0$, I applied homogeneous Neumann condition, that is, whenever $f(t,a)$ hits this boundary it reflects back; $f_a(t,0) = 0$. For the maximum possible age $a = \kappa_T$, I assumed in the beginning of my model that a person will stop aging after this age, therefore, the mortality rate will stay constant at λ_T resulting in $f_a = 0 = f_{aa}$. Also, due to the absence of aging process the problem becomes time-invariant; $f_t = 0$. Hence, a suitable condition on this boundary is $f(t,\kappa_T) = f_T$ where f_T is given by 3.25.

Therefore, the boundary conditions I am using for PDE 3.23 are:

- $f(T,a) = f_T, \forall a$
- $f_a(t,0) = 0, \forall t$
- $f(t,\kappa_T) = f_T, \forall t$. Here f_T is given by 3.25

3.3 Numerical Scheme

3.3.1 Numerical Scheme: Post-Retirement PDE

I'll begin with numerically solving the PDE 3.23 in order to solve the scaled post-retirement value function $f(t,a)$. Consider the numerical grid $(t_i, a_k); 0 \leq t_i \leq T, A1 \leq a_k \leq A2$. Let dt and da be the time step-size and bio-age step-size, respectively. $N+1$ and $K+1$ be the total number of time nodes and b-age nodes, respectively. Then $t_i = idt, i = N, N-1, ..., 0$ and $a_k = A1 + kda, k = 0, 1, ..., K$, where $t_0 = 0, t_n = T, a_0 = A1$ and $a_K = A2$. Denote $f(t_i, a_k) = f_k^i$. I used an implicit finite difference scheme to solve the post-retirement PDE 3.23. Since the problem has to be solved backwards in time, a change of variable is applied to the time variable t. Replace old time t_{old} with new time variable $t_{new} = T - t_{old}$. Call,

$$\theta = -\left[1 + \xi\left(\frac{\kappa_0 + t_{new} - a}{T - t_{new}}\right)\right]$$

Dropping the subscript from t_{new}, the new time at i^{th} grid point is $t_i, i = 0, 1, 2, ..., N$. Therefore, at the grid point (t_i, a_k)

$$\theta_k^i = -\left[1 + \xi\left(\frac{\kappa_0 + t_i - a_k}{T - t_i + \epsilon}\right)\right] \tag{3.27}$$

where ϵ is added to the denominator in order to avoid singularity arising at $t_i = T$. The chain rule replaces f_t in old time with $-f_t$ in the new time. Using forward difference for time derivative, upwind finite difference for f_a and symmetric difference for f_{aa},

$$\begin{aligned}
-f_t(t_i, a_k) &= -\frac{f_k^{i+1} - f_k^i}{dt} \\
f_a(t_{i+1}, a_k) &= \begin{cases} \frac{f_k^{i+1} - f_{k-1}^{i+1}}{da} & ,\theta_k^i \geq 0 \\ \frac{f_{k+1}^{i+1} - f_k^{i+1}}{da} & ,\theta_k^i < 0 \end{cases} \\
f_{aa}(t_{i+1}, a_k) &= \frac{f_{k+1}^{i+1} - 2f_k^{i+1} + f_{k-1}^{i+1}}{da^2}
\end{aligned}$$

gives the implicit scheme

$$\begin{aligned}
&-\frac{f_k^{i+1} - f_k^i}{dt} - \frac{\theta_k^{i+1} + |\theta_k^{i+1}|}{2}\frac{f_k^{i+1} - f_{k-1}^{i+1}}{da} - \frac{\theta_k^{i+1} - |\theta_k^{i+1}|}{2}\frac{f_{k+1}^{i+1} - f_k^{i+1}}{da} \\
&+ \frac{\sigma^2}{2}\frac{f_{k+1}^{i+1} - 2f_k^{i+1} + f_{k-1}^{i+1}}{da} + [r(1-\gamma) - (\rho + \lambda(a_k)) + \gamma B(f_k^i)^{-\tilde{\gamma}}]f_k^{i+1} = 0
\end{aligned} \tag{3.28}$$

for $i = 1, ..., N$ and $k = 1, ..., K-1$. The initial condition is $f_k^0 = f_T$. At the B-age boundary $a_0 = A1$; $f_a = 0$. The forward finite difference at $a_0 \implies \frac{f_1^i - f_0^i}{da} = 0$ gives $f_0^i = f_1^i$ for all i, while at the other boundary $a_K = A2$; $f_K^i = f_T$.

At each time step t_i; $f_k^{i+1}, f_{k-1}^{i+1}, f_{k+1}^{i+1}$ are unknowns. Carefully incorporating the boundary conditions into the pde 3.28 gives a tri-diagonal matrix system at each time step. I have used an iterative scheme to solve this tri-diagonal system.
For each time step t_i and t_{i+1}, call $f_k^{i+1}; \forall k$ a vector f and the known values $f_k^i; \forall k$ as a vector f_{old}. Collecting like terms in pde 3.28 give the coefficients of $f(k+1), f(k-1), f(k)$ as:

$$\begin{aligned}
\eta_1(k) &= \frac{D}{da^2} - \frac{\theta_k^{i+1} - |\theta_k^{i+1}|}{2da} \\
\eta_3(k) &= \frac{D}{da^2} + \frac{\theta_k^{i+1} + |\theta_k^{i+1}|}{2da} \\
\eta_2(k) &= \eta_1(k) + \eta_3(k) - r(1-\gamma) + (\rho + \lambda(a_k)) - \gamma B(fold(k))^{-\tilde{\gamma}} + \frac{1}{dt}
\end{aligned}$$

respectively. Here $D = \frac{\sigma^2}{2}$. Then pde 3.28 takes the form

$$\eta_1(k)f(k+1) + \eta_3(k)f(k-1) + \frac{fold(k)}{dt} = \eta_2(k)f(k)$$

This equation is converted to an iterative scheme for f beginning at $k = K-1$ with boundary condition at $f(K) = f_T$ and going backwards to $k = 0$.

$$\begin{aligned}
&f(K) = f_T \\
&f(k) = q(k)f(k+1) + p(k); k = K-1, K-2, ..., 0
\end{aligned} \tag{3.29}$$

The coefficients p and q are found iteratively for $k = 1, 2, ...K$ as follows:

$$\begin{aligned}
q(0) &= 1, p(0) = 0 \\
q(k) &= \frac{\eta_1(k)}{\eta_2(k) - \eta_3(k)q(k-1)} \\
p(k) &= \frac{\frac{fold(k)}{dt} + \eta_3(k)p(k-1)}{\eta_2(k) - \eta_3(k)q(k-1)}
\end{aligned} \tag{3.30}$$

$q(0) = 1, p(0) = 0$ ensures the boundary condition at $k = 0; f(1) = f(0)$. Beginning at time step $t_0 = 0$ with the boundary condition $f_k^0 = f_T, \forall k$, at each time step the unknown vector f is determined using the above mentioned iterative scheme. After finding $f(t_i, a_k), \forall k, i$ the wealth scaling given by 3.22 is used to get the post-retirement value function. That is, for any value of wealth in the domain $0 \leq w \leq W, \overline{V}(t_i, w, a_k) = f(t_i, a_k)\frac{w^{1-\gamma}}{1-\gamma}$. The wealth numerical grid will be defined in the pre-retirement numerical scheme.

3.3.2 Numerical Scheme: Pre-Retirement PDE

3.3.2.1 Mixed Explicit Implicit Scheme

As discussed in chapter 2 section 2.3, for small wealth levels $0 \leq w \leq W$, the solution behaviors change rapidly. In order to capture these behaviors I am using a log-wealth grid.

Call $y = lnw$ which gives $w = e^y$ and $V_{1w} = e^{-y}V_{1y}; 0 \leq w \leq 1$. Hence the pre-retirement pde for $0 \leq w < 1, \forall t, a$ becomes:

$$V_{1t} + \left[\left(1 + \xi\frac{\kappa_t - a}{T - t}\right)\right]V_{1a} + \frac{\sigma^2}{2}V_{1aa} + \left[e^{-y} + r - \left(\frac{e^{-y(1-\tilde{\gamma})}V_{1y}^{-\tilde{\gamma}}}{1 - \tilde{\gamma}}\right)\right]V_{1y} - (\rho + \lambda(a))V_1 = 0 \tag{3.31}$$

For $1 \leq w \leq W, \forall t, a$, consider the pre-retirement PDE 3.13 in the following form

$$V_{1t} + \left[1 + \xi\left(\frac{\kappa_t - a}{T - t}\right)\right]V_{1a} + \frac{\sigma^2}{2}V_{1aa} + \left(1 + rw - \frac{V_{1w}{}^{-\tilde{\gamma}}}{1 - \tilde{\gamma}}\right)V_{1w} - (\rho + \lambda(a))V_1 = 0 \tag{3.32}$$

As before, apply a change of time variable $t_{new} = T - t_{old}$ and drop the subscript from t_{new}. I am considering a finite difference numerical scheme which is explicit in the wealth variable but implicit in the b-age variable. Consider a numerical grid $(t_i, w_J, a_k); 0 \leq t_i \leq T, 1 \leq w_J \leq W, A1 \leq a_k \leq A2$ and a non-uniform grid $(t_i, y_j, a_k); 0 \leq t_i \leq T, 0 \leq e^{y_j} < 1, A1 \leq a_k \leq A2$. Let dt, dw, dy and da be the time step-size, wealth step-size, log-wealth step-size and bio-age step-size, respectively. $N+1, M+1, M_y+1$ and $K+1$ be the total number of time nodes, wealth nodes, log-wealth nodes and b-age nodes, respectively. Then the i^{th} time node is $t_i = idt; i = 0, 1, 2..., N$, the J^{th} wealth node is $w_J = Jdw; J =, 1, ..., M$, the j^{th} log-wealth node is $y_j = Y1 + jdy; j =, 1, ..., M_y$ and the k^{th} biological age node is $a_k = A1 + kda; k = 0, 1, ..., K$. $Y1$ is a large negative number used to cater to the need of $-\infty$ for log-wealth grid at $w = 0$. Here $t_0 = 0, t_N = T, w_0 = 1, w_M = W, y_0 = Y1, y_{M_y} = 0, a_0 = A1$ and $a_K = A2$. Let the pre-retirement value function at discretized grid points be $V_1(t_i, w_J, a_k) \equiv v^i_{J,k}; \forall i, J, k$ and $V_1(t_i, y_j, a_k) \equiv v^i_{j,k}; \forall i, j, k$.

My aim is to find the optimal boundary $\tau(t^*, w^*, a^*)$ for retirement time, wealth and biological age. I'll begin by solving the pre-retirement pde numerically using the mixed explicit-implicit finite difference numerical scheme. Discretize the wealth derivative parts of PDE 3.32 at the grid point (t_i, w_j, a_k) where the value function is known while the b-age derivative parts of this PDE are discretized at the unknown value function grid point (t_{i+1}, w_j, a_k). For time derivative, a forward difference is used at(t_i, w_j, a_k) and the 0^{th} order term is evaluated at (t_{i+1}, w_j, a_k). From previous section, θ^i_k as given in equation 3.27 is

$$\theta^i_k \equiv -\left[1 + \xi\left(\frac{\kappa_0 + t_i - a_k}{T - t_i + \epsilon}\right)\right]$$

Call

$$\beta^i_{j,k} \equiv e^{-y} + r - \left(\frac{e^{-y(1-\tilde{\gamma})}[V_{1y}(t_i, w_j, a_k)]^{-\tilde{\gamma}}}{1 - \tilde{\gamma}}\right), 0 \leq e^{y_j} < 1$$

$$\beta^i_{J,k} \equiv 1 + rw_j - \frac{[V_{1w}(t_i, w_J, a_k)]^{-\tilde{\gamma}}}{1-\tilde{\gamma}}, 1 \leq w_J \leq W$$

V_{1w} and V_{1y} inside β are evaluated using forward finite difference at current time step in order to get rid of the non-linearity of wealth derivative of the value function.

$$V_{1w}(t_i, w_J, a_k) = \frac{v^i_{J+1,k} - v^i_{J,k}}{dw}$$

$$V_{1y}(t_i, y_j, a_k) = \frac{v^i_{j+1,k} - v^i_{j,k}}{dy}$$

The time derivative is given as

$$-V_{1t}(t_i, w_j, a_k) = -\frac{v^{i+1}_{j,k} - v^i_{j,k}}{dt}$$

Using $\beta^i_{J,k}$ as the velocity of the propagating wave, the first order up-winding finite difference scheme for the wealth derivative at explicit time nodes is:

$$V_{1w}(t_i, w_J, a_k) = \begin{cases} \frac{v^i_{J+1,k} - v^i_{J,k}}{dw} & , \beta^i_{J,k} \geqslant 0 \\ \frac{v^i_{J,k} - v^i_{J-1,k}}{dw} & , \beta^i_{J,k} < 0 \end{cases}$$

Also, Using $\beta^i_{j,k}$ as the velocity of the propagating wave, the first order up-winding finite difference scheme for the log-wealth derivative at explicit time nodes is:

$$V_{1y}(t_i, y_j, a_k) = \begin{cases} \frac{v^i_{j+1,k} - v^i_{j,k}}{dy} & , \beta^i_{j,k} \geqslant 0 \\ \frac{v^i_{j,k} - v^i_{j-1,k}}{dw} & , \beta^i_{j,k} < 0 \end{cases}$$

The symmetric difference and up-winding is used for b-age derivatives at implicit time nodes for wealth nodes and log-wealth nodes i.e.,

$$V_a(t_{i+1}, w_J, a_k) = \begin{cases} \frac{v^{i+1}_{J,k} - v^{i+1}_{J,k-1}}{da} & , \theta^i_k \geq 0 \\ \frac{v^{i+1}_{J,k+1} - v^{i+1}_{J,k}}{da} & , \theta^i_k < 0 \end{cases}$$

$$V_{aa}(t_{i+1}, w_J, a_k) = \frac{v^{i+1}_{J,k+1} - 2v^{i+1}_{J,k} + v^{i+1}_{J,k-1}}{da^2}$$

$$V_a(t_{i+1}, y_j, a_k) = \begin{cases} \frac{v^{i+1}_{j,k} - v^{i+1}_{j,k-1}}{da} & , \theta^i_k \geq 0 \\ \frac{v^{i+1}_{j,k+1} - v^{i+1}_{j,k}}{da} & , \theta^i_k < 0 \end{cases}$$

$$V_{aa}(t_{i+1}, y_j, a_k) = \frac{v^{i+1}_{j,k+1} - 2v^{i+1}_{j,k} + v^{i+1}_{j,k-1}}{da^2}$$

Using these finite differences to discretize the PDE 3.31 and 3.32 gives a mixed explicit implicit scheme:

$$
\begin{aligned}
&-\frac{v^{i+1}_{J,k}-v^{i}_{J,k}}{dt}-\frac{\theta^{i+1}_{k}+|\theta^{i+1}_{k}|}{2}\frac{v^{i+1}_{J,k}-v^{i+1}_{J,k-1}}{da}-\frac{\theta^{i+1}_{k}-|\theta^{i+1}_{k}|}{2}\frac{v^{i+1}_{J,k+1}-v^{i+1}_{J,k}}{da}+\\
&\frac{\sigma^2}{2}\frac{v^{i+1}_{J,k+1}-2v^{i+1}_{J,k}+v^{i+1}_{J,k-1}}{da^2}-(\rho+\lambda(a_k))v^{i+1}_{J,k}+\\
&\frac{\beta^{i}_{J,k}+|\beta^{i}_{j,k}|}{2}\frac{v^{i}_{J+1,k}-v^{i}_{J,k}}{dy}+\frac{\beta^{i}_{J,k}-|\beta^{i}_{J,k}|}{2}\frac{v^{i}_{J,k}-v^{i}_{J-1,k}}{dy}=0
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
&-\frac{v^{i+1}_{j,k}-v^{i}_{j,k}}{dt}-\frac{\theta^{i+1}_{k}+|\theta^{i+1}_{k}|}{2}\frac{v^{i+1}_{j,k}-v^{i+1}_{j,k-1}}{da}-\frac{\theta^{i+1}_{k}-|\theta^{i+1}_{k}|}{2}\frac{v^{i+1}_{j,k+1}-v^{i+1}_{j,k}}{da}+\\
&\frac{\sigma^2}{2}\frac{v^{i+1}_{j,k+1}-2v^{i+1}_{j,k}+v^{i+1}_{j,k-1}}{da^2}-(\rho+\lambda(a_k))v^{i+1}_{j,k}+\\
&\frac{\beta^{i}_{j,k}+|\beta^{i}_{j,k}|}{2}\frac{v^{i}_{j+1,k}-v^{i}_{j,k}}{dw}+\frac{\beta^{i}_{j,k}-|\beta^{i}_{j,k}|}{2}\frac{v^{i}_{j,k}-v^{i}_{j-1,k}}{dw}=0
\end{aligned}
\tag{3.34}
$$

Group together like terms for the implicit time nodes $v^{i+1}_{J,k+1}, v^{i+1}_{J,k-1}, v^{i+1}_{J,k}$ in 3.33 and the terms for $v^{i+1}_{j,k+1}, v^{i+1}_{j,k-1}, v^{i+1}_{j,k}$ in 3.34. For these implicit time nodes, the coefficients of above mentioned terms are independent of wealth/log-wealth terms. Hence, the coefficients of $v^{i+1}_{\vartheta,k+1}, v^{i+1}_{\vartheta,k-1}, v^{i+1}_{\vartheta,k}$ with $\vartheta = j, J; \forall j, J$ respectively are as under:

$$
\begin{aligned}
\eta_1(k) &= \frac{D}{da^2}-\frac{\theta^{i+1}_{k}-|\theta^{i+1}_{k}|}{2da}\\
\eta_3(k) &= \frac{D}{da^2}+\frac{\theta^{i+1}_{k}+|\theta^{i+1}_{k}|}{2da}\\
\eta_2(k) &= \eta_1(k)+\eta_3(k)+(\rho+\lambda(a_k))+\frac{1}{dt}
\end{aligned}
$$

respectively. Here $D = \frac{\sigma^2}{2}$. Furthermore, collect all the t_i node terms in 3.33 and 3.34 and call them $g(k)$. Then, for wealth grid

$$
g(k) = \frac{\beta^{i}_{J,k}+|\beta^{i}_{J,k}|}{2}\frac{v^{i}_{J+1,k}-v^{i}_{J,k}}{dw}+\frac{\beta^{i}_{j,k}-|\beta^{i}_{J,k}|}{2}\frac{v^{i}_{J,k}-v^{i}_{J-1,k}}{dw}+\frac{v^{i}_{J,k}}{dt}
$$

and for the log-wealth grid

$$
g(k) = \frac{\beta^{i}_{j,k}+|\beta^{i}_{j,k}|}{2}\frac{v^{i}_{j+1,k}-v^{i}_{j,k}}{dy}+\frac{\beta^{i}_{j,k}-|\beta^{i}_{j,k}|}{2}\frac{v^{i}_{j,k}-v^{i}_{j-1,k}}{dy}+\frac{v^{i}_{j,k}}{dt}
$$

For every time step and wealth/log-wealth step, let $v^{i+1}_{\vartheta,k} \equiv V, v^{i}_{\vartheta,k} \equiv V_{old}, v^{i-1}_{\vartheta,k} \equiv V_{old1}$ with $\vartheta = j, J; \forall j, J; \forall k$ are vectors in k. Using all these notations equation 3.33and 3.34 are reduced to similar looking tri-diagonal systems.

$$
\eta_1(k)V(k+1)+\eta_3(k)V(k-1)+g(k)=\eta_2(k)V(k)
$$

This equation is converted to an iterative equation for vector V,

$$V(k) = q(k)V(k+1) + p(k) \tag{3.35}$$

The coefficients p and q are found recursively for $k = 1, 2, ..., K$. It is assumed that $q(0) = 1, p(0) = 0$ to ensure the boundary condition at $k = 0; V(1) = V(0)$

$$q(k) = \frac{\eta_1(k)}{\eta_2(k) - \eta_3(k)q(k-1)}$$
$$p(k) = \frac{g(k) + \eta_3(k)p(k-1)}{\eta_2(k) - \eta_3(k)q(k-1)}$$

Now I am in the position to evaluate the pre-retirement value function by using iterative scheme 3.35. But this involves finding the retirement boundary to see in what time, wealth/log-wealth and b-age region I can use these recursive equations to find pre-retirement value-function. For this purpose I'll use the fact that due to the presence of noise in b-age, value matching and smooth pasting will hold at the retirement boundary. For each time and wealth step or log-wealth step, an initial guess is made for the optimal retirement b-age, $a^*_{guess} = a_{k^*}$. At and after this age it is assumed that person has retired and $V_1(k) = \overline{V}(k); k = k^*, k^*+1, ..., K$. Then left hand (LH) b-age derivative as well as the right hand (RH) b-age derivative at this b-age node is used to determine the optimal a^*. If the LH derivative=RH derivative then smooth pasting holds and the current b-age node is marked as a^* for the current time step as well as the current wealth/log-wealth step. Otherwise, if the RH derivative>LH derivative then the slope of the value function curve is more steeper on the right hand side signaling the guess for retirement at this b-age node is not optimal and I need to move my guess one step forward, that is, take $k^* = k + 1$. Note that at the guessed b-age node, RH derivative is for post-retirement value and the LH derivative is the pre-retirement derivative. Also, at each time step, I begin with $a^*_{guess} = a_0$ for every wealth/log-wealth node. After finding the optimal a^* for this time node for all wealths, I move onto the next time step and use the a_* from previous time step as my initial guess. At every time step t_i and for the wealth grid node w_J I use the following criterion to determine the optimal b-age a^*

$$\begin{cases} V_{1a}^{+}(t_{i+1}, w_J, a_{k^*}) = V_{1a}^{-}(t_{i+1}, w_J, a_{k^*}) & , a_{k^*} \text{ is optimal b-age} \\ V_{1a}^{+}(t_{i+1}, w_J, a_{k^*}) > V_{1a}^{-}(t_{i+1}, w_J, a_{k^*}) & , k^* = k^* + 1 \end{cases} \tag{3.36}$$

where, $V_{1a}^{+}(t_{i+1}, w_J, a_{k^*}) = \frac{v^{i+1}_{J,k^*+1} - v^{i+1}_{J,k^*}}{da}$ is the right-hand derivative and $V_{1a}^{-}(t_{i+1}, w_J, a_{k^*}) = \frac{v^{i+1}_{J,k^*} - v^{i+1}_{J,k^*-1}}{da}$ is the left-hand derivative of V_1 with respect to b-age. And for the same time node t_i but for the log-wealth grid node y_j I use the following equations/inequality to determine the optimal b-age a^*

$$\begin{cases} V_{1a}^{+}(t_{i+1}, y_j, a_{k^*}) = V_{1a}^{-}(t_{i+1}, y_j, a_{k^*}) & , a_{k^*} \text{ is optimal b-age} \\ V_{1a}^{+}(t_{i+1}, y_j, a_{k^*}) > V_{1a}^{-}(t_{i+1}, y_j, a_{k^*}) & , k^* = k^* + 1 \end{cases} \tag{3.37}$$

where, $V_{1a}^{+}(t_{i+1}, y_j, a_{k^*}) = \frac{v_{j,k^*+1}^{i+1} - v_{j,k^*}^{i+1}}{da}$ is the right hand derivative and $V_{1a}^{-}(t_{i+1}, y_j, a_{k^*}) = \frac{v_{j,k^*}^{i+1} - v_{J,k^*-1}^{i+1}}{da}$ is the left hand derivative of V_1 with respect to b-age.

The life time optimal consumption $c*$ for all wealth levels $Y1 \leq y_j < 0$, $1 \leq w_J \leq W$, b-ages $0 \leq a_k \leq \kappa_T$ and all time nodes $0 \leq t_i \leq T$ is calculated as follows:

$$
\begin{aligned}
c(t_i, y_j, a_k) &= \begin{cases} B(e^{-y_j} \frac{v_{j+1,k}^{i+1} - v_{j,k}^{i+1}}{dy})^{-\tilde{\gamma}} & , t_i \geq t^* \\ (e^{-y_j} \frac{v_{j+1,k}^{i+1} - v_{j,k}^{i+1}}{dy})^{-\tilde{\gamma}} & , t_i < t^* \end{cases} \\
c(t_i, w_J, a_k) &= \begin{cases} B(\frac{v_{J+1,k}^{i+1} - v_{J,k}^{i+1}}{dw})^{-\tilde{\gamma}} & , t_i \geq t^* \\ (\frac{v_{J+1,k}^{i+1} - v_{J.k}^{i+1}}{dw})^{-\tilde{\gamma}} & , t_i < t^* \end{cases}
\end{aligned} \tag{3.38}
$$

It is saved as a 3-dimensional matrix in Matlab.

3.4 Wealth Dynamics

It is important to see under optimal consumption and retirement strategy, how does the wealth of a consumer evolves over time. I am considering a 30 years old consumer who has a certain amount of wealth w_0 in hand. He is willing to use my optimal strategy and willing to spend his wealth and retire according to this strategy. If he doesn't have large enough wealth to retire right now, he is in a labor force earning \$1 annually. As mentioned before, he is allowed to invest only at a risk free rate r. Borrowing is not allowed and once he retires, he can not go back to work. There is no pension stream and he doesn't have any bequest motives. Now, consider the post-retirement and pre-retirement wealth dynamics:

$$
\begin{aligned}
\frac{dw}{dt} &= 1 + rw(t) - c(t, w, a); t < \tau \\
\frac{dw}{dt} &= rw(t) - c(t, w, a); t \geq \tau
\end{aligned}
$$

Or,

$$
\begin{aligned}
\frac{dw}{dt} &= 1 + rw(t) - c(t, w, a); t < \tau \\
\frac{dw}{dt} &= rw(t) - B(f(t, a)w(t)^{-\gamma})^{-1/\gamma}; t \geq \tau
\end{aligned}
$$

Note that, unlike the deterministic mortality case, for stochastic mortality case consumption is a function of 3 variables time, wealth and b-age. That means for every time, I need to have a corresponding wealth and b-age at that time to find the consumption from the saved consumption matrix in Matlab. For the b-age part I need to simulate the b-age dynamics 3.3 to have a b-age path for the consumer over his whole life time.

B-age Simulation: The b-age dynamics from equation 3.3 are

$$dA_t = \left(1 + \xi \frac{\kappa_t - A_t}{T - t}\right) dt + \sigma dB_t$$

with A_0 being the b-age at $t = 0$. I am also assuming that once b-age reaches the maximum b-age κ_T it must stay there for all future times. I am using the Euler-Maruyama Method to discretize this stochastic differential equation (SDE) for b-age. Consider a partition of the the time interval $[0, T]$ into N sub-intervals of equal length dt. Then for the index i, $t_i = idt; i = 0, 1, 2, ..., N$, where $t_0 = 0$ and $t_N = T$. Then, for the intervals $[t_i, t_{i+1}]; i = 0, 1, ..., N$ use a first order approximation of $dA_t = \frac{A_{i+1} - A_i}{da}$, left-end point approximation for the b-age A_{t_i} and for the Wiener process $dB_t \approx \Delta B_{t_i} = B_{t_{i+1}} - B_{t_i}$. Here, ΔB_{t_i} are independent and identically distributed normal random variables with mean 0 and variance dt. Therefore, the b-age dynamics take the form

$$\frac{A_{i+1} - A_i}{da} = \left(1 + \xi \frac{\kappa_t - A_{t_i}}{T - t_i}\right) dt + \sigma \Delta B_{t_i}; 0 = t_0 \leq t_i \leq t_N = T$$

Let's suppose that at t_0, the consumer with c-age 30 has a b-age A_0. Then with the initial condition $A(t = 0) = A_0$, the following iterative scheme is used to simulate the consumers b-age over the course of his life time.

$$A_{i+1} = A_i + da\left(1 + \xi \frac{\kappa_t - A_i}{T - t_i}\right) dt + \sigma \sqrt{dt} Z(0, 1) \tag{3.39}$$

for $0 = t_0 \leq t_i \leq t_N = T$. Here, $Z(0, 1)$ is standard normal random variable. In Matlab, I used the in-built function *normrnd* to generate the standard normal random variables $Z(0, 1)$ [1].

Now, using the grids discussed in the beginning of this subsection: time grid $0 \leq t_i \leq T$, wealth grid $0 \leq e^{y_j} < 1, 1 \leq w_j \leq W$ and the b-age grid $0 \leq a_k \leq \kappa_T$, discretize the wealth ODEs using Euler Method. I want to emphasis that wealth and b-age are functions of time, therefore, I'll use the index i for t, w and a to show that at every t_i, $w(t_i)$ is calculated from the wealth dynamics and $a(t_i)$ is extracted from the simulated series of b-age dynamics 3.39 for $i = 0, 1, 2, ..., N$.

$$\frac{w_{i+1} - w_i}{dt} = 1 + rw_i - c(t_i, w_i, a_i); t_i < \tau$$
$$\frac{w_{i+1} - w_i}{dt} = rw(t_i) - B(f(t_i, a_i)w(t_i)^{-\gamma})^{-1/\gamma}; t_i \geq \tau$$

where $0 \leq t_i \leq T$ and $w_i = w(t_i)$. Simplification gives

$$w_{i+1} = dt\left[1 + \left(r + \frac{1}{dt}\right) w_i - c(t_i, w_i, a_i)\right]; t_i < \tau$$

$$w_{i+1} = w_i dt\left[r + \frac{1}{dt} - Bf(t_i, a_i)^{-1/\gamma}\right]; t_i \geq \tau \tag{3.40}$$

with $w(t_0) = w_0$.

Beginning at t_0 solve wealth dynamics 3.39 forwards in time. From the retirement boundary $\tau(t, w, a)$ determine if each point (t_i, w_i, a_i) falls in the post-retirement region or the pre-retirement region. Then solve the appropriate wealth dynamics iterative scheme from 3.39. Since function $f(t, a)$ and consumption $c(t, w, a)$ are saved as matrices on the grid defined above, $w(t_i)$ from equations 3.39 and b-age simulation $a(t_i)$ do not necessarily fall at those pre-defined grid points. But, the grid is same. So, I use 1-dimensional linear interpolation to extract the numbers for each t_i at the point (t_i, a_i) and 2-dimensional linear interpolation to get consumption numbers from the $c(t, w, a)$ matrix at the points (t_i, w_i, a_i). The interpolation is done using Matlab in-built functions *interp1* and *interp2* [1].

3.5 Algorithm

The algorithm for solving the stochastic mortality problem is as under.:

1. Form the numerical grid mentioned in section 3.3.
2. Solve the post-retirement PDE using the numerical scheme 3.29 and get the scaled value function $f(t, a)$.
3. In order to solve the pre-retirement problem, I need to find the retirement boundary and simultaneously solve for the pre-retirement PDE.
4. Since I am using a flipped time domain, t_0 is the first grid point referring to the time T. Solve the boundary ODE at $t_0 = T$ given by 3.14.
5. Beginning at t_1, make a guess for retirement b-age to be $a^* = a_0$ for every wealth and log-wealth grid point.
6. For this t_i begin at $w_{M_w} = W$.
7. Assume at and after this retirement b-age , $V_1(t_i, w_J, a_k) = \overline{V}(t_i, w_J, a_k)$, $V_1(t_i, y_j, a_k) = \overline{V}(t_i, y_j, a_k)$ for $k = k^*, k^* + 1, ...K + 1; \forall i, J, j$. Solve equation 3.35 recursively for $V(k)$ going backwards for $k = k^* - 1, ..., 0$ using the boundary conditions $V(k^*) = \overline{V}(t_i, e^{y_j}, k^*)$.
8. Check for the smooth pasting conditions given in 3.36. If the smooth pasting condition is fulfilled mark this k as k^* for current t_i and y_j. Otherwise, improve the guess by moving $k^*_{guess} = k^*_{guess} + 1$.
9. Keep repeating steps 7 and 8 until an optimal k^* is found. Mark that k^* for retirement boundary.
10. For this same time t_i, move down $J = M_w - 1$. Repeat steps 7-9. Keep moving down in wealth until you reach the first log-wealth grid point $y_0 = Y1$. Repeat steps 7-9 to find the optimal b-ages for every wealth and log-wealth grid point.

11. Move onto the next time step. Use a^*_{guess} equal to the optimal a^* found at previous time step for every wealth and log-wealth step.

12. Repeat the steps 6-11 until you reach the last time step t_N=0.

13. Now the matrix $V(t_i, w_j, a_k)$ represents the value of the stochastic mortality problem. In this matrix pre-retirement and post-retirement are segregated through the retirement boundary matrix $\tau(t, w, a)$.

14. For every node (t_i, w_J, a_k) and (t_i, y_j, a_k), calculate the optimal consumption c^* using 3.38.

15. Consider a 30 year old consumer willing to use my optimal retirement and consumption strategy. Say his b-age at time 0 is A_0 and he has wealth w_0 in hand. Simulate the life time b-age of this consumer using 3.39.

16. Calculate his wealth dynamics using iterative scheme 3.40.

3.6 Numerical Results

Unless mentioned, I am assuming the following parameter values $r = 2.5\%$, $\rho = r, b = 9.44, m = 88.82, \xi = 1, \alpha = 0.5, \sigma = 1, \bar{l} = 6.49$. I took the numbers for m, b, and ξ from Huang et al. article [31] and $\bar{l}$ was chosen from the efforts of calibration in chapter 2 and section 2.6.1. $r = 2.5\%$ gives a nice average risk free rate. I chose to discount the utility function at the risk free rate, that is, I considered $\rho = r$. Since my focus in this thesis is to see how varying b-age affects the retirement decision, I took a large $\sigma = 1$ to see visible effects of the presence of b-age in making an optimal retirement decision. The maximum wealth $W = 12$, time $T = 110-x$ and the maximum b-age is equal to the c-age at T, that is, $\kappa_T = 110$. The current age $x = \kappa_0$ is assumed to be 30. I took grid sizes $dw = 0.01, dy = 0.05$ and $da = 0.02\left(\frac{1}{12}\right) = dt$. The computations took almost a month. In terms of time and storage the computations are quite expensive. Due to storage issues, I ran all the computations at the fine grid mentioned but saved the retirement boundary matrix τ and the consumption matrix at a much coarser grid. For the wealth dynamics, which required both of these matrices, I used interpolation to get the results at each fine grid point. Qualitatively the results stayed the same for a relatively coarser grid $dw = 0.05, dy = 0.05$ and $da = 0.025\left(\frac{1}{12}\right) = dt$ and it took only a week to run the computations. Therefore, figures 3.11 and 3.12 are plotted using this coarse grid.

My post-retirement model reduces the retirement consumption problem for stochastic mortality force in Huang et al. article [31] article and my equation 3.23 is reduced to their equation (7) at page number 68. For numerical comparison with the Huang et al. paper [31] it is assumed that $r = 2.5\%, \rho = r, A1 = -50, A2 = 150, \xi = 1, \sigma = 0.3$ and the mortality rate is pinned $\lambda = 0.005$ at B-age 60 and $\lambda = 1.0$ at B-age 110. I used the numerical scheme

3.28 for $B = 1, \kappa_0 = 60$ and $\kappa_T = 110$ with old boundary conditions for $-\infty < a < \infty$ in order to compute the optimal retirement spending rate $f^{-\tilde{\gamma}}$. My numbers matched their numbers given in Table 3 and Table 4 [31] for both $\gamma = 8$ and $\gamma = 2$. The maximum wealth $W = 30$ and $W = 15$ didn't make any difference in the results, hence, $W = 12$ is used to save computation time and storage space.

Since I want to use the new biological-age domain and boundary conditions for the entire stochastic mortality model, I tested the post-retirement problem for $f^{-\tilde{\gamma}}$ with a new b-age domain $0 \leq a \leq \kappa_T$ using the above mentioned parameters. In figures 3.5 and 3.7, for a small $\sigma = 0.3$ I can see that changing the domain and boundary condition slightly increased the consumption levels only for large b-ages 105 and 110 for both high and low risk aversions. Increasing the volatility 5 times to $\sigma = 1.5$, figures 3.6 and 3.8, showed an increase in consumption level from age 100 to 110 and this change is more noticeable than what I saw for $\sigma = 0.3$. A possible reason for this could be that due to very low volatility, the probability of the biological age process A_t hitting the boundary $a = 110$ is quite low, it keeps wandering above this boundary, so I do not see any difference in consumption levels for both set of boundary conditions.

