Continuity of the free boundary in a non-degenerate p-obstacle problem type with monotone solution

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Abstract

We prove the continuity of the free boundary for a non-degenerate p-Obstacle problem with monotone solution. The proof uses techniques of comparison and the growth of the solution near free boundary points.

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Introduction

In this paper we are interested with the regularity of the free boundary in the problem

$$(P)\begin{cases} i) \quad u\big(-div((\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u)-f\big)=0 \qquad \text{in } \Omega,\\ ii) \quad 0\leqslant u\leqslant u_a, \qquad u_y\geqslant 0 \qquad \text{in } \Omega,\\ iii) \quad u=u_a \text{ on } \Gamma_1, \quad u=0 \text{ on } \Gamma_0,\end{cases}$$

where $\Omega=(0,1)\times(0,1)$, $\Gamma_0=(0,1)\times\{0\}$, $\Gamma_1=(0,1)\times\{1\}$, p, u_a, λ, Λ are constants such that $p>1$, $u_a>0$ and

$$-\Lambda\leqslant f(x,y)=f(x)\leqslant-\lambda<0 \quad \text{in } \Omega.$$

This problem was studied in [9] p 193 for the linear operator

$$-a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu, \qquad (a_{ij}\xi_i\xi_j > 0, \qquad \forall \xi \neq 0, \quad c \geqslant 0)$$

under the following general setting :

$$\begin{cases} \Omega = \Omega' \times (0,1), \quad \Omega' \subset \mathbb{R}^{n-1} \text{ is an open bounded set with smooth boundary} \\ f \leqslant -\lambda < 0, \quad f_{x_n} \geqslant 0 \quad \text{ and } \quad |f_{x_1} + f_{x_2} + \ldots + f_{x_{n-1}}| \leqslant C \quad \text{ in } \Omega. \end{cases}$$

The regularity of Lipschitz continuity of the free boundary $(\partial[u > 0]) \cap \Omega$ was then established. The proof used strongly the linearity of the operator. Our purpose is to extend the study to a nonlinear case. It is proved, not necessarily for monotone solution, in [8] for the p-Laplace obstacle problem ($p \geqslant 2$) and in [3] for the A-Laplace obstacle problem, including the case $1 < p < 2$, that the free boundary is porous and therefore of Lebesgue's measure zero. Here, we will prove a continuity result in the case of a non degenerate case, i.e $\kappa > 0$. The main idea to do it is to construct a suitable comparison function w and use the $C^{1,\alpha}_{loc}$ regularity of the solution. When $\kappa = 0$ and $p > 2$, the barrier function w remains good, but the non-oscillation lemma 1.1 is not entirely true. Unfortunately, the function w is not suitable for the degenerate case $\kappa = 0$ and $1 < p < 2$.

The paper is organized as follows : in Section 1, we give some outlines for the existence of a solution of (P) in a particular case, recall some regularity results, define the free boundary and establish a non oscillation lemma. In Section 2, we built the barrier function w. Finally, we establish the continuity of the free boundary in Section 3.

1 Preliminary results

In order that (P) covers different situations and because the regularity of the free boundary is a local property, we didn't specify the whole boundary conditions. We focus on the main properties needed to be satisfied by our solution, mainly, the lower and upper bound and the monotonicity in the y direction. To have an idea how to prove the existence of such solutions, let us consider the case where monotone Dirichlet conditions are satisfied on the lateral boundaries of Ω, ie : $u = g$ on $\partial\Omega$ with

$$u(0, y) = g_0(y), \qquad u(1, y) = g_1(y) \qquad y \in [0, 1]$$

with $g_0(0) = g_1(0) = 0$, $g_0(1) = g_1(1) = u_a$, g_0, g_1 are regular and non-decreasing.
We consider a family of regularized problems

$$(P_\epsilon) \begin{cases} -div((\kappa + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} \nabla u_\epsilon) = -f(x)H_\epsilon(u_\epsilon) \qquad \text{in } \Omega, \\ u_\epsilon = g \qquad \text{on } \partial\Omega, \end{cases}$$

where $H_\epsilon(t) = \min\left(1, \dfrac{t^+}{\epsilon}\right)$.

The proof of the existence of u_ϵ can be adapted from the work in [2]. It uses the Schauder fixed theorem and we only have to overcome small technical difficulties related to our operator. Then taking u_ϵ^- (resp. $(u_\epsilon - u_a)^+$) as test functions in the weak formulation associated to (P_ϵ), we get $u_\epsilon \geqslant 0$ (resp. $u_\epsilon \leqslant u_a$) in Ω. We, also, establish the uniqueness and the monotonicity of u_ϵ by comparison. For the last property, we compare for $\eta \in (0, \epsilon)$, the function $u_\epsilon^\eta(x, y) = u_\epsilon(x, y + \eta)$ with u_ϵ on the set $\Omega_\eta = (0, 1) \times (0, 1 - \eta)$. First, by the assumptions considered for the boundary conditions and by the upper and lower bounds of u_ϵ, we have $u_\epsilon^\eta \geqslant u_\epsilon$ on $\partial\Omega_\eta$. Next,we obtain, by comparison,

$$\int_{\Omega_\eta} \Big[(\kappa + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} \nabla u_\epsilon - (\kappa + |\nabla u_\epsilon^\eta|^2)^{\frac{p-2}{2}} \nabla u_\epsilon^\eta\Big] \nabla(u_\epsilon - u_\epsilon^\eta)^+ dxdy$$
$$= \int_{\Omega_\eta} f(x)(H_\epsilon(u_\epsilon) - H(u_\epsilon^\eta))(u_\epsilon - u_\epsilon^\eta)^+ dxdy \ \leqslant 0$$

