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Estimation of Shock Stand-off Distance Using the Curved Shock Theory and Its Validation via Numerical Modelling

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Abstract— Curved shocks that are locally oriented normal to the direction of the pre-shock flow vector appear on bluff and blunt bodies in supersonic flow and at the center lines of axisymmetric air intakes. There have been numerous studies to analytically approximate the shock stand-off distance associated with these curved normal shock waves; however, in view of the absence of satisfactory results across the entire range of freestream Mach numbers, further efforts are warranted. In this study, the Curved Shock Theory (CST) is used to derive analytical expressions for the pressure gradient right behind convex normal shocks in uniform upstream flow. This pressure gradient can be converted to gradients of other variables using the conservations laws and the isentropic relations. Using these gradients at the curved normal shock and an additional assumption on their variation between the shock and the blunt body it is possible to develop relations for the ratio of the shock stand-off distance to the radius of curvature of the shock surface as a function of freestream Mach number and the specific heat ratio. CST predictions are compared with experimental data from the literature and the CFD results obtained in the present study for freestream Mach numbers ranging from 1.2 to 8.

Keywords-shock stand-off distance; curved shock theory; blunt body; radius of shock curvature

I. INTRODUCTION

The shock stand-off distance from blunt bodies is an important parameter to estimate the effects of shock waves on objects moving at supersonic and hypersonic speeds. The change in the shock stand-off distance can be an effective indicator of the changes in other important properties such as the drag on an aircraft or the heat flux on a re-entry vehicle. There have been continuous efforts over many decades to come up with handy expressions to quickly calculate the shock stand-off distance from blunt bodies. However, rather limited success is achieved in formulating expressions that can be applied over a wide range of Mach numbers.

Several assumptions were implemented by different researchers to simplify the problem and formulate widely applicable expressions for the shock stand-off distance as a function of the freestream Mach number. In an earlier work by Moeckel [1], for instance, it is assumed that the detached shock wave can be represented by a hyperbola and that the sonic curve between the shock and the body is a straight line. In a recent study, Sinclair and Cui [2] made use of the variation of the size of the sonic zone bounded by the bow shock and the fore part of the body to relate the shock stand-off distance to the freestream Mach number.

In this paper, the Curved Shock Theory (CST) is used to derive an expression for the pressure gradient right behind convex bow shocks that appear in supersonic and hypersonic flows with uniform pre-shock flow. This pressure gradient relation is combined with the momentum conservation law and the isentropic relations for various flow variables, such as Mach number, density, temperature, etc. to obtain the gradients of those variables at the curved bow shock. By assuming the gradients to be constant within the shock layer between the bow shock and the blunt body or considering their average values, the ratio of the shock stand-off distance to the shock radius of curvature is obtained as a function of freestream Mach number. The aim of this study is to provide such alternative theoretical relations for the shock stand-off distance using CST and to investigate their validity. Billig's experimental correlations [3, 4], Liepmann and Roshko's experimental data [5], and our numerical simulation results are used as tools for validation.

II. THEORETICAL FORMULATION

A. Curved Shock Theory as Applied to Curved Normal Shocks in Uniform Pre-shock Flow

Theory that relates the pressure gradient, P, the streamline curvature, D, and vorticity, Γ, to the shock curvatures, S_a and S_b, on the pre-shock and post-shock sides of a doubly curved shock wave is developed in [6]. Such theories are also found in [7, 8]. When applied to a normal shock facing uniform upstream flow [8], the CST relations become:

$$0 = A_2' P_2 + C' S_a + G' S_b \tag{1}$$

$$P = \frac{\partial p/\partial s}{\rho V^2} = \frac{\partial p/\partial s}{\gamma p M^2} \tag{2}$$

$$A_2' = -\frac{M_1^2 - 1}{M_1^2} \cdot \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - \gamma + 1} \tag{3}$$

$$C' = G' = -\frac{2}{\gamma + 1} \cdot \frac{M_1^2 - 1}{M_1^2} \tag{4}$$

where subscripts 1 and 2 correspond to the upstream and downstream sides of the bow shock at the center line, respectively; M_1 is the freestream Mach number; the derivative in (2) is taken along the streamline; other notations are conventional.

