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A PROPOSED NUMERICAL METHOD FOR  
THE 3D WAVE EQUATION WITH CONSTANT SPEED

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# Abstract

This thesis implements a new numerical scheme to solve the (classical) constant speed wave equation in the three dimensional space, currently existing methods (even if more restrictive) rely on time iterations and observe the accumulation of error at each time step iteration, the new method is iteration free ! Making it a good choice for applications requiring accurate results at large times. Numerical experiments and error analysis reveal accuracy of the scheme. The principal conclusion is that the method, based on the Radon transform, must be considered, and we propose it should be developed and count among the standard methods implemented in computational software for engineering and industrial applications.

# Dedication

"

- What did water say to the boat ?

- What ? I don't know !

- Nothing. It just waved !

"

To Jasmine.

# Acknowledgements

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# 1 Introduction

In Mathematics, we are usually interested in problems, one may argue that stating a problem is equally important as solving one, so in this thesis, here is the problem we are interested in solving :

$$\frac{1}{c^2}u_{tt} - \Delta u = 0 \quad (1.1)$$

with initial conditions

$$u(x, 0) = f(x) , \quad u_t(x, 0) = g(x) \quad (1.2)$$

where  $c > 0$  is constant, and for positive times  $t \geq 0$ .

This is the Cauchy problem for the wave equation on  $\mathbb{R}^3 \times \mathbb{R}$ , the initial conditions as stated guarantee the existence of a unique solution [3].

There is a plethora of numerical methods to solve variants of the wave equation, as in [4], [7] and [12], very few deal with the case of propagation in the 3 dimensional space, in fact, a notorious scheme ( in [1] ) is a Finite elements method that requires an additional boundary condition and must be applied on a finite (bounded) space domain.

Other schemes can be based on Finite Volume methods and must proceed by time iterations, where to reach a large specific time  $T_0$ , computations need to be done for all times steps occurring before  $T_0$ .

## 1.1 Outline

This introduction continues with a discussion of the one dimensional wave equation, as a preliminary and necessary step before presenting the three dimensional case.

Chapters 2 and 3 lay the theoretical principles and necessary tools to tackle the problem, the first presents the basic properties of the Radon transform, exposing all the proofs and necessary facts, the second explains how the Radon transform can be used to solve the 3-D wave equation, then shows a very nice formula for the solution.

Chapter 4 exposes a new computational method to compute integrals over the three dimensional sphere  $S^2$ . It can be regarded as a preliminary step before implementing the main numerical method, but it can also be considered as an independent numerical method for computing all sorts of integrals over the sphere. The latter sections of the chapter present the proposed numerical method to solve the wave equation and one of the possible implementations.

Chapter 5 expose the tests to verify the accuracy of the scheme, and the computation latency, for various examples.

## 1.2 Related research

Very few recent papers discuss the Radon transform in the context of the wave equation, or other partial differential equations, we can cite the paper of Dr. Rim Donsub [9], which discusses the idea of applying the Radon transform to an acoustic equation ( two dimensions and time ) and the 'Discrete Radon transform', which seems to be applied to two dimensional 'scans' or data.

## 1.3 The one dimensional wave equation

Before we chase the solution of the 3D wave equation, let us first recall the simpler 1-D wave equation, we write this equation as:

$$u_{tt} = c^2 u_{xx} \text{ , for } x \in \mathbb{R} \quad (1.3)$$

Here the operator factors nicely:

$$u_{tt} - c^2 u_{xx} = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0$$

This means that, starting from a function  $u(x, t)$ , you compute  $u_t + cu_x$ , call the result  $v$ , then you compute  $v_t - cv_x$ , and you need to get the zero function. The general solution is

$$u(x, t) = F(x + ct) + G(x - ct) \quad (1.4)$$

where  $F$  and  $G$  are two arbitrary (twice differentiable) functions of a single variable.

We are interested in the initial-value problem, that is, (1.3) together with the initial conditions:

$$u(x, 0) = f(x) \text{ , } u_t(x, 0) = g(x) \quad (1.5)$$

where  $f$  and  $g$  are arbitrary functions of  $x$ .

There is one, and only one, solution of this problem. The solution of (1.3), (1.5) is easily found from the general formula (1.4). First, setting  $t = 0$ , we get  $f(x) = F(x) + G(x)$  then  $g(x) = cF'(x) - cG'(x)$  , regarding the last two equations for the two unknown functions  $F$  and  $G$ . To solve them, we can differentiate the first and divide the second, it is convenient

temporarily to change the name of the variable to some neutral name; we change the name of  $x$  to  $s$ . Now

$f' = F' + G'$  and  $\frac{1}{c}g = F' - G'$ . Adding and subtracting gives  $F' = \frac{1}{2}f' + \frac{g}{c}$  and  $G' = \frac{1}{2}f' - \frac{g}{c}$ .

Then integrating, we get

$$F(s) = \frac{1}{2}f(s) + \frac{1}{2c} \int_0^s g \, ds + a$$

and  $G(s) = \frac{1}{2}f(s) - \frac{1}{2c} \int_0^s g \, ds + b$

where  $a$  and  $b$  are constants such that  $a + b = 0$ . Substituting  $s = x + ct$  into the formula for  $F$  and  $s = x - ct$  into that of  $G$ , we get

$$u(x, t) = \frac{1}{2}f(x + ct) + \frac{1}{2c} \int_0^{x+ct} g + \frac{1}{2}f(x - ct) - \frac{1}{2c} \int_0^{x-ct} g$$

which simplifies to

$$u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \quad (1.6)$$

This is the solution formula for the initial-value problem, due to D'Alembert in 1746. Assuming  $F$  to have a continuous second derivative and  $G$  to have a continuous first derivative, we see that  $u$  itself has continuous second partial derivatives in  $x$  and  $t$ .

## 2 The Radon transform

### 2.1 Definitions

The Radon transform was invented by *Johann Radon* in 1917 [8], it has many applications in mathematics and other sciences, including a significant achievement in medicine. This integral transform has a close connection with the development of X-ray scans (or CAT scans) in the theory of medical imaging. The solution of the reconstruction problem, and the introduction of new algorithms and faster computers, all contributed to a rapid development of computerized tomography. In practice, X-ray scans provide a 'picture' of an internal organ, one that helps to detect and locate many types of abnormalities.

While this transform can be defined for any dimension, we recall here the basics of the Radon transform in  $\mathbb{R}^3$ .

To simplify, we shall assume that we are dealing with functions in the class  $\mathcal{S}(\mathbb{R}^3)$ . However, many of the results obtained below can be shown to be valid for much larger classes of functions. We define the Radon transform  $\mathcal{R}(f)$  of a function  $f \in \mathcal{S}(\mathbb{R}^3)$  by

$$\mathcal{R}(f)(P) = \int_P f$$

where  $P$  is a plane in the Euclidean 3-dimensional space  $\mathbb{R}^3$ .

First, we explain the integral of  $f$  over a plane. We can describe planes in  $\mathbb{R}^3$  as follows: given a unit vector  $\omega \in S^2$  and a number  $s \in \mathbb{R}$ , we define the plane

$P_{s,\omega} = \{x \in \mathbb{R}^3 : x \cdot \omega = s\}$ , so we parametrize a plane by a unit vector  $\omega$  orthogonal to it, and by the distance  $s$  from the origin. Note that  $P_{s,\omega} = P_{-s,-\omega}$ , and we allow  $s$  to take negative values.

We can then explicitly write

$$\mathcal{R}f(s, \omega) = \int_{x \cdot \omega = s} f(x) d\mu(x)$$

In particular, we see that the Radon transform is a function on the set of planes in  $\mathbb{R}^3$ . From the parametrization given for a plane, we can think of  $\mathcal{R}(f)$  as a function on the product  $\mathbb{R} \times S^2 = \{(s, \omega) : s \in \mathbb{R}, \omega \in S^2\}$ , where  $S^2$  is the unit sphere in  $\mathbb{R}^3$ . The relevant class of functions on  $\mathbb{R} \times S^2$  consists of those that satisfy the Schwartz condition in  $s$  uniformly in  $\omega$ . In other words, we define  $\mathcal{S}(\mathbb{R} \times S^2)$  to be the space of all continuous functions  $F(s, \omega)$  that are indefinitely differentiable in  $s$ , and that satisfy

$$\sup_{(s,\omega) \in \mathbb{R} \times S^2} |s|^n \left| \frac{\partial^m}{\partial s^m} F(s, \omega) \right| < \infty \quad \text{for all integers } n, m \geq 0.$$

How to answer the uniqueness problem: if  $\mathcal{R}(f) = \mathcal{R}(g)$  then  $f = g$ ? And is it possible to reconstruct  $f$  from its Radon transform  $\mathcal{R}(f)$ ?

The answer to both problems is yes, the solutions are obtained by using the Fourier transform. In fact, the key point is a very elegant and essential relation between the Radon and Fourier transforms :

**Theorem 1.** ( from [13] )

$$\mathcal{R}f(s, \omega) = \frac{1}{2\pi} \int \hat{f}(\rho\omega) e^{is\rho} d\rho \quad (2.1)$$

**Proof.** We refer to the version of the Fourier transform given by the formulas

$$\begin{aligned} \mathcal{F}f(\xi) &= \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-i\xi \cdot x} dx \\ \mathcal{F}^{-1}g(x) &= \check{g}(x) := \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} g(\xi) e^{ix \cdot \xi} d\xi \end{aligned}$$

where  $n \geq 1$  is the dimension of the underlying space, i.e.  $x, \xi \in \mathbb{R}^n$ . We note that

$$\mathcal{F} \left( \frac{\partial f}{\partial x_j} \right) (\xi) = i\xi_j \mathcal{F}f(\xi)$$

Let  $\mathcal{F}_s$  denote the Fourier transform with respect to the variable  $s$ . Then

$$\begin{aligned} (\mathcal{F}_s \mathcal{R}f)(\rho, \omega) &= \int_{s=-\infty}^{\infty} \int_{x \cdot \omega = s} f(x) d\mu(x) e^{-i\rho s} ds \\ &= \int_{\mathbb{R}^3} f(x) e^{-i\rho(x \cdot \omega)} dx \\ &= \hat{f}(\rho\omega) \end{aligned}$$

Therefore

$$\mathcal{R}f(s, \omega) = \frac{1}{2\pi} \int \hat{f}(\rho\omega) e^{is\rho} d\rho$$

■

With this result, we can deduce that  $\mathcal{R}(f)$  determines uniquely  $f$ , then we can also obtain the reconstruction formula.

## 2.2 The inversion formula

**Theorem 2.** ( from [13] or [8] )

$$f(x) = \frac{-1}{8\pi^2} \int_{\omega \in S^2} \frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) d\sigma(\omega) \quad (2.2)$$

**Proof.** Since  $\rho \mapsto \hat{f}(\rho\omega)e^{is\rho}$  is in the Schwartz space we can differentiate twice the previous equality 2.1, inside the integral, to compute :

$$\begin{aligned} \int_{\omega \in S^2} \frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) d\sigma(\omega) &= \frac{1}{2\pi} \int_{\omega \in S^2} \left( \int_{\mathbb{R}} \frac{\partial^2}{\partial s^2} \hat{f}(\rho\omega) e^{is\rho} d\rho \right) d\sigma(\omega) \\ &= \frac{1}{2\pi} \int_{S^2} \left( \int_{\mathbb{R}} (i\rho)^2 \hat{f}(\rho\omega) e^{is\rho} d\rho \right) d\sigma(\omega) \\ &= \frac{-1}{2\pi} \int_{S^2} \left( \int_{\rho=-\infty}^{\infty} \rho^2 \hat{f}(\rho\omega) e^{is\rho} d\rho \right) d\sigma(\omega) \\ &= \frac{-1}{2\pi} \int_{S^2} 2 \left( \int_{\rho=0}^{\infty} \hat{f}(\rho\omega) e^{is\rho} \rho^2 d\rho \right) d\sigma(\omega) \\ &= \frac{-1}{\pi} \int_{S^2} \int_{\rho=0}^{\infty} \hat{f}(\rho\omega) e^{is\rho} \rho^2 d\rho d\sigma(\omega) \\ &= -8\pi^2 \frac{1}{(2\pi)^3} \int_{S^2} \int_{\rho=0}^{\infty} \hat{f}(\rho\omega) e^{is\rho} \rho^2 d\rho d\sigma(\omega) \end{aligned}$$

But the Radon transform is computed over the plane  $s = x \cdot \omega$ , and if we let  $S(r) \subset \mathbb{R}^3$  denote the sphere of radius  $r$  centered at the origin, and  $\sigma_r$  the Lebesgue surface measure on this sphere, then for any Lebesgue measurable  $f : S(r) \rightarrow [0, \infty)$ , we have the known result from differential geometry

$$\int_{S(r)} f(\zeta) d\sigma_r(\zeta) = \int_{S(1)} f(r\omega) r^{n-1} d\sigma_1(\omega)$$



In fact, the claim requires just to parametrize most of the sphere by polar coordinates and then using one rule about computing determinants.

By polar coordinates parametrization, we mean a smooth injective mapping  $\gamma : U \rightarrow V_1$  where  $U$  is some open set of  $\mathbb{R}^{n-1}$ , and  $V_1$  a certain open subset of the sphere  $S_1$ , such that  $\sigma_1[S_1/V_1] = 0$ . For example the unit sphere in  $\mathbb{R}^3$ , polar spherical coordinates covers the whole sphere except for half of a meridian including the two poles.

Given a parametrization  $\gamma$  of the unit sphere, we can clearly parametrize the sphere of radius  $r$  by defining  $\gamma_r(x) := r\gamma(x)$ , so letting  $V_r = \gamma_r(U)$  we have:

$$\begin{aligned}
 \int_{S_r} f(\zeta) d\sigma_r(\zeta) &= \int_{V_r} f(\zeta) d\sigma_r(\zeta) \\
 &:= \int_U f(\gamma_r(x)) \sqrt{\left| \det \left( \frac{\partial \gamma_r}{\partial x_i}(x) \cdot \frac{\partial \gamma_r}{\partial x_j}(x) \right) \right|} d\lambda_{n-1}(x) \\
 &= \int_U f(r\gamma(x)) \sqrt{\left| \det \left( \frac{\partial \gamma}{\partial x_i}(x) \cdot \frac{\partial \gamma}{\partial x_j}(x) \right) \right|} r^{n-1} d\lambda_{n-1}(x) \\
 &:= \int_{V_1} f(r\omega) r^{n-1} d\sigma_1(\omega) = \int_{S_1} f(r\omega) r^{n-1} d\sigma_1(\omega)
 \end{aligned}$$

Here,  $\lambda_{n-1}$  is the usual Lebesgue measure on the  $(n-1)$ -dimensional open subset  $U \subset \mathbb{R}^{n-1}$ .

This remark allows us to recognize the inverse Fourier transform in the expression:

$$\begin{aligned}
\frac{1}{(2\pi)^3} \int_{S^2} \int_{\rho=0}^{\infty} \hat{f}(\rho\omega) e^{is\rho} \rho^2 d\rho d\sigma(\omega) &= \frac{1}{(2\pi)^3} \int_{S^2} \int_{\rho=0}^{\infty} \hat{f}(\rho\omega) e^{ix \cdot (\rho\omega)} \rho^2 d\rho d\sigma(\omega) \\
&= \frac{1}{(2\pi)^3} \int_{\rho=0}^{\infty} \int_{S(\rho)} \hat{f}(\xi) e^{ix \cdot (\xi)} \rho^2 d\sigma(\omega) d\rho \\
&= \frac{1}{(2\pi)^3} \int_{\rho=0}^{\infty} \int_{S(\rho)} \hat{f}(\xi) e^{ix \cdot (\xi)} d\sigma_{\rho}(\xi) d\rho \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{f}(\xi) e^{ix \cdot (\xi)} d\xi \\
&= \mathcal{F}^{-1}(\mathcal{F}f(x)) = f(x)
\end{aligned}$$

As needed to conclude the result :

$$f(x) = \frac{-1}{8\pi^2} \int_{\omega \in S^2} \frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) d\sigma(\omega)$$

■

## 2.3 Nice properties of the Radon transform

Before we go to show the interesting connection between the Radon transform and the wave equation, in the next chapter, we exhibit some properties that will be helpful for computations later.

The Radon transform is easily seen to be a linear operator, in fact

$$\begin{aligned}
 \mathcal{R}(f + ag)(s, \omega) &= \int_{x \cdot \omega = s} (f + ag)(x) d\mu(x) \\
 &= \int_{x \cdot \omega = s} f(x) d\mu(x) + \int_{x \cdot \omega = s} ag(x) d\mu(x) \\
 &= \mathcal{R}f(s, \omega) + a \mathcal{R}g(s, \omega)
 \end{aligned}$$

A translation of any function in the Schwartz space is also in the space, if  $f_T$  is the translation of  $f$  by a vector  $c$ , then  $f_T(x) = f(x - c)$  for all  $x \in \mathbb{R}^3$  and we can establish by change of variables:

$$\begin{aligned}
 \mathcal{R}(f_T)(s, \omega) &= \int_{x \cdot \omega = s} f_T(x) d\mu(x) = \int_{(x-c) \cdot \omega = s - c \cdot \omega} f(x - c) d\mu(x) \\
 &= \int_{t \cdot \omega = s - c \cdot \omega} f(t) d\mu(t + c) = \int_{t \cdot \omega = s - c \cdot \omega} f(t) d\mu(t) \\
 &= \mathcal{R}(f)(s - c \cdot \omega, \omega)
 \end{aligned}$$

As we already mentioned, negative values of  $s$  are permitted, we have

$$\mathcal{R}(f)(-s, -\omega) = \mathcal{R}(f)(s, \omega)$$

But also

$$\mathcal{R}(f)(s, -\omega) = \mathcal{R}(f)(-s, \omega)$$

Moreover, we can also talk about the Radon transform of a function  $\mathbb{R}^3 \rightarrow \mathbb{R}$  and we use the same notation, here for  $\nabla g$  in particular :

$$\mathcal{R}(\nabla g)(s, \omega) = \vec{\mathcal{R}}(\nabla g)(s, \omega) = \mathcal{R}\left(\frac{\partial g}{\partial x}\right)(s, \omega)\vec{e}_x + \mathcal{R}\left(\frac{\partial g}{\partial y}\right)(s, \omega)\vec{e}_y + \mathcal{R}\left(\frac{\partial g}{\partial z}\right)(s, \omega)\vec{e}_z$$

To conclude this section, we note an important formula exploiting the spacial symmetry of the Laplacian operator :

**Theorem 3.** ( from [13] )

$$\mathcal{R}(\Delta f)(s, \omega) = \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) \quad (2.3)$$

**Proof.** We differentiate in the formula 2.1, then we use an expression of the fourier transform of  $\Delta f$  :

$$\begin{aligned} \frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) &= \frac{1}{2\pi} \frac{\partial^2}{\partial s^2} \int \hat{f}(\rho\omega) e^{is\rho} d\rho \\ &= \frac{1}{2\pi} \int \hat{f}(\rho\omega) \frac{\partial^2}{\partial s^2} e^{is\rho} d\rho = \frac{1}{2\pi} \int \hat{f}(\rho\omega) (-\rho^2) e^{is\rho} d\rho \\ &= \frac{1}{2\pi} \int \Delta \hat{f}(\rho\omega) e^{is\rho} d\rho = \mathcal{R}(\Delta f)(s, \omega) \end{aligned}$$

■

### 3 The wave equation in light of the Radon transform

Recall our main problem is to solve the wave equation on  $\mathbb{R}^3 \times \mathbb{R}$

$$\frac{1}{c^2} u_{tt} - \Delta u = 0 \tag{1.1}$$

with initial conditions

$$u(x, 0) = f(x) , \quad u_t(x, 0) = g(x) \tag{1.2}$$

where  $c > 0$  is constant, and for positive times  $t \geq 0$ .