Table 1

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	3.638%	3.800%	3.998%	4.246%	4.563%	4.980%	5.551%	6.374%
50	3.688%	3.851%	4.051%	4.301%	4.620%	5.038%	5.611%	6.435%
55	3.752%	3.917%	4.118%	4.369%	4.690%	5.110%	5.684%	6.509%
60	3.834%	4.000%	4.203%	4.456%	4.778%	5.200%	5.775%	6.601%
65	3.942%	4.109%	4.314%	4.568%	4.892%	5.316%	5.892%	6.717%
70	4.086%	4.255%	4.461%	4.717%	5.042%	5.466%	6.042%	6.865%
75	4.285%	4.455%	4.662%	4.918%	5.243%	5.667%	6.242%	7.060%
80	4.564%	4.735%	4.941%	5.197%	5.521%	5.943%	6.513%	7.322%
85	4.969%	5.137%	5.342%	5.595%	5.915%	6.330%	6.891%	7.684%
90	5.571%	5.735%	5.934%	6.180%	6.490%	6.892%	7.433%	8.195%
95	6.502%	6.655%	6.841%	7.070%	7.359%	7.735%	8.238%	8.945%
100	7.995%	8.125%	8.284%	8.480%	8.728%	9.050%	9.482%	10.086%
105	10.489%	10.576%	10.683%	10.815%	10.982%	11.201%	11.494%	11.903%
110	15.000%	15.000%	15.000%	15.000%	15.000%	15.000%	15.000%	15.000%

Table 2

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	3.638%	3.800%	3.998%	4.246%	4.563%	4.980%	5.551%	6.374%
50	3.688%	3.851%	4.051%	4.301%	4.620%	5.038%	5.611%	6.435%
55	3.752%	3.917%	4.118%	4.369%	4.690%	5.110%	5.684%	6.509%
60	3.834%	4.000%	4.203%	4.456%	4.778%	5.200%	5.775%	6.601%
65	3.942%	4.109%	4.314%	4.568%	4.892%	5.316%	5.892%	6.717%
70	4.086%	4.255%	4.461%	4.717%	5.042%	5.466%	6.042%	6.865%
75	4.285%	4.455%	4.662%	4.918%	5.243%	5.667%	6.242%	7.060%
80	4.564%	4.735%	4.941%	5.197%	5.521%	5.943%	6.513%	7.322%
85	4.969%	5.137%	5.342%	5.595%	5.915%	6.330%	6.891%	7.684%
90	5.571%	5.735%	5.934%	6.180%	6.490%	6.892%	7.433%	8.195%
95	6.502%	6.655%	6.841%	7.070%	7.359%	7.735%	8.238%	8.945%
100	7.995%	8.125%	8.284%	8.480%	8.728%	9.050%	9.482%	10.086%
105	10.489%	10.576%	10.683%	10.815%	10.982%	11.201%	11.494%	11.903%
110	14.814%	14.824%	14.837%	14.852%	14.871%	14.894%	14.921%	14.951%

Figure 3.5: Optimal retirement spending rate. Table 1: My Model, Table 2: JEDC article, assuming γ=8, $\sigma = 0.3$

Table 1

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	3.630%	3.793%	3.993%	4.243%	4.561%	4.980%	5.552%	6.377%
50	3.677%	3.842%	4.044%	4.295%	4.616%	5.037%	5.611%	6.437%
55	3.736%	3.903%	4.106%	4.360%	4.683%	5.106%	5.683%	6.511%
60	3.811%	3.979%	4.185%	4.441%	4.767%	5.193%	5.772%	6.602%
65	3.907%	4.078%	4.286%	4.545%	4.874%	5.303%	5.885%	6.716%
70	4.034%	4.208%	4.419%	4.681%	5.013%	5.445%	6.030%	6.861%
75	4.205%	4.381%	4.596%	4.862%	5.197%	5.633%	6.220%	7.051%
80	4.439%	4.619%	4.838%	5.107%	5.447%	5.886%	6.475%	7.303%
85	4.767%	4.952%	5.175%	5.449%	5.793%	6.235%	6.824%	7.646%
90	5.240%	5.430%	5.658%	5.937%	6.285%	6.729%	7.317%	8.127%
95	5.943%	6.139%	6.372%	6.656%	7.007%	7.452%	8.032%	8.819%
100	7.028%	7.228%	7.467%	7.754%	8.105%	8.545%	9.107%	9.851%
105	8.784%	8.989%	9.230%	9.516%	9.862%	10.284%	10.807%	11.464%
110	15.000%	15.000%	15.000%	15.000%	15.000%	15.000%	15.000%	15.000%

Table 2

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	3.630%	3.793%	3.993%	4.243%	4.561%	4.980%	5.552%	6.377%
50	3.677%	3.842%	4.044%	4.295%	4.616%	5.037%	5.611%	6.437%
55	3.736%	3.902%	4.106%	4.360%	4.683%	5.106%	5.683%	6.511%
60	3.811%	3.979%	4.185%	4.441%	4.767%	5.193%	5.772%	6.602%
65	3.907%	4.078%	4.286%	4.545%	4.874%	5.303%	5.885%	6.716%
70	4.034%	4.208%	4.419%	4.681%	5.013%	5.445%	6.030%	6.861%
75	4.205%	4.381%	4.596%	4.862%	5.197%	5.633%	6.220%	7.051%
80	4.438%	4.619%	4.838%	5.107%	5.447%	5.886%	6.475%	7.303%
85	4.767%	4.952%	5.175%	5.449%	5.792%	6.235%	6.824%	7.646%
90	5.240%	5.430%	5.658%	5.937%	6.284%	6.729%	7.316%	8.127%
95	5.943%	6.138%	6.372%	6.655%	7.006%	7.451%	8.031%	8.818%
100	7.027%	7.227%	7.465%	7.752%	8.103%	8.541%	9.104%	9.848%
105	8.773%	8.976%	9.215%	9.499%	9.841%	10.260%	10.782%	11.442%
110	11.740%	11.936%	12.164%	12.428%	12.739%	13.106%	13.537%	14.029%

Figure 3.6: Optimal retirement spending rate. Table 1: My Model, Table 2: JEDC article, assuming γ=8, $\sigma = 1.5$

Table 1

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	4.242%	4.465%	4.743%	5.096%	5.560%	6.195%	7.116%	8.572%
50	4.379%	4.607%	4.889%	5.247%	5.717%	6.359%	7.289%	8.757%
55	4.559%	4.791%	5.078%	5.442%	5.918%	6.568%	7.509%	8.989%
60	4.798%	5.035%	5.328%	5.698%	6.182%	6.841%	7.792%	9.286%
65	5.123%	5.366%	5.665%	6.043%	6.535%	7.203%	8.166%	9.675%
70	5.578%	5.827%	6.133%	6.518%	7.019%	7.697%	8.672%	10.196%
75	6.233%	6.487%	6.799%	7.192%	7.700%	8.389%	9.374%	10.912%
80	7.203%	7.462%	7.779%	8.177%	8.691%	9.386%	10.379%	11.923%
85	8.691%	8.951%	9.268%	9.666%	10.180%	10.873%	11.863%	13.400%
90	11.054%	11.306%	11.614%	12.000%	12.500%	13.175%	14.138%	15.635%
95	14.937%	15.166%	15.447%	15.800%	16.258%	16.878%	17.768%	19.159%
100	21.505%	21.686%	21.910%	22.193%	22.562%	23.067%	23.798%	24.955%
105	32.817%	32.921%	33.050%	33.215%	33.432%	33.733%	34.175%	34.890%
110	52.500%	52.500%	52.500%	52.500%	52.500%	52.500%	52.500%	52.500%

Table 2

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	4.242%	4.465%	4.743%	5.096%	5.560%	6.195%	7.116%	8.572%
50	4.379%	4.607%	4.889%	5.247%	5.717%	6.359%	7.289%	8.757%
55	4.559%	4.791%	5.078%	5.442%	5.918%	6.568%	7.509%	8.989%
60	4.798%	5.035%	5.328%	5.698%	6.182%	6.841%	7.792%	9.286%
65	5.123%	5.366%	5.665%	6.043%	6.535%	7.203%	8.166%	9.675%
70	5.578%	5.827%	6.133%	6.518%	7.019%	7.697%	8.672%	10.196%
75	6.233%	6.487%	6.799%	7.192%	7.700%	8.389%	9.374%	10.912%
80	7.203%	7.462%	7.779%	8.177%	8.691%	9.386%	10.379%	11.923%
85	8.691%	8.951%	9.268%	9.666%	10.180%	10.873%	11.863%	13.400%
90	11.054%	11.306%	11.614%	12.000%	12.500%	13.175%	14.138%	15.635%
95	14.937%	15.166%	15.447%	15.800%	16.258%	16.878%	17.768%	19.159%
100	21.505%	21.686%	21.910%	22.193%	22.562%	23.067%	23.798%	24.955%
105	32.817%	32.921%	33.050%	33.215%	33.432%	33.733%	34.175%	34.890%
110	52.420%	52.421%	52.423%	52.425%	52.427%	52.431%	52.436%	52.444%

Figure 3.7: Optimal retirement spending rate. Table 1: My Model, Table 2: JEDC article, assuming γ=2, $\sigma = 0.3$

Table 1

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	4.256%	4.480%	4.757%	5.111%	5.575%	6.210%	7.132%	8.588%
50	4.394%	4.621%	4.904%	5.262%	5.733%	6.375%	7.306%	8.774%
55	4.572%	4.804%	5.092%	5.457%	5.934%	6.585%	7.526%	9.007%
60	4.807%	5.045%	5.339%	5.711%	6.196%	6.857%	7.810%	9.306%
65	5.125%	5.369%	5.670%	6.050%	6.545%	7.217%	8.183%	9.696%
70	5.565%	5.817%	6.126%	6.515%	7.021%	7.705%	8.685%	10.215%
75	6.192%	6.452%	6.770%	7.169%	7.686%	8.383%	9.379%	10.927%
80	7.112%	7.380%	7.708%	8.117%	8.645%	9.355%	10.365%	11.927%
85	8.509%	8.785%	9.120%	9.538%	10.075%	10.793%	11.811%	13.380%
90	10.714%	10.992%	11.330%	11.749%	12.286%	13.003%	14.014%	15.565%
95	14.336%	14.605%	14.930%	15.334%	15.850%	16.538%	17.507%	18.988%
100	20.510%	20.745%	21.029%	21.382%	21.834%	22.437%	23.288%	24.593%
105	31.350%	31.510%	31.705%	31.948%	32.262%	32.682%	33.280%	34.201%
110	52.500%	52.500%	52.500%	52.500%	52.500%	52.500%	52.500%	52.500%

Table 2

	C-Age							
B-Age	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**
45	4.256%	4.480%	4.757%	5.111%	5.575%	6.210%	7.132%	8.588%
50	4.394%	4.621%	4.904%	5.262%	5.733%	6.375%	7.306%	8.774%
55	4.572%	4.804%	5.092%	5.457%	5.934%	6.585%	7.526%	9.007%
60	4.807%	5.045%	5.339%	5.711%	6.196%	6.857%	7.810%	9.306%
65	5.125%	5.369%	5.670%	6.050%	6.545%	7.217%	8.183%	9.696%
70	5.565%	5.817%	6.126%	6.515%	7.021%	7.705%	8.685%	10.215%
75	6.192%	6.452%	6.770%	7.169%	7.686%	8.383%	9.379%	10.927%
80	7.112%	7.380%	7.707%	8.117%	8.645%	9.355%	10.365%	11.927%
85	8.509%	8.785%	9.120%	9.538%	10.075%	10.793%	11.811%	13.380%
90	10.714%	10.992%	11.330%	11.749%	12.286%	13.003%	14.014%	15.565%
95	14.336%	14.604%	14.930%	15.333%	15.850%	16.537%	17.506%	18.988%
100	20.509%	20.743%	21.027%	21.380%	21.832%	22.435%	23.286%	24.590%
105	31.322%	31.481%	31.675%	31.918%	32.231%	32.651%	33.249%	34.173%
110	50.442%	50.483%	50.533%	50.595%	50.676%	50.784%	50.938%	51.173%

Figure 3.8: Optimal retirement spending rate. Table 1: My Model, Table 2: JEDC article, assuming γ=2, $\sigma = 1.5$

In order to solve my retirement decision problem, post-retirement case, I have used some parameter values different than what I took above for comparison with JEDC article. T is assumed to be $110 - x$ and maximum b-age is 110. The current age κ_0 is assumed to be 30 and the units of leisure $\bar{l} = 6.49$. Figure 3.9 plots the scaled value function for post-retirement phase spanning the whole life time.

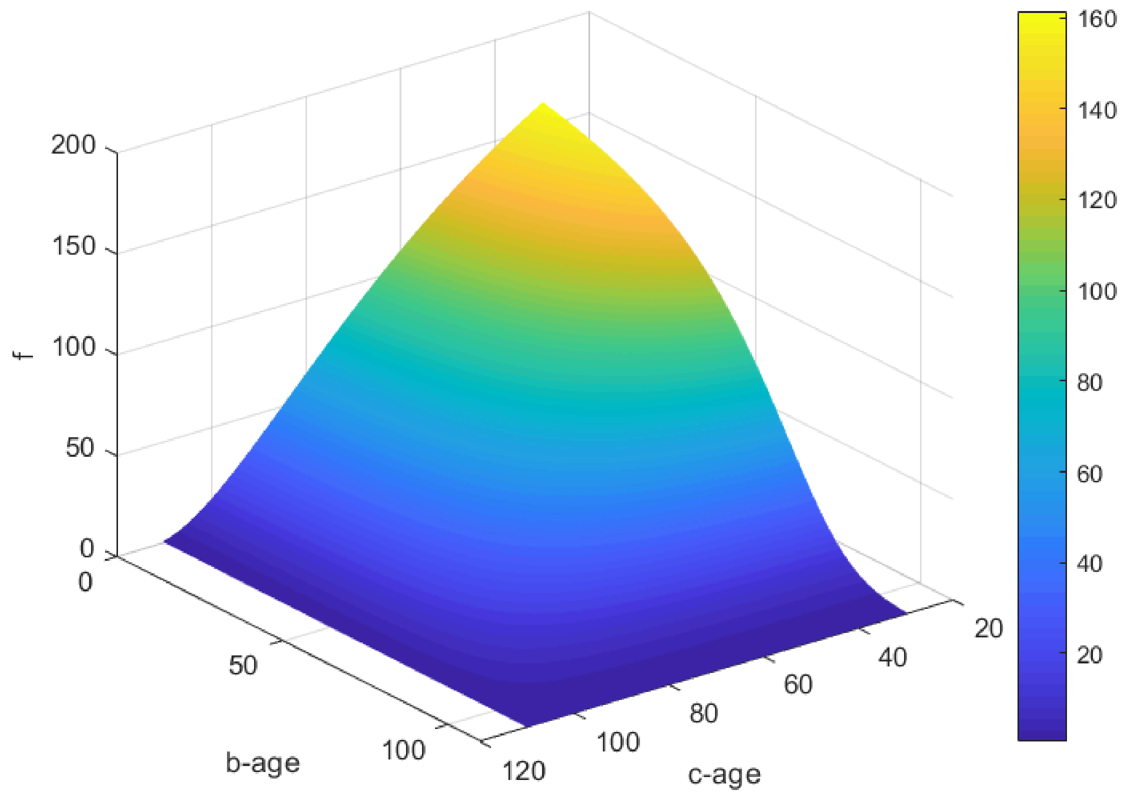


Figure 3.9: Stochastic Mortality Case: Scaled Value function $f(t, a)$ for post-retirement phase.
Assuming $r = 2.5\%, \rho = r, \gamma = 2, \sigma = 1, \bar{l} = 6.49, \alpha = 0.5$

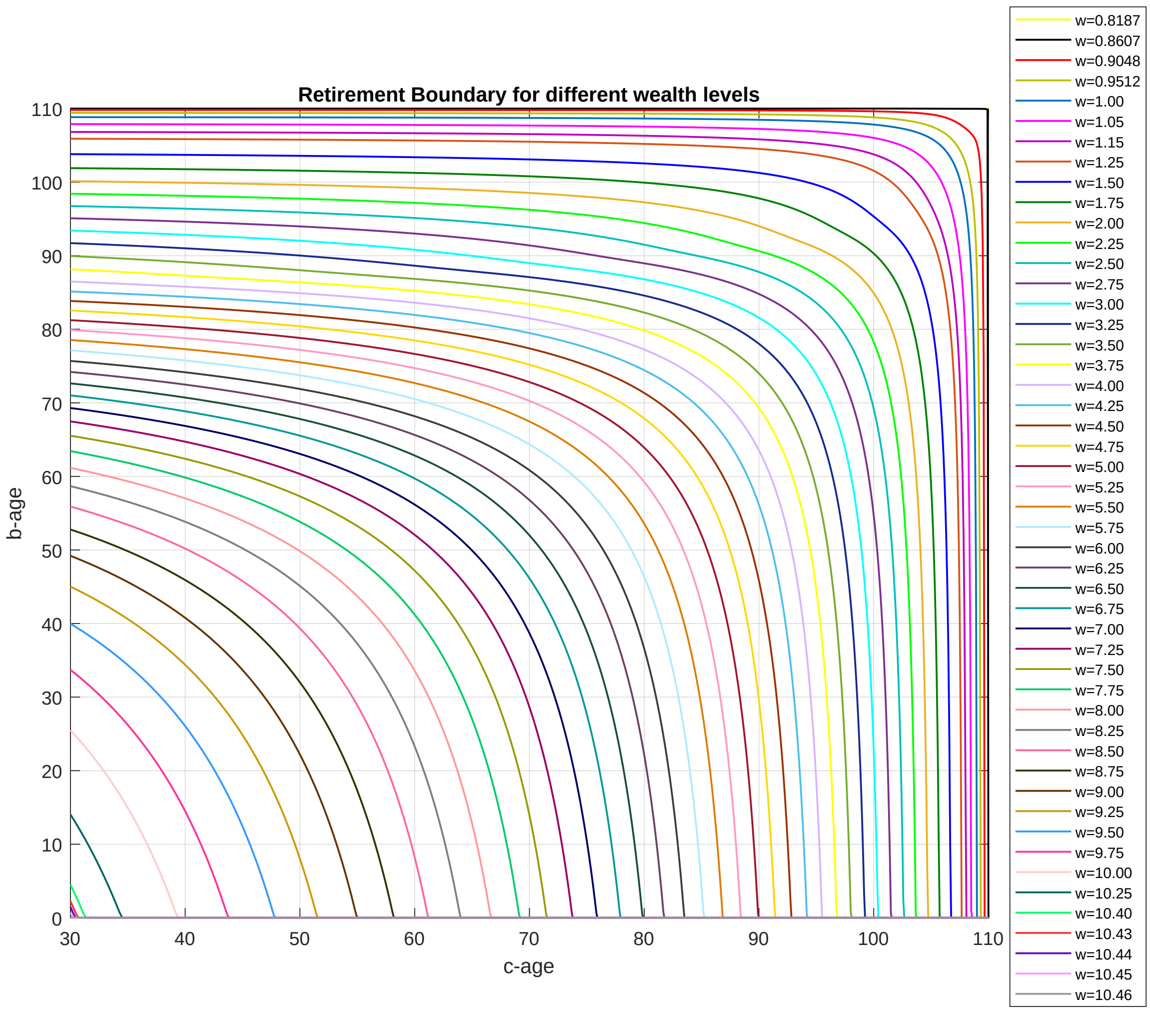


Figure 3.10: Optimal retirement boundaries for fixed wealth levels.
Assuming $\gamma = 2, r = 2.5\%, \rho = r, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

Figure 3.10 shows the optimal retirement decision as a function of biological vs chronological age for some fixed levels of wealth. For a fixed wealth the optimal retirement b-age decreases with increasing c-age. For example consider the wealth curve $w = 6.75$. On this curve, a chronologically 30 years old person wants to keep working until he is biologically 71 year old. At this b-age he thinks that his remaining life/retired life is short enough to survive his retirement savings. On the other hand, an employee who is 71 year old according to his birth date, is willing to retire at the same wealth level when he is as healthy as a 45 year old. He wants to enjoy the remaining years of his life as a retired person.

If b-age is less then c-age, the fixed wealth retirement curves are very steep depicting the sharp decline of optimal b-age with a slight increase in c-age. While, for the region where b-age is greater then c-age, with an increase in c-age the decrease in b-age is extremely slow for low wealth curves and relatively faster for wealth curves greater than and equal to $w = 6$. If a consumer has reached a wealth level of at least 10.46, he will never get into the labor force whether he is in the best possible health or his chances of survival are extremely low. This is the wealth threshold above which all consumers stay in the post-retirement consumption and utility region, while at and below this wealth, the labor income is attractive and consumers save money to retire at some point in time. For a fixed c-age, a more wealthy consumer would like to retire at an earlier b-age as compared to a consumer with low wealth level. For example a 60 year old consumer having \$7 in savings is willing to retire as early as a b-age of 57 years while another 60 year old consumer with wealth level of \$3 would like to work until his health depletes to a biological age of 91 years. Reading it the other way, an average 60 year old customer will retire at $w = 6.75$, but a 60 year old customer who is as healthy as a 23 year old will retire around $w = 8.25$ with a point of view that he might live longer then usual and needs more money saved up for a long retirement life. A 60 year old customer who did not age well and his health is like an 80 year old will retire at $w = 4.50$. He thinks that he does not have much time to live so he wants to retire at a lower wealth level then an average 60 year old and enjoy the retirement life. The same goes with consumers at same health levels; a wealthy consumer tends to retire young.

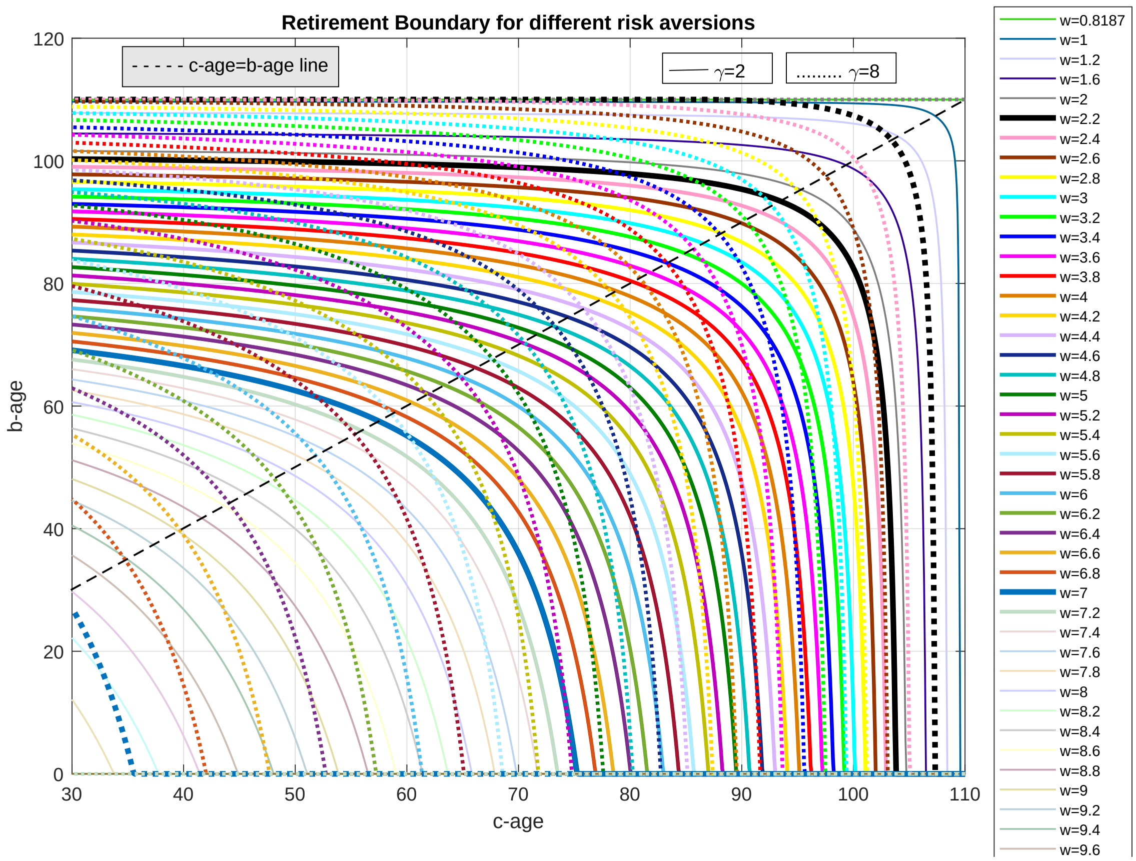


Figure 3.11: Optimal retirement boundaries comparison for different risk aversions γ and fixed wealth levels.
Assuming $\sigma = 1, r = 2.5\%, \rho = r, \bar{l} = 6.49, W = 12, \alpha = 0.5$

Figure 3.11 depicts the effect of risk aversion on retirement boundaries. The diagonal dashed black line is the $c - age = b - age$ line. The dotted level curves are for $\gamma = 8$ and the solid lined level curves are for $\gamma = 2$ consumer. The bold lines for $w = 2.2$ and $w = 7$ depict the spread of level curves for $\gamma = 8$ which are comparable to the $\gamma = 2$ curves. A more risk averse person will consume less after retirement with the fear of living longer then average, hence, he requires less money saved at the time of retirement. A moderately risk averse $\gamma = 2$ consumer requires a wealth level of $w = 10.46$ (I know from figure 3.10) while highly risk averse γ =8 consumers require a lesser amount saved (approximately) $w = 7.2$ to enjoy their life as a retired person. If I keep the wealth fixed I can see different attitudes depending upon the amount of wealth. For all wealth level less than $w = 3$, consumers with the same chronological ages and the same wealth in savings, a $\gamma = 8$ consumer with a greater fear of living a long life, who is older then an average person of the same birth year (above the b-age=c-age line), tends to retire at a higher biological age then a $\gamma = 2$ person. Obviously, he wants to work till a later b-age to save more for a longer retirement life. Similarly, people with higher c-ages (100 and above) and the same health status and higher risk aversion tend to retire at a later c-age for the same reasons mentioned before. But as the wealth level rises beyond $w = 3$ notice that the $\gamma = 2$ and $\gamma = 8$ curves cross each other. With increasing wealth the point of crossing is shifting towards lower c-ages until the curves do not intersect for wealth levels $w = 6$ and higher. For these crossing curves, at the same wealth level, towards the left hand side of the point of intersection a highly risk averse person of the same c-age tends to retire at a higher b-age and on the right hand side consumers with same b-ages and same wealth level but with moderate risk aversion retire at later c-ages. For wealth $w = 6$ and higher, customers sitting at the same wealth status but with different risk aversions, a $\gamma = 2$ person retires at later calendar ages as well as higher b-ages. This retirement age difference is widening between both consumers with increasing wealth levels. Probably, the consumer with a greater fear of living longer thinks he has enough money saved for retirement (as highly risk averse people tend to have lower spending rates post-retirement) so he should retire earlier than his moderately risk averse $\gamma = 8$ friend. Notice that, the $\gamma = 8$ and $\gamma = 2$ customers' retirement curves for the same wealth level which are crossing at some point, on the right hand side of the intersection show behaviors similar to $w \geq 6$ wealth and on the left hand side have attitudes similar to people with $w < 3$ wealth levels.

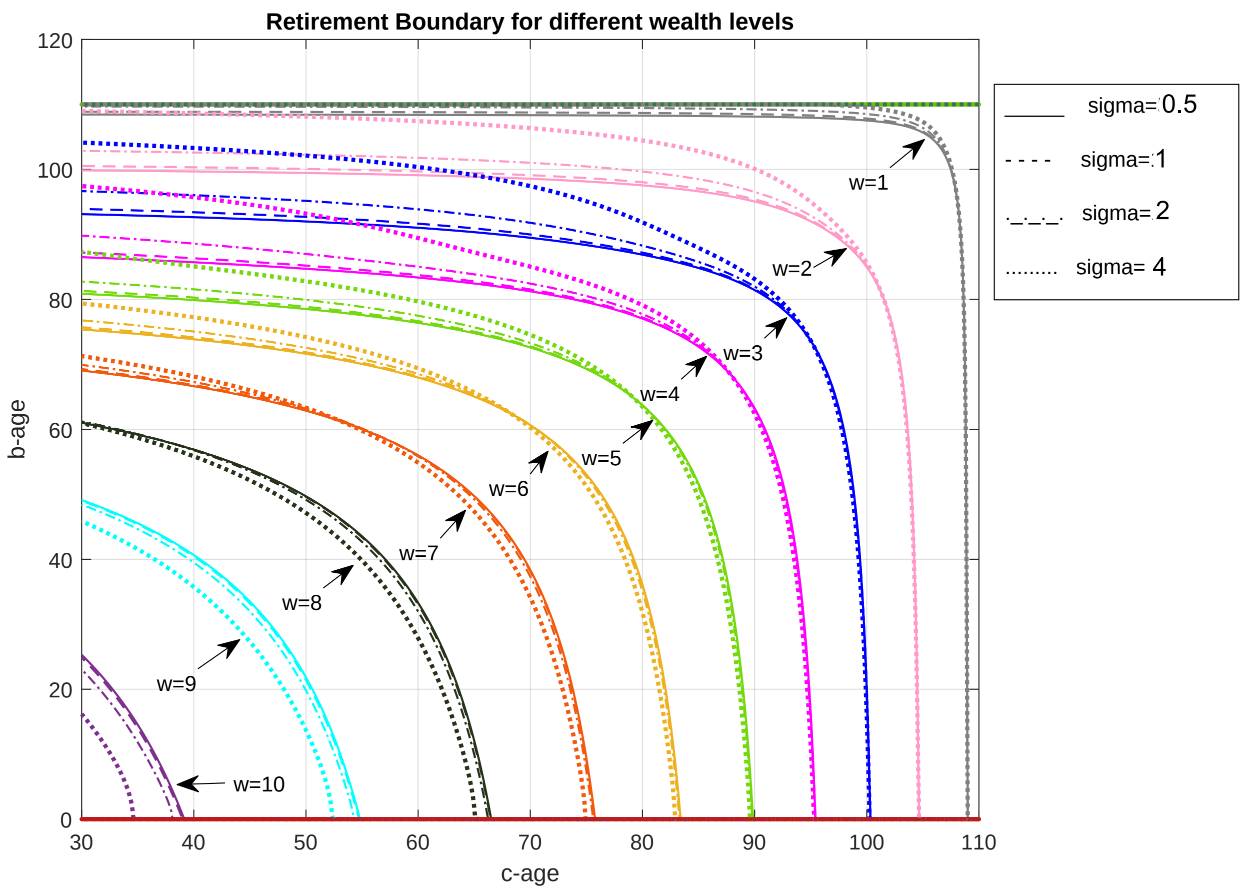


Figure 3.12: Optimal retirement boundary sensitivity to the volatility of biological age process.
Assuming $r = 2.5\%, \rho = r, \gamma = 2, \bar{l} = 6.49, W = 12, \alpha = 0.5$

Figure 3.12 shows the sensitivity of optimal retirement boundary to the volatility σ of mortality which is one of the factors driving the diffusion of biological age process. The behavior for high w is what was expected, that is, increasing volatility will cause people to retire earlier. The more fascinating effect is what happens at lower wealth levels, i.e. the curves cross. At low w with increasing σ, the difference in optimal retiring age for healthier people seems insignificant. But the people who are less healthy than normal, who also have limited savings, actually retire later (when volatility is higher). Maybe thinking *this could change, and I could get healthier, in which case I'll need extra wealth to fund my retirement.

So I better keep working, just in case*. While if volatility is low their spending needs are more predictable, so they can retire and start spending.

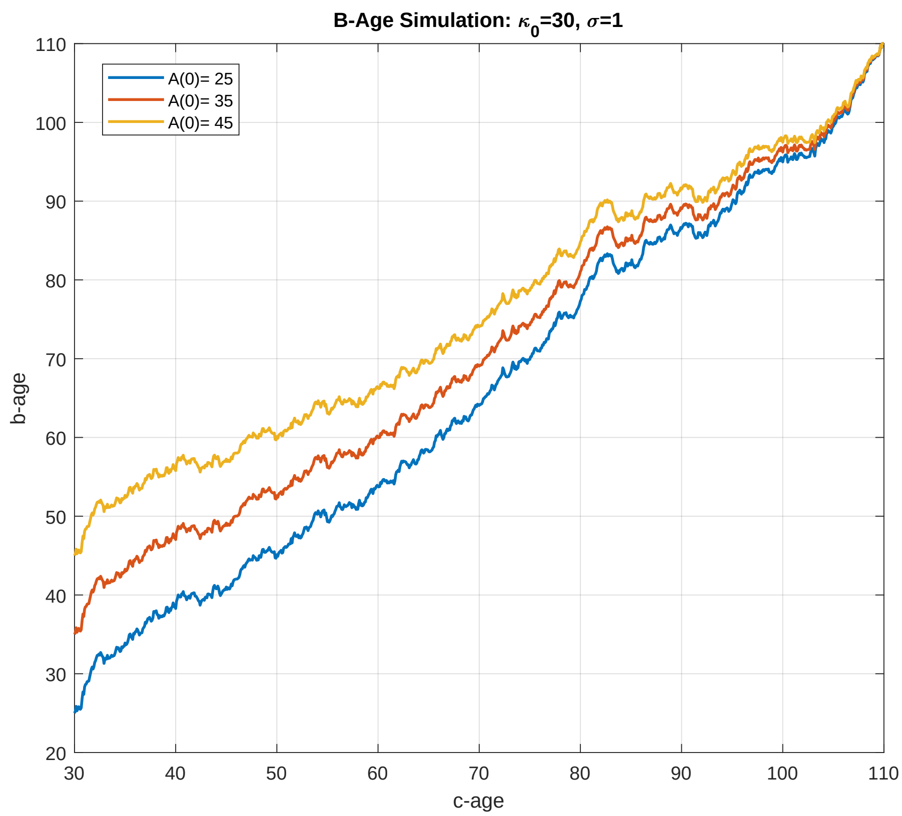


Figure 3.13: B-Age Simulations for varying A_0.
Assuming $\gamma = 2, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

Figure 3.13 is the b-age simulation of a 30 year old consumer considering different scenarios for b-ages at $t = 0$. For $A_0 = 25$, the consumer is healthier than his current c-age 30 and for $A_0 = 35, 45$ his health status corresponds to someone older than c-age 30. The upward drift increases the biological age of the consumer with growing c-age, showing a person's health deteriorates with growing c-age. The noise in the path shows volatility in the b-age. Since the maximum b-age is capped at $\kappa_T = 110$, all of the b-age paths are drifting towards this capped max b-age.

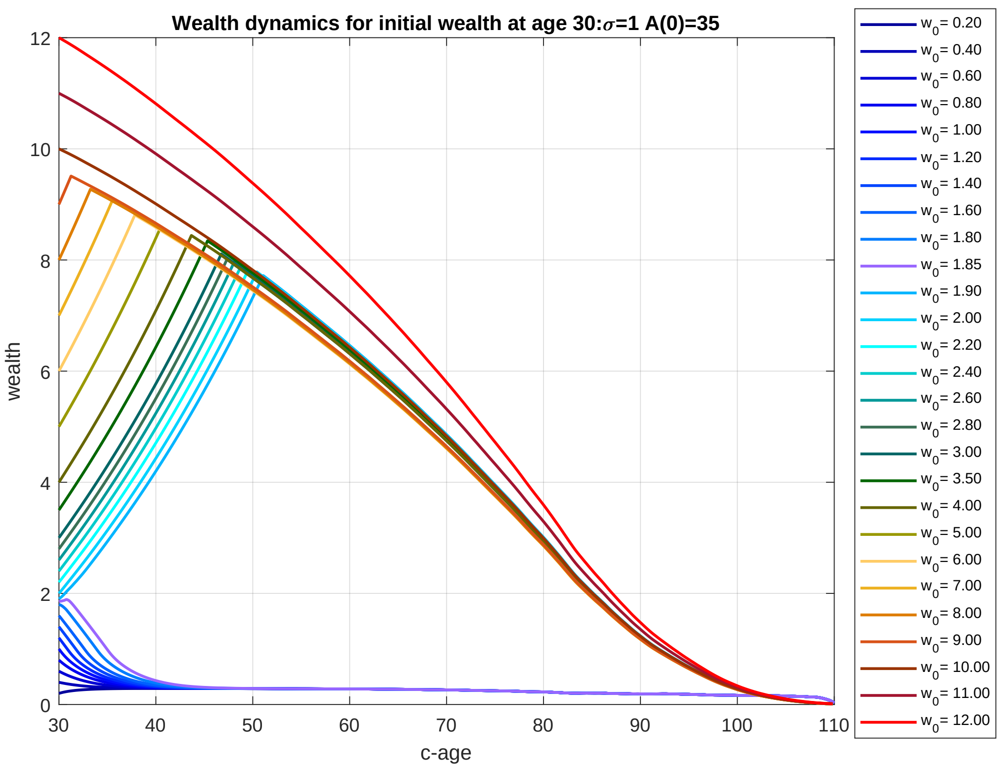


Figure 3.14: Wealth Dynamics of consumer with stochastic force of mortality for varying initial wealth w_0.
Assuming $\gamma = 2, \kappa_0 = 30, A_0 = 35, r = 2.5\%, \rho = r, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

The life time wealth dynamics of a 30 years old consumer with b-age 35 are shown in figure 3.14. The consumer uses my optimal consumption and retirement strategy. I am testing the strategy with the consumer's different levels of initial wealth w_0. Figure 3.18 plots the corresponding optimal consumption paths. Just as in the deterministic mortality case figure 2.6, for small initial wealth the consumer stays in the labor force all his life time. For wealth levels approximately less than $w = 1.85$, consumer begins at consumption levels larger than his labor income so his wealth declines at a faster pace but since he has planned on not retiring at all, soon he quickly but smoothly lowers his consumption levels to $c = 1$ so that his wealth level stays almost steady at some positive wealth near 0. The noise in the consumption plot is due to the noise in b-age simulation. Notice that the steady consumption level $c = 1$ begins to rise near the final b-ages and the corresponding wealth curves decreases to 0. The consumer knows he is unlikely to live beyond $t = T$, so he begins using up his savings as well as annual income in the hope to gain more utility from consuming towards the few final years of his life. The purple wealth curve for $w_0 = 1.85$ shows slightly different behavior for the first year or two. The corresponding purple consumption curve shows that his spending rate increases very fast in those years but soon he begins dropping his consumption levels and joins the no-retirement wealth path along with all the wealth levels less than $w = 1.85$. The next blue curve for $w_0 = 1.90$ also begins at a higher consumption level but instead of riding the no-retirement band wagon, he begins to save money for retirement and eventually retires around c-age= 52 with the retirement savings of almost \$7.5 and spends rest of his life on his savings. For all wealth levels between $1.90 \leq w_0 \leq 9.0$ the consumer saves money for retirement shown by rising wealth curves. For all these wealth levels the consumption stay almost steady at $c = 0.81$ with a slight decrease near retirement. Then we see a huge downward jump in consumption depicting the retirement and a decrease in consumption levels by the factor B in order to compensate for the lost wages. Afterwards, the consumption as well as the wealth decreases with increasing c-age in retirement life. All wealth levels above $w_0 = 9$ prefer not go in labor force at all. They are happy to lead all their life in retirement consuming around the rates of $c = 0.3$ and $c = 0.4$ for the most of their lives. And then declining their consumption levels so that their wealth saving could last till their last breath (if they stay alive till c-age=110). Figures 3.15 and 3.16 with their corresponding consumption plots figures 3.19 and 3.19 portray the wealth dynamics and consumption strategies for a 30 year old consumer with b-age 25 and 45 respectively. As compared to consumer with initial b-age 35, a consumer who enters the strategy with b-age= 25 (he is in better health than his c-age) has no-retirement wealth threshold decreased down to $w_0 = 1.50$ and for the consumer with initial b-age 45 has the no-retirement wealth threshold increased to $w_0 = 2.40$. The healthier consumer is more hopeful about living a longer life so he wants to get in the labor force and save more for his long retired life while the relatively less healthy person doesn't have high hopes about living a longer life. He needs to have a bigger amount of initial savings to give him hope about a long life to make him get into the labor force and save money for retirement. Figures 3.17 and 3.21 are merely the above wealth plots combined together and the optimal consumption plots put together to give a better visual comparison of wealth and consumption curves for the consumers with different initial b-ages.

