ONE-PARAMETER SEMIGROUPS GENERATED BY STRONGLY M-ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS ON EUCLIDEAN SPACES

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Abstract

We begin with a recall of the definitions and the basic properties of classes of elliptic pseudo-differential operators on $\mathbb{R}^n$. Then we introduce a new class of pseudo-differential operators related to the class attributed usually to Shubin [20]. We tease out from this new class of pseudo-differential operators, in particular, the strongly M-elliptic pseudo-differential operators and prove that they are the infinitesimal generators of holomorphic and hence strongly continuous one-parameter semigroups of bounded linear operators on $L^p(\mathbb{R}^n), 1 < p < \infty$. The proof is based on careful refinements of the Agmon–Douglis–Nirenberg estimates for pseudo-differential operators in the book [23] by M.W. Wong. In the case when $p = 2$, we give another proof that strongly (ρ, Λ)-elliptic pseudo-differential operators are infinitesimal generators of strongly continuous one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$ by first proving Gårding's inequality for strongly (ρ, Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$. Results on existence of inverses of (ρ, Λ)-elliptic pseudo-differential operators are also given.

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Notation

The following list of remarks, notations and formulas will be useful to us.

- $\mathbb{R}$ is the set of all real numbers and $\mathbb{C}$ is the set of all complex numbers.
- We mainly focus on the vector spaces over the field of complex numbers. All functions are assumed to be complex-valued unless otherwise specified.
- If we want to emphasize the variable x (or ξ) with respect to which we differentiate, we write ∂_x^α (or ∂_ξ^α) for ∂^α and D_x^α (or D_ξ^α) for D^α.
- The L^p norm of a function f in $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, is denoted by $\|f\|_p$.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ be any two multi-indices.

- $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for $i = 1, 2, ..., n$.
- $\alpha - \beta$ is the multi-index $(\alpha_1 - \beta_1, \alpha_2 - \beta_2, ..., \alpha_n - \beta_n)$ whenever $\beta \leq \alpha$.
- $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n}$ whenever $\beta \leq \alpha$.
- The Leibniz's formula is given by

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta f)(D^{\alpha-\beta} g).$$

- The Schwartz space, denoted by $\mathcal{S}$, is the set of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that for all multi-indices α and β, we have

$$\sup_{x\in\mathbb{R}^n} |x^\alpha(\partial^\beta\varphi)(x)| < \infty.$$

  A function in $\mathcal{S}$ is called a Schwartz function.

- Let $m \in \mathbb{R}$ and let $\rho \in \left(0, \frac{1}{\mu}\right]$, $\mu > \mu_1$. Then we define $S^m_{\rho,\Lambda}$ to be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \leq C_{\alpha,\beta}\Lambda(\xi)^{m-\rho|\beta|}, \quad x,\xi \in \mathbb{R}^n.$$

- A symbol σ in $S^m_{\rho,\Lambda}$ is said to be (ρ,Λ)-elliptic if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C\Lambda(\xi)^m, \quad |\xi| \geq R.$$

- Let $\sigma \in S^m_{\rho,\Lambda}$ be such that there exist positive constants C and R for which

$$\text{Re } \sigma(x,\xi) \geq C\Lambda(\xi)^m, \quad |\xi| \geq R.$$

  We call σ a strongly (ρ,Λ)-elliptic symbol.

- For $m \in \mathbb{R}$ and $\rho \in (0, \frac{1}{\mu}]$, we let $M^m_{\rho,\Lambda}$ be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices γ with $\gamma_j \in \{0, 1\}, j = 1, 2, ..., n$,

$$\xi^\gamma(\partial^\gamma_\xi \sigma)(x, \xi) \in S^m_{\rho,\Lambda}.$$

- A symbol σ in $M^m_{\rho,\Lambda}$ is said to be M-elliptic if there exist positive constants C and R such that

$$|\sigma(x, \xi)| \geq C\Lambda(\xi)^m, \quad |\xi| \geq R.$$

- A symbol σ in $M^m_{\rho,\Lambda}$ is said to be strongly M-elliptic if there exist positive constants C and R such that

$$\text{Re } \sigma(x, \xi) \geq C\Lambda(\xi)^m, \quad |\xi| \geq R.$$

- Let q be a fixed but arbitrary positive number and let Ω be an open subset of the complex plane $\mathbb{C}$. Then for any real number m, we define $S^m_{\rho,\Lambda,q,\Omega}$ to be the set of all functions σ on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega$ such that

$$\sigma(\cdot, \cdot, \lambda) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

for every $\lambda \in \Omega$ and for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(D^\alpha_x D^\beta_\xi \sigma)(x, \xi, \lambda)| \leq C_{\alpha,\beta} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\text{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $\lambda \in \Omega$, where $\operatorname{sgn} : \mathbb{R} \to \{\pm 1, 0\}$ is given by

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

- We denote by $M^m_{\rho,\Lambda,q,\Omega}$ the set of all functions σ in $S^m_{\rho,\Lambda,q,\Omega}$ such that for all multi-indices γ with $\gamma_j \in \{0,1\}$, $j = 1, 2, \ldots, n$,

$$\xi^\gamma (\partial_\xi^\gamma \sigma)(x,\xi,\lambda) \in S^m_{\rho,\Lambda,q,\Omega}.$$

- Let $\sigma \in M^m_{\rho,\Lambda,q,\Omega}$. Suppose that there exist positive constants C and R such that

$$|\sigma(x,\xi,\lambda)| \geq C \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\operatorname{sgn}(-m)q} \Lambda(\xi)^m$$

for all $\lambda \in \Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| + |\lambda| \geq R$. Then we say that σ is $M^m_{\rho,\Lambda,q,\Omega}$-elliptic.

$\mathbb{N}$	The set of all natural numbers including 0.
$\mathbb{R}^n$	The Euclidean space with coordinates $x = (x_1, x_2, ..., x_n)$.
$C^\infty(\mathbb{R}^n)$	The set of all infinitely differentiable functions on $\mathbb{R}^n$.
$C_0^\infty(\mathbb{R}^n)$	The set of all infinitely differentiable functions with compact support on $\mathbb{R}^n$.
$\|x\|$	$x = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$.
∂_k	The partial differential operator $\frac{\partial}{\partial x_k}$ in the direction of x_k-axis.
D_k	$D_k = -i\partial_k$ where i is the imaginary unit.
α	$\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$.
$\|\alpha\|$	$\|\alpha\| = \alpha_1 + \cdots + \alpha_n$.
$\alpha!$	$\alpha \in \mathbb{N}^n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.
∂^α	$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.
x^α	$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

1 Introduction

Let $\mathbb{R}^n$ be the usual Euclidean space give by

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_j{}'s \text{ are real numbers}\}.$$

We denote points in $\mathbb{R}^n$ by x, y, ξ, η, etc. Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be any two points in $\mathbb{R}^n$. The inner product $x \cdot y$ of x and y is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i \cdot y_i$$

and the norm $|x|$ of x is defined by

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$

On $\mathbb{R}^n$, the simplest differential operators are $\frac{\partial}{\partial x_i}, i = 1, 2, ..., n$. We sometimes denote $\frac{\partial}{\partial x_i}$ by ∂_i. We usually find the operator D_j given by $D_j = -i\partial_j, i^2 = -1$, better in expressing formulas.

The most general linear partial differential operator of order m on $\mathbb{R}^n$ is of the form

$$\sum_{\alpha_1+\alpha_2+\cdots\alpha_n\leq m} a_{\alpha_1+\alpha_2+\cdots+\alpha_n}(x)D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}, \tag{1.0.1}$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are nonnegative integers and $a_{\alpha_1+\alpha_2+\cdots+\alpha_n}(x)$ is an infinitely differentiable complex-valued function $\mathbb{R}^n$. If we let

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n),$$

then

$$|\alpha| = \sum_{j-1}^{n} \alpha_j$$

and

$$D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}.$$

The α, given by an n-tuple of nonnegative integers, is called a $multi-index$. We call $|\alpha|$ the *length* of the multi-index α. With the help of multi-indices, we can rewrite our differential operator (1.0.1) in the better form

$$\sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha. \tag{1.0.2}$$

For each fixed x in $\mathbb{R}^n$, the operator (1.0.2) is a polynomial in $D_1, D_2, ..., D_n$. Therefore it is natural to denote the operator (1.0.2) by $P(x, D)$. If we replace D in (1.0.2) by a point $\xi = (\xi_1, \xi_2, ..., \xi_n)$ in $\mathbb{R}^n$, then we obtain a polynomial

$$\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha. \tag{1.0.3}$$

in $\mathbb{R}^n$, where $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. Usually, this polynomial is denoted by $P(x,\xi)$. We call $P(x,\xi)$ the *symbol* of the operator $P(x,D)$.

In this dissertation, we study the partial differential operators (1.0.2) and their generalizations called pseudo-differential operators.

In order to get another representation of the operator $P(x,D)$, let us take any function φ in $\mathcal{S}$. Then, by (1.0.2), (1.0.3), the Fourier transformation and the Fourier inversion formula for Schwartz functions, we have

$$\begin{aligned}
&\big(P(x,D)\varphi)\big)(x)\\
&= \sum_{|\alpha|\le m} a_\alpha(x)(D^\alpha\varphi)(x)\\
&= \sum_{|\alpha|\le m} a_\alpha(x)(\widehat{D^\alpha\varphi})^\vee(x)\\
&= \sum_{|\alpha|\le m} a_\alpha(x)(\xi^\alpha\hat{\varphi})^\vee(x)\\
&= \sum_{|\alpha|\le m} a_\alpha(x)(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\xi^\alpha\hat{\varphi}(\xi)d\xi\\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha\right)\hat{\varphi}(\xi)d\xi\\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}P(x,\xi)\hat{\varphi}(\xi)d\xi. \qquad (1.0.4)
\end{aligned}$$

So, we have represented the partial differential operator $P(x,D)$ in terms of its symbol by means of the Fourier transform. This representation immediately suggests that we can get operators more general than partial differential operators if we replace the symbol $P(x,\xi)$ by more general symbols $\sigma(x,\xi)$, which are no longer polynomials in ξ. The operators we obtained are the so-called pseudo-differential operators. We should point out that in order

to get a useful and tractable class of operators, it is necessary to impose certain conditions on the function $\sigma(x,\xi)$. Many different sets of conditions have been proposed, resulting in many different classes of pseudo-differential operators. For instance, the Hörmander class $S^m_{1,0}$, also denoted by S^m can be found in the book [23] by Wong. This dissertation is devoted to the classes $S^m_{\rho,\Lambda}$ and $M^m_{\rho,\Lambda}$ studied in Garello and Morando [11] and Wong [22].

1.1 The Symbol Classes $S^m_{\rho,\Lambda}$ and $M^m_{\rho,\Lambda}$

Our starting point is based on definitions and results in the paper by Garello and Morando [11], and Wong [22]. Let $\Lambda \in C^\infty(\mathbb{R}^n)$ be a positive function such that there exist positive constants μ_0, μ_1, C_0 and C_1 for which

$$C_0(1+|\xi|)^{\mu_0} \leq \Lambda(\xi) \leq C_1(1+|\xi|)^{\mu_1}, \quad \xi \in \mathbb{R}^n. \tag{1.1.1}$$

Such a function Λ is said to be a weight with polynomial growth. Furthermore, we assume that there exists a positive constant μ with $\mu \geq \mu_1$ and for all multi-indices α and γ with $\gamma_j \in \{0,1\}, j = 1,2,...n$, we can find a positive constant $C_{\alpha,\gamma}$, depending on α and γ only, for which

$$|\xi^\gamma(\partial^{\alpha+\gamma}\Lambda)(\xi)| \leq C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n. \tag{1.1.2}$$

A weaker version of (1.1.2) is to assume that there exists a positive constant μ with $\mu \geq \mu_1$ and for all multi-indices α, we can find a positive constant C_α, depending on α, for which

$$|(\partial^\alpha\Lambda)(\xi)| \leq C_\alpha\Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n. \tag{1.1.3}$$

Let $m \in \mathbb{R}$ and let $\rho \in \left(0, \frac{1}{\mu}\right]$. Then we define $S^m_{\rho,\Lambda}$ to be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{m-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n. \tag{1.1.4}$$

A function in $S^m_{\rho,\Lambda}$ is said to be a symbol of order m and type ρ with weight Λ. It should be noted that if we let Λ be the weight defined by

$$\Lambda(\xi) = \sqrt{1 + |\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

then $S^m_{\rho,\Lambda}$ is the same as the Hörmander class $S^m_{\rho,0}$ in [15]. Let $\sigma \in S^m_{\rho,\Lambda}$. Then we define the pseudo-differential operator T_σ associated to the symbol σ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

for all φ in the Schwartz space $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

It can be shown easily that $T_\sigma : \mathcal{S} \to \mathcal{S}$ is a continuous linear mapping.

1.2 Characteristics of Pseudo-Differential Operators

The L^2-boundedness of these pseudo-differential operators, at least in the case when $\Lambda(\xi) = \sqrt{1+|\xi|^2}, \xi \in \mathbb{R}^n$, is fairly well known and can be found, for instance, in the paper [3] by Beals, the book [15] by Hörmander, the book [16] by Kumano-go, and the book [23] by Wong. However, the L^p- boundedness for $p \neq 2$ is a completely different matter when $\rho < 1$. That the L^p-boundedness is a much harder issue even for the Hörmander class is well documented in the paper [9] by Fefferman. Due to this difficulty, the following class of symbols sitting inside $S^m_{\rho,\Lambda}$ is introduced.

For $m \in \mathbb{R}$ and $\rho \in (0, \frac{1}{\mu}]$, we let $M^m_{\rho,\Lambda}$ be the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices γ with $\gamma_j \in \{0,1\}, j = 1, 2, ..., n$,

$$\xi^\gamma (\partial^\gamma_\xi \sigma)(x,\xi) \in S^m_{\rho,\Lambda}. \tag{1.2.1}$$

Pseudo-differential operators with symbols in $M^m_{\rho,\Lambda}$ have been studied in [11, 12, 22]. Standard results such as asymptotic expansions, products, and formal adjoints similar to those for pseudo-differential operators with symbols given in the book by [23] are used freely.

In fact, one can see from (1.2.1) that when $|\gamma| = 0$, we have $\sigma \in S^m_{\rho,\Lambda}$. So, $M^m_{\rho,\Lambda}$ is a subclass of $S^m_{\rho,\Lambda}$.

The following results are naturally expected and can be found in the book [16] by Kumano-go and the paper [11] by Garello and Morando. They are analogs of the corresponding results for pseudo-differential operators with symbols in $S^m_{1,0}$ given in the book [23] by Wong.

Theorem 1.2.1 *Suppose that* $\sigma_j \in S^{m_j}_{\rho,\Lambda}$, *where*

$$m_0 > m_1 > m_2 > \cdots > m_j \to -\infty$$

as $j \to \infty$. *Then there exists a symbol* σ *in* $S^{m_0}_{\rho,\Lambda}$ *such that*

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_j,$$

i.e.,

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in S^{m_N}_{\rho,\Lambda}$$

for every positive integer N. *Moreover, if* τ *is another symbol with the same asymptotic expansion, then* $\sigma - \tau \in \bigcap_{m\in\mathbb{R}} S^m_{\rho,\Lambda}$.

Theorem 1.2.2 *Let* $\sigma \in S^{m_1}_{\rho,\Lambda}$ *and* $\tau \in S^{m_2}_{\rho,\Lambda}$. *Then* $T_\sigma T_\tau = T_\lambda$, *where* $\lambda \in S^{m_1+m_2}_{\rho,\Lambda}$ *and*

$$\lambda \sim \sum_{\mu} \frac{(-i)^\mu}{\mu!} (\partial^\mu_\xi \sigma)(\partial^\mu_x \tau).$$

Here, the asymptotic expansion means that

$$\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} (\partial^\mu_\xi \sigma)(\partial^\mu_x \tau) \in S^{m_1+m_2-\rho N}_{\rho,\Lambda}$$

for every positive integer N.

Theorem 1.2.3 *Let* $\sigma \in S^m_{\rho,\Lambda}$. *Then the formal adjoint* T^*_σ *of* T_σ *is the pseudo-differential*

operator T_τ, where $\tau \in S^m_{\rho,\Lambda}$ and

$$\tau \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma}.$$

Here, the asymptotic expansion means that

$$\tau - \sum_{|\mu|<N} \frac{(-i)^{\mu}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma} \in S^{m-\rho N}_{\rho,\Lambda}$$

for every positive integer N.