We will see that the solution  $u$  can be expressed as a combination of plane waves indexed over the unit sphere  $S^2$ , enabling numerical computation of  $u(x, t)$  at large times without any time iterations. The key idea is to convert the initial data 1.2 to a single function on  $\mathbb{R} \times S^2$  that carries equivalent information.

A compactly supported function  $F$  may be recovered from its Radon transform, as shown previously, by the reconstruction formula

$$\begin{aligned} F(x) &= -\frac{1}{8\pi^2} \int_{S^2} \frac{\partial^2 \mathcal{R}F}{\partial s^2}(s, \omega) d\sigma(\omega) \\ &= -\frac{1}{8\pi^2} \int_{S^2} \mathcal{R}(\Delta F)(x \cdot \omega, \omega) d\sigma(\omega) \end{aligned} \tag{3.1}$$

Here  $d\sigma$  denotes the surface area measure on the unit sphere  $S^2$ .

We define first plane waves then we present the nice connection between the Radon transform and the wave equation.

### 3.1 Plane waves

Recall that the solution of the one dimensional wave equation can be expressed as the sum of two traveling waves, and it is natural to ask if an analogue of such traveling waves exists in higher dimensions. The answer is positive. Let  $F$  be a function of one variable, which we assume is sufficiently smooth (say  $C^2$ ), and consider  $u$  defined by

$$u(x, t) = F(x \cdot \omega - t) \quad \text{where } x \in \mathbb{R}^d \text{ and } \omega \text{ is a unit vector in } \mathbb{R}^d.$$

It is easy to verify directly that  $u$  is a solution of the wave equation  $\Delta u = u_{tt}$  in  $\mathbb{R}^d$ , here we let  $c = 1$ .

Such a solution is called a plane wave, indeed, notice that  $u$  is constant on every plane perpendicular to the direction  $\omega$ , and as time  $t$  increases, the wave travels in the  $\omega$  direction. (It should be remarked that plane waves are never Schwartz functions, not in  $\mathcal{S}(\mathbb{R}^d)$ , when

$d > 1$  because they are constant in directions perpendicular to  $\omega$ ). A more general plane wave can immediately be derived as in D'Alembert formula :

**Theorem 4.** ( from [13] ) Given initial data  $f$  and  $g$  of the form  $f(x) = F(x \cdot \omega)$  and  $g(x) = G(x \cdot \omega)$ , the plane wave given by

$$u(x, t) = \frac{F(x \cdot \omega + t) + F(x \cdot \omega - t)}{2} + \frac{1}{2} \int_{x \cdot \omega - t}^{x \cdot \omega + t} G(s) ds$$

is a solution of the Cauchy problem 1.1 & 1.2 for the  $d$ -dimensional wave equation and  $c = 1$ .

**Proof.** We check directly by computing derivatives.

For a fixed vector  $\omega$  :  $\frac{\partial}{\partial x_i} x \cdot \omega = \frac{\partial}{\partial x_i} \sum_j x_j \omega_j = \omega_i$  ,  $\frac{\partial^2}{\partial x_i^2} x \cdot \omega = 0$  . Then

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} (F(x \cdot \omega + t) + F(x \cdot \omega - t)) &= \frac{\partial}{\partial x_i} \left( \frac{\partial x \cdot \omega}{\partial x_i} F'(x \cdot \omega + t) + \frac{\partial x \cdot \omega}{\partial x_i} F'(x \cdot \omega - t) \right) \\ &= \left( \frac{\partial x \cdot \omega}{\partial x_i} \right)^2 F''(x \cdot \omega + t) + \left( \frac{\partial x \cdot \omega}{\partial x_i} \right)^2 F''(x \cdot \omega - t) \\ &= \omega_i^2 (F''(x \cdot \omega + t) + F''(x \cdot \omega - t)) \\ \frac{\partial^2}{\partial x_i^2} \left( \int_{x \cdot \omega - t}^{x \cdot \omega + t} G(s) ds \right) &= \frac{\partial}{\partial x_i} \left( \frac{\partial x \cdot \omega}{\partial x_i} G(x \cdot \omega + t) + \frac{\partial x \cdot \omega}{\partial x_i} G(x \cdot \omega - t) \right) \\ &= \left( \frac{\partial x \cdot \omega}{\partial x_i} \right)^2 G'(x \cdot \omega + t) + \left( \frac{\partial x \cdot \omega}{\partial x_i} \right)^2 G'(x \cdot \omega - t) \\ &= \omega_i^2 (G'(x \cdot \omega + t) + G'(x \cdot \omega - t)) \end{aligned}$$

So that  $\frac{\partial^2 u}{\partial x_i^2} = \frac{\omega_i^2}{2} (F''(x \cdot \omega + t) + F''(x \cdot \omega - t) + G'(x \cdot \omega + t) + G'(x \cdot \omega - t))$

And  $\Delta u = \sum_j \frac{\partial^2 u}{\partial x_j^2} = \frac{1}{2} (F''(x \cdot \omega + t) + F''(x \cdot \omega - t) + G'(x \cdot \omega + t) + G'(x \cdot \omega - t))$

On the other hand

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} (F(x \cdot \omega + t) + F(x \cdot \omega - t)) &= \frac{\partial}{\partial t} (F'(x \cdot \omega + t) - F'(x \cdot \omega - t)) \\
&= F''(x \cdot \omega + t) + F''(x \cdot \omega - t) \\
\frac{\partial^2}{\partial t^2} \left( \int_{x \cdot \omega - t}^{x \cdot \omega + t} G(s) ds \right) &= \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} G(x \cdot \omega + t) + \frac{\partial(-t)}{\partial t} G(x \cdot \omega - t) \right) \\
&= G'(x \cdot \omega + t) + G'(x \cdot \omega - t)
\end{aligned}$$

So that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} (F''(x \cdot \omega + t) + F''(x \cdot \omega - t) + G'(x \cdot \omega + t) + G'(x \cdot \omega - t))$$

The two expressions are equal so  $u$  satisfies  $\Delta u = \frac{\partial^2 u}{\partial t^2}$  and is a solution. ■

The basic fact is that when  $d > 1$ , the solution of the wave equation can be written as an integral (as opposed to sum, when  $d = 1$ ) of plane waves; this can be done via the Radon transform of the initial conditions  $f$  and  $g$ . This can be shown in general for any dimension, but for the case of interest  $d = 3$  the solution of the Cauchy problem 1.1 & 1.2 is given as a superposition of plane waves, and can be expressed in terms of the Radon transform by the formula ( from [3] )

$$u(x, t) = \frac{-1}{16\pi^2} \int_{S^2} \left( \mathcal{R} \Delta f(x \cdot \omega + t, \omega) + \mathcal{R} \Delta f(x \cdot \omega - t, \omega) + \int_{x \cdot \omega - t}^{x \cdot \omega + t} \mathcal{R} \Delta g(q, \omega) dq \right) d\sigma(\omega)$$

This formula appears also in a problem in [13].

But if we look closer at the terms inside the integral above, we notice that they are doubled, shouldn't we be able to exploit the fact that  $\mathcal{R}(f)(s, -\omega) = \mathcal{R}(f)(-s, \omega)$  ?



## 3.2 A more concise formula

For given compactly supported functions  $f, g$ , defining the initial conditions 1.2, it is more elegant to set

$$\begin{aligned}\alpha(s, \omega) &= -\frac{1}{2} \frac{\partial^2}{\partial s^2} \left( \mathcal{R}f(s, \omega) - \frac{1}{c} \int_{-\infty}^s \mathcal{R}g(q, \omega) dq \right) \\ &= \frac{1}{2} \left( \omega \cdot \frac{1}{c} \mathcal{R}(\nabla g)(s, \omega) - \mathcal{R}(\Delta f)(s, \omega) \right)\end{aligned}\tag{3.2}$$

For each  $\omega \in S^2$ ,  $\alpha(s, \omega)$  represents the plane wave component of  $u$  travelling in the direction  $\omega$  at initial time  $t = 0$ .

We are now ready to show the crucial use of the Radon transform, the function  $\alpha : \mathbb{R} \times S^2 \rightarrow \mathbb{R}$  encodes the pair  $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$  in a way that allows direct reconstruction of  $u(x, t)$  for any  $t$  as follows :

**Theorem 5.** ( *P.C. Gibson* ) *The general solution of the Cauchy problem 1.1 & 1.2 can be written as:*

$$u(x, t) = \frac{1}{4\pi^2} \int_{S^2} \alpha(x \cdot \omega - ct, \omega) d\sigma(\omega)\tag{3.3}$$

**Proof.** Letting  $V(s, \omega, t) = \mathcal{R}u$  denote the Radon transform of  $u(\cdot, t) : \mathbb{R}^3 \rightarrow \mathbb{R}$ , transforming (1.1) and (1.2) yields

$$\begin{aligned}\frac{1}{c^2} V_{tt} - \frac{\partial^2}{\partial s^2} V &= 0 \\ V(s, \omega, 0) &= \mathcal{R}f(s, \omega) , \quad V_t(s, \omega, 0) = \mathcal{R}g(s, \omega)\end{aligned}$$

which, for each fixed  $\omega \in S^2$ , is a one-dimensional wave equation in  $V$ . Thus  $V = \mathcal{R}u$  may be expressed in terms of  $\mathcal{R}f$  and  $\mathcal{R}g$  according to the D'Alembert formula

$$V(s, \omega, t) = \frac{1}{2} (\mathcal{R}f(s - ct, \omega) + \mathcal{R}f(s + ct, \omega)) + \frac{1}{2c} \int_{s-ct}^{s+ct} \mathcal{R}g(q, \omega) dq \quad (3.4)$$

Setting

$$\begin{aligned} A(s, \omega) &= \frac{1}{2} \left( \mathcal{R}f(s, \omega) - \frac{1}{c} \int_{-\infty}^s \mathcal{R}g(q, \omega) dq \right) \quad \text{and} \\ B(s, \omega) &= \frac{1}{2} \left( \mathcal{R}f(s, \omega) + \frac{1}{c} \int_{-\infty}^s \mathcal{R}g(q, \omega) dq \right), \end{aligned}$$

the D'Alembert formula (3.4) may be re-written as

$$V(s, \omega, t) = A(s - ct, \omega) + B(s + ct, \omega) \quad (3.5)$$

where  $A(s - ct, \omega)$  is a right-travelling wave (with respect to the spatial variable  $s$ ) and  $B(s + ct, \omega)$  is a left-travelling wave. According to the reconstruction formula (3.1) then :

$$\begin{aligned} u(x, t) &= -\frac{1}{8\pi^2} \int_{S^2} \frac{\partial^2 \mathcal{R}u}{\partial s^2}(x \cdot \omega, \omega, t) d\sigma(\omega) \\ &= -\frac{1}{8\pi^2} \int_{S^2} \frac{\partial^2 A}{\partial s^2}(x \cdot \omega - ct, \omega) + \frac{\partial^2 B}{\partial s^2}(x \cdot \omega + ct, \omega) d\sigma(\omega) \end{aligned}$$

Observe however that, by taking advantage of the symmetry of the Radon Transform :

$$\int_{S^2} \frac{\partial^2 A}{\partial s^2}(x \cdot \omega - ct, \omega) d\sigma(\omega) = \int_{S^2} \frac{\partial^2 B}{\partial s^2}(x \cdot \omega + ct, \omega) d\sigma(\omega) \quad (3.6)$$

In fact

$$F(x, y) = F(-x, -y) \quad \implies \quad \frac{\partial F}{\partial x}(-a, -b) = -\frac{\partial F}{\partial x}(a, b) \quad (3.7)$$

And by definition,

$$\begin{aligned}
\frac{\partial^2 A}{\partial s^2}(s, \omega) &= \frac{1}{2} \frac{\partial^2}{\partial s^2} \left( \mathcal{R}f(s, \omega) - \frac{1}{c} \int_{-\infty}^s \mathcal{R}g(q, \omega) dq \right) \\
&= \frac{1}{2} \left( \mathcal{R}\Delta f(s, \omega) - \frac{1}{c} \frac{\partial \mathcal{R}g}{\partial s}(s, \omega) \right) \\
&= \frac{1}{2} \left( \mathcal{R}\Delta f(\tilde{s}, \tilde{\omega}) + \frac{1}{c} \frac{\partial \mathcal{R}g}{\partial s}(\tilde{s}, \tilde{\omega}) \right) \text{ where } (\tilde{s}, \tilde{\omega}) = (-s, -\omega) \\
&= \frac{\partial^2 B}{\partial s^2}(\tilde{s}, \tilde{\omega}) .
\end{aligned}$$

Therefore

$$\frac{\partial^2 A}{\partial s^2}(x \cdot \omega - ct, \omega) = \frac{\partial^2 B}{\partial s^2}(x \cdot (-\omega) + ct, -\omega)$$

which implies (3.6). It follows in turn from (3.6) and the convenient definition (3.2) :

$$\begin{aligned}
u(x, t) &= -\frac{1}{4\pi^2} \int_{S^2} \frac{\partial^2 A}{\partial s^2}(x \cdot \omega - ct, \omega) d\sigma(\omega) \\
&= \frac{1}{4\pi^2} \int_{S^2} \alpha(x \cdot \omega - ct, \omega) d\sigma(\omega)
\end{aligned}$$

completing the proof. ■

This theorem expresses  $u(x, t)$  as the average value (over all the directions of travel) of the plane wave components of  $u$  passing through  $x$  at time  $t$ . Once  $\alpha$  is known, just a single integral over a compact set needs to be computed in order to recover  $u(x, t)$ . If such a numerical scheme is developed, unlike standard numerical methods, the amount of computation required would be independent of the time  $t$ , since there would be no time iterations or time-stepping involved.

## 4 Numerical implementation

### 4.1 Partitioning the sphere

In order to exploit the idea of the previous chapter to solve the 3D wave equation, we need to possess a tool to compute integrals over  $S^2$ , while many numerical methods might be available, we developed a precise, but also an economical method for the particular integration

$$\int_{S^2} f(\omega) d\sigma(\omega) , \quad \text{where } d\sigma \text{ is the Lebesgue measure on } S^2$$

Often the precise form of  $f$  is either too complicated to integrate analytically or is not known explicitly and can only be sampled at individual points on  $S^2$ . One of the most common ways to arrive at such approximations is by use of a numerical scheme, or quadrature, where we approximate the integral by a weighted sum over a finite collection of points  $x_i \in S^2$

$$\int_{S^2} f(\omega) d\sigma(\omega) = \sum_{i=0}^N w_i f(x_i) := Q[f]$$

where  $w_i$  denote the weights and  $Q[f]$  is called the quadrature of the exact integral.

The theory of quadratures for one-dimensional integrals has a long history and many results can be found in literature [2]. Some of this theory can be extended to the case of integrals over  $S^2$ , a noteworthy difference however is that the distribution of points  $\{x_i\}$  over the

surface of the unit sphere is a non-trivial problem on its own.

It is worth mentioning that the method of spherical harmonics is a common way to compute integrals of square-integrable functions on the unit sphere, as these functions can be expanded in terms of the spherical harmonics orthonormal basis on this sphere. This method requires determining both weights and nodes and is then classified as a Gauss quadrature method.

We will not use any of the already existing quadrature schemes !

Instead, we will proceed to develop our own Chebyshev quadrature, that is, a quadrature where the weights are all equal.

We will witness the power of this quadrature when we apply it to integrate wave components and solve the 3-D wave equation.

The problem of quadrature is geometric in nature, we need to partition the sphere into regions of small areas, regions relatively having the same shape as each other, in the sense that normal vectors centered at each region will be equally spread over the surface of the sphere, ideally we would want regions of equal area, we are looking for a regular 'tiling' of the sphere.

Luckily, this problem has been solved, a paper from 2006 [5] describes exactly an algorithm to do that. Based on *Zhou's* construction for  $S^2$  [14], as modified by *Saff* [10], and on *Sloan's* notes on the partition of  $S^3$  [11] ( references as they appear in the main paper [5] ). This article describes a practical algorithm for partitioning higher dimensional spheres ( $S^2, S^3, \dots$ ) into regions of equal areas and small diameter, *Paul Leopardi*, the author, calls it the 'recursive zonal equal area sphere partitioning algorithm', but we will call it **Leopardi's algorithm** going forward.

We need to settle some definitions first ;

**Definition 1.** ( from [5] ) An equal area partition of  $S^d$  is a nonempty finite set  $P$  of regions, which are closed Lebesgue measurable subsets of  $S^d$  such that

1. the regions cover  $S^d$ , that is  $\bigcup_{R \in P} R = S^d$ .
2. the regions have equal area, with the Lebesgue area measure  $\sigma$  of each  $R$  being  $\sigma(R) = \frac{\sigma(S^d)}{|P|} = \frac{4\pi}{|P|}$ , where  $|P|$  denotes the cardinality of the partition  $P$ .
3. the boundary of each region has area measure zero, that is, for each  $R \in P$ :  $\sigma(\partial R) = 0$ .

Note that this definition implies that the intersection of any two regions of  $P$  has measure zero. This in turn implies that any two regions are either disjoint or only have boundary points in common. Condition 3 excludes pathological cases.

**Definition 2.** ( from [5] ) The diameter of a region  $R \subset S^d$  is, as one could expect,  $\text{Diam}(R) = \sup\{\|x - y\| \mid x, y \in R\}$ , where the distance is in the Euclidean space  $\mathbb{R}^{d+1}$ .

It is denoted by  $EQ(d, N)$  the recursive zonal equal area partition of the unit sphere  $S^d$  into  $N$  regions.

The proof of the following theorem can be found in detail in the companion paper [6].

**Theorem 6.** ( from [5] ) For  $d \geq 1$ ,  $N \geq 1$ , the partition  $EQ(d, N)$  is an equal area partition of  $S^d$ .

