Module: Summations and Sequences

Module Outline

- Summation notation and properties
- Summations in Excel/Google sheets
- Sums of *arithmetic* and *geometric* series
- Sequences and Limits of sequences

Summation notation

- Total population of Canada:

$$\Big(\underline{P_{NFL}} + \underline{P_{PEI}} + \underline{P_{NS}} + P_{NB} + P_{QC} + P_{ON}$$
$$+ P_{MB} + P_{SK} + P_{AB} + P_{BC} + \underline{P_{YT}} + \underline{P_{NT}} + \underline{P_{NU}} \Big)$$

Summation notation

- Total population of Canada:

$$\Big(\overset{1}{P_{NFL}} + \overset{2}{P_{PEI}} + \overset{3}{P_{NS}} + \overset{4}{P_{NB}} + P_{QC} + P_{ON}$$
$$+ P_{MB} + P_{SK} + P_{AB} + P_{BC} + P_{YT} + P_{NT} + P_{NU} \Big)$$

- Writing this succinctly: $P_1 \equiv P_{NFL}$, $P_2 \equiv P_{PEI}$, ...

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6$$
$$+ P_7 + P_8 + P_9 + P_{10} + P_{11} + P_{12} + P_{13}$$

Summation notation

- Total population of Canada:

$$P_{NFL} + P_{PEI} + P_{NS} + P_{NB} + P_{QC} + P_{ON}$$
$$+ P_{MB} + P_{SK} + P_{AB} + P_{BC} + P_{YT} + P_{NT} + P_{NU}$$

- Writing this succintly: $P_1 \equiv P_{NFL}, P_2 \equiv P_{PEI}, ...$

$$P_1 + P_2 + P_3 + P_4 + P_5 + P_6$$
$$+ P_7 + P_8 + P_9 + P_{10} + P_{11} + P_{12} + P_{13}$$

- Even more succintly?

Summation notation

- $\left(P_1 + P_2 + P_3 + \ldots\ldots + P_{13} \right)$:

$i=1 \quad i=2 \quad i=3$

$$\sum_{i=1}^{13} P_i$$

Summation notation

- $P_1 + P_2 + P_3 + ... + P_{13}$:
- Sum of $P_1, P_2, ..., P_{13}$:

$$\sum_{i=1}^{13} \mathbf{P}_i = P_1 + P_2 + \cdots + P_{13}$$

Summation notation

- $P_1 + P_2 + P_3 + ... + P_{13}$:
- Sum of $P_1, P_2, ..., P_{13}$:

$$\sum_{i=1}^{13} \mathbf{P}_i$$

- Three elements: (i) $\sum$ sign **[sigma]**
  (ii) **what** are we summing over: P_i
  (iii) from where to where: i from 1 to 13 **[index]**

Summation notation: Examples

- A constituency has n districts.

  Number of votes that party L receives in district j = v_j

  $$j = 1, 2, \ldots\ldots, n$$

Summation notation: Examples

- A constituency has n districts.

  Number of votes that party L receives in district $j = v_j$

- Total number of votes for party L in the constituency=

$$v_1 + v_2 + v_3 + \cdots + v_n = \sum_{j=1}^{n} v_j = \sum_{k=1}^{n} v_k$$

Summation notation: Examples

- Total number of Covid cases last year:

$$\sum_{t=1}^{12} \mathbf{n}_t$$

where n_t = number of Covid cases in month t

$$n_1 + n_2 + n_3 + \cdots + n_{12} = \sum_{t=1}^{12} n_t$$

Summation notation: Examples

- Total number of Covid cases last year:

$$\sum_{t=1}^{12} \mathbf{n}_t$$

where n_t = number of Covid cases in month t

- Total number of Covid cases from July-Dec:

$$\sum_{t=7}^{12} \mathbf{n}_t$$

$$n_7 + n_8 + \vdots + n_{12} = \sum_{t=7}^{12} n_t$$

Clicker question 1

- Which of the following expressions denotes the total number of Covid cases during the summer months of June-August?

(a) $\sum_{t=1}^{8} \mathbf{n}_t$

(b) $\sum_{t=6}^{12} \mathbf{n}_t$

(c) $\sum_{t=6}^{8} \mathbf{n}_t$

(d) $\sum_{t=1}^{6} \mathbf{n}_t$

Solution to Clicker question 1

- Which of the following expressions denotes the total number of Covid cases during the summer months of June-August?