Definition 1.2.1 *A sequence $\{\varphi_j\}$ of functions in the Schwartz space $\mathcal{S}$ is said to be converge to zero in $\mathcal{S}$ (denoted by $\varphi_j \to 0$ in $\mathcal{S}$) if for all multi-indices α and β, we have*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta \varphi_j)(x)| \to 0$$

as $j \to \infty$.

Definition 1.2.2 *A linear functional T on $\mathcal{S}$ is called a tempered distribution if for any sequence $\{\varphi_j\}$ of functions in $\mathcal{S}$ converging to zero in $\mathcal{S}$, we have*

$$T(\varphi_j) \to 0$$

as $j \to \infty$.

Definition 1.2.3 *Let f be a measurable function defined on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^N} dx < \infty$$

for some positive integer N. Then we call f a tempered function.

Tempered distributions can be induced by tempered functions in the following way.

Proposition 1.2.4 *Let f be a tempered function defined on $\mathbb{R}^n$. Then the linear functional T_f on $\mathcal{S}$ defined by*

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \varphi \in \mathcal{S},$$

is a tempered distribution.

Using the formal adjoint, we can extend the definition of a pseudo-differential operator from the Schwartz space $\mathcal{S}$ to the space $\mathcal{S}'$ of all tempered distributions. To wit, let $\sigma \in S^m_{\rho,\Lambda}$. Then for all u in $\mathcal{S}'$, we define the linear functional $T_\sigma u$ on $\mathcal{S}$ by

$$(T_\sigma u)(\varphi) = u\overline{(T^*_\sigma \overline{\varphi})}, \quad \varphi \in \mathcal{S}.$$

It is easy to see that T_σ maps $\mathcal{S}'$ into $\mathcal{S}'$ continuously. In fact, we have the following theorem.

Theorem 1.2.5 *Let $\sigma \in S^0_{\rho,\Lambda}$. Then $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.*

The proof of Theorem 1.2.5 can be found in the book [16] by Kumano-go.

Remark 1.2.6 If

$$\Lambda(\xi) = \sqrt{1+|\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

and

$$\rho \in (0,1),$$

then $S^0_{\rho,\Lambda}$ is the Hörmander class $S^0_{\rho,0}$ and it is well known that Theorem 1.2.5 cannot be extended to $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. However, by focusing on the important subclass $M^0_{\rho,\Lambda}$ of $S^0_{\rho,\Lambda}$, Theorem 1.2.5 is true for $L^p(\mathbb{R}^n)$ for all p with $1 < p < \infty$. See [4, 5, 11, 22], among others, in this connection.

Theorem 1.2.7 *Let $\sigma \in M^0_{\rho,\Lambda}$. Then for $1 < p < \infty$, $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator.*

The proof of Theorem 1.2.7 can be found in the paper [11] by Garello and Morando. An essential ingredient of the proof is the Marcinkiewicz multiplier theorem as formulated in Theorem 1.5 on page 263 of the book [21] by Taylor. It should be noted that the Marcinkiewicz multiplier theorem plays the role of the Mihlin–Hörmander multiplier theorem in the proof of the L^p-boundedness of the classical pseudo-differential operators studied in, for instance, the book [23] by Wong. Because of the fact that the conditions of the Marcinkiewicz multiplier theorem are incorporated into the definition of the symbols in $M^m_{\rho,\Lambda}$, the letter M is used in deference to Marcinkiewicz.

To extend Theorem 1.2.7 to pseudo-differential operators with symbols in $M^m_{\rho,\Lambda}$, where m is an arbitrary real number, we need weighted Sobolev spaces. To this end, we first prove a proposition about the weight function Λ.

Proposition 1.2.8 *$\Lambda \in S^1_{\rho,\Lambda}$ and $\Lambda \in M^1_{\rho,\Lambda}$.*

Proof Let α be a multi-index. Then by (1.1.2), there exists a positive constant C_α such

that

$$|(\partial^\alpha \Lambda)(\xi)| \leq C_\alpha \Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

Since $\rho \in \left(0, \frac{1}{\mu}\right]$, it follows that

$$|(\partial^\alpha \Lambda)(\xi)| \leq C_\alpha \Lambda(\xi)^{1-\rho|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

Therefore

$$\Lambda \in S^1_{\rho,\Lambda}.$$

Now, let γ be a multi-index with $\gamma_j \in 0, 1, j = 1, ..., n$. Then for all multi-indices α,

$$\left|\partial^\alpha(\xi^\gamma \Lambda)(\xi)\right| = \left|\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta \xi^\gamma)(\partial^{\alpha-\beta+\gamma}\Lambda)(\xi)\right| = \left|\sum_{\beta \leq \alpha} \binom{\alpha}{\beta}\binom{\gamma}{\beta} \beta! \xi^{\gamma-\beta}(\partial^{\alpha-\beta+\gamma}\Lambda)(\xi)\right|.$$

Since the entries of $\gamma - \beta$ are in $\{0, 1\}$ and $\rho \in (0, \frac{1}{\mu}]$, it follows that

$$\xi^\gamma(\partial^\gamma \Lambda) \in S^1_{\rho,\Lambda}.$$

This completes the proof. □

Now, we introduce the weighted Sobolev spaces. Let $s \in \mathbb{R}$. Then we define $\Lambda(D)^s$ to be the Fourier multiplier given by

$$\Lambda(D)^s u = \mathcal{F}^{-1}\Lambda^s \mathcal{F} u, \quad u \in \mathcal{S}',$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier transform and the inverse Fourier trans-

form.

For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the Sobolev space $H^{s,p}_{\Lambda}$ of order s with weight Λ by

$$H^{s,p}_{\Lambda} = \{u \in \mathcal{S}' : \Lambda(D)^s u \in L^p(\mathbb{R}^n)\}.$$

Then $H^{s,p}_{\Lambda}$ is a Banach space in which the norm $\| \|_{s,p,\Lambda}$ is given by

$$\|u\|_{s,p,\Lambda} = \|\Lambda(D)^s u\|_{L^p(\mathbb{R}^n)}, \quad u \in H^{s,p}_{\Lambda}.$$

The following result is known as the Sobolev embedding theorem.

Remark 1.2.9 We call $H^{s,p}_{\Lambda}$, the L^p (ρ, Λ)-Sobolev spaces of order s.

Theorem 1.2.10 *For $1 < p < \infty$, $s \leq t$, we have $H^{t,p}_{\Lambda} \subseteq H^{s,p}_{\Lambda}$, and there exists a positive constant $C_{s,t,p}$ depending on s, t and p only, such that*

$$\|u\|_{s,p,\Lambda} \leq C_{s,t,p} \|u\|_{t,p,\Lambda}, \quad u \in H^{t,p}_{\Lambda}.$$

We have the following extension of Theorem 1.2.7.

Theorem 1.2.11 *Let $\sigma \in M^m_{\rho,\Lambda}$. Then for $-\infty < s < \infty$ and $1 < p < \infty$, $T_\sigma : H^{s,p}_{\Lambda} \to H^{s-m,p}_{\Lambda}$ is a bounded linear operator.*

A symbol σ in $S^m_{\rho,\Lambda}$ is said to be (ρ, Λ)-elliptic if there exist positive constants C and R such that

$$|\sigma(x, \xi)| \geq C\Lambda(\xi)^m, \quad |\xi| \geq R. \tag{1.2.2}$$

In the rest of this chapter, we focus on the pseudo-differential operators with symbols in $M^m_{\rho,\Lambda}$.

A symbol σ in $M^m_{\rho,\Lambda}$ is said to be M-elliptic if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C\Lambda(\xi)^m, \quad |\xi| \geq R.$$

It is clear that a M-elliptic symbol σ in $M^m_{\rho,\Lambda}$ is also a (ρ,Λ)-elliptic symbol in $S^m_{\rho,\Lambda}$.

Theorem 1.2.12 *Let $\sigma \in S^m_{\rho,\Lambda}$ be (ρ,Λ)-elliptic. Then there exists a symbol τ in $S^{-m}_{\rho,\Lambda}$ such that*

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R and S are pseudo-differential operators with symbols in $\bigcap_{k\in\mathbb{R}} S^k_{\rho,\Lambda}$.

Remark 1.2.13 In view of Theorem 1.2.12, the same is true for M-elliptic symbols in $M^m_{\rho,\Lambda}$.

The proof of Theorem 1.2.12 is similar to the one in the book [23] by Wong, where the author takes pseudo-differential operators with symbols $\sigma \in S^m_{1,0}$.

The pseudo-differential operator T_τ in the preceding theorem is known as a parametrix of the M-elliptic pseudo-differential operator T_σ. Let $\sigma \in M^m_{\rho,\Lambda}$, $m > 0$. Then $T_\sigma : \mathcal{S} \to \mathcal{S}$ is a continuous linear mapping. So, we can consider T_σ as a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with dense domain $\mathcal{S}$, where $1 < p < \infty$.

1.3 Minimal and Maximal Operators

Let $\sigma \in M^m_{\rho,\Lambda}$, $m > 0$. Then $T_\sigma : \mathcal{S} \to \mathcal{S}$. So, we can consider T_σ as a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with dense domain $\mathcal{S}$, where $1 < p < \infty$.

Proposition 1.3.1 *The linear operator T_σ from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with domain $\mathcal{S}$ is closable.*

A consequence of Proposition 1.3.1 is that the minimal operator $T_{\sigma,0}$ of T_σ exists. Let us recall that the domain $\mathcal{D}(T_{\sigma,0})$ of $T_{\sigma,0}$ consists of all functions f in $L^p(\mathbb{R}^n)$ for which a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{S}$ can be found such that $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma\varphi_k \to f$ in $L^p(\mathbb{R}^n)$ for some f in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Moreover, $T_{\sigma,0}u = f$.

Let u and f be functions in $L^p(\mathbb{R}^n)$. We say that u lies in $\mathcal{D}(T_{\sigma,1})$ and $T_{\sigma,1}u = f$ if and only if

$$(u, T^*_\sigma\varphi) = (f, \varphi), \quad \varphi \in \mathcal{S},$$

where

$$(u, v) = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\, dx$$

for all measurable functions u and v on $\mathbb{R}^n$, provided that the integral exists.

Proposition 1.3.2 *$T_{\sigma,1}$ is a closed linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with domain $\mathcal{D}(T_{\sigma,1})$ containing $\mathcal{S}$.*

Proposition 1.3.3 *$\mathcal{S} \subseteq \mathcal{D}(T_{\sigma,1})$, where $T^t_{\sigma,1}$ is the true adjoint of $T_{\sigma,1}$.*

Proposition 1.3.4 *$T_{\sigma,1}$ is an extension of $T_{\sigma,0}$.*

A consequence of Proposition 1.3.4 is that $T^t_{\sigma,0}$ is an extension of $T^t_{\sigma,1}$. Since $\mathcal{S} \subseteq \mathcal{D}(T^t_{\sigma,1})$, it follows that $\mathcal{S} \subseteq \mathcal{D}(T^t_{\sigma,0})$ as well. In this perspective, we have the following result.

Proposition 1.3.5 *$T_{\sigma,1}$ is the largest closed extension of T_σ having $\mathcal{S}$ contained in the domain of its adjoint. In other words, if B is a closed extension of T_σ such that $\mathcal{S} \subseteq \mathcal{D}(B^t)$, then $T_{\sigma,1}$ is an extension of B.*

In view of the preceding result, we call $T_{\sigma,1}$ the maximal operator of T_σ. We can prove that $T_{\sigma,0} = T_{\sigma,1}$ if σ is M-elliptic. We begin with the following characterization of the domain of $T_{\sigma,0}$.

Theorem 1.3.6 *If $\sigma \in M^m_{\rho,\Lambda}$ is M-elliptic, then $\mathcal{D}(T_{\sigma,0}) = H^{m,p}_\Lambda$.*

To prove Theorem 1.3.6, we use the following estimate, which is the analog of the Agmon–Douglis–Nirenberg estimate in [1] for pseudo-differential operators. In addition, we use a density result. They are given, respectively, as Theorem 1.3.7 and Proposition 1.3.8.

Theorem 1.3.7 *Let $\sigma \in M^m_{\rho,\Lambda}$ be M-elliptic. Then there exist positive constants C_1 and C_2 such that*

$$C_1\|u\|_{m,p,\Lambda} \leq \|T_\sigma u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \leq C_2\|u\|_{m,p,\Lambda}, \quad u \in H^{m,p}_\Lambda.$$

Proof By Theorem 1.2.11 and Theorem 1.2.10, there exists a positive constant C' such that

$$\|T_\sigma u\|_{0,p,\Lambda} + \|u\|_{0,p\Lambda} \leq C'\|u\|_{m,p\Lambda}, \quad u \in H^{m,p}_\Lambda.$$

By Theorem 1.2.12,

$$u = T_\tau T_\sigma u - Ru, \quad u \in H^{m,p}_\Lambda,$$

where $\tau \in M^{-m}_{\rho,\Lambda}$ and R is a pseudo-differential operator with symbol in $\bigcap_{k\in\mathbb{R}} M^k_{\rho,\Lambda}$. Hence, by Theorem 1.2.10, there is a positive constant C such that

$$C\|u\|_{m,p,\Lambda} \leq \|T_\sigma u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda}, \quad u \in H^{m,p}_\Lambda.$$

This completes the proof. □

Proposition 1.3.8 *For $-\infty < s < \infty$ and $1 < p < \infty$, $\mathcal{S}$ is dense in $H^{s,p}_\Lambda$.*

Proof Let $u \in H^{s,p}_\Lambda$. Then $\Lambda(D)^s u \in L^p(\mathbb{R}^n)$. So, there exists a sequence $\{\varphi_k\}_{k=1}^\infty$ of functions in S such that $\varphi_k \to \Lambda(D)^s u$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Let $\psi_k = \Lambda(D)^{-s}\varphi_k, k = 1, 2, ...$ Then $\psi_k \in S$, $k = 1, 2, ...$, and

$$\|\psi_k - u\|_{s,p,\Lambda} = \left\|\Lambda(D)^s\psi_k - \Lambda(D)^s u\right\|_{L^p(\mathbb{R}^n)} = \left\|\varphi_k - \Lambda(D)^s u\right\|_{L^p(\mathbb{R}^n)} \to 0,$$

as $k \to \infty$. □

We are now ready to prove the theorem 1.3.6.

Proof of Theorem 1.3.6 Let $u \in H^{m,p}_\Lambda$. By the density of S in $H^{m,p}_\Lambda$, we can find a sequence $\{\varphi_k\}_{k=1}^\infty$ of functions in S such that $\varphi_k \to u$ in $H^{m,p}_\Lambda$ as $k \to \infty$. By the Agmon–Douglis–Nirenberg estimate, $\{T_\sigma\varphi_k\}_{k=1}^\infty$ and $\{\varphi_k\}_{k=1}^\infty$ are Cauchy sequences in $L^p(\mathbb{R}^n)$. Hence $\varphi_k \to u$ and $T_\sigma\varphi_k \to f$ in $L^p(\mathbb{R}^n)$ for some u and f in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Hence, by the definition of $T_{\sigma,0}$, $u \in D(T_{\sigma,0})$, and $T_{\sigma,0}u = f$. Now, let $u \in D(T_{\sigma,0})$. Then, by the definition of $T_{\sigma,0}$ again, we can find a sequence $\{\varphi_k\}_{k=1}^\infty$ of functions in S for which $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T_\sigma\varphi_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Hence, $\{\varphi_k\}_{k=1}^\infty$ and $\{T_\sigma\varphi_k\}_{k=1}^\infty$ are Cauchy

sequences in $L^p(\mathbb{R}^n)$. So, by the Agmon–Douglis–Nirenberg estimate again, $\{\varphi_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $H_\Lambda^{m,p}$. Since $H_\Lambda^{m,p}$ is complete, it follows that $\varphi_k \to v$ in $H_\Lambda^{m,p}$ for some v in $H_\Lambda^{m,p}$ as $k \to \infty$. By Theorem 1.2.10 on embedding, $\varphi_k \to v$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Hence $u = v$ and $v \in H_\Lambda^{m,p}$. □

We now give an important fact about the minimal and maximal extensions of M-elliptic pseudo-differential operators.

