GROWING PRIOR KNOWINGS OF ZERO: GROWTH OF MATHEMATICAL UNDERSTANDING OF LEARNERS WITH DIFFICULTIES IN MATHEMATICS

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**A DISSERTATION SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY**

**GRADUATE PROGRAM IN EDUCATION
YORK UNIVERSITY
TORONTO, ONTARIO**

June 2018

Abstract

Currently, there is a large proportion of learners experiencing difficulties in mathematics. Much of the intervention research for children with great difficulty learning mathematics has focused on accommodations to the peripheral supports of mathematics, like creating step by step plans, and not on strategies to help children conceptualize mathematics or enable them to mathematize. We know very little about the conceptual development and how to effect change in conceptual understanding of mathematics for children who have great difficulty learning mathematics. At the same time, in mathematics education research, zero is a known area of difficulty for many students and misconceptions regarding zero can persist into university and adulthood. This dissertation explores growth in understanding with three learners experiencing difficulties in mathematics and their growing conceptions of zero. Utilizing the Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding and its model for tracking growth on a small scale, I ask the questions, (i) What is the process of change, the growth of understanding, that each child passes through? and (ii) What are the images and prior knowings that children experiencing difficulties in mathematics have about zero, and how do they thicken? The analysis presented here is mainly of the task-based clinical interviews in which each learner participated. Data from parental surveys, task-based interventions and classroom observations are used to support this analysis.

Results of my research indicate how learners may be thickening and revisiting their prior knowings. Thickening occurs either as a foundation to anchor growth, or as a comparative for new growth. Results of my research also indicate that on the small-scale of tracking growth there is a juncture between expectation and result where growth has the potential to occur or not occur. This research provides descriptive evidence of intervention specifically for growth in understanding that takes into account the juncture between expectation and result. Finally, because zero is a paradox, understandings in Primitive Knowing around zero require multiple revisitings.

Dedication

Rabban Yochanan Ben Zakkai told his disciples to find the best path for a person to take
Rabbi Eliezer said that a person should be charitable in his action
Rabbi Yehoshua said a person should be a good friend
Rabbi Yossi said a person should be a good neighbor
Rabbi Shimon said a person should take into account the consequences of his actions
Rabbi Elazar said a person should have a good heart
Rabbi Yochanan Ben Zakkai answered that Rabbi Elazar's words are most preferable because they include all the others.
Ethics of the Fathers 2:13

To Saar, Yitzchok and my family
whose good hearts have supported me through everything

Acknowledgements

The time I spent writing this dissertation has given me time to think about the many individuals who have inspired, helped and supported me throughout this process. To Dori Levine, my first mentor, thank you for being a living and breathing representation of belief in the potential for growth. Thank you for igniting the spark for this dissertation. To Walter Whiteley, thank you for always thinking of me, including me, and feeding my knowledge habit. To Loretta Fiorini and Laura Greco, thank you for putting up with me and for getting me out of all the quagmires I somehow got into. To my working maths and special education working group: Helen Thouless, Yan Ping Xin and Ron Tzur, thanks for creating a space to share ideas.

My sincerest gratitude to my dissertation committee members. Thank you to Steve Alsop for challenging my thinking and pushing me to think deeper about my ideas. To Sandra Schecter, my rock and compass, thank you for always guiding me in the right direction-even if I didn't see it. To Jerome Proulx, thank you for engaging with my ideas and asking me the questions that keep me thinking. To my supervisor, Lyndon Martin, the best mentor I didn't even know I needed. You knew when to push me, when to support me, when to admonish me, and when to hold back, always guiding me to do and be my best. I learned through your modelling and wisdom, truly as an apprentice, and for that I am truly grateful. Throughout this whole process I have had a cheering squad that have helped me even when it was very difficult for them. To Aaron Shier, thank you for believing in me and keeping me on track. To my mother and father, Franceen Shier and Barry Ruttenberg, for your unwavering support and encouragement. To my son Yitzchok Rozen, for providing me with the motivation to finish. Finally to my husband, Saar Rozen, thank you for doing everything else so I could do this. You have given me unwavering support from beginning to end. This is as much yours as it is mine.

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Chapter 0: Introduction

Originally, the term mathematics came from the Greek word μανθανω meaning "to learn." Consequently, from an historical point of view, mathematics and learning were often viewed as the same (Bello, 2013, p.199). Today, partly due to globalization, the underlying meanings of mathematics are complex, encompassing a myriad of personalities in society. It can be argued that the historical meaning of mathematics, "to learn," is still inextricably tied to the myriad of ways that mathematics is viewed in society today including as an academic subject, an integral part of STEM, an aesthetic experience, a cognitive experience, an economic support and predictor, a descriptor for how the world functions, a technology and science support, and as a necessary tool and life skill for participation in society.

With the abundant ways that mathematics is intertwined with learning and society, the scholarship and practice of mathematics then is beneficial and affords many resulting enriching opportunities. Beyond its scholastic utility, these opportunities can be specific to the individual, for example promoting the development of "thinking tools" for self and participation in society, and aesthetically, the learning of mathematics also introduces opportunities for "enjoyment, creativity and for personal development" (Gervasoni & Lindenskov, 2011, p.317). Aside from enriching personal benefits, the opportunities can also be situated within community or society, for example, creating or critiquing policy, or elaborating social justice issues. In our globalized society, most of our underlying societal mechanisms and systems are based in mathematics, and thus, communicated in mathematical language (Ernest, 2002). Accordingly, the learning of and achievement in mathematics has become an important means for advancement in schooling and the professions, and for participation in society (Aguilar & Zavaleta, 2012). Lack of achievement can often prevent students from graduating high school or college, or from entering many professions (Esmonde, 2009), including professions not typically associated with mathematics (Ernest, 2002). Aside from the aforementioned benefits of learning mathematics, mathematical knowledge is regarded with high esteem in society and those who possess the knowledge of mathematics may acquire cultural capital (Gutierrez, 2012). The cultural capital in turn, can provide upward mobility[1] past class or cultural position in society. At the same time, paucity of mathematics knowledge and achievement may have debilitating powers and may be in part responsible for the creation of a new "lower class" (Skovsmose, 1998, p.201) in society.