(a) $\sum_{t=1}^{8} \mathbf{n}_t$ **(b)** $\sum_{t=6}^{12} \mathbf{n}_t$

(c) $\sum_{t=6}^{8} \mathbf{n}_t$ **(d)** $\sum_{t=1}^{6} \mathbf{n}_t$

$$n_6 + n_7 + n_8 = \sum_{t=6}^{8} n_t$$

Summation notation

- $\sum_{i=1}^{n} \mathbf{Z}_i =$ "sum of Z_i from $i = 1$ to $i = n$"

Summation notation

- $\sum_{i=1}^{n} \mathbf{Z}_i =$ "sum of Z_i from $i = 1$ to $i = n$"
- $\sum_{i=5}^{n} \mathbf{Z}_i =$ $Z_5 + Z_6 + \cdots\cdots + Z_n$

Summation notation

- $\sum_{i=1}^{n} \mathbf{Z}_i =$ "sum of Z_i from $i = 1$ to $i = n$"
- $\sum_{i=5}^{n} \mathbf{Z}_i =$
- *Example*: In a village, there are 40 families, ranked in terms of their wealth: $W_1 < W_2 < W_3 < ...... < W_{40}$

  What is the total wealth held by the top 10 families? $\sum_{i=31}^{40} W_i$

Clicker question 2

- There are 24 hours in a day. C_i denotes the number of cars that pass through an intersection in hour i.

  Which of the following expressions denotes the total number of cars that pass through the intersection during a day?

(a) $\sum_{i=1}^{n} \mathbf{X}_i$

(b) $\sum_{i=1}^{12} \mathbf{C}_i$

(c) $\sum_{i=1}^{24} \mathbf{C}_i$

(d) $\mathbf{C}_{24}$

Solution to Clicker question 2

- There are 24 hours in a day. C_i denotes the number of cars that pass through an intersection in hour i.

  Which of the following expressions denotes the total number of cars that pass through the intersection during a day?

  (a) $\sum_{i=1}^{n} \mathbf{X}_i$ **(b)** $\sum_{i=1}^{12} \mathbf{C}_i$

  (c) $\sum_{i=1}^{24} \mathbf{C}_i$ **(d)** $\mathbf{C}_{24}$

$C_1 + C_2 + \cdots + C_{24}$

$= \sum_{i=1}^{24} C_i$

Summation notation: Examples

- A forest has 200 trees. What is their average height?

  Average height $=$ (*Sum* of every tree's *height*)/200

$$\text{Avg. ht.} = \frac{\sum_{i=1}^{200} h_i}{200}$$

Summation notation: Examples

- A forest has 200 trees. What is their average height?

  Average height $=$ (*Sum* of every tree's *height*)$/200$

- Total height $=$ Sum of every tree's height $= \sum_{i=1}^{200} \mathbf{h}_i$

Summation notation: Examples

- A forest has 200 trees. What is their average height?

  Average height $= (\textit{Sum}$ of every tree's $\textit{height})/200$

- Total height = Sum of every tree's height $= \sum_{i=1}^{200} \mathbf{h}_i$

- Average height:

$$\left(\sum_{j=1}^{100} \mathbf{h}_i\right) / 200 = \frac{1}{200} \sum_{i=1}^{200} \mathbf{h}_i$$

Summation notation: Examples from Statistics

- Average height in a forest with n trees:

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{h}_i$$

Summation notation: Examples from Statistics

- Average height in a forest with n trees:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{h}_i$$

- *General*: **Mean** of n data points $X_1, X_2, \ldots\ldots, X_n$:.

$$\frac{x_1 + \cdots\cdots + x_n}{n}$$

$$\bar{X} = \sum_{i=1}^{n} x_i \Big/ n$$

Summation notation: Examples from Statistics

- Average height in a forest with n trees:

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{h}_i$$

- *General*: **Mean** of n data points $X_1, X_2, \ldots.., X_n$:.

-

$$\rightarrow \quad \overline{X} = \frac{\sum_{i=1}^{n}\mathbf{X}_i}{n} \equiv \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i = \frac{X_1 + \cdots + X_n}{n}$$

Summation notation: Examples from Statistics

- **Mean** of n data points $X_1, X_2, ..., X_n$:.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$$

Summation notation: Examples from Statistics

$\frac{1}{5}\left[(10-\bar{x})^2 + (15-\bar{x})^2 + \cdots\right]$

- **Mean** of n data points $X_1, X_2, \ldots.., X_n$:.

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$$

- **Variance**:

$$\frac{1}{n}\sum_{i=1}^{n}(\mathbf{X}_i - \overline{X})^2$$

$\frac{1}{n}\left[(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2\right]$

data pts: 10, 15, 8, 4, 40

$\bar{x} = \frac{10+15+8+4+40}{5}$

Summation notation: More Applications

- *Economics:* Jim's portfolio consists of

  10 Air Canada stocks, valued at \$30 each,

  50 Ford stocks, valued at \$12 each,

  20 TD stocks, valued at \$80 each,

Summation notation: More Applications

- *Economics:* Jim's portfolio consists of

  10 Air Canada stocks, valued at \$30 each,

  50 Ford stocks, valued at \$12 each,

  20 TD stocks, valued at \$80 each,

- What is the value of his portfolio?

