Probability 3: Discrete Random Variables

Module Outline

- Review: Bayes' Rule
- Random variables: Discrete and Continuous
- Probability distributions
- Expectation and Variance
- Common discrete distributions

Review: Conditional probability

- Conditional probability: $P(B|A)$
- General multiplication rule: $P(A \cap B) = P(B|A)P(A)$
- Total/Marginal probability:
  $P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$
  $= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + ... + P(A_n)P(B|A_n)$

Review: Bayes' Rule

- **Bayes' rule**:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

- Start with $P(B) \equiv$ **prior**,

  Updating due to new information: $P(B|A) \equiv$ **posterior**

Review: Bayes' Rule example

- Prob. (infection) $= 0.3$. Do a **Test**:

  $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- **Ex**: Use Bayes' rule to find $P(\text{infected} \mid +ive) =$

Review: Bayes' Rule example

- Prob. (infection) $= 0.3$. Do a **Test**:

  $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- **Ex**: Use Bayes' rule to find $P(\text{infected} \mid +ive) =$

- $P(\text{inf} \mid +ive) = \frac{P(+ \mid \text{inf})P(\text{inf})}{P(+ \mid \text{inf})P(\text{inf}) + P(+ \mid \text{not inf})P(\text{not inf})} =$

Review contd.: A second test?

- 1*st* test: $+$ive. **Does a second test**.

  Test: $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- New prior: $P(\text{inf} \mid 1\text{st } +ive) =$

Review contd.: A second test?

- 1*st* test: +ive. **Does a second test**.

  Test: $P(+ive \mid \text{infected}) = 0.7$, $P(+ive \mid \text{not infected}) = 0.1$.

- New prior: $P(\text{inf} \mid \text{1st } +ive) =$

- $P(\text{inf} \mid \text{2nd } +ive) = \frac{P(+ \mid \text{inf})P(\text{inf})}{P(+ \mid \text{inf})P(\text{inf}) + P(+ \mid \text{not inf})P(\text{not inf})} =$

Review: Bayes' Rule extended

- **Bayes' rule**:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

- Suppose three possibilities for B : B_1, B_2, B_3

Review: Bayes' Rule extended

- **Bayes' rule**:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

- Suppose three possibilities for B : B_1, B_2, B_3
- Now, **Bayes' rule (extended)**:

$$\begin{aligned} P(B_1|A) &= \frac{P(A \cap B_1)}{P(A)} \\ &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)} \end{aligned}$$

Example: Bayes' Rule extended

- 3 *types of borrowers*:

  Good diligent: $P(\textit{default} \mid \text{good dil}) = 0.1$,

  prior: $P(\text{good dil}) = 0.4$.

  Good negligient: $P(\textit{default} \mid \text{good neg}) = 0.4$,

  prior: $P(\text{good neg}) = 0.4$.

  Bad: $P(\textit{default} \mid \text{bad}) = 1$, prior: $P(\text{bad}) = 0.2$.

Example: Bayes' Rule extended

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- **Observe a default:** $P(\text{bad} \mid \text{default}) =?$

Example: Types of borrowers

- **Good diligent**: $P(\mathit{def} \mid \text{good dil}) = 0.1$, $P(\text{good dil}) = 0.4$.

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- **Observe a default:** $P(\text{bad} \mid \text{default}) =$

$$\frac{P(\mathit{df}|\text{bad})P(\text{bad})}{P(\mathit{df}|\text{bad})P(\text{bad}) + P(\mathit{df}|\text{g dil})P(\text{g dil}) + P(\mathit{df}|\text{g neg})P(\text{g neg})}$$

Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event

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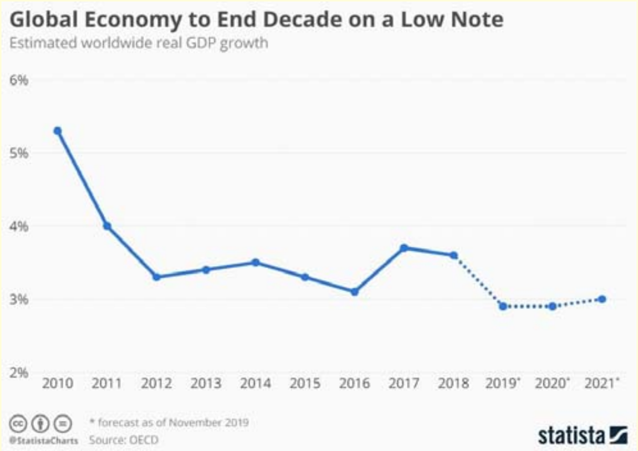
- Examples: **[Discrete]** No. of forest fires, crimes, spectators, children, minutes your phone lasts on a single charge......
- Examples: **[Continuous]** growth rate, crime rate, profits/losses, water level in a lake,.....



Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event

- Examples: **[Derived r.v.]** Letter grade based on numerical scores,……

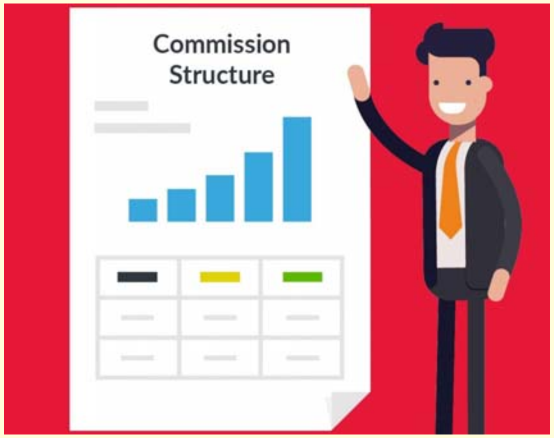
%	GPA	Letter grade
90-100%	4.0	A+
85-90%	3.9	A
80-84%	3.7	A-
77-89%	3.3	B+
73-76%	3.0	B
70-72%	2.7	B-
67-69%	2.3	C+
63-66%	2.0	C
60-62%	1.7	C-

Random Variables

- *Random variable* $\widetilde{X}$: A variable taking on *numerical values* based on the outcome of some random event

- Examples: **[Derived r.v.]** Letter grade based on numerical scores,......

- Examples: Profits based on sale numbers, commission based on sales, ...



Random Variables and Probabilities

- What are the probabilities of the different outcomes?

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- **Example 1**: A restaurant has 5 tables.

  $\widetilde{X} =$ number of tables occupied

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- **Example 1**: A restaurant has 5 tables.