since $f(x) < 0$ and H_ϵ is a nondecreasing function. We then deduce that $u_\epsilon \leqslant u_\epsilon^\eta$ in Ω_η. The last step is to show the convergence of u_ϵ as ϵ goes to zero. Arguing as in [2], again, we can prove that, up to a subsequence, we have

$$u_\epsilon \longrightarrow u \quad \text{in } W^{1,p}(\Omega), \qquad H_\epsilon(u_\epsilon) \rightharpoonup \gamma \quad \text{in } L^p(\Omega) \quad \text{with} \quad 0 \leqslant \gamma \leqslant 1$$

from which we deduce

$$\begin{cases} i) \quad div((\kappa + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u) = -f(x)\gamma \quad \text{in } \Omega, \qquad \gamma = 1 \quad \text{in } [u > 0], \\ ii) \quad 0 \leqslant u \leqslant u_a, \qquad u_y \geqslant 0 \qquad \text{in } \Omega, \\ iii) \quad u = g \quad \text{on } \partial\Omega. \end{cases}$$

To identify γ, we use the $W^{2,2}_{loc}$ regularity of u (see [1]). We have $div((\kappa + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u) = -f(x)\gamma = 0$ in $[u = 0]$, then $\gamma = 0$ in $[u = 0]$. Hence $\gamma = \chi([u > 0])$.
Finally, we link the solution to the Obstacle problem by minimizing the functionals

$$J_\epsilon(v) = \int_\Omega \Big(\frac{1}{p}(\kappa + |\nabla u|^2)^{\frac{p}{2}} - f(x)E_\epsilon(v)\Big) dxdy, \qquad E_\epsilon(s) = \int_0^s H_\epsilon(t)dt$$
$$J(v) = \int_\Omega \Big(\frac{1}{p}(\kappa + |\nabla u|^2)^{\frac{p}{2}} - f(x)u^+\Big) dxdy$$

and show that

$$J(u) \leqslant \liminf_{\epsilon \to 0} J_\epsilon(u_\epsilon) \leqslant \liminf_{\epsilon \to 0} J_\epsilon(v) = J(v) \qquad \forall v \in W^{1,p}(\Omega), \quad v = g \quad \text{on } \partial\Omega. \ \square$$

In all what follows, we consider a solution u of (P) and recall that it satisfies the following regularity result (see [6] Theorem 1.7 and [7] Theorem 1 for the interior and boundary regularity respectively)

$$u \in C^{1,\beta}_{loc}(\Omega \cup \Gamma_0 \cup \Gamma_1) \quad \text{for some } \beta \in (0, 1). \tag{1.1}$$

From the monotony of u in the y-direction, we deduce, for any point $(x_0, y_0) \in \Omega$, that :
- If $u(x_0, y_0) > 0$ then there exists $\epsilon > 0$ such that $u > 0$ in $B_\epsilon(x_0, y_0) \cup (x_0 - \epsilon, x_0 + \epsilon) \times (y_0, 1) \subset \Omega$, where $B_\epsilon(x_0, y_0)$ is the ball centered at (x_0, y_0) and with radius ϵ.
- If $u(x_0, y_0) = 0$ then $u(x_0, y) = 0 \ \forall y \in [0, y_0]$.

From this property and the regularity of u to the boundary Γ_1, the following function is well defined :

$$\Phi(x) = \inf\{y \in (0,1)/ \ u(x,y) > 0\}.$$

Moreover, we have

Proposition 1.1. *Φ is upper semi-continuous (u.s.c) on $(0,1)$ and $[u(x,y) > 0] \cap \Omega = [y > \Phi(x)]$.*

Proof. Let $x_0 \in (0,1)$ and $\epsilon \in (0, (1 - \Phi(x_0))/2)$. There exists $y_1 < \Phi(x_0) + \epsilon$ such that $u(x_0, y_1) > 0$. By continuity of u, there exists $\eta > 0$ such that $u > 0$ in $B_\eta(x_0, y_1)$. This leads to $\Phi(x) < y_1 \ \forall x \in (x_0 - \eta, x_0 + \eta)$ and the u.s.c of Φ holds.
Now let $(x_0, y_0) \in [u > 0] \cap \Omega$. If $y_0 \leqslant \Phi(x_0)$ then $u(x_0, y_0) = 0$ and we get a contradiction. So $y_0 > \Phi(x_0)$. Conversely, if $(x_0, y_0) \in [y > \Phi(x)]$, then by definition of Φ, $u(x_0, y_0) > 0$ and $(x_0, y_0) \in [u > 0] \cap \Omega$. □

The main result of this paper is the following theorem :

Theorem 1.1. *Φ is continuous on $(0,1) \cap [\Phi(x) > 0]$.*

The proof of the theorem is established in section 3. First, we will establish the following non-oscillation lemma.

Lemma 1.1. *Let u be a solution of (P). Let $(x_0, y_0) \in \Omega$ and $r > 0$ such that $B_r = B_r(x_0, y_0) \subset\subset \Omega$. Then we cannot have the following situations :*

$$\begin{array}{lllll} (i) & u = 0 & \text{on } S = \{x_0\} \times [y_0 - r, y_0 + r] & \text{and} & u > 0 \quad \text{in } B_r \setminus S \\ (ii) & u > 0 & \text{in } B_r \cap [x > x_0] = B_r^+ & \text{and} & u = 0 \quad \text{in } B_r \cap [x \leqslant x_0] \\ (iii) & u > 0 & \text{in } B_r \cap [x < x_0] = B_r^- & \text{and} & u = 0 \quad \text{in } B_r \cap [x \geqslant x_0]. \end{array}$$

We have the following preliminary result.