$$\begin{array}{r} 30 \times 10 \\ + \ 12 \times 50 \\ + \ 80 \times 20 \\ \hline \end{array}$$

Summation notation: More Applications

- *More generally:* Jim's portfolio consists of

  $\mathbf{x}_1$ stocks of Company 1, valued at p_1 each,

  $\mathbf{x}_2$ stocks of Company 2, valued at p_2 each,

  ....$\mathbf{x}_n$ stocks of Company n, valued at p_n each,

Q: What is the value of his portfolio?

$$p_1 x_1 + p_2 x_2 + p_3 x_3 + \cdots + p_n x_n = \sum_{i=1}^{n} p_i x_i$$

Summation notation: More Applications

- *More generally:* Jim's portfolio consists of

  $\mathbf{x}_1$ stocks of Company 1, valued at p_1 each,

  $\mathbf{x}_2$ stocks of Company 2, valued at p_2 each,

  ....$\mathbf{x}_n$ stocks of Company n, valued at p_n each,

  What is the value of his portfolio?

- $p_1 x_1 + p_2 x_2 + ... + p_n x_n = \sum_{i=1}^{n} p_i x_i$

Summation notation: More Applications

- There are 5 assignments in a course

  $\mathbf{w}_1$: weightage on Assignment 1, score: s_1

  $\mathbf{w}_2$: weightage on Assignment 2, score: s_2......

  What is your overall score for the course?

$$w_1 s_1 + w_2 s_2 + \cdots + w_5 s_5 = \sum_{i=1}^{5} w_i s_i$$

Summation notation: More Applications

- There are 5 assignments in a course

  $\mathbf{w}_1$: weightage on Assignment 1, score: s_1

  $\mathbf{w}_2$: weightage on Assignment 2, score: s_2......

  What is your overall score for the course?

- $w_1 s_1 + w_2 s_2 + \ldots\ldots + w_5 s_5 = \sum_{i=1}^{5} w_i s_i$ **[weighted average]**

Summation: Properties

- $10 + 20 + \ldots\ldots + 100 =$ $10\times 1 + 10\times 2 + \cdots\cdots + 10\times 10$

$$= 10(1 + 2 + \cdots\cdots + 10)$$

Summation: Properties

- $10 + 20 + ... + 100 =$

- $10x_1 + 10x_2 + ... + 10x_n = 10(x_1 + x_2 + ... + x_n) =$

$$\sum_{i=1}^{n} (10x_i) = 10 \sum_{i=1}^{n} x_i$$

Summation: Properties

- $10 + 20 + ... + 100 =$

- $10x_1 + 10x_2 + ... + 10x_n = 10(x_1 + x_2 + ... + x_n) =$

- *More generally:*

$$cx_1 + cx_2 + ... + cx_n = c(x_1 + x_2 + ... + x_n) = c\sum_{i=1}^{n} x_i$$

1

$$\sum_{i=1}^{n}(cx_i) = c\sum_{i=1}^{n} x_i$$

Summation: Properties

- $10 + 20 + ... + 100 =$
- $10x_1 + 10x_2 + ... + 10x_n = 10(x_1 + x_2 + ... + x_n) =$
- *More generally:*
  $cx_1 + cx_2 + ... + cx_n = c(x_1 + x_2 + ... + x_n) = c\sum_{i=1}^{n} x_i$
- $$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

Clicker question 3

- Aya measured the heights of trees in a forest in metres and found that the total height of all the trees was 9000 m. Maya measured the same, but in feet.

  Which of the following expressions denotes the total height that Maya found? Note that 1 $m = 3.3$ ft.

  (a) 9000 ft **(b)** 3.3 X 9000 ft

  (c) $9000/3.3$ ft **(d)** 33000 ft

Solution to Clicker question 3

Aya $\sum_{i=1}^{n}(h_i)$ Maya: $\sum_{i=1}^{n}(3.3\,h_i) = (3.3)\sum_{i=1}^{n} h_i$

- Aya measured the heights of trees in a forest in metres and found that the total height of all the trees was 9000 m. Maya measured the same, but in feet.

  Which of the following expressions denotes the total height that Maya found? Note that $1\ m = 3.3\ ft$.

  (a) 9000 ft **(b)** 3.3 X 9000 ft

  (c) 9000/3.3 ft **(d)** 33000 ft

Summation: Properties

- *Adding up two sets of numbers*:

  $v_i =$ number of violent crimes in province i

  $z_i =$ number of non-violent crimes in province i

Total no. of crimes in prov. i $= v_i + z_i$

Summation: Properties

- *Adding up two sets of numbers*:

  $v_i =$ number of violent crimes in province i

  $z_i =$ number of non-violent crimes in province i

- Total number of crimes in province i: $v_i + z_i$.