  $\widetilde{X}$ = number of tables occupied

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$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

Example 1

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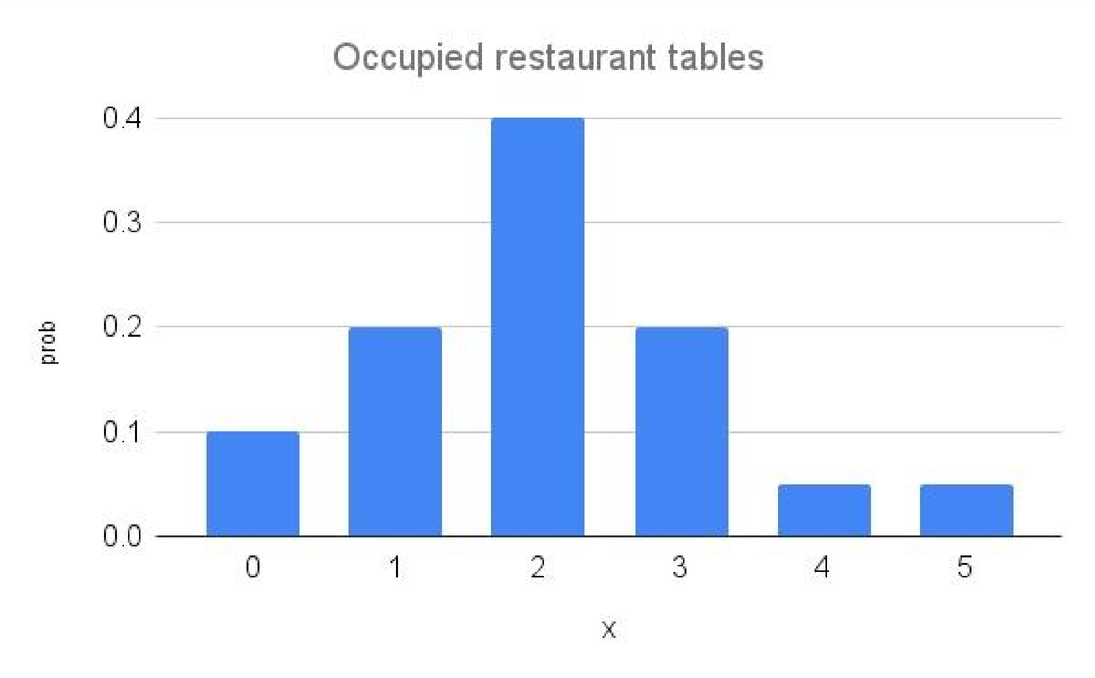

Random Variables and Probabilities: Example

 

- **Example 2**: 2 dice rolled. $\widetilde{X}$ = sum of their outcome.

$\widetilde{X}$	2	3	4	5	6	7	8	9	10	11	12
$p(x)$											

Example 2

- 2 dice rolled. $\widetilde{X}$ = sum of their outcome.

$\widetilde{X}$	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Example 1 contd.

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$\widetilde{X}$	0	1	2	3	4	5
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- *Profit per table* = \$50; *fixed costs* = \$100

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- *Profit per table* $= \$50$; *fixed costs* $= \$100$
- $\widetilde{Z} =$ total profit for the restaurant

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$						

Example 1 contd.

- $\widetilde{X}$ = number of tables occupied, $\widetilde{Z}$ = total profit

$\widetilde{X}$	0	1	2	3	4	5
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-

$\widetilde{Z}$	-100	-50	0	50	100	150
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- *Other probs.*: Prob. that restaurant is occupied?

Example 1 contd.

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- *Other probs.*: Prob. that restaurant is occupied?
- Prob. that restaurant makes $+$*ive* profits?

Clicker question 1

- $\widetilde{X}$ = number of tables occupied

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

Q: What is prob. that an odd number of tables are occupied?

(a) 0.1 **(b)** 0.2

(c) 0.4 **(d)** 0.45

Solution to Clicker question 1

- $\widetilde{X}$ = number of tables occupied

$\widetilde{X}$	0	1	2	3	4	5
$p(x)$	0.1	0.2	0.4	0.2	0.05	0.05

Q: What is prob. that an odd number of tables are occupied?

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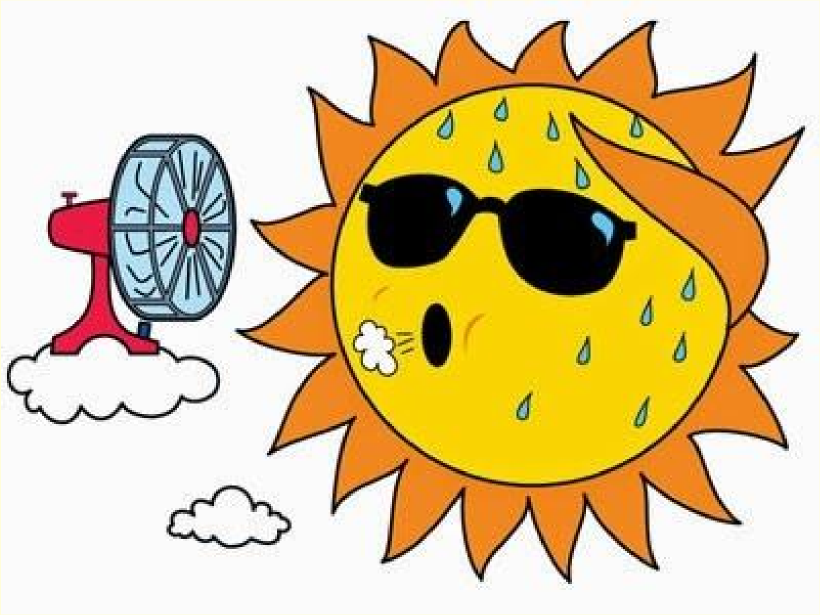
Example 2

- An ice-cream parlour's sales depends on the temp. $\widetilde{t}$

$\widetilde{t}$	< 20	$20-25$	$25-30$	$30-35$	$35-40$	$40+$
$p(t)$	0.15	0.15	0.2	0.2	0.25	0.05



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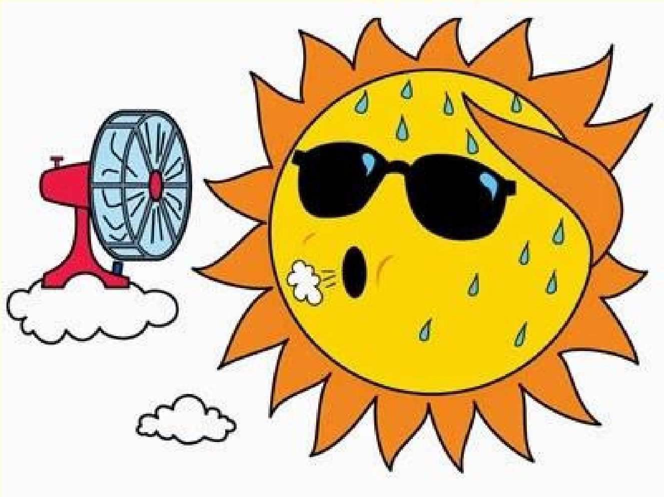
$p(t)$	0.15	0.15	0.2	0.2	0.25	0.05