For the blunt body case both shock surface curvatures, S_a and S_b, are negative and the corresponding radii of curvature, $R_a = -1/S_a$ and $R_b = -1/S_b$, are positive. The total curvature is $S_a + S_b$, the 'average curvature' is $\overline{S}=(S_a + S_b)/2$ and the 'harmonic average radius of curvature' is $\overline{R}$, where $1/\overline{R} =(1/R_a +1/R_b)/2$, so that $\overline{S}= -1/\overline{R}$.

Substituting (3) and (4) into (1), it is possible to show that,

$$P_2 = -\frac{4\overline{S}}{(\gamma+1)M_2^2} = \frac{4}{(\gamma+1)M_2^2\overline{R}} \tag{5}$$

Furthermore, applying (2) at the downstream face of the shock wave it can be written as,

$$\left(\frac{\mathrm{d}p}{p}\right)_2 = \frac{4\gamma}{(\gamma+1)\overline{R}}\,\mathrm{d}s \tag{6}$$

B. *Shock Stand-off Distance from a Blunt Body in Uniform Pre-shock Flow*

Expressions for the shock stand-off distance, Δ, from blunt and bluff bodies in a uniform pre-shock flow can be obtained using the post-shock pressure gradient (6) from CST, along with appropriate assumptions regarding flow variables in the downstream region. One such assumption is the constant pressure gradient assumption, in other words – the linear pressure profile in the shock layer. Equation (6) can be then written as,

$$\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_2 = \frac{p_0 - p_2}{\Delta} = \frac{4\gamma p_2}{(\gamma+1)\overline{R}} \tag{7}$$

$$\frac{\Delta}{\overline{\overline{R}}} = \frac{\gamma+1}{4\gamma}\left(\frac{p_0}{p_2} - 1\right) \tag{8}$$

$$\frac{p_0}{p_2} = \left(1 + \frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}} \tag{9}$$

$$\frac{\Delta}{\overline{\overline{R}}} = \frac{\gamma+1}{4\gamma}\left[\left(\frac{M_1^2(\gamma+1)^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{10}$$

Here, p_0 is the stagnation pressure for the post shock flow at the centerline. Equation (10) gives a relation of the ratio of the shock stand-off distance to the radius of curvature as a function of the freestream Mach number and the specific heat ratio only.

A similar equation can be obtained by using an average pressure gradient instead of the constant gradient assumption. It can be shown that the pressure gradient at the stagnation point is zero, i.e., the pressure gradient varies across the shock layer from the value given by (6) at the shock to zero at the stagnation point. Therefore, the average pressure gradient within the shock layer can be taken as an arithmetic average, i.e., one half of the value given by (6),

$$\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_{\mathrm{avg}} = \frac{1}{2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_2 + 0\right] = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_2 \tag{11}$$

It is easy to show that such an assumption is mathematically equivalent to the parabolic pressure profile in the shock layer. This leads to,

$$\frac{\Delta}{\overline{\overline{R}}} = \frac{\gamma+1}{2\gamma}\left[\left(\frac{M_1^2(\gamma+1)^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{12}$$

Another estimation of $\Delta/\overline{\mathrm{R}}$ can be obtained by assuming a constant velocity gradient in the downstream flow and using the conservation of momentum equation for steady, inviscid, one-dimensional flow,

$$\rho V\left(\frac{\mathrm{d}V}{\mathrm{d}s}\right) = -\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right) \tag{13}$$

Substituting the pressure gradient from CST into the right-hand side of (13) and applying the constant velocity gradient assumption on the left-hand side result in,

$$\frac{\Delta}{\overline{\overline{R}}} = \frac{\gamma+1}{4}\left(\frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2 - \gamma + 1}\right) \tag{14}$$