Across all provinces : $\sum_{i=1}^{n} (v_i + z_i)$

Summation: Properties

- *Adding up two sets of numbers*:

  v_i = number of violent crimes in province i

  z_i = number of non-violent crimes in province i

- Total number of crimes in province i:
- Total number of crimes across all provinces:

2

$$\sum_{i=1}^{n}(v_i + z_i) = \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} z_i$$

	V	Z	
Prov. 1	10	5	15
2	100	500	600
3	40	80	120

150 + 585

15 + 600 + 120

Summation: Properties

- *General property*:

$$\sum_{i=1}^{n}(a_i + b_i + c_i + d_i \cancel{+}) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} d_i$$

Summation in Excel/Google Sheets

How to do summation in **Excel/Google Sheets**?

- Use SUM[.....*Range*....] for Summation
- Use AVERAGE[.....*Range*....] for Average
- Use VAR[.....*Range*....] for Variance

Common sums: Arithmetic Series

- $1 + 2 + 3 + 4 + 5 + ... + 100?$

Common sums: Arithmetic Series

- $1 + 2 + 3 + 4 + 5 + \ldots\ldots + 100?$ $= \sum_{i=1}^{100} i$
- $1 + 2 + 3 + \ldots\ldots + 100 = \sum_{i=1}^{100} i$

Common sums: Arithmetic Series

- $1 + 2 + 3 + 4 + 5 + ... + 100?$
- $1 + 2 + 3 + ... + 100 = \sum_{i=1}^{100} i$
- $1 + 2 + 3 + 4 + ... + 100 = 50 \text{ X } 101 = 5050$

$1 + 2 + 3 + \cdots + 99 + 100$

101

101

101

$= 50 \times 101$

Common sums: Gauss' formula

- $1 + 2 + 3 + 4 + \ldots\ldots + n = \sum_{i=1}^{n} i$?

$$= \frac{n}{2}(n+1)$$

$1 + 2 + \cdots\cdots + n-1 + n$

$n+1$

$n+1$

$n+1$

Common sums: Gauss' formula

- $1+2+3+4+\dots\dots+n=\sum_{i=1}^{n} i$?
- $$1+2+3+4+\dots\dots+n=\frac{n(n+1)}{2}$$

Common sums: Gauss' formula

- $1 + 2 + 3 + 4 + ... + n = \sum_{i=1}^{n} i$?
-
$$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$$
- $1 + 2 + 3 + 4 + ... + 1000 =$ $\frac{\cancel{1000}^{500} \times 1001}{\cancel{2}} = 500500$

Common sums: Gauss' formula

- $$1 + 2 + 3 + 4 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

Quick proof:

$$\begin{cases} S = 1 + 2 + 3 + \cdots\cdots + n \\ S = n + n-1 + n-2 + \cdots\cdots + 1 \end{cases}$$

$$2S \quad (n+1) + \underbrace{(n+1) + (n+1) + \cdots\cdots + (n+1)}$$

$$2S = n(n+1) \Rightarrow S = \frac{n(n+1)}{2}$$

Common sums: Gauss' formula

- $$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$$

  Quick proof:

- What if $51 + 52 + 53 + ... + 78 + 79 + 80$?

Common sums: Gauss' formula

$$51 + 52 + \cdots\cdots + 79 + 80$$

$$131 \qquad 131 \qquad 131$$

- $51 + 52 + ... + 79 + 80 = \sum_{i=51}^{80} i$?

$$= \frac{30}{2} \times 131 = 15 \times 131 = \ldots$$

Common sums: Gauss' formula

- $51 + 52 + ... + 79 + 80 = \sum_{i=51}^{80} i$?

$$\frac{30}{2} \times (51 + 80)$$

- *General formula for arithmetic series:*

$$a_1 + a_2 + a_3 + ... + a_n = \frac{n(a_1 + a_n)}{2}$$

where $a_{i+1} = a_i + 1$ c

$$405 = \frac{3\cancel{6}}{\cancel{2}}(55 + 80) =$$

$$\underbrace{55 + \underbrace{60 + 65 + \cdots + 75}_{135} + 80}_{135}$$

Gauss' formula: Applications

- Suppose Kim joins a company at the starting salary of \$501.
  Every year his salary goes up by \$1.
  If he works for 40 years, what is lifetime earnings?

Q: $501 + 502 + 503 + \cdots + 540$

$$= \frac{40}{2}(501 + 540)$$

$$= 20 \times 1041 = \$20820$$

Gauss' formula: Applications

- Suppose Kim joins a company at the starting salary of \$501.
  Every year his salary goes up by \$1.
  If he works for 40 years, what is lifetime earnings?
- $501 + 502 + 503 + \ldots\ldots + 540 =$

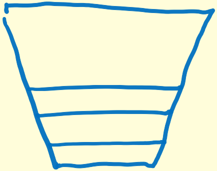
Common sums: Gauss' formula

- *Tiered stadium with* 10 *tiers* :

Tier 1 = 6 seats

Each tier + 4

Q: Total seats ?