Moreover, this partition is diameter-bounded in the sense that there exists a  $k \in \mathbb{R}_+$  (called a diameter bound) for which for all  $R \in P$ :

$$\text{Diam}(R) \leq \frac{k}{|P|^{1/d}}$$

It is important to underline here that the last statement is important to us, since the diameter of the regions is bounded, we are sure to have regions of the same shape relatively, so this

sphere partition will yield points evenly spread over the sphere.

We describe now the algorithm briefly, for the specific case of  $S^2$  :

### 4.1.1 Leopardi's algorithm

For each natural number  $N$ , the algorithm outputs  $EQ(2, N)$  which is a partition of the unit sphere  $S^2$  into  $N$  regions of equal area  $A(R) = \frac{4\pi}{N}$

**1-** Determine the colatitude of polar caps :

$\theta_c = \Theta(A(R))$  where  $\Theta = V^{-1}$  and  $V(\theta) = 2\pi \int_0^\theta \sin x \, dx = 2\pi(1 - \cos \theta)$  , giving

$$\theta_c = \arccos \left( 1 - \frac{2}{N} \right)$$

**2-** Determine the ideal collar angle :

$$\delta_I = A(R)^{1/2} = 2\sqrt{\frac{\pi}{N}}$$

**3-** Determine the ideal number of collars :

$$n_I = \frac{\pi - 2\theta_c}{\delta_I}$$

**4-** Determine the actual number of collars :

We use a rounding procedure to obtain an integer close to the ideal number of collars. Here  $N > 2$  ,

$$n = \max(1, \text{round}(n_I)) \quad \text{where} \quad \text{round}(x) = \lfloor x + 0.5 \rfloor$$

**5-** Create a list of the ideal number of regions in each collar :

We number the zones southward, from 1 for the North polar cap to  $n + 2$  for the South polar cap, and number the collars so that collar  $i$  is zone  $i + 1$  . The 'fitting' collar angle is

$$\delta_F = \frac{\pi - 2\theta_c}{n}$$

We use  $\delta_F$  to produce an increasing list of 'fitting' colatitudes of caps, defined by

$$\theta_{F,i} = \theta_c + (i - 1)\delta_F , \quad i = 1, \dots, n + 1$$

The area of each corresponding 'fitting' collar is given by successive colatitudes in this list.

The ideal number of regions,  $y_i$  , in each collar  $i \in \{1, 2, \dots, n\}$ , is then

$$y_i = \frac{V(\theta_{F,i+1}) - V(\theta_{F,i})}{A(R)}$$

**6-** Create a list of the actual number of regions in each collar :

We use a rounding procedure with  $n$  the number of collars, we define  $m_i$ , the required number of regions in collar  $i$  as follows. Define the sequences  $a_i$  and  $m_i$  by starting with  $a_0 = 0$  and

$$a_i = \sum_{j=1}^i y_j - m_j , \quad m_i = \text{round}(y_i + a_{i-1})$$

**7-** Create a list of colatitudes of each zone :

We now define  $\theta_0 = 0$  and  $\theta_{n+2} = \pi$  and  $\theta_i = \Theta \left( \left( 1 + \sum_{j=1}^{i-1} m_j \right) A(R) \right)$  for  $i \in \{1, 2, \dots, n + 1\}$

We then define zone  $i + 1$  to be the spherical sector  $Z(\theta_i, \theta_{i+1})$  , and of course for

$i \in \{1, 2, \dots, n\}$  we define collar  $i$  to be zone  $i + 1$  .



**8-** Partition each spherical collar into regions of equal area :

Using our algorithm for the circle  $S^1$ , we partition collar  $i$  into  $m_i$  regions, each corresponding to a region of the partition  $EQ(1, m_i)$ , we assume that each region of  $EQ(1, m_i)$  is rectilinear in spherical polar coordinates.

**End**

Notice how this last step is a direct computation, as we know how to split a circle ( $S^1$ ) into equal length arcs, so we are just doing one recursion.

We can divide the sphere for example into 50 regions of equal area :

50 equally distributed points on the sphere

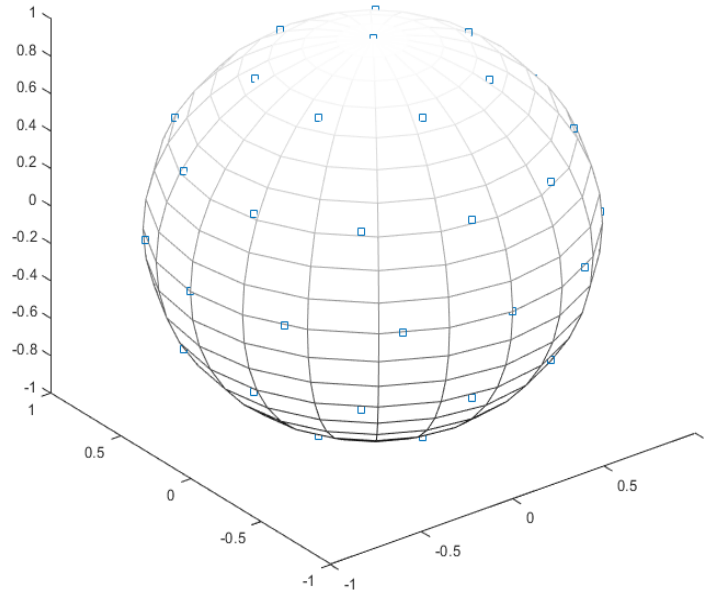


FIGURE 4.1: The points are exactly in the centers of the regions given by Leopardi's algorithm

Now we just need to adapt this algorithm to our problem, we are only interested in computing points that are the centers of each region in the partition  $EQ(2, N)$ . Then we can assign a normal unit vector to each of the obtained points. We will see that our method turns out to be much simpler, since for each collar, all of the central points lie in the same latitude.

### 4.1.2 ESP algorithm (short for Equally Spread Points)

For each given  $N > 20$  (we are only interested in fine partitions), the ESP algorithm outputs the  $N$  unit vectors, that are given by  $\omega_i = (\theta_i, \phi_i)$  in spherical coordinates, where  $\theta$  represents the colatitude angle and  $\phi$  the azimuthal angle.

**1-**  $\omega_1 = e_z$  ,  $\theta_1 = 0$  and  $\omega_N = -e_z$ ,  $\theta_N = \pi$

Then compute  $\theta_c = \arccos\left(1 - \frac{2}{N}\right)$  ,  $\delta_I = A(R)^{1/2} = 2\sqrt{\frac{\pi}{N}}$

**2-** Determine the number of collars :  $n_I = \frac{\pi - 2\theta_c}{\delta_I}$  ,  $n = \lfloor n_I + 0.5 \rfloor$

**3-** Create a list of the actual number of regions in each collar :

The 'fitting' collar angle is  $\delta_F = \frac{\pi - 2\theta_c}{n}$  then

$$\theta_{F,i} = \theta_c + (i - 1)\delta_F , \quad i = 1, \dots, n + 1$$

The ideal number of regions,  $y_i$ , in each collar  $i \in \{1, 2, \dots, n\}$  :

$$y_i = \frac{V(\theta_{F,i+1}) - V(\theta_{F,i})}{A(R)} = \frac{N}{2}(\cos \theta_{F,i} - \cos \theta_{F,i+1})$$

The required number of regions in collar  $i$  is then  $m_i$  computed using the recursion method with  $a_0 = 0$  and

$$a_i = \sum_{j=1}^i y_j - m_j, \quad m_i = \text{round}(y_i + a_{i-1})$$

**4-** Create a list of colatitudes of each zone :

We now define  $\alpha_0 = 0$  and  $\alpha_{n+2} = \pi$  and for  $i \in \{1, 2, \dots, n+1\}$

$$\alpha_i = \Theta \left( \frac{4\pi}{N} \left( 1 + \sum_{j=1}^{i-1} m_j \right) \right) = \arccos \left( 1 - \frac{2}{N} \left( 1 + \sum_{j=1}^{i-1} m_j \right) \right)$$

At this point, ordering the  $\omega_i$ 's from North to South:

$\omega_2, \omega_3, \dots, \omega_{m_1+1}$  will have all the same  $\theta_2 = \theta_3 = \dots = \theta_{m_1+1} = \frac{\alpha_1 + \alpha_2}{2}$

The next  $m_2$  vectors will have  $\theta_{m_1+2} = \theta_{m_1+3} = \dots = \theta_{m_1+m_2+1} = \frac{\alpha_2 + \alpha_3}{2}$

... continue

Until finally the last  $m_n$  vectors will have  $\theta_{\sum_{j=1}^{n-1} m_j+2} = \theta_{\sum_{j=1}^{n-1} m_j+3} = \dots = \theta_{N-1} = \frac{\alpha_n + \alpha_{n+1}}{2}$

**5-** Partition the directions on each spherical collar :

We partition collar  $i$  into  $m_i$  regions, we assume that for large  $N$  it is not relevant to care about the positioning of each direction in relation with the position in the neighbouring collars.

So each direction  $\omega_2, \omega_3, \dots, \omega_{m_1+1}$  will have  $\phi_2 = 0, \phi_3 = \frac{2\pi}{m_1}, \dots, \phi_{m_1+1} = \frac{2\pi(m_1-1)}{m_1}$

The next  $m_2$  vectors will have  $\phi_{m_1+2} = 0, \phi_{m_1+3} = \frac{2\pi}{m_2}, \dots, \phi_{m_1+m_2+1} = \frac{2\pi(m_2-1)}{m_2}$

... continue

Until finally the last  $m_n$  vectors will have  $\phi_{\sum_{j=1}^{n-1} m_j+2} = 0, \phi_{\sum_{j=1}^{n-1} m_j+3} = \frac{2\pi}{m_n}, \dots$

and  $\phi_{N-1} = \frac{2\pi(m_n-1)}{m_n}$

**End**

We illustrate the method with the the following figure.

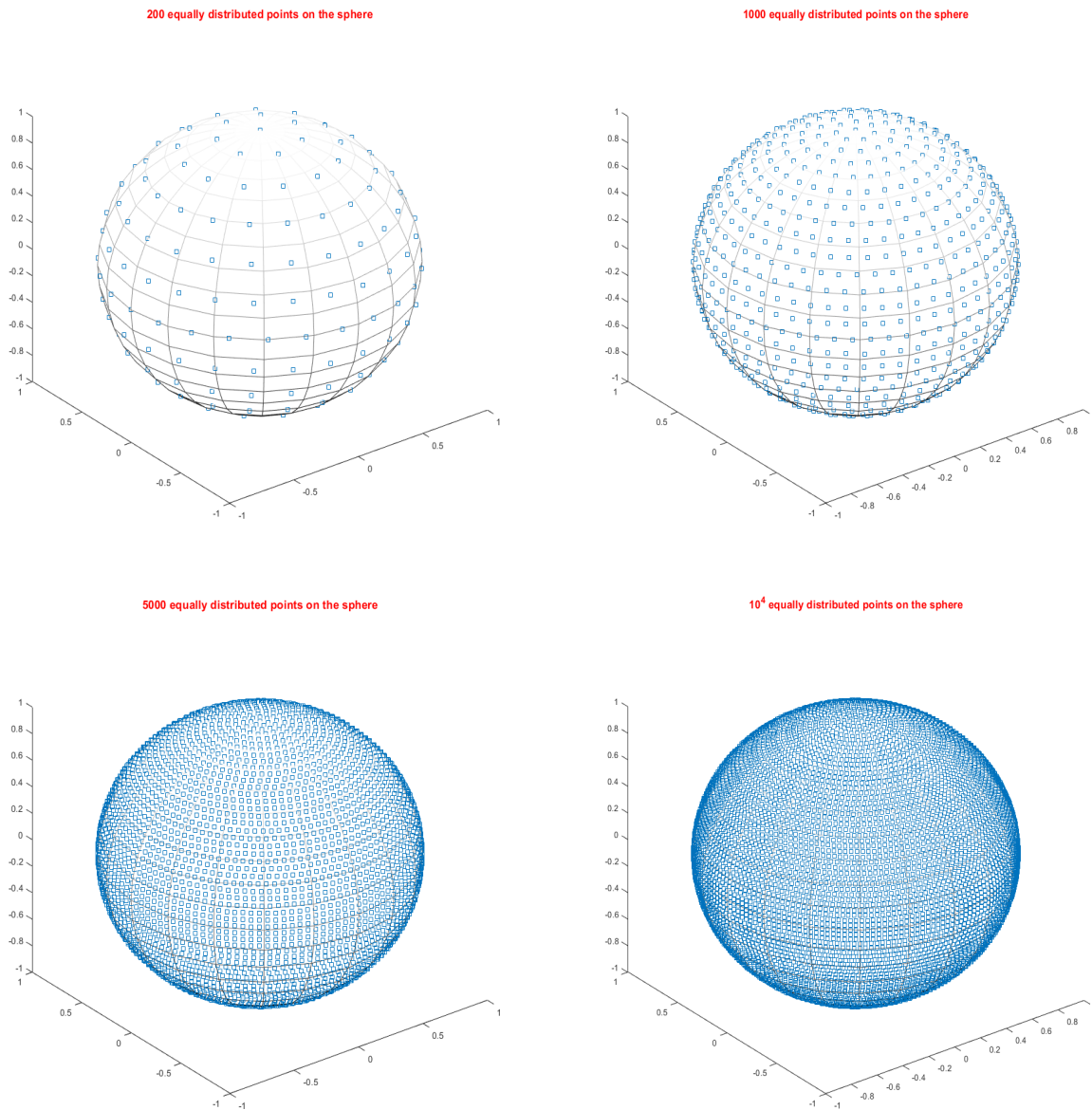


FIGURE 4.2: Plots of 200, 1000, 5000 and 10 000 points

It is clear that this algorithm has complexity  $\mathcal{O}(N)$ , returning to our problem, given  $N$  normal unit vectors, covering equally the directions of space, we can now compute our integral via the Riemann sum

$$\int_{S^2} f(w) d\sigma(w) = \lim_{N \rightarrow \infty} \frac{4\pi}{N} \sum_{i=1}^N f(\omega_i)$$

In fact, this regular 'tiling' simplify computation immensely in comparison with other quadrature methods, that is because all the weights are equal, the area of each region  $A = \frac{4\pi}{N}$  is constant, thus pulled out of the summation to be done by the machine. Moreover, this method can be used for various types of functions, however their variations can be, even for highly variant functions, one can take a finer distribution of points on the sphere and get a good approximation.

Thanks to Leopardi's algorithm, we can exploit this regular 'tiling' to compute many integrals of the sort. Particularly, we are interested in computing the integral of the plane wave component to solve the wave equation.

$10^4$  equally distributed points on the sphere

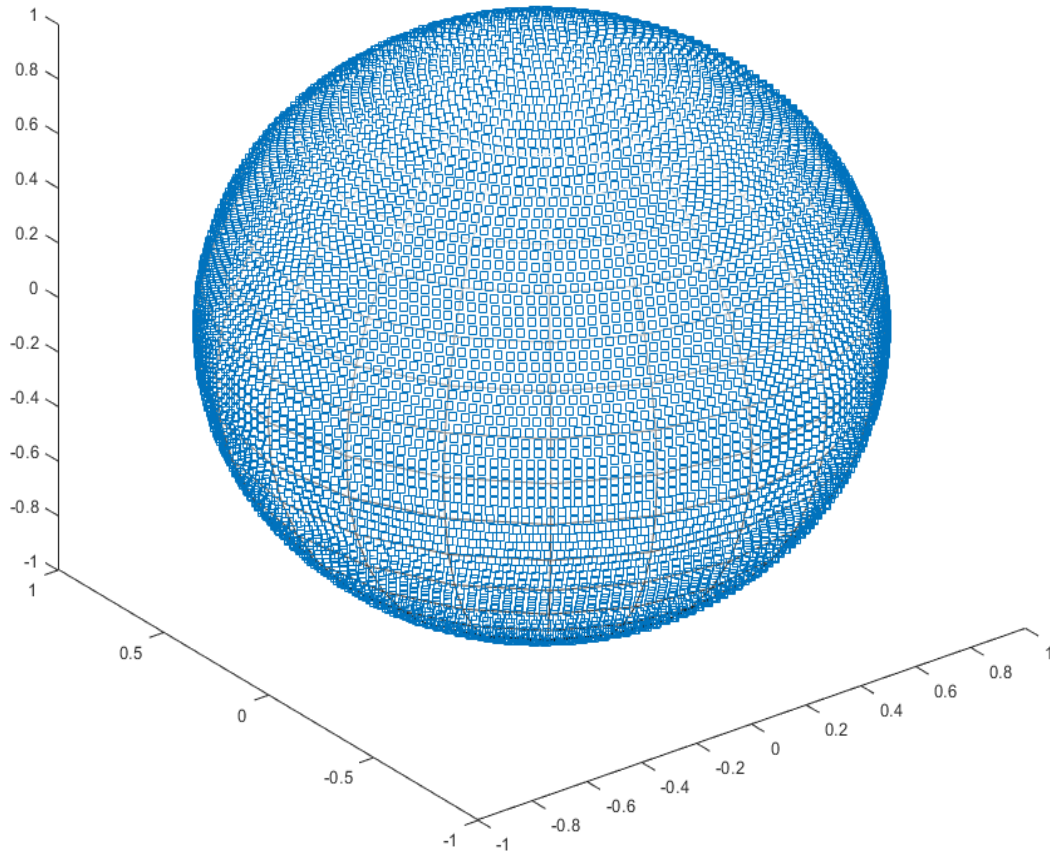


FIGURE 4.3: 10 000 points are already challenging the resolution of the screen

## 4.2 A new scheme to solve the 3D wave equation

We established that if we consider the plane wave component of  $u$  For each  $\omega \in S^2$ ,

$$\alpha(s, \omega) = \frac{1}{2} \left( \omega \cdot \frac{1}{c} \mathcal{R}(\nabla g)(s, \omega) - \mathcal{R}(\Delta f)(s, \omega) \right) \quad (3.2)$$

we can find solutions by direct reconstruction of  $u(x, t)$  for any time  $t$  from the formula

$$u(x, t) = \frac{1}{4\pi^2} \int_{S^2} \alpha(x \cdot \omega - ct, \omega) d\sigma(\omega) \quad (3.3)$$

We will use the ESP algorithm to compute precise approximations to this integral.

Clearly by Riemann integration, we have the convergence of the sequence :

$$\sum_{i=1}^N \alpha(x \cdot \omega_i - ct, \omega_i) A(\omega_i) \xrightarrow{N \rightarrow \infty} \int_{S^2} \alpha(x \cdot \omega - ct, \omega) d\sigma(\omega)$$

Where the  $\omega_i$ 's are equally distributed over  $S^2$  and each one is at the center of a region of the same area  $A(\omega_i) = \frac{4\pi}{N}$ .