- If $\tilde{t} < 20$, sales $\tilde{s} = 0$; $20 \le \tilde{t} < 30$, sales $\tilde{s} = 30$;

  $30 \le \tilde{t} < 40$, sales $\tilde{s} = 60$; $40 \le \tilde{t}$, sales $\tilde{s} = 100$



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  $\underline{30 \leq \tilde{t} < 40}$, sales $\tilde{s} = 60$; $\underline{40 \leq \tilde{t}}$, sales $\tilde{s} = 100$

-

$\tilde{s}$	0	30	60	100
$p(s)$				

Cumulative distribution function

- **Cumulative distribution function** $(c.d.f)$: Prob. that $\widetilde{X}$ takes values less than or equal to x :

$$F(x) = P(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$

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$$F(x) = P(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$

- **Example 1**:

$\widetilde{X}$	0	1	2	3	4	5
$p.m.f.$	0.1	0.2	0.4	0.2	0.05	0.05
$c.d.f.$						

C.d.f. for Example 1 contd.

-

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05
$F(z)$						

C.d.f. for Example 1 contd.

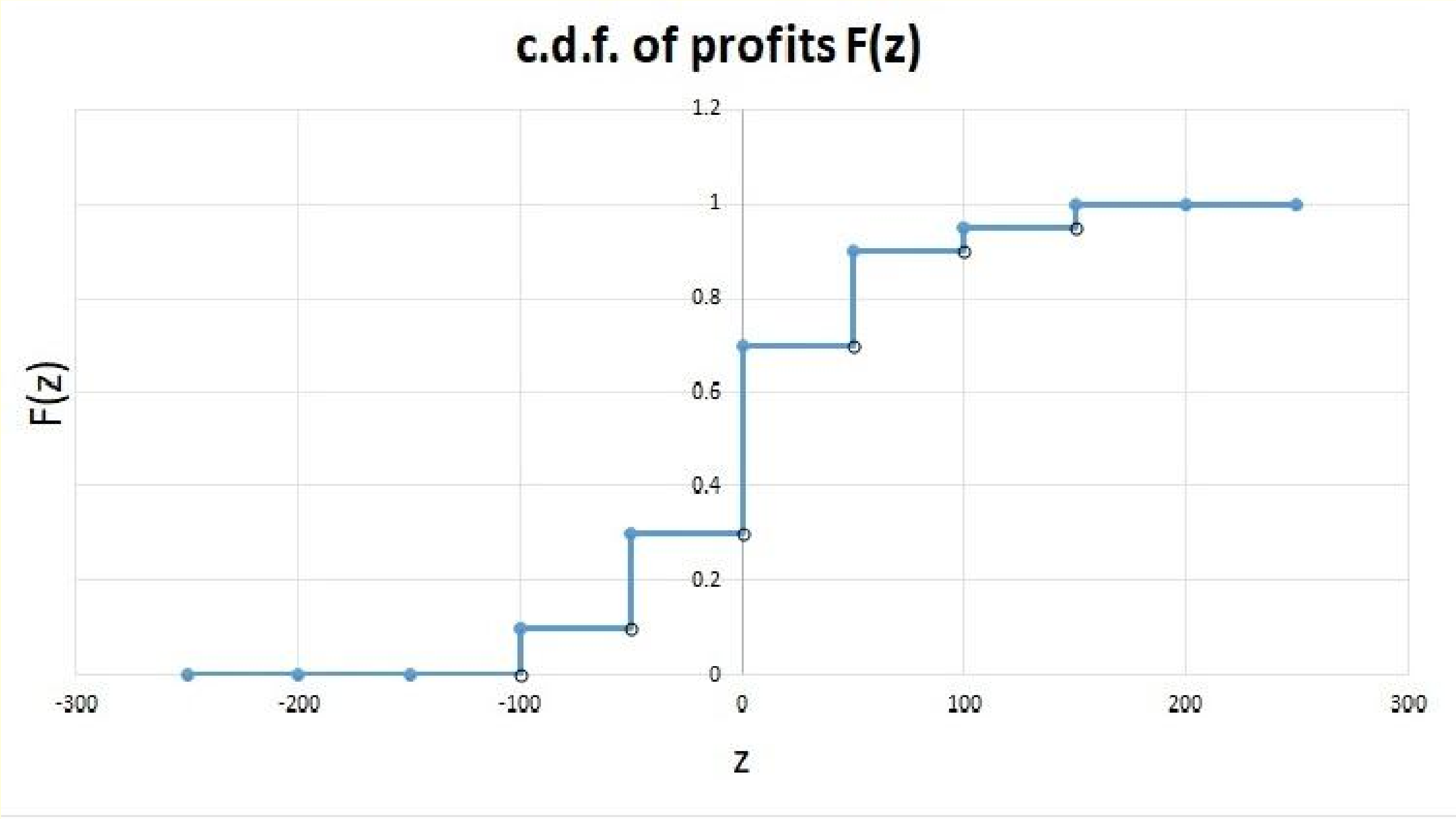
$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05
$F(z)$	0.1	0.3	0.7	0.9	0.95	1

- What is $F(22)$? $F(124)$? $F(200)$? $F(-200)$?

Graph of c.d.f. for Example 1 contd.

$\widetilde{Z}$	-100	-50	0	50	100	150
$F(z)$	0.1	0.3	0.7	0.9	0.95	1

- Graph: step-function



Cumulative distribution function: Properties

- $0 \leq F(x) \leq 1$

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  $F(x) = 1$ for $x \geq x^{\text{max}}$.

Cumulative distribution function: Properties

- $0 \leq F(x) \leq 1$
- $F(x)$ is non-decreasing
- If maximum possible value of $\widetilde{X}$ is $x^{\max}$:

  $F(x) = 1$ for $x \geq x^{\max}$.
- If minimum possible value of $\widetilde{X}$ is $x^{\min}$:

  $F(x) = 0$ for $x < x^{\min}$.

Clicker question 2

- A random variable $\widetilde{X}$ has the following p.m.f.:

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2

Q: What is $F(60)$?

(a) 0.25 **(b)** 0.45

(c) 0.65 **(d)** 1

Solution to Clicker question 2

- A random variable $\widetilde{X}$ has the following p.m.f.:

$\widetilde{X}$	10	20	30	40	50	60
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Q: What is $F(60)$?