6

Q: $6 + 10 + 14 + \cdots\cdots + 42 = \frac{10}{2}(6 + 42)$

$= \frac{10}{\cancel{2}} \times \cancel{48}^{\,24} = 240$

Common sums: Gauss' formula

- *Tiered stadium with* 10 *tiers* :
- *Arithmetic series*: **add a constant** to get the next term

Common sums: Gauss' formula

- *Tiered stadium with* 10 *tiers* :

- *Arithmetic series*: **add a constant** to get the next term
- *General formula*:

$$\frac{\text{(no. of terms)}}{2}(1^{st}\text{ term } + \text{ last term})$$

Common sums: Gauss' formula

- *Tiered stadium with* 10 *tiers* :

- *Arithmetic series*: **add a constant** to get the next term
- *General formula*:

$$\frac{\text{(no. of terms)}}{2}(1^{st} \text{ term } + \text{ last term})$$

- $33 + 36 + \ldots\ldots + 60?$ $= 3(11 + 12 + \cdots + 20)$

$= \frac{10}{2} \times (33 + 60) = 5 \times 93$

$= 465$

Clicker question 4

- What is $4 + 8 + 12 + ... + 100$?.

  (a) 325

  (b) 400

  (c) 800

  (d) 1300

Solution to Clicker question 4

$$4(1 + 2 + 3 + \dots + 25)$$

- What is $4 + 8 + 12 + \dots + 100$?.

$$= \frac{25}{2} \times (4 + 100)$$

(a) 325

(b) 400

(c) 800

(d) 1300

$$= \frac{25}{\cancel{2}} \times \cancel{104}^{52} = 1300$$

Common sums: Geometric Series

- [Model of infections]
  $T{=}1$: 1 $T{=}2$: R_0 $T{=}3$: R_0^2....... $T{=}10$: R_0^9

  $R_0 = (np)$ $R_0^2 = (np)^2$

Common sums: Geometric Series

- [Model of infections]
  $T{=}1$: 1 $\quad T{=}2$: $R \quad T{=}3$: R^2....... $T{=}10$: R^9
- Total number of infections till day 10 :

$$\underset{R^0}{\underset{\|}{1}} + \underset{R^1}{\underset{\|}{R}} + \underset{R^2}{\underset{\|}{R^2}} + \ldots\ldots + \underset{R^9}{\underset{\|}{R^9}} = ? \qquad \sum_{i=0}^{9} R^i$$

Common sums: Geometric Series

- [Model of infections]
  T=*1*: 1 $\quad$ T=*2*: R $\quad$ T=*3*: R^2....... $\quad$ T=*10*: R^9
- Total number of infections till day 10 :

$$1 + R + R^2 + ... + R^9 = ?$$

- $\sum_{i=1}^{9} R^i = ?$

Common sums: Geometric Series

- *Geometric series*: **multiply by a constant** to get next term

$$1 + R + R^2 + ... + R^{n-1} = \frac{1 - R^n}{1 - R}$$

Common sums: Geometric Series

- *Geometric series*: **multiply by a constant** to get next term

$$1 + R + R^2 + R^3 + \cdots + R^{n-1} = \frac{1 - R^n}{1 - R}$$

- If $R = 2$, $\displaystyle\sum_{i=\cancel{1}\,0}^{9} R^i = \frac{1-2^{10}}{1-2} = \frac{1-1024}{1-2} = \frac{-1023}{-1} = 1023$

$1 + R + \cdots + R^9$

Common sums: Geometric Series

- *Geometric series*: **multiply by a constant** to get next term

$$1 + R + R^2 + \ldots\ldots + R^{n-1} = \frac{1-R^n}{1-R}$$

- If $R = 2$, $\quad \sum_{i=1}^{9} R^i =$

- General formula:

$$a\left(\underbrace{1+R+R^2+\cdots+R^{n-1}}\right) = \quad a + \underbrace{aR + aR^2 + aR^3}+\ldots + aR^{n-1} = a\frac{1-R^n}{1-R}$$

Geometric Series: Applications

- *Chess board story*: Put 1 grain of rice on square 1

  Put 2 grains on square 2

  Put 4 grains on square 3

  .....Put double the amount on the next square

$$1 + 2 + 2\times 2 + 2\times(2\times 2) + \cdots\cdots$$

$$= \underset{\underset{2^0}{\shortparallel}}{1} + \underset{\underset{2^1}{\shortparallel}}{2} + 2^2 + 2^3 + \cdots\cdots + 2^{63}$$

Geometric Series: Applications

- *Chess board story*: Put 1 grain of rice on square 1

  Put 2 grains on square 2

  Put 4 grains on square 3

  .....Put double the amount on the next square

- How many grains in total?

$$1 + 2 + 4 + 8 + 16 + \dots\dots ?$$

Geometric Series: Applications

- *Chess board story*: Put 1 grain of rice on square 1

  Put 2 grains on square 2

  Put 4 grains on square 3

  .....Put double the amount on the next square

- How many grains in total?