For all these reasons, access to learning and the creation of opportunities for learners in mathematics becomes a concern of equity. The issue of equitable access is all the more pronounced when considering the estimates of between 5% and 17% of children in North American schools who are experiencing mathematics difficulties (MD) and thus, low achievement in mathematics (Geary, 2004; Geary, Hoard, Nugent & Bailey, 2012; Bartelet, Ansari, Vaessen & Blomert, 2014)[2]. The children's lack of achievement in mathematics is likely to affect their future economic and academic prospects, and importantly, leaves them without access to the myriad benefits, some already discussed, of learning and interacting with mathematics. Children who experience MDs include those with labeled learning disabilities, as well as those with sustained low mathematics achievement despite interventions and teaching (Lewis, 2014).

As will be described below, there has been a gap in research in the area of MDs in general and specifically from a mathematics education perspective. While the main fields conducting research on MDs-special education, psychology and neuroscience-have made important contributions to our understanding of difficulties children experience, they often position interventions for children with MDs from behaviorist (Lambert, 2015) and deficit

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[1] Upward mobility is determined by many factors. For an elaborated discussion of the relationship between mathematics, culture, country and upward mobility see Uri https://www.youtube.com/watch?v=J6VEyCA1pN0

[2] Note these estimates, as evidenced by the cited sources, were made by researchers in psychology and special education. The estimates that mathematics education researchers would make, would likely be higher. See chapter 1 for a full discussion of the differences between the fields in identifying and labeling MDs.

(Gervasoni et al., 2011) perspectives. Thus, there has been a hyper-focus on rote fluency[3] and procedural methods for computational procedures as interventions for children with MDs, often at the expense of conceptual understanding development and other mathematical areas. Ironically, for children with MDs it may very well be conceptual understanding of mathematics that leads them to better fluency in procedures in mathematics (Landerl, Bevin & Butterworth, 2004). The efficacy of typical interventions for children with MDs, such as teaching rules to memorize steps, may receive mixed results (i.e. Zentall, 2007) ***because*** the child does not yet have conceptual understanding. It may also very well be that educators confuse teaching explicitly with teaching rules without understanding for rote procedural fluency. Explicit teaching, that is the modeling and explanation of steps in procedural fluency, has been an important teaching strategy in teaching those with difficulties (i.e. Mills & Goos, 2011). Explicit teaching does not preclude understanding; in fact the modeling done in explicit teaching should include the modeling of thinking for conceptual understanding.

Added to this, many of the other interventions for learning mathematics for children with MDs proscribed by these fields centre around the peripheral supports (i.e. accommodations such as the way work is presented and discourse used by the teacher) to learning mathematics and not on the learning of mathematics itself (e.g. Houssart, 2004). Gervasoni and colleague (2011), considering international input, summarized the state of affairs of mathematics interventions for students with special learning needs from an international perspective:

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[3] This is the first instance I use the word fluency. Fluency has different meanings to the various entities involved in mathematics education practice and research. These meanings can range from a purely procedural activity associated with speed of recall, like what Landerl and colleagues (2004), mentioned above, refer to, all the way to a flexible understanding of the relationships within and between numbers, properties of the computations and the computations themselves (e.g. Fosnot & Dolk, 2001). In this chapter, unless explicitly written otherwise, I use the first meaning of fluency because it is more common to fields outside of mathematics education research, and my purpose is in comparing the field of mathematics education to other fields.

> …many mathematics programs and learning activities for students with ‘special needs’ attempt to teach mathematics using a conventional approach, but at a slower pace and with a more tunnelled view of a limited range of mathematics. In these cases, instructional innovations were based on deficit models of learners and focused mainly on designing tools to aid communication between the teacher and student that enabled students to access classroom mathematics programs and teaching. (p. 308).

Indeed, through focusing on deficit models and by placing a hyper-focus on fluency and supports peripheral to mathematics, research and teaching for children with MDs has become fixated on the child as the object of study, not the mathematics.

Educators cannot look to policy to rectify the current hyper-focus on fluency and the discrepancy between procedural and conceptual interventions in mathematics for children with special learning needs. Neoliberal educational policy documents and curricula, with their focus on skills, do not aid a move towards conceptual understanding either. These documents are usually focused on a narrow view of achievement as acquiring skills. Educators, in turn, interpret these policies by equating procedural fluency with mathematical understanding. As a result, interventions leading to conceptual understanding for children who are experiencing difficulty, can be viewed as superfluous, especially when there are time constraints on teaching. However, in consideration of time constraints, fluency practice does not have to be taught from a solely behaviorist method. Of note, there are other methods for developing fluency in computational procedures that extend beyond behavioralist methods. Building relational understandings (Fosnot & Dolk, 2001) in computation can also develop fluency while at the same time strengthening conceptual understanding (Landerl et al., 2004). Importantly, Fosnot (2010) has edited a book containing anecdotes of how building relational understandings has aided the growth of students experiencing MDs.