The third flow variable that can be used to estimate the ratio of the shock stand-off distance to the shock radius of curvature is the Mach number. The equation that allow to achieve that follows from (9),

$$\left(\frac{\mathrm{d}M}{M}\right)_2 = -\frac{2+(\gamma-1)M_2^2}{2\gamma M_2^2}\left(\frac{\mathrm{d}p}{p}\right)_2 \tag{15}$$

Equation (15) used with (6), the normal shock relation for Mach number, and the constant Mach number gradient assumption results in,

$$\frac{\Delta}{\overline{\overline{R}}} = \frac{(\gamma-1)M_1^2 + 2}{2(\gamma+1)M_1^2} \tag{16}$$

It is also possible to relate the density gradient at the downstream face of the shock wave with the pressure gradient using the fact that the flow in this region is isentropic:

$$\left(\frac{\mathrm{d}p}{p}\right)_2 = \gamma\left(\frac{\mathrm{d}\rho}{\rho}\right)_2 \tag{17}$$

Equation (17) can be used either with the constant density gradient assumption or the average density gradient assumption to find relations for $\Delta/\overline{\mathrm{R}}$. The respective results are,

$$\frac{\Delta}{\overline{\overline{R}}} = \frac{\gamma+1}{4}\left[\left(\frac{M_1^2(\gamma+1)^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\frac{1}{\gamma-1}} - 1\right] \tag{18}$$

$$\frac{\Delta}{\overline{\overline{R}}}=\frac{\gamma+1}{2}\left[\left(\frac{M_1^2(\gamma+1)^2}{4\gamma M_1^2-2(\gamma-1)}\right)^{\frac{1}{\gamma-1}}-1\right] \tag{19}$$

The isentropic nature of the flow in the downstream flowfield can also be used to obtain an equation that relates the temperature to the pressure gradient,

$$\left(\frac{\mathrm{d}T}{T}\right)_2=\frac{\gamma-1}{\gamma}\left(\frac{\mathrm{d}p}{p}\right)_2 \tag{20}$$

Applying (20) at the downstream face of the shock wave and employing the constant temperature gradient assumption and the average temperature gradient assumption in the downstream flow field, it is possible to show that the respective equations that relate $\Delta/\overline{R}$ to the freestream Mach number are,

$$\frac{\Delta}{\overline{\overline{R}}}=\frac{\gamma+1}{8}\left(\frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-\gamma+1}\right) \tag{21}$$

$$\frac{\Delta}{\overline{\overline{R}}}=\frac{\gamma+1}{4}\left(\frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-\gamma+1}\right) \tag{22}$$

III. Numerical Investigation of Ratio of Shock Stand-Off distance to Radius of Curvature

Flows over a sphere, representing an axisymmetric flow case, and over a cylinder, representing a planar flow case, are numerically simulated for Mach numbers ranging from 1.2 to 8 using an adaptive unstructured total variation diminishing (TVD) finite volume Euler flow solver [9]. The shock stand-off distance and the radius of curvature for each case are obtained, and their ratio is compared against the CST results and Billig's experimental correlations.

The relations for $\Delta/\overline{R}$ presented in Section II are adapted for different flow types by substituting the average radius of curvature, $\overline{R}$, with R for axisymmetric flows and with $2R$ for planar flows, where R is the radius of shock curvature.

Experimental correlations of the shock stand-off distance and the radius of curvature with the freestream Mach number are presented in [3, 4]. These correlations, in addition to the numerical studies conducted, are used to validate the CST-based estimations of $\Delta/\overline{\mathrm{R}}$. Billig's experimental correlations for spherical and cylindrical bodies, respectively, are,

$$\frac{\Delta}{R}=\frac{0.143\exp\left[3.24/M_1^2\right]}{1.143\exp\left[0.54/(M_1-1)^{1.2}\right]} \tag{23a}$$

$$\frac{\Delta}{R}=\frac{0.386\exp\left[4.67/M_1^2\right]}{1.386\exp\left[1.8/(M_1-1)^{0.75}\right]} \tag{23b}$$