18, 446, 744, 073, 709,
551, 615

$$1 + 2 + 4 + 8 + 16 + ...?$$

- $1 + 2 + 2^2 + 2^3 + 2^4 .... + 2^{63} = \sum_{i=0}^{63} 2^i = \frac{1-2^{64}}{1-2}$

2^0

Geometric Series: Other Applications

- *Crime rate grows by 5% every year*:

  year 1 : 100 crimes

$Y2: \underline{100} + \frac{5}{100} \times 100 = 0.0\underline{5} \times 100 = 100(1+0.05)$

Geometric Series: Other Applications

- *Crime rate grows by 5% every year*:

  year 1 : 100 crimes

- year 2 : $100(1 + 0.05)$

  year 3 : $100(1 + 0.05)(1 + 0.05)$........

  $= 100(1+0.05)^2$

Y3 : $100(1+0.05)$

$+ \; 0.05 \times 100(1+0.05)$

$= 100(1+0.05)(1+0.05)$

Y4 : $100(1+0.05)^3$

Y5 : $100(1+0.05)^4$ . . . . . .

Geometric Series: Other Applications

$$a + aR + aR^2 + \cdots + aR^{n-1} = a \cdot \frac{1-R^n}{1-R}$$

- *Crime rate grows by 5% every year*:

  year 1 : 100 crimes

- year 2 : $100(1 + 0.05)$

  year 3 : $100(1 + 0.05)(1 + 0.05)$........

$$= 100 \, \frac{1-(1.05)^{10}}{1-1.05}$$

$$= 100 \times 12.578$$

$$= 1257.8$$

- How many crimes over 10 years?

$$\rightarrow \underbrace{100}_{y1} + \underbrace{100(1.05)}_{y2} + \underbrace{100(1.05)^2}_{y3} + \ldots\ldots \underbrace{100(1.05)^9}_{y10} =$$

Geometric Series Application: Annuities

- *Annuity:* a stream of N equal cash flows paid at regular intervals (years, months, quarters)

  Examples: lottery payouts, retirement benefits, car loan payments,.......

Geometric Series Application: Annuities

- *Annuity:* a stream of N equal cash flows paid at regular intervals (years, months, quarters)

  Examples: lottery payouts, retirement benefits, car loan payments,.......

- Say, you receive C every year for years $1, 2, \ldots.., N$

  Q: What is the *present value* of this annuity?

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- What is the present value of receiving C a year from now?

Annuities: Present value calculation

- **Q**: What is the present value of receiving C a year from now?

Today — Year from now

$$\$1 \xrightarrow{\text{bank}} \$(1+r)$$

$$\$\frac{1}{(1+r)} \longrightarrow \$1$$

$$\frac{1}{\cancel{1+r}} \cdot \cancel{(1+r)} = \$1$$

$$PV = \frac{C}{1+r} \longrightarrow \$C$$

Annuities: Present value calculation

- **Q**: What is the present value of receiving C a year from now?
- **A**: $\frac{C}{1+r}$

Annuities: Present value calculation

- **Q**: What is the present value of receiving C a year from now?
- **A**: $\frac{C}{1+r}$
- **Q**: What is the present value of receiving C two years from now?

$T=1$ $\qquad$ $T=2$

$\$1 \longrightarrow \$(1+r) \longrightarrow \$(1+r)(1+r) = (1+r)^2$

$PV = \frac{C}{(1+r)^2} \longleftarrow \$C$

Annuities: Present value calculation

- **Q**: What is the present value of receiving C a year from now?
- **A**: $\frac{C}{1+r}$
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Annuities: Present value calculation

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- **Q**: What is the present value of receiving C two years from now?
- **A**: $\frac{C}{(1+r)^2}$

$\vdots$

- The PV of receiving C N years from now: $\frac{C}{(1+r)^N}$

Annuities: Present value calculation

- *Annuity*: receive C every year for years $1, 2, \ldots.., N$

Annuities: Present value calculation

- *Annuity*: receive C every year for years $1, 2, ..., N$
- **Present value (PV):**

$$\underbrace{\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + ... + \frac{C}{(1+r)^N}}_{} = ?$$

$T=1$ $T=2$ $\cdots\cdots$ $T=N$

$$\frac{C}{1+r} \cdot \boxed{\frac{1}{(1+r)}} \cdot \boxed{\frac{1}{(1+r)}} \cdots\cdots$$ Geometric series

Annuities: Present value calculation

- *Annuity*: receive C every year for years $1, 2, \ldots\ldots, N$
- **Present value (PV):**

$$\frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots\ldots + \frac{C}{(1+r)^N} = ?$$

$$= \frac{C}{(1+r)}\left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{N-1}}\right]$$