Theorem 1.3.9 *Let $\sigma \in M_{\rho,\Lambda}^m$ be M-elliptic. Then $T_{\sigma,0} = T_{\sigma,1}$.*

Proof

Let $\sigma \in M_{\rho,\lambda}^m$ be M-elliptic. By Theorem 1.3.6, we have $H_\Lambda^{m,p} = D(T_{\sigma,0})$ and by Proposition 1.3.4, since $T_{\sigma,0}$ is the smallest closed extension of T_σ, it is sufficient to prove that $D(T_{\sigma,1}) \subseteq H_\Lambda^{m,p}$. Let $u \in D(T_{\sigma,1})$. Then

$$u = T_\tau T_\sigma u - Ru,$$

where $\tau \in M_{\rho,\Lambda}^{-m}$ and R is a pseudo-differential operator with symbol in $\bigcap_{k\in\mathbb{R}} M_{\rho,\Lambda}^k$. Since

$$T_\sigma u = T_{\sigma,1} u \in L^p(\mathbb{R}^n),$$

it follows from Theorem 1.2.11 that

$$T_\tau T_\sigma u \in H_\Lambda^{m,p}.$$

Since $u \in L^p(\mathbb{R}^n)$ and R has symbol in $M_{\rho,\Lambda}^{-m}$, we use Theorem 1.2.11 again to conclude that

$Ru \in H^{m,p}_{\Lambda}$. Thus, $u \in H^{m,p}_{\Lambda}$ and we are done.

□

Remark 1.3.10 The above theory for $M^m_{\rho,\Lambda}$ in $L^p(\mathbb{R}^n), 1 < p < \infty$, works equally well for $S^m_{\rho,\Lambda}$ in $L^2(\mathbb{R}^n)$.

The overarching contribution in this dissertation is the detailed proof of the existence of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n), 1 < p < \infty$, generated by a strongly M-elliptic pseudo-differential operator on $\mathbb{R}^n$. This result is equivalent to the existence and uniqueness of solutions to the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -(T_{\sigma,0}u)(x,t), \quad x \in \mathbb{R}^n, t > 0, \\ \\ u(\cdot,0) = f, \end{cases} \tag{1.3.1}$$

where $\sigma \in M^m_{\rho,\Lambda}, m > 0$ and f is a function in $L^p(\mathbb{R}^n), 1 < p < \infty$. The proof given in Chapter 3 entails a sharpening of one side of the Agmon– Douglis–Nirenberg inequalities used in the book [23] by Wong for the Hörmander class. In addition, a modification of a functional-analytic result in the book [17] by Pazy is needed. When $p = 2$, we give in Chapter 2 a related result to the effect that a strongly (ρ, Λ)-elliptic pseudo-differential operator on $\mathbb{R}^n$ is the infinitesimal generator of a strongly continuous one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$. While this is a special case of the abovementioned result, we give another proof. To that end, we first establish the Gårding's inequality for strongly (ρ, Λ)-elliptic pseudo-differential operator on $\mathbb{R}^n$, which is a useful result in its own right. In

Chapter 4, results on the regularity of the solutions of the initial value problem described above are given.

2 Initial Value Problems for Heat Equations Generated by Strongly Elliptic (ρ, Λ)-Pseudo-Differential Operators on $\mathbb{R}^n$

The aim of this Chapter is to use Gårding's inequality to prove that strongly (ρ, Λ)-elliptic pseudo-differential operators are infinitesimal generators of strongly continuous one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$. In Section 2.1, we give Gårding's inequality for strongly (ρ, Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$. In Section 2.3, we aim at giving the key result on the existence and uniqueness of weak (and hence strong) solutions of Poisson's equations modelled by (ρ, Λ)-elliptic pseudo-differential operators on $\mathbb{R}^n$. The key result in Section 2.3 and the Hille–Yosida–Phillips theorem are used in Section 2.4 to prove that strongly (ρ, Λ)-elliptic pseudo-differential operators generate strongly continuous one-parameter semigroups of bounded linear operators on $L^2(\mathbb{R}^n)$. We give in Section 2.5 solutions of initial value problems for heat equations governed by strongly (ρ, Λ)-elliptic pseudo-differential operators.

Related spectral results can be found in [4, 5, 22]. Results on microlocal analysis of M-elliptic pseudo-differential operators can be found in [13, 14].

2.1 Gårding's Inequality

The main result in this section is Gårding's inequality for strongly (ρ, Λ)-elliptic operators. It is an extension of Gårding's inequality in Chapter 17 of the book [23].

Theorem 2.1.1 (Gårding's Inequality) *Let $\sigma \in S^{2m}_{\rho,\Lambda}$ be such that there exist positive constants C and R for which*

$$\operatorname{Re}\sigma(x,\xi) \geq C\Lambda(\xi)^{2m}, \quad |\xi| \geq R.$$

Then we can find a positive constant C' and a constant C_s for every real number $s \in \left[\frac{\rho}{2}, \infty\right)$ such that

$$\operatorname{Re}\left(T_\sigma\varphi, \varphi\right) \geq C'\|\varphi\|^2_{m,2,\Lambda} - C_s\|\varphi\|^2_{m-\rho s,2,\Lambda}, \quad \varphi \in \mathcal{S}.$$

A symbol satisfying the hypothesis of the theorem is said to be strongly (ρ, Λ)-elliptic. In order to prove the theorem, we need some preliminary results, some of which are of interest in their own right.

Lemma 2.1.2 *Let $F \in C^\infty(\mathbb{C})$. Then for every σ in $S^0_{\rho,\Lambda}$, $F \circ \sigma \in S^0_{\rho,\Lambda}$.*

Proof We need to prove that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(\partial^\alpha_x \partial^\beta_\xi (F \circ \sigma))(x,\xi)| \leq C_{\alpha,\beta}\Lambda(\xi)^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n. \tag{2.1.1}$$

(2.1.1) is true for all multi-indices α and β with $|\alpha + \beta| = 0$. Indeed, there exists a positive constant C such that

$$|\sigma(x,\xi)| \leq C, \quad x, \xi \in \mathbb{R}^n.$$

Thus, $F \circ \sigma$ is in fact a C^∞ function on a compact subset of $\mathbb{R}^{2n}$. Hence there exists another positive constant C' such that

$$|(F \circ \sigma)(x, \xi)| \leq C', \quad x, \xi \in \mathbb{R}^n.$$

Now, suppose that (2.1.1) is valid for all F in $C^\infty(\mathbb{C})$, σ in $S^0_{\rho,\Lambda}$ and multi-indices α and β with $|\alpha + \beta| = l$. Let α and β be multi-indices with

$$|\alpha + \beta| = l + 1.$$

We first suppose that

$$\partial_x^\alpha \partial_\xi^\beta = \partial_x^\alpha \partial_\xi^\gamma \partial_{\xi_j}$$

for some multi-index γ and some $j = 1, 2, \ldots, n$. Then, by the chain rule,

$$(\partial_x^\alpha \partial_\xi^\beta (F \circ \sigma))(x, \xi) = (\partial_x^\alpha \partial_\xi^\gamma (F_1 \circ \sigma) \partial_{\xi_j} \sigma + (F_2 \circ \sigma) \partial_{\xi_j} \sigma))(x, \xi)$$

for all x and ξ in $\mathbb{R}^n$, where F_1 and F_2 are the partial derivatives of F with respect to, respectively, the first and second variables. Now, by Leibniz' formula and the induction

hypothesis, there exist positive constants $C_{\lambda,\delta}$ and $C_{\alpha,\lambda,\gamma,\delta,j}$ such that

$$\begin{aligned}
& |\partial_x^\alpha \partial_\xi^\gamma ((F_1 \circ \sigma)\partial_{\xi_j}\sigma)(x,\xi)| \\
\leq & \sum_{\lambda\leq\alpha,\delta\leq\gamma} \binom{\alpha}{\lambda}\binom{\gamma}{\delta} |(\partial_x^\lambda \partial_\xi^\delta (F_1\circ\sigma))(x,\xi)|\, |(\partial_x^{\alpha-\lambda}\partial_\xi^{\gamma-\delta}(\partial_{\xi_j}\sigma))(x,\xi)| \\
\leq & \sum_{\lambda\leq\alpha,\delta\leq\gamma} \binom{\alpha}{\lambda}\binom{\gamma}{\delta} C_{\lambda,\delta}\Lambda(\xi)^{-\rho|\delta|} C_{\alpha,\lambda,\gamma,\delta,j}\Lambda(\xi)^{-\rho|\gamma|+\rho|\delta|-\rho} \\
= & C_{\alpha,\gamma,j}\Lambda(\xi)^{-\rho(|\gamma|+1)}, \quad x,\xi\in\mathbb{R}^n,
\end{aligned}$$

where

$$C_{\alpha,\gamma,j} = \sum_{\lambda\leq\alpha,\delta\leq\gamma} \binom{\alpha}{\lambda}\binom{\gamma}{\delta} C_{\lambda,\delta} C_{\alpha,\lambda,\gamma,\delta,j}.$$

Similarly, there exists a positive constant $C'_{\alpha,\gamma,j}$ such that

$$|(\partial_x^\alpha \partial_\xi^\gamma ((F_2\circ\sigma)\partial_{\xi_j}\sigma)(x,\xi)| \leq C'_{\alpha,\gamma,j}\Lambda(\xi)^{-\rho(|\gamma|+1)}$$

for all x and ξ in $\mathbb{R}^n$. Therefore

$$|(\partial_x^\alpha \partial_\xi^\beta (F\circ\sigma))(x,\xi)| \leq (C_{\alpha,\gamma,j} + C'_{\alpha,\gamma,j})\Lambda(\xi)^{-\rho|\beta|}$$

for all x and ξ in $\mathbb{R}^n$. Now, we suppose that

$$\partial_x^\alpha \partial_\xi^\beta = \partial_x^\gamma \partial_{x_j}\partial_\xi^\beta$$

for some multi-index γ and some $j = 1, 2, \ldots, n$. Then as before, there exists a positive

constant $C''_{\alpha,\gamma,j}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta (F \circ \sigma))(x,\xi)| \leq C''_{\alpha,\gamma,j} \Lambda(\xi)^{-\rho|\beta|}$$

for all x and ξ in $\mathbb{R}^n$. Thus, by the principle of mathematical induction, (2.1.1) follows. $\square$

Lemma 2.1.3 *Let σ be a strongly elliptic symbol in $S^{2m}_{\rho,\Lambda}$. Then there exist positive constants γ and κ such that*

$$\operatorname{Re}\sigma(x,\xi) \geq \gamma \Lambda(\xi)^{2m} - \kappa \Lambda(\xi)^{2m-1}, \quad x,\xi \in \mathbb{R}^n.$$

Proof By strong ellipticity, there exist positive constants C and R such that

$$\operatorname{Re}\sigma(x,\xi) \geq C \Lambda(\xi)^{2m}, \quad |\xi| \geq R.$$

Since $\sigma \in S^{2m}_{\rho,\Lambda}$, we can find a positive constant K such that

$$|\sigma(x,\xi)| \leq K \Lambda(\xi)^{2m}, \quad x,\xi \in \mathbb{R}^n.$$

Therefore

$$|\operatorname{Re}\sigma(x,\xi)| \leq K \Lambda(\xi)^{2m} \leq K(1+R^2)^{2m}, \quad |\xi| \leq R.$$

Hence there exists a positive constant M such that

$$\operatorname{Re}\sigma(x,\xi) \geq -M, \quad |\xi| \leq R.$$

Since $\frac{\operatorname{Re}\sigma}{\Lambda^{2m-1}}$ is a continuous function on the compact set $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, we can find a positive constant κ such that

$$\frac{\operatorname{Re}\sigma(x,\xi)}{\Lambda(\xi)^{2m-1}} \geq -\kappa, \quad |\xi| > R.$$

Therefore

$$\operatorname{Re}\sigma(x,\xi) + \kappa\Lambda(\xi)^{2m-1} > 0, \quad |\xi| \leq R.$$

Since $\frac{\operatorname{Re}\sigma+\kappa\Lambda^{2m-1}}{\Lambda^{2m}}$ is a positive and continuous function on the compact set $\{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, there is a positive constant δ such that

$$\frac{\operatorname{Re}\sigma(x,\xi) + \kappa\Lambda(\xi)^{2m-1}}{\Lambda(\xi)^{2m}} \geq \delta, \quad |\xi| \leq R.$$

So, the lemma is proved if we let $\gamma = \min(C, \delta)$. □

Proof of Theorem 2.1.1 Let $T_\tau = \Lambda(D)^{-m} T_\sigma \Lambda(D)^{-m}$. Then using the asymptotic expansion for the product of two pseudo-differential operators,

$$T_\sigma \Lambda(D)^{-m} = T_{\tau_1},$$

where

$$\tau_1 - \Lambda^{-m}\sigma \in S^{m-\rho}_{\rho,\Lambda}. \tag{2.1.2}$$

Similarly,

$$T_\tau = \Lambda(D)^{-m} T_{\tau_1}$$

and

$$\tau - \Lambda^{-m}\tau_1 \in S^{-\rho}_{\rho,\Lambda}. \tag{2.1.3}$$

Multiplying (2.1.2) by Λ^{-m} and adding the result to (2.1.3), we get

$$\tau - \Lambda^{-2m}\sigma \in S^{-\rho}_{\rho,\Lambda}.$$

Therefore

$$\tau = \Lambda^{-2m}\sigma + r,$$

where $r \in S^{-\rho}_{\rho,\Lambda}$. So, by Lemma 2.1.3,

$$\begin{aligned}
\operatorname{Re}\tau &= \Lambda^{-2m}\operatorname{Re}\sigma + \operatorname{Re} r \\
&\geq \Lambda^{-2m}[\gamma\Lambda^{2m} - \kappa\Lambda^{2m-\rho}] + \operatorname{Re} r \\
&= \gamma - \kappa\Lambda^{-\rho} + \mathrm{r} \\
&\geq \gamma - \kappa'\Lambda^{-\rho},
\end{aligned}$$

where κ' is another positive constant. Therefore τ satisfies the conclusion of Lemma 2.1.3 with $m = 0$. Let us suppose for a moment that Gårding's inequality is valid for $m = 0$. Then we can find a positive constant C' and a positive constant C_s for every real number

$s \in \left[\frac{\rho}{2}, \infty\right)$ such that

$$\begin{aligned}
\operatorname{Re}(T_\sigma \varphi, \varphi) &= \operatorname{Re}(\Lambda(D)^m T_\tau \Lambda(D)^m \varphi, \varphi) \\
&= \operatorname{Re}(T_\tau \Lambda(D)^m \varphi, \Lambda(D)^m \varphi) \\
&\geq C' \|\Lambda(D)^m \varphi\|_{0,2}^2 - C_s \|\Lambda(D)\varphi\|_{-\rho s,2}^2 \\
&= C' \|\varphi\|_{m,2,\Lambda}^2 - C_s \|\varphi\|_{m-\rho s,2,\Lambda}^2
\end{aligned}$$

for all φ in $\mathcal{S}$. We are now ready to prove Gårding's inequality for $m = 0$. By Lemma 2.1.3, we have positive constants γ and κ such that

$$\operatorname{Re}\sigma + \kappa \Lambda^{-\rho} \geq \gamma.$$

Let $F \in C^\infty(\mathbb{C})$ be such that

$$F(z) = \sqrt{\frac{\gamma}{2} + z}, \quad z \in [0, \infty).$$

Let τ be the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\tau(x, \xi) = F(2(\operatorname{Re}\sigma(x, \xi) + \kappa \Lambda(\xi)^{-\rho} - \gamma)), \quad x, \xi \in \mathbb{R}^n.$$

Then by Lemma 2.1.3, $\tau \in S^0_{\rho,\Lambda}$ and for all x and ξ in $\mathbb{R}^n$,

$$\begin{aligned} \tau(x,\xi) &= \sqrt{\frac{\gamma}{2} + 2\mathrm{Re}\,\sigma(x,\xi) + 2\kappa\Lambda(\xi)^{-\rho} - 2\gamma} \\ &= \sqrt{2\mathrm{Re}\,\sigma(x,\xi) + 2\kappa\Lambda(\xi)^{-\rho} - \frac{3}{2}\gamma}. \end{aligned}$$

Using the asymptotic expansion for the formal adjoint of a pseudo-differential operator, we have

$$T^*_\tau = T_{\tau^*},$$

where $\tau^* \in S^0_{\rho,\Lambda}$ and $\tau - \tau^* \in S^{-\rho}_{\rho,\Lambda}$. Using also the asymptotic expansion for the product,

$$T^*_\tau T_\tau = T_\lambda,$$

where

$$\lambda - \tau^*\tau \in S^{-\rho}_{\rho,\Lambda}.$$

If we let r_1 and r_1' in $S^{-\rho}_{\rho,\Lambda}$ be such that

$$\tau^* = \tau + r_1$$

and

$$\lambda = \tau^*\tau + r_1',$$

then with $r_2 = r_1\tau + r' \in S^{-\rho}_{\rho,\Lambda}$, we get

$$\begin{aligned}\lambda &= (\tau + r_1)\tau + r_1' \\ &= \tau^2 + r_1\tau + r_1' \\ &= 2\mathrm{Re}\,\sigma + 2\kappa\Lambda(\xi)^{-\rho} - \frac{3}{2}\gamma + r_2.\end{aligned}$$