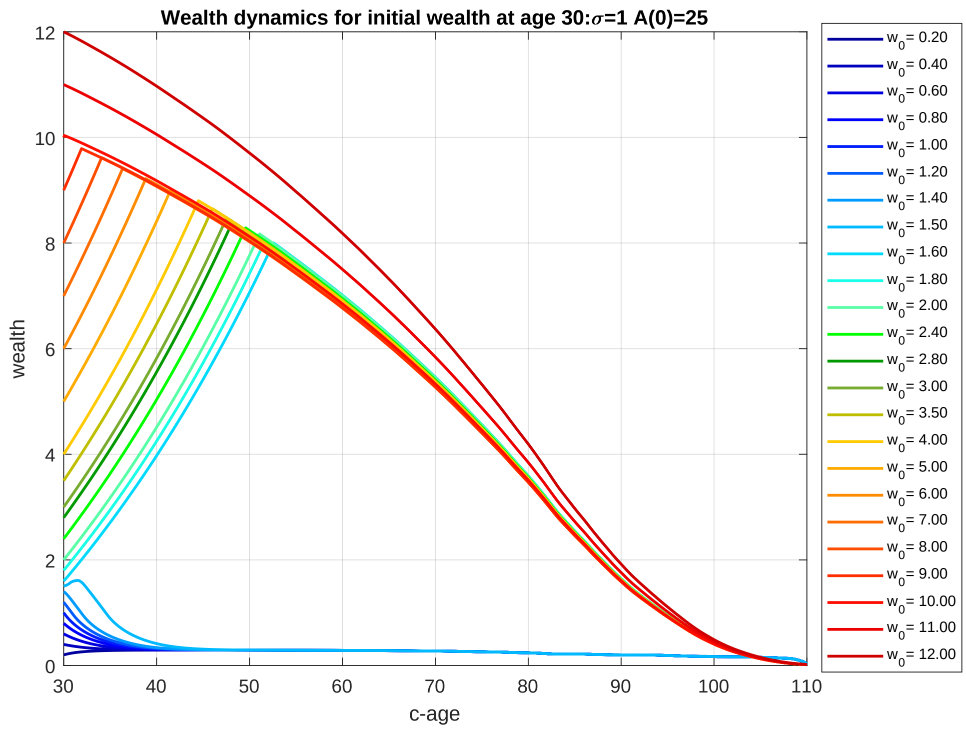


Figure 3.15: Wealth dynamics of a consumer with $A_0 = 25$.
Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

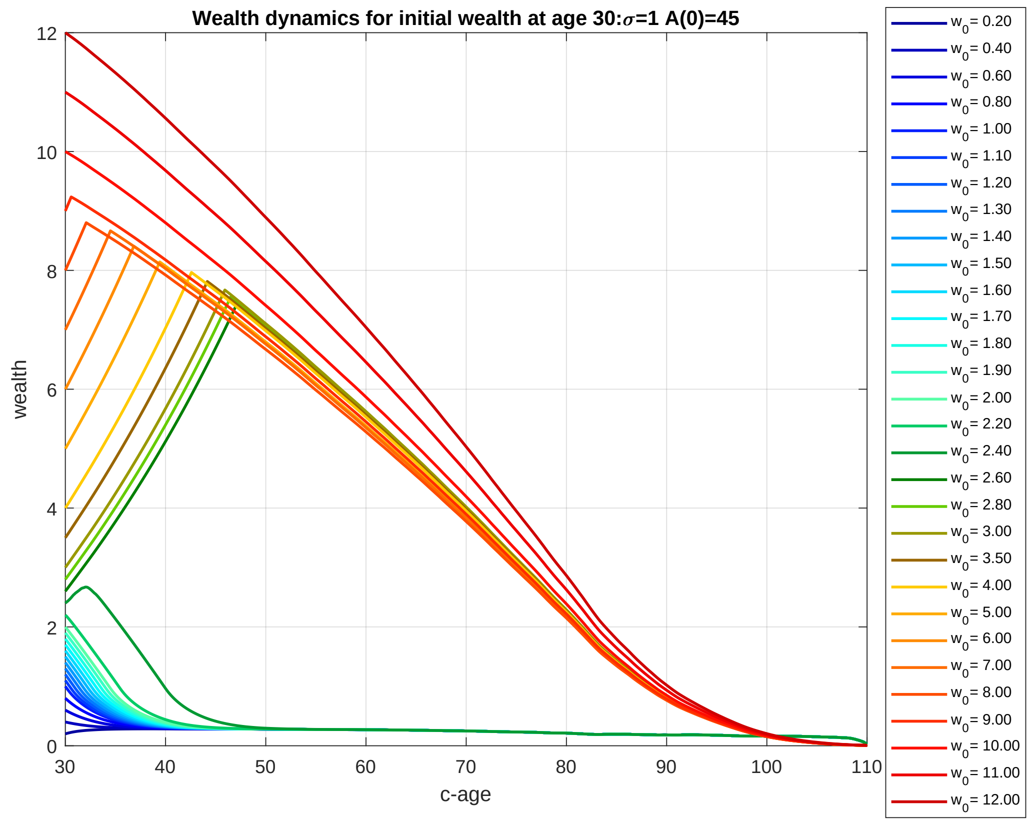


Figure 3.16: Wealth dynamics of a consumer with $A_0 = 45$.
Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

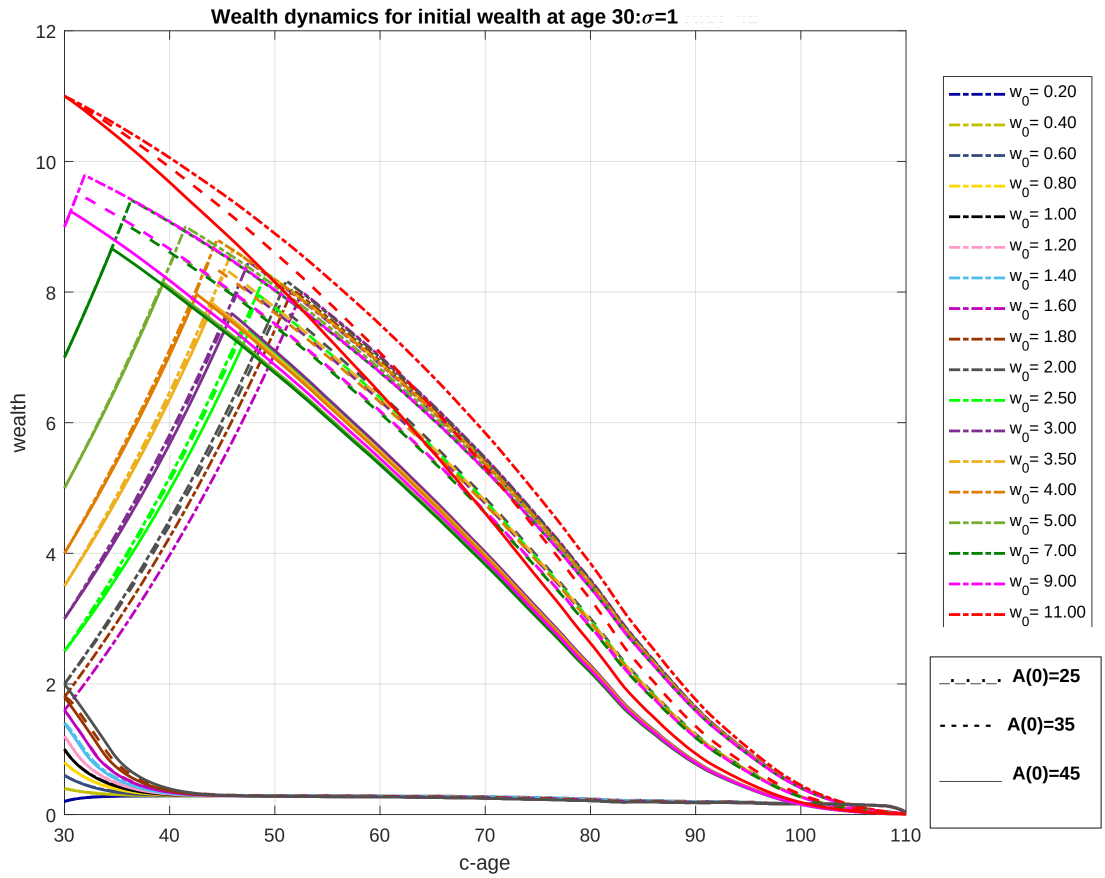


Figure 3.17: Comparison of wealth dynamics of consumers with different A_0. Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

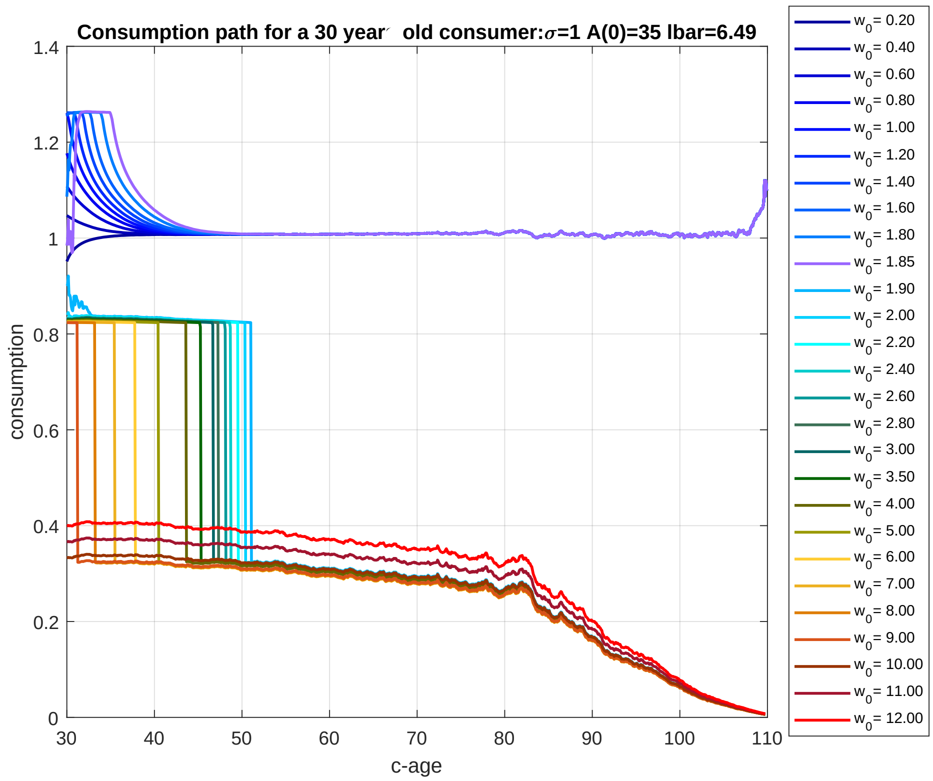


Figure 3.18: Life time consumption path of a consumer.
Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, A_0 = 35, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

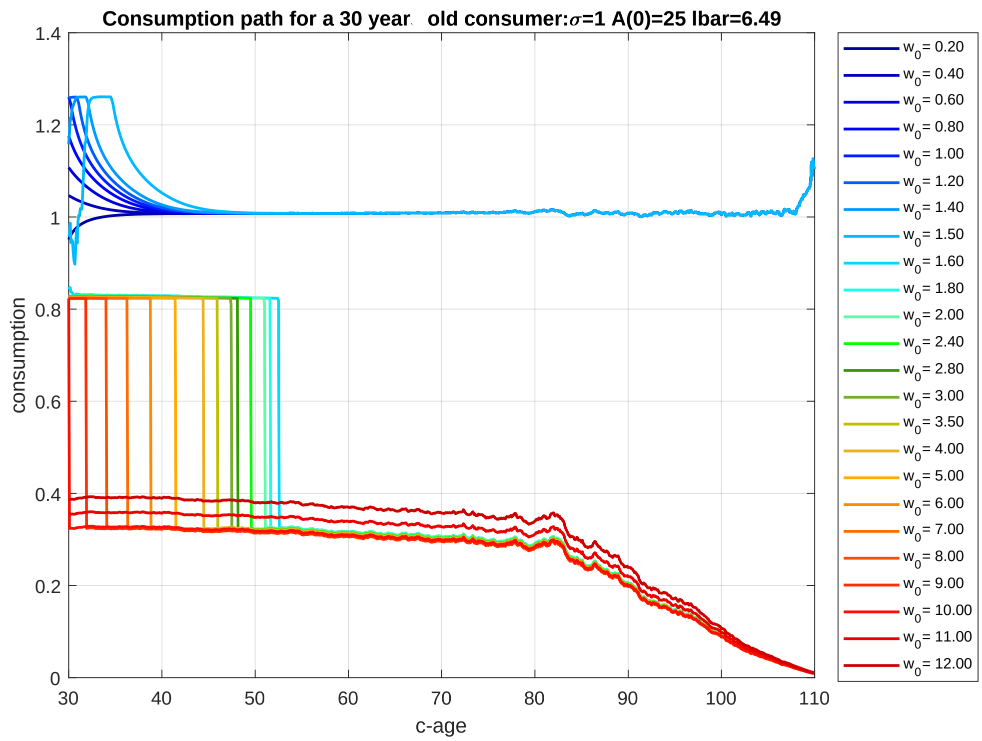


Figure 3.19: Life time Consumption of a consumer with $A_0 = 25$.
Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

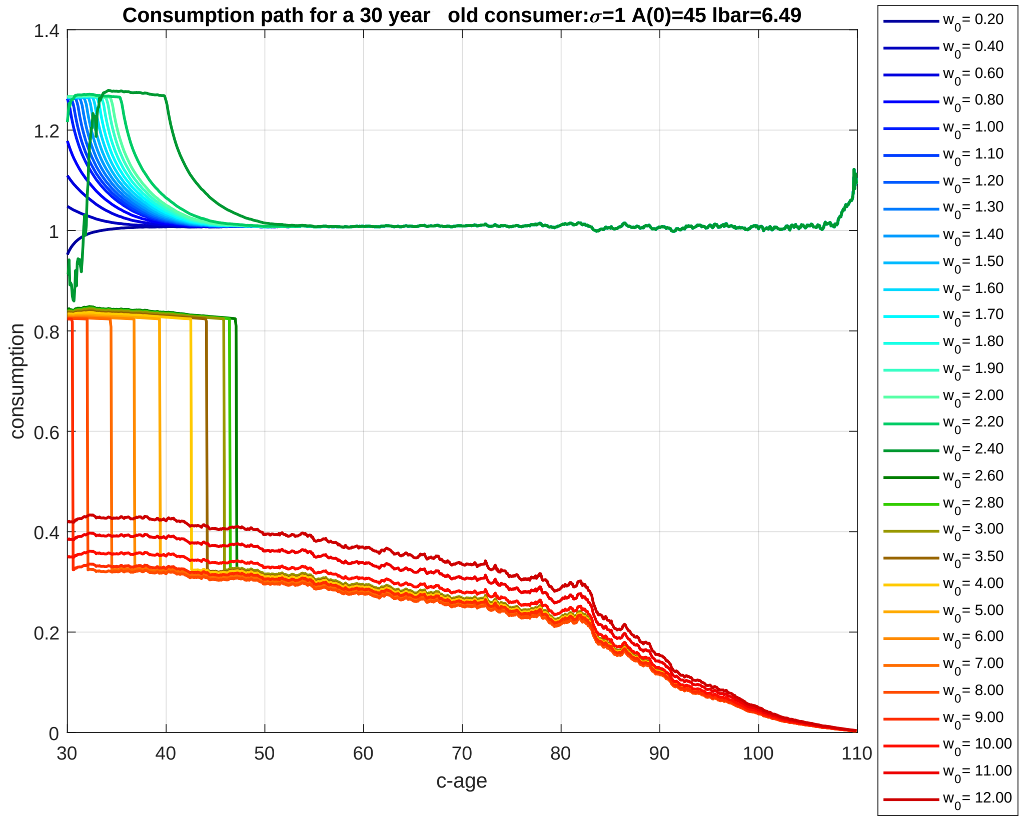


Figure 3.20: Life time Consumption of a consumer with $A_0 = 45$.
Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

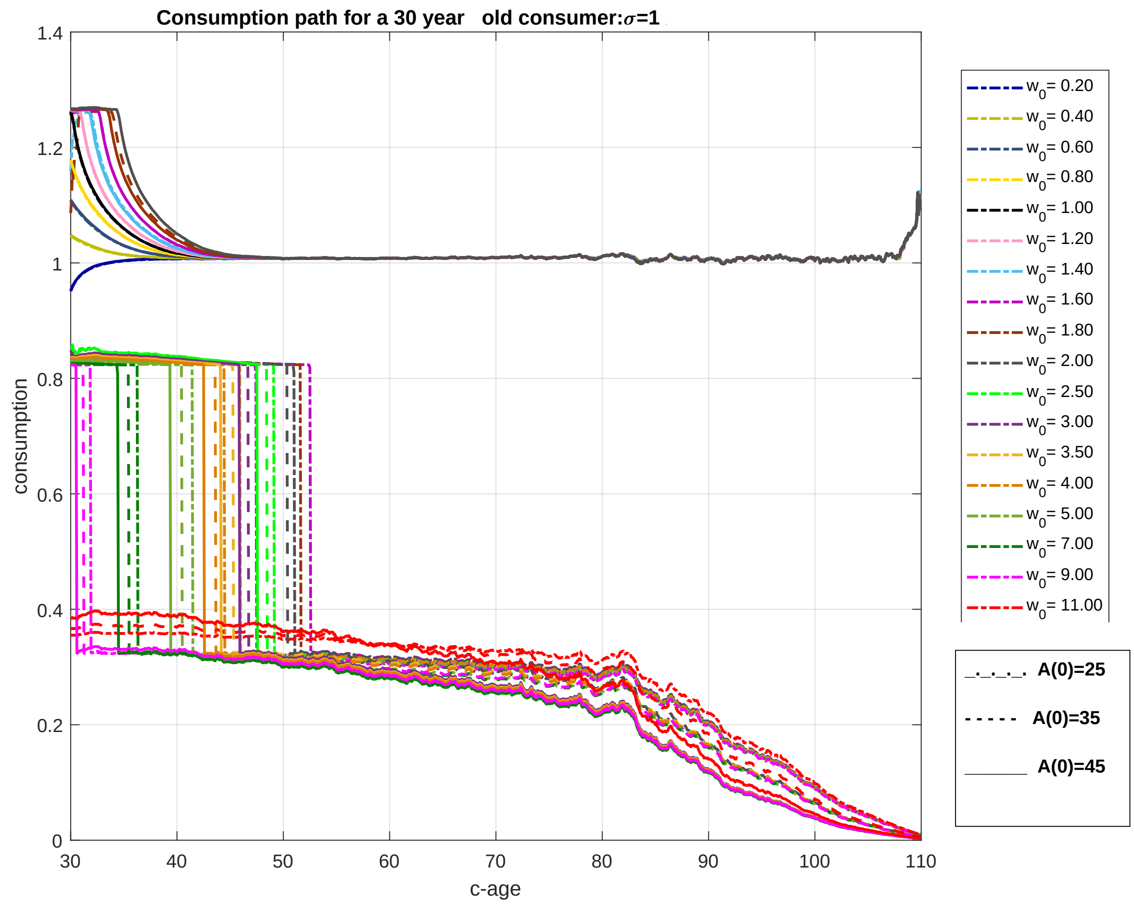


Figure 3.21: Comparison of life time consumption of consumers with different A_0. Assuming $\gamma = 2, r = 2.5\%, \rho = r, \kappa_0 = 30, \sigma = 1, \bar{l} = 6.49, W = 12, \alpha = 0.5$

3.7 Pre-Retirement PDE: Explicit Numerical Scheme

Initially I used an Explicit Finite Difference Scheme to solve the pre-retirement problem due to the ease of implementing explicit schemes. I was not able to solve the problem with great accuracy but it gave me a good idea about the shape of the pre-retirement value function and the problems I need to focus on. For example, I saw that around large time retirement at extremely small wealth was causing instability issues in the code. I had to fix this issue with the use of log-wealth grid. I think it's a good idea to mention this scheme here as well.

Consider the numerical grid (t_i, w_j, a_k); $0 \leq t_i \leq T, 1 \leq w_j \leq W, A1 \leq a_k \leq A2$ and a non-uniform grid (t_i, y_j, a_k);$0 \leq t_i \leq T, 0 \leq e^{y_j} < 1, A1 \leq a_k \leq A2$. Let dt, dw, dy and da be the time step-size, wealth step-size, log-wealth step-size and bio-age step-size, respectively. $N+1, M+1, M_y+1$ and $K+1$ be the total number of time nodes, wealth nodes, log-wealth nodes and b-age nodes, respectively. I am considering an explicit finite difference scheme evaluating all derivatives at old time step. For this scheme I am considering the original forward time (not using the time transformation). Then the i^{th} time node is $t_i = idt; i = N, N-1, ..., 0$, the j^{th} wealth node is $w_j = jdw; j =, 1, ..., M$, the j^{th} log-wealth node is $y_j = Y1 + jdy; j =, 1, ..., M_y$ and the k^{th} biological age node is $a_k = A1 + kda; k = 0, 1, ..., K$. The pre-retirement value function at discretized grid points be $V_1(t_i, w_j, a_k) \equiv v^i_{j,k}; \forall i, j, k$. Assume that everyone is forced to retire on reaching some large c-age $t_{retI} + x$ and not allowed to go back to work. This constraint is used to avoid the instability arising due to very small values of life expectancy close to this age. A small retirement bonus '*bonus*' is offered to the retiree to stabilize the numerical scheme when people having very small wealth are forced to retire at T.

Consider the pre-retirement problem PDEs mentioned in the previous Section 3.2.1 4.12 and 3.32:

$$V_{1t} + \left[\left(1 + \xi\frac{\kappa_t - a}{T-t}\right)\right]V_{1a} + \frac{\sigma^2}{2}V_{1aa} + \left[e^{-y} + r - \left(\frac{e^{-y(1-\tilde{\gamma})}{V_{1y}}^{-\tilde{\gamma}}}{1-\tilde{\gamma}}\right)\right]V_{1y} - (\rho + \lambda(a))V_1 = 0$$

For $1 \leq w \leq W, \forall t, \forall a$, consider the pre-retirement PDE 3.13 in the following form

$$V_{1t} + \left[1 + \xi(\frac{\kappa_t - a}{T-t})\right]V_{1a} + \frac{\sigma^2}{2}V_{1aa} + \left(1 + rw - \frac{{V_{1w}}^{-\tilde{\gamma}}}{1-\tilde{\gamma}}\right)V_{1w} - (\rho + \lambda(a))V_1 = 0$$

along with the boundary conditions:

$$\begin{aligned}
&v^i_{M,k} = \overline{V}(t_i, W + bonus, a_k); \forall i, k \\
&v^i_{j,0} = v^i_{j,1}; \forall i, j \\
&v^i_{j,K} = \overline{V}(t_i, w_j + bonus, K), v^i_{j,K} = \overline{V}(t_i, e^{y_j} + bonus, K), \forall i, j \\
&V_1(\tau) = \overline{V}(\tau), \text{where } \tau(t^*, w^*, a^*) \text{ is the optimal retirement boundary.}
\end{aligned}$$

The the time derivative is discretized at (t_{i+1}, w_j, a_k) using backward finite difference.

$$V_{1t}(t_{i+1}, w_j, a_k) = \frac{v^{i+1}_{j,k} - v^i_{j,k}}{dt}$$

Call

$$\theta^{i+1}_k \equiv -\left[1 + \xi\left(\frac{\kappa_0 + t_{i+1} - a_k}{T - t_{i+1} + \epsilon}\right)\right]$$

and

$$\beta_{j,k}^{i+1} \equiv \begin{cases} e^{-y_j} + r - (\frac{e^{-y_j(1-\tilde{\gamma})}[V_{1y}(t_{i+1},w_j,a_k)]^{-\tilde{\gamma}}}{1-\tilde{\gamma}}), & 0 \leq e^{y_j} < 1 \\ 1 + rw_j - \frac{[V_{1w}(t_{i+1},w_j,a_k)]^{-\tilde{\gamma}}}{1-\tilde{\gamma}}, & 1 \leq w_j \leq W \end{cases}$$

For β, V_{1w} and V_{1y} are evaluated at current time-step (t_{i+1}, w_j, a_k) or at (t_{i+1}, y_j, a_k) using forward difference. Then, using β up-winding is used for the wealth derivative at explicit time nodes i.e.,

$$V_{1w}(t_{i+1}, w_j, a_k) = \begin{cases} \frac{v_{j+1,k}^{i+1} - v_{j,k}^{i+1}}{dw} & , \beta_{j,k}^{i+1} \geqslant 0 \\ \frac{v_{j,k}^{i+1} - v_{j-1,k}^{i+1}}{dw} & , \beta_{j,k}^{i+1} < 0 \end{cases}$$

Symmetric difference and up-winding is used for b-age derivatives i.e.,

$$V_a(t_{i+1}, w_j, a_k) = \begin{cases} \frac{v_{j,k+1}^{i+1} - v_{j,k}^{i+1}}{da} & , \theta_k^{i+1} \geq 0 \\ \frac{v_{j,k}^{i+1} - v_{j,k-1}^{i+1}}{da} & , \theta_k^{i+1} < 0 \end{cases}$$

$$V_{aa}(t_{i+1}, w_j, a_k) = \frac{v_{j,k+1}^{i+1} - 2v_{j,k}^{i+1} + v_{j,k-1}^{i+1}}{da^2}$$

The $0^{th}-$order term is evaluated at the un-known grid point (t_i, w_j, a_k). Denote $P_k = 1 + dt(\rho + \lambda(a_k))$ and $D = \frac{\sigma^2}{2}$. Then the discretized version of both PDEs 4.12 and 3.32; as per their respective β; becomes

$$\begin{aligned} &\frac{v_{j,k}^{i+1} - v_{j,k}^{i}}{dt} + \frac{\theta_k^{i+1} + |\theta_k^{i+1}|}{2}\frac{v_{j,k+1}^{i+1} - v_{j,k}^{i+1}}{da} + \frac{\theta_k^{i+1} - |\theta_k^{i+1}|}{2}\frac{v_{j,k}^{i+1} - v_{j,k-1}^{i+1}}{da} + \\ &D\frac{v_{j,k+1}^{i+1} - 2v_{j,k}^{i+1} + v_{j,k-1}^{i+1}}{da^2} - (\rho + \lambda(a_k))v_{j,k}^{i} + \\ &\frac{\beta_{j,k}^{i+1} + |\beta_{j,k}^{i+1}|}{2}\frac{v_{j+1,k}^{i+1} - v_{j,k}^{i+1}}{dw} + \frac{\beta_{j,k}^{i+1} - |\beta_{j,k}^{i+1}|}{2}\frac{v_{j,k}^{i+1} - v_{j-1,k}^{i+1}}{dw} = 0 \end{aligned}$$

This solves explicitly for the unknown $v_{j,k}^{i}; \forall i, j, k$ as

$$\begin{aligned} v_{j,k}^{i} =& \frac{1}{P_k}\Bigg[v_{j,k}^{i+1} + dt\Bigg(D\frac{v_{j,k+1}^{i+1} - 2v_{j,k}^{i+1} + v_{j,k-1}^{i+1}}{da^2} + \\ &\frac{\theta_k^{i+1} + |\theta_k^{i+1}|}{2}\frac{v_{j,k+1}^{i+1} - v_{j,k}^{i+1}}{da} + \frac{\theta_k^{i+1} - |\theta_k^{i+1}|}{2}\frac{v_{j,k}^{i+1} - v_{j,k-1}^{i+1}}{da} + \\ &\frac{\beta_{j,k}^{i+1} + |\beta_{j,k}^{i+1}|}{2}\frac{v_{j+1,k}^{i+1} - v_{j,k}^{i+1}}{dw} + \frac{\beta_{j,k}^{i+1} - |\beta_{j,k}^{i+1}|}{2}\frac{v_{j,k}^{i+1} - v_{j-1,k}^{i+1}}{dw}\Bigg)\Bigg] \end{aligned} \tag{3.41}$$

3.7.0.1 Algorithm

The algorithm for solving the optimal retirement boundary is given below:

1. Assume that everyone is forced to retire on reaching the age of 100 and not allowed to go back to work. Then, $t_{retI} + x = 100$ and set

$$v^i_{j,k} = \overline{V}(i, e^{y_j} + bonus, a_k), v^i_{j,k} = \overline{V}(i, w_j + bonus, a_k)$$

$$v^K_{j,k} = \overline{V}(K, e^{y_j} + bonus, a_k), v^K_{j,k} = \overline{V}(K, w_j + bonus, a_k)$$

$$v^i_{M,k} = \overline{V}(i, M, a_k)$$

for $i = retI, retI + 1, ..., N\ \forall j, k$

2. Begin at $i = retI - 1$ and $k = K - 1$.

3. Begin at $j = 1$ for the log-wealth grid.

4. Solve 3.41 for $v^{retI-1}_{j,K-1}$.

5. The post-retirement value at this grid point is $\overline{v} = f(t_{retI-1}, a_{K-1})\frac{(e^{y_j}+bonus)^{1-\gamma}}{1-\gamma}$

6. Evaluate $max(v^{retI-1}_{j,K-1}, \overline{v})$. If the maximum turns out to be equal to $\overline{v}$, mark this i, j, k for optimal retirement, that is, $\tau(i^*, j^*, k^*)$.

7. Move onto next j and repeat steps $3 - 5$.

8. After the last grid point in log-wealth grid begin at $j = 1$ for uniform wealth grid.

9. Solve 3.41 for $v^{retI-1}_{j,K-1}$. The post-retirement value at this grid point is $\overline{v} = f(t_{retI-1}, a_{K-1})\frac{(w_j+bonus)^{1-\gamma}}{1-\gamma}$

10. Repeat step 6 until the final grid point on wealth grid.

11. Iterate through all b-age steps going backwards from $k = K - 1, ..., 2$. For each k repeat steps 3-10. Finally set $v^1_{j,k} = v^2_{j,k}$.

12. Walk to the next time step, solve for all $j's$ and $k's$ while repeating steps 3-13. The marked values of $\tau(i^*, j^*, k^*)$ form the optimal retirement boundary.

I used Matlab to program this numerical scheme. The maximum wealth is $W = 15$. The time $T = 110 - x$ and the maximum b-age is 110. The current age κ_0 is assumed to be 30. The units of leisure $\bar{l} = 7$, $Y1 = -4, Y2 = ln1, W1 = 1, W2 = W$. To control the instability arising due to $\lambda(a)$ near age 110 age, everyone reaching the calendar age of 100 is forced to retire. For the stability conditions of explicit finite difference scheme, the time grid size has to be quite small, that is, $dt = 10^{-4}$ in order to get a reasonable wealth step size, $dw = 0.05 = dy$. The b-age step size is considered to be 1 month, $da = 1/12$.

Figure 3.22 shows the optimal retirement decision as a function of b-age vs c-age for some fixed levels of wealth. Figures 3.23– 3.30 plot the value function of a person at different ages making optimal retirement decision. Due to the extremely small time step, it took almost 20 hours to run the code.

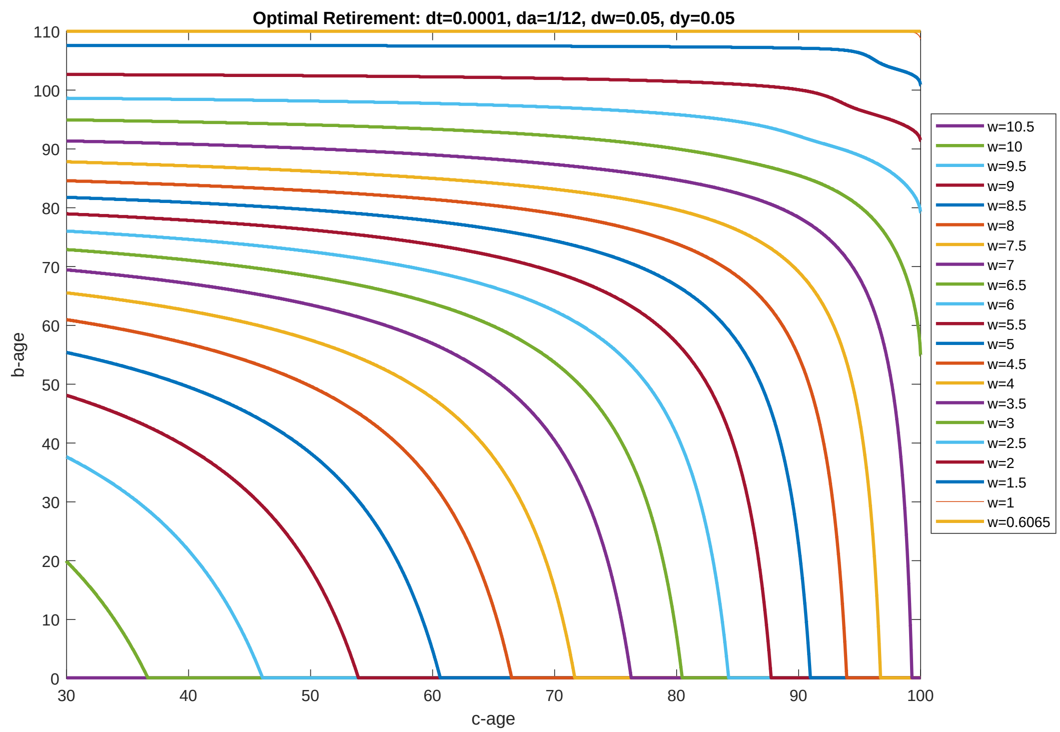


Figure 3.22: Optimal retirement decision for fixed wealths, Explicit Finite Difference Scheme. Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

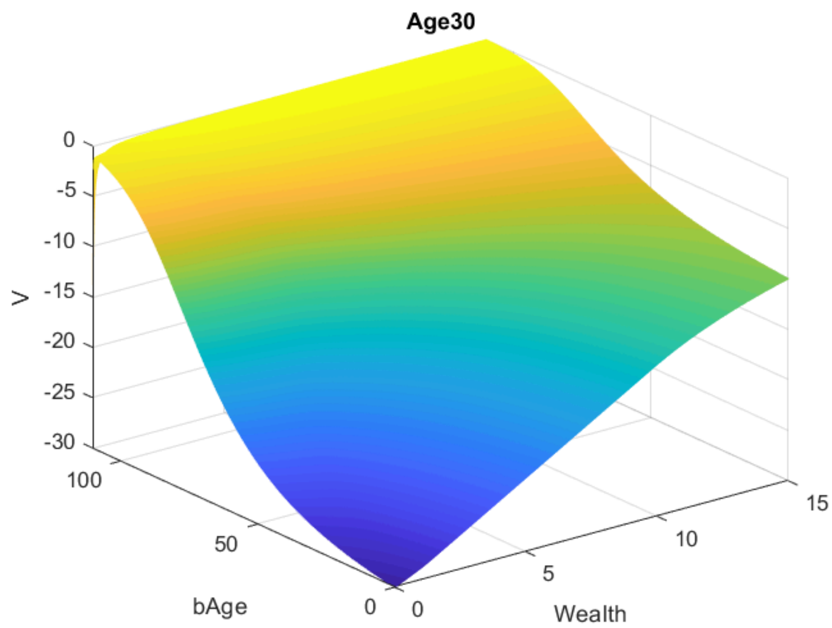


Figure 3.23: Value Function for Optimal Retirement decision at age 30.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

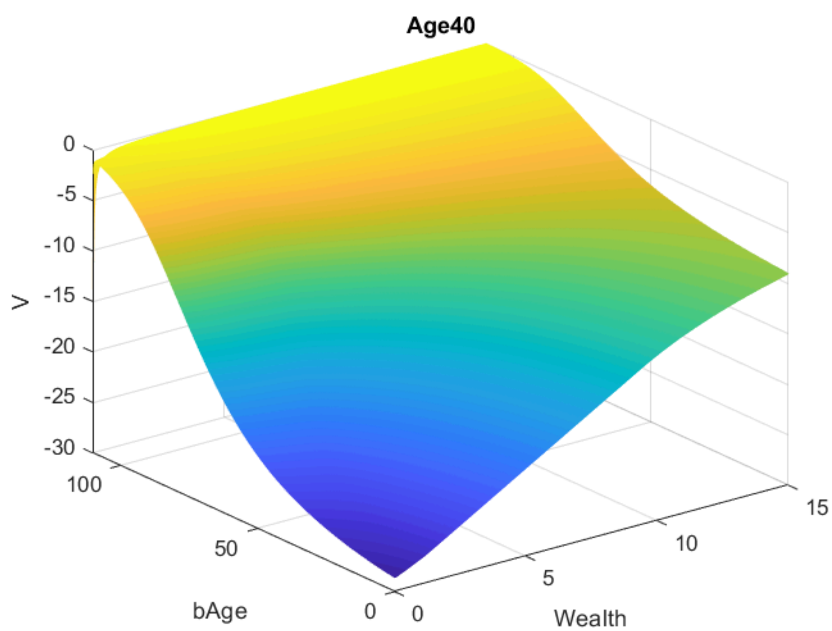


Figure 3.24: Value Function for Optimal Retirement decision at age 40.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