Lemma 1.2. *Assume that one of the situations (i), (ii) or (iii) of Lemma 1.1 holds. Then, we have*

$$\frac{\partial u}{\partial y} = 0 \qquad \text{in } B_{r/4}(x_0, y_0).$$

Proof. We consider situations (i) and (ii) only. Situation (iii) is similar to (ii).

Case (i) : From (P), we have

$$div\big((\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\big) = -f(x)\chi([u>0]) \qquad \text{in } \Omega.$$

Since $u>0$ in $B_r\setminus S$ and $|S|=0$ then $\chi([u>0])=1$ a.e in B_r. Then, we have

$$\int_{B_r}(\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u.\nabla\zeta dxdy = \int_{B_r}-f(x)\zeta dxdy \qquad \forall\zeta\in W_0^{1,p}(B_r).$$

Let $\eta\in(0,r/4)$ and $u_\eta(x,y)=u(x,y-\eta)=u(x',y')$. Then u_η satisfies

$$\begin{aligned}
&\int_{B_{r/4}}(\kappa+|\nabla u_\eta|^2)^{\frac{p-2}{2}}\nabla u_\eta.\nabla\zeta(x,y)dxdy\\
&=\int_{B_{r/4}-\eta e_y}(\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u(x',y').\nabla(\zeta(x',y'+\eta))dx'dy'\\
&=\int_{B_{r/4}-\eta e_y}-f(x')\zeta(x',y'+\eta)dx'dy' = \int_{B_{r/4}}-f(x)\zeta(x,y)dxdy\\
&=\int_{B_{r/4}}(\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u.\nabla\zeta(x,y)dxdy \qquad \forall\zeta\in\mathcal{D}(B_{r/4}).
\end{aligned}$$

Case (ii) : In this case, we have $\chi([u>0])=\chi(B_r^+)$. Then, we have

$$\int_{B_r}(\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u.\nabla\zeta dxdy = \int_{B_r^+}-f(x)\zeta dxdy \qquad \forall\zeta\in W_0^{1,p}(B_r).$$

Considering the function u_η as above, one can check that we also obtain, as before,

$$\int_{B_{r/4}}\big((\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u-(\kappa+|\nabla u_\eta|^2)^{\frac{p-2}{2}}\nabla u_\eta\big).\nabla\zeta dxdy = 0 \qquad \forall\zeta\in W_0^{1,p}(B_{r/4}).$$

In both cases, we write the equality as :

$$\int_{B_{r/4}}a(x,y)\nabla(u-u_\eta).\nabla\zeta dxdy = 0 \qquad \forall\zeta\in W_0^{1,p}(B_{r/4}),$$

where $a(x,y)=\Big(\int_0^1\frac{\partial A^i}{\partial h_j}(\nabla u_t)dt\Big)_{i,j=1,2}$
with $A(h)=(\kappa+|h|^2)^{\frac{p-2}{2}}h$, $h\in\mathbb{R}^2$, $(x,y)\in B_{r/4}$, $u_t=tu+(1-t)u_\eta$ $t\in[0,1]$.
The matrix $a(x,y)$ satisfies

$$\min(1,p-1)\lambda(x,y)|\xi|^2\leqslant a(x,y)\xi.\xi\leqslant\max(1,p-1)\lambda(x,y)|\xi|^2 \qquad \forall\xi\in\mathbb{R}^2$$

with $\lambda(x,y)=\displaystyle\int_0^1(\kappa+|\nabla u_t|^2)^{\frac{p-2}{2}}dt$. So the function $(u-u_\eta)$ satisfies

$$\begin{cases} div(a(x,y)\nabla(u-u_\eta))=0 & \text{in} \quad \mathcal{D}'(B_{r/4}),\\ u-u_\eta\geqslant 0 & \text{in} \quad B_{r/4}\\ u-u_\eta=0 & \text{on} \quad \begin{cases} S_{r/4} & \text{in case } i)\\ B_{r/4}\cap[x\leqslant x_0] & \text{in case } ii).\end{cases}\end{cases}$$

Now,there exists two constants $c_p, C_p>0$, depending only on p (see [5] Lemma 8.3 p 366), such that

$$c_p(\kappa+|\nabla u_\eta|^2+|\nabla u|^2)^{\frac{p-2}{2}}\leqslant\lambda(x,y)\leqslant C_p(\kappa+|\nabla u_\eta|^2+|\nabla u|^2)^{\frac{p-2}{2}}.$$

Since ∇u_η and ∇u are bounded, we deduce that the matrix $a(x,y)$ is strictly elliptic and uniformly bounded. Therefore,we have $u-u_\eta=0$ in $B_{r/4}$ by the strong maximum principle ([4] Theorem 8.19 p 198). This holds for all $\eta\in(0,r/4)$. Hence $\dfrac{\partial u}{\partial y}=0$ in $B_{r/4}(x_0,y_0)$. $\square$

Proof of Lemma 1.1. Here also, we only consider cases i) and ii) since case iii) is similar to ii). From Lemma 1.2, we deduce that $u(x,y)=\tau(x)$ in $B_{r/4}$. In particular, in both two cases, we have

$$\big((\kappa+(\tau'(x))^2)^{\frac{p-2}{2}}\tau'(x)\big)'=-f(x)\qquad\text{in } B^+_{r/4}.$$

