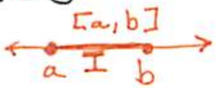
Antiderivatives Introduced

Set-Up: $\left\{ \begin{array}{l} f \text{ function AND} \\ I \text{ an interval} \end{array} \right\}$



Definition: $\boxed{\text{Antiderivative}}$ of f on I:

A function F so that

$F'(x) = f(x)$ for all x in I

Next Time:

Q: How are the different antiderivatives for a function related?

Q: Antiderivative of $\cos(x)$?

$\sin(x)$

$\sin(x)+1$

$\sin(x)-\pi$

$\sin(x)+3e^2$

Q: Antiderivative of e^x?

$e^x+\sqrt{\pi}$

Q: Antiderivative of e^{3x}?

$\frac{1}{3}e^{3x}$

$\frac{1}{3}e^{3x}-\frac{1}{2}$

Test: $\frac{d}{dx}\left[\frac{1}{3}e^{3x}\right] = \frac{1}{3}\frac{d}{du}\left[e^u\right]\frac{d}{dx}\left[3x\right]$

$= \frac{1}{3}e^u \cdot 3$

$= \frac{1}{3}\cdot 3e^{3x}$

$= e^{3x}$ ✓

Antiderivatives for a Function Are Related

Important Facts (Theorem):

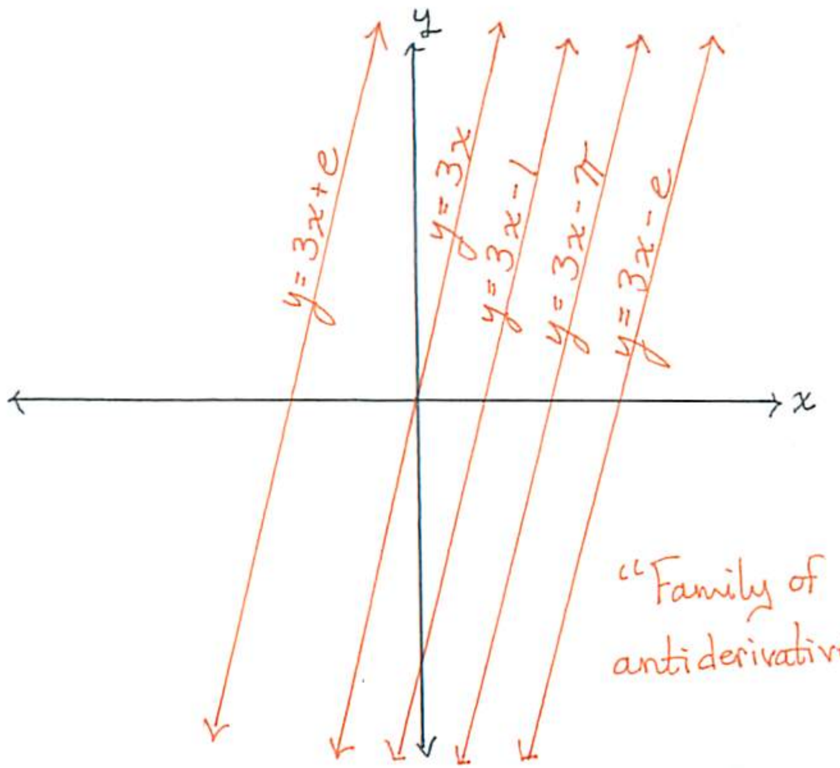
① Antiderivatives are never unique!

② They precisely differ by all constants

Call $F + C$ a $\boxed{\text{generalized antiderivative}}$

Example: $f(x) = 3$

Then $F(x) = 3x + C$ is a generalized antiderivative



"Family of antiderivatives"

Note: The graphs of the various antiderivatives in the family are all vertical translates of each other (& all parallel!)

Antiderivative Rules and Examples

Important Fact: Antidifferentiation is linear!

SUPPOSE $\left\{ \begin{array}{l} F(x) \text{ antiderivative of } f(x) \\ G(x) \text{ antiderivative of } g(x) \\ c \text{ is a constant} \end{array} \right\}$

Function	Antiderivative
$c f(x)$	$c F(x)$
$f(x) + g(x)$	$F(x) + G(x)$

Other Examples and Rules

Function	Antiderivative
$x^n \ (n \neq 1)$	$\frac{1}{n+1} x^{n+1}$
$\frac{1}{x}$	$\ln(\lvert x \rvert)$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$

These antiderivatives only work if nothing is on the "inside" replacing x (because of chain rule)

Fundamental Theorems of Calculus

SUPPOSE f is continuous on $[a,b]$, THEN:

Fundamental Theorem of Calculus 1 (FTC1):

IF $F(x) = \int_a^x f(t)\,dt$,

THEN

$$F'(x) = f(x) \text{ for each } x \text{ in } [a,b]$$

Example:

$$F(x) = \int_1^x \sin(t)\,dt \rightsquigarrow \boxed{F'(x) = \sin(x)}$$

(Just replaced "t" with "x" in the integrand!)