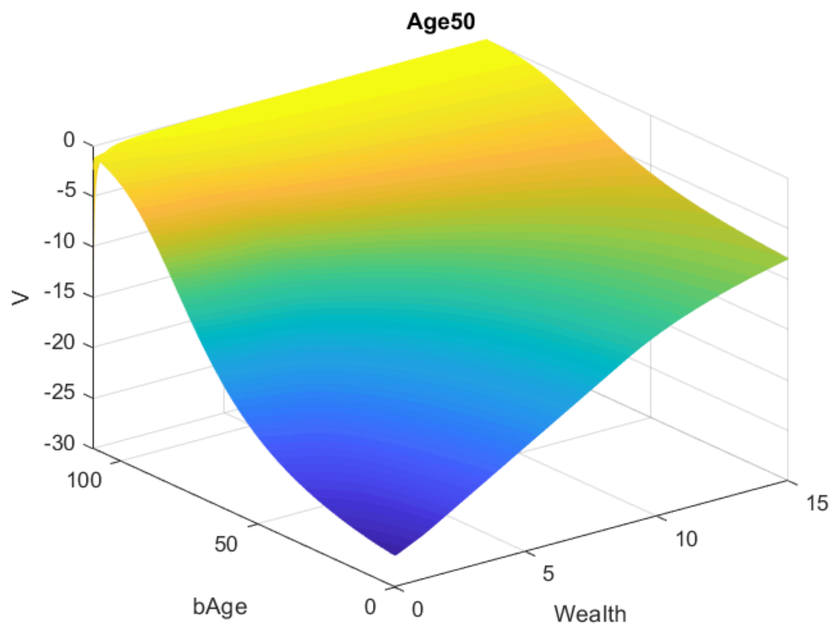


Figure 3.25: Value Function for Optimal Retirement decision at age 50.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

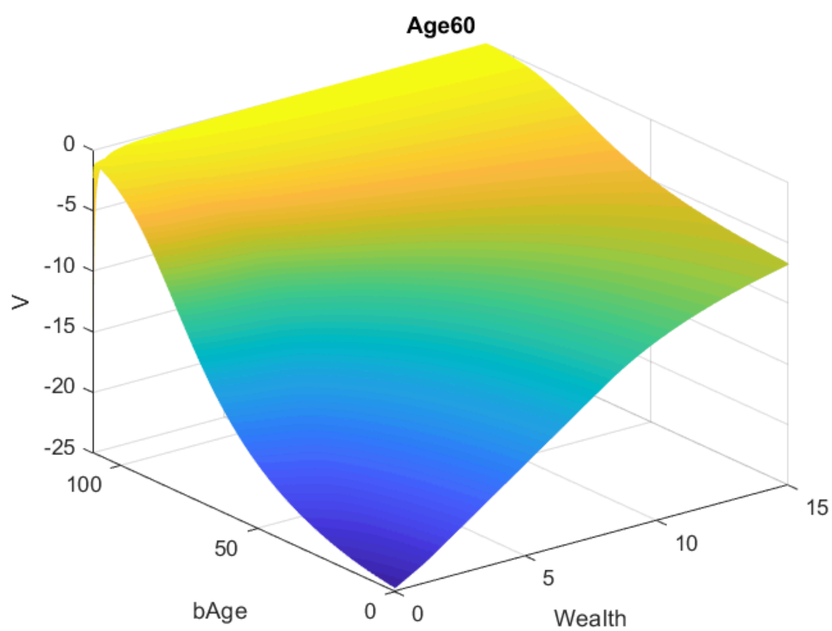


Figure 3.26: Value Function for Optimal Retirement decision at age 60.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

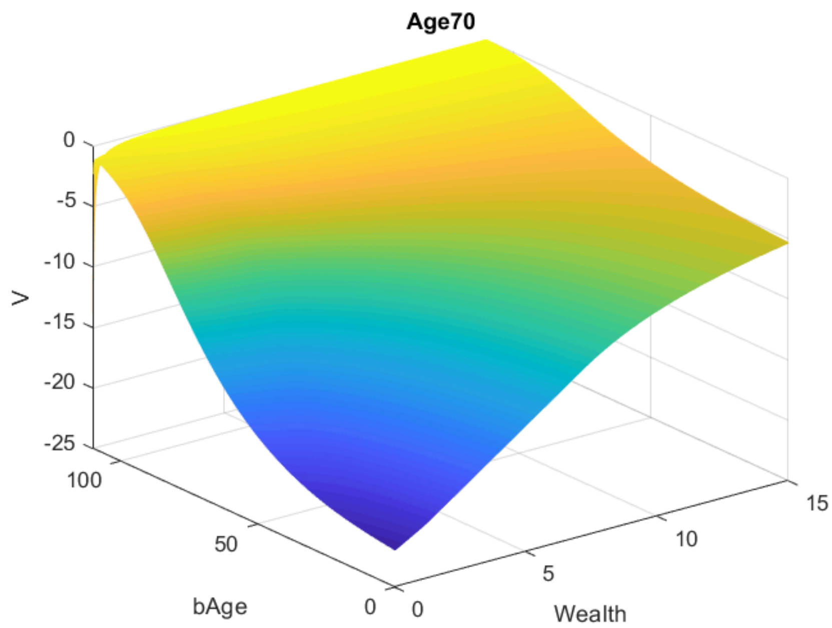


Figure 3.27: Value Function for Optimal Retirement decision at age 70.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

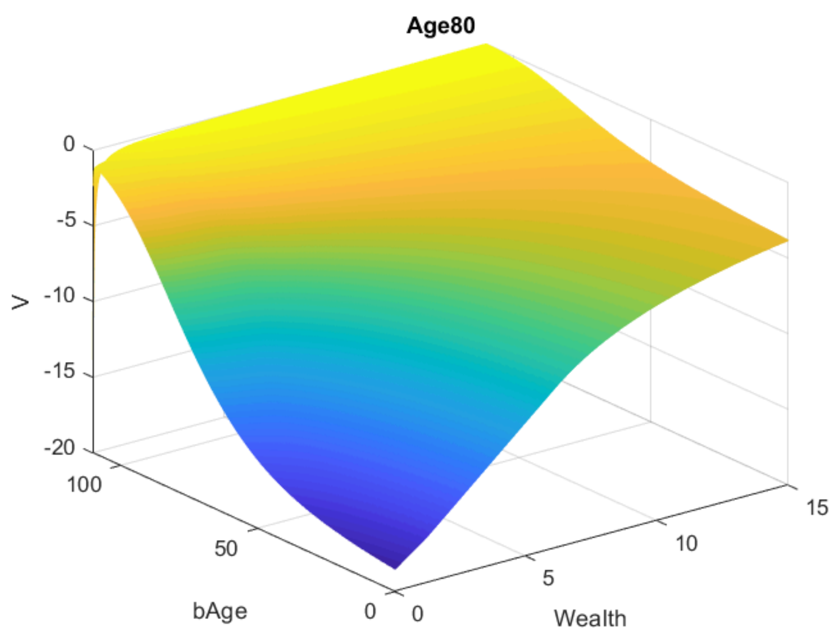


Figure 3.28: Value Function for Optimal Retirement at age 80.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

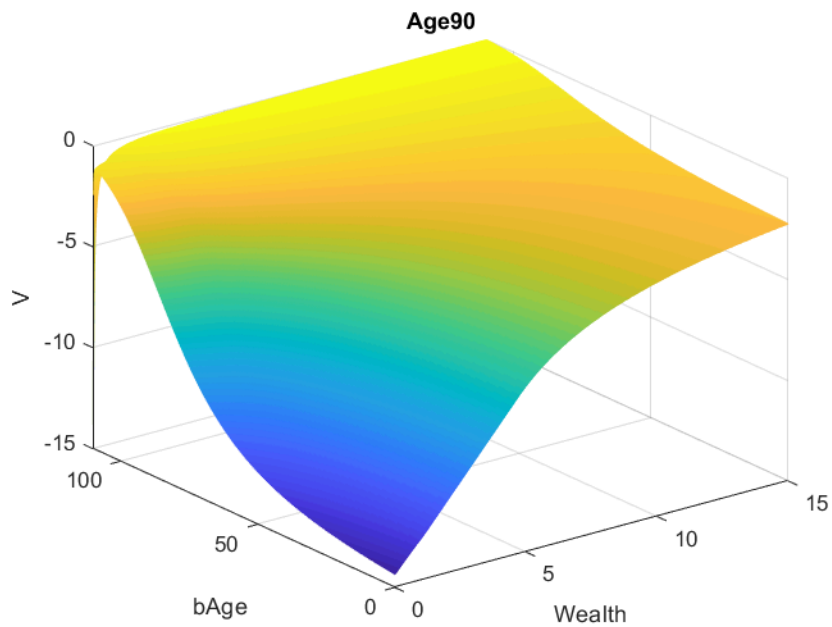


Figure 3.29: Value Function for Optimal Retirement decision at age 90.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

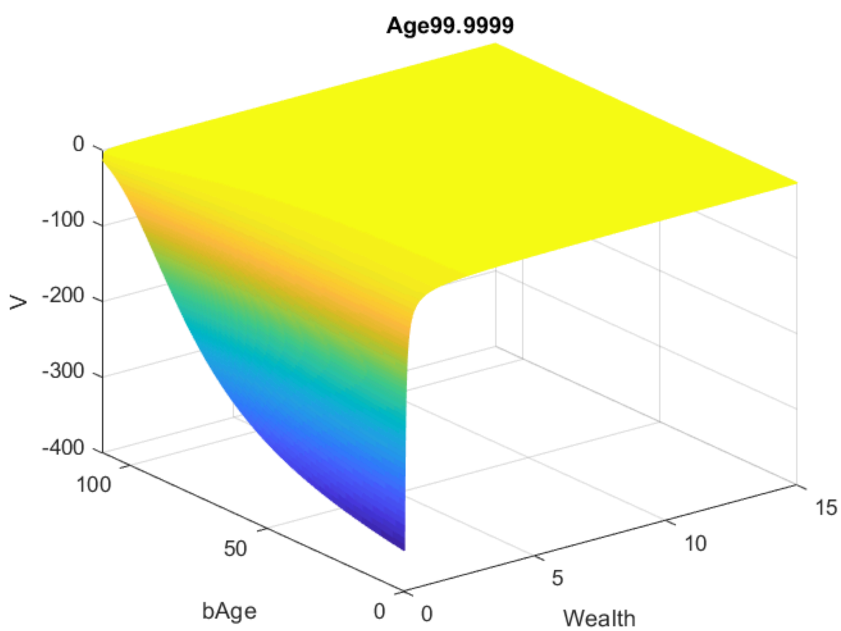


Figure 3.30: Value Function for Optimal Retirement decision at age 99.999.
Assuming γ=8, $\sigma = 0.3, \bar{l} = 7, bonus = 1/52, W = 15, \alpha = 0.5$, Forced Retirement age= 100

Chapter 4

Stochastic Mortality Rate with Exogenous Pension Income

4.1 Problem

Once again consider a consumer with chronological age (C-age) x. At any time t his C-age is κ_t and his biological age (B-age) is A_t. The consumer is earning a labor income of \$1 annually and will invest his wealth at a risk-free rate r with no bequest motives. He is interested in determining the optimal B-age a^* and wealth w^* to retire at an optimal time t^* while having an optimal consumption plan for his entire lifetime.

4.2 Mathematical Setup

The mortality law is based on biological age as discussed in chapter 3 section 3.2 and T is a time beyond which aging ceases and the mortality plateaus [6]. Recall that in chapter 2 section 2.2 equation 2.6 the utility of consumer at any time $0 \leq t \leq \infty$ was given as

$$u(c_t, l_t) = \begin{cases} u_1 = \frac{c_t^{1-\gamma}}{1-\gamma} & \text{, pre-retirement} \\ \overline{u} = B^{\gamma} \frac{c_t^{1-\gamma}}{1-\gamma} & \text{, post- retirement} \end{cases} \tag{4.1}$$

where $B \equiv \bar{l}^{\,\tilde{\gamma}(1-\alpha)(1-\gamma^*)}$. That is, the consumer gets to choose a time to retire, and earn a utility of $u_1(c_t)$ for pre-retirement consumption and $B^{\gamma} u_1(c_t)$ for post-retirement consumption. I will take the risk aversion parameter $\gamma > 1$ which makes $u_1(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ negative. Therefore, I also take $0 < B < 1$, which means that a consumption of c_t earns more utility post-retirement then pre-retirement. Let $\overline{V^{\pi}}(t, w, a)$ be the post-retirement value function and $V_1^{\pi}(t, w, a)$ be the pre-retirement value function.

4.2.1 POST-RETIREMENT ($t \geq t^*$)

Let the consumer be entitled to an exogenous pension income of π times the labor income, annually, after retirement, where $0 < \pi < 1$. Then the wealth dynamics for post-retirement are

$$dw_t = (\pi + rw_t - c_t)dt \tag{4.2}$$

Consider the expected total benefit given in equation 3.18 under stochastic mortality in the post-retirement case.

$$Z_t = \int_0^t e^{-\int_0^s (\rho+\lambda_q)dq} u(c_s, l_s)ds + e^{-\int_0^t (\rho+\lambda_q)dq}\overline{V^\pi}(t, w_t, l, A_t)$$

Applying Ito's lemma, substituting for wealth dynamics and B-age dynamics, and setting drift equal to zero, gives

$$\sup_c \Bigg[\overline{u}(c,\bar{l}) - (\rho + \lambda(a))\overline{V^\pi} + \overline{V^\pi}_t + \overline{V^\pi}_w(\pi + rw - c) + \\ \overline{V^\pi}_a\left[\left(1 + \xi\frac{\kappa_t - a}{T-t}\right)\right] + \frac{\sigma^2}{2}\overline{V^\pi}_{aa}\Bigg] = 0 \tag{4.3}$$

Recall that, the optimal post-retirement consumption is $c = B(\overline{V^\pi}_w)^{-\tilde{\gamma}}$ and the optimization of c is developed as a convex dual of $\overline{u}(c,\bar{l})$ given in equation 2.18

$$\sup_c\{\overline{u}(c) - c\overline{V^\pi}_w\} = \tilde{\overline{u}}(\overline{V^\pi}_w) = -\frac{B(\overline{V^\pi}_w)^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} \tag{4.4}$$

The optimal retirement boundary $\tau(t^*, w^*, a^*); 0 \leq t^* \leq T, 0 \leq w^* \leq W, 0 \leq a^* \leq \kappa_T$ is found using a numerical scheme. This optimal choice of retirement is a complex balance of a consumer's age, health, savings and consumption patterns as well as the utility both post-retirement and pre-retirement. Hence, for an optimal choice of t^* the HJB for $t < T$ is:

$$\overline{V^\pi}_t + (\pi + rw)\overline{V^\pi}_w - \frac{B}{1-\tilde{\gamma}}\overline{V^\pi}_w{}^{1-\tilde{\gamma}} \\ -(\rho + \lambda(a))\overline{V^\pi} + \left[\left(1 + \xi\frac{\kappa_t - a}{T-t}\right)\right]\overline{V^\pi}_a + \frac{\sigma^2}{2}\overline{V^\pi}_{aa} = 0 \tag{4.5}$$

Boundary Condition at wealth $w = 0$

Post-retirement consumption should not rise over time as mortality adds up to the discounting parameter ρ. That means, discounting is done at a rate higher than the risk free rate (as I am considering $\rho = r$). So a good choice of consumption at $w = 0$ is $c_t = \pi$, which is sustainable in the sense that the consumer can consume at this rate forever. The only reason for not consuming as much as you are allowed to now (i.e. π, when $w = 0$) is if you want to build up wealth in order to consume extra later. And the higher discounting makes that sub-optimal.

Hence, it is optimal to consume at the rate $c_t = \pi$ when wealth is zero. This will keep the wealth constant at $w = 0$ (see the wealth dynamics for post-retirement, I get $dw_t/dt = 0$). In other words, we won't forego consumption today in order to consume later at a higher rate! This will help me derive the boundary condition for my ODE at $w = 0$. I am expecting that once a person hits the zero wealth level $w = 0$, he consumes at the exogenous pension level π . This gives a constant post-retirement utility of $\overline{u}(c_t, \bar{l}) = B^{\gamma}\frac{\pi^{1-\gamma}}{1-\gamma}$. At $w = 0$ and $c_t = \pi$ the wealth dynamics 4.2 become $dw = 0$ which makes the wealth stay constant at $w = 0$. Hence, the wealth derivative terms from Ito expansion of Z_t vanish. Therefore, the HJB for $\overline{V^{\pi}}(t, a)$ at the boundary $w = 0$ for all $t < T$ is given by,

$$\overline{V^{\pi}}_t + \left[\left(1 + \xi\frac{\kappa_t - a}{T - t}\right)\right]\overline{V^{\pi}}_a + \frac{\sigma^2}{2}\overline{V^{\pi}}_{aa} - (\rho + \lambda(a))\overline{V^{\pi}} + B^{\gamma}\frac{\pi^{1-\gamma}}{1-\gamma} = 0 \tag{4.6}$$

The boundary condition for this PDE are: at $t = T$; $\overline{V^{\pi}} = \overline{V^{\pi}_T}(w = 0)$, a reflecting boundary at b-age $a = 0$; $\overline{V^{\pi}}_a = 0$ and an absorbing boundary at maximum b-age $a = \kappa_T$; $\overline{V^{\pi}} = \overline{V^{\pi}_T}(w = 0)$, where $\overline{V^{\pi}_T}$ is the solution of ODE 4.7 below.

Boundary Condition at time $t = T$

For all times $t \geq T$ I am assuming that the hazard rate λ_T is constant which makes the problem time invariant, that is, $\overline{V^{\pi}}_t = 0$ and the aging stops which gives $\overline{V^{\pi}}_a = 0 = \overline{V^{\pi}}_{aa}$. Hence, the value function is dependent on the wealth variable only. Call $\overline{V^{\pi}}(T, w, \bar{l}) \equiv \overline{V^{\pi}_T}(w)$. This reduces the PDE 4.5 to an ODE:

$$(\pi + rw)(\overline{V^{\pi}_T})_w - \frac{B}{1-\tilde{\gamma}}[(\overline{V^{\pi}_T})_w]^{1-\tilde{\gamma}} - (\rho + \lambda_T)\overline{V^{\pi}_T} = 0, 0 \leq w \leq W \tag{4.7}$$

For this ODE, I need a boundary condition at $w = 0$ as discussed above. For post-retirement, I am optimizing the total value

$$\overline{V^{\pi}_T}(w) = \int_0^{\infty} e^{-(\rho+\lambda_T)t}\overline{u}(c_t)dt = \int_0^{\infty} e^{-(\rho+\lambda_T)t}B^{\gamma}u_1(c_t)dt \tag{4.8}$$

At $w = 0$ I have $c_t = \pi$ and

$$\begin{aligned}
\overline{V^{\pi}_T}(0) &= \int_0^{\infty} e^{-(\rho+\lambda_T)t}B^{\gamma}u_1(\pi)dt \\
&= \int_0^{\infty} e^{-(\rho+\lambda_T)t}B^{\gamma}\frac{\pi^{1-\gamma}}{1-\gamma}dt \\
&= B^{\gamma}\frac{\pi^{1-\gamma}}{1-\gamma}\left[\frac{e^{-(\rho+\lambda_T)t}}{-(\rho+\lambda_T)}\right]_0^{\infty} \\
&= B^{\gamma}\frac{\pi^{1-\gamma}}{1-\gamma}\left[\frac{1}{(\rho+\lambda_T)}\right]
\end{aligned} \tag{4.9}$$

Therefore the boundary condition for the PDE 4.7 is $\overline{V_T^\pi}(0) = \frac{B^\gamma \pi^{1-\gamma}}{(1-\gamma)(\rho+\lambda_T)}$. If $z \equiv (\overline{V_T^\pi})_w$ and the function $H(z) = (\pi + rw)z - \frac{B}{1-\tilde{\gamma}} z^{1-\tilde{\gamma}}$, then I can write the ODE (4.7) as a pure algebraic equation

$$H(z) = (\rho + \lambda_T)\overline{V_T^\pi} \tag{4.10}$$

When defining the function H, I am not showing it's dependence on w. When I need the dependence of H on wealth, I'll write $H(w, z)$. I need to analyze this equation for given values of $\overline{V_T^\pi}$(and w). For the function $H(z) = (\pi + rw)z - \frac{B}{1-\tilde{\gamma}} z^{1-\tilde{\gamma}}$, I have $H'(z) = (\pi + rw) - Bz^{-\tilde{\gamma}}$. Setting this equals to zero gives the critical point $\hat{z} = \left[\frac{\pi+rw}{B}\right]^{-\gamma}$. It's a nice concave function, hence, the minima for $H(z)$ is $H_0 = H(\hat{z}) = \frac{B^\gamma}{(1-\gamma)}(\pi + rw)^{1-\gamma}$. The zeros of $H(z)$ are $z_{LHS} = 0$ and $z_{RHS} = \left[\frac{(\pi+rw)(1-\tilde{\gamma})}{B}\right]^{-\gamma}$.

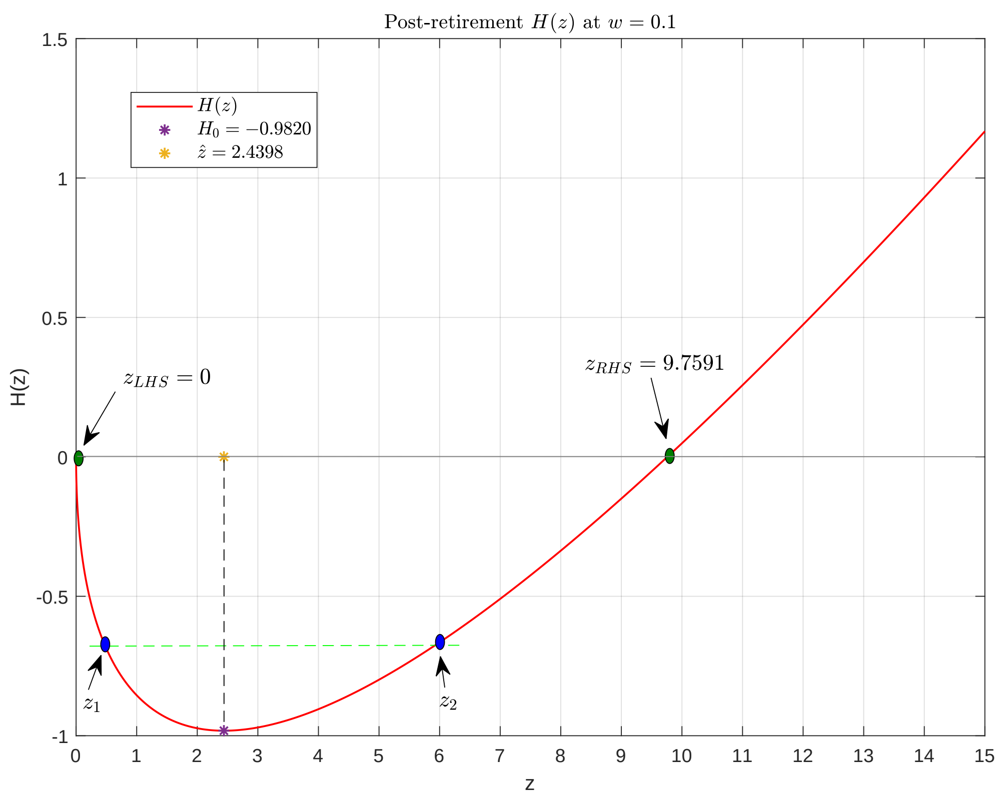


Figure 4.1: Example for function $H(z)$ at $w = 0.1, \pi = 40\%, \gamma = 2, r = 2.5\%, \rho = r, \bar{l} = 3$

If I look at the plot for $H(z)$, when $H(z) > 0$ there is 1 solution z to the algebraic equation (4.10) when $\overline{V_T^\pi} > 0$ (using horizontal line test). Since, $\gamma > 1$ so $\overline{V_T^\pi}$ is a negative quantity, therefore, $H(z) > 0$ solution to the algebraic equation doesn't concern me. If $H(z) < H_0$ there are no solutions for the inequality $(\rho + \lambda_T)\overline{V_T^\pi} < H_0$, and if $H_0 < H(z) \leq 0$ there are 2 solutions for $H_0 < (\rho + \lambda_T)\overline{V_T^\pi} \leq 0$. Call them z_1 and z_2; $z_1 < \hat{z}$ and $z_2 > \hat{z}$.

Now the question is, which solution to the equation 4.10 I want. Consider the function

$$H(w,z) = (\pi + rw)z - \frac{B}{1-\tilde{\gamma}} z^{1-\tilde{\gamma}}$$

Then the algebraic equation is $H(w,z) = (\rho + \lambda_T)\overline{V_T^\pi}$. Taking $\frac{d}{dw}$ on both sides give

$$\frac{dH(w,z)}{dw} = (\rho + \lambda_T)(\overline{V_T^\pi})_w$$

$$H_w + H_z(\overline{V_T^\pi})_{ww} = (\rho + \lambda_T)(\overline{V_T^\pi})_w$$

Since, $H_w = r(\overline{V_T^\pi})_w$, therefore,

$$r(\overline{V_T^\pi})_w + H_z(\overline{V_T^\pi})_{ww} = (\rho + \lambda_T)(\overline{V_T^\pi})_w$$

I am considering discounting at the risk free rate $\rho = r$, therefore, the r and ρ terms from both sides of the equation cancel out and I am left with

$$H_z(\overline{V_T^\pi})_{ww} = \lambda_T(\overline{V_T^\pi})_w$$

When I increase w, consumption $c = B[(\overline{V_T^\pi})_w]^{-\tilde{\gamma}}$ would increase, so $(\overline{V_T^\pi})_w$ should decrease. In other words, $(\overline{V_T^\pi})_w$ being a decreasing function gives $(\overline{V_T^\pi})_{ww} < 0$. Since, $\overline{V_T^\pi}$ is an increasing function of wealth w, then $(\overline{V_T^\pi})_w > 0$ which implies that $H_z(\overline{V_T^\pi})_{ww} = \lambda_T \overline{V_T^\pi}_w > 0$. The inequality on the right hand side forces $H_z(w,z) < 0$ because $(\overline{V_T^\pi})_{ww} < 0$. Hence, $H(z)$ is a decreasing function. For my choice of parameters I want H to be on decreasing side, that is I refer z_1 the smaller of the two solutions to the algebraic equation. That establishes the post-retirement HJB at $t = T$, which is now easy to solve numerically.

More generally for any wealth $w \in [0, W]$, the consumption of $\hat{c} = \pi + rw$ keeps wealth constant over time ($\frac{dw}{dt} = 0$ from the post-retirement wealth dynamics). This strategy gives a utility of

$$\begin{aligned}
\overline{V_T^\pi}\Big|_{c=\hat{c}} &= \int_0^\infty e^{-(\rho+\lambda_T)t} B^\gamma u_1(\pi + rw) dt \\
&= \int_0^\infty e^{-(\rho+\lambda_T)t} B^\gamma \frac{(\pi + rw)^{1-\gamma}}{1-\gamma} dt \\
&= B^\gamma \frac{(\pi + rw)^{1-\gamma}}{1-\gamma} \left[\frac{e^{-(\rho+\lambda_T)t}}{-(\rho+\lambda_T)} \right]_0^\infty \\
&= B^\gamma \frac{(\pi + rw)^{1-\gamma}}{1-\gamma} \left[\frac{1}{(\rho+\lambda_T)} \right] = \frac{H_0}{\rho + \lambda_T}
\end{aligned} \tag{4.11}$$

If I take the smaller solution $z_1 < \hat{z}$ then, $B{z_1}^{-\tilde{\gamma}} > B\hat{z}^{-\tilde{\gamma}}$ and $c(z_1) > c(\hat{z})$. Consumption will exceed this subsistence level making wealth decline over time.

Boundary Condition for B-Age

The minimum b-age $a = 0$ is a reflecting boundary, that is, $\overline{V^\pi}_a(t, w, 0) = 0$ while at maximum b-age $a = \kappa_T$ there is an absorbing boundary with $\overline{V^\pi}(t, w, \kappa_T) = \overline{V_T^\pi}$ discussed above.

4.2.2 PRE-RETIREMENT $(t < t^*)$

For the optimal retirement boundary $\tau(t^*, w^*, a^*)$; a consumer is in the pre-retirement phase of life for $0 \le t < t^*; 0 \le t^* \le T$. He earns a labor income of $1 annually until it is optimal for him to retire. The exogenous pension pre-retirement problem is same as the stochastic mortality case-no pension pre-retirement problem discussed in chapter 3 section 3.2.1. The optimal consumption is $c_t = [(V_1^\pi)_w]^{-\tilde{\gamma}}$ and the HJB equation is as discussed in chapter 3 section 3.2.1 equation 3.13:

$$
\begin{aligned}
(V_1^\pi)_t + &(1 + rw)(V_1^\pi)_w - \frac{{(V_1^\pi)_w}^{1-\tilde{\gamma}}}{1 - \tilde{\gamma}} \\
&+ \left[(1 + \xi \frac{\kappa_t - a}{T - t})\right](V_1^\pi)_a + \frac{\sigma^2}{2}(V_1^\pi)_{aa} - (\rho + \lambda(a))V_1^\pi = 0
\end{aligned}
\tag{4.12}
$$

But, the set of boundary conditions are different than what I had in section 3.2.1.

Boundary Condition at time $t = T$

Just like the post-retirement problem, at time T the mortality rate is constant λ_T. This results in time invariance for all times $t \ge T$; $(V_1^\pi)_t = 0$ and the aging stops resulting in $(V_1^\pi)_a = 0 = (V_1^\pi)_{aa}$. Call $V_1^\pi(T, w) \equiv V_T^\pi(w)$. This reduces the PDE 4.12 to an ODE:

$$
(1 + rw)V_T^\pi - \frac{1}{1 - \tilde{\gamma}}[V_T^\pi]^{1-\tilde{\gamma}} - (\rho + \lambda_T)V_T^\pi = 0, 0 \le w \le W \tag{4.13}
$$

Consider a wealth level $\overline{w}$ at which a person at c-age κ_T or a person at maximum b-age would retire. Then, his pre-retirement utility V_T^π should agree with the post-retirement utility $\overline{V_T^\pi}$ above this wealth $\overline{w}$, that is,

$$
V_T^\pi(w) = \overline{V_T^\pi}(w); \forall w \ge \overline{w}
$$

Below the wealth $w < \overline{w}$ I need to look for the solution of ODE 4.13. A closer look at the ODE shows that the algebraic equation (4.10) mentioned in section 4.2.1 also holds with $\overline{V_T^\pi}$ replaced by V_T^π and $H(z)$ replaced by $S(\nu) = (1 + rw)\nu - \frac{1}{1-\tilde{\gamma}}\nu^{1-\tilde{\gamma}}$. Now, the variable $\nu \equiv \frac{dV_T^\pi}{dw}$ and the algebraic equation for which I am seeking a solution is

$$
S(\nu) = (\rho + \lambda_T)V_T^\pi \tag{4.14}
$$

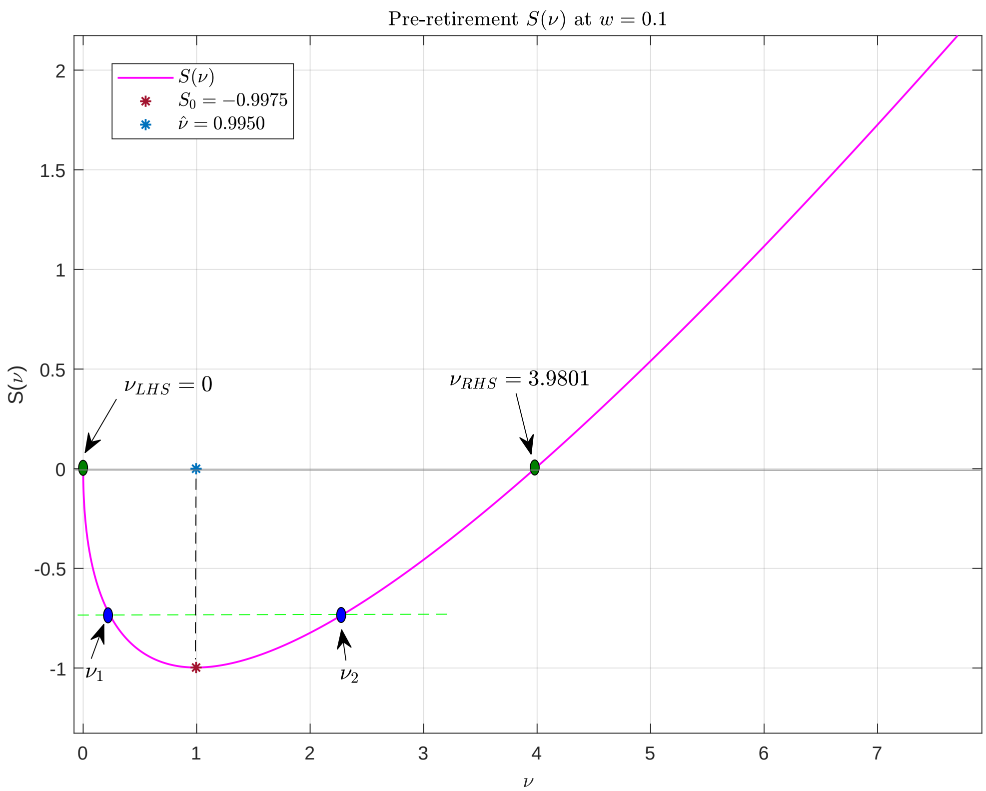


Figure 4.2: Example for function $S(\nu)$ at $w = 0.1, \pi = 40\%, \gamma = 2, r = 2.5\%, \rho = r, \bar{l} = 3$

Using the same arguments as discussed for the post-retirement equation 4.10, function S has a critical point at $\hat{\nu} = (1 + rw)^{-\gamma}$ and minima is $S_0 = \frac{(1+rw)^{1-\gamma}}{1-\gamma}$. The zeros for $S(\nu)$ are $\nu_{LHS} = 0$ and $\nu_{RHS} = ((1 + rw)(1 - \tilde{\gamma}))^{-\gamma}$. The subsistence consumption level is $\tilde{c} = 1 + rw$. There are 2 possible solutions to the algebraic equation 4.14 in an interval $(S_0, 0]$; ν_1 and ν_2. The smaller solution $\nu_1 < \hat{\nu}$ corresponds to the pre-retirement consumption rising above the subsistence level $c(\nu_1) > \tilde{c}$ resulting in a declining wealth over time (the pre-retirement wealth dynamics give $dw/dt < 0$). The larger solution $\nu_2 > \hat{\nu}$ results in a $c(\nu_2) < \tilde{c}$ and rising wealth.

Let the pre-retirement values $V^{\pi}_{T\,1}$ and $V^{\pi}_{T\,2}$ be the solutions for ODE 4.13 corresponding to ν_1 and ν_2, respectively. Unlike the post-retirement case, it is no longer clear (and in fact not always true) that the optimal consumption c always increase with w and hence, I cannot pinpoint the solution to the equation 4.14 I always need. I'll be working with both solutions as per the need of pre-retirement optimal consumption.

Therefore, consider both solutions $V^{\pi}_{T\,1}(w)$ and $V^{\pi}_{T\,2}(w)$. Since both have different slopes ν_1 and ν_2, they will definitely intersect at some point. Let both solutions cross at a wealth w^*. The maximum of two solutions; $max(V^{\pi}_{T\,1}, V^{\pi}_{T\,2})$ will have a lower $(V^{\pi}_{T})_w$ (i.e. the smaller solution ν_1 to 4.14) and so a higher consumption to the left of w^* and higher $(V^{\pi}_{T})_w$ (the larger solution ν_2 to 4.14) resulting in a lower c to the right of w^*. In other words, on the curve for maximum of two solutions, with increasing wealth there is a possibility of switching transversely from high consumption to low consumption but not vice versa. That means, this transverse crossing can occur only at one point, so, there can be at most one such wealth w^*. Having wealth w^* one can consume at a higher rate so as to lower wealth, in which case optimality of the smaller solution $V^{\pi}_{T\,1}$shows the supermartingale property. Or one can choose a lower consumption rate to raise wealth, in which case supermartingale property follows from optimality of the larger solution $V^{\pi}_{T\,2}$. Thus such transverse crossings can be optimal. This is in contrast to the problems where there is noise in wealth which drives the wealth in both directions across w^*, forcing smooth pasting to hold.

Of course I do know that (V^{π}_{T}) is continuous, as otherwise I could use either a small consumption c or a large c to make the supermartingale jump up. As remarked above, if there was noise in this problem I could rely on smooth pasting to give me that $(V^{\pi}_{T})_w$ is continuous everywhere. But in fact, I have seen that there is the potential for there to be a $w^* < \overline{w}$ (but only one such w^* exists) at which $(V^{\pi}_{T})_w$ fails to be continuous.

Now, let's have a look at (V^{π}_{T}) near $\overline{w}$. At retirement wealth $w = \overline{w}$ post-retirement and pre-retirement values match; $\overline{V^{\pi}_{T}}(\overline{w}) = V^{\pi}_{T}(\overline{w})$. If $(V^{\pi}_{T})_w$ is continuous around this region and smooth pasting is valid then $(\overline{V^{\pi}_{T}})_w(\overline{w}) = (V^{\pi}_{T})_w(\overline{w})$. That is, at $w = \overline{w}$, the post-retirement ODE 4.7 and pre-retirement ODE 4.13 give

$$(\pi + r\overline{w})(\overline{V^{\pi}_{T}})_w\Big|_{w=\overline{w}} - \frac{B}{1-\tilde{\gamma}}[(\overline{V^{\pi}_{T}})_w]^{1-\tilde{\gamma}}\Big|_{w=\overline{w}} - (\rho + \lambda_T)\overline{V^{\pi}_{T}}(\overline{w}) = 0$$

$$(1 + r\overline{w})(V^{\pi}_{T})_w\Big|_{w=\overline{w}} - \frac{1}{1-\tilde{\gamma}}[(V^{\pi}_{T})_w]^{1-\tilde{\gamma}}\Big|_{w=\overline{w}} - (\rho + \lambda_T)V^{\pi}_{T}(\overline{w}) = 0$$

Subtracting the equations result in

$$(\overline{V^{\pi}_{T}})_w\Big|_{w=\overline{w}}\left[(1-\pi) - \frac{1}{1-\tilde{\gamma}}(1-B)[(\overline{V^{\pi}_{T}})_w]^{-\tilde{\gamma}}\Big|_{w=\overline{w}}\right] = 0$$

For my choice of parameters $0 < B < 1, \pi < 1, 0 < \gamma < 1$ I have $(\overline{V^{\pi}_{T}})_w \neq 0$ (unless the consumer consumes at an infinite rate, which is not possible). Therefore,

$$(\overline{V^{\pi}_{T}})_w\Big|_{w=\overline{w}} = \nu_{\overline{w}} = \left[\frac{(1-\pi)(1-\tilde{\gamma})}{1-B}\right]^{-\gamma} \tag{4.15}$$

My plan is to find $(\overline{V_T^\pi})_w$ numerically via algebraic equation 4.15. This lets me find a wealth w° numerically for which $\nu_{\overline{w}} = (\overline{V_T^\pi})_w(w^\circ)$ (or $w^\circ = 0$ if there is no such value). In fact, a careful examination of the argument shows that the pre-retirement supermartingale property holds for the post-retirement function $\overline{V_T^\pi}$ to the right of w°. In the case of smooth pasting at $\overline{w}$, I know that $\overline{w} = w^\circ$.