Then we obtain a formula for the solution  $u$  :

$$\frac{1}{N\pi} \sum_{i=1}^N \alpha(x \cdot \omega_i - ct, \omega_i) \xrightarrow{N \rightarrow \infty} u(x, t)$$

or, for a sufficiently large  $N$  we have the approximation

$$u(x, t) \approx \frac{1}{N\pi} \sum_{i=1}^N \alpha(x \cdot \omega_i - ct, \omega_i)$$

We can then implement this in any computational software, of course we can fix the approximation error and tune accordingly. The algorithm is simple, for the sake of precise exposition, we present an implementation in Matlab.

### 4.3 Implementation in Matlab

We first compute (manually) the Radon transforms of the initial conditions, for example, polynomials are particularly easy to compute, as we will see in the next chapter.

Given initial conditions  $f, g$ , they determine a unique example to be tested, we will name our test cases (or examples) and we can show here how the implementation goes for an example  $A$ .

#### Preparation

Depending on computation power, one can start by storing the units vectors, at once, in a file, before using them, (or including them in a library or a package that should be called before running the scheme). We will store them in a standard Matlab table which we save under a '.mat' file.

- We use the method 'ESP(N)' that outputs the list of unit vectors  $\omega_i$ 's for each  $N$  ranging from 400, 800, ... to 205200, by iterating the steps described in the ESP algorithm.
- Store the unit vectors in a Matlab table, corresponding to each  $N$  in the range from 400, 800, 1200, ... to 205200 (by steps of 400).

#### Input

- The wave component  $\alpha\text{-}A(s, \omega)$  as a Matlab function
- The wave speed  $c$  as a scalar



- An approximation error, or tolerance  $\epsilon > 0$
- The point in space  $X = (x_1, x_2, x_3)$ , and time  $t$

**Processing**

Try  $u(X, t) = \frac{1}{N\pi} \sum_{i=1}^N \alpha(X \cdot \omega_i - ct, \omega_i)$  (the vectors  $\omega_i$  are now available in the file), increment  $N$  by 500 while the difference between two successive approximations is still bigger than  $\epsilon$ .

**Output**

The solver, which is the Matlab function, `radonSolver-A( $\epsilon, c, X, t$ )`, is a method that returns the value of  $u$  at time  $t$  and position  $X$ , which is  $\epsilon$  close to the exact solution, i.e.

$$|u_{exact}(X, t) - u(X, t)| < \epsilon$$

---

### Our implementation of the new scheme in Matlab

---

*Input*      The wave component  $\alpha\text{-A}(s, \omega)$  as a Matlab function

The wave speed  $c$  as a scalar

An approximation error, or tolerance  $\epsilon > 0$

The point in space  $X = (x_1, x_2, x_3)$ , and time  $t$

*Step 1:*      Store the units vectors in a file that should be 'loaded' before running the script.

*Step 2:*      initialize with  $N = 1200$  and  $\text{value1} = \text{value} = \frac{1}{N\pi} \sum_{i=1}^N \alpha(X \cdot \omega_i - ct, \omega_i)$   
 and a second value with  $N = 2000$  and  $\text{value2} = f = \frac{1}{N\pi} \sum_{i=1}^N \alpha(X \cdot \omega_i - ct, \omega_i)$ .

Looping: While ( $|\text{value2} - \text{value1}| > \epsilon$ ) do {

$\text{value1} \leftarrow \text{value2}$ ,  $N = N + 1600$ ,  $\text{value2} \leftarrow \frac{1}{N\pi} \sum_{i=1}^N \alpha(X \cdot \omega_i - ct, \omega_i)$

return  $\text{value2}$  }.

*Output*      The desired value of  $u(X, t)$

---

## 5 Testing the new scheme

Now that the numerical scheme is settled, we should dispose of some exact data against which we can test it, we will be able to verify that the algorithm works correctly (though we already proved it works) when compared to 'ground truth' solutions.

We can invoke known analytic solutions to the wave equation for simple Cauchy problems, which we develop in the following section.

### 5.1 Exact solutions to some particular wave equations

**Theorem 7.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a real function, and denote  $x \in \mathbb{R}^3$ ,  $r = |x|$ , if  $C : \mathbb{R} \rightarrow \mathbb{R}$  is a real function then  $u(x, t) = C(r)F(r - ct)$  is a solution to the wave equation [1.1](#) if and only if  $C(r) = \frac{a}{r}$  for some real number  $a$*

**Proof.** We note that  $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$ ,  $\frac{\partial^2 r}{\partial x_j^2} = \frac{r^2 - x_j^2}{r^3}$  and for  $u(x, t) = C(r)F(r - ct)$  :

$$\begin{aligned} \frac{\partial^2 u}{\partial x_j^2} &= F(r - ct) \frac{\partial^2 C(r)}{\partial x_j^2} + C(r) \frac{\partial^2 F(r - ct)}{\partial x_j^2} + 2 \frac{\partial C(r)}{\partial x_j} \frac{\partial F(r - ct)}{\partial x_j} \\ &= F(r - ct) \frac{\partial}{\partial x_j} \left( \frac{\partial r}{\partial x_j} C'(r) \right) + C(r) \frac{\partial}{\partial x_j} \left( \frac{\partial r}{\partial x_j} F'(r - ct) \right) + 2 \frac{\partial r}{\partial x_j} C'(r) \frac{\partial r}{\partial x_j} F'(r - ct) \end{aligned}$$

$$\begin{aligned}
&= F(r - ct) \left( \frac{\partial^2 r}{\partial x_j^2} C'(r) + \left( \frac{\partial r}{\partial x_j} \right)^2 C''(r) \right) + C(r) \left( \frac{\partial^2 r}{\partial x_j^2} F'(r - ct) + \left( \frac{\partial r}{\partial x_j} \right)^2 F''(r - ct) \right) \\
&\quad + 2 \left( \frac{\partial r}{\partial x_j} \right)^2 C'(r) F'(r - ct) \\
&= F(r - ct) \left( \frac{r^2 - x_j^2}{r^3} C'(r) + \left( \frac{x_j}{r} \right)^2 C''(r) \right) + C(r) \left( \frac{r^2 - x_j^2}{r^3} F'(r - ct) + \left( \frac{x_j}{r} \right)^2 F''(r - ct) \right) \\
&\quad + 2 \left( \frac{x_j}{r} \right)^2 C'(r) F'(r - ct) \\
&= F(r - ct) \left( \frac{r^2 - x_j^2}{r^3} C'(r) + \frac{x_j^2}{r^2} C''(r) \right) + F'(r - ct) \left( C(r) \frac{r^2 - x_j^2}{r^3} + 2 \frac{x_j^2}{r^2} C'(r) \right) \\
&\quad + F''(r - ct) \frac{x_j^2}{r^2} C(r)
\end{aligned}$$

Then we can have the Laplacian of  $u$ , noting that  $\sum_{j=1}^3 \frac{x_j^2}{r^2} = 1$  and  $\sum_{j=1}^3 \frac{r^2 - x_j^2}{r^3} = \frac{2}{r}$  :

$$\begin{aligned}
\Delta u &= \sum_{j=1}^3 \frac{\partial^2 u}{\partial x_j^2} = F(r - ct) \left( C'(r) \sum_{j=1}^3 \frac{r^2 - x_j^2}{r^3} + C''(r) \sum_{j=1}^3 \frac{x_j^2}{r^2} \right) \\
&\quad + F'(r - ct) \left( c(r) \sum_{j=1}^3 \frac{r^2 - x_j^2}{r^3} + 2C'(r) \sum_{j=1}^3 \frac{x_j^2}{r^2} \right) + F''(r - ct) C(r) \sum_{j=1}^3 \frac{x_j^2}{r^2} \\
&= F(r - ct) \left( c'(r) \frac{2}{r} + C''(r) \right) + F'(r - ct) \left( c(r) \frac{2}{r} + 2C'(r) \right) + F''(r - ct) C(r)
\end{aligned}$$

On the other hand we have  $\frac{\partial^2 u}{\partial t^2} = c^2 C(r) F''(r - ct)$ , so for  $u$  to be a solution we need

$$C(r) F''(r - ct) = F(r - ct) \left( C'(r) \frac{2}{r} + C''(r) \right) + F'(r - ct) \left( C(r) \frac{2}{r} + 2C'(r) \right) + F''(r - ct) C(r)$$

That is

$$\begin{aligned}
C'(r) \frac{2}{r} + C''(r) &= 0 \\
C(r) \frac{2}{r} + 2C'(r) &= 0
\end{aligned}$$

We can solve either of the two differential equation to find that all  $C(r) = \frac{a}{r}$  for  $a \in \mathbb{R}$  are solutions of the two conditions combined. ■

### Expressing $F$ in term of the initial conditions of our Cauchy problem

For a non trivial solution of the wave equation,  $a \neq 0$ , if  $u$  satisfies as well the two initial conditions 1.2 then :

$$\begin{aligned} u(x, 0) = \frac{a}{r} F(r) = f &\implies F(r) = \frac{r}{a} f \\ u_t(x, 0) = -c \frac{a}{r} F'(r) = g &\implies F'(r) = \frac{-r}{ac} g \end{aligned}$$

Which yields that  $f$  and  $g$  depend solely on  $r$  with :

$$g(r) = \frac{-ac}{r} F'(r) = \frac{-ac}{r} \left( \frac{r}{a} f(r) \right)' = \frac{-ac}{r} \left( \frac{r}{a} f'(r) + \frac{1}{a} f(r) \right) = -c(f'(r) + \frac{1}{r} f(r))$$

So we need necessarily that  $f, g$  must obey the following conditions :

$$f(r) = \frac{a}{r} F(r) \quad , \quad g(r) = -c \left( f'(r) + \frac{1}{r} f(r) \right)$$

We can then come up with some nice examples of 3 dimensional waves propagating only 'radially outward', we start by compactly supported piece-wise polynomials' cases before we show that we can also compute some nice formulas for non compactly supported functions as well, namely  $e^{-r^2}$  and  $e^{-r}$  as initial states.

In each of the following, first we find an expression of the exact solution, then we proceed to compute

$$\omega \cdot \mathcal{R}(\nabla g)(s, \omega) = \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) \quad \text{and} \quad \mathcal{R}(\Delta f)(s, \omega) = \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega)$$

So we can retrieve the expression of  $\alpha(s, \omega)$  to be put in our numerical scheme :

$$\begin{aligned}\alpha(s, \omega) &= \frac{1}{2} \left( \omega \cdot \frac{1}{c} \mathcal{R}(\nabla g)(s, \omega) - \mathcal{R}(\Delta f)(s, \omega) \right) \\ &= \frac{1}{2c} \left( \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) - c \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) \right)\end{aligned}$$

As we will see, we apply some calculus, and we take advantage of the radial nature of the examples.

## 5.2 Generating the first experiments

The motivation for piece-wise polynomials comes from the following two examples that we first experimented with, each represent a pair of initial conditions, we note that these examples doesn't need  $u(x, 0)$  to be  $C^2$ , nor  $u_t(x, 0)$  to be  $C^1$ , but they still work! We found the data to be exactly matching the exact solution obtained analytically. We hint to a potential generalization of our scheme to include discontinuous cases, distribution theory is a natural way to go. But for now, let us draw some plots :

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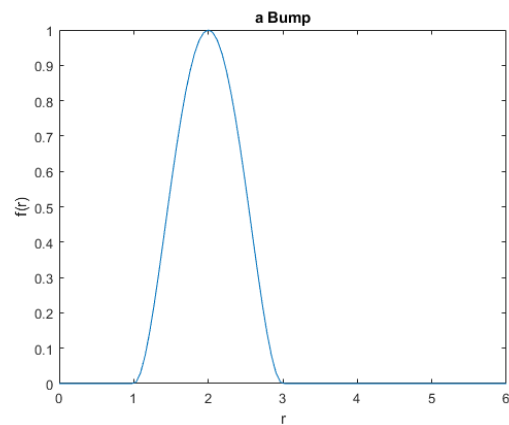
### Starting examples

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Example B  
Bump

$$(r-1)^2(r-3)^2$$

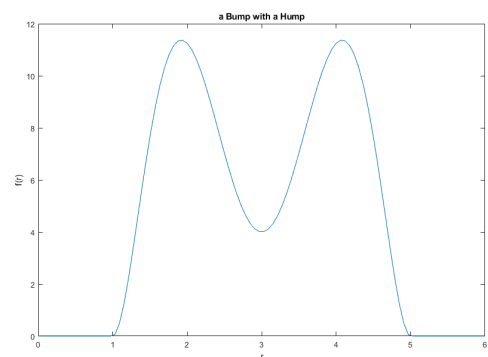
for  $1 \leq r \leq 3$



Example BH  
Bump with a Hump

$$(r-1)^2(r-5)^2((r-3)^2 + 0.25)$$

for  $1 \leq r \leq 5$



### Example B, (B for Bump)

Take the function  $f$  that is differentiable for all  $r > 0$  :

$$f(r) = \begin{cases} (r-1)^2(r-3)^2 = r^4 - 8r^3 + 22r^2 - 24r + 9 & , \text{ if } 1 \leq r \leq 3 \\ 0 & , \text{ otherwise} \end{cases}$$

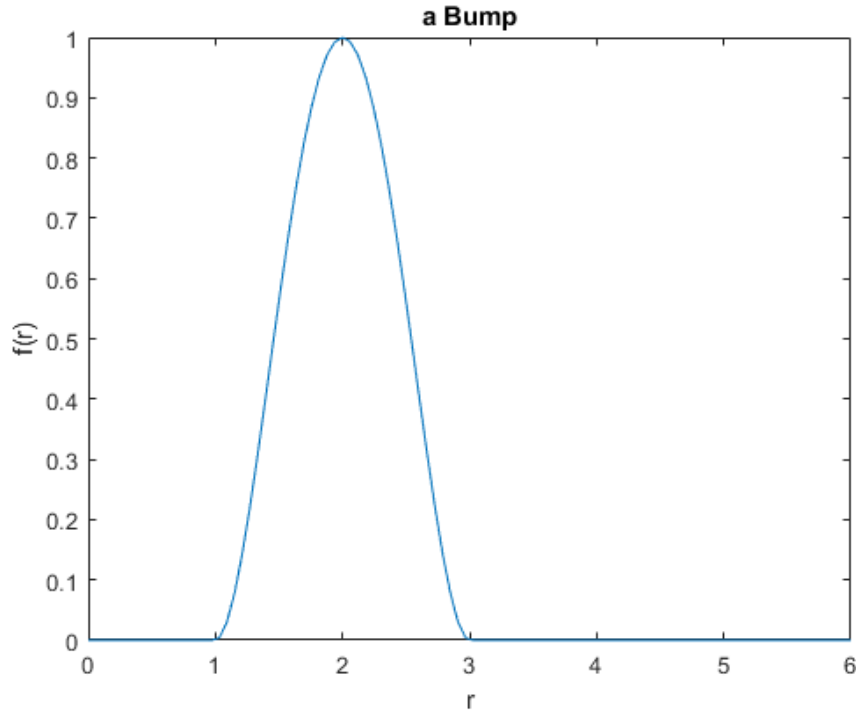


FIGURE 5.1: Plot of the Bump

For  $1 \leq r \leq 3$ :  $f'(r) = 4r^3 - 24r^2 + 44r - 24$  and

$$\begin{aligned} g(r) &= -c(4r^3 - 24r^2 + 44r - 24 + \frac{1}{r}(r^4 - 8r^3 + 22r^2 - 24r + 9)) \\ &= -c(5r^3 - 32r^2 + 66r - 48 + \frac{9}{r}) \end{aligned}$$

In this case

$$F(r) = \begin{cases} \frac{r}{a}(r-1)^2(r-3)^2 & , \text{if } 1 \leq r \leq 3 \\ 0, & , \text{otherwise} \end{cases}$$



and we can work with the exact analytic solution :

$$u(x, t) = \begin{cases} \frac{r-ct}{r}(r-ct-1)^2(r-ct-3)^2 & , \text{ if } 1 \leq r-ct \leq 3 \\ 0 & , \text{ otherwise} \end{cases}$$

Now to compute the wave component: We can start by computing first for  $s \geq 0$ , since  $f, g$  are symmetrical radially, we can immediately retrieve the values of negative  $s$  from  $\mathcal{R}(f)(-s, \omega) = \mathcal{R}(f)(s, -\omega) = \mathcal{R}(f)(s, \omega)$  .

Any plane distant by  $s \leq 1$  from the origin and normal to the radial vector  $\omega$ , intersect the density function  $f$  on a circular sector of an internal radius  $\sqrt{1-s^2}$  and external radius  $\sqrt{9-s^2}$ , any such plane distant by more than or equal to 3 from the origin gives a Radon transform value of 0, so we know that  $\mathcal{R}(f)$  vanishes for all  $s \geq 3$ , and for  $s \leq 1$  it evaluates to :

$$\begin{aligned} \mathcal{R}(f)(s, \omega) &= \int_C f(r) d\mu = 2\pi \int_{\sqrt{1-s^2}}^{\sqrt{9-s^2}} r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\ &\quad \text{(where } C \text{ is the circular sector of intersection)} \\ &= 2\pi \int_1^3 r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\ &= 2\pi \left[ \frac{r^6}{6} - \frac{8r^5}{5} + \frac{22r^4}{4} - \frac{24r^3}{3} + \frac{9r^2}{2} \right]_1^3 \\ &= \frac{64}{15}\pi \end{aligned}$$

then for  $1 < s < 3$  :

$$\begin{aligned}
\mathcal{R}(f)(s, \omega) &= \int_D f(r) d\mu = 2\pi \int_0^{\sqrt{9-s^2}} r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
&\quad \text{(where } D \text{ is the disk of intersection)} \\
&= 2\pi \int_{|s|}^3 r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= 2\pi \left[ \frac{r^6}{6} - \frac{8r^5}{5} + \frac{22r^4}{4} - \frac{24r^3}{3} + \frac{9r^2}{2} \right]_s^3 \\
&= \pi \left( 782,992,224 - \frac{s^6}{3} + \frac{16s^5}{5} - 11s^4 + \frac{48s^3}{3} - 9s^2 \right)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= \pi (-10s^4 + 64s^3 - 132s^2 + 96s - 18) \quad \text{for } 1 < s < 3 \\
\frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(-s, \omega) = \frac{\partial^2}{\partial s^2} \pi \left( 782,992,224 - \frac{s^6}{3} - \frac{16s^5}{5} - 11s^4 - \frac{48s^3}{3} - 9s^2 \right) \\
&= \pi (-10s^4 - 64s^3 - 132s^2 - 96s - 18) \quad \text{for } -3 < s < -1 \\
\text{and } \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= 0 \quad \text{for } |s| \notin [1, 3]
\end{aligned}$$

On the other hand we have for  $g$  ;  $\mathcal{R}(g)$  vanishes for all  $s \geq 3$  and for  $s \leq 1$  it evaluates to :

$$\begin{aligned}
\mathcal{R}(g)(s, \omega) &= \int_C g(r) d\mu = 2\pi \int_{\sqrt{1-s^2}}^{\sqrt{9-s^2}} r_1 g\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
&\quad \text{(where } C \text{ is the circular sector of intersection)} \\
&= 2\pi \int_1^3 r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= -2c\pi [r^5 - 8r^4 + 22r^3 - 24r^2 + 9r]_1^3 = 0
\end{aligned}$$

and for  $1 < s < 3$  :

$$\begin{aligned}
\mathcal{R}(g)(s, \omega) &= \int_D g(r) d\mu = 2\pi \int_0^{\sqrt{9-s^2}} r_1 g\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
&\quad (\text{where } D \text{ is the disk of intersection}) \\
&= 2\pi \int_{|s|}^3 r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= -2c\pi \left[ r^5 - 8r^4 + 22r^3 - 24r^2 + 9r \right]_s^3 \\
&= 2c\pi (s^5 - 8s^4 + 22s^3 - 24s^2 + 9s)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 2c\pi (5s^4 - 32s^3 + 66s^2 - 48s + 9) \quad \text{for } 1 < s < 3 \\
\frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= \frac{\partial}{\partial s} \mathcal{R}(g)(-s, \omega) = \frac{\partial}{\partial s} 2c\pi (-s^5 - 8s^4 - 22s^3 - 24s^2 - 9s) \\
&= -2c\pi (5s^4 + 32s^3 + 66s^2 + 48s + 9) \quad \text{for } -3 < s < -1 \\
\text{and } \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 0 \quad \text{for } |s| \notin [1, 3]
\end{aligned}$$

We obtain the expression of the plane wave component for  $f$  and  $g$  :

$$\begin{aligned}
\alpha(s, \omega) &= \frac{1}{2c} \left( \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) - c \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) \right) \\
&= \begin{cases} \pi (10s^4 - 64s^3 + 132s^2 - 96s + 18) & , \text{ if } 1 \leq s \leq 3 \\ 0 & , \text{ otherwise} \end{cases}
\end{aligned}$$

We can then run a comparison with the exact solution of our new numerical method in Matlab, specifically for this example, a plot of relative errors is below where we took  $c = 1$

and we computed the solution at the  $(30, 30, 30)$  point in space and for times ranging from 45 seconds to 55 seconds, corresponding to the time where the ‘perturbation’ passes through the point.