While fluency and procedural competency are important for learning mathematics, conceptual understanding is essential, especially for access to higher order mathematical reasoning, and for appreciating aesthetic components of mathematics. Mathematics education research, with its specific theoretical perspectives around education and the thinking and understanding of mathematics (Schoenfeld, 2000) including on the "activity" of doing mathematics, offers the possibility to explore "the dynamic "knowing" that portrays the growth of understanding" (Pirie, 1996, p.xv) of children with MDs. Importantly, it is the exploration of mathematical activity that may create more equitable spaces (further discussed in chapter 3) and negates deficit beliefs (Moschkovich, 2010a) about mathematics learners.

Some special educators (e.g. Cawley, 2002) have called for a shift from only "doing mathematics" (i.e. procedural fluency) to also "knowing mathematics" (p. 3) (i.e. conceptual understanding). Mathematics education researchers, already theorizing the knowing and doing of mathematics, can aid this call by helping shift the focus of mathematics interventions towards being inclusive of "knowing mathematics." Additionally, delving into the growth of understanding and exploring the changes that lead to mathematical understandings can help to shift the focus of research from the behaviorist lens of the child as object of study towards the study of mathematical outputs and "*mathematizing*."

"*Mathematizing*" is the verb of the noun "*mathematization*" (Barnes, 2005) and can be defined as "a human activity," that uses mathematics to organize the world (Gravemeijer, 1994). Thus, mathematics is the object or tool enacted by the learner, through which broader understanding takes place. In this way, through mathematizing, the whole of mathematics is considered. This means not only computational procedures, but computational procedures and other threads of mathematics, along with conceptual and aesthetic aspects, the larger environment and the human activity that allows individuals to connect with and appropriate their understandings to interact with and through a mathematical world. For this reason, procedural learning alone cannot create equitable opportunities for children with MDs, as procedural learning is intricately connected to the rest of mathematics. There needs to be a greater focus on mathematizing and the whole of mathematics.

0.1 Implicating gatekeepers: A closer look at my role and how research(ers) and systems of schooling act to ensure the narrowing of the entrance at the gate

A large population of children is experiencing difficulty learning mathematics in North American elementary classrooms. Considering that mathematics acts as a gatekeeper for future economic and academic success, achievement in mathematics for children experiencing mathematical difficulties has become a concern for a growing population of researchers in special education (e.g. Case, Harris & Graham, 1992; Jitendra & Hoff, 1996; Gersten, Chard, Jayanthi, Baker, Morphy, & Flojo, 2009), psychology (e.g. Geary et al., 2012; Mazzocco & Grimm, 2013), and medicine and neuroscience (e.g. Shalev, Auerbach, Manor & Gross-Tsur, 2000; von Aster & Shalev, 2007). In fact, an edited book (Berch & Mazzocco, 2007) purported to be the "first comprehensive and multidisciplinary examination of the study of learning difficulties" and mathematics (p. xxiv) recounts the collaboration of all the above disciplines. Conspicuously absent from the lists are the fields of mathematics and mathematics education. While the mathematics difficulties and learning disabilities fields are still in their infancy, having less research conducted than other areas of special education (Siegler, 2007), it is growing with not enough input from mathematics education researchers. To be fair, there are mathematics education researchers who have or are conducting research about MDs and mathematics (e.g. Boyd & Bargerhuff, 2009; Heyd-Metzuyanim, 2013), but their numbers are small and systematic inquiry has not yet built a research base like the other fields. Thus, their influence is really yet to be felt outside of their small circle.

Each disciplinary field has a different frame for analysis and point of view and while the field of mathematics education can learn a lot from studies conducted by researchers in special education, psychology, neuroscience and medicine, researchers in mathematics education have much to contribute to the conversation. In the realm of research into mathematics and students with MDs, much of the focus of the research from special education researchers has been on fluency interventions, and compensations or remediation to aid achievement (e.g. Fuchs, Fuchs, Powell, Seethaler, Cirino, & Fletcher, 2008; Gersten et al., 2009) with a large focus in problem solving of word problems (e.g. Kroesbergen & Van Luit, 2003); psychology has focused on identification, delineation, and categorization of difficulties, ability and measurement of performance (e.g. Landerl et al., 2004; Silver, Ring, Pennett & Black, 2007); and medical researchers and neuroscientists have focused on identification, occurrence, genetic origins, internal limitations and brain function of dyscalculia (one of a myriad of labels for a mathematical difficulty, which will be discussed in chapter 2) (e.g. Gross-Tsur, Manor & Shalev, 1996; Piazza, Facoetti, Trussardi, Berteletti, Conte, Lucangeli, Dehaene & Zorzi 2010). Coming from their various perspectives, researchers outside of mathematics education have taken their various areas of expertise and applied their findings onto mathematics education. However, without the input of mathematics education researchers, blatantly missing from these research foci is the mathematizing-the actual act of doing mathematics, the coming to know the world in a mathematical way (Gravemeijer, 1994). Thus, an important voice is missing from conversations on the teaching and learning of mathematics for children with MDs.

Translated into the school system, currently a prevalent conversation about learners with MDs surrounds achievement and remediation (i.e. identification, accommodations, modification and strategies for difficulties imposed by classroom constructs that are not necessarily mathematical in nature) for mathematics learning conditions primarily only found in school settings. However, as described in the introduction, mathematics is larger than just what is found in schooling. Thus, only focusing on achievement in school, while important, narrows the opportunities for mathematics learners. Something more is needed.