Since $u\in C^1(B_r)$, $u=0$ and $\nabla u=0$ on S_r, we then have $\tau(x_0)=\tau'(x_0)=0$. Therefore, we deduce that

$$(\kappa+(\tau'(x))^2)^{\frac{p-2}{2}}\tau'(x)=\int_{x_0}^x -f(s)ds\quad\text{ and }\quad\tau'(x)>0\quad\forall x\in(x_0,x_0+\frac{r}{4}).$$

By monotonicity of u, we have

$$u(x,y)\geqslant\tau(x)>0\qquad\forall(x,y)\in D=(x_0,x_0+\frac{r}{4})\times(y_0,1).$$

By comparing u and τ in D, we have for $\zeta\in\mathcal{D}(D)$

$$\int_D(\kappa+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u.\nabla\zeta dxdy=\int_D f(x)\zeta dxdy=\int_D(\kappa+|\nabla\tau|^2)^{\frac{p-2}{2}}\nabla\tau.\nabla\zeta dxdy$$

which can be written

$$\int_D\Big(\int_0^1\frac{d}{dt}\big(|\nabla\tilde{u}_t|^{p-2}\nabla\tilde{u}_t\big)dt\Big).\nabla\zeta dxdy=0,\qquad\tilde{u}_t=t\tau+(1-t)u,\qquad t\in[0,1],$$

or

$$\int_D \tilde{a}(x,y)\nabla(u-\tau).\nabla\zeta dxdy = 0 \qquad \forall\zeta \in W_0^{1,p}(D),$$

where $\quad \tilde{a}(x,y) = \Big(\int_0^1 \frac{\partial A^i}{\partial h_j}(\nabla\tilde{u}_t)dt\Big)_{i,j=1,2}, \qquad (x,y) \in D.$

We also have

$$\min(1,p-1)\tilde{\lambda}(x,y)|\xi|^2 \leqslant \tilde{a}(x,y)\xi.\xi \leqslant \max(1,p-1)\tilde{\lambda}(x,y)|\xi|^2 \qquad \forall\xi \in \mathbb{R}^2$$

with $\tilde{\lambda}(x,y) = \displaystyle\int_0^1 (\kappa + |\nabla\tilde{u}_t|^2)^{\frac{p-2}{2}} dt.$

Further more, $(u-\tau)$ satisfies

$$\begin{cases} div(\tilde{a}(x,y)\nabla(u-\tau)) = 0 & \text{in} \quad \mathcal{D}'(D), \\ u-\tau \geqslant 0 & \text{in} \quad D \\ u-\tau = 0 & \text{on} \quad B^+_{r/4}. \end{cases}$$

Arguing as in the proof of Lemma 1.1, we deduce by the strong maximum principle (see [4] Theorem 8.19 p 198) that $u-\tau = 0$ in D. By continuity to the boundary Γ_1, we deduce that $u_a = u(x,1) = \tau(x)$ $\forall x \in (x_0, x_0 + r/4)$ and we get a contradiction with $\tau'(x) > 0$. $\square$

2 A Barrier Function and Comparison

Let $x_0, \underline{y} \in (0,1)$ and $\epsilon_0 = \min(x_0, 1-x_0)/6$. There exists $\alpha \in (0,1)$ such that

$$u \in C^{1,\alpha}\big([x_0-\epsilon_0, x_0+\epsilon_0] \times [0,1]\big). \tag{2.1}$$

Assume that there exists $\epsilon_1 > 0$ such that

$$\forall\epsilon \in (0,\epsilon_1), \qquad \underline{y} - \epsilon > 0.$$

($\underline{y}$ may depend on ϵ)

Let $x_1 \in (0,1)$ such that $x_1 < x_0$ and $|x_1 - x_0| < \epsilon^{1+\frac{1}{\alpha}}$ where $\epsilon \in \big(0, \min(\epsilon_0^{\frac{\alpha}{\alpha+1}}, \epsilon_1)\big)$. Set

$$Z = (x_1, x_0) \times (\underline{y} - \epsilon, \underline{y}), \qquad D = (x_1, x_0) \times (0, \underline{y}),$$
$$\psi(x) = \underline{h}\int_{x_1}^{x} \frac{ds}{h(s)} + \underline{y} - \epsilon,$$

where

$\underline{h} > 0$ is fixed, $\qquad h(x) = \underline{h} - \displaystyle\int_x^{x_0} f(s)ds,$

$-\Lambda \leqslant h'(x) = f(x) \leqslant -\lambda < 0, \qquad \underline{h} \leqslant h(x) \leqslant \underline{h} + \Lambda(x_0 - x_1) \leqslant \underline{h} + \Lambda\epsilon_0^{\frac{\alpha}{\alpha+1}} \leqslant \underline{h} + \Lambda = \overline{h}.$

Then ψ satisfies

$$\psi'(x) = \frac{\underline{h}}{h(x)} > 0, \qquad \psi''(x) = -\underline{h}\frac{h'(x)}{h^2(x)} > 0, \qquad \underline{y} - \epsilon \leqslant \psi(x) \leqslant \underline{y} - \epsilon + \epsilon^{1+\frac{1}{\alpha}}.$$

Note that for $\epsilon < (1/2)^\alpha$, we have $[y > \psi(x)] \cap Z \neq \emptyset$ since

$$0 < \epsilon - 2\epsilon^{1+\frac{1}{\alpha}} \leqslant y - \psi(x) \leqslant \epsilon \qquad \forall (x,y) \in (x_1, x_0) \times (\underline{y} - \epsilon^{1+\frac{1}{\alpha}}, \underline{y}).$$