So, if we let $r_3 = 2\kappa\Lambda^{-1} + r_2 \in S^{-1}_{\rho,\Lambda}$, then we get

$$\lambda = 2\mathrm{Re}\,\sigma - \frac{3}{2}\gamma + r_3.$$

But

$$2\mathrm{Re}\,\sigma = \sigma + \overline{\sigma} = \sigma + \sigma^* + r_4$$

for some r_4 in $S^{-\rho}_{\rho,\Lambda}$. Therefore

$$\lambda = \sigma + \sigma^* - \frac{3}{2}\lambda + r_5$$

for some r_5 in $S^{-\rho}_{\rho,\Lambda}$. Thus,

$$\sigma + \sigma^* = \lambda + \frac{3}{2}\gamma + r_5.$$

Now, for all φ in $\mathcal{S}$,

$$\begin{aligned}2\mathrm{Re}\,(T_\sigma\varphi,\varphi) &= (T_\sigma\varphi,\varphi) + (T_\sigma^*\varphi,\varphi) = (T_{\sigma+\sigma^*}\varphi,\varphi) \\ &= (T_\lambda\varphi,\varphi) + \frac{3}{2}\gamma\|\varphi\|^2_{0,2,\Lambda} + (T_{r_5}\varphi,\varphi) \\ &\geq \gamma\|\varphi\|^2_{0,2,\Lambda} + \left\{\frac{\gamma}{2}\|\varphi\|^2_{0,2,\Lambda} - \|T_{r_5}\varphi\|_{\frac{1}{2},2,\Lambda}\|\varphi\|_{-\frac{1}{2},2,\Lambda}\right\}.\end{aligned}$$

Using the L^2-boundedness of (ρ, Λ)-pseudo-differential operators, we get a positive constant μ such that

$$2\mathrm{Re}\,(T_\sigma\varphi, \varphi) \geq \gamma\|\varphi\|^2_{0,2,\Lambda} + \left\{\frac{\gamma}{2}\|\varphi\|^2_{0,2,\Lambda} - \mu\|\varphi\|^2_{-\frac{1}{2},0,\Lambda}\right\}, \quad \varphi \in \mathcal{S}.$$

But

$$\mu\|\varphi\|^2_{-\frac{1}{2},0,\Lambda} = \int_{\mathbb{R}^n} \mu\Lambda(\xi)^{-1}|\hat{\varphi}(\xi)|^2\, d\xi = I + J,$$

where

$$I = \int_{\mu\Lambda(\xi)^{-1}\leq\frac{\gamma}{2}} \mu\Lambda(\xi)^{-1}|\hat{\varphi}(\xi)|^2 d\xi$$

and

$$J = \int_{\mu\Lambda(\xi)^{-1}\geq\frac{\gamma}{2}} \mu\Lambda(\xi)^{-1}|\hat{\varphi}(\xi)|^2 d\xi.$$

Obviously,

$$I \leq \frac{\gamma}{2}\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 d\xi = \frac{\gamma}{2}\|\varphi\|^2_{0,2}.$$

To estimate J, we note that

$$\mu\Lambda(\xi)^{-1} \geq \frac{\gamma}{2} \Rightarrow \Lambda(\xi) \leq \frac{2\mu}{\gamma}.$$

So, for $\mu\Lambda(\xi)^{-1} \geq \frac{\gamma}{2}$, we get for every nonnegative real number s with $s \geq \frac{\rho}{2}$.

$$\begin{aligned}\mu\Lambda(\xi)^{-1} &= \mu\Lambda(\xi)^{2\rho s-1}\Lambda(\xi)^{-2\rho s} \\ &\leq \mu\left(\frac{2\mu}{\gamma}\right)^{2\rho s-1}\Lambda(\xi)^{-2\rho s}.\end{aligned}$$

So, for every nonnegative real number s with $s \geq \frac{\rho}{2}$,

$$J \leq \mu \left(\frac{2\mu}{\gamma}\right)^{2\rho s-1} \int_{\mathbb{R}^n} \Lambda(\xi)^{-2\rho s} |\hat{\varphi}(\xi)|^2 d\xi = C_s' \|\varphi\|^2_{-\rho s,2,\Lambda},$$

where $C_s' = \mu \left(\frac{2\mu}{\gamma}\right)^{2\rho s-1}$. Therefore

$$2\mathrm{Re}\,(T_\sigma \varphi, \varphi) \geq \gamma \|\varphi\|^2_{0,2,\Lambda} - C_s' \|\varphi\|^2_{-\rho s,2,\Lambda}, \quad \varphi \in \mathcal{S},$$

and the theorem follows with $C_0 = \frac{\gamma}{2}$ and $C_s = \frac{C_s'}{2}$.

2.2 Poisson's Equations

The following theorem is a well-known result that follows from the corresponding result in functional analysis. See, for example, Theorem 16.3 in [23].

Theorem 2.2.1 *Let $\sigma \in S^m_{\rho,\Lambda}$, $m > 0$. Let $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. Then the Poisson equation, $T_\sigma u = f$ has a weak solution u in $L^p(\mathbb{R}^n)$ if and only if there exists a positive constant C such that*

$$|(f, \varphi)| \leq C \|T^*_\sigma \varphi\|_{p'}, \quad \varphi \in \mathcal{S},$$

where p' is the conjugate index of p and

$$(f, \varphi) = \int_{\mathbb{R}^n} f(x) \overline{\varphi(x)}\, dx.$$

Theorem 2.2.2 *Let $\sigma \in S^m_{\rho,\Lambda}, m > 0$, be such that there exists a positive constant C for*

which

$$\|\varphi\|_{p'} \leq C\|T_\sigma^*\varphi\|_{p'}, \quad \varphi \in \mathcal{S}.$$

Then the pseudo-differential equation $T_\sigma u = f$ on $\mathbb{R}^n$ has a weak solution u in $L^p(\mathbb{R}^n)$ for every function $f \in L^p(\mathbb{R}^n), 1 < p < \infty$.

Proof Let $\sigma \in S^m_{\rho,\Lambda}, m > 0$, be such that there exists a positive constant C for which

$$\|\varphi\|_{p'} \leq C\|T_\sigma^*\varphi\|_{p'}, \quad \varphi \in S.$$

By the inequality in the assumption and Hôlder's inequality, we have for all functions $\varphi \in \mathcal{S}$, and $f \in L^p(\mathbb{R}^n)$,

$$|(f, \varphi)| \leq \|f\|_p\|\varphi\|_{p'} \leq C\|f\|_p\|T_\sigma^*\varphi\|_{p'}, \quad \varphi \in \mathcal{S}.$$

Let $C' = C\|f\|_p$. Then

$$|(f, \varphi)| \leq C'\|T_\sigma^*\varphi\|_{p'}, \quad \varphi \in \mathcal{S}.$$

So, by Theorem 2.2.1, we conclude that the pseudo-differential equation $T_\sigma u = f$ on $\mathbb{R}^n$ has a weak solution u in $L^p(\mathbb{R}^n)$.

□

The focus from now on is on strong solutions in $L^2(\mathbb{R}^n)$ of Poisson's equations modelled by (ρ, Λ)-elliptic pseudo-differential operators.

Theorem 2.2.3 *Let $\sigma \in S^{2m}_{\rho,\Lambda}$, $m > 0$, be a (ρ, Λ)-elliptic symbol such that there exists a*

positive constant C for which

$$\mathrm{Re}\,(T_\sigma\varphi,\varphi) \geq C\|\varphi\|^2_{m,2,\Lambda}, \quad \varphi \in \mathcal{S}. \tag{2.2.1}$$

Then for every function f in $L^2(\mathbb{R}^n)$, the Poisson equation

$$T_\sigma u = f$$

has a unique strong solultion u in $L^2(\mathbb{R}^n)$.

Proof Using the Sobolev embedding theorem, (2.2.1), and the Cauchy–Schwarz inequality, there exists a positive constant C' such that for all functions φ in $\mathcal{S}$,

$$\|\varphi\|_2^2 \leq C'\|\varphi\|^2_{m,2,\Lambda} \leq \frac{C'}{C}\|\varphi\|_2\|T^*\varphi\|_2$$

and hence

$$\|\varphi\|_2 \leq \frac{C'}{C}\|T^*\varphi\|_2.$$

So, for all functions f in $L^2(\mathbb{R}^n)$,

$$|(f,\varphi)| \leq \|f\|_2\|\varphi\|_2 \leq \frac{C'}{C}\|f\|_2\|T^*\varphi\|_2, \quad \varphi \in \mathcal{S}.$$

Then by Theorem 2.2.1, the Poisson equation

$$T_\sigma u = f$$

on $\mathbb{R}^n$ has a weak solution u in $L^2(\mathbb{R}^n)$. By (ρ, Λ)-ellipticity, u is a strong solution in $L^2(\mathbb{R}^n)$. To prove uniqueness, we first note that by (2.2.1) and a density argument,

$$\mathrm{Re}\,(T_\sigma u, u) \geq C\|u\|^2_{m,2,\Lambda}, \quad u \in H^{2m,2}_\Lambda.$$

Let v be another strong solution in $L^2(\mathbb{R}^n)$ of the Poisson equation

$$T_\sigma u = f$$

on $\mathbb{R}^n$. Then

$$\|u - v\|^2_{m,2,\Lambda} \leq \frac{1}{C}\mathrm{Re}\,(T_\sigma(u - v), u - v) = 0.$$

Therefore $u = v$ and this completes the proof of the theorem. □

Corollary 2.2.4 *Under the condition of Theorem 2.2.3, the inverse of $T_{\sigma,0}u$ (or $T_{\sigma,1}u$) exists.*

2.3 Strongly Continuous One-Parameter Semigroups

Let $\{T(t) : t \geq 0\}$ be a family of bounded linear operators on a complex Banach space X such that

- $T(s + t) = T(s)T(t)$ for all $s, t \in [0, \infty)$ with $s + t \in [0, \infty)$,
- for all $x \in X$,

$$\lim_{t \to 0^+} T(t)x = x.$$

Then we call $\{T(t) : t \geq 0\}$ a strongly continuous one-parameter semigroup of bounded linear operators on X. The linear operator A on $\left\{x \in X : \lim_{t\to 0^+} \frac{T(t)x-x}{t} \text{ exists}\right\}$ defined by

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}$$

is called the infinitesimal generator of the strongly continuous one-parameter semigroup $\{T(t) : t \geq 0\}$ on X.

We often denote $T(t)$ by e^{At}, $t \geq 0$. It is worth mentioning that the parameter t in $T(t)$ plays the role of time.

In this section, we give an application of Gårding's inequality in the strongly continuous one-parameter semigroups in Theorem 2.3.2.

We begin with an existence and uniqueness theorem.

Theorem 2.3.1 *Let $\sigma \in S^{2m}_{\rho,\Lambda}$, $m > 0$, be an elliptic symbol such that we can find a positive constant C and a constant λ_0 for which*

$$\operatorname{Re}(-T_\sigma\varphi, \varphi) \geq C\|\varphi\|^2_{m,2,\Lambda} - \lambda_0\|\varphi\|^2_2, \quad \varphi \in \mathcal{S}. \tag{2.3.1}$$

Let $\lambda > \lambda_0$. Then for every $f \in L^2(\mathbb{R}^n)$, there exists a unique solution $u \in H^{2m,2}_\Lambda$ of the equation

$$(\lambda I - T_{\sigma,0})u = f,$$

where I is the identity operator on $L^2(\mathbb{R}^n)$. Moreover,

$$\|(\lambda I - T_{\sigma,0})u\|_2 \geq (\lambda - \lambda_0)\|u\|_2, \quad u \in H^{2m,2}_\Lambda. \tag{2.3.2}$$

Proof By (2.3.1), we get

$$\begin{aligned} \mathrm{Re}\,((\lambda I - T_\sigma)\varphi, \varphi) &= \mathrm{Re}\,((\lambda_0 I - T_\sigma)\varphi, \varphi) + (\lambda - \lambda_0)\|\varphi\|_2^2 \\ &\geq C\|\varphi\|_{m,2,\Lambda}^2 + (\lambda - \lambda_0)\|\varphi\|_2^2 \\ &\geq C\|\varphi\|_{m,2,\Lambda}^2 \end{aligned} \tag{2.3.3}$$

and

$$\mathrm{Re}\,((\lambda I - T_\sigma)\varphi, \varphi) \geq (\lambda - \lambda_0)\|\varphi\|_2^2 \tag{2.3.4}$$

for all functions φ in $\mathcal{S}$. By (2.3.3), Theorem 1.3.6, and Theorem 2.2.3, we conclude that the equation

$$(\lambda I - T_{\sigma,0})u = f$$

on $\mathbb{R}^n$ has a unique solution u in $H_\Lambda^{2m,2}$. Moreover, by (2.3.4) and a limiting argument,

$$\|(\lambda I - T_{\sigma,0})u\|_2 \geq (\lambda - \lambda_0)\|u\|_2, \quad u \in H_\Lambda^{2m,2}.$$

□

We conclude with the following theorem.

Theorem 2.3.2 *Let $\sigma \in S_{\rho,\Lambda}^{2m}$, $m > 0$, be a strongly (ρ, Λ)-elliptic symbol. Then $-T_{\sigma,0}$ is the infinitesimal generator of a strongly continuous one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$.*

Proof By Gårding's inequality, we can find a positive constant C and a constant C_s for

every real number s in $\left[\frac{\rho}{2}, \infty\right)$ such that

$$\mathrm{Re}\,(T_\sigma \varphi, \varphi) \geq C\|\varphi\|^2_{m,2,\Lambda} - C_s\|\varphi\|^2_{m-\rho s,\Lambda}, \quad \varphi \in \mathcal{S}.$$

By choosing s such that $\rho s = m$, we get a constant λ_0 such that (2.3.1) is satisfied. By Theorem 2.3.1, the resolvent set of the pseudo-differential operator $-T_{\sigma,0}$ is the same as (λ_0, ∞) and

$$\|(-T_{\sigma,0} - \lambda I)^{-1}\|_* \leq (\lambda - \lambda_0)^{-1}, \quad \lambda > \lambda_0,$$

where $\| \|_*$ denotes the norm in the C^*-algebra of bounded linear operators on $L^2(\mathbb{R}^n)$. Hence

$$\|(-T_{\sigma,0} - \lambda I)^{-n}\|_* \leq (\lambda - \lambda_0)^{-n}, \quad n = 1, 2, \ldots.$$

The Hille-Yosida–Phillips theorem then completes the proof. □

The Hille–Yosida–Phillips theorem is one of the basic theorems in one-parameter semigroups of bounded linear operators on Banach spaces. See, for instance, [6, 7, 17, 23, 24] in this connection.

We state explicitly the Hille-Yosida-Phillips theorem here for the sake of being self-contained.

Theorem 2.3.3 *let A be a closed linear operator from a complex Banach space X into X with dense domain $D(A)$. Then A is the infinitesimal generator of a strongly continuous one-parameter semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators on X if and only if we*

can find a positive number M and a real number ω such that

$$\{\lambda \in \mathbb{R} : \lambda > \omega\} \subseteq \rho(A)$$

and

$$\left\|R(\lambda; A)^n\right\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all λ in (ω, ∞) and all positive integers n.

3 Holomorphic One-Parameter Semigroups of Bounded Linear Operators on $L^p(\mathbb{R}^n), 1 < p < \infty$, Generated by Strongly M-Elliptic Pseudo-Differential Operators

We prove in this chapter that for a privileged class of elliptic symbols in $M^m_{\rho,\Lambda}, m > 0$, the corresponding pseudo-differential operator generates a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n), 1 < p < \infty$. To this end, we first introduce a class of symbols, attributed to Shubin [20], that is related to the fundamental class of strongly M-elliptic symbols. Moreover, we show that a strongly M-elliptic pseudo-differential operator generates a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n), 1 < p < \infty$.

It is worth mentioning that the results in this chapter improve the results in Chapter 2 from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n), 1 < p < \infty$. The techniques used in this chapter, however, are drastically different from those in the previous chapter in the sense that Gårding's inequality, valid only for $L^2(\mathbb{R}^n)$, is no longer applicable.