This will in fact be the case if immediately to the left of $\overline{w}$, the solution V_T^π is using a consumption c below subsistence $\tilde{c}$. To see this, suppose not. Then if we continue to use this same c but (suboptimally) don't retire upon reaching level $\overline{w}$, then wealth will continue to increase and the transverse crossing will give an extra push to the supermartingale process that forces it to increase. This is impossible. This leaves open the possibility of a transverse crossing at $\overline{w}$, provided the solution to the left of $\overline{w}$ uses a c above subsistence. In that case however, I would need $w^\circ \leq \overline{w}$ in order to obtain the supermartingale property.

To summarize, the following possibilities remain for the pre-retirement utility V_T^π. I will show by example that any of them can in fact occur. Consider the notation $\tilde{V_T^\pi}$ (blue) for the solution to post-retirement HJB at large T but with $B = 1$ and $\pi = 1$. ${V_T^\pi}_2$ (green) is the pre-retirement HJB solution corresponding to larger ν_2. $\overline{V_T^\pi}$ (black) is the post-retirement ODE solution. Let $\tilde{w}$ be the wealth value at which $\overline{V_T^\pi}$ and $\tilde{V_T^\pi}$ cross (and $\tilde{w} = 0$ if $\overline{V_T} > \tilde{V}_T$ everywhere). w^* is the wealth where $\tilde{V_T^\pi}$ and ${V_T^\pi}_2$ meet (remember that for post-retirement ODE I am considering the solution to smaller $z_1 < \hat{z}$. Therefore $\tilde{V_T^\pi}$ corresponds to the smaller slope z_1 and ${V_T^\pi}_2$ has larger slope z_2, so the solutions will intersect) . At w°, smooth pasting holds, that is, $\overline{V}_T^\pi$ and ${V_T^\pi}_2$ intersect smoothly. The pre-retirement solution V_T^π is the maximum of $\tilde{V_T^\pi}, {V_T^\pi}_2$ and $\overline{V}_T^\pi$ over the wealth domain $[0, W]$. The retirement wealth $\overline{w}$ is maximum of $\tilde{w}, w^\circ$.

- CASE I $\tilde{w} = 0 = w^\circ$. That is, $\overline{w} = 0$. In this case $V_T^\pi = \overline{V_T^\pi}$ and it is optimal to retire at any wealth. See figure 4.3.
- CASE II $w^\circ > \tilde{w}$. In this case, $\overline{w} = w^\circ$ and smooth pasting holds, with V_T^π is equal to the larger solution ${V_T^\pi}_2$ to the pre-retirement HJB to the left of $\overline{w}$. In this situation two things may occur:
    1. Beginning at $w = w^\circ$ the pre-retirement solution may extend all the way to $w = 0$ without crossing $\tilde{V_T^\pi}$. With pre-retirement consumption below subsistence level, we always either retire or save for retirement. See figure 4.4.
    2. That ${V_T^\pi}_2$ solution crosses $\tilde{V_T^\pi}$ at a point $w^* > 0$. In that case, $V_T^\pi = \tilde{V_T^\pi}$ to the left of w^*. To the left of w^* we consume above subsistence level and give up on retirement, but between w^* and $\overline{w}$ we save for retirement (with consumption below subsistence level). See figure 4.5.
- CASE III $w^\circ \leq \tilde{w}$ and $\tilde{w} > 0$. In this case $\overline{w} = \tilde{w}$ and $V_T^\pi = \tilde{V_T^\pi}$ to the left of $\overline{w}$. We either retire immediately, or we give up on retirement. See figure 4.6.

Note that there is an additional case $\tilde{w} = 0 = w^\circ$ (CASE I) in this chapter which was not present in the no-pension case section 3.2.1. Because in the no-pension case, there is always some wealth below which we do not retire.

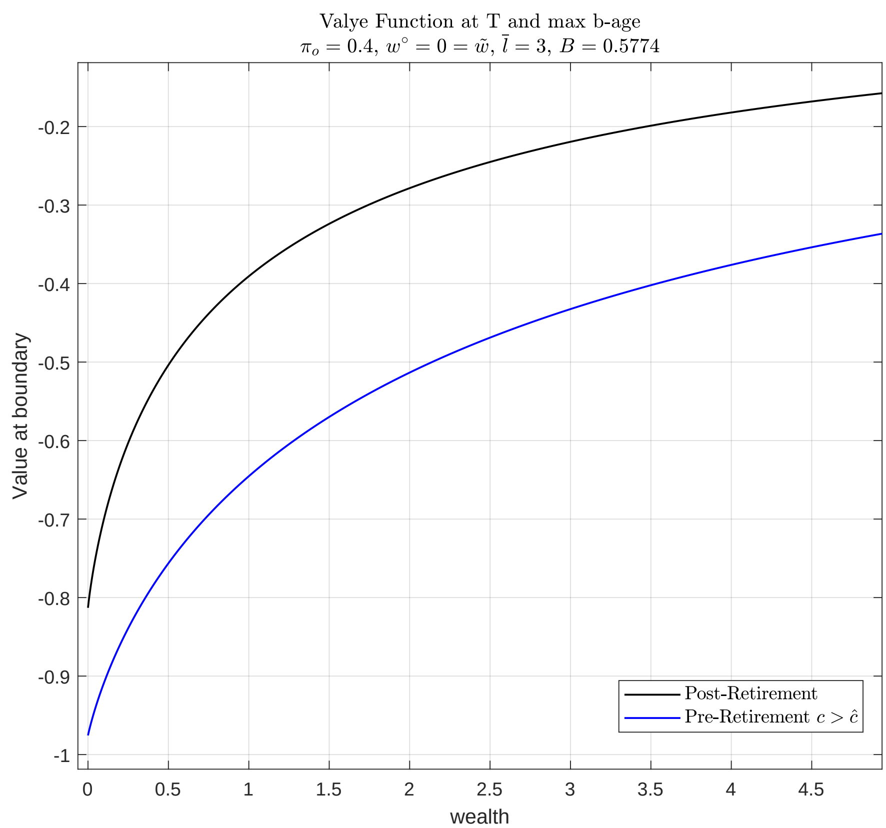


Figure 4.3: Example for Case I for Value function at time T and maximum b-age κ_T. Assuming $\pi = 40\%, \gamma = 2, \sigma = 0.5, r = 2.5\%, \bar{l} = 3$

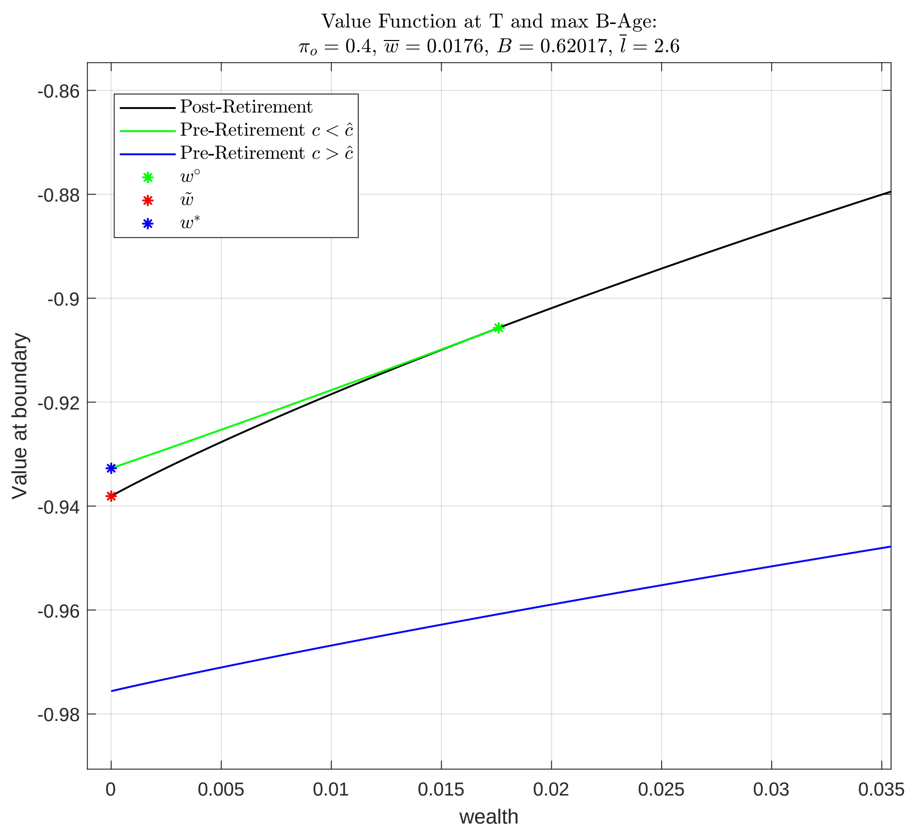


Figure 4.4: Example for Case II-1 for Value function at time T and maximum b-age κ_T. Assuming $\pi = 40\%, \gamma = 2, \sigma = 1, r = 2.5\%, \bar{l} = 2.6$

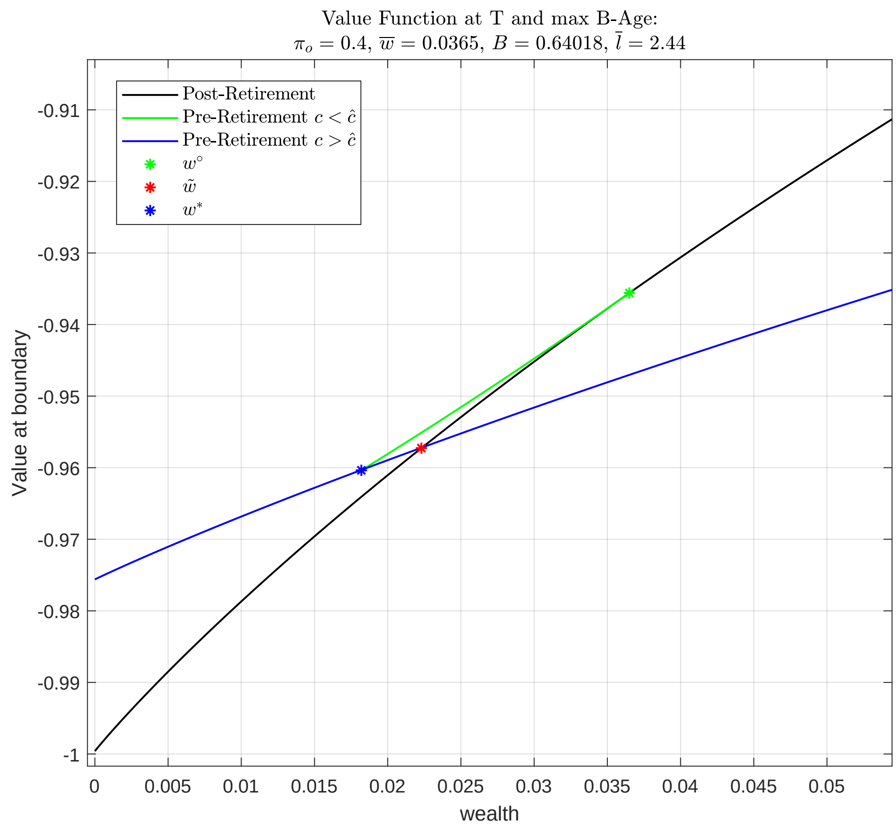


Figure 4.5: Example for Case II-2 for Value function at time T and maximum b-age κ_T. Assuming $\pi = 40\%, \gamma = 2, \sigma = 0.5, r = 2.5\%, \bar{l} = 2.44$

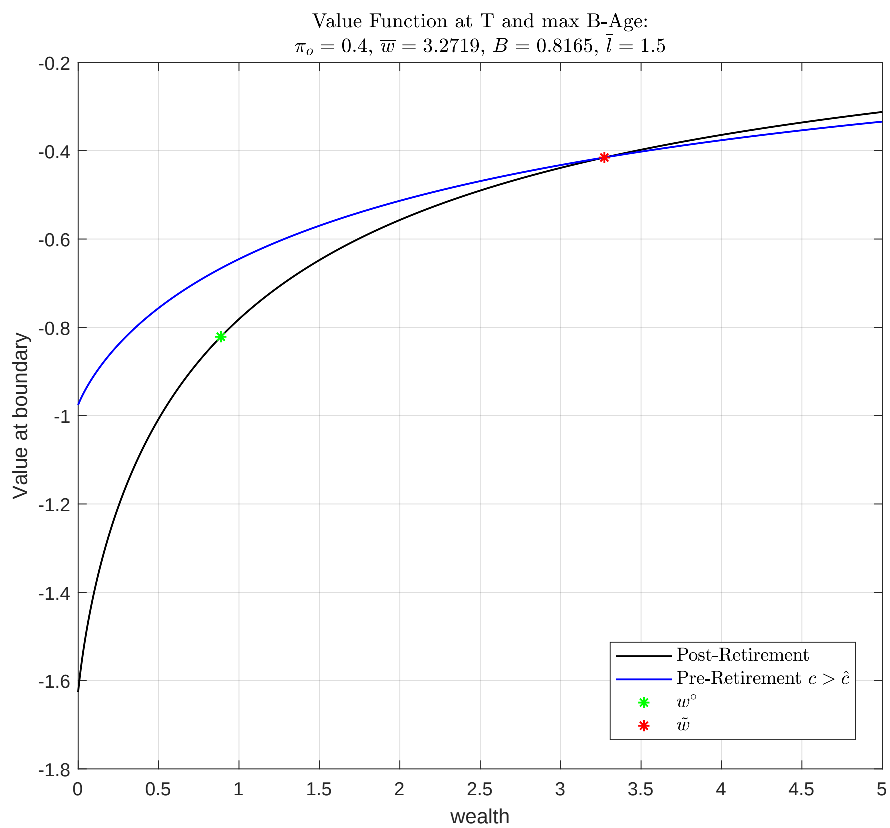


Figure 4.6: Example for Case III for Value function at time T and maximum b-age κ_T. Assuming $\pi = 40\%, \gamma = 2, \sigma = 1, r = 2.5\%, \bar{l} = 1.5$

In particular, I can determine the circumstances under which it is always optimal to retire. That is, I want $\overline{w} = max(\tilde{w}, w^{\circ}) = 0$. To have $\tilde{w} = 0$ I need $\tilde{V}_T^{\pi}$ and $\overline{V_T^{\pi}}$ intersect at $w = 0$ or $\tilde{V}_T^{\pi} < \overline{V_T^{\pi}}$, that is, the post-retirement utility is always higher and preferable as compared to the pre-retirement utility. From 4.9 $\overline{V_T^{\pi}}(0) = \frac{B^{\gamma}\pi^{1-\gamma}}{(1-\gamma)(\rho+\lambda_T)}$. $\tilde{V}_T^{\pi}$ is the post-retirement solution at T for $B = 1 = \pi$. Therefore, $\tilde{V}_T^{\pi}(0) = \frac{1}{(1-\gamma)(\rho+\lambda_T)}$. Equating both values at $w = 0$ results in

$$\frac{B^{\gamma}\pi^{1-\gamma}}{(1-\gamma)(\rho+\lambda_T)} \geq \frac{1}{(1-\gamma)(\rho+\lambda_T)}$$

For $\gamma > 1$, the condition for $\tilde{w} = 0$

$$B^{\gamma}\pi^{1-\gamma} \leq 1 \tag{4.16}$$

To have $w^{\circ} = 0$ consider post-retirement consumption $c|_{w=0} = \pi$. Also,

$$\begin{aligned} c &= B[(\overline{V_T^{\pi}})_w]^{-\tilde{\gamma}}; \forall w \\ \implies \pi &= B[(\overline{V_T^{\pi}})_w]^{-\tilde{\gamma}}|_{w=0} \\ \implies (\overline{V_T^{\pi}})_w\Big|_{w=0} &= \left(\frac{\pi}{B}\right)^{-\gamma} \end{aligned}$$

I know that post-retirement consumption rises with increasing wealth. That is, $c \geq \pi$ for all $w \geq 0$. This implies that

$$\begin{aligned} B[(\overline{V_T^{\pi}})_w]^{-\tilde{\gamma}}\Big|_{w\geq 0} &\geq B[(\overline{V_T^{\pi}})_w]^{-\tilde{\gamma}}\Big|_{w=0} \\ B[(\overline{V_T^{\pi}})_w]^{\tilde{\gamma}}\Big|_{w\geq 0} &\leq B[(\overline{V_T^{\pi}})_w]^{\tilde{\gamma}}\Big|_{w=0} \\ (\overline{V_T^{\pi}})_w\Big|_{w\geq 0} &\leq (\overline{V_T^{\pi}})_w\Big|_{w=0} \\ \implies \quad (\overline{V_T^{\pi}})_w &\leq \left(\frac{\pi}{B}\right)^{-\gamma} \end{aligned}$$

for all $w \geq 0$. From equation 4.15 I know that at $w = w^{\circ}$

$$(\overline{V_T^{\pi}})_w\Big|_{w=w^{\circ}} = \left[\frac{(1-\pi)(1-\tilde{\gamma})}{1-B}\right]^{-\gamma}$$

Remember that if smooth pasting holds, $\overline{w} = w^{\circ}$. Now I have

$$\begin{aligned} &\left[\frac{(1-\pi)(1-\tilde{\gamma})}{1-B}\right]^{-\gamma} \leq \left(\frac{\pi}{B}\right)^{-\gamma} \\ \implies &\left[\frac{1-B}{(1-\pi)(1-\tilde{\gamma})}\right]^{\gamma} \leq \left(\frac{B}{\pi}\right)^{\gamma} \\ \implies &\left(\frac{1-B}{B}\right) \leq \left[\frac{(1-\pi)(1-\tilde{\gamma})}{\pi}\right] \\ \implies &\frac{1}{B} - 1 \leq \left(\frac{1}{\pi} - 1\right)\left(1 - \frac{1}{\gamma}\right) \end{aligned}$$

$$\Longrightarrow 1-\frac{1}{B} \geq \left(1-\frac{1}{\pi}\right)\left(1-\frac{1}{\gamma}\right) \tag{4.17}$$

So for it to be always optimal to retire, I require that both of these conditions 4.16 and 4.17 hold. But in fact, 4.17 implies 4.16. So all I need do is verify the 2nd condition 4.17. To see this, rewrite the 1st condition as $B\pi^{\frac{1}{\gamma}-1} \leq 1$. Or, $\left(\frac{1}{\pi}\right)^{1-\frac{1}{\gamma}} \leq \frac{1}{B}$. From 4.17

$$1-\left(1-\frac{1}{\pi}\right)\left(1-\frac{1}{\gamma}\right) \geq \frac{1}{B}$$

So I'll be done if I show that

$$1+\left(\frac{1}{\pi}-1\right)\left(1-\frac{1}{\gamma}\right) \geq \left(\frac{1}{\pi}\right)^{1-\frac{1}{\gamma}}$$

To see this, consider the function $x \mapsto x^{1-\frac{1}{\gamma}}$ which is a power function with $0 < 1-\frac{1}{\gamma} < 1$. Hence, it is a nice concave function. The tangent line to this function at $x = 1$ is $1+(x-1)\left(1-\frac{1}{\gamma}\right)$. And as a consequence of concavity I have $1+(x-1)\left(1-\frac{1}{\gamma}\right) > x^{1-\frac{1}{\gamma}}$. That is, this function's graph does indeed lie below the tangent line. Therefore, at $x = \frac{1}{\pi}$

$$1+\left(\frac{1}{\pi}-1\right)\left(1-\frac{1}{\gamma}\right) \geq \left(\frac{1}{\pi}\right)^{1-\frac{1}{\gamma}}$$

and the 2nd condition is verified.

Boundary Conditions for B-Age

The minimum b-age $a = 0$ is a reflecting boundary, that is, $(V_1^\pi)_a(t, w, 0) = 0; 0 \leq t < t^*, 0 \leq w < W$ while at maximum b-age $a = \kappa_T$ there is an absorbing boundary with $V_1^\pi(t, w, \kappa_T) = V_T^\pi; 0 \leq t < t^*, 0 \leq w < W$.

Boundary Condition at wealth $w = 0$

For now I am not considering any boundary conditions for the wealth domain of pre-retirement. The details of how a wealth boundary is avoided are mentioned in the Section 4.3 for my numerical scheme.

4.3 NUMERICAL SCHEME:

Since there is a boundary condition at the time T which requires PDE 4.5 to be solved backwards in time, therefore, I apply a change of time variable $t_{new} = T - t_{old}; 0 \leq t_{new} \leq T$ and drop the subscript from t_{new}. The optimal retirement time t^* can be anywhere between

$t = 0$ and $t = T$, the post-retirement PDE has to be solved for the whole time domain and t^* will be determined numerically while solving the pre-retirement retirement problem.

Once again, I am considering a finite difference numerical scheme which is explicit in the wealth variable but implicit in the b-age variable. Consider a numerical grid $(t_i, w_j, a_k); 0 \leq t_i \leq T, 0 \leq w_j \leq W, A1 \leq a_k \leq A2$. Let dt, dw and da be the time step-size, wealth step-size, and bio-age step-size, respectively. $N+1, M+1$ and $K+1$ be the total number of time nodes, wealth nodes and b-age nodes, respectively. Then, the i^{th} time node is $t_i = idt; i = 0, 1, 2..., N$, the j^{th} wealth node is $w_j = jdw; j =, 1, ..., M$ and the k^{th} biological age node is $a_k = A1 + kda; k = 0, 1, ..., K$. Here $t_0 = 0, t_N = T, w_0 = 0, w_M = W, a_0 = A1$ and $a_K = A2$.

4.3.1 Numerical Scheme: Post-Retirement PDE

Let the post-retirement value function at (i, j, k) discretized grid points be $\overline{V^\pi}(t_i, w_j, a_k) \equiv \overline{v}^i_{j,k}; \forall i, j, k$. I am using the mixed explicit-implicit finite difference numerical scheme to solve both the post-retirement PDE and pre-retirement PDE. For that, discretize the wealth part of PDE 4.5 at the grid point (t_i, w_j, a_k) where the value function is known while the b-age part of this PDE is discretized at the unknown value function grid point (t_{i+1}, w_j, a_k). For time derivative, a forward difference is used at(t_i, w_j, a_k) and the 0^{th} order term is evaluated at (t_{i+1}, w_j, a_k). Using an up-winding scheme finite difference the wealth derivative at explicit time nodes is discretized based on $\beta^i_{j,k}$; the coefficient of $\overline{V^\pi}_w$ in the PDE.

$$\overline{V^\pi}_w(t_i, w_j, a_k) = \begin{cases} \frac{\overline{v}^i_{j+1,k} - \overline{v}^i_{j,k}}{dw} & , \beta^i_{j,k} \geqslant 0 \\ \frac{\overline{v}^i_{j,k} - \overline{v}^i_{j-1,k}}{dw} & , \beta^i_{j,k} < 0 \end{cases}$$

for $j = 1, 2, 3, ..., M-1$. For $j = M$ I'll use backward difference to evaluate the wealth derivative of post-retirement value function.

$$\overline{V^\pi}_w(t_i, w_M, a_k) = \frac{\overline{v}^i_{M,k} - \overline{v}^i_{M-1,k}}{dw}$$

$\beta^i_{j,k}$ is defined as

$$\beta^i_{j,k} \equiv \pi + rw_j - \frac{B}{1-\tilde{\gamma}} [\overline{V^\pi}_w(t_i, w_j, a_k)]^{-\tilde{\gamma}}, 0 \leq w_j \leq W$$

$\overline{V^\pi}_w$ inside $\beta^i_{j,k}$ is evaluated using a forward finite difference at the current time step in order to get rid of the non-linearity of the wealth derivative of the value function. The time derivative is gives as

$$-\overline{V^\pi}_t(t_i, w_j, a_k) = -\frac{\overline{v}^{i+1}_{j,k} - \overline{v}^i_{j,k}}{dt}$$

The symmetric difference is used for the second order b-age derivative and and up-winding is used for the first order b-age derivative at implicit time nodes i.e.

$$\overline{V^\pi}_a(t_{i+1}, w_j, a_k) = \begin{cases} \frac{\overline{v}^{i+1}_{j,k} - \overline{v}^{i+1}_{j,k-1}}{da} & , \theta^i_k \geq 0 \\ \frac{\overline{v}^{i+1}_{j,k+1} - \overline{v}^{i+1}_{j,k}}{da} & , \theta^i_k < 0 \end{cases}$$

$$\overline{V^\pi}_{aa}(t_{i+1}, w_j, a_k) = \frac{\overline{v}^{i+1}_{j,k+1} - 2\overline{v}^{i+1}_{j,k} + \overline{v}^{i+1}_{j,k-1}}{da^2}$$

where, θ^i_k is the discretized version of coefficient of $\overline{V^\pi}_a$ defined as

$$\theta^i_k \equiv -\left[1 + \xi\left(\frac{\kappa_0 + t_i - a_k}{T - t_i + \epsilon}\right)\right] \tag{4.18}$$

Using these finite differences to discretize the post-retirement HJB 4.5 gives a mixed explicit implicit scheme.

$$\begin{aligned} &-\frac{\overline{v}^{i+1}_{j,k} - \overline{v}^{i}_{j,k}}{dt} - \frac{\theta^{i+1}_k + |\theta^{i+1}_k|}{2}\frac{\overline{v}^{i+1}_{j,k} - \overline{v}^{i+1}_{j,k-1}}{da} - \frac{\theta^{i+1}_k - |\theta^{i+1}_k|}{2}\frac{\overline{v}^{i+1}_{j,k+1} - \overline{v}^{i+1}_{j,k}}{da} + \\ &\frac{\sigma^2}{2}\frac{\overline{v}^{i+1}_{j,k+1} - 2\overline{v}^{i+1}_{j,k} + \overline{v}^{i+1}_{j,k-1}}{da^2} - (\rho + \lambda(a_k))\overline{v}^{i+1}_{j,k} + \\ &\frac{\beta^i_{j,k} + |\beta^i_{j,k}|}{2}\frac{\overline{v}^{i}_{j+1,k} - \overline{v}^{i}_{j,k}}{dw} + \frac{\beta^i_{j,k} - |\beta^i_{j,k}|}{2}\frac{\overline{v}^{i}_{j,k} - \overline{v}^{i}_{j-1,k}}{dw} = 0 \end{aligned} \tag{4.19}$$

Group together like terms and label the coefficients of $\overline{v}^{i+1}_{j,k+1}, \overline{v}^{i+1}_{j,k-1}, \overline{v}^{i+1}_{j,k}$ as follows:

$$\begin{aligned} \eta_1(k) &= \frac{D}{da^2} - \frac{\theta^{i+1}_k - |\theta^{i+1}_k|}{2da} \\ \eta_3(k) &= \frac{D}{da^2} + \frac{\theta^{i+1}_k + |\theta^{i+1}_k|}{2da} \\ \eta_2(k) &= \eta_1(k) + \eta_3(k) + (\rho + \lambda(a_k)) + \frac{1}{dt} \end{aligned} \tag{4.20}$$

respectively. Here $D = \frac{\sigma^2}{2}$. Collect all the t_i node terms and call them

$$g(k) = \frac{\beta^i_{j,k} + |\beta^i_{j,k}|}{2}\frac{\overline{v}^{i}_{j+1,k} - \overline{v}^{i}_{j,k}}{dw} + \frac{\beta^i_{j,k} - |\beta^i_{j,k}|}{2}\frac{\overline{v}^{i}_{j,k} - \overline{v}^{i}_{j-1,k}}{dw} + \frac{\overline{v}^{i}_{j,k}}{dt} \tag{4.21}$$

For every time step and wealth step, let $\overline{v}^{i+1}_{j,k} \equiv \overline{V}, \overline{v}^{i}_{j,k} \equiv \overline{V}_{old}, \overline{v}^{i-1}_{j,k} \equiv \overline{V}_{old1}; \forall k$ are vectors in k. Using all these notations equation 4.19 is reduced to a tri-diagonal system for every t_i and w_j.

$$\eta_1(k)\overline{V}(k+1) + \eta_3(k)\overline{V}(k-1) + g(k) = \eta_2(k)\overline{V}(k)$$

This equation is converted to an iterative equation for vector $\overline{V}$,

$$\overline{V}(k) = q(k)\overline{V}(k+1) + p(k) \tag{4.22}$$

where,

$$\begin{aligned} q(k) &= \frac{\eta_1(k)}{\eta_2(k) - \eta_3(k)q(k-1)} \\ p(k) &= \frac{g(k) + \eta_3(k)p(k-1)}{\eta_2(k) - \eta_3(k)q(k-1)} \end{aligned} \tag{4.23}$$

The coefficients p and q are found recursively for $k = 1, 2, ..., K$. It is assumed that $q(0) = 1, p(0) = 0$ to ensure the reflecting boundary condition at $k = 0; \overline{V}(1) = \overline{V}(0)$.

Numerical scheme to evaluate the boundary conditions:

At the reflecting boundary of b-age $a = 0$, I have $\overline{V^{\pi}}_a = 0$. Using first order forward finite difference on the numerical grid, the b-age derivative of post-retirement value function is discretized as: $\overline{V^{\pi}}_a(t_i, w_j, a_0) = \frac{\overline{v}^i_{j,1} - \overline{v}^i_{j,0}}{da}, \forall i = 1, ..., N, j = 0, 1, ..., M$. Putting this derivative equal to zero results in

$$\overline{v}^i_{j,1} = \overline{v}^i_{j,0}, \forall i = 1, ..., N, j = 0, 1, ..., M$$

At the absorbing boundary $a = \kappa_T$, I have $\overline{V^{\pi}}(t, w, \kappa_T) = \overline{V^{\pi}_T}(w) \forall t, w$ which corresponds to the boundary condition on the numerical grid as

$$\overline{V^{\pi}}_a(t_i, w_j, a_K) = \overline{v}^i_{j,K} = \overline{V^{\pi}_T}(w_j); i = 0, 1, ..., N, j = 0, 1, 2, ..., M$$

At the time T, the ODE 4.7 is solved using an Euler Newton method. At the boundary $w = 0$, the PDE 4.6 is solved for $\overline{V^{\pi}}(t, a)$ with the same numerical grid established for the post-retirement pde but only the $w_j = 0$ slice is used, that is, $(t_i, 0, a_k); 0 \le t_i \le T, A1 \le a_k \le A2$. Using above mentioned finite differences and discretization, PDE 4.6 gives the fully implicit finite difference scheme for $\overline{V^{\pi}}(t_i, a_k)$

$$\begin{aligned} &-\frac{\overline{v}^{i+1}_k - \overline{v}^i_k}{dt} - \frac{\theta^{i+1}_k + |\theta^{i+1}_k|}{2}\frac{\overline{v}^{i+1}_k - \overline{v}^{i+1}_{k-1}}{da} - \frac{\theta^{i+1}_k - |\theta^{i+1}_k|}{2}\frac{\overline{v}^{i+1}_{k+1} - \overline{v}^{i+1}_k}{da} + \\ &\frac{\sigma^2}{2}\frac{\overline{v}^{i+1}_{k+1} - 2\overline{v}^{i+1}_k + \overline{v}^{i+1}_{k-1}}{da^2} - (\rho + \lambda(a_k))\overline{v}^{i+1}_k = 0 \end{aligned} \tag{4.24}$$

Using the above mentioned coefficients from equations 4.20 and $g(k) = \frac{\overline{v}^i_k}{dt}, \overline{v}^{i+1}_k \equiv \overline{V}, \overline{v}^i_k \equiv \overline{V}_{old}, \overline{v}^{i-1}_k \equiv \overline{V}_{old1}; \forall k$, these equations are converted to the iterative equation 4.25

$$\overline{V}(k) = q(k)\overline{V}(k+1) + p(k) \tag{4.25}$$

$p(k)$ and $q(k)$ are defined in equations 4.23.

Beginning at the boundary condition for time T, I have $\overline{V^\pi} = \overline{V_T^\pi}(w = 0)$. On the numerical grid (the time reversed grid) $\overline{V^\pi}(t_0, a_k) = \overline{V_T^\pi}(w_0)$. Then for every $t_i; i = 1, ..., N$ I solve equation 4.25 iteratively backwards for $k = K-1, ..., 1$. At $k = K$ and $k = 0$ I have the b-age boundary conditions $\overline{V^\pi}(t_i, a_K) = \overline{V_T^\pi}(w0)$ and $\overline{V^\pi}(t_i, a_0) = \overline{V^\pi}(t_i, a_1)$, respectively.

4.3.2 Numerical Scheme: Pre-Retirement PDE

I want to solve the pre-retirement HJB:

$$\begin{aligned}(V_1^\pi)_t + &\left[(1+rw) - \frac{{(V_1^\pi)_w}^{-\tilde{\gamma}}}{1-\tilde{\gamma}}\right](V_1^\pi)_w \\ &+ \left[(1+\xi\frac{\kappa_t - a}{T-t})\right](V_1^\pi)_a + \frac{\sigma^2}{2}(V_1^\pi)_{aa} - (\rho + \lambda(a))V_1^\pi = 0 \qquad (4.26)\end{aligned}$$

for all $(t, w, a) \leq \tau(t^*, w^*, a^*); 0 \leq w^* \leq W, 0 \leq t^* \leq T, 0 \leq a^* \leq \kappa_T$. I'll use the same numerical grid as I used for the post-retirement problem along with the time change applied ($t_{new} = T - t_{old}; 0 \leq t_{new} \leq T$ and drop the subscript from t_{new}). Let the pre-retirement value function at discretized grid points be $V_1^\pi(t_i, w_j, a_k) \equiv v_{j,k}^i; i = 1, ..., N, j = 0, ..., M, k = 0, ..., K$.

My plan is to find this optimal boundary τ for retirement time, wealth and biological age. In order to do that; solve the pre-retirement pde using the mixed explicit-implicit finite difference numerical scheme. Discretize the wealth part of PDE 4.12 at the grid point (t_i, w_j, a_k) where the value function is known while the b-age part of this PDE is discretized at the unknown value function grid point (t_{i+1}, w_j, a_k). For the time derivative, a forward difference is used at(t_i, w_j, a_k) and the 0^{th} order term is evaluated at (t_{i+1}, w_j, a_k). The coefficient of $(V_1^\pi)_w$ in PDE 4.12 is $\beta_{j,k}^i$ defined as

$$\beta_{j,k}^i \equiv 1 + rw_j - \frac{[(V_1^\pi)_w(t_i, w_j, a_k)]^{-\tilde{\gamma}}}{1-\tilde{\gamma}}, 0 \leq w_j \leq W$$

$(V_1^\pi)_w$ inside $\beta_{j,k}^i$ is evaluated using forward finite differences at the current time step in order to get rid of the non-linearity of wealth derivative of the value function. The time derivative is gives as

$$-(V_1^\pi)_t(t_i, w_j, a_k) = -\frac{v_{j,k}^{i+1} - v_{j,k}^i}{dt}$$

An up-winding scheme is used for the wealth derivative at explicit time nodes for wealth nodes $j = 1, 2, ..., M-1$:

$$(V_1^\pi)_w(t_i, w_j, a_k) = \begin{cases} \frac{v_{j+1,k}^i - v_{j,k}^i}{dw} & , \beta_{j,k}^i \geqslant 0 \\ \frac{v_{j,k}^i - v_{j-1,k}^i}{dw} & , \beta_{j,k}^i < 0 \end{cases}$$

For $j = 0$ I'll use a forward difference and for $j = M$ I'll use a backward difference to evaluate the wealth derivative of post-retirement value function.

$$(V_1^\pi)_w(t_i, w_0, a_k) = \frac{v^i_{1,k} - v^i_{0,k}}{dw}$$

$$(V_1^\pi)_w(t_i, w_M, a_k) = \frac{v^i_{M,k} - v^i_{M-1,k}}{dw}$$

Since, all wealth derivatives are calculated at the previous time step t_i at which the value function is known, therefore, I do not feel the need to have a wealth boundary condition for the pre-retirement value function.