Now, set

$$w(x,y) = \int_x^{\widetilde{\psi}(y)} k(y - \psi(s))^+ ds \qquad (x,y) \in D, \qquad k > 0$$

where

$$\widetilde{\psi}(y) = \begin{cases} x_1 & \text{if} \quad y \in [0, \underline{y} - \epsilon] \\ \psi^{-1}(y) & \text{if} \quad y \in [\underline{y} - \epsilon, \psi(x_0)] \\ x_0 & \text{if} \quad y \in [\psi(x_0), \underline{y}]. \end{cases}$$

Let

$$G(x,y) = \Big[\big((y - \psi(x))^+\big)^2 + \big((\widetilde{\psi}(y) - x)^+\big)^2\Big]^{1/2}.$$

Note that $w \in C^1(D)$ and we have

$$\begin{array}{ll} \nabla w(x,y) = k\langle -(y - \psi(x))^+, (\widetilde{\psi}(y) - x)^+\rangle, & \nabla w = 0 \quad \text{on } [y = \psi(x)], \\ |\nabla w(x,y)| = kG(x,y), & (\kappa + |\nabla w|^2)^{1/2} = (\kappa + G^2k^2)^{1/2}. \end{array}$$

Lemma 2.1. *Let $\epsilon \in \big(0, \min((\frac{1}{2})^\alpha, \epsilon_0^{\frac{\alpha}{\alpha+1}}, \epsilon_1)\big)$ and $\theta > 0$. Then, there exists $k_0 > 0$ independent of ϵ such that $\forall k \in (0, k_0)$, we have*

$$\int_D \big[(\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w.\nabla\zeta + \theta\chi([w > 0])\zeta\big] dxdy \geqslant 0 \qquad \forall \zeta \in W_0^{1,p}(D), \quad \zeta \geqslant 0.$$

Proof. Let $\zeta \in W_0^{1,p}(D)$, $\zeta \geqslant 0$. We have

$$\int_D \big[(\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w.\nabla\zeta + \theta\chi([w > 0])\zeta\big] dxdy = \int_{[y>\psi(x)]} \big[(\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w.\nabla\zeta + \theta\zeta\big] dxdy$$

$$= \int_{[y>\psi(x)]} \big[-div((\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w) + \theta\big]\zeta dxdy + \int_{[y=\psi(x)]} (\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w.\nu\zeta$$

$$= \int_{[y>\psi(x)]} \big[-div((\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w) + \theta\big]\zeta dxdy \qquad \text{since} \quad \nabla w = 0 \text{ on } [y = \psi(x)]$$

where ν is the unit normal to the curve $[y=\psi(x)]$ pointing towards the set $[y<\psi(x)]$. It is given by $\nu = \langle \nu_x, \nu_y \rangle = \dfrac{1}{\sqrt{1+\psi'^2(x)}}\langle \psi'(x), -1\rangle$.
We have, for $(x,y) \in [y>\psi(x)]$,

$$div\big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\nabla w\big) = \frac{\partial}{\partial x}\Big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\frac{\partial w}{\partial x}\Big) + \frac{\partial}{\partial y}\Big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\frac{\partial w}{\partial y}\Big),$$

$$\frac{\partial}{\partial x}\Big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\frac{\partial w}{\partial x}\Big) = \frac{\partial}{\partial x}\Big[-(\kappa+G^2k^2)^{\frac{p-2}{2}}k(y-\psi(x))\Big]$$
$$= k\Big[(\kappa+G^2k^2)^{\frac{p-2}{2}}\psi'(x) + (p-2)(\kappa+G^2k^2)^{\frac{p-4}{2}}k^2\big[(y-\psi(x))^2\psi'(x)$$
$$+(\widetilde{\psi}(y)-x)(y-\psi(x))\big]\Big],$$

$$\frac{\partial}{\partial y}\Big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\frac{\partial w}{\partial y}\Big) = \frac{\partial}{\partial y}\Big[(\kappa+G^2k^2)^{\frac{p-2}{2}}k(\widetilde{\psi}(y)-x)\Big]$$

$$= \begin{cases} (p-2)k(\kappa+G^2k^2)^{\frac{p-4}{2}}k^2(x_0-x)(y-\psi(x)) & \text{if} \quad y \in (\psi(x_0), \underline{y}) \\[2ex] k\Big[(\kappa+G^2k^2)^{\frac{p-2}{2}}\frac{1}{\psi'(\psi^{-1}(y))} + \\ \qquad (p-2)(\kappa+G^2k^2)^{\frac{p-4}{2}}k^2\big\{(\psi^{-1}(y)-x)^2\frac{1}{\psi'(\psi^{-1}(y))} + (\widetilde{\psi}(y)-x)(y-\psi(x))\big\}\Big] \\ & \text{if} \quad y \in (\underline{y}-\epsilon, \psi(x_0)). \end{cases}$$

Hence, we have, on $[y>\psi(x)] \cap [\psi(x_0) < y < \underline{y}]$,

$$div\big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\nabla w\big) = k\Big[(\kappa+G^2k^2)^{\frac{p-2}{2}}\psi'(x) +$$
$$+(p-2)(\kappa+G^2k^2)^{\frac{p-4}{2}}k^2\big\{(y-\psi(x))^2\psi'(x) + 2(\widetilde{\psi}(y)-x)(y-\psi(x))\big\}\Big],$$

and, we have, on $[y>\psi(x)] \cap [\underline{y}-\epsilon < y < \psi(x_0)]$,

$$div\big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\nabla w\big) = k\Big[(\kappa+G^2k^2)^{\frac{p-2}{2}}\big\{\psi'(x) + \frac{1}{\psi'(\psi^{-1}(y))}\big\}$$
$$+(p-2)(\kappa+G^2k^2)^{\frac{p-4}{2}}k^2\big\{(y-\psi(x))^2\psi'(x) + 2(\widetilde{\psi}(y)-x)(y-\psi(x))$$
$$+(\widetilde{\psi}(y)-x)^2\frac{1}{\psi'(\psi^{-1}(y))}\big\}\Big].$$