Pseudo-differential operators are quantities that can be measured by experiments in quantum mechanics. One-parameter semigroups of bounded linear operators on $L^p(\mathbb{R}^n), 1 < p < \infty$, are solutions of evolution equations

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -(T_{\sigma,0}u)(x,t), \quad x \in \mathbb{R}^n, t > 0, \\ \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where the initial date $f \in L^p(\mathbb{R}^n), 1 < p < \infty$, governed by the generators $-T_{\sigma,0}$. Therefore this research will contribute to a deeper understanding of the way the initial state evolves as time goes to infinity.

3.1 Related Symbols and Strongly M-Elliptic Symbols

The pseudo-differential operators for classical symbols have been studied in great detail in the books [17], [23], and [20]. Then we move on to the L^p-theory, $1 < p < \infty$. In the theory of the pseudo-differential operators, it is useful to focus on those operators depending on a parameter $\lambda \in \Omega$, where Ω is a subset of the complex plane; in our work, this will be an angle with the vertex at the origin. The parameter-dependence in the symbol allows for additional flexibility in the operator, making it a more powerful tool for solving differential equations in a wide range of applications. An example of such an operator is the resolvent $(A - \lambda I)^{-1}$. In fact, many of the theorems related to a pseudo-differential operator without a parameter can also be proved for the case with a parameter λ, see [20]. We call symbols in $M^m_{\rho,\Lambda,q,\Omega}$ parametrized symbols. We note that the whole theory of asymptotic expansions carries over to parametrized symbols. The equivalent formulations are obtained by changing $M^m_{\rho,\Lambda}$ to $M^m_{\rho,\Lambda,q,\Omega}$, and the proofs are almost verbatim repetitions of the same arguments. In this chapter, we are only interested in a very simple class of parametrized symbols, i.e.,

$\Big\{\sigma + \lambda : \sigma \in M^m_{\rho,\Lambda}, \lambda \in \mathbb{C}\Big\}$.

Let q be a fixed but arbitrary positive number and let Ω be an open subset of the complex plane $\mathbb{C}$. Then for any real number m, we define $S^m_{\rho,\Lambda,q,\Omega}$ to be the set of all functions σ on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega$ such that

$$\sigma(\cdot, \cdot, \lambda) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

for every $\lambda \in \Omega$ and for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$|(D^\alpha_x D^\beta_\xi \sigma)(x, \xi, \lambda)| \leq C_{\alpha,\beta} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\mathrm{sgn}(-m)q} \Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $\lambda \in \Omega$, where $\mathrm{sgn} : \mathbb{R} \to \{\pm 1, 0\}$ is given by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

We call any function $\sigma \in \bigcup_{m\in\mathbb{R}} S^m_{\rho,\Lambda,q,\Omega}$, a parametrized symbol. As usual,

$$S^{-\infty}_{\rho,\Lambda,q,\Omega} = \bigcap_{m\in\mathbb{R}} S^m_{\rho,\Lambda,q,\Omega}.$$

We denote by $M^m_{\rho,\Lambda,q,\Omega}$ the set of all functions σ in $S^m_{\rho,\Lambda,q,\Omega}$ such that for all multi-indices

γ with $\gamma_j \in \{0,1\}$, $j = 1, 2, \ldots, n$,

$$\xi^\gamma(\partial_\xi^\gamma \sigma)(x,\xi,\lambda) \in S^m_{\rho,\Lambda,q,\Omega}. \tag{3.1.1}$$

The condition (3.1.1) is equivalent to the fact that for all multi-indices α, β, and γ with $\gamma_j \in \{0,1\}$, $j = 1, 2, \ldots, n$, there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$\left|\xi^\gamma(\partial_x^\alpha \partial_\xi^{\beta+\gamma}\sigma)(x,\xi,\lambda)\right| \leq C_{\alpha,\beta,\gamma}\Lambda(\xi)^{m-\rho|\beta|}. \tag{3.1.2}$$

We need a new characterization of elements in $M^m_{\rho,\Lambda,q,\Omega}$.

Proposition 3.1.1 *$\sigma \in M^m_{\rho,\Lambda,q,\Omega}$ if and only if for all multi-indices α, β and γ with $\gamma_j \in \{0,1\}$, $j = 1, 2, \ldots, n$, there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that*

$$|\xi^\gamma(D_x^\alpha D_\xi^{\beta+\gamma}\sigma)(x,\xi,\lambda)| \leq C_{\alpha,\beta,\gamma}\left(\frac{\Lambda(\xi)}{\Lambda(\xi)+|\lambda|^{1/q}}\right)^{\mathrm{sgn}(-m)q}\Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$.

Proof Let $\sigma \in M^m_{\rho,\Lambda,q,\Omega}$. Then $\sigma \in S^m_{\rho,\Lambda,q,\Omega}$. We have for all multi-indices γ with $\gamma_j \in \{0,1\}$, $j = 1, 2, \ldots, n$,

$$\xi^\gamma(\partial_\xi^\gamma \sigma)(x,\xi,\lambda) \in S^m_{\rho,\Lambda,q,\Omega}.$$

Therefore for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$\begin{aligned}
& |(D_x^\alpha D_\xi^\beta(\xi^\gamma(D_\xi^\gamma\sigma)))(x,\xi,\lambda)| \\
\leq & \sum_{\beta'\leq\beta,\beta'\leq\gamma}\binom{\beta}{\beta'}|(D_\xi^{\beta'}\xi^\gamma)(D_x^\alpha D_\xi^{\beta-\beta'+\gamma}\sigma)(x,\xi,\lambda)| \\
= & \sum_{\beta'\leq\beta,\beta'\leq\gamma}\binom{\beta}{\beta'}\binom{\gamma}{\beta'}\beta'!|\xi^{\gamma-\beta'}(D_x^\alpha D_\xi^{\beta+(\gamma-\beta')}\sigma)(x,\xi,\lambda)| \\
\leq & C_{\alpha,\beta,\gamma}\left(\frac{\Lambda(\xi)}{\Lambda(\xi)+|\lambda|^{1/q}}\right)^{\mathrm{sgn}(-m)q}\Lambda(\xi)^{m-\rho|\beta|}. \qquad (3.1.3)
\end{aligned}$$

Conversely, suppose that for all multi-indices α, β and γ with $\gamma_j \in \{0,1\}$, $j = 1,2,\ldots,n$, there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\xi^\gamma(D_x^\alpha D_\xi^{\beta+\gamma}\sigma)(x,\xi,\lambda)| \leq C_{\alpha,\beta,\gamma}\left(\frac{\Lambda(\xi)}{\Lambda(\xi)+|\lambda|^{1/q}}\right)^{\mathrm{sgn}(-m)q}\Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$. Let γ be the zero multi-index. Then there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(D_x^\alpha D_\xi^\beta\sigma)(x,\xi,\lambda)| \leq C_{\alpha,\beta}\left(\frac{\Lambda(\xi)}{\Lambda(\xi)+|\lambda|^{1/q}}\right)^{\mathrm{sgn}(-m)q}\Lambda(\xi)^{m-\rho|\beta|}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega$. Therefore $\sigma \in S^m_{\rho,\Lambda,q,\Omega}$. Now, for all multi-indices α, β and γ with $\gamma_j \in \{0,1\}$, $j = 1,2,\ldots,n$, the derivation of (3.1.3) shows that $\xi^\gamma(D_\xi^\gamma\sigma)(x,\xi,\gamma) \in S^m_{\rho,\Lambda,q,\Omega}$. Thus, $\sigma \in M_{\rho,\Lambda,q,\Omega}$. □

Since elliptic symbols are of particular importance in the theory of pseudo-differential operators, we need to incorporate this concept into the theory of pseudo-differential operators with parametrized symbols.

Let $\sigma \in M^m_{\rho,\Lambda,q,\Omega}$. Suppose that there exist positive constants C and R such that

$$|\sigma(x,\xi,\lambda)| \geq C \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/q}} \right)^{\mathrm{sgn}(-m)q} \Lambda(\xi)^m$$

for all $\lambda \in \Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| + |\lambda| \geq R$. Then we say that σ is $M^m_{\rho,\Lambda,q,\Omega}$-elliptic.

Of course, its associated pseudo-differential operator T_σ is then said to be $M^m_{\rho,\Lambda,q,\Omega}$-elliptic.

In fact, if σ is a strongly M-elliptic symbol in $M^m_{\rho,\Lambda}, m > 0$, and λ is in a region Ω of the complex plane, then $\sigma + \lambda$ belongs to the class $M^m_{\rho,\Lambda,m,\Omega}$. By specializing Ω to open sectors with center at the origin of the complex plane $\mathbb{C}$, we have the following lemma.

Lemma 3.1.2 *Let σ be a strongly M-elliptic symbol in $M^m_{\rho,\Lambda}$, $m > 0$. Assume there exists a number θ in $(0, \pi/2)$ such that*

$$\Omega(\theta) = \{\lambda \in \mathbb{C} : \theta - \pi < \arg\, \lambda < \pi - \theta\},$$

then $\sigma_{\Omega(\theta)} = \sigma(x,\xi) + \lambda \in M^m_{\rho,\Lambda,m,\Omega(\theta)}$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$.

Proof We first assume that $|\alpha| + |\beta| = 0$. Then there exists a positive constant C such that for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$,

$$\begin{aligned}
|\sigma(x,\xi) + \lambda| &\leq C \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m + |\lambda| \\
&\leq C \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m + \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m \\
&\leq (C+1) \left(\frac{\Lambda(\xi) + |\lambda|^{1/m}}{\Lambda(\xi)} \right)^m \Lambda(\xi)^m.
\end{aligned}$$

Now, for all multi-indices α and β with $|\alpha| \neq 0$ or $|\beta| \neq 0$, we can find a positive constant $C_{\alpha,\beta}$ such that for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$,

$$\begin{aligned} |(D_x^\alpha D_\xi^\beta(\sigma+\lambda))(x,\xi)| &= |(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \\ &\leq C_{\alpha,\beta}\left(\frac{\Lambda(\xi)+|\lambda|^{1/m}}{\Lambda(\xi)}\right)^m \Lambda(\xi)^{m-\rho|\beta|}. \end{aligned}$$

Therefore, $\sigma_{\Omega(\theta)} = \sigma(x,\xi) + \lambda \in M^m_{\rho,\Lambda,m,\Omega(\theta)}$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$.

□

The next theorem is about the $M^m_{\rho,\Lambda,m,\Omega(\theta)}$-ellipticity of $\sigma_{\Omega(\theta)}$.

Theorem 3.1.3 *Let σ be a strongly M-elliptic symbol in $M^m_{\rho,\Lambda}$, $m > 0$. Then there exists a number θ in $(0,\pi/2)$ such that $\sigma_{\Omega(\theta)}$ is $M^m_{\rho,\Lambda,m,\Omega(\theta)}$-elliptic, where*

$$\Omega(\theta) = \{\lambda \in \mathbb{C} : \theta - \pi < \arg\,\lambda < \pi - \theta\}$$

and $\sigma_{\Omega(\theta)}$ is the function on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega(\theta)$ defined by

$$\sigma_{\Omega(\theta)}(x,\xi,\lambda) = \sigma(x,\xi) + \lambda$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta)$.

Proof By Lemma 3.1.2, we know that for $\lambda \in \Omega(\theta)$, $\sigma + \lambda$ is a symbol in $M^m_{\rho,\Lambda,m,\Omega(\theta)}$, Now, we need to show there exists an angle θ_1, thus a region $\Omega = \Omega(\theta_1)$, and two positive constants

C_1 and R_1 that may depend on θ_1 such that for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq R_1, \lambda \in \Omega$,

$$\begin{aligned} |\sigma(x,\xi)+\lambda| &\geq C\left(\frac{\Lambda(\xi)+|\lambda|^{\frac{1}{m}})}{\Lambda(\xi)}\right)^m (\Lambda(\xi))^m \\ &= C(\Lambda(\xi)+|\lambda|^{\frac{1}{m}})^m. \end{aligned}$$

To this end, we let λ be any complex number and write for all x and ξ in $\mathbb{R}^n$,

$$\begin{aligned} |\sigma(x,\xi)+\lambda| &= |\text{Re } \sigma(x,\xi) + i \text{ Im } \sigma(x,\xi) + |\lambda|\cos\theta + i|\lambda|\sin\theta| \\ &= |\text{Re } \sigma(x,\xi) + |\lambda|\cos\theta + i(\text{Im } \sigma(x,\xi) + |\lambda|\sin\theta)| \\ &= \sqrt{(\text{Re } \sigma(x,\xi) + |\lambda|\cos\theta)^2 + ((\text{Im } \sigma(x,\xi) + |\lambda|\sin\theta))^2} \\ &= \sqrt{|\sigma(x,\xi)|^2 + 2|\lambda|(\text{Re } \sigma(x,\xi)\cdot\cos\theta + \text{Im } \sigma(x,\xi)\cdot\sin\theta) + |\lambda|^2}. \end{aligned}$$

Since σ is a strongly M-elliptic symbol in $M^m_{\rho,\Lambda}$, there exist positive constants A and R such that

$$\text{Re } \sigma(x,\xi) \geq A\Lambda(\xi)^m, \quad |\xi| \geq R.$$

Since σ is a symbol in $S^m_{\rho,\Lambda}$, there is a positive constant C such that

$$|\sigma(x,\xi)| \leq C\Lambda(\xi)^m, \quad x,\xi \in \mathbb{R}^n.$$

Thus, for $|\xi| \geq R$,

$$\frac{|\text{Im } \sigma(\xi)|}{\text{Re } \sigma(x,\xi)} < \frac{|\sigma(x,\xi)|}{\text{Re } \sigma(x,\xi)} \leq \frac{C(\Lambda(\xi))^m}{A(\Lambda(\xi))^m} = \frac{C}{A} = \tan\omega, \tag{3.1.4}$$

where $\omega = \arctan \frac{C}{A}$.

Let $E(\theta)$ be given by

$$E(\theta) = \text{Re } \sigma(x,\xi) \cdot \cos\,\theta + \text{Im } \sigma(x,\xi) \cdot \sin\,\theta + |\sigma(x,\xi)| \sin\,(\omega + \delta),$$

where δ is chosen such that $\omega + 2\delta = \frac{\pi}{2}$. Our goal is to show that for $|\xi| \geq R$ and $\theta \in (-\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta), E(\theta) > 0$. To this end, we first show that for $|\xi| \geq R$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $E(\theta) > 0$. Since for these values of ξ and θ, we have for $\delta = 0$,

$$\begin{aligned} \text{Re } \sigma(x,\xi) \cdot \cos\,\theta &\geq 0, \\ \text{Im } \sigma(x,\xi) \cdot \sin\,\theta &\geq -|\text{Im } \sigma(x,\xi)|, \end{aligned}$$

and

$$\sin\,(\omega + \delta) = \sin\,(\omega).$$

Therefore

$$\begin{aligned} E(\theta) &= \text{Re } \sigma(x,\xi) \cdot \cos\,\theta + \text{Im } \sigma(x,\xi) \cdot \sin\,\theta + |\sigma(x,\xi)| \sin\,(\omega + \delta) \\ &\geq -|\text{Im } \sigma(x,\xi)| + |\sigma(x,\xi)| \sin\,(\omega). \end{aligned}$$

As $\delta = 0$, we have

$$\omega = \frac{\pi}{2}.$$

Hence

$$E(\theta) \geq -|\text{Im}\sigma(x,\xi)| + |\sigma(x,\xi)|.$$

Since $|\sigma(x,\xi)| \geq |\text{Im } \sigma(x,\xi)|$, we have

$$E(\theta) > 0.$$

Next, we show that for $|\xi| \geq R$ and $\theta \in (-\frac{\pi}{2} - \delta, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{\pi}{2} + \delta)$, we also have $E(\theta) > 0$. By letting $\theta' = |\theta| - \frac{\pi}{2}$, we have $\theta' \in (0, \delta)$. For these values of ξ, θ, we have

$$\begin{aligned} E(\theta') &= \text{Re } \sigma(x,\xi) \cdot \cos\left(\theta' + \frac{\pi}{2}\right) + \text{Im } \sigma(x,\xi) \cdot \sin\left(\pm\left(\theta' + \frac{\pi}{2}\right)\right) \\ &\quad + |\sigma(x,\xi)| \sin\left(\theta' + \delta\right) \\ &\geq -\text{Re } \sigma(x,\xi) \cdot \sin\,\theta' - |\text{Im } \sigma(x,\xi)| \cdot \cos\,\theta' + |\sigma(x,\xi)| \sin\,(\omega + \delta). \end{aligned}$$