A. Comparison of Analytical and Numerical Results

The ratios of the shock stand-off distance to the radius of shock curvature obtained for the axisymmetric and planar flow cases are presented in Fig. 1 and Fig. 2, respectively. It is seen that the theoretical curves form two distinct groups. One of the groups shows reasonable agreement with the present numerical results and Billig's experimental correlations for freestream Mach numbers higher than ~2 for axisymmetric flow and higher than ~3 for planar flow. The lines of this group are for the average gradient (parabolic profile) assumption applied to the flow variables having zero gradient at the stagnation point (pressure, density, temperature) or for the constant gradient (linear profile) assumption applied to the flow variables with non-zero gradient at the stagnation point (Mach number, velocity). The second group of theoretical lines gives two times lower values of Δ/R. The lines of this group are for the constant gradient (linear profile) assumption applied to the flow variables having zero gradient at the stagnation point (pressure, density, temperature). It is clear that the constant gradient (linear profile) assumption is not suitable for flow variables having zero gradients at the stagnation point because for gradient to become zero at the stagnation point, it should significantly deviate from its value at the shock.

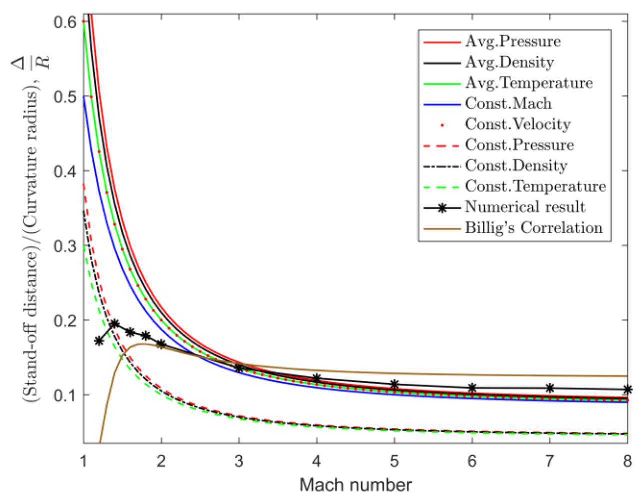

Figure 1. Ratio of the shock stand-off distance to the radius of shock curvature vs. the freestream Mach number for axisymmetric flow (over a sphere)

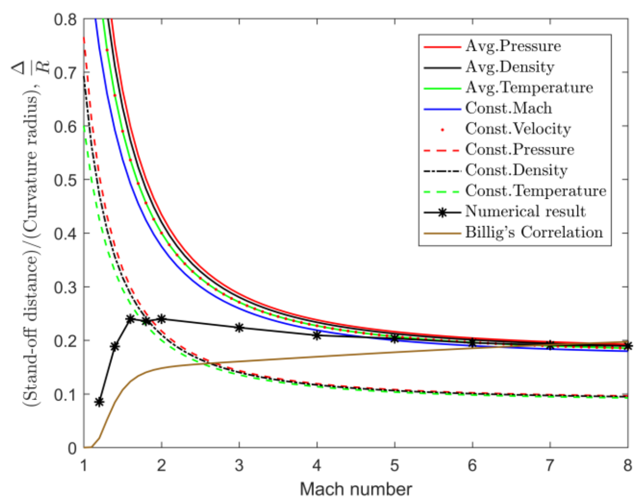

Figure 2. Ratio of the shock stand-off distance to the radius of shock curvature vs. the freestream Mach number for planar flow (over a cylinder)

B. Evaluation of Assumptions Using Numerical Solutions

To shed some light on the accuracy of the constant pressure gradient (linear pressure profile) and average pressure gradient (parabolic pressure profile) assumptions for various Mach numbers, the average pressure gradients in the flow downstream from the bow shock at the center line are obtained from the numerical solutions. The deviation of the constant and average pressure gradient assumptions from these numerical pressure gradient values are shown in Figs. 3 and 4. It is evident from these figures that for higher Mach numbers (> 2 or 3) the average pressure gradient assumption results in the pressure gradients which are much closer to the numerically determined values. On the other hand, at lower Mach numbers the constant pressure gradient assumption is more accurate. This is consistent with the results for the shock stand-off distance in Figs. 1 and 2.