(a) 0.25 **(b)** 0.45

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C.d.f. for Clicker question

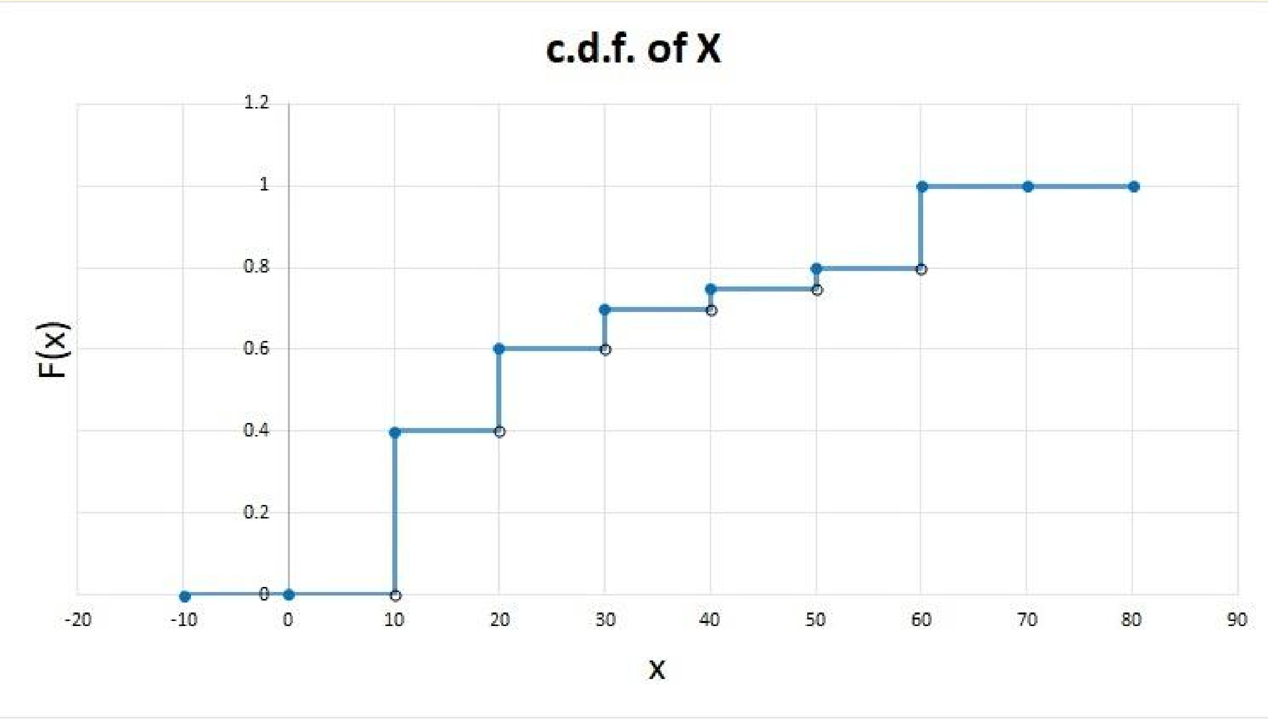
- *Clicker question*: start with p.m.f.; derive c.d.f.:

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2
$F(x)$						

C.d.f. for Clicker question

$\widetilde{X}$	10	20	30	40	50	60
$p(x)$	0.4	0.2	0.1	0.05	0.05	0.2
$F(x)$	0.4	0.6	0.7	0.75	0.8	1

- Graph: step-function



Deriving p.m.f from c.d.f.

- *Clicker question*: start with p.m.f.; derive c.d.f.:

$\widetilde{X}$	10	20	30	40	50	60
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- **Opposite:** Start with $c.d.f.$ $F(x)$; how to get $p.m.f.$ $p(x)$?

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- $F(x) = P(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$

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- *Clicker question*: start with p.m.f.; derive c.d.f.:

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- **Opposite:** Start with $c.d.f.$ $F(x)$; how to get $p.m.f.$ $p(x)$?
- $F(x) = P(\widetilde{X} \leq x) = \sum_{y:\, y \leq x} p(y)$
- $\Rightarrow$ p.m.f. $=$ difference between adjacent c.d.f.s

Deriving p.m.f from c.d.f: Examples

- *Start with c.d.f.:*

$\underset{\sim}{X}$	10	20	30	40	50	60
$F(x)$	0.4	0.6	0.7	0.75	0.8	1
$p(x)$						

Deriving p.m.f from c.d.f: Examples

- *Start with c.d.f.*:

$\widetilde{X}$	10	20	30	40	50	60
$F(x)$	0.4	0.6	0.7	0.75	0.8	1
$p(x)$						

- **Example 1 contd.**: derive *pmf* from *cdf*

$\widetilde{Z}$	-100	-50	0	50	100	150
$F(z)$	0.1	0.3	0.7	0.9	0.95	1
$p(z)$						

Clicker question 2

- A random variable $\widetilde{X}$ has the following c.d.f.:

$\widetilde{X}$	1	2	3	4	5	6
$F(x)$	0.1	0.2	0.4	0.5	0.75	1

Q: What is $p(6)$?

(a) 0.1 **(b)** 0.2

(c) 0.25 **(d)** 0.75

Solution to Clicker question 2

- A random variable $\widetilde{X}$ has the following c.d.f.:

$\widetilde{X}$	1	2	3	4	5	6
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Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

$\widetilde{Z}$	-100	-50	0	50	100	150
$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

Expected profits?

Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

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$p(z)$	0.1	0.2	0.4	0.2	0.05	0.05

  Expected profits?

- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} xp(x)$$

Expected value

- **Example 1** (contd.) $\widetilde{Z}$ = profit

$\widetilde{Z}$	-100	-50	0	50	100	150
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- ! *Not the same* as average
  only when $p(x) = \frac{1}{N}$

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  only when $p(x) = \frac{1}{N}$
- **Weighted average**

Example 3

- Diameter $\widetilde{d}$ of tree-trunks in a forest:

$\widetilde{d}$ (in m)	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1

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$\widetilde{d}$ (in m)	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1

- Mean diameter $\mu_d =$

- **Expected value** of a function of $\widetilde{X}$:

$$Eg(\widetilde{X}) = \sum_{\text{all } x} g(x)p(x)$$

Example 3 contd.

- Diameter $\widetilde{d}$ of tree-trunks in a forest;

  Lumber yield y from a tree of diameter d : $10\pi d^2$

  Expected lumber yield?

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  Expected lumber yield?

$\widetilde{d}$	1	2	3	4
$p(d)$	0.3	0.4	0.2	0.1
$\widetilde{y} = 10\pi\widetilde{d}^2$				

  Mean yield $\mu_d =$

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- **Expected value** (*mean*) of $\widetilde{X}$:

$$E\widetilde{X} = \mu_X = \sum_{\text{all } x} xp(x)$$

- $E(a + b\widetilde{X}) = a + bE\widetilde{X}$
- $E(b\widetilde{X} + c\widetilde{Y}) = bE(\widetilde{X}) + cE(\widetilde{Y})$

Example 4

- **Example 4:** Cake sales in two locations:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1

$\widetilde{s}_2$	0	20	50	100
$p(s_2)$	0.1	0.2	0.5	0.2

Expected sales: $E\widetilde{s}_1 =$, $E\widetilde{s}_2 =$

Example 4

- **Example 4:** Cake sales in two locations:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1

$\widetilde{s}_2$	0	20	50	100
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Expected sales: $E\widetilde{s}_1 =$, $E\widetilde{s}_2 =$

- Fixed costs = \$25 for location 1, \$100 for location 2.