Now, assume by contradiction that for some $|\xi| \geq R$, we have

$$|\sigma(x,\xi)| \sin\,(\omega + \delta) \leq \text{Re } \sigma(x,\xi) \cdot \sin\,\theta' + |\text{Im } \sigma(x,\xi)| \cdot \cos\,\theta'.$$

That is,

$$\begin{aligned} & \sin^2(\omega + \delta)((\text{Re } \sigma(x,\xi))^2 + (\text{Im } \sigma(x,\xi))^2) \\ \leq\ & (\text{Re}\sigma(x,\xi))^2 \cdot \sin^2\,\theta' + |(\text{Im } \sigma(x,\xi))^2| \cdot \cos^2\ \theta' \\ & + 2\text{Re } \sigma(x,\xi) \cdot \sin\,\theta' \cdot |\text{Im } \sigma(x,\xi)| \cdot \cos\,\theta'. \end{aligned}$$

Rearranging terms, we have

$$(\sin^2\ \theta' - \sin^2\ (\omega+\delta)) \cdot (\text{Re}\ \sigma(x,\xi))^2 + (\cos^2\ \theta' - \sin^2\ (\omega+\delta)) \cdot (\text{Im}\ \sigma(x,\xi))^2$$

$$+2\text{Re}\ \sigma(x,\xi) \cdot \sin\ \theta' \cdot \cos\ \theta' \cdot |\text{Im}\ \sigma(x,\xi)| \geq 0.$$

Now, dividing both sides by $(\text{Re}\ \sigma(x,\xi))^2$, we have

$$(\cos^2\ \theta' - \sin^2\ (\omega+\delta)) \cdot \left(\frac{|\text{Im}\ \sigma(x,\xi)|}{\text{Re}\ \sigma(x,\xi)}\right)^2 + 2\sin\ \theta' \cdot \cos\ \theta' \cdot \frac{|\text{Im}\ \sigma(x,\xi)|}{\text{Re}\ \sigma(x,\xi)}$$

$$+(\sin^2\ \theta' - \sin^2\ (\omega+\delta)) \geq 0.$$

Now, let $x = \frac{|\text{Im}\ \sigma(x,\xi)|}{\text{Re}\ \sigma(x,\xi)}$, we have

$$(\cos^2\ \theta' - \sin^2\ (\omega+\delta))x^2 + (2\sin\ \theta' \cdot \cos\ \theta')x + (\sin^2\ \theta' - \sin^2(\omega+\delta)) \geq 0.$$

We have two roots by solving the quadratic equation associated to the above, i.e.,

$$\begin{aligned} x &= \frac{-2\sin\ \theta' \cdot \cos\ \theta' \pm \sqrt{4\sin^2\ (\omega+\delta) - 4\sin^4\ (\omega+\delta)}}{2(\cos^2\ \theta' - \sin^2\ (\omega+\delta))} \\ &= \frac{-\sin\ \theta' \cdot \cos\ \theta' \pm \sin\ (\omega+\delta)\cos\ (\omega+\delta)}{\cos^2\ \theta' - \sin^2\ (\omega+\delta)}. \end{aligned}$$

Since for $\theta' \in (0,\delta)$, the leading coefficient $\cos^2\ \theta' - \sin^2\ (\omega+\delta)$, is positive and $x = \frac{|\text{Im}\ \sigma(x,\xi)|}{\text{Re}\ \sigma(x,\xi)}$

is positive, so we get

$$\frac{|\text{Im } \sigma(x,\xi)|}{\text{Re } \sigma(x,\xi)} \geq \frac{\sin\ (\omega+\delta)\cos\ (\omega+\delta) - \sin\ \theta' \cdot \cos\ \theta'}{\cos^2 \theta' - \sin^2\ (\omega+\delta)}.$$

Next,

$$\sin\left(\frac{\pi}{2}\right) = \sin\ (\omega+2\delta) > \sin\ (\omega+2\theta').$$

Since

$$\sin\ (2\omega+2\delta-\omega) > \sin\ (\omega+2\theta'),$$

we get

$$(\sin\ (2\omega+2\delta) - \sin\ (2\theta'))\cos\ \omega > (\cos\ (2\theta') + \cos\ (2\omega+2\delta))\sin\ \omega.$$

Now, by some trigonometric identities, we have

$$(\sin\ (\omega+\delta)\cos\ (\omega+\delta) - \sin\ (\theta')\cos\ (\theta'))\cos\ \omega > (\cos^2\ \theta' - \sin^2\ (\omega+\delta))\sin\ \omega.$$

This gives

$$\frac{\sin\ (\omega+\delta)\cos\ (\omega+\delta) - \sin\ (\theta') \cdot \cos^2\ (\theta')}{\cos^2\ (\theta') - \sin^2\ (\omega+\delta)} > \frac{\sin\ (\omega)}{\cos\ (\omega)} = \tan\ \omega.$$

By (3.1.4), we obtain

$$\tan\ \omega > \frac{|\text{Im } \sigma(x,\xi)|}{\text{Re } \sigma(x,\xi)} \geq \frac{\sin\ (\omega+\delta)\cos\ (\omega+\delta) - \sin\ (\theta') \cdot \cos\ (\theta')}{\cos^2\ (\theta') - \sin^2\ (\omega+\delta)} > \tan\ \omega,$$

which is a contradiction. When $|\xi| \geq R$ and $\theta \in (-\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$, we have

$$E(\theta) > 0.$$

Let $\theta_1 = \frac{\pi}{2} - \delta$ and let Ω be the region given by

$$\Omega = \left\{ \lambda \in \mathbb{C} : \ -\frac{\pi}{2} - \delta < \arg \lambda < \frac{\pi}{2} + \delta \right\},$$

where θ_1 and Ω depend only on the fraction $\frac{A}{C}$. Then for $\lambda \in \Omega$ and $|\xi| \geq R$,

$$\begin{aligned} & |\sigma(x,\xi) + \lambda| \\ = \ & \sqrt{|\sigma(x,\xi)|^2 + 2|\lambda|(\text{Re } \sigma(x,\xi) \cdot \cos\theta) + \text{Im } \sigma(x,\xi) \cdot \sin\theta) + |\lambda|^2} \\ \geq \ & \sqrt{|\sigma(x,\xi)|^2 - 2|\lambda||\sigma(x,\xi)|\sin(\omega + \delta) + |\lambda|^2}. \end{aligned}$$

Since $0 \leq \omega + \delta \leq \frac{\pi}{2}$, we have $\sin(\omega + \delta) \geq 0$ and

$$\begin{aligned} 2|\lambda||\sigma(x,\xi)|\sin(\omega + \delta) &= 2|\lambda||\sigma(x,\xi)|\sqrt{\sin(\omega + \delta)}\sqrt{\sin(\omega + \delta)} \\ &\leq |\lambda|^2 \sin(\omega + \delta) + |\sigma(x,\xi)|^2 \sin(\omega + \delta). \end{aligned}$$

This implies that

$$\begin{aligned}
& |\sigma(x,\xi)+\lambda| \\
\geq\ & \sqrt{|\sigma(x,\xi)|^2+|\lambda|^2-|\sigma(x,\xi)|^2\sin\,(\omega+\delta)-|\lambda|^2\sin\,(\omega+\delta)} \\
=\ & \sqrt{(1-\sin\,(\omega+\delta))(|\sigma(x,\xi)|^2+|\lambda|^2)} \\
\geq\ & (1-\sin\,(\omega+\delta))^{\frac{1}{2}}(A^2(\Lambda(\xi))^{2m}+(|\lambda|^{\frac{1}{m}})^{2m})^{\frac{1}{2}} \\
\geq\ & (1-\sin\,(\omega+\delta))^{\frac{1}{2}}\min\{A^2,1\}2^{-m}(\Lambda(\xi)+|\lambda|^{\frac{1}{m}})^m.
\end{aligned}$$

The last line is obtained using the fact that $(p+q)^{\frac{1}{2}} \leq 2^{\frac{1}{2}}(p^{\frac{1}{2}}+q^{\frac{1}{2}})$ for $p \geq 0$ and $q \geq 0$. Finally, by letting R_1 to be equal to R, and C_1 to be equal to $(1-\sin\,(\omega+\delta))^{\frac{1}{2}}\min\{A^2,1\}2^{-m}$, we have

$$|\sigma(x,\xi)+\lambda| \geq C_1(\Lambda(\xi)+|\lambda|^{\frac{1}{m}})^m, \quad |\xi|+|\lambda| \geq R_1.$$

That is, $\sigma+\lambda$ is a $M^m_{\rho,\Lambda,m,\Omega(\theta)}$-elliptic symbol. □

Theorem 3.1.4 *Under the hypotheses of Theorem* 3.1.3*, there exists a symbol* τ *in* $M^{-m}_{\rho,\Lambda,m,\Omega(\theta_1)}$ *such that*

$$T_\tau T_{\sigma_{\Omega(\theta_1)}} = I + R,$$

where R *is a pseudo-differential operator with symbol in* $\bigcap_{k\in\mathbb{R}} M^k_{\rho,\Lambda,m,\Omega(\theta_1)}$

The proof is similar to that of the existence of a parametrix of a M-elliptic pseudo-differential operator in [11].

In the next section, will use the fact that $\sigma(x,\xi)+\lambda$ is a $M^m_{\rho,\Lambda,m,\Omega(\theta)}$-elliptic symbol to show that the operator $-T_{\sigma,0}$ is the infinitesimal generator of a one-parameter holomorphic

semigroup of bounded linear operator on a complex Banach space.

3.2 Holomorphic One-Parameter Semigroups

To make the paper more self-contained, let us first recall what a holomorphic one-parameter semigroup of bounded linear operators on a complex Banach space is. Good references are the books [2], [8] and [17], among others.

Definition 3.2.1 Let Ω be the sector in the complex plane given by

$$\Omega = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2, \text{ where } \varphi_1 < 0 < \varphi_2\}.$$

Let $\{T(z) : z \in \Omega\}$ be a family of bounded linear operators on a complex Banach space X such that

- the mapping $\mathbb{C} \ni z \mapsto T(z) \in B(X)$, where $B(X)$ is the Banach space of all bounded linear operators on X, is holomorphic,
- $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Omega$ with $z_1 + z_2 \in \Omega$,
- for all $x \in X$,

$$\lim_{z \to 0} T(z)x = x.$$

Then we call $\{T(z) \in B(X) : z \in \Omega\}$ a holomorphic one-parameter semigroup of bounded linear operators on the complex Banach space X and the linear operator A defined on

$\left\{x \in X : \lim_{z\to 0} \frac{T(z)x-x}{z} \text{ exists}\right\}$ by

$$Ax = \lim_{z\to 0} \frac{T(z)x - x}{z}$$

is called the infinitesimal generator of the holomorphic one-parameter semigroup

$$\big\{T(z) \in B(X) : z \in \Omega\big\}$$

of bounded linear operators on X.

Holomorphicity here means that for every complex number $z_0 \in \Omega$ and every $x \in X$,

$$\lim_{z\to z_0} \frac{T(z)x - T(z_0)x}{z - z_0}$$

exists in the norm topology of X. The parameter z in $T(z)$ can be thought of as complex time, which deserves further investigation to be understood.

We now begin with a result on infinitesimal generators of holomorphic one-parameter semigroups of bounded linear operators on complex Banach spaces in [17].

Theorem 3.2.2 *Let A be a closed linear operator densely defined on a complex Banach space X such that $\Omega(\theta) \cup \{0\}$ is a subset of the resolvent set $\rho(A)$ of A for some θ in $(0, \pi/2)$ and there exists a positive constant M for which*

$$\|(A - \lambda I)^{-1}\| \leq \frac{M}{|\lambda|}$$

for all λ in $\rho(A)$ with $\theta - \pi < \arg\ \lambda < \pi - \theta$ and $\lambda \neq 0$, where $\| \|$ is the norm in the Banach algebra of all bounded linear operators on X and I is the identity operator on X. Then A is the infinitesimal generator of a holomorphic one-parameter semigroup $\{T(z) : |\arg\ z| < \theta\}$ of bounded linear operators on X.

We need to modify Theorem 3.2.2 for our need.

Theorem 3.2.3 *Let A be a closed linear operator densely defined on a complex Banach space X such that*

$$\Omega(\theta) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq R\} \subset \rho(A)$$

for some θ in $(0, \pi/2)$ and some positive number R. In addition, we suppose that

$$\|(A - \lambda I)^{-1}\| \leq \frac{M}{|\lambda|}$$

for all λ in $\rho(A)$ with

$$\theta - \pi < \arg\ \lambda < \pi - \theta$$

and $|\lambda| \geq R$. Then A is the infinitesimal generator of a holomorphic one-parameter semigroup $\{T(z) : |\arg\ z| < \theta\}$ of bounded linear operators on X.

Proof Let $\tilde{A} = A - RI$. Then for all $\lambda \in \Omega(\theta) \cup \{0\}$,

$$\tilde{A} - \lambda I = (A - RI) - \lambda I = A - (\lambda + R)I,$$

which is bijective. So, $\lambda \in \rho(\tilde{A})$. Therefore

$$\Omega(\theta) \cup \{0\} \subset \rho(\tilde{A}).$$

Since

$$\tilde{A} - \lambda I = A - (\lambda + R)I,$$

we see that

$$\|(\tilde{A} - \lambda I)^{-1}\| \leq \frac{M}{|\lambda + R|} \leq \frac{M}{|\lambda|}.$$

By Theorem 3.2.2, $\tilde{A}$ is the infinitesimal generator of a holomorphic one-parameter semigroup $\{\tilde{T}(z) : |\arg z| < \theta\}$ of bounded linear operators on X. For all $z \in \mathbb{C}$ with $|\arg z| < \theta$, let

$$T(z) = e^{Rz}\tilde{T}(z).$$

Then A is the infinitesimal generator of a holomorphic one-parameter semigroup $\{T(z) : |\arg z| < \theta\}$ of bounded linear operators on X. □

The next theorem is an estimate that captures the essence of the Agmon–Douglis–Nirenberg estimates for pseudo-differential operators on $\mathbb{R}^n$.