C. Weighted Average Pressure Gradient Approximation

The above discussion suggests that the CST predictions for the shock stand-off distance would be more accurate if one would come up with a better approximation of gradients in the downstream flow field. To demonstrate that, a weighting factor, *W*, is introduced in the calculation of the average pressure gradient. The weighting factor is calculated as the ratio of the average *numerical* pressure gradient downstream from the shock to the pressure gradient at the downstream face of the shock wave. The results are presented in Fig. 5.

Using the weighting factor, the averaging equation (11) is modified as follows,

$$\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_{\mathrm{avg}} = \frac{1}{2}\left[W\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_2 + 0\right] = \frac{W}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}s}\right)_2 \qquad (24)$$

This allows to get (compare with (12)),

$$\frac{\Delta}{\overline{R}} = \frac{1}{W}\cdot\frac{\gamma+1}{2\gamma}\left[\left(\frac{M_1^2(\gamma+1)^2}{4\gamma M_1^2-2(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \qquad (25)$$

The values $\Delta/\overline{R}$ based on the weighted average pressure gradient approximation for axisymmetric and planar flows are shown in Fig. 6 and 7, respectively. It is seen that this approximation leads

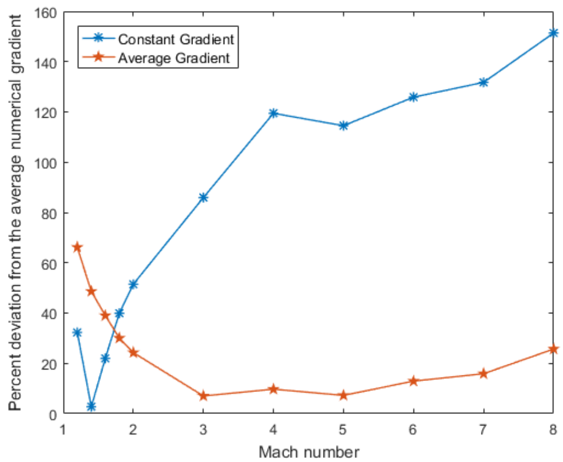

Figure 3. Percent deviation of the average and constant pressure gradient assumptions from the numerical average pressure gradient for axisymmetric flow

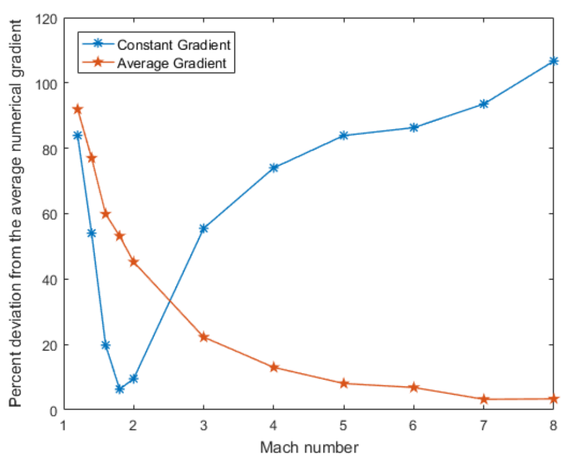

Figure 4. Percent deviation of the average and constant pressure gradient assumptions from the numerical average pressure gradient for planar flow