Fundamental Theorem of Calculus 2 (FTC2):

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f

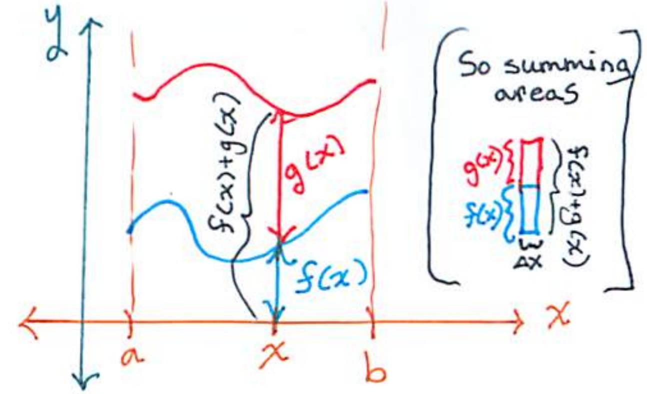
i.e. $\frac{d}{dx}(F) = f$

Example:

$$\int_2^3 x^2\,dx = \left.\frac{x^3}{3}\right|_{x=3} - \left.\frac{x^3}{3}\right|_{x=2}$$

$$= \frac{(3)^3}{3} - \frac{(2)^3}{3} = \boxed{19/3}$$

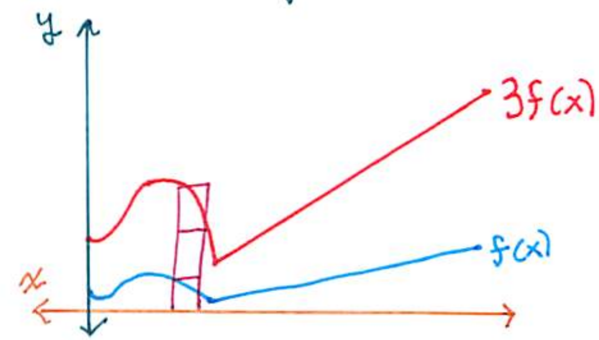
The Definite Integral is Linear



Linearity Means:

① $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

② $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

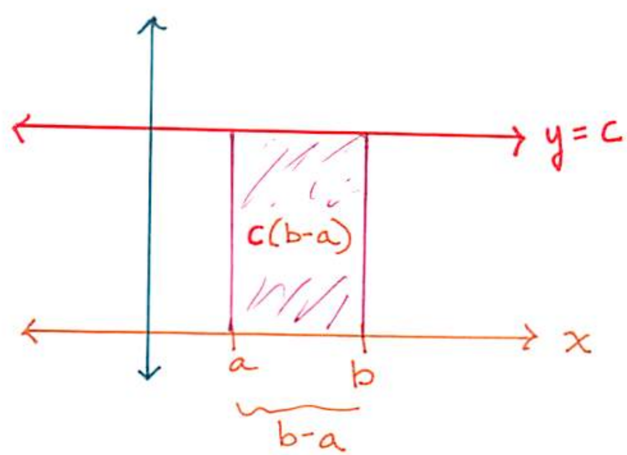


Rectangles 3x as tall (so have 3x the area)

③ $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

(Follows from ① ; ②)

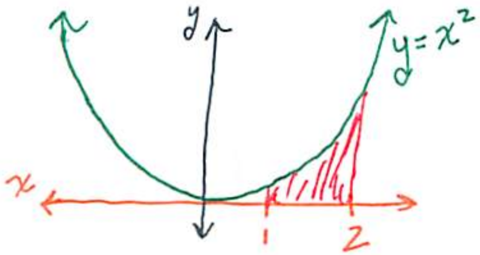
④ $\int_a^b c\,dx = c(b-a)$



Further Definite Integral Properties

(I) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

ex: $\int_1^2 x^2\,dx = -\int_2^1 x^2\,dx$

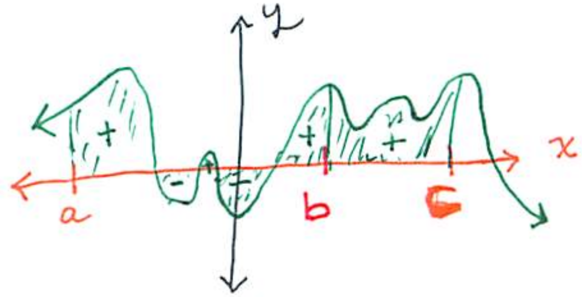


(II) $\int_a^a f(x)\,dx = 0$

ex: $\int_1^1 x^2\,dx = 0$

(III) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Why believable:



ex: $\int_1^2 x^2\,dx + \int_2^3 x^2\,dx = \int_1^3 x^2\,dx$

$\int_a^b f(x)g(x)\,dx \neq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$

NO!

Indefinite Integrals Introduced

Recall Antiderivatives:

Derivatives:

$\frac{d}{dx}[x^2] = 2x$

$\frac{d}{dx}[x^2+1] = 2x$

$\frac{d}{dx}[x^2-\pi] = 2x$

$\frac{d}{dx}[x^2+C] = 2x$

Andiderivative:

What is $2x$ the derivative of? $x^2, x^2+1, x^2-\pi, x^2+C$

$\boxed{x^2+C}$ is the $\boxed{\text{(generalized) antiderivative}}$

$\boxed{\text{Indefinite Integral}}$ of $2x$ is $\int 2x\,dx = x^2+C$

Convention: When a formula for an indefinite integral is given, it's only valid when it's defined

Example 1: Write $\int \frac{1}{x^3}\,dx = \frac{-2}{x^2} + C$

(even though only defined on intervals without zero)

Example 2: $\int \frac{1}{x}\,dx = \ln(|x|) + C$

FIND AN INTEGRAL TABLE TO USE FOR REFERENCE!

Relationship Between Definite & Indefinite Integrals

SUPPOSE $F(x)$ is the antiderivative of $f(x)$
(example: $F(x) = 2e^{3x}$ and $f(x) = 6e^{3x}$)

To get a specific function, must solve for (using initial conditions)

Indefinite Integral

$\int f(x)\,dx = F(x) + C$ is a family of functions

$$\int 6e^{3x}\,dx = 2e^{3x} + C$$

Definite Integral

$\int_a^b f(x)\,dx = F(b) - F(a)$ is a number

$$\int_1^2 6e^{3x}\,dx = \left[2e^{\overbrace{3(2)}^{6}}\right] - \left[2e^{\overbrace{3(1)}^{3}}\right]$$

$$= 2(e^6 - e^3)$$

$$\boxed{\text{Relationship}: \int_a^b f(x)\,dx = \int f(x)\,dx \Big|_{x=a}^{x=b}}$$

(First) Properties of the Indefinite Integral

The Indefinite Integral is also Linear:

① $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

② $\int k f(x)\,dx = k \int f(x)\,dx$

This makes sense since:

① $(F+G)' = F' + G'$

② $(kf)' = k(f)'$

DANGER! DANGER!

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$$

One of our goals will be:

What do

Product Rule: $(f \cdot g)' = fg' + gf'$ AND

Chain Rule: $(f \circ g)' = (f' \circ g) \cdot g'$

tell us about integrals?

Solving For the Constant in an Indefinite Integral

Recall Indefinite Integral: $\int f(x)\,dx = \underbrace{F(x)}_{\text{antiderivative}} + \underbrace{C}_{\text{Don't forget! Must solve for!}}$

Ex Set-Up:

$M(x) = \frac{d}{dx}[R(x)]$

SUPPOSE

- the marginal revenue for a ski resort is $\boxed{M(x) = 50 - .01x}$

AND

- at \$50 per ticket, the ski resort will have $\boxed{2000}$ skiers

$x = 2000$ units

$p = 50$

Task: What is the revenue equation?

$R(x) = \underbrace{p}_{\text{price}}\underbrace{x}_{\text{units}}$

Solution:

① $R(x) = \int M(x)\,dx = \int [50 - .01x]\,dx$

$R(x) = 50x - 0.005x^2 + C \quad (*)$

② Use "initial condition" to solve for C:

$x_0 = 2000, \quad p_0 = 50 \rightsquigarrow$

$R(x_0) = p_0 x_0 = 50 \cdot 2000 = 100{,}000$

$R(x_0) = 50(x_0) - 0.005(x_0)^2 + C$

$100{,}000 = 50(2000) - 0.005(2000)^2 + C$

$C = 920{,}000$

③ Plug C into equation $(*)$

$R(x) = 50x - 0.005x^2 + C$:

$$\boxed{R(x) = 50x - 0.005x^2 + 920{,}000}$$

Substitution for Indefinite Integrals Introduced

ONE OF OUR GOALS:

What do

Product Rule: $(fg)' = fg' + gf'$

AND

Chain Rule: $(f \circ g)' = (f' \circ g) \cdot g'$

tell us about integrals?