Now, since we have the following inequalities :

$$\big[(y-\psi(x))^2\psi'(x)+2(\widetilde{\psi}(y)-x)(y-\psi(x))\big]\leqslant(1+\psi'(x))G^2$$
$$\leqslant\frac{1}{k^2}(1+\psi'(x))(\kappa+G^2k^2)$$

$$(y-\psi(x))^2\psi'(x)+2(\widetilde{\psi}(y)-x)(y-\psi(x))+(\widetilde{\psi}(y)-x)^2\frac{1}{\psi'(\psi^{-1}(y))}$$
$$\leqslant\Big(1+\psi'(x)+\frac{1}{\psi'(\psi^{-1}(y))}\Big)G^2\leqslant\frac{1}{k^2}\Big(1+\psi'(x)+\frac{1}{\psi'(\psi^{-1}(y))}\Big)(\kappa+G^2k^2)$$

$$\underline{h}/\overline{h}\leqslant\psi'(x)\leqslant 1,\quad 0<y-\psi(x)<\epsilon<1,\quad 0<\widetilde{\psi}(y)-x\leqslant x_0-x\leqslant\epsilon^{1+\frac{1}{\alpha}}<1,$$

we deduce that we have, in $[y>\psi(x)]$,

$$|div\big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\nabla w\big)|\leqslant\sigma_p(k)=\begin{cases} k(\kappa+4k^2)^{\frac{p-2}{2}}\big(2+\frac{\overline{h}}{\underline{h}}\big)(1+|p-2|) & \text{if } p\geqslant 2\\ k\big(2+\frac{\overline{h}}{\underline{h}}\big)(1+|p-2|)\kappa^{\frac{p-2}{2}} & \text{if } 1<p<2.\end{cases}$$

Remark that we have $\lim\limits_{k\to 0}\sigma_p(k)=0$. Then, there exists $k_0>0$ such that for $k\in(0,k_0)$ we have $0<\sigma_p(k)<\theta/2$.
To conclude the lemma, it suffices to note that

$$\int_{[y>\psi(x)]}\Big(-div\big((\kappa+|\nabla w|^2)^{\frac{p-2}{2}}\nabla w\big)+\theta\Big)\zeta dxdy\geqslant(-\sigma_p(k)+\theta)\int_{[y>\psi(x)]}\zeta dxdy\geqslant 0.\square$$

Remark 2.1. *The constant k_0 doesn't depend on ϵ. It depends on θ, p, κ, $\overline{h}$, and $\underline{h}$ when $p\neq 2$. It doesn't depend on κ when $p=2$, which corresponds to the Laplace operator.*

The following lemma, compares u and w near a free boundary point, along the x direction. First, from (2.1), there exists a positive constant C^* depending on p, Λ, u_a such that $\forall(x,y),(x',y')\in[x_0-\epsilon_0,x_0+\epsilon_0]\times[0,1]$

$$|u(x,y)-u(x',y')|\leqslant C^*\big(|x-x'|^2+|y-y'|^2\big)^{\frac{1+\alpha}{2}}. \tag{2.2}$$

Lemma 2.2. *Assume that $u(x_0,\underline{y})=0$, then for $\epsilon\in\Big(0,\min\big((\frac{k}{2C^*})^{\frac{1}{\alpha}},(\frac{1}{2})^{\alpha},\epsilon_0^{\frac{\alpha}{\alpha+1}},\epsilon_1\big)\Big)$, we have*

$$u(x,\underline{y})\leqslant w(x,\underline{y})\qquad\qquad\forall x\in[x_1,x_0].$$

Proof. Since $u(x_0, \underline{y}) = 0$, we have by (2,2) in particular

$$u(x, \underline{y}) \leqslant C^* |x - x_0|^{1+\alpha} \qquad \text{for all } x \in [x_0 - \epsilon_0, x_0 + \epsilon_0].$$

On another hand, we have for $x \in [x_1, x_0]$,

$$w(x, \underline{y}) = \int_x^{\widetilde{\psi}(\underline{y})} k(\underline{y} - \psi(s))^+ ds = \int_x^{x_0} k(\underline{y} - \psi(s))^+ ds \geqslant \frac{\epsilon}{2} k(x_0 - x).$$

Indeed, we have

$$\underline{y} - \psi(x) = \epsilon - \underline{h} \int_{x_1}^{x} \frac{ds}{h(s)} \geqslant \epsilon - \epsilon^{1+\frac{1}{\alpha}} > \epsilon/2.$$

Now, we can write

$$u(x, \underline{y}) \leqslant C^* |x - x_0|^{1+\alpha} = C^* |x - x_0|^{\alpha} |x - x_0| \leqslant C^* \epsilon^{1+\alpha} |x - x_0| \leqslant \frac{\epsilon}{2} k(x_0 - x) \Longleftrightarrow \epsilon \leqslant \Big(\frac{k}{2C^*}\Big)^{\frac{1}{\alpha}}. \square$$

3 Proof of Theorem 1.1

Let $(x_0, \Phi(x_0)) = (x_0, y_0) \in \Omega$. Since $\lim\limits_{\epsilon \to 0} \epsilon + \epsilon^{1+\frac{1}{\alpha}} = 0$, there exists $\epsilon_1 > 0$ such that

$$\forall \epsilon \in (0, \epsilon_1), \qquad 3\epsilon + 3\epsilon^{1+\frac{1}{\alpha}} < y_0 = \Phi(x_0).$$