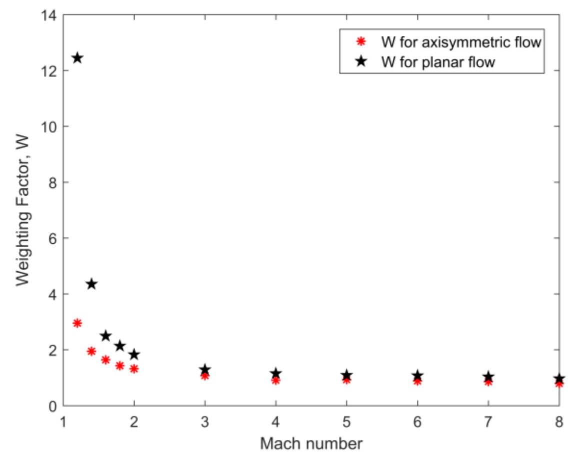

Figure 5. Weighting factor at different Mach numbers for axisymmetric and planar flows

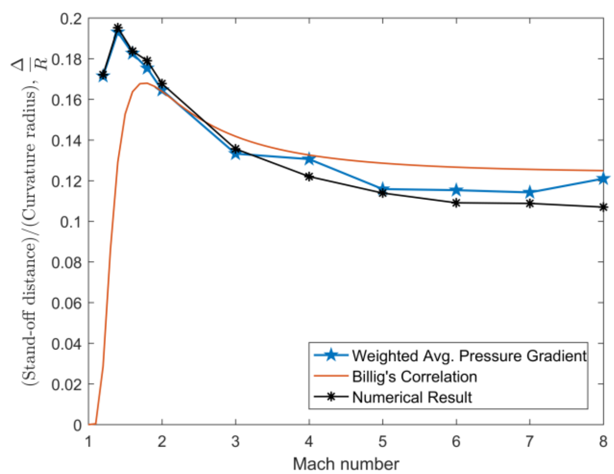

Figure 6. Ratio of the shock stand-off distance to the radius of shock curvature vs. freestream Mach number for axisymmetric flow using the weighted average pressure gradient

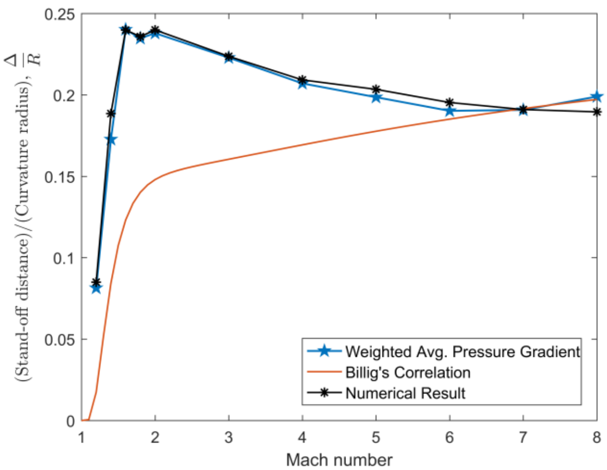

Figure 7. Ratio of the shock stand-off distance to the radius of hock curvature vs. freestream Mach number for planar flow using weighted average pressure gradient

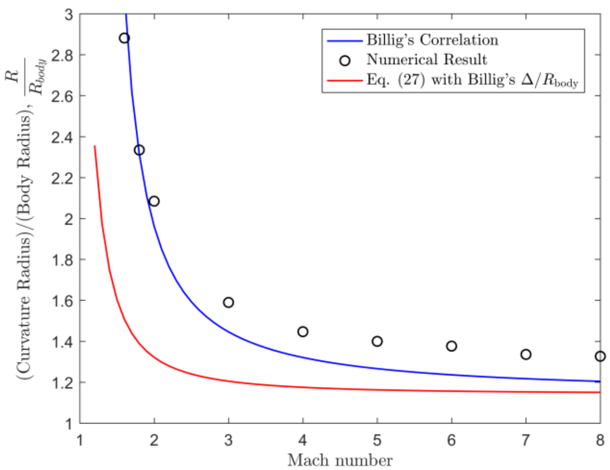

Figure 8. Comparison of the radius of shock curvature given by (27) with experimental data and numerical results for axisymmetric flow

to the results which are close to the values produced via numerical modelling (the difference is less than 10% for the whole Mach number range considered). The predictions based on the weighted average pressure gradient approximation cannot be considered as an independent theoretical treatment because numerical simulation data are used when calculating the weighting factor. However, Figs. 6 and 7 demonstrate clearly that better approximation of downstream flow gradients would certainly lead to more accurate results for $\Delta/\overline{R}$.