Profit per ice-cream: Location 1: \$1; Location 2: \$1.50.

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Clicker question 4

- A random variable $\widetilde{Y}$ has expected value of \$20.

  Q: What is $E(40 + 5\widetilde{Y})$?

  (a) 20 **(b)** 40

  (c) 100 **(d)** 140

Solution to Clicker question 4

- A random variable $\widetilde{Y}$ has expected value of \$20.

  Q: What is $E(40 + 5\widetilde{Y})$?

(a) 20 **(b)** 40

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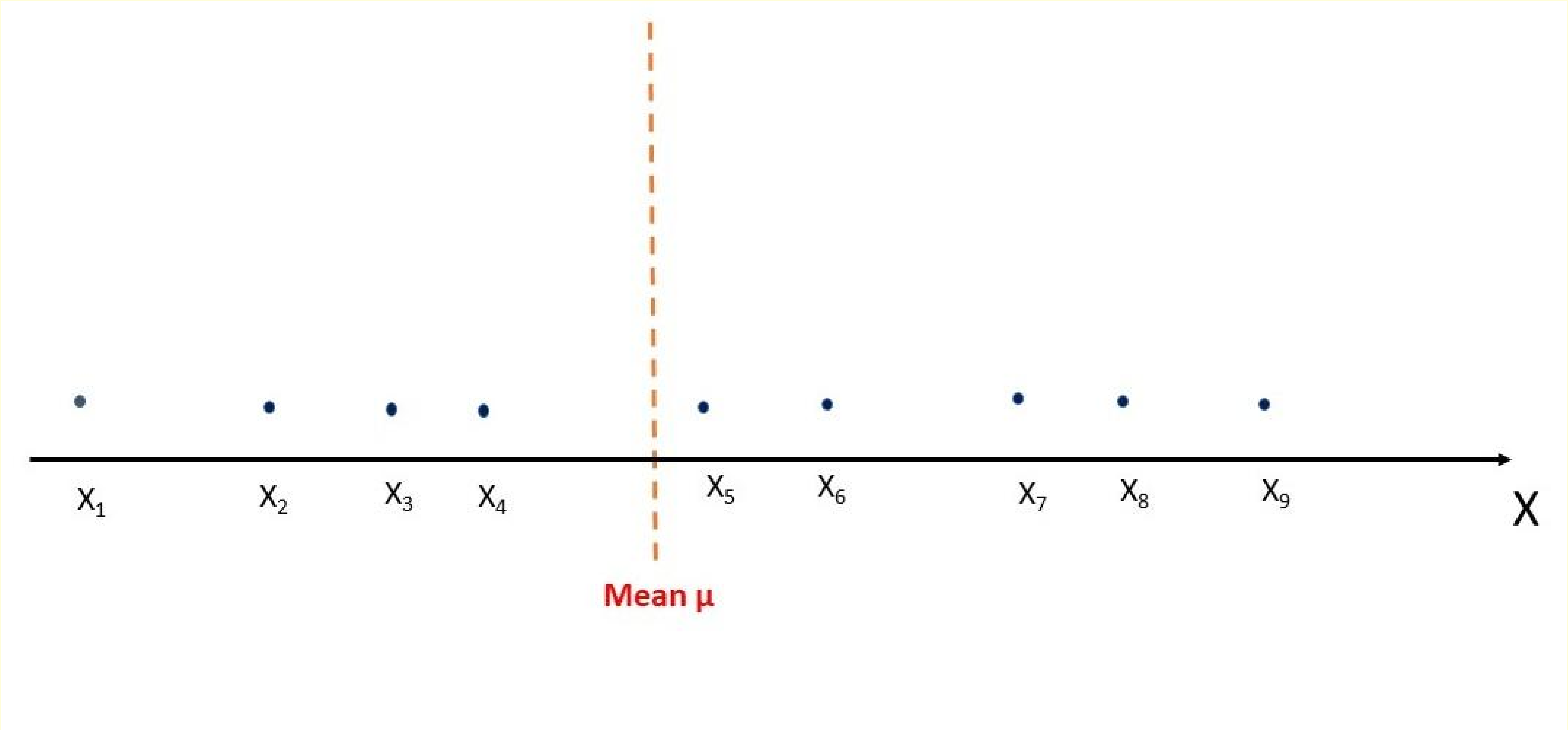
Variance: Measure of variability



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- **Standard deviation of** $\widetilde{X} = \sqrt{Variance} = \sqrt{\sigma_X^2} = \sigma_X$

Variance

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- Cake sales in location 1:

$\widetilde{s}_1$	0	10	20	30
$p(s_1)$	0.1	0.6	0.2	0.1
$(\widetilde{s}_1-\mu_S)^2$				

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Variance: an alternate (easier) formula

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Variance: an alternate (easier) formula

- **Variance of** $\widetilde{X} = \sigma_X^2 = E[(X - \mu_X)^2] =$
- $E(X - \mu_X)^2 = E(X^2) - (\mu_X)^2 = E(X^2) - (EX)^2$

$$\sigma_X^2 = E(X^2) - (EX)^2$$

Variance: Example 4

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Variance: Example 4

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$E\widetilde{s} = \mu_S = 13$, $E\widetilde{s}^2 =$

Variance of $\widetilde{s} = \sigma_s^2 =$

Common Discrete Distributions 1: Bernoulli

- A single trial is conducted:

  outcome: *Success* $(\widetilde{Z} = 1)$ or *Failure* $(\widetilde{Z} = 0)$

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$\widetilde{Z}$	1	0
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- $\widetilde{Z}$ is an **indicator variable**. Examples?

Bernoulli Distribution: Examples

- **Bernoulli distribution**: any 2 outcome event

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- e.g. *pick a random tree in a forest*:

  is it taller than 20m?

  is it diseased?

Common Discrete Distributions 2: Binomial distribution

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  Each trial can either be *Success* (with prob. p) or *Failure*

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- Examples:

Binomial distribution

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Binomial distribution

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- For $x \in \{0, 1, 2, ...n\}$,

$$Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

and $x! = x(x-1)(x-2).......1.$

Binomial distribution: Example

- **Example 5**: Pick 5 people at random.

  Prob. of anyone being university-educated $= 0.3$

  Q: Prob. that exactly 3 in this sample are univ-educated?