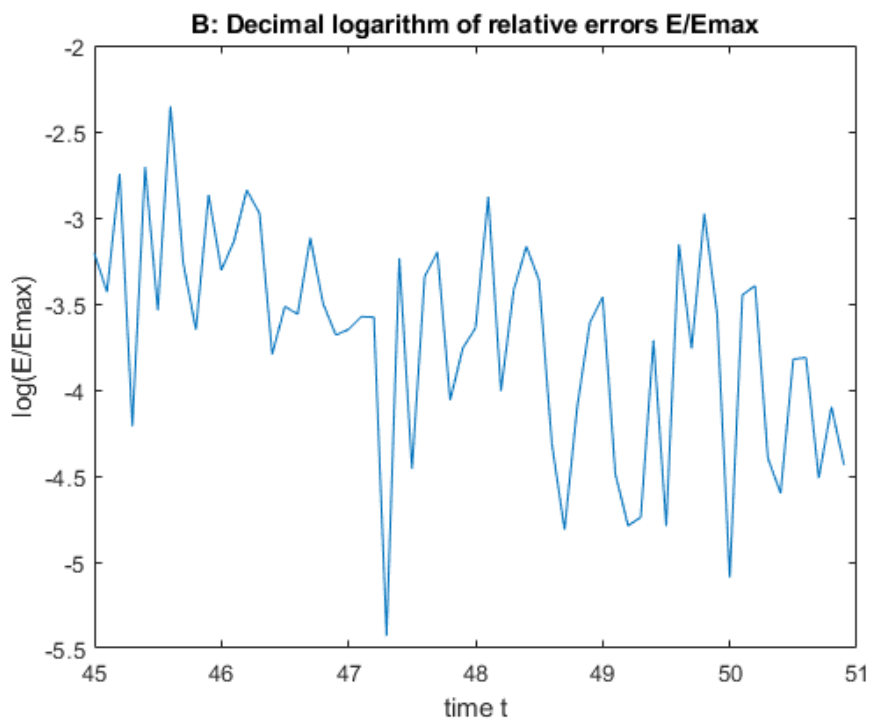


FIGURE 5.2: Decimal logarithm of relative errors for  $t$  from 45 to 55

We can see that the relative error is not more than 0.002 of the value of the exact solution, meaning that our new scheme in this case is very precise.

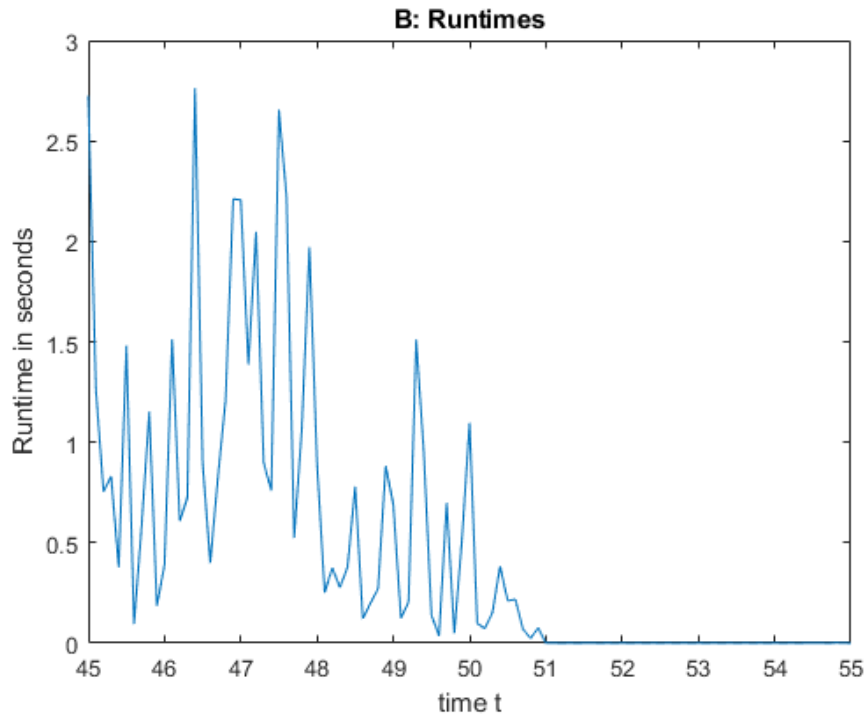
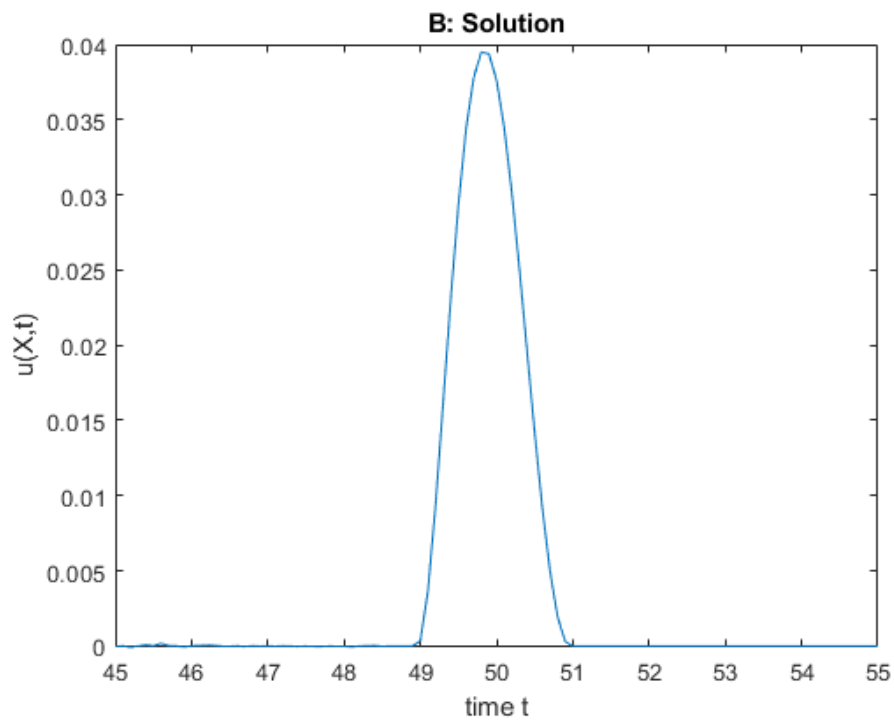


FIGURE 5.3: Runtimes as measured in Matlab

We can measure the computation time, which is dependent on the machine in where the scheme is being run. On a personal laptop with CPU Intel i5, under Windows, x64, Matlab version R2020b, we have less than 3 seconds for each obtained value. Not to mention that other applications are running, meaning that the total power of the personal laptop is not fully exploited.

Clearly this performance can be exceeded by just 'tuning' the scheme to skip unnecessary iterations to reach the tolerance of  $10^{-6}$ ; an example of that can be to modify the loop increments, instead of making steps of 1600 in the equally distributed points of  $S^2$  we can take 3200 to reduce the runtime by almost a half !

FIGURE 5.4: The solution  $u$  for  $t$  from 45 to 55

One can follow the Bump as it travels through space, as in the above figure, notice how  $30\sqrt{3} \approx 43$ .

**Example BH, (BH for Bump with a Hump)**

Take the function  $f$  that is differentiable for all  $r \geq 0$  :

$$f(r) = \begin{cases} (r-1)^2(r-5)^2((r-3)^2 + 0.25) = \\ r^6 - 18r^5 + 127.25r^4 - 447r^3 + 810.5r^2 - 705r + 231.25 & , \text{if } 1 \leq r \leq 5 \\ 0 & , \text{otherwise} \end{cases}$$

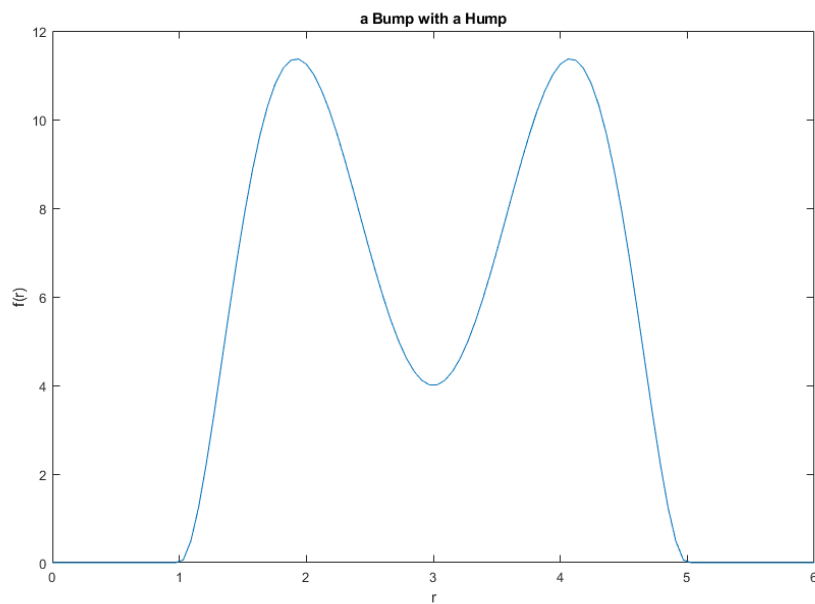


FIGURE 5.5: Plot of the Bump with a Hump

For  $1 \leq r \leq 5$  :  $f'(r) = 6r^5 - 90r^4 + 509r^3 - 447 \times 3r^2 + 1621r - 705$  and

$$\begin{aligned} g(r) &= -c(6r^5 - 90r^4 + 509r^3 - 447 \times 3r^2 + 1621r - 705) \\ &\quad + \frac{1}{r}(r^6 - 18r^5 + 127.25r^4 - 447r^3 + 810.5r^2 - 705r + 231.25) \\ &= -c(7r^5 - 108r^4 + 636.25r^3 - 1788r^2 + 2431.5r - 1410 + 231.25\frac{1}{r}) \end{aligned}$$

In this case

$$F(r) = \begin{cases} \frac{r}{a}(r-1)^2(r-5)^2((r-3)^2 + 0.25) & , \text{if } 1 \leq r \leq 5 \\ 0 & , \text{otherwise} \end{cases}$$

and we can work with the exact analytical solution :

$$u(x, t) = \begin{cases} \frac{r-ct}{r}(r-ct-1)^2(r-ct-5)^2((r-ct-3)^2 + 0.25) & , \text{if } 1 \leq r-ct \leq 5 \\ 0 & , \text{otherwise} \end{cases}$$

We can proceed now, differently, to illustrate more than one way of computation. Just by considering the absolute value of  $s$ , the computation is straight forward.

Any plane distant by  $|s| \leq 1$  from the origin and normal to the radial vector  $\omega$ , intersect the density function  $f$  on a circular sector of an internal radius  $\sqrt{1-s^2}$  and external radius  $\sqrt{5^2-s^2}$ , any such plane distant by more than or equal to 5 from the origin gives a Radon transform value of 0, so we know that  $\mathcal{R}(f)$  vanishes for all  $|s| \geq 5$ , and for  $|s| \leq 1$  it



evaluates to :

$$\begin{aligned}
\mathcal{R}(f)(s, \omega) &= \int_C f(r) d\mu = 2\pi \int_{\sqrt{1-s^2}}^{\sqrt{5^2-s^2}} r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
&\quad (\text{where } C \text{ is the circular sector of intersection}) \\
&= 2\pi \int_1^5 r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= C^{ste} \quad (\text{we don't care what constant is that since its derivative is zero})
\end{aligned}$$

For  $1 < |s| < 5$  :

$$\begin{aligned}
\mathcal{R}(f)(s, \omega) &= \int_D f(r) d\mu = 2\pi \int_0^{\sqrt{5^2-s^2}} r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
&\quad (\text{where } D \text{ is the disk of intersection}) \\
&= 2\pi \int_{|s|}^5 r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= 2\pi \left[ r^8/8 - 18r^7/7 + 127.25r^6/6 - 447r^5/5 + 810.5r^4/4 - 705r^3/3 + 231.25r^2/2 \right]_{|s|}^5 \\
&= \pi \left( \frac{8125}{42} - \frac{s^8}{4} + \frac{36}{7}|s|^7 - \frac{127.25}{3}s^6 + \frac{894}{5}|s|^5 - \frac{810.5}{2}s^4 + \frac{1410}{3}|s|^3 - 231.25s^2 \right)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= \frac{\partial}{\partial s} \pi \left( -2s^7 + 36s^6 - 2(127.25)s^5 + 894s^4 - 2(810.5)s^3 + 1410s^2 - 2(231.25)s \right) \\
&= \pi \left( -14s^6 + 6(36)s^5 - 1272.5s^4 + 4(894)s^3 - 6(810.5)s^2 + 2(1410)s - 2(231.25) \right) \\
&\quad \text{for } 1 < s < 5
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= \frac{\partial^2}{\partial s^2} \pi \left( \frac{8125}{42} - s^8/4 - 36s^7/7 - 127.25s^6/3 \right. \\
&\quad \left. - 894s^5/5 - 810.5s^4/2 - 1410s^3/3 - 231.25s^2 \right) \\
&= \frac{\partial}{\partial s} \pi \left( -2s^7 - 36s^6 - 2(127.25)s^5 - 894s^4 - 2(810.5)s^3 - 1410s^2 - 2(231.25)s \right) \\
&= \pi \left( -14s^6 - 6(36)s^5 - 1272.5s^4 - 4(894)s^3 - 6(810.5)s^2 - 2(1410)s - 2(231.25) \right) \\
&\quad \text{for } -5 < s < -1
\end{aligned}$$

$$\text{And } \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) = 0 \quad \text{for } |s| \notin [1, 5]$$

On the other hand we have for  $g : \mathcal{R}(g)$  vanishes for all  $|s| \geq 5$  and for  $|s| \leq 1$  it evaluates to :

$$\begin{aligned}
\mathcal{R}(g)(s, \omega) &= \int_C g(r) d\mu = 2\pi \int_{\sqrt{1-s^2}}^{\sqrt{5^2-s^2}} r_1 g \left( \sqrt{r_1^2 + s^2} \right) dr_1 \\
&\quad (\text{where } C \text{ is the circular sector of intersection}) \\
&= 2\pi \int_1^5 r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= C^{ste}
\end{aligned}$$

and for  $1 < |s| < 5$  :

$$\begin{aligned}
\mathcal{R}(g)(s, \omega) &= \int_D g(r) d\mu = 2\pi \int_0^{\sqrt{5^2 - s^2}} r_1 g\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
&\quad (\text{where } D \text{ is the disk of intersection}) \\
&= 2\pi \int_{|s|}^5 r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
&= -2c\pi \left[ r^7 - 18r^6 + 636.25r^5/5 - 447r^4 + 2431.5r^3/3 - 705r^2 + 231.25r \right]_{|s|}^5 \\
&= 2c\pi \left( |s|^7 - 18s^6 + 636.25|s|^5/5 - 447s^4 + 2431.5|s|^3/3 - 705s^2 + 231.25|s| \right)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 2c\pi \left( 7s^6 - 6(18)s^5 + 636.25s^4 - 4(447)s^3 + 2431.5s^2 - 1410s + 231.25 \right) \\
&\quad \text{for } 1 < s < 5 \\
\frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= \frac{\partial}{\partial s} 2c\pi \left( -s^7 - 18s^6 - 636.25s^5/5 - 447s^4 - 2431.5s^3/3 - 705s^2 - 231.25s \right) \\
&= -2c\pi \left( 7s^6 + 6(18)s^5 + 636.25s^4 + 4(447)s^3 + 2431.5s^2 + 1410s + 231.25 \right) \\
&\quad \text{for } -5 < s < -1 \\
\text{And } \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 0 \quad \text{for } |s| \notin [1, 5]
\end{aligned}$$

We obtain the expression of the plane wave component for  $f$  and  $g$  :

$$\begin{aligned}
\alpha(s, \omega) &= \frac{1}{2c} \left( \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) - c \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) \right) \\
&= \begin{cases} \pi (14s^6 - 216s^5 + 1272.5s^4 - 3576s^3 + 4863s^2 - 2820s + 462.5) & , \text{ if } 1 \leq s \leq 5 \\ 0 & , \text{ otherwise} \end{cases}
\end{aligned}$$

We run the comparison with the exact solution again in Matlab, specifically for this example, a plot of relative errors is below where we took  $c = 1$  and we computed the solution at the  $(30, 30, 30)$  point in space and for times ranging from 45 seconds to 55 seconds again.