Mathematics education researchers can disrupt some of these underlying assumptions by shifting some of the focus from achievement in school mathematics to focusing on an area already being currently researched-analyzing and synthesizing underlying mathematical understandings that lead to mathematizing. Importantly, for current conversations about learners with MDs that are mainly concerned with identification and remediation, an analysis from a mathematics education perspective offers a different perspective ***because*** they are “less concern(ed) with the knowledge that a child possesses at any given instant; (instead) our focus…(is) on the "coming to know," the dynamic "knowing" that portrays the growth of understanding" (Pirie, 1996, p.xv).

"Everyone is a product of discourses," (Llewellyn, 2014, p.126) and as a researcher, I need to recognize my role in being a part of creating and actualizing discourses about learners. Thus, one difficulty that this research has presented is the difficulty of language around discussing and describing difficulties, and the learners experiencing those difficulties. In the previous section I discussed a deficit perspective underlying research about MDs and the potential for mathematics education research to create a new more equitable discourse. However, this argument is problematic and needs to be expanded upon because any research conducted, including this present study, builds on the research before. Much of the language already used to describe children with MDs traditionally stems from the medical model and, therefore, has its roots in a deficit perspective. Importantly, this deficit language is so ubiquitous in usage, it is often difficult to identify. As such, speakers and/or users, whose intentions might not be to objectify the learner, do not always even recognize that implicit meanings of a deficit perspective underlie their language use. In the next sections I outline some problematic aspects of the language of MDs, and then introduce a potential reconciliation of the problems.

Whenever one group of learners are identified, whether implicitly or explicitly, as the norm and then compared to another group of learners, a deficit model is being used (Moschkovich, 2010a). Thus, the very identification of a group as having mathematical difficulties implies a comparison to a normal group without difficulties. The group identified as having mathematical difficulties then become constructed as deficient. Through this lens of deficiency, the child, or group, not achieving in mathematics becomes synonymous with the problem. As this metamorphosis occurs, the end result becomes not only the child being viewed as "the problem" and therefore in need of fixing, but the child is also blamed as being the cause of the deficiency (Buenrostro & Figueras, 2011). Moreover, the very act of placing a label of "mathematics difficulty" may have adverse consequences. Labeling serves to construct (Chronaki, 2011) future and past "truths" about learners.

Labels declare what a child will be as opposed to what we think they are, and we build scenarios around the child to make the label become true. The act of examining and labeling difficulties is a disciplinary technique, it:

> combines the techniques of an observing hierarchy and those of a normalizing judgement. It is a normalizing gaze, a surveillance that makes it possible to qualify, to classify and to punish. It establishes over individuals a visibility through which one differentiates them and judges them. (Foucault, 1995, p.184)

Thus, labels common within the field both originate from a space of discipline and direct our gaze onto the child as object of study. This is all the more significant when combining special education labels, for example "difficulties" or "disabilities," with tactics of mathematical funneling. As discussed above, mathematics education itself already acts as a funnel, creating a "new lower class" (Skovsmose, 1998, p. 201). Thus, a special learning needs label layered onto the effects of categorization already inherent within mathematics, further "disciplines" those already disciplined.

While my main concern is with the way that my use of labeling and language negates and creates (deficit) truths about a population of students, deficit labels may be a necessary evil. Some (e.g. Ho, 2004) argue that the education system often requires labels to be in place in order to procure resources for students who need them. Moreover, some schools and school boards will not sanction the use of interventions unless the interventions have been associated as effective for a particular group of labeled children. However, this is a backwards argument. Does the system require labels, or does the system create the labels? McDermott (1996), in his seminal article, "The acquisition of a child by a learning disability," argues that schools are structured for the very purpose of categorizing and finding deficits within children. Thus, deficit labels serve to both perpetuate the aims of the school, and to continuously reinforce the school's authority and mandate. One way to combat the pervasive negative effects of labeling (Moschkovich, 2010b) inherent to the system, is to consider the strengths that a child has instead of just their "deficits."

In order to create more equitable labels, identifying language other than mathematics difficulties has been proposed by researchers concerned with equity for this population of learners. Alternative labels that have been used include differences (Lewis, 2014), and special needs or special rights (Gervasoni et al., 2011). While these re-namings help to deflect some of the disciplinary effects of the medical model, it does not deflect our gaze from the child as the source of the problem, and therefore, although labels may change, there is still a risk of equating the child with the problem.

In this study I do not attempt to overcome the problems of language. Instead, I acknowledge these problems and, for the purposes of communication to the research community, still use the language with deficit roots that is prevalent in the field: for example, difficulties, understanding and achievement. Sfard (1998) reconciles the problem of using language already prevalent in the field while trying to change the underlying inequities of its meaning, "endowing old notions with new definitions is a usual practice. In this way, the continuity of the scientific endeavor may be preserved in spite of the apparently unbridgeable breaches" (p.12 Note 8). Since this present research is situated in a field with language already constructed, in order to be a part of the research conversation I have chosen to use the discourse already in use in the field. However, while I use the language of the field, at the same time, by directing my analytical gaze away from the child towards analyzing the mathematizing, I endeavor to change the conceptual underpinnings of the language I am using.