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  Q: Prob. that exactly 3 in this sample are univ-educated?

- Binomial: $n = 5$, $p = 0.3$

- $Prob(\widetilde{X} = 3) = \binom{5}{3}(0.3)^3(1 - 0.3)^{5-3} =$

  $\binom{5}{3} = \frac{5!}{3!2!} =$

Binomial distribution

- Set-up: n *independent* trials, each with 2 possible outcomes

  identical trials: same prob. of success $= p$

  $\widetilde{X} =$ total number of successes (out of n)

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- **Mean** $= np$

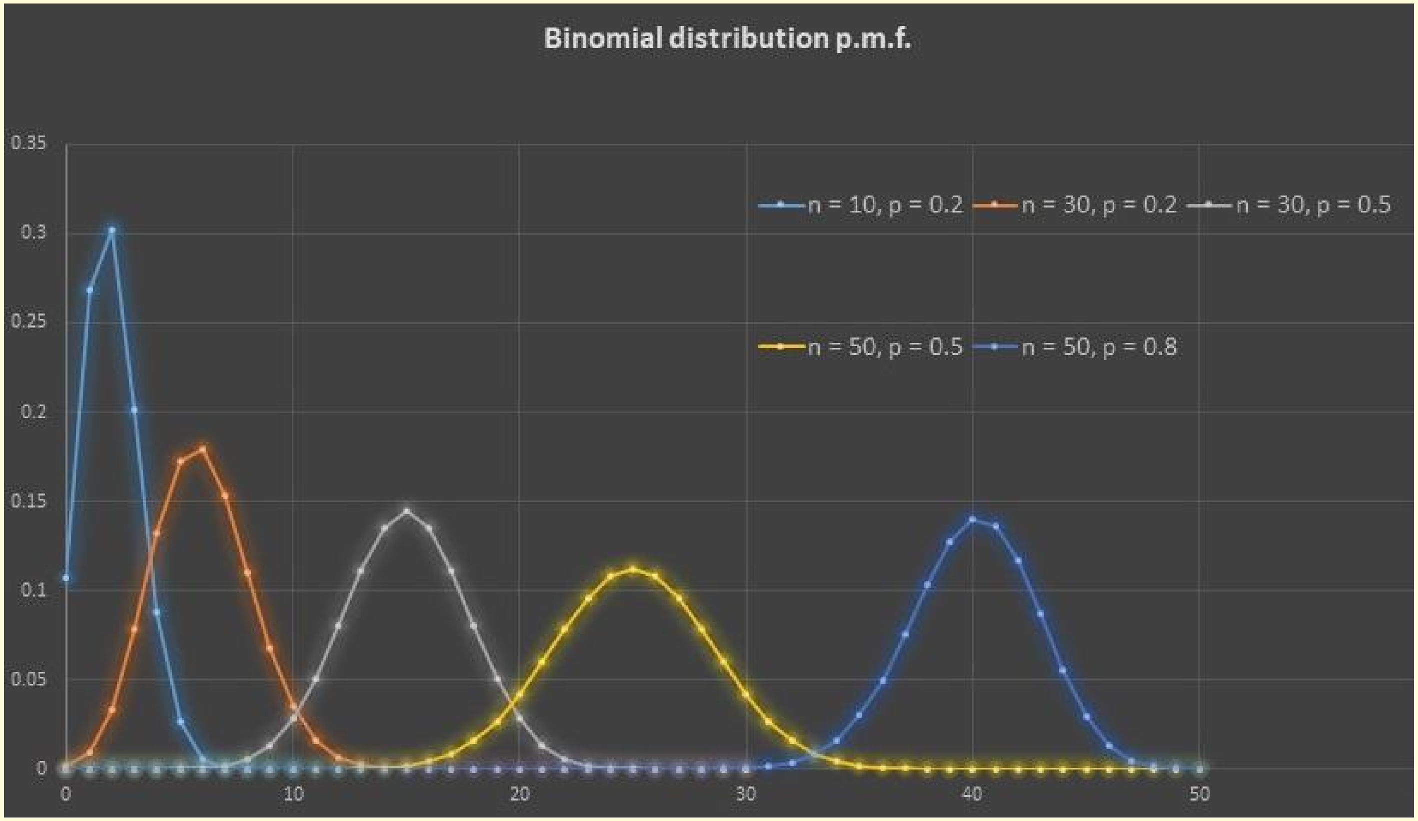
  Variance $= np(1-p)$

Binomial distribution: Shape of p.m.f.

- **p.m.f.**: $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$. How do they look?

Binomial distribution: Shape of p.m.f.

- **p.m.f.**: $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$. How do they look?
- Depends on n and p : **parameters**



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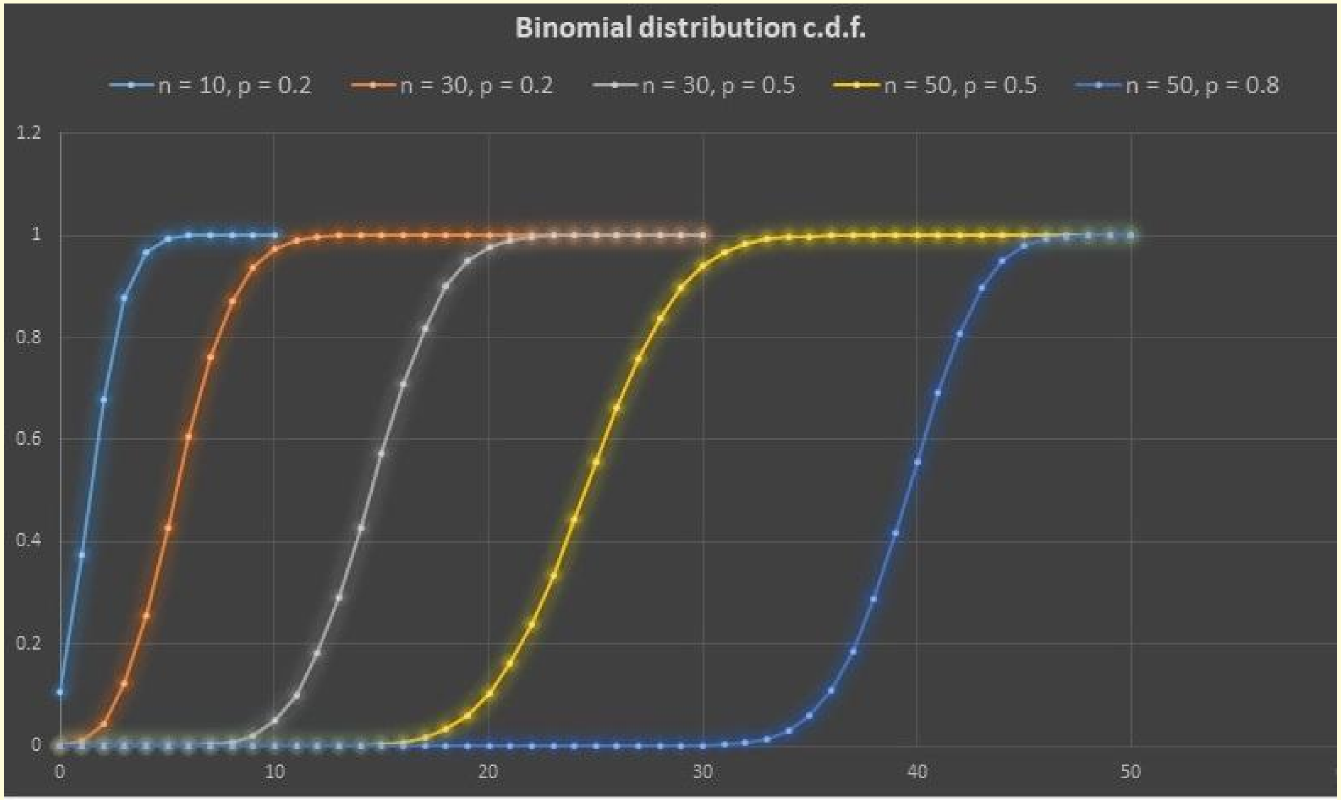
- 