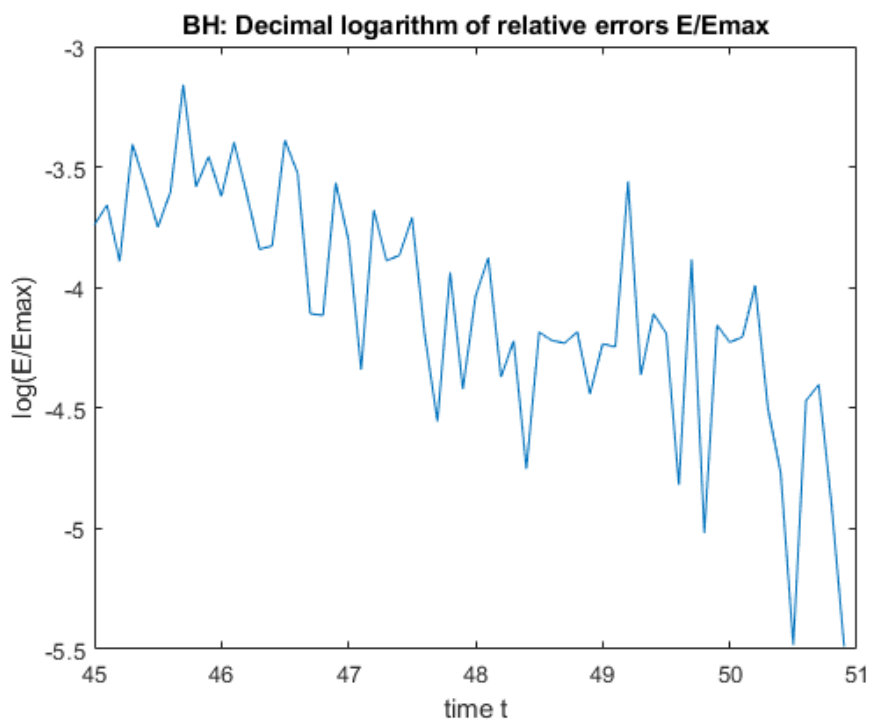


FIGURE 5.6: Decimal logarithm of relative errors for  $t$  from 45 to 55

We notice the same order of relative errors, not more than 0.002 for this case as well.

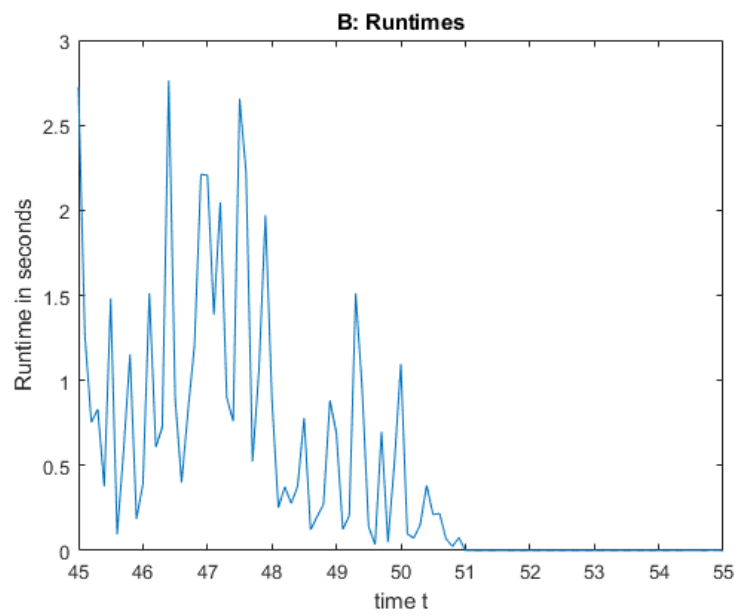
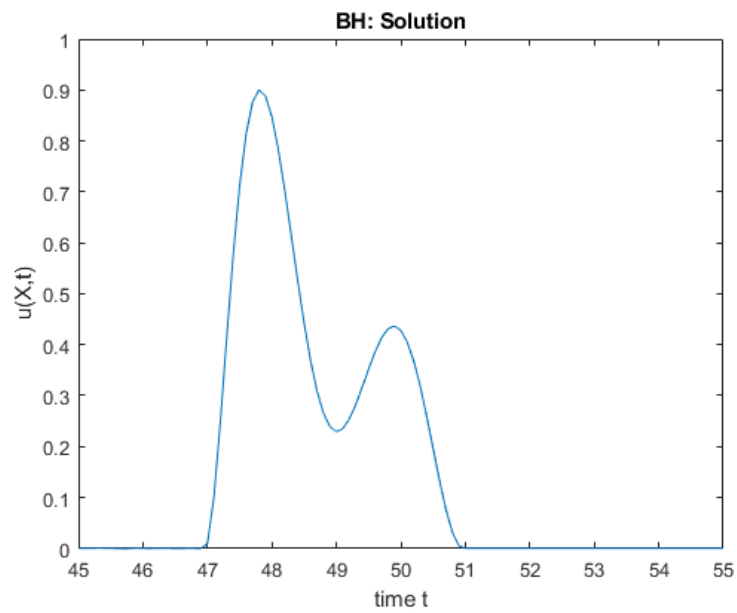


FIGURE 5.7: Runtimes as measured in Matlab

FIGURE 5.8: The solution  $u$  for  $t$  from 45 to 55

We like to make things more simple, so we generalize our calculus for any polynomial expression in the following analysis.

### 5.3 Piece-wise Polynomials

Now we make the computation more automated; we consider any polynomial, up to a vanishing point at  $r = M$

$$f(r) = P(r) = \sum_{k=0}^n a_k r^k$$

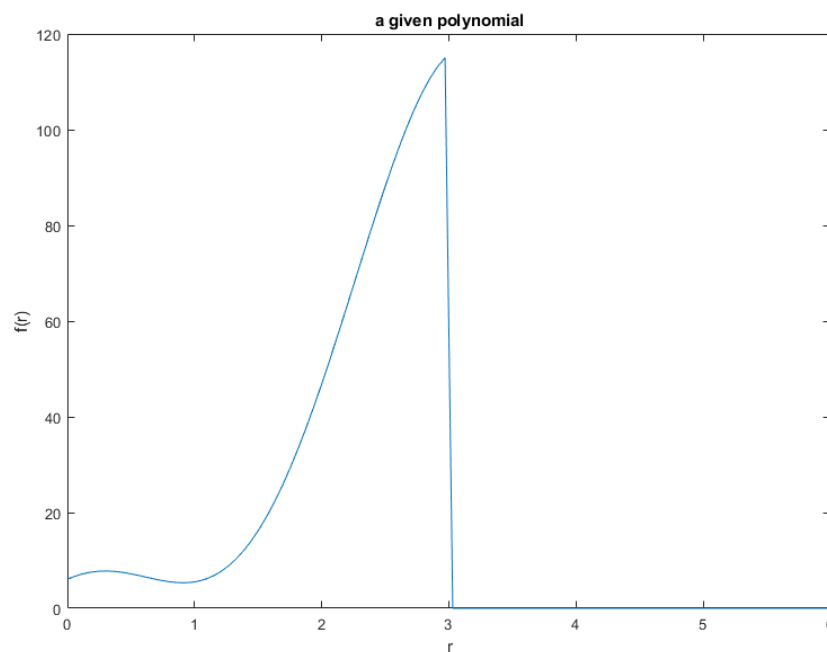


FIGURE 5.9: A polynomial where  $r = 3$  is the vanishing point

Question: can we consider density functions which are discontinuous like that example above?

Let us explore this later.

For now, to continue in the spirit of taking only differentiable initial states (  $f'$  exists ), we will choose only polynomials whose derivative is zero at the vanishing point  $r = M$ .

Here  $f'(r) = \sum_{k=1}^n k a_k r^{k-1}$  and

$$\begin{aligned} g(r) &= -c \left( \sum_{k=1}^n k a_k r^{k-1} + \frac{1}{r} \sum_{k=0}^n a_k r^k \right) \\ &= -c \left( \sum_{k=1}^n (k+1) a_k r^{k-1} + \frac{a_0}{r} \right) \end{aligned}$$

In this case

$$F(r) = \frac{r}{a} \left( \sum_{k=0}^n a_k r^k \right) = \sum_{k=0}^n \frac{a_k}{a} r^{k+1} \text{ for } r \leq M, \text{ 0 otherwise}$$

and the exact solution has the formula:

$$u(x, t) = \begin{cases} \frac{a}{r} F(r - ct) = \frac{r-ct}{r} f(r - ct) = \sum_{k=0}^n \frac{a_k}{r} (r - ct)^{k+1} & , \text{ if } 0 \leq r - ct \leq M \\ 0 & , \text{ otherwise} \end{cases}$$

Proceeding in a similar fashion, any plane distant by  $|s|$  from the origin and normal to the radial vector  $\omega$ , intersects the density function  $f$  everywhere up to  $M$ , meaning as long as

$|s| < M$ , so we know that  $\mathcal{R}(f)$  evaluates to :

$$\begin{aligned}
 \mathcal{R}(f)(s, \omega) &= \int_P f(r) d\mu = 2\pi \int_0^{\sqrt{M^2 - s^2}} r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
 d\mu &= r_1 d\theta dr_1 \text{ with polar calculus} \\
 &= 2\pi \int_{|s|}^M r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
 &= 2\pi \left[ \sum_{k=0}^n \frac{a_k}{k+2} r^{k+2} \right]_{|s|}^M = 2\pi \sum_{k=0}^n \frac{a_k}{k+2} (M^{k+2} - |s|^{k+2})
 \end{aligned}$$

Then, we can compute for positive  $s$  :

$$\begin{aligned}
 \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= \frac{\partial}{\partial s} \left( -2\pi \sum_{k=0}^n a_k s^{k+1} \right) \\
 &= -2\pi \sum_{k=0}^n (k+1) a_k s^k
 \end{aligned}$$

and for negative  $s$  :

$$\begin{aligned}
 \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) &= \frac{\partial}{\partial s} \left( 2\pi \sum_{k=0}^n a_k (-s)^{k+1} \right) \\
 &= -2\pi \sum_{k=0}^n (k+1) a_k (-s)^k
 \end{aligned}$$



On the other hand, we have for  $g$ , for  $|s| \leq M$ ,  $\mathcal{R}(g)$  evaluates to :

$$\begin{aligned}
 \mathcal{R}(g)(s, \omega) &= \int_P g(r) d\mu = 2\pi \int_0^{\sqrt{M^2 - s^2}} r_1 g\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
 d\mu &= r_1 d\theta dr_1 \text{ with polar calculus} \\
 &= 2\pi \int_{|s|}^M r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
 &= -2\pi c \left[ \sum_{k=1}^n a_k r^{k+1} + a_0 r \right]_{|s|}^M = 2\pi c \sum_{k=0}^n a_k (|s|^{k+1} - M^{k+1})
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 2c\pi \sum_{k=0}^n (k+1) a_k s^k \quad \text{for } s \geq 0 \\
 \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= -2c\pi \sum_{k=0}^n (k+1) a_k (-s)^k \quad \text{for } s < 0
 \end{aligned}$$

We obtain the expression of the plane wave component for  $f$  and  $g$  :

$$\begin{aligned}
 \alpha(s, \omega) &= \frac{1}{2c} \left( \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) - c \frac{\partial^2}{\partial s^2} \mathcal{R}(f)(s, \omega) \right) \\
 &= \begin{cases} 2\pi \sum_{k=0}^n (k+1) a_k s^k & , \text{ if } 0 < s \leq M \\ 0 & , \text{ otherwise} \end{cases}
 \end{aligned}$$

Now we possess a powerful tool to compute as many polynomials as we wish.

But we can do better. If we look carefully at the expression of  $\alpha(s, \omega)$  we can make a useful observation. The expression looks like the polynomial  $f$  and this can shortcut the computations, so we can immediately retrieve the wave component without relying on the

Radon transform, only for radial waves:

**Theorem 8.** *Let  $M > 0$  and  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  be a real differentiable function such that  $f = 0$  for all  $r \geq M$ ,  $f$  is a polynomial over  $[0, M]$  and  $f'(0^+) = 0$ , then  $f$  gives rise to a unique radial wave function*

$$u(x, t) = \begin{cases} \frac{r-ct}{r} f(r-ct) & , \text{ if } 0 < r-ct \leq M \\ 0 & , \text{ otherwise} \end{cases}$$

such that  $u(x, 0) = f(r)$ , where  $r = |x|$ .

Moreover  $u_t(x, 0) = g(r) = -c(f'(r) + \frac{1}{r}f(r))$  and the wave component is retrieved by the formula

$$\alpha(s, \omega) = \begin{cases} \frac{-2\pi}{c} sg(s) = 2\pi(s f'(s) + f(s)) & , \text{ if } 0 < s \leq M \\ 0 & , \text{ otherwise} \end{cases}$$

**Proof.** As we computed, we have for  $0 < s \leq M$ ,

$$\begin{aligned} \alpha(s, \omega) &= 2\pi \sum_{k=0}^n (k+1) a_k s^k = \frac{-2\pi}{c} \left( -c \left( \sum_{k=0}^n (k+1) a_k s^k \right) \right) \\ &= \frac{-2\pi}{c} s \left( -c \left( \frac{a_0}{s} + \sum_{k=1}^n (k+1) a_k s^{k-1} \right) \right) = \frac{-2\pi}{c} sg(s) \end{aligned}$$

And recall that  $g(s) = -c(f'(s) + \frac{f(s)}{s})$ . ■

Now that we have this nice 'mechanical' formula for producing the wave component for any initial condition of this sort, we can just take any polynomial we want to test our scheme.

**Example S, (S for sinus)**

Take the function  $f$  that is differentiable for all  $r \geq 0$  :

$$\begin{aligned} f(r) &= r^2(r-4)^2(r-2) = \sum_{k=0}^5 a_k r^k \quad \text{up to the vanishing point } r = 4 \\ &= r^5 - 10r^4 + 32r^3 - 32r^2 \end{aligned}$$

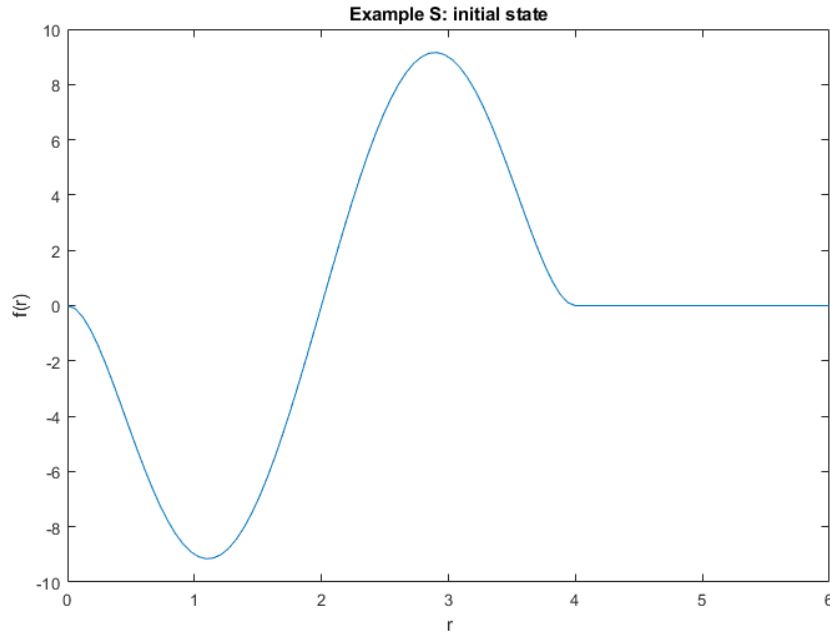


FIGURE 5.10: Plot of example S

For  $r \leq 4$  :  $f'(r) = 5r^4 - 40r^3 + 96r^2 - 64r$  and

$$\begin{aligned} g(r) &= -c \left( 5r^4 - 40r^3 + 96r^2 - 64r + \frac{1}{r}(r^5 - 10r^4 + 32r^3 - 32r^2) \right) \\ &= -c(6r^4 - 50r^3 + 128r^2 - 96r) \end{aligned}$$

In this case

$$F(r) = \begin{cases} \frac{r}{a} r^2 (r-4)^2 (r-2) & , \text{if } r \leq 4 \\ 0 & , \text{otherwise} \end{cases}$$

and the exact analytical solution is :

$$u(x, t) = \begin{cases} \frac{r-ct}{r} (r-ct)^2 (r-ct-4)^2 (r-ct-2) & , \text{if } 0 \leq r-ct \leq 4 \\ 0 & , \text{otherwise} \end{cases}$$

We have  $sg(s) = -c(6s^5 - 50s^4 + 128s^3 - 96s^2)$  and we apply the observation that we made:

$$\alpha(s, \omega) = \begin{cases} 2\pi(6s^5 - 50s^4 + 128s^3 - 96s^2) & , \text{if } 0 \leq s \leq 4 \\ 0 & , \text{otherwise} \end{cases}$$

We run the comparison again in a similar fashion as the first two examples. And we obtain the plots:

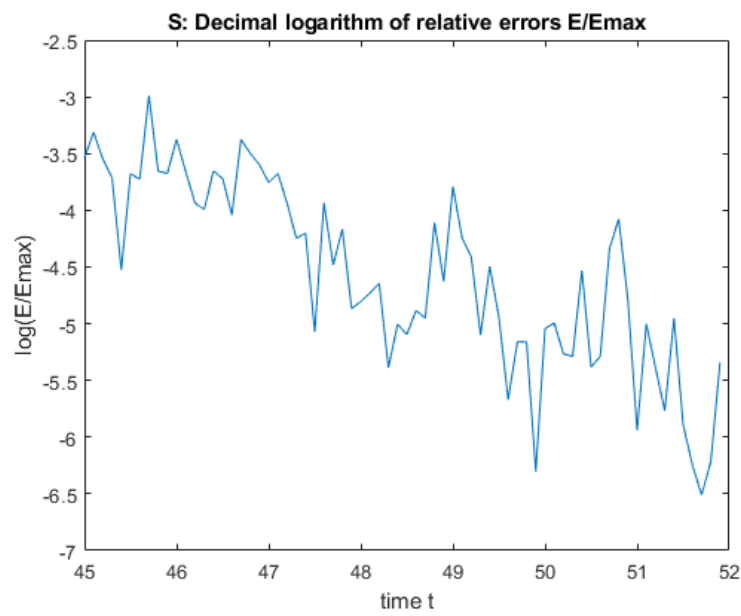
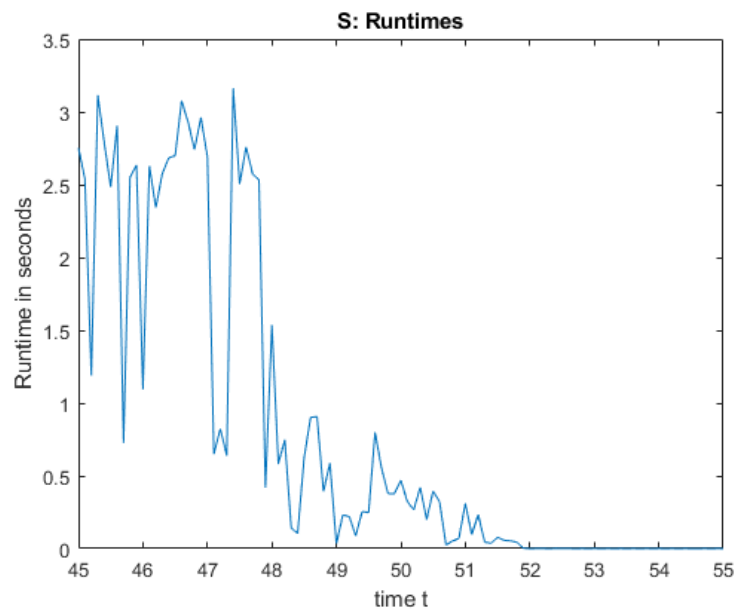
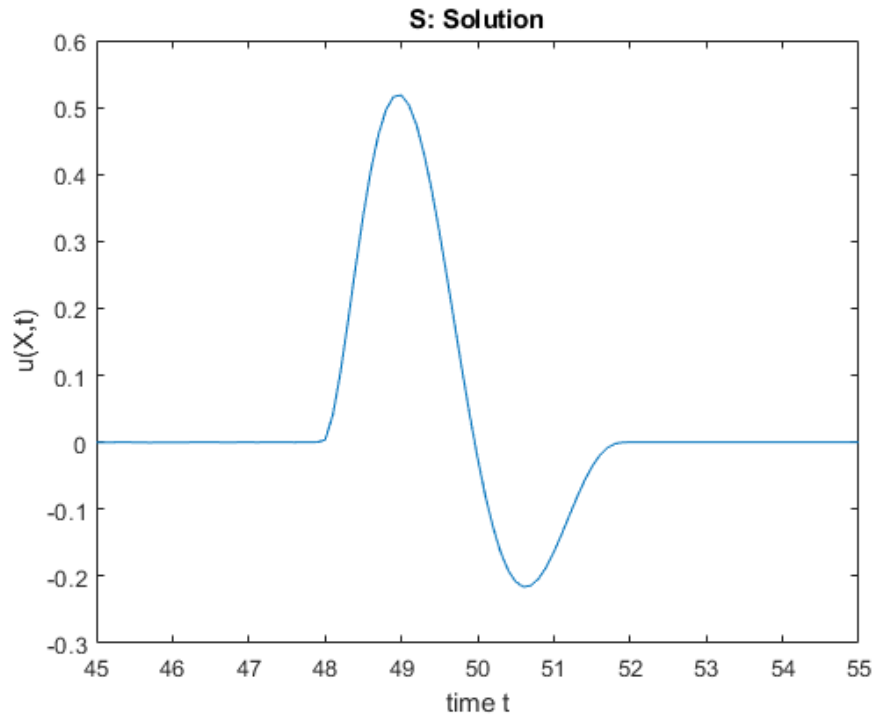
FIGURE 5.11: Decimal logarithm of relative errors for  $t$  from 45 to 55

FIGURE 5.12: Runtimes as measured in Matlab

FIGURE 5.13: The solution  $u$  for  $t$  from 45 to 55**Example W, (W for Wiggly)**

Consider now the function  $f$  (differentiable for all  $r \geq 0$ ) :

$$\begin{aligned}
 f(r) &= (r^2 - 1)(r^2 - 4)(r^2 - 9)(r^2 - 16)(r^2 - 25)(r^2 - 36)(r^2 - 49)^2 \quad \text{up to the vanishing point } r = 7 \\
 &= \sum_{k=0}^{16} a_k r^k = r^{16} - 189r^{14} + 14322r^{12} - 557258r^{10} + 11864853r^8 - 136589817r^6 \\
 &\quad + 787692424r^4 - 1907102736r^2 + 1244678400
 \end{aligned}$$

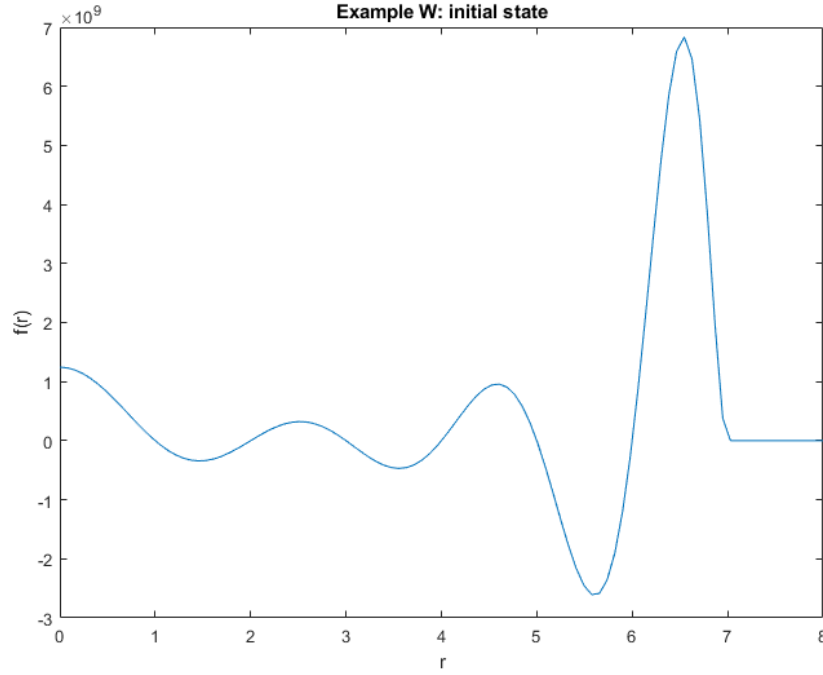


FIGURE 5.14: Plot of the example W

For  $r \leq 7$  :  $f'(r) = 16r^{15} - 189(14)r^{13} + 14322(12)r^{11} - 5572580r^9 + 11864853(8)r^7 - 136589817(6)r^5 + 787692424(4)r^3 - 1907102736(2)r$  and

$$\begin{aligned}
 g(r) &= -c \left( 16r^{15} - 189(14)r^{13} + 14322(12)r^{11} - 5572580r^9 + 11864853(8)r^7 - 136589817(6)r^5 \right. \\
 &\quad \left. + 787692424(4)r^3 - 1907102736(2)r + \frac{1}{r}(r^{16} - 189r^{14} + 14322r^{12} - 557258r^{10} \right. \\
 &\quad \left. + 11864853r^8 - 136589817r^6 + 787692424r^4 - 1907102736r^2 + 1244678400) \right) \\
 &= -c \left( 17r^{15} - 189(15)r^{13} + 14322(13)r^{11} - 557258(11)r^9 + 11864853(9)r^7 \right. \\
 &\quad \left. - 136589817(7)r^5 + 787692424(5)r^3 - 1907102736(3)r + \frac{1244678400}{r} \right)
 \end{aligned}$$

In this case

$$F(r) = \begin{cases} \frac{r}{a}(r^2 - 1)(r^2 - 4)(r^2 - 9)(r^2 - 16)(r^2 - 25)(r^2 - 36)(r^2 - 49)^2 & , \text{if } r \leq 7 \\ 0 & , \text{otherwise} \end{cases}$$

and the exact analytical solution is

$$u(x, t) = \begin{cases} \frac{r-ct}{r}((r-ct)^2 - 1)((r-ct)^2 - 4)((r-ct)^2 - 9)((r-ct)^2 - 16)... \\ ...((r-ct)^2 - 25)((r-ct)^2 - 36)((r-ct)^2 - 49)^2 & , \text{if } 0 \leq r - ct \leq 7 \\ 0 & , \text{otherwise} \end{cases}$$

We have  $sg(s) = -c(17s^{16} - 189(15)s^{14} + 14322(13)s^{12} - 557258(11)s^{10} + 11864853(9)s^8 - 136589817(7)s^6 + 787692424(5)s^4 - 1907102736(3)s^2 + 1244678400)$

and we apply our theorem :

$$\alpha(s, \omega) = \begin{cases} 2\pi(17s^{16} - 15(189)s^{14} + 13(14322)s^{12} - 11(557258)s^{10} + 9(11864853)s^8 \\ -7(136589817)s^6 + 5(787692424)s^4 - 3(1907102736)s^2 + 1244678400) & , \text{if } 0 < s \leq 7 \\ 0 & , \text{otherwise} \end{cases}$$

Below are the plots obtained;



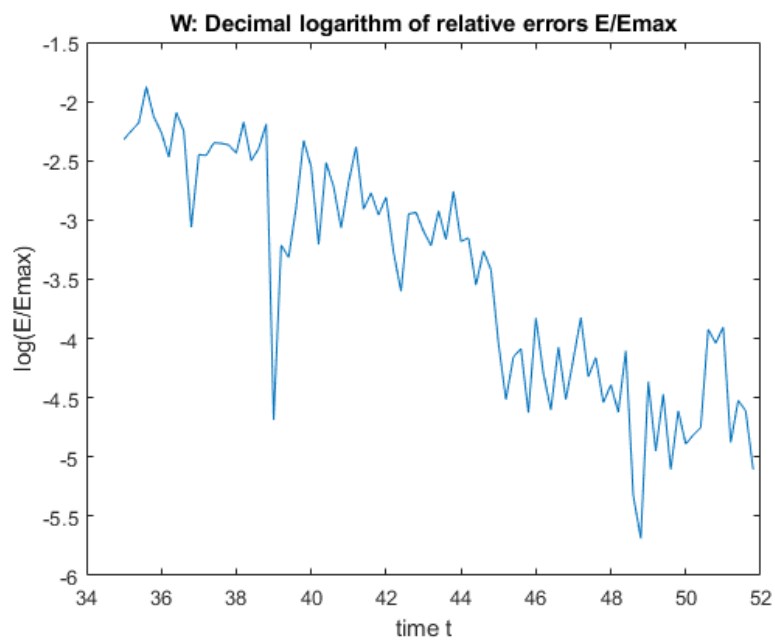
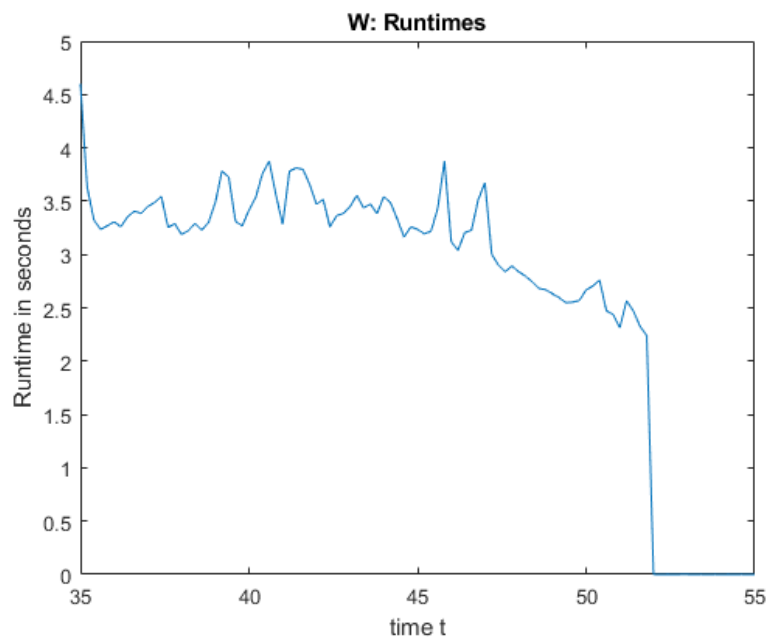
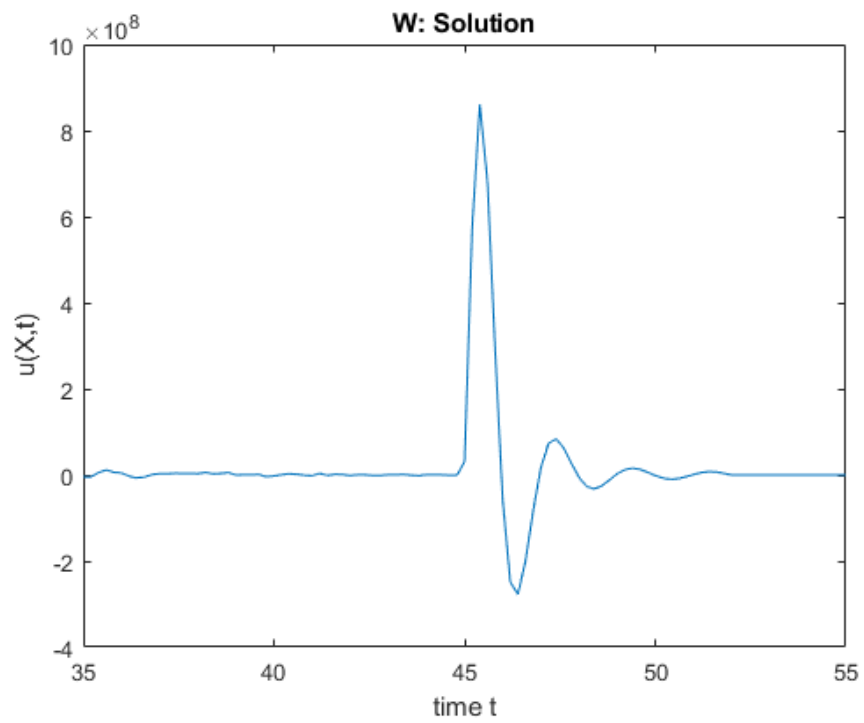
FIGURE 5.15: Decimal logarithm of relative errors for  $t$  from 45 to 55

FIGURE 5.16: Runtimes as measured in Matlab

FIGURE 5.17: The solution  $u$  for  $t$  from 45 to 55

## 5.4 Non-compactly supported initial states

### Gaussian density

Take the function  $f$  that is differentiable for all  $r > 0$  :  $f(r) = e^{-r^2}$

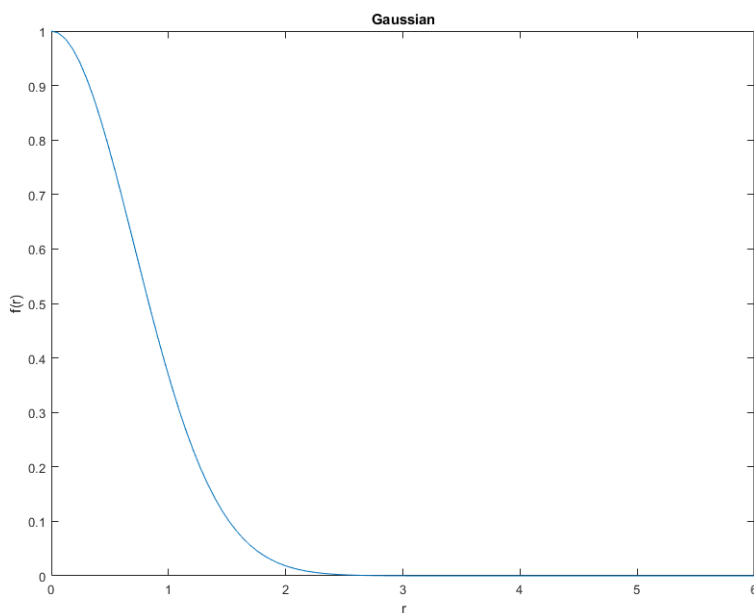


FIGURE 5.18: Plot of the Gaussian

Here  $f'(r) = -2re^{-r^2}$  and

$$g(r) = -c(-2re^{-r^2} + \frac{1}{r}e^{-r^2}) = (2cr - \frac{c}{r})e^{-r^2}$$

In this case

$$F(r) = \frac{r}{a}e^{-r^2}$$

and we can work with the following

$$u(x, t) = \begin{cases} \frac{a}{r} F(r - ct) = \frac{r-ct}{r} e^{-(r-ct)^2} & , \text{if } r - ct \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Integrating :

$$\begin{aligned} \mathcal{R}(f)(s, \omega) &= \int_D f(r) d\mu = 2\pi \int_0^\infty r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\ &\quad (\text{where } D \text{ is the infinite plane of intersection}) \\ &= 2\pi \int_{|s|}^\infty r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\ &= 2\pi \left[ \frac{-1}{2} e^{-r^2} \right]_{|s|}^\infty \\ &= -\pi \left( 0 - e^{-s^2} \right) = \pi e^{-s^2} \end{aligned}$$

Then

$$\frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) = \frac{\partial}{\partial s} \left( -2\pi s e^{-s^2} \right) = 2\pi (2s^2 - 1) e^{-s^2}$$

On the other hand we have for  $g$  :

$$\begin{aligned}
 \mathcal{R}(g)(s, \omega) &= \int_D g(r) d\mu = 2\pi \int_0^\infty r_1 g\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
 &\quad (\text{where } D \text{ is the infinite plane of intersection}) \\
 &= 2\pi \int_{|s|}^\infty r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
 &= 2\pi \int_{|s|}^\infty (2cr^2 - c)e^{-r^2} dr = 2c\pi \left[ -re^{-r^2} \right]_{|s|}^\infty \\
 &= 2c\pi \left( |s|e^{-s^2} - 0 \right) = 2c\pi |s|e^{-s^2}
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 2c\pi(1 - 2s^2)e^{-s^2} \quad \text{for } s \geq 0 \\
 \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= -2c\pi(1 - 2s^2)e^{-s^2} \quad \text{for } s < 0
 \end{aligned}$$

We obtain the expression of the plane wave component for  $f$  and  $g$  :

$$\alpha(s, \omega) = \begin{cases} 2\pi(1 - 2s^2)e^{-s^2} & , \text{ if } s \geq 0 \\ 0 & , \text{ if } s < 0 \end{cases}$$