This study is a study of the recursive changes and growth in understandings that occur, as learners who experience MDs develop more connected understandings between concepts of and related to zero. I use everyday understandings of zero as catalysts to create opportunities for growth of mathematizing of formal mathematical understandings of zero and, through zero, other number sense concepts. This study is situated as a bridge between mathematics education research and the other fields of research, as described above. Mathematics, when viewed as fluency skills from an acquisitionist perspective (Sfard, 1998), becomes a binary of acquisition and non-acquisition. Mathematical understanding from this perspective, is viewed as a linear progression between points of acquisitions, and learners are constructed as either able or unable to do mathematics at each point on the linear continuum. In practice, educators, psychologists and those doing the labeling tend to focus on inabilities. Thus, the prevailing discourse for children with MDs is one of deficiency, as they are most often recorded, in this linear model, with an inability to do mathematics (McDermott, Goldman & Varenne, 2006). However, there are also points in this linear model where learners will display moments of "ability," however inconsistently. A dichotomy arises when learners with MDs display moments of capacity amidst these expectations of deficiency. In her work, Houssart (2004) noted that teachers called this "surprising" display of mathematical ability a gift from the "maths fairy."

Because of the strength of the prevailing discourse, moments of capacity are often dismissed as insignificant, with explanations of deficiency used to clarify learners' ability. Siegler (1996) argues that in order to advance an alternative discourse, researchers should consider processes of change and the fluid, non-linear movement and moments of change. The study of processes of change should be, according to Siegler, the "the deepest goal of cognitive development" research (p.6). Also, because the process of change is individualized with commonalities across groups, the process of intervention is as well. In this dissertation, I seek to understand the processes of individual change and commonalities of change in growth of understanding of zero for children with mathematics difficulties.

0.2 Why Study Zero?

Zero as a mathematical concept is tied to many areas of lower and higher order mathematics. Some examples of this connection include exponents, zero-degree angles, zero in place value, mathematical logic, points, and topology. This list is by no means exhaustive; in fact this list is only a small fraction of the concepts to which zero is tied. In light of zero's connectedness to understanding so many other areas in mathematics, Gervasoni and colleagues' (2011) lament about the "more tunneled view of a limited range of mathematics" (p.308) to which learners with special needs in mathematics are generally exposed, becomes all the more problematic for children experiencing MDs and their learning about zero. Growth in understanding of mathematical zero and all its conceptual underpinnings, thus has the potential to support opportunities for learning a wider range of mathematics. Conversely, lack of opportunity to learn zero and its conceptual underpinnings has the potential to limit opportunities for learning.

Developing a mathematical conceptual understanding of zero that extends beyond colloquial ideas of "nothing" are difficult for both children and adults (Anthony & Walshaw, 2004). The North American curricula do not include the explicit teaching of conceptual understandings of zero. This may be one reason why adults continue to experience difficulty and have misconceptions about the number zero. Zero has been referenced in research concerning MDs (e.g. Parmar & Cawley, 1997), but the learning of zero has not been formally focused on and explored for this population of learners. Even more so, we have not explored the prior understandings children with MDs have about zero and how these understandings can be leveraged to create growth and change in mathematical concepts connected to zero.

There are those who question the objective of focusing on the specific population of students with mathematics difficulties as opposed to all children, as conceptions of zero and developing growth in understanding affect all children. It can be argued that the findings from this research should be applicable to all students, and therefore all students should be included in the population to study. However, this relationship between findings and applicability is not reciprocal. While findings from studies with children with MDs will most often apply to the broader population (Lloyd & Hallahan, 2005), findings from the broader population of students will not necessarily apply to students with MDs. Thus, it is for this rationale that the population I study with for this dissertation is students with MDs.

0.3 Discovering the Gate: My Background; and Unblocking the Gate: My Aims

Experience and research are intertwined, one informing the other. “Teachers are constantly faced with “felt difficulties” or dilemmas as they reflect in and on their acts of teaching. As a result, these “felt difficulties” are direct concerns that emerge from one’s own teaching experiences” (Dana & Yendol-Silva, 2003, p. 15). As Dana and Yendol-Silva iterated, my background as a teacher and my “felt difficulties” provided the impetus for this dissertation project. Although I can trace a number of experiences that influenced my research questions, one specific teaching experience had the greatest impact. In the following paragraphs I will describe my experiences surrounding teaching David. Then I will use the description to outline my aims for this dissertation. Please note that while I do describe dialogue, the dialogue is from memory and paraphrased the way that I remember.

I was a mathematics and science teacher in a self-contained grade 4/5 split special education classroom. Before the start of the year I met with the school principal to discuss the children I was about to teach. Most of the children were either diagnosed with a learning disability or ADHD or both, and some children had other diagnoses, i.e. deletion 22q11.2. Then there was David. David was diagnosed as a “slow learner.” The label of slow learner was meant to convey low intelligence, certainly below average, and falling somewhere between 75 and 90 on a normal curve of intelligence (Williamson & Paul, 2012). I will discuss in chapter 2 the arbitrariness of committing to a spot on a normal curve of intelligence and labeling those at that spot with “low intelligence” or whatever the popular term is nowadays. However, for this section it suffices to conclude that those with educational authority around David had little hope for him. They believed that teaching would not matter for him, especially in mathematics. He would not be able to progress. This was the explicit description of David conveyed to me by the principal before I began the school year. I was told that my responsibility was to focus on the other students who may possibly have potential to learn mathematics, but not to “waste” too much time and effort on David. David just could not understand.