- Use Geometric series formula:

$$\Rightarrow \quad 1 + R + R^2 + \ldots\ldots + R^{n-1} = \frac{1-R^n}{1-R}$$

$$= \frac{C}{(1+r)} \frac{1-\left(\frac{1}{1+r}\right)^N}{1-\frac{1}{1+r}}$$

$$= \frac{C}{\cancel{(1+r)}} \cdot \frac{1-\frac{1}{(1+r)^N}}{\frac{1+r-1}{\cancel{1+r}}}$$

$$= \frac{C}{r}\left[1 - \frac{1}{(1+r)^N}\right]$$

PV of an Annuity

- **PV** of an annuity:

$$\boxed{PV = \frac{C}{r}[1 - \frac{1}{(1+r)^N}]}$$

PV of an Annuity

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- Say $C = 1000$, $N = 10$, $r = 3\%$; $PV = \frac{1000}{0.03}\left[1 - \frac{1}{(1+0.03)^{10}}\right]$

$= \$8530.20$

PV of an Annuity

- **PV** of an annuity:

$$\rightarrow PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^N}\right]$$

- Say $C = 1000$, $N = 10$, $r = 3\%$; $PV =$
- Suppose $r = 5\%$; $PV =$ $\frac{1000}{0.05}\left[1 - \frac{1}{(1+0.05)^{10}}\right]$

$= \$7721.7$

PV of an Annuity

- **PV** of an annuity:

$$PV = \frac{C}{r}[1 - \frac{1}{(1+r)^N}]$$

- Say $C = 1000$, $N = 10$, $r = 3\%$; $PV =$
- Suppose $r = 5\%$; $PV =$
- Upfront payment or annuity?

PV of a Perpetuity

- PV of an annuity: $\frac{C}{r}\left[1 - \frac{1}{(1+r)^N}\right]$

PV of a Perpetuity

- PV of an annuity: $\frac{C}{r}\left[1 - \frac{1}{(1+r)^N}\right] \rightarrow N = \infty$
- **Perpetuity**: Receive C forever i.e. $N = \infty$

$$PV = \frac{C}{r}$$

$$\left(\frac{1}{1+r}\right)^N$$

$$r = 1 \quad \left(\frac{1}{2}\right)^N \longrightarrow 0$$

PV of a Perpetuity

- PV of an annuity: $\frac{C}{r}[1 - \frac{1}{(1+r)^N}]$
- **Perpetuity**: Receive C forever i.e. $N = \infty$
- PV of a perpetuity:

$$\boxed{PV = \frac{C}{r}}$$

$C = \$1000$

$r = 0.03$

$PV = \frac{1000}{0.03} = \$33,333.33$

PV of a Perpetuity

- PV of an annuity: $\frac{C}{r}[1 - \frac{1}{(1+r)^N}]$
- **Perpetuity**: Receive C forever i.e. $N = \infty$
- PV of a perpetuity:
$$PV = \frac{C}{r}$$
- Say $C = 1000$, $r = 3\%$; PV of perpetuity =

Sequences

- *Sequence*: Numbers in a particular pattern

Sequences

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- (Previous example) *Kim's salary*: 501, 502, 503, 504........

  Infections: $1, R_0, R_0^2, R_0^3, \ldots\ldots$

Sequences

- *Sequence*: Numbers in a particular pattern
- (Previous example) *Kim's salary*: $501, 502, 503, 504........$

  Infections: $1, R_0, R_0^2, R_0^3, ...$
- More general: $a_1, a_2, a_3, ..., a_n, ...$

Some common sequences

- *Arithmetic sequence* pattern:

  next number is found by adding a constant: $a_{n+1} = a_n + c$

4, 9, 14, 19, 24, ...

$a_{n+1} = a_n + 5$

Some common sequences

- *Arithmetic sequence* pattern:

  next number is found by adding a constant: $a_{n+1} = a_n + c$

- *Geometric sequence* pattern:

  next number is found by multiplying by a constant:
  $a_{n+1} = a_n r$

$4, \ 4R, \ 4R^2, \ 4R^3, \ \ldots\ldots$

$a_{n+1} = a_n R$

Some common sequences

- *Arithmetic sequence* pattern:

  next number is found by adding a constant: $a_{n+1} = a_n + c$

- *Geometric sequence* pattern:

  next number is found by multiplying by a constant: $a_{n+1} = a_n r$

- *Fibonacci sequence* pattern:

  next number is found by adding previous two: $a_{n+1} = a_n + a_{n-1}$

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

[The Fibonacci Sequence: Nature's Code – YouTube](https://www.youtube.com/watch?v=wTlw7fNcO-0) :
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Limits of sequences

- *Limit of a sequence*: $\lim_{n\to\infty} a_n$
  where does the number go toward when n becomes very large?

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Limits of sequences

- *Limit of a sequence*: $\lim_{n \to \infty} a_n$

  where does the number go toward when n becomes very large?