Let $\epsilon \in \big(0, \min((\frac{k}{2C^*})^{\frac{1}{\alpha}}, (\frac{1}{2})^{\alpha}, \epsilon_0^{\frac{\alpha}{\alpha+1}}, \epsilon_1)\big)$. From Lemma 1.1, one of the following situations occurs :

i) there exists $(x_1, y_1) \in B_{\epsilon^{1+\frac{1}{\alpha}}}(x_0, y_0) \cap [x < x_0]$ such that $u(x_1, y_1) = 0$.

ii) there exists $(x_1', y_1') \in B_{\epsilon^{1+\frac{1}{\alpha}}}(x_0, y_0) \cap [x > x_0]$ such that $u(x_1', y_1') = 0$.

Assume that $i)$ holds. We distinguish two cases :

$*$ $\underline{y_1 \geqslant y_0 = \Phi(x_0)}$. Set $\underline{y} = y_0$. We have

$$u(x_1, y) = u(x_0, y) = 0 \qquad \forall y \in [0, \underline{y}].$$

Definning D, ψ and w as in section 2, we have

$$u(x, \underline{y}) \leqslant w(x, \underline{y}) \qquad \forall x \in [x_1, x_0].$$

Since $u(x, 0) = 0$ for all $x \in (0, 1)$, we deduce that $(u - w)^+ \in W_0^{1,p}(D)$. Hence, by Lemma 2.1, we have for $\theta > 0$ and $k \in (0, k_0)$

$$\int_D \left[(\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w.\nabla(u-w)^+ + \theta\chi([w>0])(u-w)^+\right] dxdy \geqslant 0.$$

From (P), we have

$$\int_D \left[(\kappa + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u.\nabla(u-w)^+ - f\chi([u>0])(u-w)^+\right] dxdy = 0.$$

Hence, we get

$$\begin{aligned}
&\int_D \big((\kappa + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u - (\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w\big).\nabla(u-w)^+ dxdy \\
&\quad \leqslant \int_D (f\chi([u>0]) + \theta\chi([w>0])(u-w)^+ dxdy \\
&\quad \leqslant \int_{D\cap[w=0]} f\chi([u>0])(u-w)^+ dxdy + \int_{D\cap[w>0]} (f+\theta)(u-w)^+ dxdy \\
&\quad \leqslant \int_{D\cap[w>0]} (-\lambda+\theta)(u-w)^+ dxdy \qquad\qquad \text{since } f \leqslant -\lambda < 0.
\end{aligned}$$

Choosing $\theta \in (0,\lambda)$, we deduce that

$$\int_D \big((\kappa + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u - (\kappa + |\nabla w|^2)^{\frac{p-2}{2}} \nabla w\big).\nabla(u-w)^+ dxdy \leqslant 0$$

from which we deduce that $\nabla(u-w)^+ = 0$ a.e. in D, and therefore $u \leqslant w$ in D.
As a consequence, we have $u(x, \underline{y} - \epsilon) = 0$ for all $x \in (x_1, x_0)$. So

$$\Phi(x) \geqslant \underline{y} - \epsilon = \Phi(x_0) - \epsilon \qquad \forall x \in (x_1, x_0). \tag{3.1}$$

Now, since we have $u = 0$ in $B_{\epsilon^{1+\frac{1}{\alpha}}}(x_0, \underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}}) \cap [x \leqslant x_0]$, there exists, by Lemma 1.1,

$$(x_2, y_2) \in B_{\epsilon^{1+\frac{1}{\alpha}}}(x_0, \underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}}) \cap [x > x_0] \quad \text{such that} \quad u(x_2, y_2) = 0.$$

Set

$$\begin{aligned}
&\underline{y}' = y_2, \qquad x_1' = x_0, \qquad x_0' = x_2, \\
&Z' = (x_1', x_0') \times (\underline{y}' - \epsilon, \underline{y}'), \qquad D' = (x_1', x_0') \times (0, \underline{y}'), \\
&\psi_r(x) = \underline{h} \int_{x_1'}^{x} \frac{ds}{h_r(s)} + \underline{y}' - \epsilon, \qquad r \text{ is for right side,}
\end{aligned}$$

where

$$h_r(x) = \underline{h} - \int_x^{x_0'} f(s)ds, \qquad -\Lambda \leqslant h_r'(x) = f(x) \leqslant -\lambda < 0,$$
$$\underline{h} \leqslant h(x) \leqslant \underline{h} + \Lambda(x_0' - x_1') \leqslant \underline{h} + \Lambda\epsilon_0^{\frac{\alpha}{\alpha+1}} \leqslant \underline{h} + \Lambda = \overline{h}.$$

Then ψ_r satisfies

$$\psi_r'(x) = \frac{\underline{h}}{h_r(x)} > 0, \qquad \psi_r''(x) = -\underline{h}\frac{h_r'(x)}{h_r^2(x)} > 0, \qquad \underline{y}' - \epsilon \leqslant \psi_r(x) \leqslant \underline{y}' - \epsilon + \epsilon^{1+\frac{1}{\alpha}}$$
$$\underline{y}' - \epsilon > \underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}} - \epsilon^{1+\frac{1}{\alpha}} - \epsilon > 0 \Longleftrightarrow 2\epsilon + 2\epsilon^{1+\frac{1}{\alpha}} < \underline{y} \quad \text{which is satisfied for } \epsilon \in (0, \epsilon_1).$$