Chain Rule $(F \circ g)' = (F' \circ g) \cdot g'$ says

antiderivative of $(F' \circ g) \cdot g'$ is $F \circ g$

SO $\int (F' \circ g) \cdot g' \, dx = F(g(x)) + C$

In other words:

Substitution Rule:

$$\boxed{\int f(\underline{u}) \cdot \underline{\frac{du}{dx}} \, dx = \int f(u) \, du}$$

How to identify: Function on Inside; Derivative next door

Identifying Substitution Situations

Substitution Rule:

$$\int f(u)\cdot\frac{du}{dx}\,dx=\int f(u)\,du$$

Function on Inside

Derivative next door

Ex 1: $\int \frac{2x}{(x^2-11)}\,dx=\int \frac{1}{(x^2-11)}(2x)\,dx$

Ex 2: $\int 3e^{3x}\cos(e^{3x})\,dx=\int \cos(e^{3x})\cdot(e^{3x})\,dx$

Ex 3: $\int 2xe^{x^2}\,dx=\int e^{x^2}\cdot(2x)\,dx$

Substitution for Indefinite Integrals Example 1

$$\boxed{\int f(u)\frac{du}{dx}dx = \int f(u)\,du}$$

derivative next door

Example 1: $\int 3x^2\sqrt{x^3-7}\,dx$

$\sqrt{\ }$ outside

$u = x^3 - 7$ inside

Step 1: Identify $f(u)$ & $u(x)$

Outside: $f(u) = \sqrt{u}$

Inside: $u(x) = x^3 - 7$

Step 2: $du = u'(x)dx$

$du = 3x^2 dx$

Step 3: Solve for dx

$$\frac{du}{3x^2} = dx$$

Step 4: Sub $u(x)$ & dx into $\int 3x^2\sqrt{x^3-7}\,dx$

$$\int \cancel{3x^2}\sqrt{u}\,\frac{du}{\cancel{3x^2}}$$

Can't integrate yet, more than 1 variable

Step 5: Cancel until everything's in terms of u

$$\int \sqrt{u}\,du$$

Step 6: Integrate (wrt u)

$$\int \sqrt{u}\,du = \int u^{1/2}du = \frac{u^{1/2+1}}{1/2+1} = \frac{2}{3}u^{3/2} + C$$

$\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

Step 7: Substitute $u(x) = x^3 - 7$ into $\frac{2}{3}u^{3/2} + C$

$$\frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^3-7)^{3/2} + C$$

Potential solution check!

Step 8: Check (Take derivative to get integrand)

$$\frac{d}{dx}\left[\frac{2}{3}(x^3-7)^{3/2} + C\right] = \frac{2}{3}\left[\frac{3}{2}(x^3-7)^{1/2}\cdot\frac{d}{dx}[x^3-7]\right]$$

$$= (x^3-7)^{1/2}\cdot 3x^2$$

Integrand from original integral so happy!

$$\boxed{\textbf{Answer: } \int 3x^2\sqrt{x^3-7}\,dx = \frac{2}{3}(x^3-7)^{3/2}}$$

Substitution for Indefinite Integrals Example 2

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

Example 2: $\displaystyle\int \overbrace{x^3}^{u'(x)=4x^3}\,\underbrace{\sin}_{\sin(u)}(\overbrace{x^4-3}^{u=x^4-3})\,dx$

Step 1: Identify $f(u)$ & $u(x)$

① $f(u)=\sin(u)$, $u(x)=x^4-3$

Step 2: $du = u'(x)\,dx$

② $du = 4x^3\,dx$

Step 3: Solve for dx

③ $dx = \dfrac{du}{4x^3}$

Step 4: Substitute $u(x)$ & dx into integral

④ $\displaystyle\int x^3 \sin(u)\,\frac{du}{4x^3}$

Step 5: Cancel until everything's in terms of u

⑤ $\displaystyle\frac{1}{4}\int \sin(u)\,du$

Step 6: Integrate (wrt u)