- *Some common limits*:

- ~~K~~im: $501, 502, 503, ...500 + n, ... \to \infty$

Yoda

$$\lim_{n \to \infty} \text{salary}_n = \infty$$

Some common limits of sequences

- $1, 2, 3, \ldots\ldots., n, \ldots. \to \infty$

$1, 2^2, 3^2, \ldots\ldots, n^2, \ldots\ldots \longrightarrow \infty$

Some common limits of sequences

- $1, 2, 3, ..., n, ... \rightarrow$ ∞
- $1, 0, -1, -2, -3, ..., -n, ... \rightarrow$ $-\infty$

Some common limits of sequences

- $1, 2, 3, \ldots\ldots, n, \ldots. \longrightarrow$
- $1, 0, -1, -2, -3, \ldots\ldots, -n, \ldots. \longrightarrow$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots. \longrightarrow$ 0 $\quad \frac{1}{n}$

$$\lim_{n \to \infty} a_n = 0$$

Some common limits of sequences

- $1, 2, 3, \ldots\ldots., n, \ldots. \to$
- $1, 0, -1, -2, -3, \ldots\ldots., -n, \ldots. \to$
- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots. \to$
- Implications for other sequences: $\lim_{n\to\infty} \frac{10+n}{n}$?

$\frac{\infty}{\infty} = 1$

$\frac{11}{1}, \frac{12}{2}, \frac{13}{3}, \frac{14}{4}, \ldots, \frac{10+n}{n} \;\ldots\ldots$

$$\lim_{n\to\infty} \frac{10+n}{n} = \frac{10}{n} + \frac{n}{n}$$

$$= \lim_{n\to\infty} \frac{10}{n} + 1 = 1$$

$\frac{10}{n} \to 0$

Some common limits of sequences

- $1, 2, 3, \ldots\ldots., n, \ldots. \longrightarrow$
- $1, 0, -1, -2, -3, \ldots\ldots., -n, \ldots. \longrightarrow$
- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots. \longrightarrow$
- Implications for other sequences: $\lim_{n\to\infty} \frac{10+n}{n}$?
- $\lim_{n\to\infty} \frac{n}{n+1}$? : $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.. \longrightarrow$ $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n/n}{n/n + 1/n}$

$$= \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1$$

$\frac{1}{n} \to 0$

$\equiv$ 1

Some common limits of sequences

- *Interesting sequence*: 1, R, R^2, R^3, ..., R^n, ... $\to$

Geometric sequence

$\lim_{n\to\infty} R^n$?

$R=1$: 1, 1, 1, 1, ... $\to 1$

$R=\frac{1}{2}$: 1, $\frac{1}{2}$, $\frac{1}{2^2}=\frac{1}{4}$, $\frac{1}{2^3}=\frac{1}{8}$, $\frac{1}{2^4}=\frac{1}{16}$, ..., $\frac{1}{2^{n-1}}$ $\to 0$

$R<1$: $\lim = 0$

$R>1$:

e.g. $R=5$: 1, 5, 25, 125, 625, ..., 5^{n-1}, ... $\to \infty$

Some common limits of sequences

- <u>*Interesting sequence*</u>: $1, R, R^2, R^3, \ldots\ldots, R^n, \ldots. \rightarrow$
- $\lim_{n\to\infty} R^n = 0$ if $R < 1$, $\lim_{n\to\infty} R^n = \infty$ if $R > 1$, $\lim_{n\to\infty} R^n = 1$ if $R = 1$

Some common limits of sequences

- *Interesting sequence*: $1, R, R^2, R^3, \ldots\ldots, R^n, \ldots\ldots \rightarrow$
- $\lim_{n\to\infty} R^n = 0$ if $R < 1$, $\qquad \lim_{n\to\infty} R^n = \infty$ if $R > 1$,
  $\lim_{n\to\infty} R^n = 1$ if $R = 1$
- Not all sequences have limits: $2, -2, 3, -3, 4, -4, \ldots\ldots\ldots$

Clicker question 5

- What is the limit of the following sequence:
  $7, \frac{9}{2}, \frac{11}{3}, \frac{13}{4}, \ldots\ldots., \frac{2n+5}{n}, \ldots.$

  (a) 0

  (b) 2

  (c) 5

  (d) ∞

Solution to Clicker question 5

- What is the limit of the following sequence:
  $7, \frac{9}{2}, \frac{11}{3}, \frac{13}{4}, \ldots\ldots., \frac{2n+5}{n}, \ldots.$

  (a) 0

  (b) 2 ✓

  (c) 5

  (d) ∞

$$\lim_{n\to\infty} \frac{\overset{\to\infty}{2n+5}}{\underset{\to\infty}{n}} = \lim_{n\to\infty} \frac{2n}{n} + \frac{5}{n}$$

$$= \lim_{n\to\infty} 2 + \underset{\to 0}{\frac{5}{n}}$$

$$= 2$$