David's IEP (individual education plan) contained a list of things he could not do that inhibited his growth. While I can pause here in my writing to discuss the list of things that David could not do in this section, I will not, as it will detract from what I found David could do. I created a special curriculum for David that year. However, when we did group activities, especially mathematical problem-solving, David was always included in the lesson. It was David's participation during one of those lessons and the principal's subsequent response that led me to the research questions for this project.

We were exploring student generated algorithms for multiplication. Through constructing their own algorithms, learners develop multiplicative reasoning, specifically reasoning with sets of numbers. This is an especially abstract reasoning to develop and use (Barmby, Harries, Higgins & Suggate, 2009). This reasoning, is at the same time, necessary for more advanced mathematics. I gave the class the question of what is 16 x 4. I told the class that they could use anything in the room as a tool to help them solve the problem. Some students gravitated to various materials, such as base ten rods and snap cubes; others didn't use manipulatives at all, they drew pictures. David was different. He did not gravitate to anything right away. Instead, he paused and thought about the task for a while. David, then proceeded to the abacus. At this point, I was drawn away from my observations of David, as other students required help. While helping the other students I heard David call out in a proud voice "64. The answer is 64." I finished with the students I was working with and immediately made my way to David's desk. I wanted to interact with him, to question him and discover what reasoning he was using.

When I came to David's desk I asked him, "David, why do you think the answer is 64? Can you show me your reasoning?" David pushed all the beads on the abacus to one side. He said, "You see, 16 is 10 and 6." I nodded to show I was listening. David continued, "Well, 10 times 4 is this." David pushed 4 rows of beads aside on the abacus, counting 1,2,3,4 as he pushed each row. "And, then you have 4 sixes." David counted out six beads on 4 rows and pushed them aside to align with the rows of ten. "Then you count them all together, I know that these are 10, so 10, 20, 30, 40, that's 40. Then this is 6 so 46." Then

David counted each bead, one-by-one from 46 to 64 to get the answer. I had just witnessed David do something he wasn't supposed to do-David showed abstract reasoning and mental flexibility with numbers, two important foundations to number sense and higher order mathematics (Fosnot et al., 2001). I wanted to share the news with my principal.

I waited until after school to tell the news to my principal. When I approached the principal after school, she told me "David really can't do mathematics. He is really low. What you think you saw is not really what David can do-he is a slow learner." I will stop my description of the incident here. More than what David did, the principal's response had a profound effect on me. How could she negate what David did? How could she rely on a static, inequitable, arbitrary statistic for what David was capable of? David had just done a complex mathematical maneuver. Why was it not recognized? I was witnessing the veracity of Siegler's (1998) assertion about the strength of the prevailing negative discourse causing moments of capacity to be dismissed as insignificant with explanations of deficiency used to clarify a learner's ability. Thus, the aims of this research are: (i) to explore the pathways of change that occur for students experiencing MDs; (ii) to explore strengths that lead to contexts allowing for growth and change, and (iii) to explore the fluidity and recursiveness of mathematical understanding and learning.

There is an essential underlying tension in this dissertation that I would like to make explicit from this first chapter. This tension is between my experiences, my habits, my aims, my questions and my research design. As described in the previous paragraphs, it was a surprising moment of inequity that led me to this dissertation journey. I want to discover why, but more than that, I want to see change. At the same time that I am situating this work within an equity framework, my learned habits direct me to what Sfard (1998) calls an "acquisitionist" stance. I have to constantly be on guard to keep myself from (re)visiting these linear notions of understanding and learning. Because of my focus, and my experience I am constantly on the line between acquisition and non-acquisition.

> "…one finds it extremely difficult to avoid the acquisitionist language altogether. Whenever we try to comprehend a change, the perpetual, bodily roots of all our thinking compel us to look for structure-imposing invariants and to talk in terms of objects and abstracted properties. We seem to know no other route to understanding. No wonder, therefore, that those who oppose objectification and try to exorcise abstraction and generalization from the discourse on learning find themselves entangled in conflicting statements. They may be making heroic efforts to free themselves from the idea of learning as acquisition, but the metaphor- engraved in the language- would invariably bounce back… As I argue in the concluding section, even if one cannot solve the dilemma, one can- and probably should- learn to live with it." (Sfard, 1998, p. 10)

0.4 Research Questions

The rationale for this study, already outlined above, is the paucity of research in mathematics education for children experiencing difficulty in mathematics and their understandings of zero. The focus of my study is in exploring the understandings of zero learners bring with them, and the processes of change that are evoked by mathematizing through discursive exchange and tasks, that is, the growth of understanding. The following research questions are proposed:

1) What are the images and prior knowings that children experiencing difficulties in mathematics have about zero?
2) What is the process of change, the growth of understanding that each child passes through?
    a. Are there commonalities between the processes and mappings of developing understandings?
3) How do images and previous understandings about zero thicken through interactions and mathematizing around mathematical tasks?
    a. What specific conceptual areas about zero thicken through mathematizing?
    b. Does specific intervention interrupt pathways and initiate change? And if so how?

Chapter 1.0: Literature Review

Three research areas are of import to this project: (i) equity and ability labeling (ii) Mathematics Learning Disabilities and Difficulties, and (iii) Zero. Consequently, in what follows, I begin with a discussion of equity in mathematics education. Through this discussion I identify a useful equity framework, Gutierrez's (2012) four dimensions of equity, and explore this framework as it pertains to ability labeling. Next, I give an overview and synthesis of the research on Mathematics Learning Disabilities (MLD) and the labeling of a Disability. I then utilize this discussion of MLD in order to create the parameters for defining the term "Mathematics Difficulties" (MD) used for this present study. Last, I discuss zero, first through connecting its historical progression to research concerning developmental progressions of zero, and then through synthesizing research of the teaching, learning and understanding of zero.