D. Comparison of Shock Stand-off Distance with Results in Literature Using Approximated Radius of Curvature

Most published data on the shock stand-off distance is in terms of the body radius, $R_{\rm body}$, rather than the shock radius of curvature, R. Hence, an approximation of the radius of curvature is needed to facilitate the comparison of the CST results with other data. In the present work, the radius of curvature is assumed to be the sum of the body radius and the shock stand-off distance, i.e.,

$$R = R_{\rm body} + \Delta \tag{26}$$

or

$$\frac{R}{R_{\rm body}} = 1 + \frac{\Delta}{R_{\rm body}} \tag{27}$$

Then the CST predictions for Δ/R can be converted to $\Delta/R_{\rm body}$ as follows,

$$\frac{\Delta}{R_{\rm body}} = \frac{\Delta}{R} \times \frac{R}{R_{\rm body}} \tag{28}$$

As seen from Figs. 8 and 9 it is a reasonable approximation at high Mach numbers when the bow shock lies close to the body surface. At low Mach numbers, equation (27) significantly underestimates the radius of shock curvature.

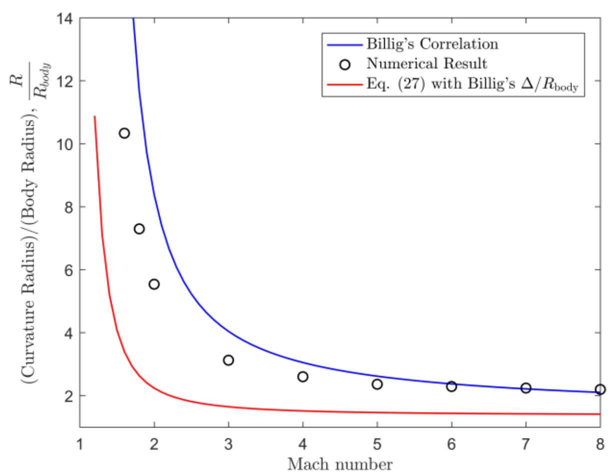

Figure 9. Comparison of the radius of shock curvature given by (27) with experimental data and numerical results for planar flow

The results for $\Delta/R_{\rm body}$ are shown in Figs. 10 and 11 for axisymmetric and planar flows, respectively. The CST predictions are represented by the same two groups of curves as in Figs. 1 and 2. Experimental results include Billig's correlations and data from Liepmann & Roshko [5]. Numerical results from the present work are also shown. The analytical prediction by Sinclair & Cui [2] is provided as well (for the planar case only).

It is seen in Figs. 10 and 11 that the CST relations correctly predict the overall monotonic variation of the stand-off distance (in terms of the body radius) with increasing Mach number (as opposed to Figs. 1 and 2 where non-monotonic behavior of the ratio of the stand-off distance to the radius of shock curvature is not reproduced by the CST predictions). This is because for low Mach numbers CST overpredicts the ratio of Δ/R (see Figs. 1 and 2) while assumption (26-27) underestimates the radius of curvature $R/R_{\rm body}$ (see Figs. 8 and 9) so that their product (28) serendipitously exhibits correct behavior (Figs. 10 and 11).

As seen in Figs. 10 and 11, among various assumptions regarding the gradients in the post-shock flow along the center line the average pressure gradient assumption (in other words, the parabolic pressure distribution in the shock layer) leads to the best CST-based result in comparison with experimental and numerical simulation data. Nevertheless, CST somewhat underestimates the shock stand-off distance for the whole range of Mach number considered. This may be attributed to underprediction of the radius of shock curvature by (27). It is to be noted that the alternative theory by Sinclair & Cui [2] appears to overestimate the shock stand-off distance.