Binomial distribution: Example

- **Example 6**: Voter turnout rate in 2021 fed. election = 60%.

  Pick 8 people at random.

  Q: Prob. that exactly 4 in this sample voted?

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Binomial distribution: Example

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  Q: Prob. that exactly 4 in this sample voted?

- Binomial: $n = 8$, $p = 0.6$
- $Prob(\widetilde{X} = 4) = \binom{8}{4}(0.6)^4(1 - 0.6)^{8-4} =$

  $\binom{8}{4} = \frac{8!}{4!4!} =$

Binomial distribution: Example

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- Prob. that 4 or fewer voted =

- Mean number who voted $= np =$

  $Variance = np(1 - p) =$

Clicker question 5

- The probability that any given e-mail is spam is 0.25. Suppose you receive 80 e-mails in a day.

  Q: What is the mean number of spam e-mails that you will receive in a day?

(a) 20 **(b)** 25

(c) 40 **(d)** 80

Soluton to Clicker question 5

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  Q: What is the mean number of spam e-mails that you will receive in a day?

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Common Discrete Distributions 3: Poisson distribution

- $\widetilde{X} =$ number of times an event occurs within a given time interval/area

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- For $x \in \{0, 1, 2, 3, \ldots., \infty\}$,

$$Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

where $e \approx 2.71828$

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where $e \approx 2.71828$

- Only one *parameter*: $\boldsymbol{\lambda}$ = mean number of times the event occurs within the given time interval/area

Poisson distribution: Example

- **Example 7**: On average, Pam receives 5 e-mails in an hour.

  Q: Prob. that she will receive 6 e-mails in the next hour?

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- Here: $\lambda = 5$; $Prob(\widetilde{X} = 6) = \frac{e^{-6}5^{6}}{6!} =$

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- Prob. that she gets no e-mail: $Prob(\widetilde{X} = 0) = \frac{e^{-5}5^0}{0!} =$

- Prob. she gets 6 e-mails or less:

  $Prob(\widetilde{X} = 0) + Prob(\widetilde{X} = 1) + ... + Prob(\widetilde{X} = 6)$

Poisson distribution: Examples

- On average, 10 crimes in a city per day

  Q: Prob. that there will be 8 crimes tomorrow?

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- **Mean** $= \lambda$

  Variance $= \lambda$

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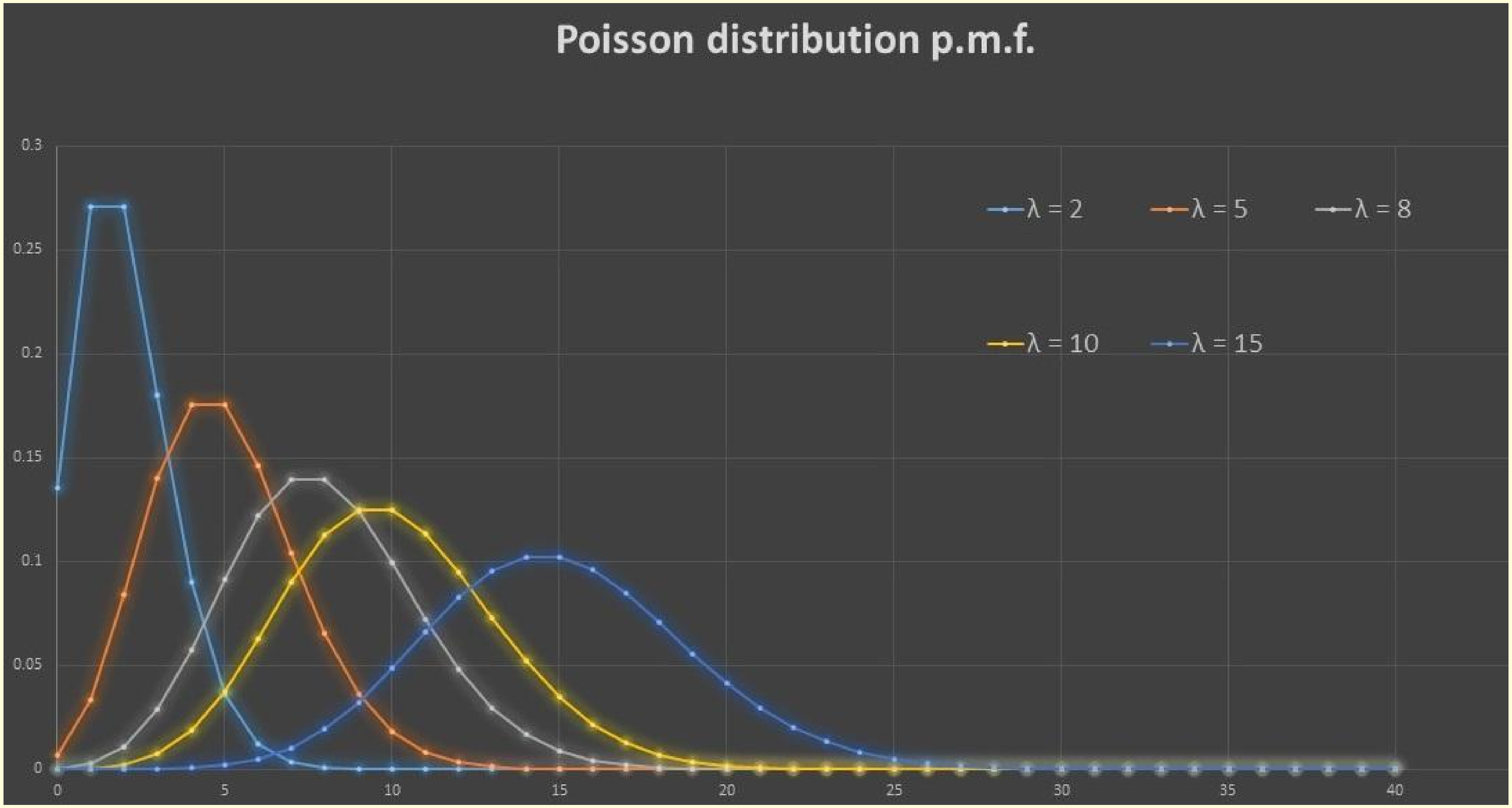