⑥ $-\frac{1}{4}\cos(u)+C$

Step 7: Substitute $u(x)$ back in

⑦ $-\frac{1}{4}\cos(x^4-3)+C$

Step 8: Check that the derivative of the answer is the integrand

⑧ Check $\dfrac{d}{dx}\left[-\frac{1}{4}\cos(x^4-3)+C\right]$

$$= \frac{-1}{4}\left(-\sin(x^4-3)\cdot\frac{d}{dx}(x^4-3)\right)$$

$$= \frac{1}{4}\sin(x^4-3)\cdot 4x^3 = x^3\sin(x^4-3) \checkmark$$

$$\boxed{\text{Answer: } \int x^3\sin(x^4-3)\,dx = x^3\sin(x^4-3)+C}$$

Substitution Rule for Definite Integrals

Substitution Rule for Definite Integrals

SUPPOSE $\left\{ \begin{array}{l} g' \text{ is continuous on } [a,b] \text{ AND} \\ f \text{ is continuous on the range of } u = g(x) \end{array} \right\}$

THEN $\displaystyle\int_{x=a}^{x=b} f(g(x))\,g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$

Example: $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln(x)}} = \int_{x=e}^{x=e^4} \overbrace{(\underbrace{\ln x}_{u=g(x)})^{1/2}}^{u^{1/2}} \cdot \frac{1}{x}\,dx$

Step 1: Identify $f(u)$, $u(x)$

$f(u) = (u)^{-1/2}$, $u(x) = \ln(x)$

Step 2: $du = u'(x)\,dx$

$du = \frac{1}{x}\,dx$

Step 3: Solve for dx

$dx = x\,du$

Step 4: Substitute $\left\{ \begin{array}{l} u(x) = \ln(x) \\ x = x\,du \end{array} \right\}$ into integral

& change bounds

$\displaystyle\int_{u(e) = \ln(e) = 4}^{u(e^4) = \ln(e^4) = 4} (u)^{-1/2} \cdot \frac{1}{x}\,x\,du$

This is the last time we ever see x !!!

Step 5: Cancel until everything is in terms of u

$\displaystyle\int_{u=1}^{u=4} u^{-1/2}\,du$

Step 6: Integrate with respect to u

$\displaystyle\int_{u=1}^{u=4} (u)^{-1/2}\,du = \frac{1}{-\frac{1}{2}+1}\,u^{-\frac{1}{2}+1}\Big|_{u=1}^{u=4} = 2u^{1/2}\Big|_{u=1}^{u=4}$

$= \left[2(4)^{1/2}\right] - \left[2(1)^{1/2}\right] = \boxed{2}$

x never returns, we integrate in terms of u, & sub in the u-bounds

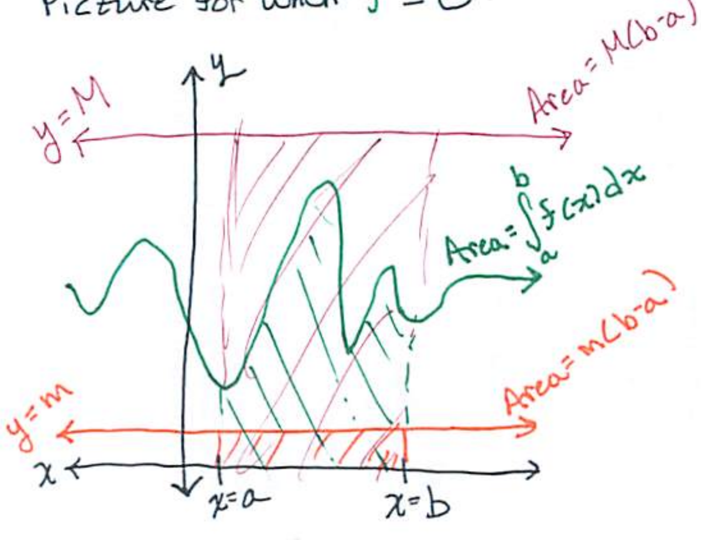
Comparison Properties for the Integral (for f & g Integrable Functions)

SUPPOSE $m \le f(x) \le M$ for each $a \le x \le b$

Theorem: IF $m \le f(x) \le M$ for every $a \le x \le b$

THEN $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Picture for when $f \ge 0$:



Picture holds even when f not always nonnegative!

Ex: $-1 \le \sin(x) \le 1$ on $[-\pi/2, \pi]$

$$-1\left(\pi - \left(\tfrac{-\pi}{2}\right)\right) \le \int_{-\pi/2}^{\pi} \sin(x)\,dx \le 1\left(\pi - \left(\tfrac{-\pi}{2}\right)\right)$$

$$-\frac{3\pi}{2} \le \int_{-\pi/2}^{\pi} \sin(x)\,dx \le \frac{3\pi}{2}$$