Note that for $\epsilon < (1/2)^\alpha$, we have $[y > \psi_r(x)] \cap Z' \neq \emptyset$ since

$$0 < \epsilon - 2\epsilon^{1+\frac{1}{\alpha}} \leqslant y - \psi_r(x) \leqslant \epsilon \qquad \forall (x,y) \in (x_1', x_0') \times (\underline{y}' - \epsilon^{1+\frac{1}{\alpha}}, \underline{y}').$$

We define

$$w_r(x,y) = \int_x^{\widetilde{\psi}_r(y)} k(y - \psi_r(s))^+ ds \qquad (x,y) \in D',$$

where

$$\widetilde{\psi}_r(y) = \begin{cases} x_1' & \text{if} \quad y \in [0, \underline{y}' - \epsilon] \\ \psi_r^{-1}(y) & \text{if} \quad y \in [\underline{y}' - \epsilon, \psi_r(x_0')] \\ x_0' & \text{if} \quad y \in [\psi(x_0'), \underline{y}']. \end{cases}$$

Arguing as we did on the left of x_0, we have $u \leqslant w_r$ in D' from which we deduce that $u(x, \underline{y}' - \epsilon) = 0$ for all $x \in (x_1', x_0')$. So

$$\Phi(x) \geqslant \underline{y}' - \epsilon > \underline{y} - 2\epsilon - 2\epsilon^{1+\frac{1}{\alpha}} = \Phi(x_0) - 2\epsilon - 2\epsilon^{1+\frac{1}{\alpha}} \qquad \forall x \in (x_0, x_2). \qquad (3.2)$$

Hence, from (3.1) and (3.2), Φ is l.s.c at x_0. Therefore, it is continuous at x_0.

** $\underline{y_1 < y_0 = \Phi(x_0)}$. Set $\underline{y} = y_1$.
First, note that, since $\epsilon \in (0, \epsilon_1)$, we have

$$\underline{y} - \epsilon = y_1 - \epsilon > y_0 - \epsilon^{1+\frac{1}{\alpha}} - \epsilon > 0 \Longleftrightarrow \epsilon + \epsilon^{1+\frac{1}{\alpha}} < y_0.$$

We define D, ψ and w as in section 2. We check that for $\epsilon < (1/2)^\alpha$, we have $[y > \psi(x)] \cap Z \neq \emptyset$ since for $(x, y) \in (x_1, x_0) \times (\underline{y} - \epsilon^{1+\frac{1}{\alpha}}, \underline{y})$, we have

$$y - (\underline{y} - \epsilon + \epsilon^{1+\frac{1}{\alpha}}) \leqslant y - \psi(x) \leqslant \epsilon$$
$$y - (\underline{y} - \epsilon + \epsilon^{1+\frac{1}{\alpha}}) \geqslant \underline{y} - \epsilon^{1+\frac{1}{\alpha}} - (\underline{y} - \epsilon + \epsilon^{1+\frac{1}{\alpha}}) = \epsilon - 2\epsilon^{1+\frac{1}{\alpha}} > 0 \Longleftrightarrow \epsilon < \left(\frac{1}{2}\right)^\alpha.$$

We get $u \leqslant w$ in D, from which we deduce that $u(x, \underline{y} - \epsilon) = 0 \ \forall x \in (x_1, x_0)$. So

$$\Phi(x) \geqslant \underline{y} - \epsilon = y_1 - \epsilon > \Phi(x_0) - \epsilon^{1+\frac{1}{\alpha}} - \epsilon \qquad \forall x \in (x_1, x_0). \tag{3.3}$$

Now, note that we have

$$\underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}} > y_0 - \epsilon^{1+\frac{1}{\alpha}} - \epsilon - \epsilon^{1+\frac{1}{\alpha}} > 0 \Longleftrightarrow \epsilon + 2\epsilon^{1+\frac{1}{\alpha}} < y_0 \quad \text{since } \epsilon \in (0, \epsilon_1).$$

Since we have $u = 0$ in $B_{\epsilon^{1+\frac{1}{\alpha}}}(x_0, \underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}}) \cap [x \leqslant x_0]$, there exists, by Lemma 1.1,

$$(x_2, y_2) \in B_{\epsilon^{1+\frac{1}{\alpha}}}(x_0, \underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}}) \cap [x > x_0] \quad \text{such that} \quad u(x_2, y_2) = 0.$$

Define, as in the first case, Z', D', $\underline{y}'$, x_1', x_0', and check that

$$\underline{y}' - \epsilon = y_2 - \epsilon > \underline{y} - \epsilon - \epsilon^{1+\frac{1}{\alpha}} - \epsilon^{1+\frac{1}{\alpha}} - \epsilon > y_0 - \epsilon^{1+\frac{1}{\alpha}} - 2\epsilon - 2\epsilon^{1+\frac{1}{\alpha}} > 0 \Longleftrightarrow 2\epsilon + 3\epsilon^{1+\frac{1}{\alpha}} < y_0$$

which is satisfied for $\epsilon \in (0, \epsilon_1)$.
We deduce that $u(x, \underline{y}' - \epsilon) = 0 \ \forall x \in (x_1', x_0')$. So

$$\Phi(x) \geqslant \underline{y}' - \epsilon = y_2 - \epsilon > \Phi(x_0) - 3\epsilon^{1+\frac{1}{\alpha}} - 2\epsilon \qquad \forall x \in (x_0, x_2). \tag{3.4}$$

We conclude from (3.3) and (3.4) that Φ is lower semi continuous at x_0 and therefore it is continuous at this point. $\square$

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