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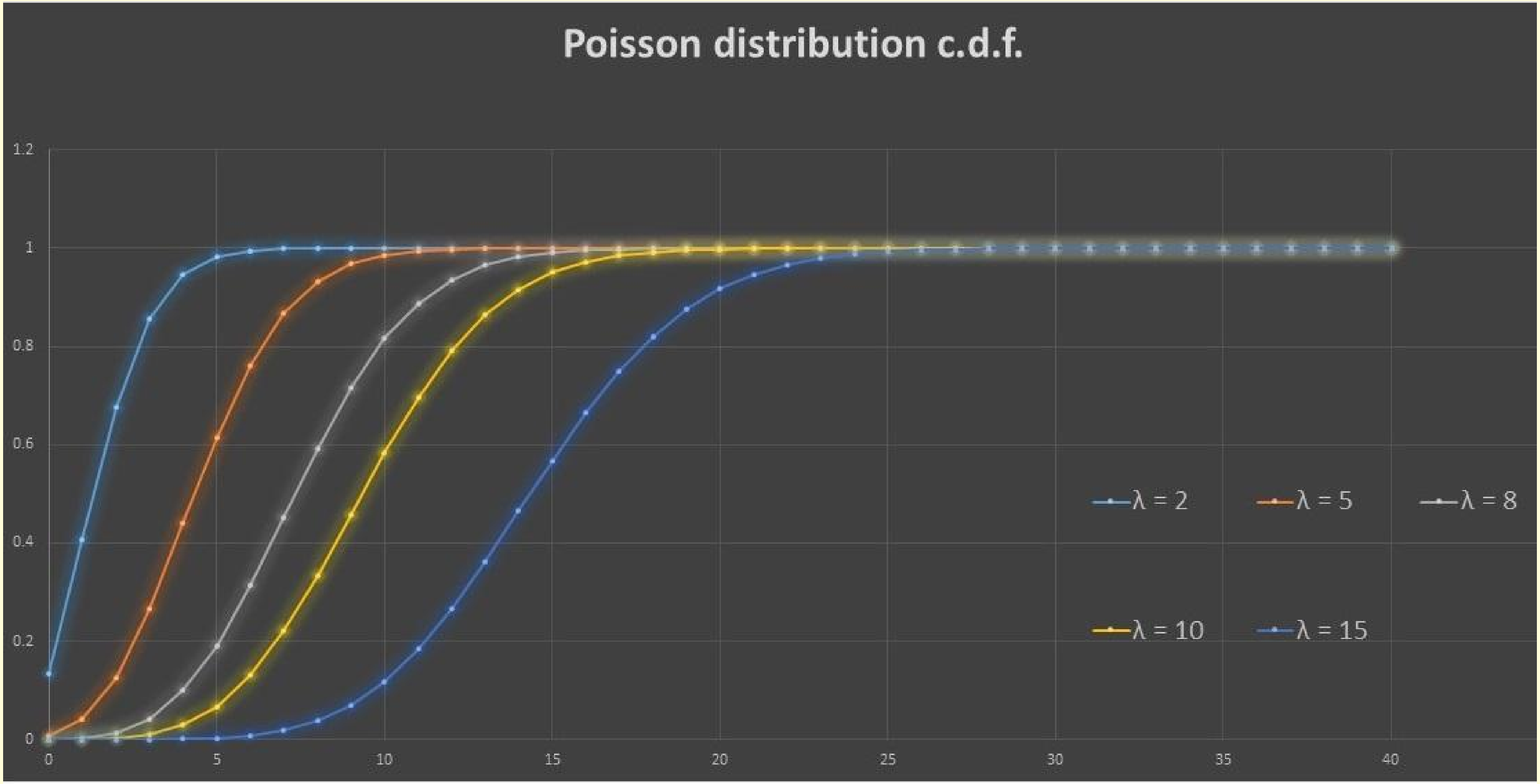

Excel: Computing Expectation and Variance

- **Column A**: possible x **Column B**: corresponding prob. $p(x)$
- **Column C**: $xp(x)$ $E\widetilde{X} = \sum_{\text{all } x} xp(x) = \text{SUM(C2: C11)}$
- Variance: $\sigma_X^2 = E[(X - \mu_X)^2] = E(X^2) - (EX)^2$
- **Column D**: $x^2p(x)$ $E(X^2) = \sum_{\text{all } x} x^2p(x) = \text{SUM(D2: D11)}$
- Variance $= D12 - (C12)\hat{}2$: **Cell** $E12$

Excel: Binomial and Poisson Distributions

- *Binomial*: $Prob(\widetilde{X} = x) = \binom{n}{x} p^x (1-p)^{n-x}$

  Poisson: $Prob(\widetilde{X} = x) = \frac{e^{-\lambda}\lambda^x}{x!}$
- **Column C**: possible $x \in \{0, 1, 2, 3, ...\}$ generate using $C4 = C3 + 1$ etc.
- *Binomial pmf*: **Column D**: $=$ BINOM.DIST($x, n, p, FALSE$)
- *Binomial cdf*: **Column H**: $=$ BINOM.DIST($x, n, p, TRUE$)

  In *Google Sheets*: BINOMDIST
- *Poisson pmf*: **Column D**: $=$ POISSON.DIST($x, n, p, FALSE$)
- *Poisson cdf*: **Column H**: $=$ POISSON.DIST($x, n, p, TRUE$)

  In *Google Sheets*: POISSON.DIST

Module recap

- Random variables: Discrete
- Probability mass function (p.m.f.) $p(x)$

  Cumulative distribution function (c.d.f.):
  $Prob(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$

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- Random variables: Discrete
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- Expectation, Variance (next module: Correlation)

Module recap

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- Probability mass function (p.m.f.) $p(x)$

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  $$Prob(\widetilde{X} \leq x) = \sum_{y:\ y \leq x} p(y)$$
- Expectation, Variance (<u>next module</u>: Correlation)
- Common discrete distributions:

  Bernoulli (single trial)

  Binomial (number of successes in n trials)

  Poisson (number of occurrences in a time period/area)