1.1 Equity, An Introduction

I argue in this dissertation that creating opportunities for students experiencing MDs is an imperative of equity. Although mathematics education research has addressed issues of equity for a quarter century already (Ellington & Prime, 2011), and equity is a concern for every aspect of schooling, there is still a paucity of research (Cobb & Hodge, 2002). Many inequities that were discussed 25 years ago, sadly still exist in mathematics education today (Ellington et al., 2011).

Wagner and colleagues (2012) assert that conversations about equity in mathematics education became more robust in recent years because of research using sociocultural theoretical frameworks. Sociocultural frameworks are important to equity research because these frameworks shift the considerations from personal access to include social considerations as well. As a consequence of these considerations the view of the child shifts from being an object to be studied, to being a part of complex and situated interactions. Thus, through this lens, the environment becomes at least as important to understanding equity and inequities, as the individual (Moschkovich, 2010a). One earlier example of the use of sociocultural theoretical frameworks in mathematics education equity research is Cobb and Hodge (2002). Through the lens of sociocultural theory, they situated equity within a social and affective context. Aside from issues of personal access, Cobb and Hodge include in their conception of equity students' relationships and interactions with academic and real-life mathematics, and the mathematics identities they build as a result (Cobb et al., 2011). Thus, equity is not a tangible possession but social relationships in and through people and their environment (Walshaw, 2011).

Although defining equity is important for identifying opportunities and considering efficacy for equity (Ellington et al., 2011), equity is difficult to define and is often confused with equality. According to Allexsaht-Snider and Hart (2001) equity means access to mathematics for all learners, regardless of their cultural or learning background. To them, equitable practice is taking into account then attempting to diminish the differing backgrounds of students in order to provide equal opportunity. Aside from, and maybe because of, viewing differences in a negative light-as something that needs to be diminished-this understanding of equity, though, is incomplete and requires further elaboration. Allexsaht-Snider and Hart are confusing equity with equality. Through utilizing one meaning of access for all learners, they have run the risk of potential inequity through "unequal outcomes."

In defining equity, it is important to consider and differentiate it from another close term: equality. Equity and equality are very different with very different potential consequences: "Equity refers to the unequal treatment of students (or people more generally) in order to produce more equal outcomes. In contrast, equality means the equal treatment of students with the potential of unequal outcomes" (Zevenbergen, 2001, p.14). Thus, equity and equality are really opposites. Equality looks to create the same initial conditions for everyone regardless of whether they are needed, ignoring the elaborate and complex differences that produce individuals in the first place. Whereas equity attempts to take into account these elaborate differences to nevertheless try to create at least similar results. As will be argued later, the same results for everyone is really an impossibility, subsequently making the realization of equity an impossibility. However just the acknowledgement of the potential for more equitable outcomes, creates more opportunity for equity (Gutierrez, 2012).

In defining equity, scholars like Allexsaht Snider and Hart (2001) and Bose and Remillard (2011) each take into account only a few aspects of equity. A more complete definition that takes into account multiple aspects of equity including experience, can be found in the work of mathematics education equity scholar, Rochelle Gutierrez. Gutierrez (2012, p.18) gathered multiple aspects of equity and categorized them into four dimensions:

(i) access,
(ii) achievement,
(iii) identity, and
(iv) power.

Each of Gutierrez's dimensions are complex, interrelated and relate to inherent systemic inequities within the school system and the teaching and learning of mathematics. All of these dimensions are integral to equity but equity does not exist when only one is present, as each has discrepancies that are filled by another.

1.1.1 The Four Dimensions of Equity Elaborated

Of Gutierrez's (2012) four dimensions of equity, access is the first category. Access is related to 'opportunity to learn,' a rhetoric popular thirty years ago that assumes everyone is equivalent. It addresses the learning supports that create opportunities for learning in the classroom. Stemming from access, Gutierrez's second dimension is achievement. Achievement describes the outcomes from access. These achievement outcomes include not only school based, both K-12 and post-secondary success in mathematics, but also postacademic, including access to mathematics careers. Achievement as equity originated as a discourse twenty years ago as a way to view equity in relation to standardized assessment.

The third dimension of equity, identity, has arisen as an important aspect of achievement. School-based mathematics often implicitly negates the identities of those in the non-dominant culture by forcing them to conform to sometimes opposing perspectives of mathematics (Wagner & Borden, 2011). Thus, in order to achieve and participate in school based and societal mathematics, students may begin to construct deficit views of themselves and their cultural backgrounds. These deficit views may then lead them to negate their identity. Thus, considerations of mathematical identity sensitive to non-dominant populations should be built into any equitable mathematics curriculum.

Finally, even if all the other dimensions of equity-access, achievement and identity-are in place, issues of power may still disrupt equity. Wagner and colleagues (2012) discuss that each learner experiences constant "positioning" (p.3) in relation to the structures, including mathematical structures and social structures, within and without the classroom. Consequently, power may have an impact on equity concerning the personalized, social impact and social justice components of mathematics: "voice in the classroom," "opportunities for students to use math as an analytical tool to critique society," "alternative notions of knowledge," and "rethinking the field of mathematics as a more humanistic enterprise" (Gutierrez, 2012, p.20).

To sum up the dimensions: a learner has to have access to the mathematics that allows for school and life achievement, societal participation **and** societal contribution (Gutierrez, 2002, p.158), while simultaneously gaining the opportunity to critique the inequities of the dominant mathematics and their situation at the foundations of their learning and achievement.