$e^{-r}$  density

This density is defined by, for all  $r > 0$  :  $f(r) = e^{-r}$

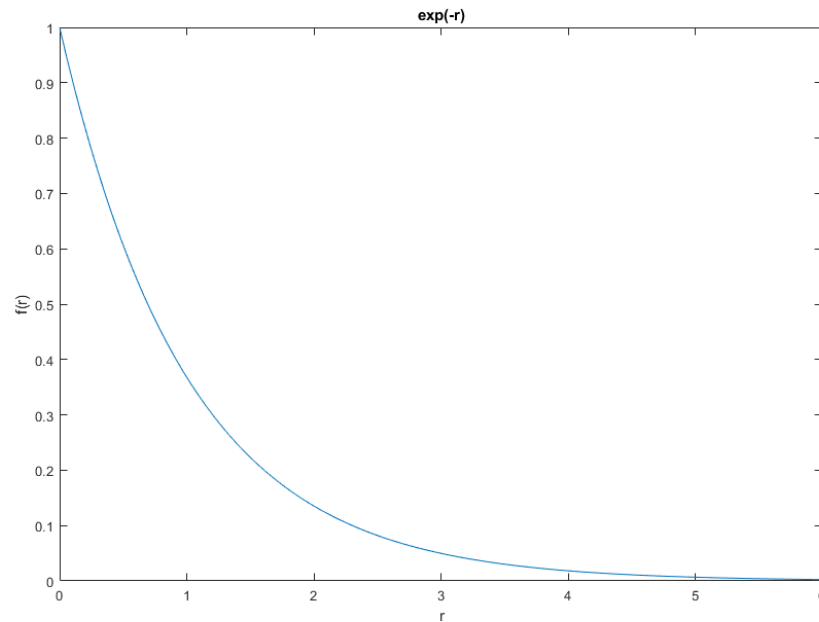


FIGURE 5.19: Plot of the  $e^{-r}$  density

Here  $f'(r) = -e^{-r}$  and

$$g(r) = -c(-e^{-r} + \frac{1}{r}e^{-r}) = (c - \frac{c}{r})e^{-r}$$

In this case

$$F(r) = \frac{r}{a}e^{-r}$$

and we can work with the following

$$u(x, t) = \begin{cases} \frac{a}{r} F(r - ct) = \frac{r-ct}{r} e^{-(r-ct)} & , \text{if } r - ct \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Integrating :

$$\begin{aligned} \mathcal{R}(f)(s, \omega) &= \int_D f(r) d\mu = 2\pi \int_0^\infty r_1 f\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\ &\quad (\text{where } D \text{ is the infinite plane of intersection}) \\ &= 2\pi \int_{|s|}^\infty r f(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\ &= 2\pi \left[ -re^{-r} \right]_{|s|}^\infty - 2\pi \int_{|s|}^\infty -e^{-r} dr \\ &= 2\pi \left[ -re^{-r} - e^{-r} \right]_{|s|}^\infty \\ &= 2\pi (0 + |s|e^{-|s|} + e^{-|s|}) = 2\pi(1 + |s|)e^{-|s|} \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) &= \frac{\partial}{\partial s} 2\pi (-se^{-s}) = 2\pi(s-1)e^{-s} \text{ if } s \geq 0 \\ \frac{\partial^2}{\partial s^2} \mathcal{R}f(s, \omega) &= \frac{\partial}{\partial s} 2\pi(-s)e^s = -2\pi(1+s)e^s \text{ if } s < 0 \end{aligned}$$

On the other hand we have for  $g$  :

$$\begin{aligned}
 \mathcal{R}(g)(s, \omega) &= \int_D g(r) d\mu = 2\pi \int_0^\infty r_1 g\left(\sqrt{r_1^2 + s^2}\right) dr_1 \\
 &\quad (\text{where } D \text{ is the infinite plane of intersection}) \\
 &= 2\pi \int_{|s|}^\infty r g(r) dr \quad \text{by the change of variable } r = \sqrt{r_1^2 + s^2} \\
 &= 2\pi \int_{|s|}^\infty (cr - c)e^{-r} dr = 2\pi \left[-cre^{-r}\right]_{|s|}^\infty = 2c\pi|s|e^{-|s|}
 \end{aligned}$$

Then

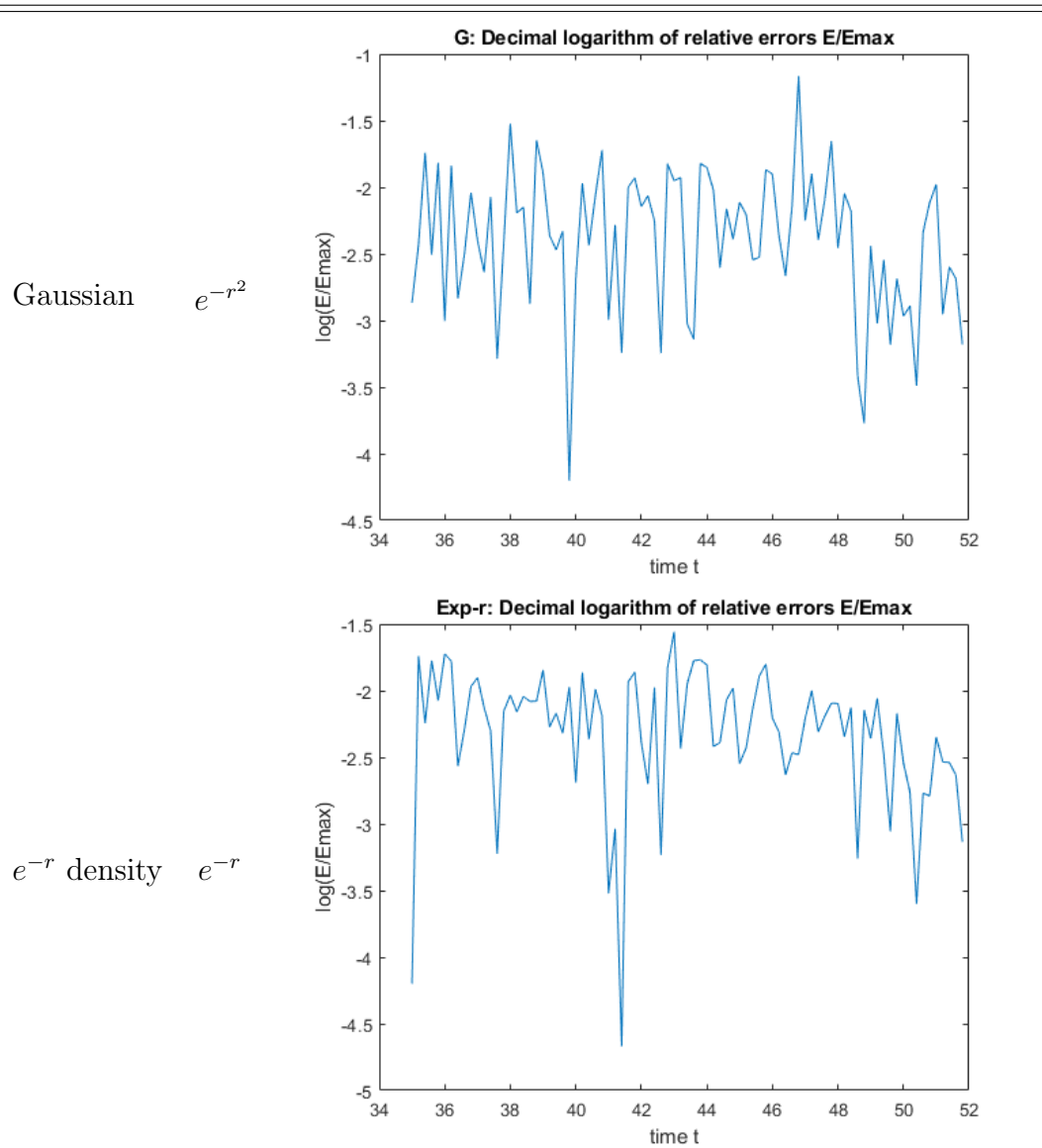
$$\begin{aligned}
 \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= 2c\pi(1 - s)e^{-s} \quad \text{for } s \geq 0 \\
 \frac{\partial}{\partial s} \mathcal{R}(g)(s, \omega) &= -2c\pi(1 + s)e^s \quad \text{for } s < 0
 \end{aligned}$$

We obtain the expression of the plane wave component for  $f$  and  $g$  :

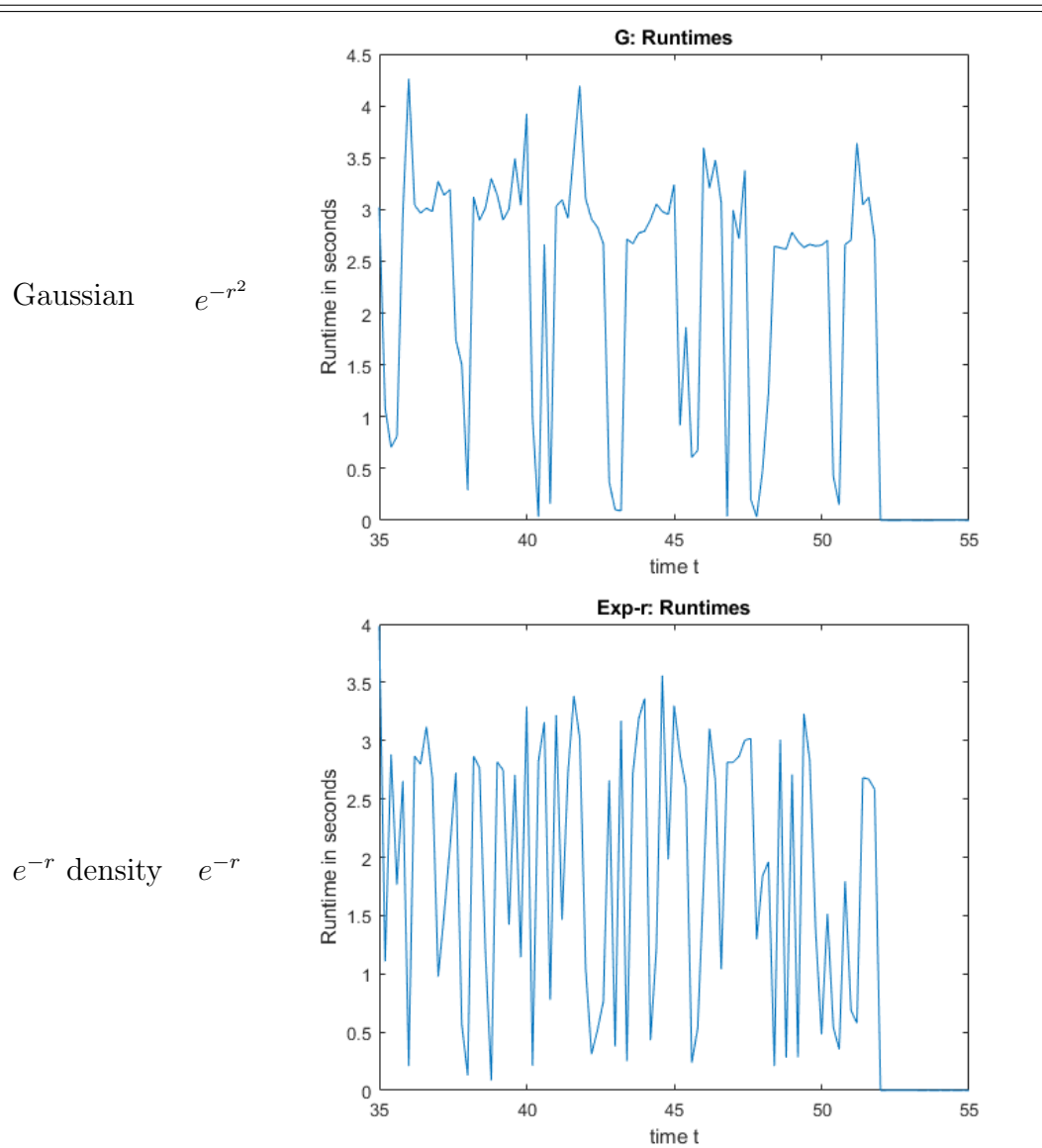
$$\alpha(s, \omega) = \begin{cases} 2\pi(1 - s)e^{-s} & , \text{ if } s \geq 0 \\ 0 & , \text{ if } s < 0 \end{cases}$$

We note the relative errors obtained in Matlab for these two situations :

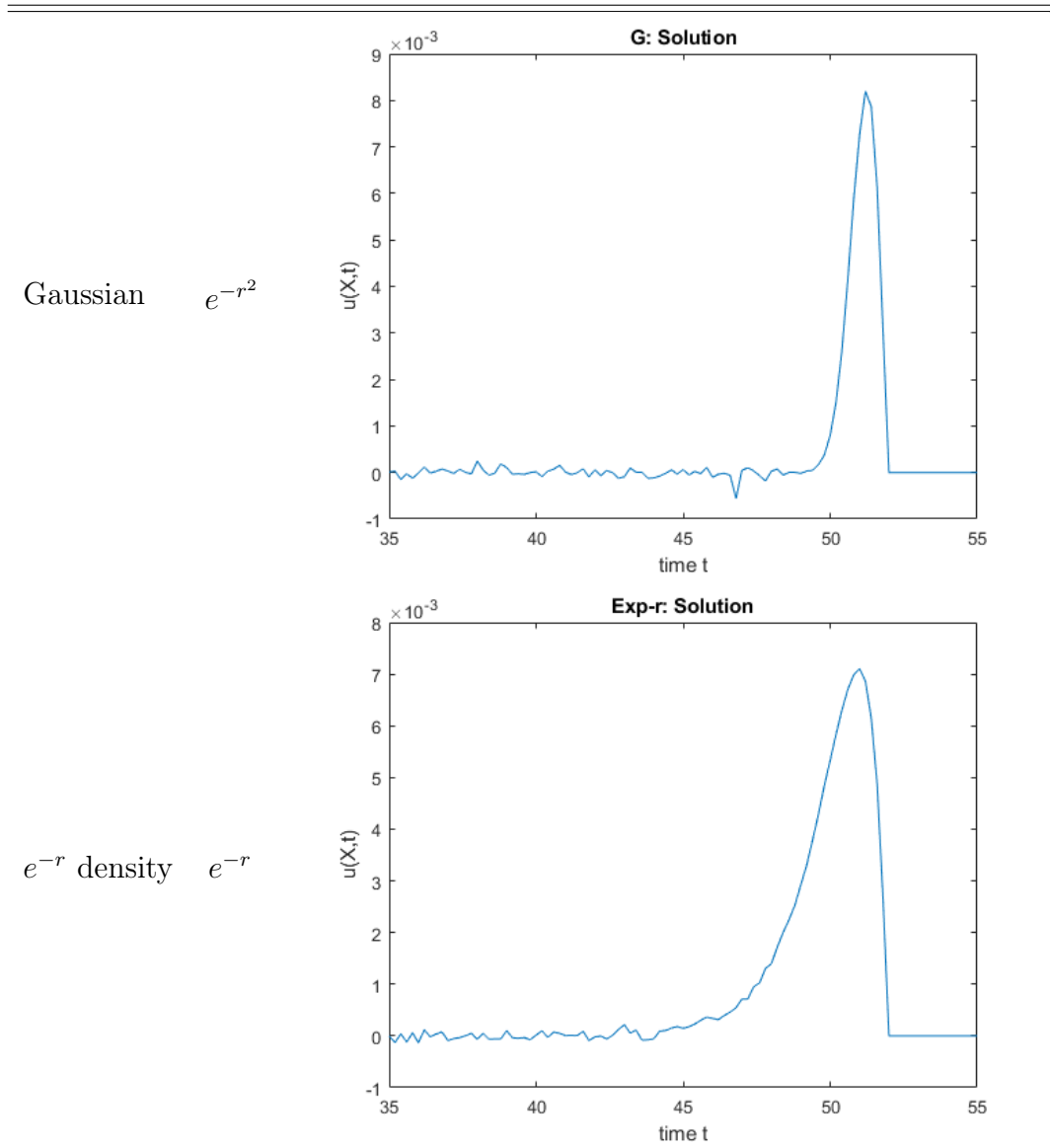




Then the runtimes, under 4 seconds always;



And the solutions which are not different from the exact solutions;



We can also compute any other values we want, we recap in the following the results obtained for different points in space-time. Here  $(300, 30, 30, 50)$  correspond to the point  $(30, 30, 30)$  in space and time  $t = 50$  seconds.

Relative errors, $\log E_r$						
Initial state	B	BH	S	W	G	Expr
(30, 30, 30, 50)	-5.06919	-3.90426	-3.69285	-2.10977	-1.95904	-2.41681
(300, 300, 300, 500)	Inf	Inf	Inf	Inf	164.82807	4.06320
(3000, 3000, 3000, 5000)	Inf	Inf	Inf	Inf	Inf	82.41244
( $3 \cdot 10^4$ , $3 \cdot 10^4$ , $3 \cdot 10^4$ , $5 \cdot 10^4$ )	NaN	NaN	NaN	NaN	Inf	Inf

Runtimes (in seconds)						
Initial state	B	BH	S	W	G	Expr
(30, 30, 30, 50)	2.08400	2.40600	0.39000	2.60200	2.96300	0.38500
(300, 300, 300, 500)	4.85600	2.26700	2.28200	2.41900	0.09300	2.19400
(3000, 3000, 3000, 5000)	2.34700	2.32900	2.31800	2.34900	0.86900	2.22900
( $3 \cdot 10^4$ , $3 \cdot 10^4$ , $3 \cdot 10^4$ , $5 \cdot 10^4$ )	0.00100	0.00100	0.00100	0.00200	0.00000	0.00300

The higher values 164.82 and 82.41 are understandable if you consider the limitations of Matlab, when the values are closer to 0, we can easily get  $\log(\frac{10^{-8}}{\text{Inf}}) = +\infty$ .

Notice also how 'Nan' is given when Matlab tries to divide by 0. We leave it at the conclusion that the new scheme gives very accurate results.

## 6 Conclusion

We have defined the Radon transform and showed how it can be used in solving the 3 dimensional wave equation, we have proved the general form of solution:

$$u(x, t) = \frac{1}{4\pi^2} \int_{S^2} \alpha(x \cdot \omega - ct, \omega) d\sigma(\omega)$$

Then we have exploited this formula to implement an accurate numerical scheme. The scheme doesn't rely on time iterations, hence it can be used to compute solutions for large times with a very low latency. The new scheme gave promising results and hopefully opens up a new idea in the field of computational science for the wave equation.

We sketch some ideas for future work that would incorporate the results of the thesis into a broader framework of solving numerically the general stated Cauchy problem. A natural starting point is to assume that initial data  $f, g$  is given discretely, at points  $x_j$  on an equally-spaced grid in  $\mathbb{R}^3$ . The question is: how to obtain a function  $\alpha(s, \omega)$  corresponding to the given discrete initial data?

If  $x_j$  is a grid point, one idea is to replace the given data  $f(x_j), g(x_j)$  with scaled characteristic functions of a ball centred at  $x_j$  (call them “pixel functions”). This way the discrete data

is replaced by a sum of pixel functions, defined on all of space. One can then compute the corresponding  $\alpha$ . A technical difficulty inherent in the aforementioned scheme is that the characteristic function of a ball is discontinuous. Therefore one has to evaluate the Radon transform in a distributional sense. With the resulting  $\alpha(s, \omega)$  in general being a distribution. Thus a basic open question is to extend the Radon transform to the more general setting of tempered distributions, and to express the general solution  $u(x, t)$  in terms of a distributional version of  $\alpha$ . In the cases where  $u$  is a regular function, one needs to be able to recover its values at specific points. Provided the distribution  $\alpha(s, \omega)$  is integrable with respect to the variable  $\omega$ , the proposed numerical method can be used to recover  $u$ . The analysis of precisely when the distribution  $\alpha$  is integrable with respect to  $\omega$ , depending on given initial distributions  $f$  and  $g$ , is an open question for future research.

The hope is that the requisite integrability can be established for pixel functions (or a suitable modification to this idea), thereby incorporating the proposed scheme into a fully-fledged numerical solution to the (discrete) Cauchy problem for the wave equation.

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