From section 4.3.1 the post-retirement numerical scheme, θ^i_k as given in equation 4.18 is

$$\theta^i_k \equiv -\left[1 + \xi\left(\frac{\kappa_0 + t_i - a_k}{T - t_i + \epsilon}\right)\right]$$

The symmetric difference and up-winding is used for b-age derivatives at implicit time nodes. i.e.,

$$(V_1^\pi)_a(t_{i+1}, w_j, a_k) = \begin{cases} \frac{v^{i+1}_{j,k} - v^{i+1}_{j,k-1}}{da} & , \theta^i_k \geq 0 \\ \frac{v^{i+1}_{j,k+1} - v^{i+1}_{j,k}}{da} & , \theta^i_k < 0 \end{cases}$$

$$(V_1^\pi)_{aa}(t_{i+1}, w_j, a_k) = \frac{v^{i+1}_{j,k+1} - 2v^{i+1}_{j,k} + v^{i+1}_{j,k-1}}{da^2}$$

These finite differences discretize the PDE 4.12 and give a mixed explicit implicit scheme.

$$\begin{aligned}
&-\frac{v^{i+1}_{j,k} - v^i_{j,k}}{dt} - \frac{\theta^{i+1}_k + |\theta^{i+1}_k|}{2}\frac{v^{i+1}_{j,k} - v^{i+1}_{j,k-1}}{da} - \frac{\theta^{i+1}_k - |\theta^{i+1}_k|}{2}\frac{v^{i+1}_{j,k+1} - v^{i+1}_{j,k}}{da} + \\
&\frac{\sigma^2}{2}\frac{v^{i+1}_{j,k+1} - 2v^{i+1}_{j,k} + v^{i+1}_{j,k-1}}{da^2} - (\rho + \lambda(a_k))v^{i+1}_{j,k} + \\
&\frac{\beta^i_{j,k} + |\beta^i_{j,k}|}{2}\frac{v^i_{j+1,k} - v^i_{j,k}}{dw} + \frac{\beta^i_{j,k} - |\beta^i_{j,k}|}{2}\frac{v^i_{j,k} - v^i_{j-1,k}}{dw} = 0
\end{aligned} \tag{4.27}$$

Group together like terms and label the coefficients of $v^{i+1}_{j,k+1}, v^{i+1}_{j,k-1}, v^{i+1}_{j,k}$ as follows:

$$\begin{aligned}
\eta_1(k) &= \frac{D}{da^2} - \frac{\theta^{i+1}_k - |\theta^{i+1}_k|}{2da} \\
\eta_3(k) &= \frac{D}{da^2} + \frac{\theta^{i+1}_k + |\theta^{i+1}_k|}{2da} \\
\eta_2(k) &= \eta_1(k) + \eta_3(k) + (\rho + \lambda(a_k)) + \frac{1}{dt}
\end{aligned}$$

respectively. Here $D = \frac{\sigma^2}{2}$. Collect all the t_i node terms and call them

$$g(k) = \frac{\beta^i_{j,k} + |\beta^i_{j,k}|}{2}\frac{v^i_{j+1,k} - v^i_{j,k}}{dw} + \frac{\beta^i_{j,k} - |\beta^i_{j,k}|}{2}\frac{v^i_{j,k} - v^i_{j-1,k}}{dw} + \frac{v^i_{j,k}}{dt}$$

For every time step t_i and wealth step w_j, let $v^{i+1}_{j,k} \equiv V, v^i_{j,k} \equiv V_{old}; \forall k$ are vectors in k. Using all these notations, equation 4.27 is reduced to a tri-diagonal system for every t_i and w_j.

$$\eta_1(k)V(k+1) + \eta_3(k)V(k-1) + g(k) = \eta_2(k)V(k)$$

This equation is converted to an iterative equation for the vector V using p and q defined in equations 4.23

$$V(k) = q(k)V(k+1) + p(k) \tag{4.28}$$

For every time step t_i and wealth step w_j, beginning at the retirement b-age $a^* = a_{k^*}; V(k^*) = \overline{V}(k^*)$ as the final condition, the iterative equation is solved backwards for $k = k^* - 1, ..., 0$. $\overline{V}(k^*)$ is the post-retirement value calculated from equation 4.25.

Numerical scheme to evaluate the boundary conditions:

At the reflecting boundary of b-age $a = 0$, I have $(V_1^\pi)_a = 0$. Using first order forward finite difference on the numerical grid, the b-age derivative of pre-retirement value function is discretized as: $(V_1^\pi)_a(t_i, w_j, a_0) = \frac{v^i_{j,1} - v^i_{j,0}}{da}, \forall i = 1, ..., N, j = 0, 1, ..., M$. Putting this derivative equal to zero results in

$$v^i_{j,1} = v^i_{j,0}, \forall i = 1, ..., N, j = 0, 1, ..., M$$

At the absorbing boundary $a = \kappa_T$, I have $V_1^\pi(t, w, \kappa_T) = V_T^\pi(w) \forall t, w$ which corresponds to the boundary condition on numerical grid as

$$V_1^\pi(t_i, w_j, a_K) = v^i_{j,K} = V_T^\pi(w_j); i = 0, 1, ..., N, j = 0, 1, 2, ..., M$$

At the time T, the ODE 4.7 is solved using an Euler Newton method.

Numerical scheme to find the optimal retirement free boundary $\tau(t^*, w^*, a^*)$:

At the retirement boundary, value matching and smooth pasting must hold due to the presence of noise in b-age. For every time step t_i and wealth step w_j, an initial guess is made for the optimal retirement b-age, $a^*_{guess} = a_0$. Then smooth pasting is used to determine the optimal a^* by checking the b-age derivatives of V_1^π and updating a^*_{guess} until $a^*_{guess} = a_{k^*} = a^*$:

$$\begin{cases} (V_1^\pi)^+_a(t_{i+1}, w_j, a_{k^*}) = (V_1^\pi)^-_a(t_{i+1}, w_j, a_{k^*}) & , a_{k^*} \text{ is optimal b-age} \\ (V_1^\pi)^+_a(t_{i+1}, w_j, a_{k^*}) > (V_1^\pi)^-_a(t_{i+1}, w_j, a_{k^*}) & , k^* = k^* + 1 \end{cases} \tag{4.29}$$

where, $(V_1^\pi)_a^+(t_{i+1}, w_j, a_{k^*}) = \frac{v_{j,k^*+1}^{i+1} - v_{j,k^*}^{i+1}}{da}$ is the right hand derivative and $(V_1^\pi)_a^-(t_{i+1}, w_j, a_{k^*}) = \frac{v_{j,k^*}^{i+1} - v_{j,k^*-1}^{i+1}}{da}$ is the left hand derivative of V_1^π with respect to b-age. At and after this age it is assumed that person has retired and $(V_1^\pi)(k) = \overline{V^\pi}(k); k = k^*, k^*+1, ..., K$. Using this as boundary condition the iterative scheme 4.28 gives the pre-retirement value for all k=0,1,2,...,k*.

Optimal Consumption:

The pre-retirement optimal consumption $c_{pre}^*(t_i, w_j, a_k)$ and the post-retirement optimal consumption $c_{post}^*(t_i, w_j, a_k)$, for all wealth levels $0 \leq w_j \leq W$, b-ages $0 \leq a_k \leq \kappa_T$, and all time nodes $0 \leq t_i \leq T$ are calculated as follows:

$$\begin{aligned} c_{pre}^*(t_i, w_j, a_k) &= \left(\frac{v_{j+1,k}^{i+1} - v_{j,k}^{i+1}}{dw}\right)^{-\tilde{\gamma}}, t_i < t^* \\ c_{post}^*(t_i, w_j, a_k) &= B\left(\frac{\overline{v}_{j+1,k}^{i+1} - \overline{v}_{j,k}^{i+1}}{dw}\right)^{-\tilde{\gamma}}, t_i \geq t^* \end{aligned} \tag{4.30}$$

Save these consumptions as two separate 3-dimensional matrices in Matlab [1].

4.4 Wealth Dynamics

Consider a 30 year old consumer having wealth w_0 in savings. Just as in the stochastic mortality no-pension case in chapter 3 section 3.4, I want to see how his wealth and consumption evolves over the course of his life if he uses my optimal retirement and spending strategy. In this section, this person has an exogenous pension income stream for post-retirement life.

The post-retirement and pre-retirement wealth dynamics:

$$\begin{aligned} \frac{dw}{dt} &= 1 + rw(t) - c_{pre}^*(t, w, a); t < \tau \\ \frac{dw}{dt} &= \pi + rw(t) - c_{post}^*(t, w, a); t \geq \tau \end{aligned}$$

Once again, consumption is a function of 3 variables; time, wealth and b-age. The biological age needs to be simulated using dynamics 3.3 to have a b-age path for the consumer over his whole life time. This is discussed in chapter 3 section 3.4. Given the initial condition $A(t = 0) = A_0$, the following iterative scheme is used to simulate the consumer's b-age:

$$A_{i+1} = A_i + da\left(1 + \xi\frac{\kappa_t - A_i}{T - t_i}\right)dt + \sigma\sqrt{dt}Z(0, 1)$$

for $0 = t_0 \leq t_i \leq t_N = T$. Here, $Z(0, 1)$ is a standard normal random variable.

Discretize the wealth ODEs using an Euler Method over the time grid $0 \leq t_i \leq T$. The subscript i for wealth and b-age; w_i and a_i shows that wealth and b-age are functions of time. That is, at each time node t_i wealth is w_i and b-age is a_i.

$$\frac{w_{i+1} - w_i}{dt} = 1 + rw_i - c^*_{pre}(t_i, w_i, a_i); t_i < \tau$$
$$\frac{w_{i+1} - w_i}{dt} = \pi + rw(t_i) - c^*_{post}(t_i, w_i, a_i); t_i \geq \tau$$

where $0 \leq t_i \leq T$. Simplification gives

$$w_{i+1} = dt\left[1 + \left(r + \frac{1}{dt}\right)w_i - c^*_{pre}(t_i, w_i, a_i)\right]; t_i < \tau$$
$$w_{i+1} = dt\left[\pi + \left(r + \frac{1}{dt}\right)w_i - c^*_{post}(t_i, w_i, a_i)\right]; t_i \geq \tau \tag{4.31}$$

with $w(t_0) = w_0$.

Beginning at t_0 solve the wealth dynamics 4.31 forwards in time. Use the retirement matrix $\tau(t, w, a)$ to determine if each point (t_i, w_i, a_i) falls in the post-retirement region or the pre-retirement region and solve the appropriate wealth dynamics from 4.31. Use 2-dimensional linear interpolation to get consumption numbers from the $c^*_{pre}(t, w, a)$ and $c^*_{post}(t, w, a)$ matrices at the points (t_i, w_i, a_i) calculated from 4.30. The interpolation is done using the Matlab in-built functions *interp2* [1].

4.5 Algorithm

The algorithm for solving the pension income optimal retirement boundary is given below:

1. Solve the post-retirement value $\overline{V^\pi_T}(w_j); j = 1, 2, ..., M$ at the boundary $t = T$ using the boundary condition $\overline{V^\pi_T}(w_0)$. Set $\overline{v}^0_{j,k} = \overline{V^\pi_T}(w_j), \forall j, k$

2. Set the post-retirement boundary at $a_K = \kappa_T; \overline{v}^i_{j,K} = \overline{V^\pi_T}(w_j), \forall i, j$.

3. Solve the pre-retirement value at the boundary $t = T, V^\pi_T(w_j); j = 1, 2, ..., M$. Set $v^0_{j,k} = V^\pi_T(w_j), \forall j, k$

4. Set the pre-retirement boundary at $a_K = \kappa_T; v^i_{j,K} = V^\pi_T(w_j), \forall i, j$.

5. Now using the boundary values from step (1) and step (2), for every $t_i; i = 1, .., N$ and $w_j; j = 1, ..., M$, solve the iterative equation 4.25 backwards for $k = K - 1, ..., 0$ to get the post-retirement value function $\overline{v}^i_{j,k}$.

6. Move onto solving the pre-retirement value function. Begin at $i = 2$ for time grid.

7. Begin at $j = 0$ for the wealth grid and set $k^* = a_0$.

8. Solve equation 4.28 recursively for $V(k)$ going backwards for $k = k^* - 1, ..., 0$ using the boundary conditions $V(k^*) = \overline{v}^i_{j,k^*}$.

9. Check for the smooth pasting conditions given in 4.29.
If $(V_1^\pi)_a^+(t_{i+1}, w_j, a_{k^*}) = (V_1^\pi)_a^-(t_{i+1}, w_j, a_{k^*})$, mark a_k^* as optimal for given i and j.
If $(V_1^\pi)_a^+(t_{i+1}, w_j, a_{k^*}) > (V_1^\pi)_a^-(t_{i+1}, w_j, a_{k^*})$, make the guessed k^* move up one step.

10. Keep repeating steps 8 and 9 until an optimal k^* is found. Mark a_k^* as optimal b-age for the current time and wealth steps.

11. Now $V(k, k = 0, 1, ..., k^* - 1)$ is pre-retirement value function. Assign the post retirement value to $V(k = k^*, k^* + 1, ..., K) = \overline{v}^i_{j,k}; k = k^*, ..., K$.

12. Move onto next j. Repeat steps 8-11 for every j in wealth grid. This solves the value function problem for first time step $i = 2$

13. Walk to the next time-step and once again begin at the first wealth step $j = 0$.

14. Repeat steps 7-13. Iterating through each time step and every wealth step reach the final time step t_N. Keep marking optimal k^* for every time and wealth step. The array of time, wealth and the correspondingly marked k^* forms the optimal retirement boundary $\tau(t, w, a)$.

15. For every node (t_i, w_j, a_k), calculate the pre-retirement and post-retirement optimal consumption using 4.30.

16. Simulate the lifetime b-age of a 30 year old consumer using 3.39.

17. Calculate his wealth dynamics using iterative scheme 4.31.

4.6 Numerical Results

I am using the following parameter values $r = 2.5\%$, $\rho = r, b = 9.44, m = 88.82, \xi = 1, \alpha = 0.5, \sigma = 1, \bar{l} = 2.53$, unless otherwise mentioned. The maximum wealth $W = 5$. The c-age at time T and the maximum b-age are $\kappa_T = 110$. The current age κ_0 is assumed to be 30. The pension level is 0.4 times the labor income (which is \$1 in my case). Since both the post-retirement problem and the pre-retirement problem are 3 dimensional, computationally this problem is expensive in terms of time and requires a lot of storage. I took grid sizes $dw = 0.05 = dy, da = 0.0025 = dt$. It took almost a week to run the code for one set of parameters. I tried reducing the grid sizes for wealth to get better accuracy but even after 1 month but I was not even close to getting results. With changing grid-sizes the qualitative results stayed the same, therefore, I used the above mentioned grid to run all computations. Also, it is worth mentioning that I showed the results for only one set of parameters because the computations are really expensive. Also, the qualitative behavior of results for varying r and γ is not different from the results mentioned in the previous chapter for the no-pension case. Moreover, my code is working for a specific set of $\bar{l}$ and π. With the $\bar{l}$ used, I couldn't reduce pension beyond $\pi = 0.4$ and above this level at $\pi = 0.5$ everyone prefers to retire. Hence, for this set of leisure and pension, it is reasonable to show the results for only one set of values.

The optimal retirement boundary level curves in figure 4.7 for fixed wealth as a function of b-age and c-age show qualitatively similar behaviors as the retirement boundary plots for the stochastic mortality no-pension case in figure 3.10. I won't repeat that discussion here. Rather my focus is on the effects of pension income. Notice that with pension income level 0.4 which is almost half the amount of annual labor income, people with quite low wealth like $w = 3.30$ are not willing to enter the labor force at all. They prefer retirement life over earning labor income. But looking at figure 4.8 for the value function at the boundary $t = T$ and max B-Age $a = \kappa_T$ for the same leisure $\bar{l} = 2.53$ I can see that $\overline{w}$ is non-zero. That means a consumer with such small wealth likes to work till time T when their life expectancy is exactly 1 year and they are not willing to retire before. That means the retirement boundary curves change quite fast between $\overline{w}$ (no retirement at $t < T$) to not entering the labor force at $w = 3.30$. With such high pension income my hope was to to decrease the post-retirement leisure $\bar{l}$ in order to see retirement curves with $w > 3.30$ but my code is not stable for this combination of $\bar{l}$ and π.

With increasing values of $\bar{l}$ I am making retirement more and more attractive by adding more leisure to it, therefore, people with smaller wealth levels are willing to never enter the labor force. In figure 4.7 for $\bar{l} = 2.53$ consumer with $w = 3.30$ won't ever work while in the figure 4.9 for $\bar{l} = 2.55$ a person with $w = 3.10$ won't enter the labor force.

For the wealth dynamics in figure 4.10 notice that due to the presence of pension income, consumers exhaust their wealth even before reaching c-age= 80. They lead the rest of their retired life spending the pension income π without saving any wealth. People with wealth 2.80 and above with biological age 35 will like to retire right away at c-age 30 and spend rest of their lives enjoying post-retirement utility. If I compare this with the no-labor wealth level from optimal retirement curves in figure 4.7, I see that a 30 year old with $w = 2.80$ likes to retire at b-age 24. Hence, with c-age 30 and b-age 35, the consumer with wealth $w = 2.80$ will retire right away thinking he has passed the retirement b-age. Also note that, because of the presence of noise there are no uniqueness-wealth-curves for the pension case as opposed to what I saw for deterministic mortality wealth dynamics in figure 2.5 and stochastic mortality wealth dynamics in the figure 3.14. Another difference I notice is that with the presence of pension income, every single person with any wealth level is willing to retire sooner or later. Looking at the corresponding optimal consumption plots in figure 4.11, people with wealth levels below 2.80 who are working and saving money for retirement, their pre-retirement consumption levels keep rising with increasing c-age until they retire. Once they retire, their post-retirement consumption gradually begins decreasing up till c-age between 70 and 80. After that there is a steep decline in the consumption levels before reaching the constant subsistence level of $c = \pi$. Obviously the fluctuation in consumption curves is due to the noise in b-age.

4.6.1 Efforts to Calibrate $\bar{l}$

I discussed (in chapter 2 and section 2.6.1) about the calibration of the post-retirement leisure process $\bar{l}$ required to make retirement from labor attractive enough for the consumer to give up his pre-retirement consumption. I had no success in calibrating the deterministic income wealth dynamics numbers with Fidelity numbers. In this section I am trying once again to calibrate for $\bar{l}$ using a different set of numbers which take into consideration the pension income.

According to [2] the average Canada Pension Plan (CPP) payment at retirement in 2021 is \$714.21 per month, or \$8570 per year. According to [3] in 2021 the basic Old Age Security (OAS) plus Guaranteed Income Supplement (GIS) payment to a senior with other income at \$8570 per year is \$1091.08 per month, or $\$13,093$ per year. Combining the two gives a pension of $\$21,663$ per year.

According to [4] the median 2018 income of people in the 55-64 age bracket was $\$42,400$ per year. Adjusting the latter for inflation (and trying to come up with round numbers), let's imagine a typical senior at retirement earning a pension of $\$22,000$ and pre-retirement income of $\$44,000$. So in my scale (which makes the pre-retirement income =1), I'd have a pension of 0.5.

According to Table 4 in the latter, in 2019 the median liquid wealth of Canadian senior households was $\$107,000$. Of course, not all those families consist of 2 people, and not all those seniors are age 65, but to a first approximation I might guess that the median individual senior has liquid assets of $\$53,500$. With that in mind, I will consider 3 wealths - $1.25, 2.5$ and 3.75 times labor income, respectively. I'll try calibrating so that someone who has biological age and chronological age 65, and a pension of 0.5, will have wealth 2.5 as the cutoff between retirement and continuing to work. My hope is that someone with B-age and C-age 55 then won't retire at wealth 3.75 and that someone with B-age and C-age 70 would retire at wealth 1.25.

With these calibration numbers in mind I plotted the optimal retirement boundary curves figure 4.7 but with a pension of 0.4 and $\bar{l} = 2.53$, a person c-age 65 and wealth level 1.25 will retire at approximately b-age 60. While a 70 year old consumer with b-age 70 prefers to retire at wealth 0.80. My retirement boundary numbers are quite different than what I was expecting given the factual numbers above. For higher pension like 0.5 consumers retire earlier than the numbers I saw for pension 0.4. Also, my code is working for a certain set of $\bar{l}$ and π combinations. For $\pi = 0.4$ it is working only for $\bar{l} = 2.53$ and higher. Basically for this $\bar{l}$ the large T retirement wealth $\overline{w}$ is very small close to 0. The smaller $\bar{l}$ is the larger retirement wealth $\overline{w}$ is. When you provide less leisure post-retirement people will need more wealth to retire. When I take $\bar{l} < 2.53$ hoping to see some curves for higher wealth levels in retirement boundary plots, my code stops working. So, I can't fully calibrate $\bar{l}$ once again!

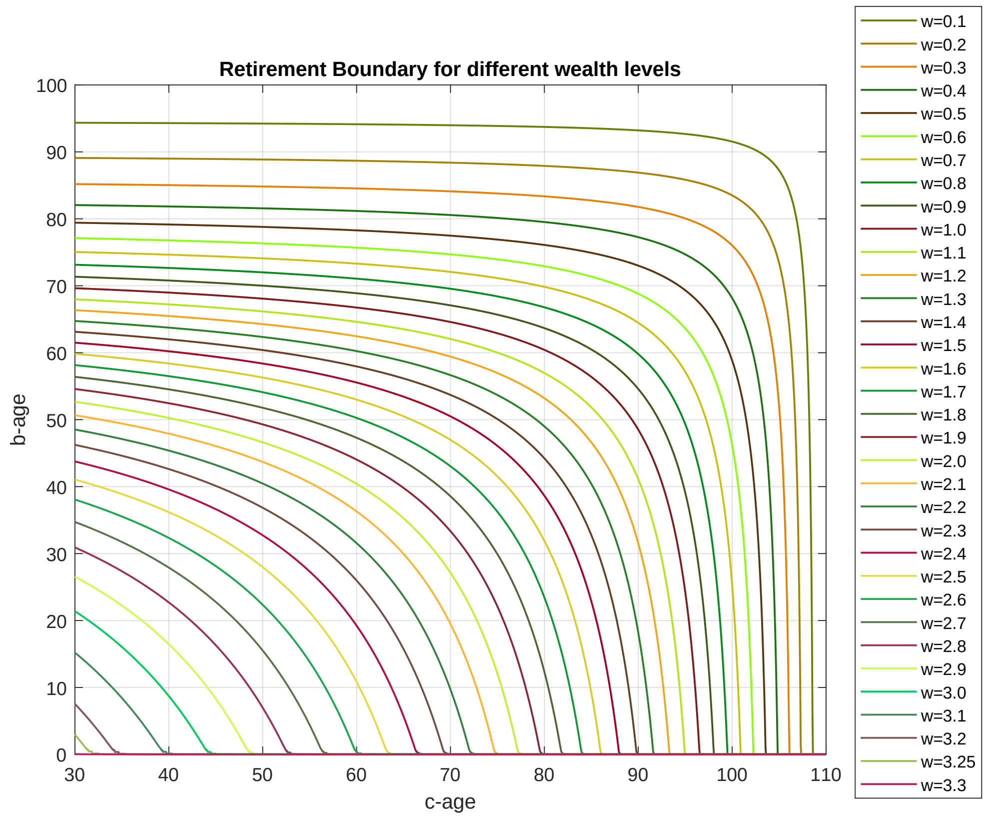


Figure 4.7: Optimal Boundary curves for $\bar{l} = 2.53$ and $B = 0.6287$. Assuming $\pi = 40\%, \gamma = 2, \sigma = 0.5, r = 2.5\%, \rho = r, \alpha = 0.5$

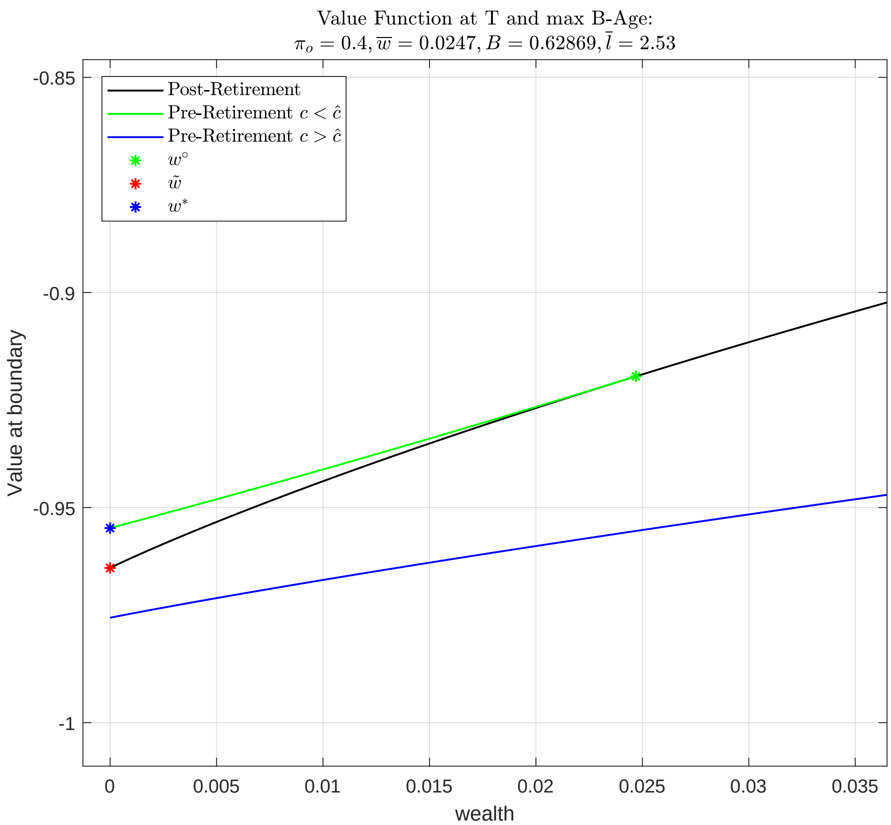


Figure 4.8: Value at T and max B-Age κ_T.
Assuming $\pi = 40\%, \gamma = 2, \sigma = 1, r = 2.5\%, \rho = r, \bar{l} = 2.53, \alpha = 0.5$

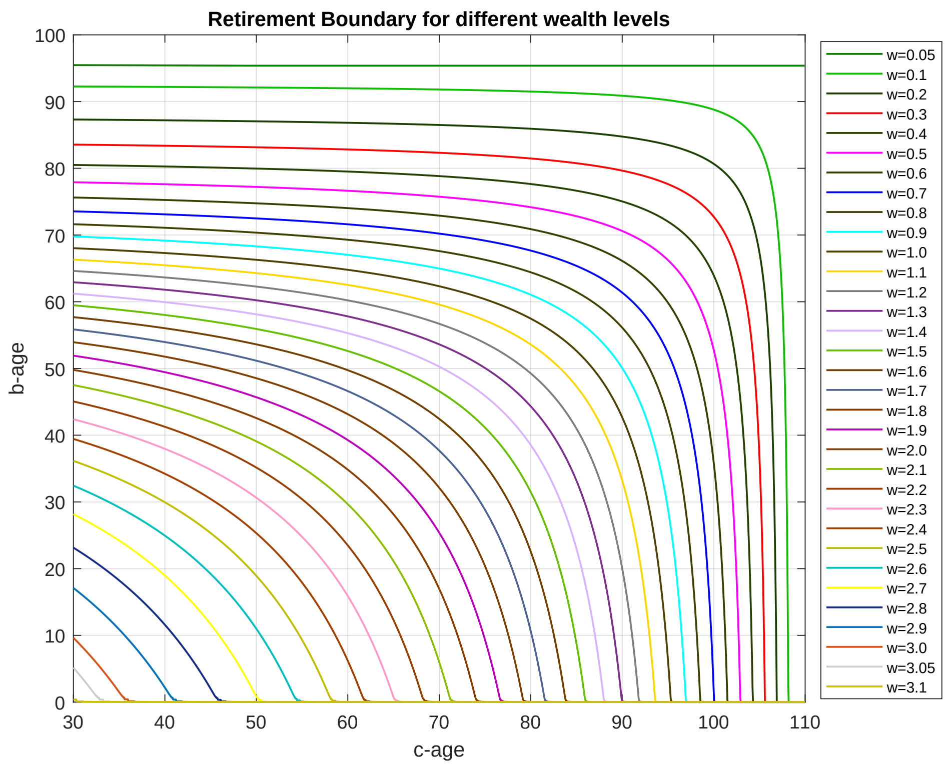


Figure 4.9: Optimal Boundary curves for $\bar{l} = 2.55$ and $B = 0.6262$. Assuming $\pi = 40\%, \gamma = 2, \sigma = 0.5, r = 2.5\%, \rho = r, \alpha = 0.5$

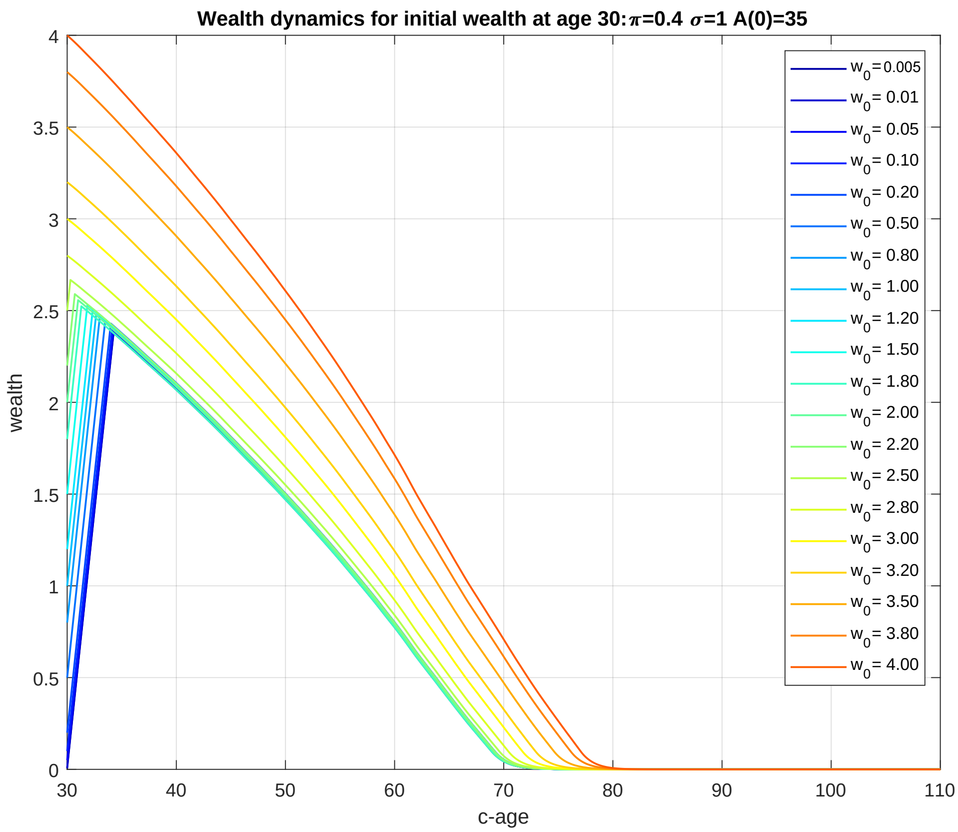


Figure 4.10: Wealth Dynamics for a 30 year old consumer with varying initial wealth w_0. Assuming $A_0 = 35, \pi = 40\%, \gamma = 2, \sigma = 1, r = 2.5\%, \rho = r, \bar{l} = 2.53, \alpha = 0.5$

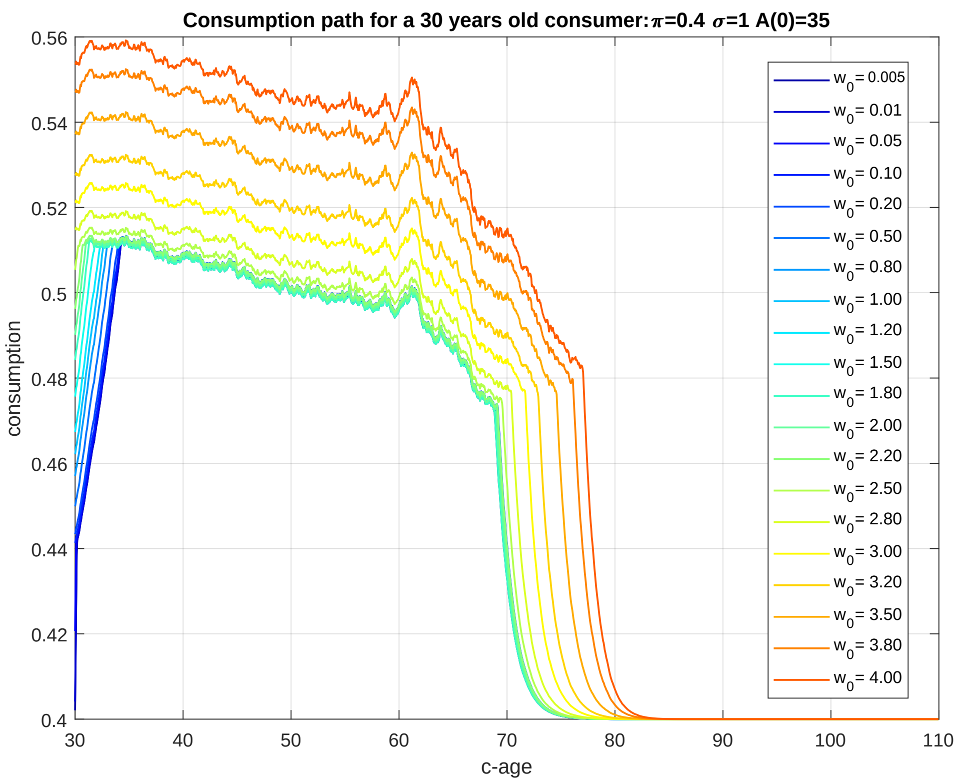


Figure 4.11: Lifetime Consumption for a 30 year old consumer with varying initial wealth w_0. Assuming $A_0 = 35, \pi = 40\%, \gamma = 2, \sigma = 1, r = 2.5\%, \rho = r, \bar{l} = 2.53, \alpha = 0.5$

Chapter 5

Conclusions

This thesis focuses on a life cycle model of a consumer with a focus on two types of mortality: deterministic mortality which is given by the famous Gompertz mortality law and stochastic mortality for which I have used a Gompertz law based on the consumer's biological age. The consumer is looking to retire at an optimal age and wealth while consuming optimally all his life. His utility is described by the Cobb-Douglas utility framework and it is assumed that he is endowed with $\bar{l}$ units of leisure to make the retirement attractive.

Chapter 2 deals with the case of deterministic mortality where the retirement decision is based on the consumer's calendar age and his savings. The requirement for savings in hand decreases with increasing retirement age. The wealth dynamics of the consumer show interesting behaviors. If the initial wealth is small, the consumer will like to stay in the labor force all his life. A consumer with a certain amount of initial wealth in savings has the lifetime wealth dynamics follow one of the three paths; retire right away, save with the intention of retiring at a future time or stay in the labor force all his life. If the initial wealth is a special value $\tilde{w}$, the consumer's wealth path follows an uncommitted curve where all his consumption choices give the same utility.

Chapter 3 focuses on a consumer whose age follows a stochastic mortality law based on the biological age. The consumer's retirement decision is based on his calendar age, biological age and his wealth. It is also an infinite horizon problem but I have assumed that the mortality plateaus at a certain time T, beyond that time the consumer stops aging. I noticed that for a fixed wealth the optimal retirement b-age decreases with increasing c-age. For a fixed c-age, a more wealthy consumer will retire at an earlier b-age. Similarly, consumers at same health levels; a consumer with more savings tends to retire young. I also saw some fascinating effects of the volatility of biological age on retirement curves. At lower wealth levels, high volatility drives less healthier people to retire later.

In chapter 4 I have introduced an exogenous pension income along with the stochastic mortality law mentioned in chapter 3. Unlike the cases of deterministic mortality and stochastic mortality no-pension case, with the presence of pension income every consumer at any wealth level is willing to retire sooner or later. For moderate wealth my results seem consistent with actual behavior. But for high wealth people retire unusually early, and for low wealth they retire unreasonably late. I have arrived at the conclusion that this model doesn't explain observed behaviors perhaps due to the factors like a model based purely on consumption is not rich enough to fully explain the retirement decision. In particular, the leisure process l_t and specifically post-retirement leisure $\bar{l}$, which I considered as a constant, should probably be a function of c-age, b-age and possibly pension. It can't be a constant number in order to have a practical retirement boundary close to the factual numbers.

Nevertheless, over a fair range of age, health status and wealth, my results are applicable in the real world. The optimal retirement boundary curves provide some reliable guidance for a to-be pensioner if he is interested in using my optimal retirement strategy. For a moderately risk averse consumer with c-age and b-age with in a modest range around 65 years, who expects to have an exogenous pension of 40% of his annual labor income upon retirement, if he has savings equal to almost double the amount of his annual labor income he can consider retirement.

For future research projects I plan to work on the following:

1. The free boundary $\tau(t^*, a^*, w^*)$ has kind of almost singular behavior around $t = T$. I can try some kind of parametric approximation to the free boundary around $t = T$ which might reduce the instability.
2. wealth boundary conditions for pension problem.
3. use a stochastic wealth.
4. use leisure as a function of age (c-age as well as b-age) and possibly pension.
5. public policy-certainty equivalent: how much utility is lost if we fix a retirement age?
6. premium to find b-age. Is it worth paying for knowing b-age?

Chapter 6

Impact of MVA on Cash Surrender Values

6.1 Introduction

The following is a summary of the project I did for my internship at CANNEX Financial Exchanges Limited. This project deals with the impact of market value adjustment (MVA) on cash surrender value. A sample set of Fixed Index Annuities (FIA) is considered for the analysis. The Vasicek Model is used as a mean reverting model to calibrate the historical MVA reference asset rates. The equity index for the indexed interest crediting strategy is modeled using Geometric Brownian motion. The correlated Vasicek model and Geometric Brownian motion are used to project MVA reference asset rates and equity index values, respectively, over the time horizon of the surrender charge period. The Monte Carlo simulation method is employed to generate a distribution of Cash Surrender Value with MVA and Cash Surrender Value without MVA. Histograms, Quantile-Quantile plots and central moments are used to examine the similarities and differences between both distributions. Box and Whisker plots are utilized to see the percentage change in Cash Surrender Value caused by Market Value Adjustment.

6.1.1 Market Value Adjustment

Market value adjustment (MVA) is an adjustment applied to the cash surrender value; the account value withdrawn or surrendered in excess of the free withdrawal allowance (if any), during the time when a surrender charge schedule is in effect. It protects the insurance company against any losses incurred due to premature unwinding of the investment strategy caused by early withdrawal. MVA takes into account the fluctuation in bond yields due to changing market conditions and as a consequence increases or decreases the cash surrender value (CSV). In general, if reference yields go up on the surrender date in comparison to the yields when the contract was issued, MVA decreases the surrender value and if yields go down on the surrender date, CSV is adjusted upwards.

6.1.2 The Reference Asset

Market value adjustment calculations are based on a fixed income reference asset; for example constant maturity treasury yields, corporate bond indices etc., taking into consideration the change between reference asset yields between the issue date of the contract and the surrender date, provided the surrender date lies within the surrender charge period.

6.2 Stochastic Models

Stochastic models are used to represent processes having some sort of randomness or uncertain situation. Most of the financial market data is influenced by unpredictable events in the general economic environment. Stochastic models provide a way to handle this randomness in the financial data series and hence carry out the required analysis. For this project, there are two types of financial data series to be dealt with; equity index and fixed income index.