Theorem 3.2.4 *Let $\sigma \in M^m_{\rho,\Lambda}$, $m > 0$, be a strongly M-elliptic symbol. Then we can find positive constants K_1, R_1 and θ_1 with $\theta_1 \in (0, \pi/2)$ such that*

$$K_1\|u\|_{m,p,\Lambda} \leq \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda}, \quad u \in H^{m,p}_{\Lambda},$$

for all $\lambda \in \Omega(\theta_1)$ *with* $|\lambda| > R_1$, *where*

$$\Omega(\theta_1) = \{\lambda \in \mathbb{C} : \theta_1 - \pi < \arg\, \lambda < \pi - \theta_1\},$$

I *is the identity operator on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Proof Let $\sigma_{\Omega(\theta_1)}$ be the function on $\mathbb{R}^n \times \mathbb{R}^n \times \Omega(\theta_1)$ defined by

$$\sigma_{\Omega(\theta_1)}(x, \xi, \lambda) = \sigma(x, \xi) + \lambda, \quad (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \Omega(\theta_1).$$

Since σ is strongly M-elliptic, it follows from Theorem 3.1.3 that $\sigma_{\Omega(\theta_1)}$ is $M^m_{\rho,\Lambda,m,\Omega(\theta_1)}$-elliptic. By Theorem 3.1.4, there exists a symbol τ in $M^{-m}_{\rho,\Lambda,m,\Omega(\theta_1)}$ such that

$$T_\tau T_{\sigma_{\Omega(\theta_1)}} = I + R,$$

where R is a pseudo-differential operator with symbol in $\bigcap_{k\in\mathbb{R}} M^k_{\rho,\Lambda,m,\Omega(\theta_1)}$. For all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(D_x^\alpha D_\xi^\beta \tau)(x, \xi, \lambda)| \leq C_{\alpha,\beta} \left(\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} \right)^m \Lambda(\xi)^{-m-\rho|\beta|}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta_1)$. Since

$$\frac{\Lambda(\xi)}{\Lambda(\xi) + |\lambda|^{1/m}} < 1, \quad \xi \in \mathbb{R}^n, \lambda \in \Omega(\theta_1),$$

it follows that

$$|(D_x^\alpha D_\xi^\beta \tau)(x,\xi,\lambda)| \leq C_{\alpha,\beta}\Lambda(\xi)^{-m-\rho|\beta|}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \in \Omega(\theta_1)$. By the L^p-boundedness of pseudo-differential operators with symbols in Theorem 1.2.11, there exist two positive constants C' and C'' such that

$$\begin{aligned}
\|u\|_{m,p,\Lambda} &= \|(T_\tau T_{\sigma_{\Omega(\theta_1)}} + R)u\|_{m,p,\Lambda} \\
&\leq \|T_\tau T_{\sigma+\lambda} u\|_{m,p,\Lambda} + \|Ru\|_{m,p,\Lambda} \\
&\leq C'\|T_{\sigma+\lambda}u\|_{0,p,\Lambda} + C''\|u\|_{0,p,\Lambda}, \quad u \in H^{m,p}_\Lambda,
\end{aligned}$$

for all $\lambda \in \Omega(\theta_1)$ Let $Q = \max(C', C'')$. Then

$$\|u\|_{m,p,\Lambda} \leq Q(\|T_{\sigma+\lambda}u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda}), \quad u \in H^{m,p}_\Lambda.$$

Let $K = 1/Q$. Then

$$K\|u\|_{m,p,\Lambda} \leq \|T_{\sigma+\lambda}u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda}, \quad u \in H^{m,p}_\Lambda. \tag{3.2.1}$$

Since $\sigma \in M^m_{\rho,\Lambda}$ is M-elliptic, it follows from the Agmon–Douglis–Nirenberg estimates that we can find two positive constants C_1 and C_2 for which

$$C_1\|u\|_{m,p,\Lambda} \leq \|T_{\sigma,0}u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \leq C_2\|u\|_{m,p,\Lambda}, \quad u \in H^{m.p}_\Lambda. \tag{3.2.2}$$

By (3.2.1) and (3.2.2), we get

$$\|T_{\sigma,0}u\|_{0,p,\Lambda} + \|u\|_{0,p.\Lambda} \leq \frac{C_2}{K}\|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \frac{C_2}{K}\|u\|_{0,p,\Lambda}.$$

Therefore

$$\|(T_{\sigma,0} + \lambda I)u - \lambda u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \leq \frac{C_2}{K}\|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \frac{C_2}{K}\|u\|_{0,p,\Lambda},$$

which implies that

$$\begin{aligned} & |\lambda|\,\|u\|_{0,p,\Lambda} - \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \|u\|_{0,p,\Lambda} \\ \leq\ & \frac{C_2}{K}\|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda} + \frac{C_2}{K}\|u\|_{0,p,\Lambda}. \end{aligned}$$

Thus,

$$\left(|\lambda| + 1 - \frac{C_2}{K}\right)\|u\|_{0,p,\Lambda} \leq \left(1 + \frac{C_2}{K}\right)\|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda}.$$

Now, let $\omega = \frac{C_2}{K} - 1$ and $L = 1 + \frac{C_2}{K}$. Then

$$\frac{|\lambda| - \omega}{L}\|u\|_{0,p,\Lambda} \leq \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda}.$$

Let $\tilde{R}$ be a positive number such that

$$1 - \frac{\omega}{\tilde{R}} > 0.$$

Then for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| \geq \tilde{R}$, we have

$$\tilde{C}|\lambda| \, \|u\|_{0,p,\Lambda} \leq \|(T_{\sigma,0} + \lambda I)u\|_{0,p,\Lambda}, \quad u \in H^{m,p}_{\Lambda}, \tag{3.2.3}$$

for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| \geq \tilde{R}$ if we pick $\tilde{C}$ to be any positive number such that

$$\tilde{C} < \frac{1 - \frac{\omega}{\tilde{R}}}{L}.$$

□

The same proof of Theorem 3.2.4 gives a similar estimate for the adjoint $T^t_{\sigma,0}$ of the pseudo-differential operator $T_{\sigma,0}$.

Theorem 3.2.5 *Let $\sigma \in M^m_{\rho,\Lambda}$, $m > 0$, be a strongly M-elliptic symbol. Then we can find positive constants K_2, R_2 and θ_2 with $\theta_2 \in (0, \pi/2)$ such that*

$$K_2\|u\|_{m,p',\Lambda} \leq \|(T^t_{\sigma,0} + \overline{\lambda} I)u\|_{0,p',\Lambda} + \|u\|_{0,p',\Lambda}, \quad u \in H^{m,p'}_{\Lambda},$$

for all $\lambda \in \Omega(\theta_2)$ with $|\lambda| > R_2$, where

$$\Omega(\theta_2) = \{\lambda \in \mathbb{C} : \theta_2 - \pi < \arg \lambda < \pi - \theta_2\},$$

I is the identity operator on $L^{p'}(\mathbb{R}^n)$ and

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We can now state and prove the main result of this chapter.

Theorem 3.2.6 *Let $\sigma \in M^m_{\rho,\Lambda}$, $m > 0$, be a strongly M-elliptic symbol. Then the pseudo-differential operator $-T_{\sigma,0}$ is the infinitesimal generator of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

Proof By Theorem 3.2.4, $T_{\sigma,0} + \lambda I$ is injective and has closed range for all $\lambda \in \Omega(\theta_1)$ with $|\lambda| > R_1$. By Theorem 3.2.5, $T^t_{\sigma,0} + \overline{\lambda} I$ is injective and has closed range for all $\lambda \in \Omega(\theta_2)$ with $|\lambda| > R_2$. Let

$$\theta = \min(\theta_1, \theta_2),$$

$$R = \max(R_1, R_2),$$

and

$$\Omega(\theta) = \{\lambda \in \mathbb{C} : \theta - \pi < \arg\, \lambda < \pi - \theta\}.$$

Let $\lambda \in \Omega(\theta)$ be such that $|\lambda| > R$. Then for any function v in $H^{0,p'}_{\Lambda}$ annihilating the range of $T_{\sigma,0} + \lambda I$, we have

$$v((T_{\sigma,0} + \lambda I)u) = 0, \quad u \in H^{m,p}_{\Lambda}.$$

Therefore

$$u((T^t_{\sigma,0} + \overline{\lambda} I)v) = 0, \quad u \in H^{m,p}_{\Lambda}.$$

So, $(T^t_{\sigma,0} + \overline{\lambda} I)v = 0$. Using the injectivity of $T^t_{\sigma,0} + \overline{\lambda} I$, we see that $v = 0$. Thus, the range of $T_{\sigma,0} + \lambda I$ is dense in $H^{0,p}_{\Lambda}$. Since the range of $T_{\sigma,0} + \lambda I$ is closed in $H^{0,p}_{\Lambda}$, it follows that the range of $T_{\sigma,0} + \lambda I$ is equal to $H^{0,p}_{\Lambda}$. Therefore $T_{\sigma,0} + \lambda I$ is surjective. This implies that

$-T_{\sigma,0} - \lambda I$ is also surjective. Hence for all $\lambda \in \Omega(\theta)$ with $|\lambda| > R$,

$$\lambda \in \rho(-T_{\sigma,0})$$

and by (3.2.3),

$$\|(-T_{\sigma,0} - \lambda I)^{-1}\| \leq \frac{M}{|\lambda|},$$

where $\| \ \|$ is the norm in the Banach algebra of all bounded linear operators on $L^p(\mathbb{R}^n)$ and

$$M = \frac{1}{\tilde{C}}.$$

Therefore by Theorem 3.2.3, $-T_{\sigma,0}$ is the infinitesimal generator of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$.

□

We conclude this chapter by giving an estimate for this holomorphic one-parameter semigroup. See details on page 134 of [23], and Theorem 2.2 in Chapter 1 of [17].

Theorem 3.2.7 *For $t \geq 0$, $e^{-tT_{\sigma,0}}$ is a bounded linear operator on $L^p(\mathbb{R}^n)$ and we have for every $\omega_1 \in (0, \infty)$,*

$$\|e^{-tT_{\sigma,0}}\| \leq Me^{t\omega_1},$$

Moreover, if we take $\omega_1 \to 0$, then we have $\|e^{-tT_{\sigma,0}}\| \leq M$.

4 Solutions of Evolution Equations in $L^p(\mathbb{R}^n), 1 < p < \infty$

Let us recall the following initial value problem for the evolution equation governed by a strongly M-elliptic pseudo-differential operator, i.e.,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -(T_{\sigma,0}u)(x,t), \quad x \in \mathbb{R}^n, t > 0, \\ \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $\sigma \in M^{2m}_{\rho,\Lambda}$ is a strongly M-elliptic symbol with $m > 0$, f is a given function in $L^p(\mathbb{R}^n), 1 < p < \infty$, and $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{C}$ is to be determined.

We introduce the classical solution and the mild solution of the initial value problem (4.1).

A classical solution u of the initial value problem (4.1) is a continuous function $u : [0,\infty) \to L^p(\mathbb{R}^n)$ such that u is continuously differentiable on $(0,\infty)$ and satisfies both equations in (4.1). We have the following theorem, which follows from the definition of an infinitesimal generator of a holomorphic one-parameter semigroup of bounded linear operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Theorem 4.0.1 *The initial value problem* (4.1) *has a unique classical solution u for every*

$f \in L^p(\mathbb{R}^n)$. *Moreover,*

$$u(t) = e^{-T_{\sigma,0}t} f, \quad t \in [0, \infty).$$

Remark 4.0.2 However, the proof of Theorem 2.3.2 gives us only a strongly continuous one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$. So $u(t) = e^{-T_{\sigma,0}t} f$ is a classical solution if $f \in H^{2m,2}_{\rho,\Lambda}$. If $f \in L^2(\mathbb{R}^n) \setminus H^{2m,2}_{\rho,\Lambda}$, then the function $u : [0, \infty) \to L^2(\mathbb{R}^n)$ defined by

$$u(t) = e^{-T_{\sigma,0}t} f, \quad t \in [0, \infty), \tag{4.2}$$

may not be a classical solution of the initial value problem (4.1). The best we can say about u is given in the following theorem.

Theorem 4.0.3 *Let $\{T(t) : t > 0\}$ be the strongly continuous one-parameter semigroup of bounded linear operators on $L^2(\mathbb{R}^n)$ generated by $-T_{\sigma,0}$. Then the function $u : [0, \infty) \to L^2(\mathbb{R}^n)$ defined by*

$$u(t) = T(t) f, \quad t \geq 0,$$

is a mild solution of the initial value problem (4.1) *in the sense that there exists a sequence $\{f_k\}_{k=1}^{\infty}$ in $H^{2m,2}_{\Lambda}$ such that $f_k \to u(0)$ in $L^2(\mathbb{R}^n)$ and $T(\cdot) f_k \to u$ in $L^2(\mathbb{R}^n)$ uniformly on every compact subset of $[0, \infty)$ as $k \to \infty$.*

Proof Since $H^{2m,2}_{\Lambda}$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k=1}^{\infty}$ in $H^{2m,2}_{\Lambda}$ such that $f_k \to f$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. So, $f_k \to u(0)$ in $L^2(\mathbb{R}^n)$ as $k \to \infty$. By Theorem 2.2 in Chapter 1 of the book [17], we can find constants ω and M with $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq M e^{\omega t}, \quad t \in [0, \infty),$$

where $\|\ \|$ denotes the norm in the Banach algebra of bounded linear operators on $L^2(\mathbb{R}^n)$. Let K be a compact subset of $[0,\infty)$. Then for all t in K,

$$\|T(t)f_k - u(t)\|_2 = \|T(t)f_k - T(t)f\|_2 \leq \|T(t)\|\|f_k - f\|_2 \leq Me^{\omega t}\|f_k - f\|_2.$$

Let $S = \sup_{t\in K} e^{\omega t}$. Then

$$\|T(t)f_k - T(t)f\|_2 \leq MS\|f_k - f\|_2 \to 0$$

for all $t \in K$ as $k \to \infty$. Therefore

$$T(\cdot)f_k \to u$$

in $L^2(\mathbb{R}^n)$ uniformly on every compact subset of $[0,\infty)$ as $k \to \infty$. This completes the proof that $u : [0,\infty) \to L^2(\mathbb{R}^n)$ is a mild solution of the initial value problem (4.1). □

Remark 4.0.4 It is clear that the proof of Theorem 4.0.3 is independent of the choice of the sequence $\{f_k\}_{k=1}^{\infty}$ in $H_{\Lambda}^{2m,2}$. So the mild solution is unique.

Remark 4.0.5 If $p = 2$ and $\sigma \in S_{\rho,\Lambda}$ is a strongly (ρ,Λ)-elliptic symbol with $m > 0$, then Theorem 4.0.1, Remark 4.0.2, Theorem 4.0.3 and Remark 4.0.4 are all true with the same arguments.

5 Examples

5.1 Heat Operator

We begin with our first example on the heat operators. The symbol σ of the heat operator $\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2}$ on $\mathbb{R}^2$ is given by

$$\sigma(\xi_1, \xi_2) = i\xi_1 + \xi_2^2, \quad \xi_1, \xi_2 \in \mathbb{R}.$$

Let

$$\Lambda(\xi_1, \xi_2) = 1 + \xi_1^2 + \xi_2^4, \quad \xi_1, \xi_2 \in \mathbb{R}.$$

We now show that $\Lambda(\xi_1, \xi_2)$ is a weight function and σ is an M-elliptic symbol in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$.

Step 1 $\Lambda(\xi_1, \xi_2) = 1 + \xi_1^2 + \xi_2^4$ is a weight.

$$\frac{1 + \sqrt{\xi_1^2 + \xi_2^2}}{1 + \xi_1^2 + \xi_2^4} \leq \frac{1 + \sqrt{\xi_1^2 + \xi_2^2}}{\sqrt{1 + \xi_1^2 + \xi_2^4}} = \frac{1}{\sqrt{1 + \xi_1^2 + \xi_2^4}} + \frac{\sqrt{\xi_1^2 + \xi_2^2}}{\sqrt{1 + \xi_1^2 + \xi_2^4}} \leq 2,$$

since $\xi_1^2 + \xi_2^2 < 1 + \xi_1^2 + \xi_2^4$ for any $(\xi_1, \xi_2) \in \mathbb{R}^2$. Hence we get

$$\frac{1}{2}\left(1 + \sqrt{\xi_1^2 + \xi_2^2}\right) \leq \Lambda(\xi_1, \xi_2), \tag{5.1}$$

So, in (1.1.1), $C_0 = \frac{1}{2}, \mu_0 = 1$. Next, we have for all $|\xi| > R$,

$$\begin{aligned}
\frac{1+\xi_1^2+\xi_2^4}{(1+\sqrt{\xi_1^2+\xi_2^2})^4} &\leq \frac{1+\xi_1^2+\xi_2^4}{(\xi_1^2+\xi_2^2)^2} \\
&= \frac{1+(\xi_1^2+\xi_2^2)+(\xi_1^2+\xi_2^2)^2}{(\xi_1^2+\xi_2^2)^2} \\
&= \frac{1}{(\xi_1^2+\xi_2^2)^2}+\frac{1}{\xi_1^2+\xi_2^2}+1 \\
&< 2.
\end{aligned}$$

Now, for all $|\xi| \leq R$, we use the fact that it is a compact set to conclude that $\frac{1+\xi_1^2+\xi_2^4}{(1+\sqrt{\xi_1^2+\xi_2^2})^4}$ attains a maximum C_2 on $|\xi| \leq R$. Let $C_1 = \max\{C_2, 2\}$ and $\mu_1 = 4$. Then we have

$$C_1\left(1+\sqrt{\xi_1^2+\xi_2^2}\right)^4 \geq \Lambda(\xi_1,\xi_2), \quad (\xi_1,\xi_2) \in \mathbb{R}^2. \tag{5.2}$$

Hence by (5.1) and (5.2), we have

$$\frac{1}{2}\left(1+\sqrt{\xi_1^2+\xi_2^2}\right) \leq \Lambda(\xi_1,\xi_2) \leq C_1\left(1+\sqrt{\xi_1^2+\xi_2^2}\right)^4, \quad (\xi_1,\xi_2) \in \mathbb{R}^2. \tag{5.3}$$

Next, for all multi-indices α and γ with $\gamma_j \in \{0,1\}, j = 1,2,...,n$, we can find a positive constant $C_{\alpha,\gamma}$, depending on α and γ only, for which $\Lambda(\xi_1,\xi_2) = 1+\xi_1^2+\xi_2^4$, satisfies equation (1.1.2) with $\mu = 4$.

There are only four possibilities for γ, i.e., $\gamma = (0,0), \gamma = (1,0), \gamma = (0,1)$ and $\gamma = (1,1)$.

Let $\alpha+\gamma = (\alpha_1+\gamma_1, \alpha_2+\gamma_2)$. It is easy to see that if $\alpha_1+\gamma_1 \geq 1$ and $\alpha_2+\gamma_2 \geq 1$, then

$$|\xi^\gamma(\partial^{\alpha+\gamma}\Lambda)(\xi)| \leq C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi \in \mathbb{R}^2.$$

.