IV. Concluding Remarks

It may be concluded that the CST-based approach shows a good promise in predicting the shock stand-off distance. Further improvements might be achieved by finding better approximations for the post-shock flow gradients and the radius of shock curvature.

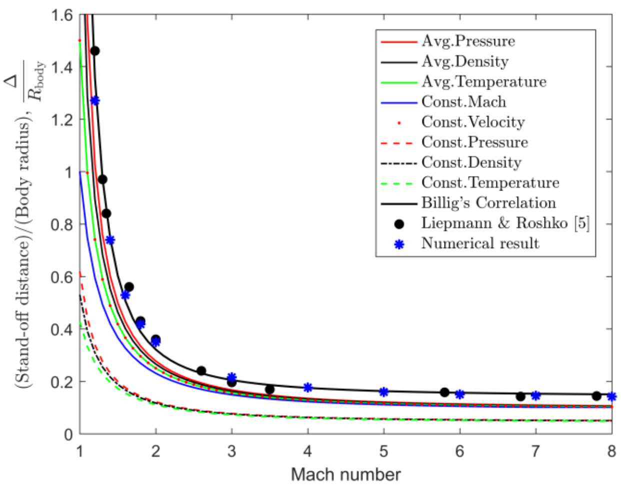

Figure 10. Comparison of CST estimates of the shock stand-off distance with experimental data and numerical results for axisymmetric flow

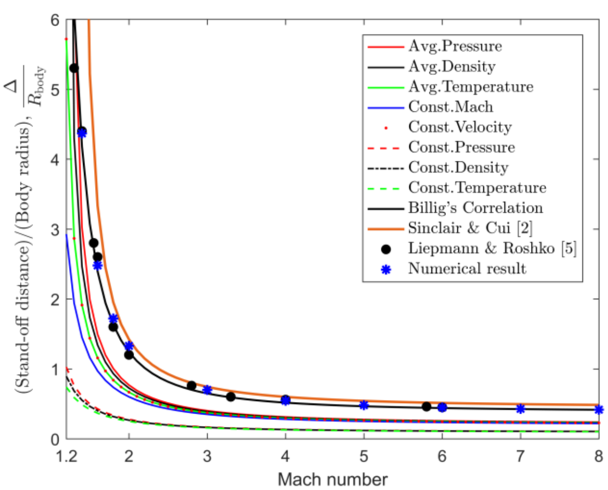

Figure 11. Comparison of CST estimates of the shock stand-off distance with experimental data and numerical results for planar flow

References

[1] Moeckel, W.E.: Approximate method for predicting form and location of detached shock waves ahead of plane or axially symmetric bodies, NACA-TN-1921 (1949)

[2] Sinclair, J. and Cui, X.: A theoretical approximation of the shock standoff distance for supersonic flows around a circular cylinder, Physics of Fluids **29**, 026102 (2017)

[3] Billig, F.S.: Shock-wave shape around spherical- and cylindrical-nosed bodies, Journal of Spacecraft and Rockets, Vol.4, No.6, 822-823 (1967)

[4] Anderson Jr., J.D.: Hypersonic and High-Temperature Gas Dynamics, 2nd Ed., AIAA, Reston (2006)

[5] Liepmann, H.W., and Roshko, A.: Elements of Gas Dynamics, Wiley, London (1956)

[6] Mölder, S.: Curved shock theory, Shock Waves Journal **26**, 337-353 (2016)

[7] Emanuel, G.: Analytical Fluid Dynamics, 3rd Ed., CRC Press, Boca Raton (2016)

[8] Mölder, S.: Curved Aerodynamic Shock Waves, Ph.D. Thesis, McGill University, Montreal (2012)

[9] `Masterix` (2003-2015), Version 3.40, RBT Consultants, Toronto, Ontario