6.2.1 Equity Index Model

Across all the FIAs under consideration, S&P 500 Stock Index is used as the reference asset for the indexed interest crediting option of the accumulation account. We have used Geometric Brownian motion to model the stock index level. Let B_{1t} be a standard Brownian motions under the physical measure causing the randomness in stock price. The dynamics of the stock Index level are given as

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dB_{1t} \tag{6.1}$$

Here, S_t is the stock index level at time t, where t represent years. μ_1 is the annual mean and σ_1 is the volatility of stock index. The solution to the stochastic differential equation 6.1 given the knowledge of the stock index at time t_0 is

$$S_t = S_{t_0} e^{(\mu_1 - \frac{1}{2}{\sigma_1}^2)(t-t_0)+\sigma_1(B_{1t}-B_{1t_0})} \tag{6.2}$$

6.2.2 MVA Reference Rate Model

Note: Interest rates and MVA reference asset yields are used interchangeably throughout this chapter.

Interest rates are Ornstein–Uhlenbeck process. In finance these processes are modeled using the Vasicek model. Interest rates cannot increase indefinitely; a very high level of rates trigger some economic factors which bring down the interest rates hence forcing them to revert back to their long term mean value. The MVA reference rates are yields and we want to model them using the most simple mean reverting model, an obvious choice is the Vasicek Model. Here, the Vasicek model is used as a piece of mathematics to describe the dynamics of mean reverting rates. The modeled rates are the yields of the MVA reference asset and they should not be confused with the short term rates/ instantaneous rates for which the Vasicek model is used frequently. If r_t denotes the MVA reference asset yields, t is the time in years, then their dynamics using the Vasicek model are given by:

$$dr_t = \alpha(\mu_2 - r_t)dt + \sigma dB_t \tag{6.3}$$

where, μ_2 is the long-term mean, α gives the rate at which r_t reverts back to its long-term mean and σ is the local volatility. B_t is the standard Brownian motion under the physical measure and α, μ_2, σ are strictly positive constants. We assume that bond index yields are correlated to the equity asset and are driven by two independent sources of randomness. Let B_{1t} and B_{2t} be two independent standard Brownian motions under the physical measure causing the randomness in bond yields. Here B_{1t} is the same Brownian motion discussed in the previous section driving the stock index process and B_{2t} is representing some other market factors. $B_t = \frac{1}{\sigma}(\sigma_1' B_{1t} + \sigma_2 B_{2t})$, where, $\sigma = \sqrt{{\sigma_1'}^2 + {\sigma_2}^2}$ and the correlation of interest rate with stock price is given as $\rho = \frac{\sigma_1'}{\sigma}$. The variance in the interest rate caused by stock index is σ_1' and the rest of the variance is captured by σ_2. The closed form solution of the Vasicek model given the knowledge of the bond yield at time t_0 is given as

$$\begin{aligned} r_t = e^{-\alpha(t-t_0)} r_{t_0} + \mu_2(1 - e^{-\alpha(t-t_0)}) \\ + \int_{t_0}^{t} e^{-\alpha(t-s)} \sigma_2 dB_{2s} + \int_{t_0}^{t} e^{-\alpha(t-s)} \sigma_1' dB_{1s} \end{aligned} \tag{6.4}$$

The expected value of yield is

$$E[r(t)] = \mu_2 - (\mu_2 - r_{t_0})e^{-\alpha t} \tag{6.5}$$

and the variance is

$$Var[r(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \tag{6.6}$$

The first integral in the solution of r_t

$$I_1 = \int_{t_0}^{t} e^{-\alpha(t-t_0)} \sigma_2 dB_{2s}$$

is a Gaussian process with mean 0 and variance $\int_{t_0}^{t} e^{-2\alpha(t-s)} {\sigma_2}^2 ds$. The second integral

$$I_2 = \int_{t_0}^{t} e^{-\alpha(t-t_0)} \sigma_1' dB_{1s}$$

is correlated to the stock index process. It is also a Gaussian process with mean 0 and variance $\int_{t_0}^{t} e^{-2\alpha(t-s)} {\sigma_1'}^2 ds$.

6.3 Calibration

For calibration of the above mentioned models, the Euler-Maruyama method is used to discretize the stochastic differential equations 6.1 and 6.5. Daily historical data for the reference assets is used for this purpose because the Euler discretization scheme requires a smaller step size Δt to keep numerical error small. The Euler-Maruyama scheme strongly converges with order $\frac{1}{2}$ and weakly converges with order 1. This is equivalent to saying for sufficiently small time step size Δt, $error = O((\Delta t)^{\frac{1}{2}})$ for strong convergence and $error = O((\Delta t)^{1})$ for weak convergence. In order to avoid large discretization errors, daily data is preferred over monthly data or annual data.

6.3.1 Stock Index Model Parameters

Using daily historical data for the stock index, the parameters in the stock model are calculated. If μ_H and σ_H are the mean and standard deviation of historical daily returns, then the GBM parameters are $\mu_1 = \frac{\mu_H}{\Delta t}$ and $\sigma_1 = \frac{\sigma_H}{\sqrt{\Delta t}}$, where Δt is the time step size measured in years. Since, daily data for stock returns is used, $\Delta t = \frac{1}{252}$, considering 252 trading days (on average) in one year.
In-order to get a sense of how daily stock returns are fluctuating about the mean, we applied regression analysis to the stock index process using a simple Euler scheme to discretize the model.

$$\frac{\hat{S}_{t+\Delta t} - \hat{S}_t}{\hat{S}_t} = \mu_1 \Delta t + \sigma_1 Z_{1t}\sqrt{\Delta t} \tag{6.7}$$

where Z_{1t} has a standard normal distribution. Equation 6.1 can be written as

$$\frac{\hat{S}_{t+\Delta t} - \hat{S}_t}{\sqrt{\Delta t}\hat{S}_t} = \mu_1 \sqrt{\Delta t} + \sigma_1 Z_1 \tag{6.8}$$

Running linear regression on the historical data for S&P500, no dividends, this equation generates the standardized residuals $\hat{Z}_1$, which gives a sense of how stock prices are scattered around their mean value. We need these residuals for calibration of the interest rate model.

6.3.2 Interest Rate Model Parameters

Consider interest rate model 6.5. We use the basic Euler scheme for discretization in order to calibrate model parameters with the historical data of the MVA reference asset.

$$r_{t+\Delta t} - r_t = \alpha(\mu_2 - r_t)\Delta t + \sigma_1' \sqrt{\Delta t} Z_{1t} + \sigma_2 \sqrt{\Delta t} Z_{2t}$$

$$\frac{r_{t+\Delta t} - r_t}{\sqrt{\Delta t}} = \alpha\mu_2\sqrt{\Delta t} - \alpha\sqrt{\Delta t} r_t + \sigma_1' Z_{1t} + \sigma_2 Z_{2t} \tag{6.9}$$

where, Z_{1t} are exactly equal to $\hat{Z}_{1t}$; the residuals from stock model regression and Z_{2t} is an independent standard normal variable. This equation is in the form to run multi-linear

regression. Call $y_t = \frac{r_{t+\Delta t} - r_t}{\sqrt{\Delta t}}$ the response variable, $b_0 = \alpha\mu_2\sqrt{\Delta t}, b_1 = -\alpha\sqrt{\Delta t}, b_2 = \sigma_1'$ are the coefficients for the predictors $x_{1t} = r_t, x_{2t} = Z_{1t}$. Therefore running regression on $y_t = b_0 + b_1 x_{1t} + b_2 x_{2t} + \epsilon_t$ give the model parameter estimates as:
$\alpha = -\frac{b_1}{\sqrt{\Delta t}}$
$\mu_2 = -\frac{b_0}{b_1}$
$\sigma_1' = b_2$
$\sigma_2 = \sqrt{MSE}$ where MSE is mean squared error. The parameters generated using this methodology are conditioned on the stock index Brownian motion.

The regression results are given in table 6.4. The 95% confidence interval and the p-values mentioned are for the regression coefficients (they are not reliable statistics for the model parameters) . The parameters for the stock index and MVA reference assets pulled out of the regression results are tabulated in 6.3.

6.4 Simulation

Once, the model parameters are estimated, the GBM and Vasicek model are ready to project equity index levels and bond yields, respectively. Let T be the surrender charge period in years and $0 = t_0 < t_1 < t_2 < ... < t_m = T$ be the equidistant fixed set of dates measured in years. We want to simulate the stock path $S(t_1), ..., S(t_m)$ and the interest rate path $r(t_1), ..., r(t_m)$ for these points in time. The Euler-Maruyama discretization scheme does not work well with large time step, the presence of a closed form solution for Geometric Brownian Motion and the Vasicek model makes it easier to avoid the Euler-Maruyama discretization scheme and use a larger time step size like a month or an year to simulate the asset paths.

6.4.1 Stock path Simulation

The equation 6.2 along with initial condition S_{t_0} can be used to simulate $S(t_j)$ given the knowledge of $S(t_{j-1})$ for all $j = 1, ..., m$:

$$S_{tj} = S_{t_{j-1}} e^{(\mu_1 - \frac{1}{2}{\sigma_1}^2)(t_j - t_{j-1}) + \sigma_1\sqrt{t_j - t_{j-1}} Z_{1j}} \tag{6.10}$$

where $Z_{11}, Z_{12}, ..., Z_{1m}$ are independent standard normal random variables.

6.4.2 Interest rate path Simulation

Consider equation 6.4. For the same set of dates the interest rate path can be simulated using the initial state of the interest rate r_{t_0} and the equation

$$\begin{aligned} r_{t_j} = e^{-\alpha(t_j - t_{j-1})} r_{t_{j-1}} + \mu_2(1 - e^{-\alpha(t_j - t_{j-1})}) \\ + \int_{t_{j-1}}^{t_j} e^{-\alpha(t_j - s)} \sigma_1' dB_{1s} + \int_{t_{j-1}}^{t_j} e^{-\alpha(t_j - s)} \sigma_2 dB_{2s} \end{aligned} \tag{6.11}$$

where $j = 1, 2, ..., m$.

The first integral I_{1j} is Gaussian with mean 0 and variance $\int_{t_{j-1}}^{t_j} e^{-2\alpha(t_j - s)}{\sigma_1'}^2 ds$ and is correlated to the stock model's Brownian motion increment $\int_{t_{j-1}}^{t_j} dB_{1s}$. The second integral I_{2j} is also a normal random variable with mean 0 and variance $\int_{t_{j-1}}^{t_j} e^{-2\alpha(t_j - s)}{\sigma_2}^2 ds$. Let,

$$I_1 = a(B_{1t_j} - B_{1t_{j-1}}) + bW \tag{6.12}$$

for some constants a and b, where W has a standard normal distribution independent of everything else. In-order to find these constants, consider

$$\begin{aligned} Cov[I_1, \int_{t_{j-1}}^{t_j} dB_{1s}] &= E[I_1 \int_{t_{j-1}}^{t_j} dB_{1s}] \\ &= E[\int_{t_{j-1}}^{t_j} e^{-\alpha(t_j - s)}\sigma_1' dB_{1s} \int_{t_{j-1}}^{t_j} dB_{1s}] \end{aligned}$$

Using Ito's isometry the above expectation is reduced to

$$= \int_{t_{j-1}}^{t_j} e^{-\alpha(t_j - s)}\sigma_1' ds \tag{6.13}$$

Also, using 6.12 this covariance can be written as

$$\begin{aligned} &Cov[I_1, \int_{t_{j-1}}^{t_j} dB_{1s}] \\ &= E[\{a(B_{1t_j} - B_{1t_{j-1}}) + bW\}\{\int_{t_{j-1}}^{t_j} dB_{1s}\}] \\ &= aE[(B_{1t_j} - B_{1t_{j-1}})^2] \\ &= a(t_j - t_{j-1}) \end{aligned} \tag{6.14}$$

Using 6.13 and 6.14

$$a(t_j - t_{j-1}) = \int_{t_{j-1}}^{t_j} e^{-\alpha(t_j - s)}\sigma_1' ds \tag{6.15}$$

Once again using 6.12

$$Var[I_1] = a^2(t_j - t_{j-1}) + b^2 = \int_{t_{j-1}}^{t_j} e^{-2\alpha(t_j - s)}{\sigma_1'}^2 ds \tag{6.16}$$

Equations 6.15 and 6.16 can be used to calculate a and b. For $\omega = e^{-\alpha(t_j - t_{j-1})}$

$$a = \frac{1}{(t_j - t_{j-1})}\frac{\sigma_1'}{\alpha}(1 - w) \tag{6.17}$$

$$b = \sqrt{\frac{{\sigma_1'}^2}{2\alpha^2}(1 - w)[(\alpha - 2) + w(\alpha + 2)]} \tag{6.18}$$

Going back to r_{t_j}, 6.10 can be written as

$$r_{t_j} = e^{-\alpha(t_j - t_{j-1})} r_{t_{j-1}} + \mu_2(1 - e^{-\alpha(t_j - t_{j-1})})$$

$$+ a(B_{1t_j} - B_{1t_{j-1}}) + bW + N[0, Var(I_2)]$$

Call $c^2 = Var[I_2] + b^2$, where $Var(I_2) = \frac{\sigma_2{}^2}{2\alpha}(1 - w^2)$. Now, using the stock model's set of IID standard normal random variables Z_{1j} and another set of independent standard normal distributed variables Z_{2j},

$$r_{t_j} = \omega r_{t_{j-1}} + \mu_2(1 - \omega) + a\sqrt{t_j - t_{j-1}} Z_{1j} + cZ_{2j} \tag{6.19}$$

Initial Distribution

We are assuming that the bond yields are initially in their steady state. With the passage of time the yields follow the equation dynamics and in the long run get back to the steady state. In order to get the steady state distribution, let $t \to \infty$ in above equations 6.5 and 6.6

$$\lim_{t\to\infty} E[r(t)] = \mu_2 \tag{6.20}$$

and

$$\lim_{t\to\infty} Var[r(t)] = \frac{\sigma^2}{2\alpha} \tag{6.21}$$

This process is a stationary process. Hence the steady state distribution of $r(t)$ is Gaussian with mean and variance given in equations 6.20 and 6.22, respectively. Then $r_{t_0} \sim N(\mu_2, \frac{\sigma^2}{2\alpha})$. If Z_3 is standard normal independent of Z_1 and Z_2, then r_{t_0} is to be sampled from

$$r_{t_0} \sim \mu_2 + \sqrt{\frac{\sigma^2}{2\alpha}} Z_3 \tag{6.22}$$

6.5 Algorithm

The algorithm for Monte-Carlo Simulations for a given FIA product is as under:

1. Generate two independent set of standard normal random variables $\{Z_{1j}\}_1^m$, $\{Z_{2j}\}_1^m$ and using the same seed, sample Z_3 from standard normal distribution.

2. Take S_{t_0} be the last data point of historical data for the stock index and r_{t_0} follows the steady state distribution 6.22.

3. Using the parameters determined in the previous section, $\{Z_{1j}\}_1^m$ and equation 6.10, simulate the stock path over $t_1 < t_2 < ... < t_m = T$.

4. Generate the MVA reference rates over the same time periods $t_1 < t_2 < ... < t_m = T$ using $\{Z_{1j}\}_1^m, \{Z_{2j}\}_1^m$, the Vasicek model parameters determined in previous section and the interest rate equation 6.19.

5. Get cash surrender values with MVA and without MVA for each time period, assuming the annuity was surrendered in full at that time.

6. This completes one Monte-Carlo Simulation. Repeat the same process for $10,000$ simulations. This gives a distribution of CSV with MVA and CSV without MVA.

7. Using both distributions, find the distribution for the percentage change in CSV due to MVA, that is

$$\%change = \Big(\frac{\text{CSV with MVA} - \text{CSV without MVA}}{\text{CSV without MVA}}\Big) \text{ x } 100$$

8. Analyze the distributions using Box-whisker plots, histograms, Quantile-Quantile Plots and the central moments.

6.6 Fixed Index Annuity Products

For this internship project I have used 7 Fixed index annuity (FIA) products with MVA to analyze the behavior of cash surrender value during the surrender charge period. However, I am mentioning the contract specifications and numerical results for only one product in this thesis.

There are some assumptions made to simplify each product in order to highlight the behavior of MVA.

1. Time t_0 is the date of issue of the contract. All the premium is paid at that time, there are no other premiums deposited afterwards.

2. All the premium is deposited in only one indexed interest crediting strategy.

3. Cap rates and guaranteed minimum rates are fixed for the surrender charge period.

4. The indexed crediting strategy is decided on the date of issue, and does not change during the surrender charge period.

5. The purchaser of annuity remains alive through out this time.

6. There is only one full cash surrender at the contract anniversary date. There are no partial surrenders or other withdrawals.

7. There are no riders attached to the FIAs except for the market value adjustment rider if it is not a part of the basic contract.

8. There are no applicable fees.

6.6.1 Product A

Contract Specifications

Premium: $\$100,000$
Surrender Charge Period: 10 years
Indexed Interest crediting Strategy:
  i. Annual Point-to-Point crediting with a cap of 4.5%
  ii. Monthly Sum crediting with a cap of 2%
Guaranteed Minimum Value: 87.5% of the premium earning an annual interest rate of 1.35%
Free Withdrawal Amount: None
MVA Reference Rate: Yield of the Bloomberg Barclays US Intermediate Corporate Bond Index
MVA factor: $(\frac{A}{B})^t - 1$, where $A =$ 1+MVA reference rate on the issue date, $B =$ 1+MVA reference rate on the surrender date, t is number of whole years remaining in the surrender charge period.
MVA limits: Minimum of pre-MVA cash surrender value minus guaranteed minimum value and full surrender charge.
Bounds for CSV with MVA: Bounded below by Guaranteed Minimum Value and bounded above by the Account Value.
MVA Calculations: See statement of understanding for details about MVA calculation.
Note: Only 3 years of historical data is available for the MVA reference rate, this small time range is not enough to produce reliable model parameters.

Numerical Findings

The Box and Whisker plot in figure 1 (see Appendix for details about the plot) for percentage change in CSV due to MVA for the annual point-to-point interest crediting strategy shows that the median of the distribution is at zero and the maximum change in surrender value occurs on the third contract anniversary. The impact of MVA is gradually decreasing with each passing year and completely eliminating by the end of the surrender charge period. The first and 3rd quartiles for all years are above -4% and below $+4\%$, respectively. The histograms, show that for each year the CSV without MVA has a left skewed distribution with a few very sharp spikes. These spikes are caused due to the cap on interest rates. The MVA is smoothing out these spikes for the distribution of CSV with MVA (orange bars). For year 9, the CSV with MVA turned into a muti-modal distribution and coincided with the CSV without MVA distribution for the final year of the surrender charge period. The Quantile-Quantile plots (see Appendix for details) indicate that for surrender in early years, CSV without MVA has a distribution quite different from the distribution of CSV with MVA. The piece-wise plots are due to the presence of an interest crediting cap, equal in number to the contract anniversary year. For the advanced years, both distributions are quite similar and completely identical on the 10th contract anniversary. The box-whisker plot for a monthly sum crediting strategy is almost similar to the annual crediting strategy except for outliers in first three years. The histograms for a monthly sum crediting show that the major impact of MVA is only in the first two years, after which both distributions are almost similar and right skewed. Similar results can be drawn from the quantile-quantile plots for the monthly sum crediting strategy.

6.7 Conclusions

This report summarizes the analysis done to find the impact of Market Value Adjustment on Cash Surrender Values across a sample set of fixed index annuities. Geometric Brownian motion is used to model the equity index $S\&P500$ and the Vasicek Model is used as a mean reverting model for the MVA reference asset. Monte-Carlo Simulations generated the distributions for Cash Surrender Values with MVA and without MVA across all time horizons within the surrender charge period, for each FIA contract.

The contract specification for all FIAs are not same, for instance there is a difference in MVA reference asset rates, guaranteed minimum surrender value factor, free account value available at the time of surrender, surrender charge schedule, interest crediting cap rates, MVA reference rates, MVA formulation, bounds for cash surrender value, etc. Hence, there is no direct comparison of two FIA contracts, but a generalized perspective is given below.

For each FIA sample, Box and Whisker plots are used to report the percentage change in Cash Surrender Value due to MVA. For most of the products the median for percentage change is lying on 0 with values scattered symmetrically on both sides of the median.

Combined histograms are plotted for Cash Surrender Value with MVA and Cash Surrender Value without MVA. Across all contracts, CSV without MVA has visible spikes (blue bars) while the presence of MVA is smoothing out the distribution (orange bars) with a wider variety of surrender values. For Annual point-to-point strategy the CSV without MVA and CSV with MVA, both distributions are quite symmetrical, with symmetry more prominent in later years of surrender charge period. But for other crediting strategies considered for this report; monthly sum and annual point to point with participation rate , both distributions are skewed towards right with one major spike. This spike smooths out with the passage of years. Also for these strategies, CSV without MVA and CSV with MVA have almost similar distributions with histograms overlapping each other. The impact of MVA for annual point-to-point with a cap strategy is more pronounced as compared to the other three strategies, where MVA doesn't have a very significant effect.

For Quantile-Quantile plots, the quantiles for CSV without MVA and quantiles for CSV with MVA are graphed against each other. For annual point-point strategy, with the decrease of years remaining in the surrender charge period the Q-Q plot is getting smoother and by the last year both distributions coincide and are perfectly identical. For other strategies, the Q-Q plots are almost lying on the red line except for few outliers depicting almost similar quantiles and distributions.

The first four central moments of all FIA samples for CSV with MVA and CSV without MVA across all time horizon within surrender charge period are mentioned in tables below.

6.8 Recommendations

As per the requirement, the simplest continuous stochastic models are used to model the equity index and the MVA reference rate. Basic techniques are used for model calibration and simulation. A few options for further improvements are listed below:

- The Vasicek Model used to model MVA reference rates has a drawback of generating negative interest rates though the probability of getting negative rates in quite small. A more sophisticated model like Cox–Ingersoll–Ross model can be used to model the MVA reference assets in order to avoid negative rates.

- As per the requirement, the stock index model parameters are not calibrated for this project. The parameters used for Geometric Brownian Motion are simply historical mean and variance of the equity index data series. Since there is a correlation between stock indices and fixed income indices, a better approach is the simultaneous calibration of the Stochastic model and the Interest rate model using a linear programming approach in linear regression.

- More complicated techniques to model the bond yields in MVA reference assets can be used. For example modeling short rates first and then estimating bond yields using the simulated short rates, but this approach can get really complicated due to the presence of large set of bonds in a bonds index.

6.9 Figures and Tables

BOX-WHISKER PLOT

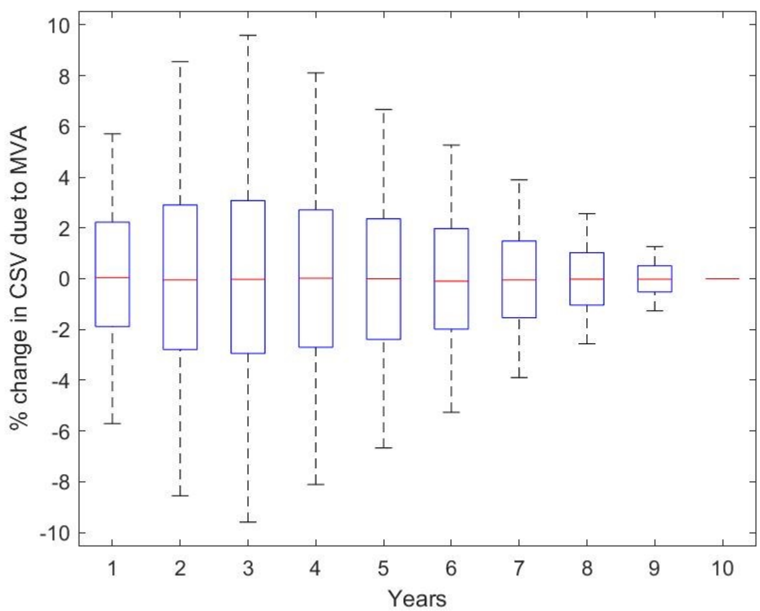
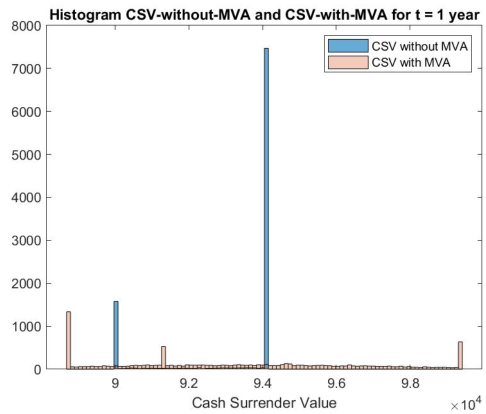
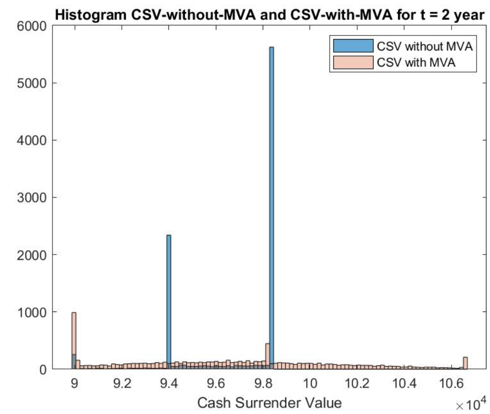


Figure 6.1: Box and Whisker Plot: Percentage Change in Cash Surrender Value due to MVA for Product A. *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

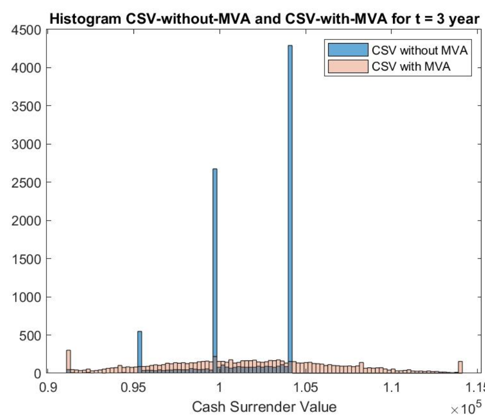
HISTOGRAMS



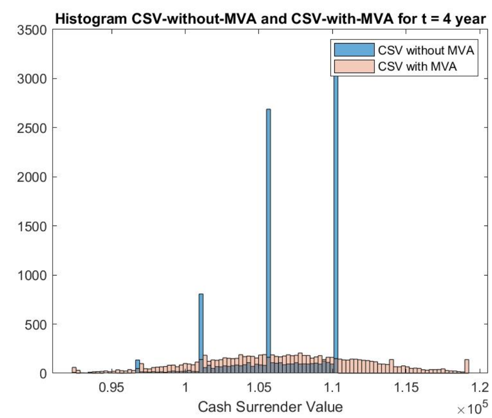
1st contract anniversary



2nd contract anniversary



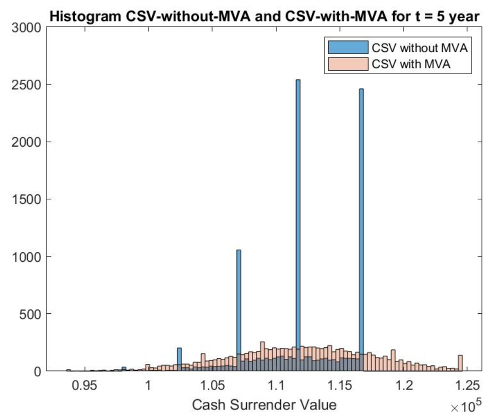
3rd contract anniversary



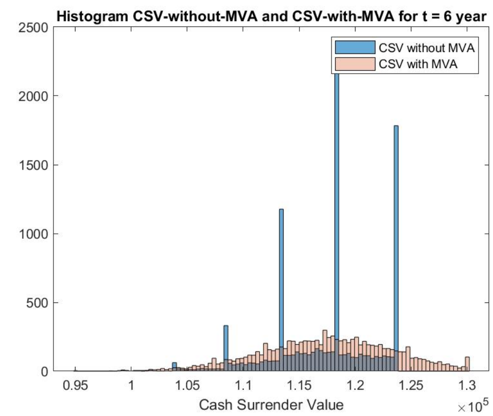
4th contract anniversary

Figure 6.2: Combined Histogram of CSV without MVA and CSV with MVA for full surrender on a contract anniversary (1st to 4th). Product A, *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

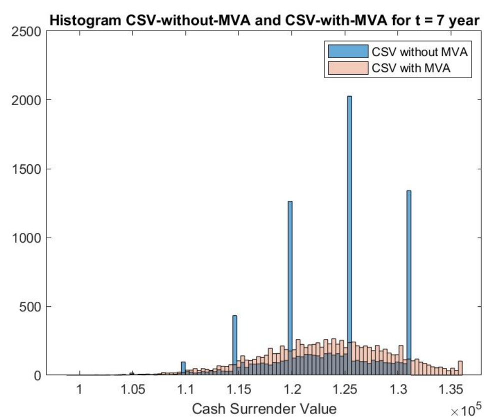
HISTOGRAMS



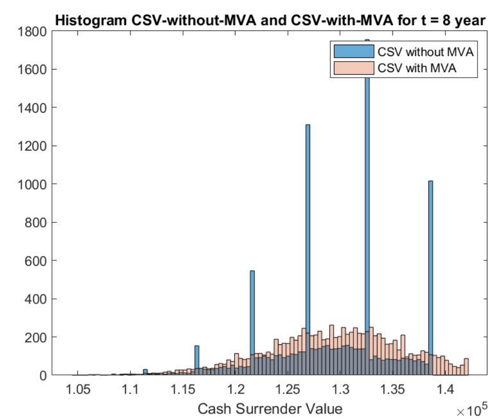
5th contract anniversary



6th contract anniversary



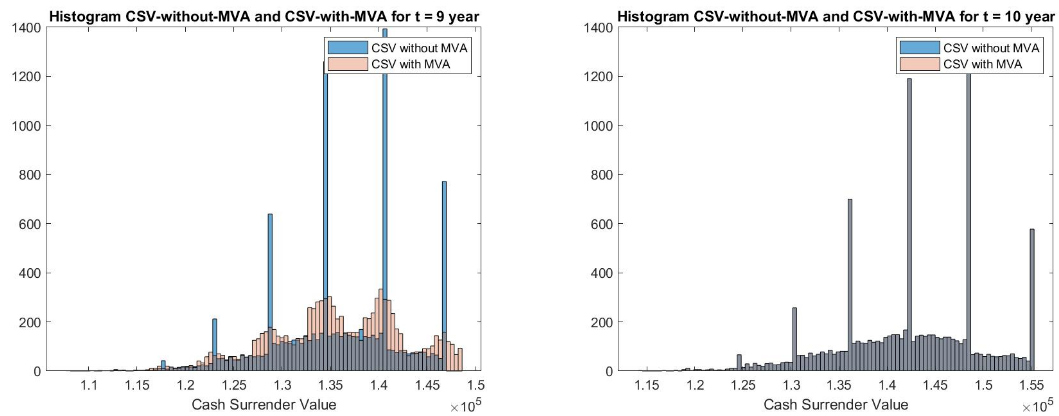
7th contract anniversary



8th contract anniversary

Figure 6.3: Combined Histogram of CSV without MVA and CSV with MVA for full surrender on a contract anniversary (5th to 8th). Product A, *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

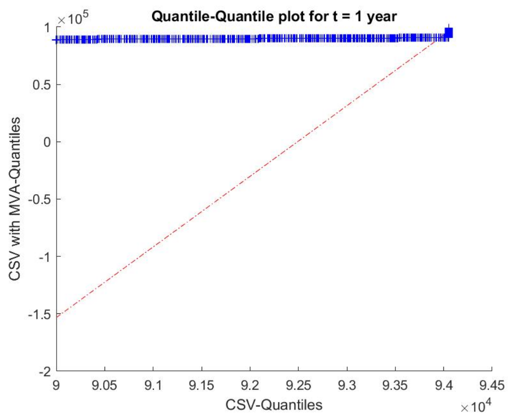
HISTOGRAMS



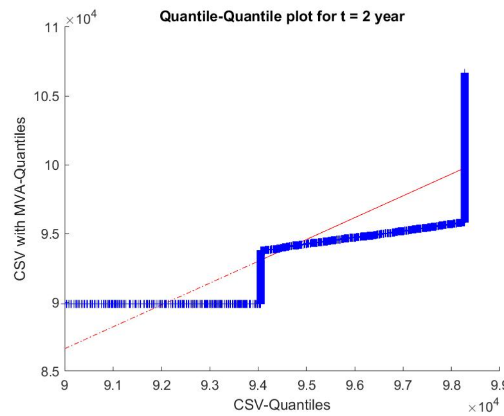
9th contract anniversary 10th contract anniversary

Figure 6.4: Combined Histogram of CSV without MVA and CSV with MVA for full surrender on a contract anniversary (9th and 10th). Product A, *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

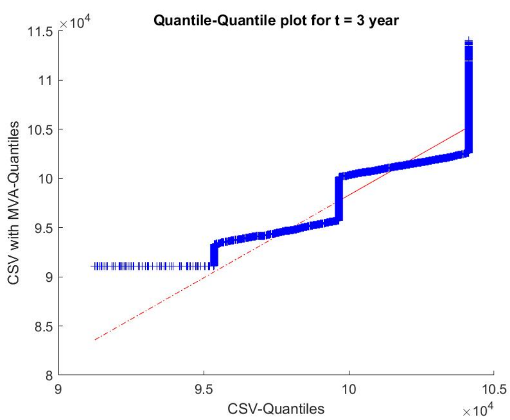
QUANTILE-QUANTILE PLOTS



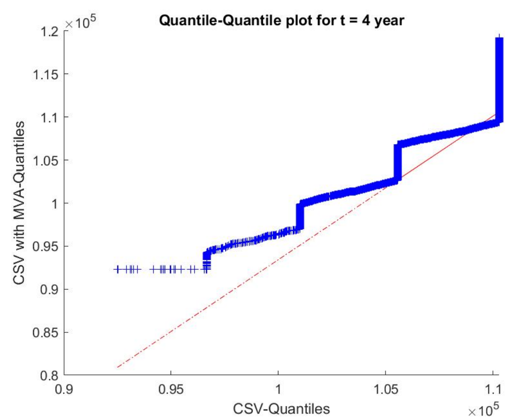
1st contract anniversary



2nd contract anniversary



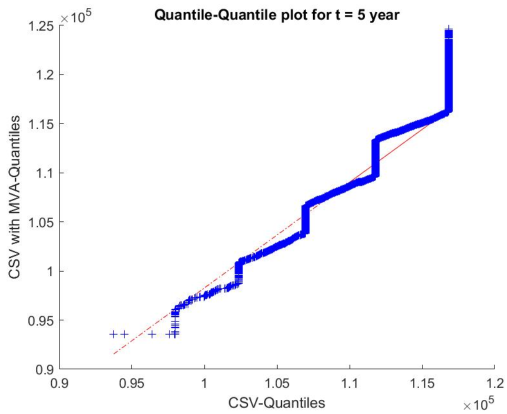
3rd contract anniversary



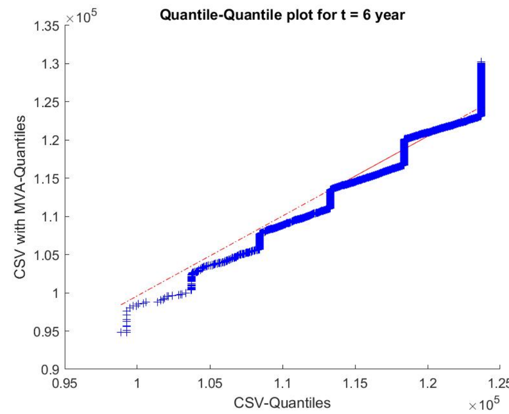
4th contract anniversary

Figure 6.5: Quantile-Quantile Plot of CSV without MVA and CSV with MVA for full surrender on contract anniversary (1st to 4th). Product A, *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

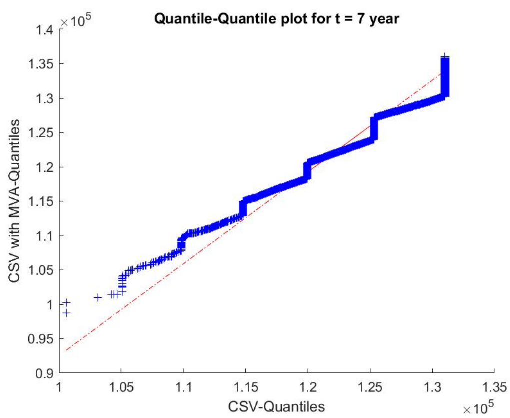
QUANTILE-QUANTILE PLOTS



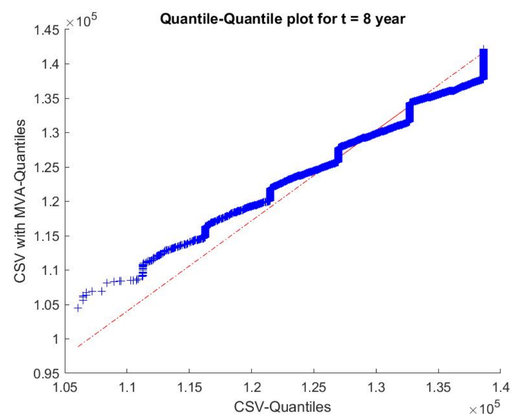
5th contract anniversary



6th contract anniversary



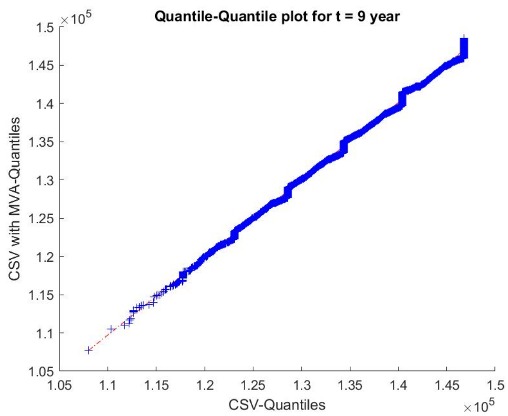
7th contract anniversary



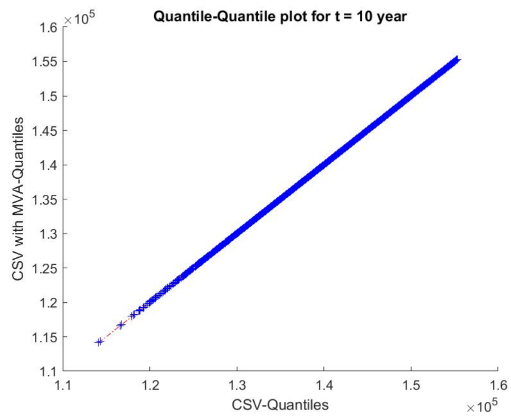
8th contract anniversary

Figure 6.6: Quantile-Quantile Plot of CSV without MVA and CSV with MVA for full surrender on contract anniversary (5th to 8th). Product A, *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