Now let $\alpha+\gamma=(0,\alpha_2+\gamma_2)$. Then

Case 1: If $\alpha+\gamma=(0,\alpha_2)$ with $\gamma=(0,0)$, $\alpha=(0,\alpha_2), 0\leq\alpha_2\leq 4$, then we have

$$\begin{aligned}
|\xi^{\gamma}(\partial^{\alpha+\gamma}\Lambda)(\xi)| &= \left|\partial_{\xi_2}^{\alpha_2}(1+\xi_1^2+\xi_2^4)\right| \\
&= \left|\frac{4!}{(4-\alpha_2)!}\xi_2^{4-\alpha_2}\right| \\
&\leq \left|\frac{4!}{(4-\alpha_2)!}(1+\xi_1^2+\xi_2^4)^{\frac{4-\alpha_2}{4}}\right| \\
&= C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{|\alpha|}{4}}, \quad \xi\in\mathbb{R}^2.
\end{aligned}$$

Case 2: If $\alpha+\gamma=(0,\alpha_2+1)$ with $\gamma=(0,1)$, $\alpha=(0,\alpha_2), 0\leq\alpha_2\leq 4$, then we have

$$\begin{aligned}
|\xi^{\gamma}(\partial^{\alpha+\gamma}\Lambda)(\xi)| &= \left|\xi_2^1\partial_{\xi_2}^{\alpha_2+1}(1+\xi_1^2+\xi_2^4)\right| \\
&= \left|\frac{4!}{(4-\alpha_2)!}\xi_2^{4-(\alpha_2+1)+1}\right| \\
&\leq \left|\frac{4!}{(4-\alpha_2)!}(1+\xi_1^2+\xi_2^4)^{\frac{4-\alpha_2}{4}}\right| \\
&= C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{|\alpha|}{4}}, \quad \xi\in\mathbb{R}^2.
\end{aligned}$$

Next consider $\alpha+\gamma=(\alpha_1+\gamma_1,0)$. Then

Case 1: If $\alpha+\gamma=(\alpha_1,0)$ with $\gamma=(0,0)$, $\alpha=(\alpha_1,0), 0\le\alpha_1\le 2$, then we have

$$\begin{aligned}
|\xi^{\gamma}(\partial^{\alpha+\gamma}\Lambda)(\xi)| &= \left|\partial_{\xi_1}^{\alpha_1}(1+\xi_1^2+\xi_2^4)\right| \\
&= \left|\frac{2!}{(2-\alpha_1)!}\xi_1^{2-\alpha_1}\right| \\
&\le \left|\frac{2!}{(2-\alpha_1)!}(1+\xi_1^2+\xi_2^4)^{\frac{4-\alpha_1}{4}}\right| \\
&= C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{|\alpha|}{4}}, \quad \xi\in\mathbb{R}^2.
\end{aligned}$$

Case 2: If $\alpha+\gamma=(\alpha_1+1,0)$ with $\gamma=(1,0)$, $\alpha=(\alpha_1,0)$, $0\le\alpha_1\le 2$, then we have

$$\begin{aligned}
|\xi^{\gamma}(\partial^{\alpha+\gamma}\Lambda)(\xi)| &= \left|\xi_1^1\partial_{\xi_1}^{\alpha_1+1}(1+\xi_1^2+\xi_2^4)\right| \\
&= \left|\frac{2!}{(2-\alpha_1)!}\xi_1^{2-(\alpha_1+1)+1}\right| \\
&\le \left|\frac{2!}{(2-\alpha_1)!}(1+\xi_1^2+\xi_2^4)^{\frac{4-\alpha_1}{4}}\right| \\
&= C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{|\alpha|}{4}}, \quad \xi\in\mathbb{R}^2.
\end{aligned}$$

Therefore, for $\mu=\mu_1=4$ and for all multi-indices α and γ with $\gamma_j\in\{0,1\}, j=1,2$, we can find a positive constant $C_{\alpha,\gamma}$, depending on α and γ only, for which

$$|\xi^{\gamma}(\partial^{\alpha+\gamma}\Lambda)(\xi)|\le C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|}, \quad \xi\in\mathbb{R}^2. \tag{5.4}$$

By equations (5.3) and (5.4), we see that $\Lambda(\xi_1,\xi_2)=1+\xi_1^2+\xi_2^4, \quad \xi_1,\xi_2\in\mathbb{R}$, is a weight function.

Our next objective is to show that σ is in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$.

Step 2 $\sigma\in M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$, where $\Lambda(\xi_1,\xi_2)=1+\xi_1^2+\xi_2^4$.

We want to show that for $m = \frac{1}{2}$, $\rho = \frac{1}{4}$, and for multi-indices γ with $\gamma_j \in \{0, 1\}, j = 1, 2$, we have $\sigma(\xi_1, \xi_2) = i\xi_1 + \xi_2^2$ satisfies the condition $\xi^\gamma \left(\partial_\xi^\gamma \sigma\right)(\xi) \in S^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$.

In fact, we need to show that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α, β only, for which

$$\left|(\partial_x^\alpha \partial_\xi^\beta (\xi^\gamma (\partial_\xi^\gamma \sigma)))(\xi)\right| \leq C_{\alpha,\beta}(1 + \xi_1^2 + \xi_2^4)^{\frac{1}{2} - \frac{1}{4}|\beta|} \tag{5.5}$$

Note σ is a function of $\xi \in \mathbb{R}^2$ only. There are only three possibilities.

Case 1: If $|\alpha| \geq 1$, $|\beta| \geq 3$, or $|\gamma| = 2$, then left hand side of inequality (5.5) is 0 and so this inequality holds automatically.

Case 2: If $|\alpha| = 0$, $|\beta| \leq 2$ with $\beta = (\beta_1, 0)$, and $\gamma = (\gamma_1, 0)$, then we have for all $\xi \in \mathbb{R}^2$,

$$\begin{aligned}
\left|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\left(\xi_1^{\gamma_1}\xi_2^{\gamma_2}\left(\partial_{\xi_1}^{\gamma_1}(i\xi_1 + \xi_2^2)\right)\right)\right| &= \left|\partial_{\xi_1}^{\beta_1}\left(\xi_1^{\gamma_1}\frac{1!}{(1-\gamma_1)!}i\xi_1^{1-\gamma_1}\right)\right| \\
&= \left|\partial_{\xi_1}^{\beta_1}\left(\frac{1!}{(1-\gamma_1)!}i\xi_1\right)\right| \\
&= \left|\frac{1!}{(1-\gamma_1)!}\frac{1!}{(1-\beta_1)!}\xi_1^{1-\beta_1}\right| \\
&= \left|\frac{1!}{(1-\beta_1)!}\xi_1^{1-\beta_1}\right| \\
&= C_{0,\beta}|\xi_1|^{1-\beta_1}, \\
&= C_{0,\beta}|\xi_1^2|^{\frac{1-\beta_1}{2}} \\
&\leq C_{0,\beta}(1 + \xi_1^2 + \xi_2^4)^{\frac{1-\beta_1}{2}} \\
&\leq C_{0,\beta}(1 + \xi_1^2 + \xi_2^4)^{\frac{1}{2} - \frac{1}{4}|\beta|},
\end{aligned}$$

where $C_{0,\beta} = \frac{1!}{(1-\beta_1)!}$.

Case 3: If take $|\alpha| = 0$, $|\beta| \leq 2$ with $\beta = (0, \beta_2)$, and $\gamma = (0, \gamma_2)$, then for all $\xi \in \mathbb{R}^2$,

$$\begin{aligned}
\left|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\left(\xi_1^{\gamma_1}\xi_2^{\gamma_2}\left(\partial_{\xi_1}^{\gamma_1}(i\xi_1+\xi_2^2)\right)\right)\right| &= \left|\partial_{\xi_2}^{\beta_2}\left(\xi_2^{\gamma_2}\frac{2!}{(2-\gamma_2)!}\xi_2^{2-\gamma_2}\right)\right| \\
&= \left|\partial_{\xi_2}^{\beta_2}\left(\frac{2!}{(2-\gamma_2)!}\xi_2\right)\right| \\
&= \left|\frac{2!}{(2-\gamma_2)!}\frac{2!}{(2-\beta_2)!}\xi_2^{2-\beta_2}\right| \\
&= C_{0,\beta}|\xi_2|^{2-\beta_2}, \\
&= C_{0,\beta}|\xi_2^4|^{\frac{2-\beta_2}{4}} \\
&\leq C_{0,\beta}(1+\xi_1^2+\xi_2^4)^{\frac{2-\beta_2}{4}} \\
&= C_{0,\beta}(1+\xi_1^2+\xi_2^4)^{\frac{1}{2}-\frac{1}{4}|\beta|},
\end{aligned}$$

where $C_{0,\beta} = \frac{2!}{(2-\gamma_2)!}\frac{2!}{(2-\beta_2)!}$.

Therefore for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which (5.5) holds. So, $\sigma \in M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$.

Our next step is to show that σ is a M-elliptic symbol in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$.

Step 3 $\sigma \in M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$ is M-elliptic.

We need to show there exist positive constants C and R such that for all $|\xi| \geq R$,

$$|\sigma(\xi)| \geq C\Lambda(\xi)^{\frac{1}{2}}.$$

Indeed,

$$\begin{aligned}\frac{|\sigma(\xi)|}{\Lambda(\xi)^{\frac{1}{2}}} &= \frac{|i\xi_1+\xi_2^2|}{\sqrt{1+\xi_1^2+\xi_2^4}} \\ &= \sqrt{\frac{\xi_1^2+\xi_2^4}{1+\xi_1^2+\xi_2^4}} \geq \frac{1}{2},\end{aligned}$$

for all $|\xi| \geq R$ and $C = \frac{1}{2}$.

Therefore $\Lambda(\xi_1, \xi_2)$ is a weight function and σ is a M-elliptic symbol in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$.

Unfortunately, $\sigma(\xi_1, \xi_2) = i\xi_1 + \xi_2^2$ is not a strongly M-elliptic symbol. Since $\operatorname{Re}\sigma(\xi_1, \xi_2) = \xi_2^2 \geq C\sqrt{1+\xi_1^2+\xi_2^4}$ for $|\xi| \geq R$ is not true when $\xi_1 \to \infty$ and $\xi_2 = 0$.

Conclusions 5.1.1 The symbol of the heat operator is a M-elliptic symbol in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$, but not strongly M-elliptic. By Theorem 1.3.6, the domain of the minimal and hence maximal extension of the heat operator on $L^p(\mathbb{R}^2)$ is equal to the Sobolev space $H^{\frac{1}{2},p}_{\Lambda}$ for $1 < p < \infty$.

5.2 Wave Operator

The wave operator is

$$\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2},$$

where $(x_1, x_2) \in \mathbb{R}^2$. Since $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} = \partial_1^2 - \partial_2^2 = -(-i\partial_1)^2 + (-i\partial_2)^2 = -D_1^2 + D_2^2$, its symbol is

$$\sigma(\xi_1, \xi_2) = -\xi_1^2 + \xi_2^2, \quad \xi \in \mathbb{R}^2.$$

Here, we are wondering if this symbol is M-elliptic. More specifically, assume there exists a weight function $\Lambda(\xi)$ such that such that for all multi-indices γ with $\gamma_j \in \{0,1\}, j = 1,2$.

$$\xi^\gamma(\partial_\xi^\gamma \sigma)(x,\xi) \in S^m_{\rho,\Lambda}.$$

Can the symbol of wave operator be M-elliptic? The answer is No.

Let us assume that there exists a weight function $\Lambda(\xi_1,\xi_2)$ such that $\sigma(\xi_1,\xi_2) \in M^m_{\rho,\Lambda}$. If it is M-elliptic, then we are able to find two positive constants C and R, such that

$$|\sigma(\xi_1,\xi_2)| \geq C\Lambda(\xi)^m, \quad |\xi| \geq R. \tag{5.1}$$

Now, take the set $W = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : \xi_1 = \xi_2\}$. Then for any $(\xi_1,\xi_2) \in W$, we have $\sigma(\xi_1,\xi_2) = 0$. Therefore (5.1) fails. So there is no weight function that can make the wave operator to be M-elliptic.

Conclusions 5.2.1 The wave operator is not M-elliptic of any order, any ρ, and any weight.

5.3 Cauchy−Riemann Operator

The Cauchy−Riemann operator is a linear partial differential operator that arises in the study of complex analysis. It is defined as

$$\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}, \quad (x_1,x_2) \in \mathbb{R}^2.$$

One can easily find its symbol to be $\sigma(\xi_1, \xi_2) = i\xi_1 - \xi_2$ and the zero set is just a single point $(0, 0)$. It can be shown that the symbol σ is M-elliptic in $M^{\frac{1}{2}}_{\frac{1}{2},\Lambda}$ with respect to the weight Λ on $\mathbb{R}^2$ given by

$$\Lambda(\xi_1, \xi_2) = 1 + \xi_1^2 + \xi_2^2, \quad \xi_1, \xi_2 \in \mathbb{R}.$$

Indeed, for $|\xi| \geq R$, we have

$$\frac{|\sigma(x, \xi)|}{\Lambda(\xi)^{\frac{1}{2}}} = \sqrt{\frac{\xi_1^2 + \xi_2^2}{1 + \xi_1^2 + \xi_2^2}} \geq \frac{1}{2}.$$

By picking $C = \frac{1}{2}$, we have

$$|\sigma(x, \xi)| \geq \frac{1}{2}\Lambda(\xi)^{\frac{1}{2}}, \quad |\xi| \geq R.$$

Now, take the set $W = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 > 0\}$. Then for any $(\xi_1, \xi_2) \in W$, we have Re $\sigma(x, \xi) = -\xi_2 < 0$. Therefore there is no weight function that can make the Cauchy−Riemann operator to be strongly M-elliptic.

Conclusions 5.3.1 The symbol of the Cauchy−Riemann operator is a M-elliptic symbol in $M^{\frac{1}{2}}_{\frac{1}{2},\Lambda}$, but not strongly M-elliptic.

5.4 Bi-Heat Operator

The bi-Heat operator is given by

$$-\frac{\partial^2}{\partial x_1^2} + \frac{\partial^4}{\partial x_2^4}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

One can show that its symbol is $\sigma(\xi_1, \xi_2) = \xi_1^2 + \xi_2^4$. Let

$$\Lambda(\xi_1, \xi_2) = 1 + \xi_1^2 + \xi_2^4, \quad \xi_1, \xi_2 \in \mathbb{R}.$$

This is the same weight function $\Lambda(\xi_1, \xi_2)$ associated with the heat operator. We can write $\sigma(\xi_1, \xi_2)$ as

$$\sigma(\xi_1, \xi_2) = (i\xi_1 + \xi_2^2)(-i\xi_1 + \xi_2^2).$$

Let $\pi(\xi_1, \xi_2) = i\xi_1 + \xi_2^2$. Then $\sigma = \pi\bar{\pi}$. We note that π is the symbol of the heat operator, so π is M-elliptic in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$. Obviously, $\bar{\pi}$ is also M-elliptic in $M^{\frac{1}{2}}_{\frac{1}{4},\Lambda}$. Moreover, by using the product rule in Theorem 1.2.2, we see that $\sigma \in M^1_{\frac{1}{4},\Lambda}$ is M-elliptic. Since, σ is real-valued and non-negative, it follows that σ is a strongly M-elliptic symbol in $M^1_{\frac{1}{4},\Lambda}$.

Conclusions 5.4.1 The bi-heat operator is a strongly M-elliptic symbol in $M^1_{\frac{1}{4},\Lambda}$ and hence also a M-elliptic symbol.

5.5 Weights

Let Λ be the weight defined in Chapter 1. By Proposition 1.2.8, we know that $\Lambda \in S^1_{\rho,\Lambda}$ and $\Lambda \in M^1_{\rho,\Lambda}$.

Also, by letting $C = 1$ and $m = 1$, we see $\Lambda(\xi) \geq C\Lambda(\xi)^m$ holds for all $\xi \in \mathbb{R}^n$. That is, $\Lambda \in M^1_{\rho,\Lambda}$ is also M-elliptic.

Finally, $\Lambda(\xi)$ is a positive real-valued function. So it is also a strongly M-elliptic symbol in $M^1_{\rho,\Lambda}$.

Conclusions 5.5.1 The symbol Λ is a strongly M-elliptic symbol in $M^1_{\rho,\Lambda}$ and hence also M-elliptic.

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