QUANTILE-QUANTILE PLOTS



9th contract anniversary



10th contract anniversary

Figure 6.7: Figures: Quantile-Quantile Plot of CSV without MVA and CSV with MVA for full surrender on contract anniversary (9th and 10th). Product A, *Index Crediting Strategy: Annual point-to-point crediting with cap of* 4.5%

BOX-WHISKER PLOT

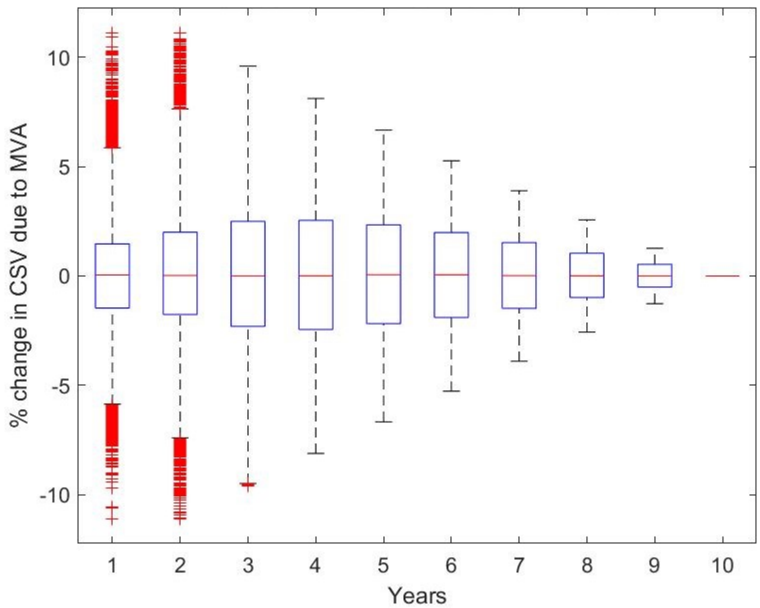
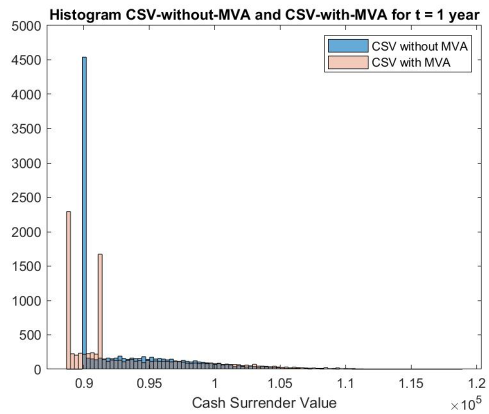
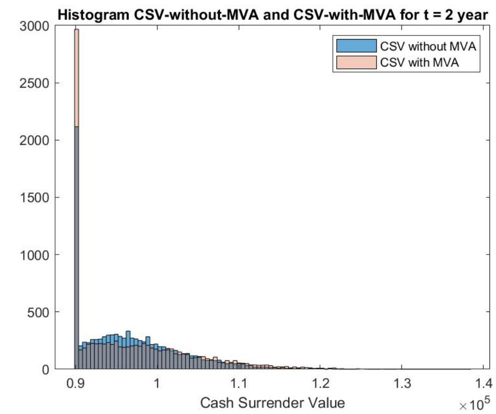


Figure 6.8: Box and Whisker Plot: Percentage Change in Cash Surrender Value due to MVA for Product A. *Index Crediting Strategy: Monthly Sum with cap of* 2%

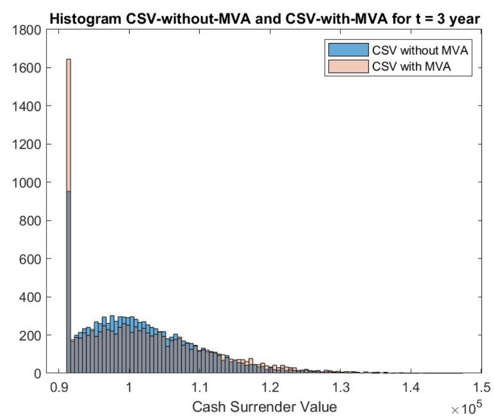
HISTOGRAMS



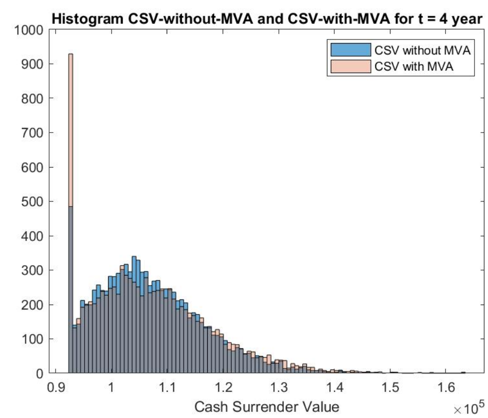
1st contract anniversary



2nd contract anniversary



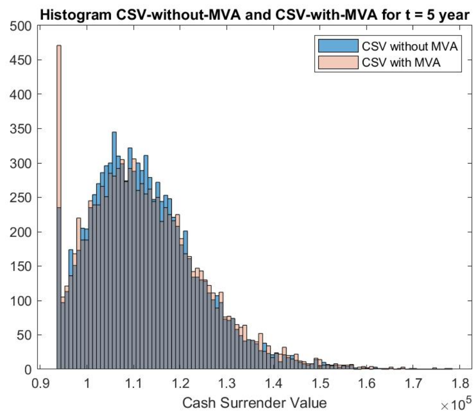
3rd contract anniversary



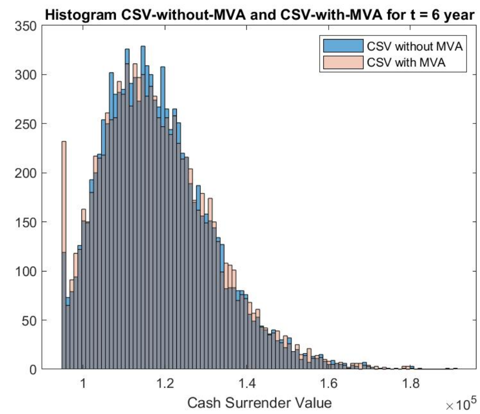
4th contract anniversary

Figure 6.9: Combined Histogram of CSV without MVA and CSV with MVA for full surrender on contract anniversary (1st to 4th). Product A, *Index Crediting Strategy: Monthly Sum with cap of* 2%

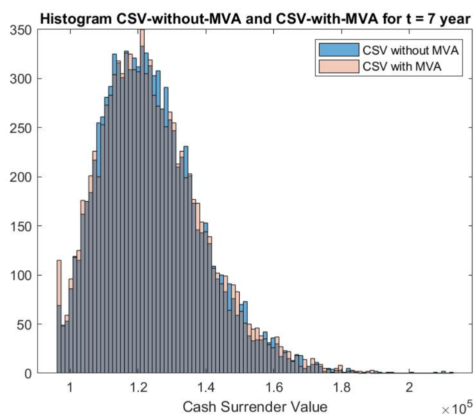
HISTOGRAMS



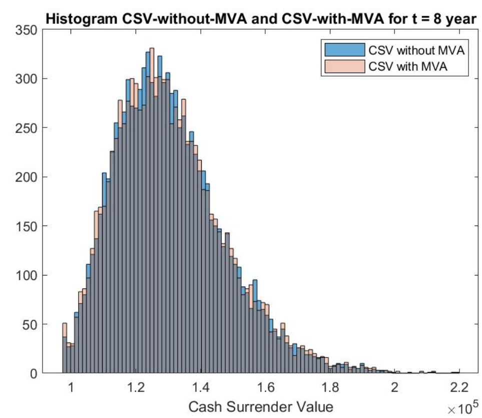
5th contract anniversary



6th contract anniversary



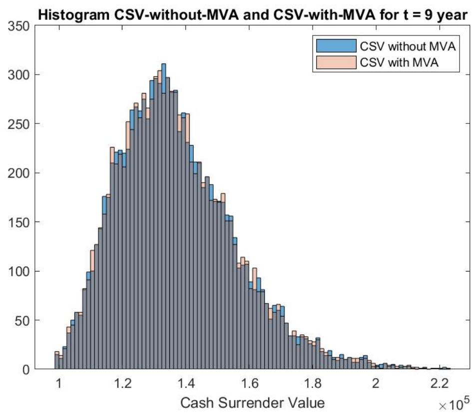
7th contract anniversary



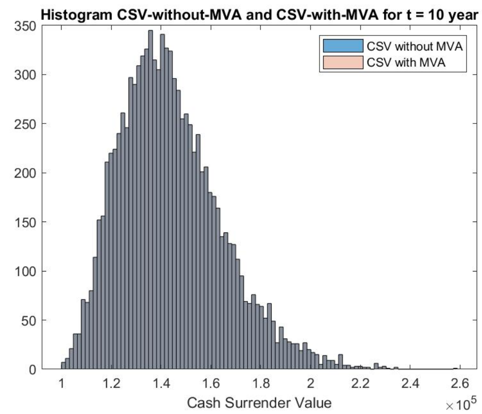
8th contract anniversary

Figure 6.10: Combined Histogram of CSV without MVA and CSV with MVA for full surrender on contract anniversary (5th to 8th). Product A, *Index Crediting Strategy: Monthly Sum with cap of* 2%

HISTOGRAMS



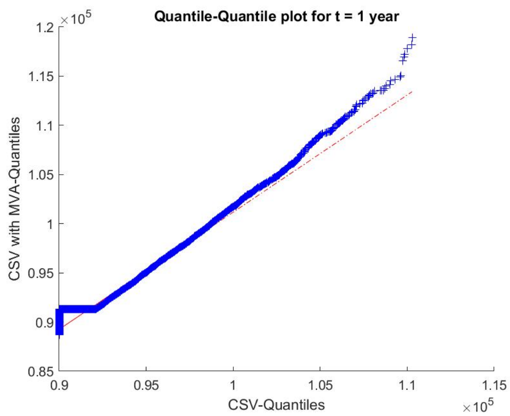
9th contract anniversary



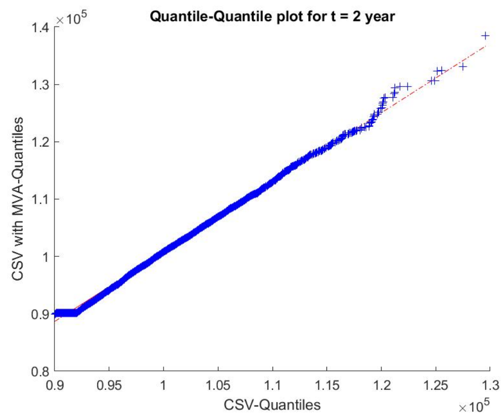
10th contract anniversary

Figure 6.11: Combined Histogram of CSV without MVA and CSV with MVA for full surrender on contract anniversary (9th and 10th). Product A, *Index Crediting Strategy: Monthly Sum with cap of* 2%

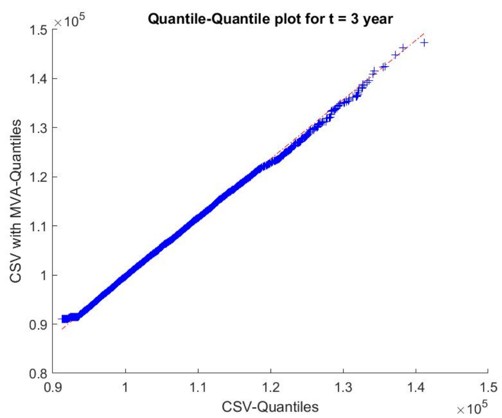
QUANTILE-QUANTILE PLOTS



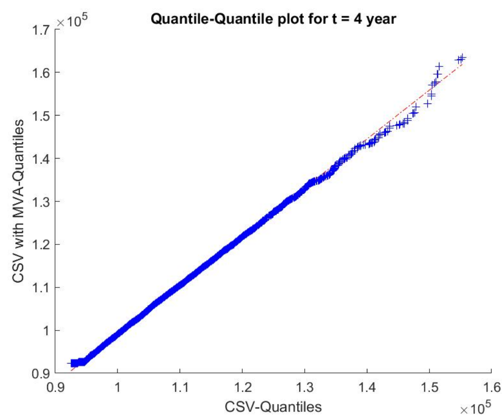
1st contract anniversary



2nd contract anniversary



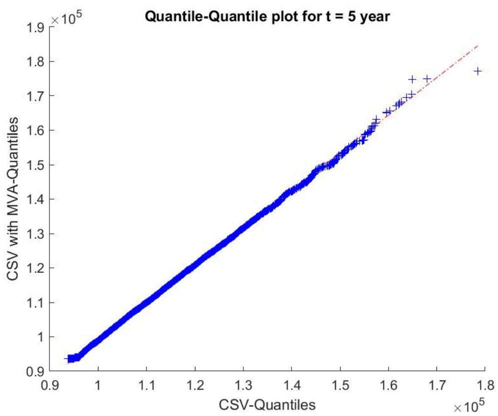
3rd contract anniversary



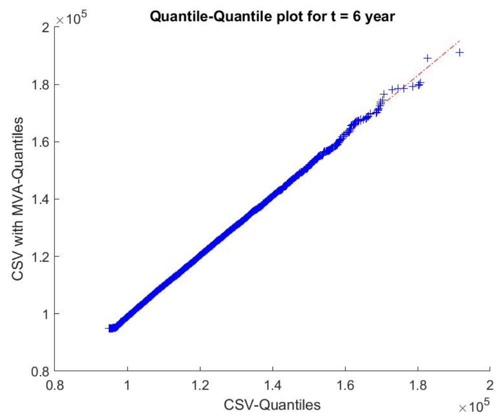
4th contract anniversary

Figure 6.12: Quantile-Quantile Plot of CSV without MVA and CSV with MVA for full surrender on contract anniversary (1st to 4th). Product A, *Index Crediting Strategy: Monthly Sum with cap of* 2%

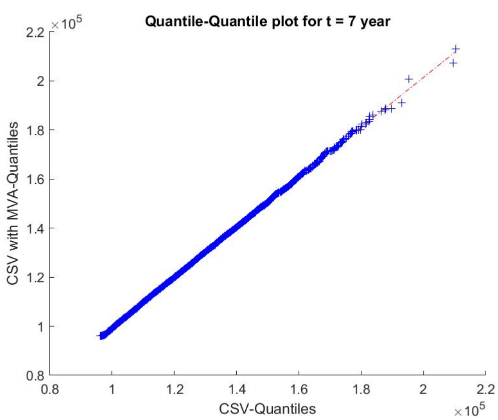
QUANTILE-QUANTILE PLOTS



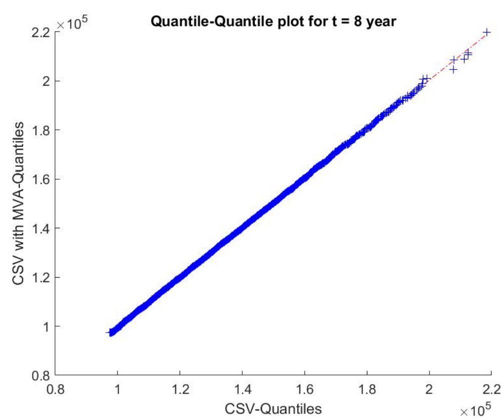
5th contract anniversary



6th contract anniversary



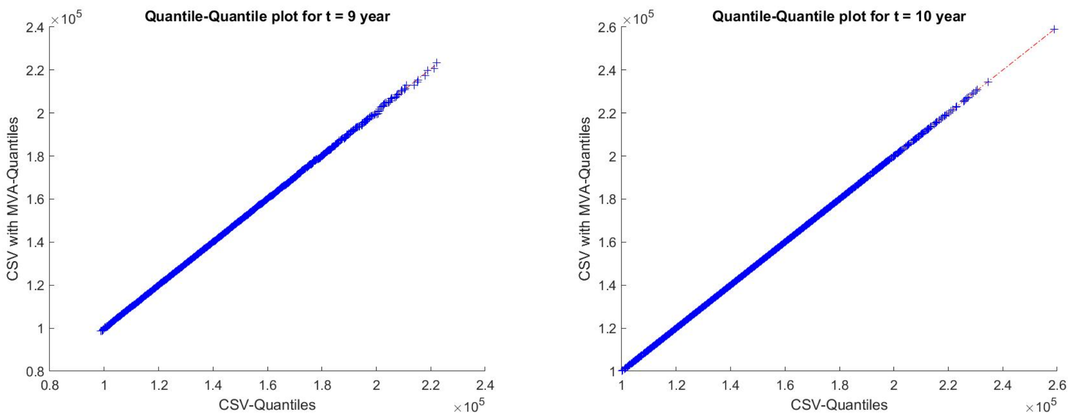
7th contract anniversary



8th contract anniversary

Figure 6.13: Quantile-Quantile Plot of CSV without MVA and CSV with MVA for full surrender on contract anniversary (5th to 8th). Product A, *Index Crediting Strategy: Monthly Sum with cap of* 2%

QUANTILE-QUANTILE PLOTS



9th contract anniversary 10th contract anniversary

Figure 6.14: Quantile-Quantile Plot of CSV without MVA and CSV with MVA for full surrender on contract anniversary (9th and 10th). Product A, *Index Crediting Strategy: Monthly Sum with cap of* 2%

CENTRAL MOMENTS for Product A

	Central Moments of CSV without MVA			Central Moments of CSV with MVA		
Year	**2nd**	**3rd**	**4th**	**2nd**	**3rd**	**4th**
1	2.37353E+06	-5.45975E+09	1.89273E+13	1.12684E+07	9.04759E+09	2.44942E+14
2	5.17644E+06	-1.19711E+10	8.07466E+13	2.04361E+07	2.20967E+10	9.47483E+14
3	8.64209E+06	-2.09916E+10	2.25012E+14	2.85662E+07	1.43952E+10	2.03875E+15
4	1.24316E+07	-3.03891E+10	4.59018E+14	3.15151E+07	-9.39086E+09	2.60095E+15
5	1.70314E+07	-4.16952E+10	8.50710E+14	3.31341E+07	-2.64831E+10	3.00467E+15
6	2.24401E+07	-5.91123E+10	1.50311E+15	3.45779E+07	-4.87507E+10	3.44231E+15
7	2.90486E+07	-7.47711E+10	2.44481E+15	3.71524E+07	-6.39928E+10	3.96236E+15
8	3.63593E+07	-9.13421E+10	3.76864E+15	4.09359E+07	-8.78198E+10	4.80853E+15
9	4.51313E+07	-1.07210E+11	5.67116E+15	4.67177E+07	-1.07926E+11	6.10033E+15
10	5.47962E+07	-1.34426E+11	8.37648E+15	5.47962E+07	-1.34426E+11	8.37648E+15

Table 6.1: Central Moments of Cash Surrender value with and without MVA for Product A. *Index Crediting strategy: Annual Point-to-Point crediting with cap of* 4.5%

	Central Moments of CSV without MVA			Central Moments of CSV with MVA		
Year	**2nd**	**3rd**	**4th**	**2nd**	**3rd**	**4th**
1	1.73994E+07	8.64735E+10	1.08564E+15	2.65446E+07	1.80949E+11	3.01261E+15
2	3.75682E+07	2.21912E+11	5.25889E+15	5.30155E+07	4.36555E+11	1.13725E+16
3	6.28749E+07	4.25098E+11	1.44734E+16	8.41617E+07	7.00659E+11	2.57974E+16
4	9.36531E+07	7.37567E+11	3.36149E+16	1.16602E+08	1.04134E+12	5.13528E+16
5	1.30512E+08	1.22373E+12	6.68752E+16	1.52208E+08	1.52464E+12	9.01709E+16
6	1.72192E+08	1.73793E+12	1.12173E+17	1.89258E+08	1.96144E+12	1.34510E+17
7	2.22806E+08	2.53590E+12	1.87975E+17	2.34721E+08	2.68366E+12	2.06632E+17
8	2.84739E+08	3.66335E+12	3.06033E+17	2.92163E+08	3.78729E+12	3.21152E+17
9	3.45832E+08	4.72634E+12	4.40267E+17	3.48968E+08	4.80228E+12	4.49688E+17
10	4.25550E+08	6.39933E+12	6.63581E+17	4.25550E+08	6.39933E+12	6.63581E+17

Table 6.2: Central Moments of Cash Surrender value with and without MVA for Product A. *Index Crediting strategy: Monthly Sum with a cap of* 2%

PARAMETERS for EQUITY INDEX MODEL and MVA REFERENCE ASSET MODEL

	Product A	Product B	Product C	Product D	Product E	Product F		Product G	
						Asset 1	Asset 2	Asset 1	Asset 2
μ_1	0.1347	0.087446	0.078984	0.077988	0.1421427	0.079228	0.079228	0.052042	0.052042
σ_1	0.124949	0.180231	0.190239	0.160854	0.1545029	0.191121	0.191121	0.190513	0.190513
α	0.356574	0.148447	0.135695	0.056122	1.7316643	0.298945	0.145539	0.270208	0.172422
μ_2	0.034155	0.050263	0.038511	0.056621	0.0325285	0.026271	0.042167	0.023786	0.017303
σ_1'	-5.52E-04	0.001464	0.00204	-3.49E-04	0.0016616	0.003126	0.001813	0.003697	-0.00108
σ_2	0.004791	0.007788	0.008792	0.010421	0.0066065	0.012012	0.008479	0.008932	0.003204
σ	0.004823	0.007924	0.009025	0.010426	0.0068122	0.012412	0.008671	0.009667	0.003382
ρ	-0.11452	0.184758	0.226008	-0.03347	0.2439183	0.251828	0.209089	0.382455	-0.31992

Table 6.3: Parameters for Equity Index Model and MVA Reference Asset Model as a result of Linear Regression

RESULTS from LINEAR REGRESSION

Carrier	Coefficients for the Predictors				Model	
		b_0	b_1	b_2	R-squared	Adjusted R-squared
Product A	estimate	0.000767	-0.02246	-0.0005523	0.0136	0.011
	p-value	0.43152	0.47823	0.0014452		
	confidence interval	[-0.0011 0.0027]	[-0.0846 0.0397]	[-0.0009 -0.0002]		
Product B	estimate	0.00047	-0.00935	0.0014641	0.0345	0.0342
	p-value	0.27052	0.16964	1.56E-51		
	confidence interval	[-0.0004 0.0013]	[-0.0227 0.0040]	[0.0013 0.0017]		
Product C	estimate	0.000329	-0.00855	0.0020398	0.0514	0.0511
	p-value	0.39015	0.24338	4.70E-66		
	confidence interval	[-0.0004 0.0011]	[-0.0229 0.0058]	[0.0018 0.0023]		
Product D	estimate	0.0002	-0.00354	-0.000349	0.00121	0.00107
	p-value	0.33238	0.24337	6.42E-05		
	confidence interval	[-0.0002 0.0006]	[-0.0095 0.0024]	[-0.0005 -0.0002]		
Product E	estimate	0.003548	-0.10908	0.0016616	0.0706	0.0698
	p-value	3.90E-08	3.07E-09	1.84E-35		
	confidence interval	[0.0023 0.0048]	[-0.1450 -0.0731]	[0.0014 0.0019]		
Product F Asset 1	estimate	0.000495	-0.01883	0.0031258	0.0642	0.0639
	p-value	0.15129	0.047763	5.46E-81		
	confidence interval	[-0.0002 0.0012]	[-0.0375 -0.0002]	[0.0028 0.0034]		
Product F Asset 2	estimate	0.000387	-0.00917	0.001813	0.0441	0.0437
	p-value	0.34486	0.22635	7.78E-56		
	confidence interval	[-0.0004 0.0012]	[-0.0240 0.0057]	[0.0016 0.0020]		
Product G Asset 1	estimate	0.000405	-0.01702	0.0036971	0.147	0.147
	p-value	0.25191	0.11193	6.99E-161		
	confidence interval	[-0.0003 0.0011]	[-0.0380 0.0040]	[0.0034 0.0040]		
Product G Asset 2	estimate	0.000188	-0.01086	-0.0010818	0.103	0.102
	p-value	0.22416	0.18054	1.27E-110		
	confidence interval	[-0.0001 0.0005]	[-0.0268 0.0050]	[-0.0012 -0.001]		

Table 6.4: Linear Regression Results for MVA Reference Assets

6.10 Appendix: Statistical Visualization

Box-Whisker Plot [1]

The tops and bottoms of each *box* are the 25th and 75th percentiles of the distribution, respectively. The distances between the tops and bottoms are the inter-quartile ranges. The line in the middle of each box is the sample median. The whiskers are the dashed lines extending above and below each box. Whiskers are drawn from the ends of the inter-quartile ranges to the furthest observations within the whisker length (the adjacent values). The adjacent values are defined as the lowest and highest observations that are still inside the region defined by the following limits:

- Lower Limit: 25^{th} percentile - (1.5 x interquartile range)
- Upper Limit: 75^{th} percentile + (1.5 x interquartile range)

Observations beyond the whisker length are marked as outliers (displayed with a red + sign), the values more than 1.5 times the inter-quartile range away from the top or bottom of the box.

Quantile-Quantile Plot [1]

A Quantile-Quantile plot is used to analyze if two data sets are coming from the same distribution. The Q-Q plot first sorts both data samples from smallest to largest and then places quantiles of one data set along x-axis and the corresponding quantiles of other data along y-axis. The quantiles are based on the number of values in the data set. The blue + are for the distribution quantiles and the red line is the reference line for both distributions being identical. If the resulting plot coincides with red reference line, then the two sets of sample data likely come from the same distribution.

Central Moments

Central Moments are the moments of a distribution of a random variable about its mean. The central moment of order 'k' for the distribution of a population X is defined as

$$m_k = E(X - \mu_k)$$

where μ is the population mean of the random variable X. For discrete version of a population of size the sample version of central moments is computed as:

$$m_k = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^k$$

where $\overline{X}$ is the average or sample mean of the population X. Note that the first-order central moment is zero.

Bibliography

[1] URL: `www.mathworks.com/help`.

[2] URL: `https://www.canada.ca/en/services/benefits/publicpensions/cpp/payment-amounts.html`.

[3] URL: `https://www.canada.ca/en/services/benefits/publicpensions/cpp/old-age-security/payments/tab1-28.html`.

[4] URL: `https://www.canada.ca/en/services/benefits/publicpensions/cpp/old-age-security/payments/tab1-28.html`.

[5] D. Babbel and C. Merrill. *Rational Decumulation.* Working Paper, Wharton Financial Institutions Center. 2006.

[6] E. Barbi et al. "The plateau of human mortality: Demography of longevity pioneers". In: *Science* 360.6396 (2018), pp. 1459–1461.

[7] R. E. Beard. "Some aspects of theories of mortality, cause of death analysis, forecasting and stochastic processes". In: *Biological aspects of demography.* Taylor, Francis, 1971, pp. 57–68.

[8] R. E. Beard, G. E. W. Wolstenholme, and M. O'Connor. "The lifespan of animals". In: *Note on some mathematical mortality models.* Little, Brown, Boston, 1959, pp. 302–311.

[9] M. Bebbington, L. D. Chin, and R. Zitizkis. "Modelling Deceleration in Senescent Mortality". In: *Mathematical Population Studies* 18.1 (2011), pp. 18–37.

[10] D. W. Belsky et al. "Quantification of the pace of biological aging in humans through a blood test, the DunedinPoAm DNA methylation algorithm". In: *eLife* 9 (2020), pp. 1–56.

[11] L. R. Belzile et al. "Is there a cap on longevity? A statistical review". In: *Annual Review of Statistics and Its Application* 9.1 (2022), pp. 21–45.

[12] Z. Bodie, R. C. Merton, and W. F. Samuelson. "Labor supply flexibility and portfolio choice in a life cycle model". In: *Journal of Economic Dynamics and Control* 16.3-4 (1992), pp. 427–449.

[13] Z. Bodie et al. "Optimal consumption–portfolio choices and retirement planning". In: *Journal of economic dynamics and control.* Journal of Economic Dynamics and Control 28.6 (2004), pp. 1115–1148.

[14] M. Butler. “Neoclassical Life-Cycle Consumption: A Textbook Example”. In: *Economic theory* 17.1 (2001), pp. 209–221.

[15] A. J. G. Cairns, D. Blake, and K. Dowd. “A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration”. In: *The Journal of risk and insurance* 73.4 (2006), pp. 687–718.

[16] J. B. Davies. “Uncertain lifetime, consumption, and dissaving in retirement”. In: *The Journal of political economy* 89.3 (1981), pp. 561–577.

[17] A. Deaton. “Saving and Liquidity Constraints”. In: *Econometrica* 59.5 (1991), pp. 1221–1248.

[18] L. Dothan. “On the term structure of interest rates”. In: *Journal of financial economics*. Journal of Financial Economics 6.1 (1978), pp. 59–69.

[19] T. L. Dubina, A. Y. Mints, and E. V. Zhuk. “Biological age and its estimation. III. Introduction of a correction to the multiple regression model of biological age and assessment of biological age in cross-sectional and longitudinal studies”. In: *Experimental gerontology* 19.2 (1984), pp. 133–143.

[20] P. H. Dybvig and H. Liu. “Lifetime consumption and investment: Retirement and constrained borrowing”. In: *Journal of economic theory*. Journal of Economic Theory 145.3 (2010), pp. 885–907.

[21] E. Farhi and S. Panageas. “Saving and Investing for Early Retirement: A Theoretical Analysis”. In: *Journal of Financial Economics* 83.1 (2007), pp. 87–121.

[22] J. Feigenbaum. “Can mortality risk explain the consumption hump?” In: *Journal of macroeconomics*. Journal of Macroeconomics 30.3 (2008), pp. 844–872.

[23] S. Fischer. “A Life Cycle Model of Life Insurance Purchases”. In: *International economic review (Philadelphia)* 14.1 (1973), pp. 132–152.

[24] I. Fisher. *The theory of interest as determined by impatience to spend income and opportunity to invest it*. The Macmillan company, 1930.

[25] J. Gampe. “Human mortality beyond age 110”. In: *Supercentenarians*. Demographic Research Monographs. Springer Berlin Heidelberg, 2010, pp. 219–230.

[26] N. S. Gavrilova and L. A. Gavrilov. “Are We Approaching a Biological Limit to Human Longevity?” In: *The journals of gerontology. Series A, Biological sciences and medical sciences* 75.6 (2020), pp. 1061–1067.

[27] B. Gompertz. “On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies”. In: *Philosophical Transactions of the Royal Society of London* 115 (1825), pp. 513–583.

[28] A. L. Gustman and T. L. Steinmeier. *Retirement and the stock market bubble*. NBER Working Paper No. w9404. 2002.

[29] *How much do I need to retire?* URL: `https://www.fidelity.com/viewpoints/retirement/how-much-money-do-i-need-to-retire`.

[30] H. Huang, M. A. Milevsky, and T. S. Salisbury. “Optimal retirement consumption with a stochastic force of mortality”. In: 51.2 (2012), pp. 282–291.

[31] H. Huang, M. A. Milevsky, and T. S. Salisbury. “Retirement spending and biological age”. In: *Journal of economic dynamics and control* 84 (2017), pp. 58–76.

[32] S. H. Jackson, M. R. Weale, and R. A. Weale. “Biological age—what is it and can it be measured?” In: *Archives of gerontology and geriatrics* 36.2 (2003), pp. 103–115.

[33] J. Jylhävä, N. L. Pedersen, and S. Hägg. “Biological Age Predictors”. In: *EBioMedicine* 21.C (2017), pp. 29–36.

[34] G. Kingston and S. Thorp. “Annuitization and asset allocation with HARA utility”. In: *Journal of pension economics and finance* 4.3 (2005), pp. 225–248.

[35] J. L. Koo, B. L. Koo, and Y. H. Shin. “An optimal investment, consumption, leisure, and voluntary retirement problem with Cobb–Douglas utility: Dynamic programming approaches”. In: *Applied Mathematics Letters* 26.4 (2013), pp. 481–486.

[36] G. Kula. *Optimal retirement decision, PhD Thesis*. 2003.

[37] M. Lachance. “Optimal onset and exhaustion of retirement savings in a life-cycle model”. In: *Journal of pension economics and finance* 11.1 (2012), pp. 21–52.

[38] S. Laskow. *How retirement was invented?* Accessed: 2022-09. URL: `https://www.theatlantic.com/business/archive/2014/10/how-retirement-was-invented/381802/`.

[39] R. D. Lee and L. R. Carter. “Modeling and Forecasting U.S. Mortality”. In: *Journal of the American Statistical Association* 87.419 (1992), pp. 659–671.

[40] S. F. Leung. “Uncertain Lifetime, The Theory of the Consumer, and the Life Cycle Hypothesis”. In: *Econometrica* 62.5 (1994), pp. 1233–1239.

[41] D. Levhari and L. J. Mirman. “Savings and Consumption with an Uncertain Horizon”. In: *The Journal of political economy* 85.2 (1977), pp. 265–281.

[42] X. Liu and X. S. Lin. “A subordinated Markov model for stochastic mortality”. In: *European actuarial journal* 2.1 (2012), pp. 105–127.

[43] W. M. Makeham. “On the Law of Mortality and the Construction of Annuity Tables”. In: *The Assurance magazine and Journal of the Institute of Actuaries* 8.6 (1860), pp. 301–310.

[44] F. Menoncin and L. Regis. “Longevity-linked assets and pre-retirement consumption/portfolio decisions”. In: *Insurance, Mathematics and Economics* 76 (2017), pp. 75–86.

[45] F. Menoncin and S. Vergalli. “Optimal stopping time, consumption, labour, and portfolio decision for a pension scheme”. In: *Journal of Economics* 132 (2021), pp. 67–98.

[46] M. A. Milevsky. *Longevity Insurance for a Biological Age: Why Your Retirement Plan Shouldn't Be Based on the Number of Times You Circled the Sun*. Independently Published, 2019. ISBN: 9781790658268.

[47] M. A. Milevsky. *The calculus of retirement income : financial models for pension annuities and life insurance*. Cambridge University Press, 2006. ISBN: 9780511191794.

[48] M. A. Milevsky and H. Huang. "Spending Retirement on Planet Vulcan: The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates". In: *Financial analysts journal* 67.2 (2011), pp. 45–58.

[49] F. Modigliani. "Life Cycle, Individual Thrift, and the Wealth of Nations". In: *The American economic review* 76.3 (1986), pp. 297–313.

[50] F. Modigliani and R. H. Brumberg. "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data". In: *Kurihara, K.K., Ed., Post-Keynesian Economics* (1954), pp. 388–436.

[51] M. H. Park. "An analytical solution to the inverse consumption function with liquidity constraints". In: *Economics letters*. Economics Letters 92.3 (2006), pp. 389–394.

[52] M. Pearce and A. E. Raftery. "Probabilistic forecasting of maximum human lifespan by 2100 using Bayesian population projections". In: *Demographic research* 44 (2021), pp. 1271–1294.

[53] W. Perks. "On Some Experiments in the Graduation of Mortality Statistics". In: *Journal of the Institute of Actuaries* 63 (1932), pp. 12–57.

[54] G. Peskir and A. Shiryaev. *Optimal stopping and free-boundary problems*. Springer, 2006.

[55] E. Pitacco et al. *Modelling Longevity Dynamics for Pensions and Annuity Business*. OUP Catalogue. Oxford University Press, Incorporated, 2009. ISBN: 0199547270.

[56] F. P. Ramsey. "A Mathematical Theory of Saving". In: *The Economic journal (London)* 38.152 (1928), pp. 543–559.

[57] W. Ries and D. Pöthig. "Chronological and biological age". In: *Experimental gerontology* 19.3 (1984), pp. 211–216.

[58] L. C. G. Rogers. *Optimal Investment*. 2013th ed. Springer Briefs in Quantitative Finance. Berlin, Heidelberg: Springer Berlin Heidelberg.

[59] H. Rootzén and D. Zholud. "Human life is unlimited – but short". In: *Extremes (Boston)* 20.4 (2017), pp. 713–728.

[60] J. S. Rosenthal. *A first look at rigorous probability theory*. Second Edition. World Scientific, 2006.

[61] A. R. Thatcher. "The Long-Term Pattern of Adult Mortality and the Highest Attained Age". In: *Journal of the Royal Statistical Society. Series A (Statistics in Society)* 162.1 (1999), pp. 5–43.

[62] A. R. Thatcher, V. Kannisto, and J. W. Vaupel. *The Force of Mortality at Ages 80 to 120*. Monographs on population aging. University Press of Southern Denmark, 1998. ISBN: 9788778383815.

[63] J. W. Vaupel, K. G. Manton, and E. Stallard. “The impact of heterogeneity in individual frailty on the dynamics of mortality”. In: *Demography* 16.3 (1979), pp. 439–54.

[64] M. Wallmeier and F. Zainhofer. “How to invest over the life cycle: Insights from theory”. In: *Journal für Betriebswirtschaft* 56.4 (2006), pp. 219–244.

[65] M. E. Yaari. “Uncertain Lifetime, Life Insurance, and the Theory of the Consumer”. In: *The Review of economic studies* 32.2 (1965), pp. 137–150.

[66] R. Zhou, J. S.-H. Li, and K.S. Tan. “Economic Pricing of Mortality-linked Securities in the Presence of Population Basis Risk”. In: *Geneva papers on risk and insurance. Issues and practice* 36.4 (2011), pp. 544–566.