1.1.2 The Four Dimensions: An example of Ability Labeling

One belief underlying inequitable social practice in the learning and teaching of mathematics is that learners should be categorized and thus labeled based on ability. Then, based on their category the learners should be provided with "the appropriate" form of instruction. Ability labeling, then, and its inequitable practices and effects are partly at the root of the research questions and some methodological choices for this project. Because a strong example of the interplay of Gutierrez's (2012) dimensions with mathematics difficulties can be seen in the practice of labeling perceived lack of "ability" in mathematics and the resulting proscribed instruction, in what follows I analyze the construct of ability and then the practices of ability labeling. Throughout the discussion, I utilize the dimensions of equity to illuminate the analysis.

The term "ability" does not have the same meaning in North America and some other Western nations as it does in other areas of the world. In North America, mathematics ability is typically viewed as located within the child and as fixed and predetermined at birth. This is in contrast to other places where mathematics ability is equated with effort (Gutierrez, 2002)[4]. Skovsmose and Penteado (2011) further elaborate upon this conception of ability and its inherent problems:

> "However, the term 'ability' is a strikingly misleading concept—most dangerous because it has come to assume an almost universal currency. It designates some phenomena as personal or as individual characteristics of the students, while these phenomena more realistically represent characteristics of the students' learning conditions. 'Ability' is thus a social construct, not a psychological one." (p.83)

Thus, in North America, mathematical ability is seen to describe the inner workings of a child as opposed to describing the opportunities and structures that support learning.

[4] Some western countries are even more egregious than those in North America. The UK, for instance, practices tracking based on ability from a very young age (Boylan & Povey, 2014).

Consequently, “ability” becomes a reifying, hierarchical, and elitist societal construct, which is then used to categorize and subsequently funnel students. The construct of ability as typically conceptualized in North America is inequitable and dangerous. In fact, Gutierrez (2002) states that it is that belief that only some learners have the ability to do mathematics that threatens equity.

If ability is viewed as fixed, then, the logical argument that follows is that interventions and teacher effort will not really help those with ‘low ability.’ Thus, even in differentiated classrooms, without an explicit belief that all children can learn and achieve (Howery, McClellan & Pedersen-Bayus, 2013), and that ability in mathematics is not inborn, children experiencing MDs will remain excluded from learning higher order mathematics (Gervasoni et al., 2011). Those who perpetuate the belief that children do not have the ability to do mathematics, may in fact do so in order to create more opportunities for those deemed worthy to achieve in mathematics (Gates, 2014). Importantly through the labeling of ability, learners are continuously disempowered (Gutierrez, 2012).

1.1.2.1 Inequitable Outcomes of Ability Labeling

Even if the intention of labeling is directed at only one group, namely those of “low-ability,” it is through the very practice of labeling one group that two groups become categorized: those who receive the label–a negation relative to a second group-and those who do not–a positive position relative to the negated group. Because higher order mathematics is often determined as “inappropriate” for their learning level, the “appropriate” instruction for children with low ability-type labels rarely includes higher mathematics past fluency (Gervasoni et al., 2011, p.307). Many students who have been categorized with ability deficiencies have their “appropriate” curriculum of learning outlined as modifications and accommodations on individualized education plans (IEPS). Aligned with the low expectations, the accommodations and modifications often proscribe similar methodologies (Cuban, 1989), fluency and practice. These prescriptions may be at the expense of other affordances that may provide much needed language practice and support for students experiencing MDs. For example, participation in the reasoning and problem-solving practices of higher order mathematics often provides academic and mathematical language support to students (Allsopp, McHatton, & Farmer, 2010). In other words, the accommodations and modifications on IEPs that claim to create equitable opportunity, or access, to support student learning, may actually have the opposite effect by limiting participation, or achievement, in mathematical activities that may help the learner. And, as the child moves on in the system, and their growing "gap" is identified by each new teacher, then confirmed by the IEP, or vice versa, the mathematics the labeled child learns is further dumbed down (Boylan et al., 2014) creating a circle of deficiency (Gates, 2014).

This all occurs because the "appropriate curriculum," lacking in higher order thinking, widens the distance between those categorized as learning deficient and those categorized as not learning deficient. As a result, the belief that children categorized with "low-ability" are incapable of learning mathematics is further solidified. Additionally, the means of reversing these deficit beliefs, through viewing the learner actually participate and reason in mathematical activity (Moschkovich, 2010a), cannot occur because the "appropriate" instruction takes precedence. The instruction deemed "appropriate," especially for those categorized at or near the bottom end of the hierarchy, further impede those already marginalized by society, and further stigmatize them as incapable of succeeding in mathematics (Allexsaht-Snider et al., 2001). In turn, these practices then result in an "undermin(ing of) their sense of self " (Walshaw, 2011, p.96). In this sense, these established practices have become discourses of truths, continuously constructing and objectifying the learner as unable (Foucault, 1972).

The constant comparison between those categorized with an inability and those with an ability, others often already marginalized students (Ross, Hogaboam-Gray & McDougall, 2002) and serves to continuously subjugate students academically (Gates, 2014). Some argue (e.g. Mather & Gregg, 2006) that it is the act of categorization, meaning testing and labeling, that allows those involved with the education of these labeled students to know how students learn. And only then can they prepare appropriate curricula and interventions for the learners. However, the way students learn, like the multifaceted aspects of equity, are also tied to a multiplex of inputs, causes, reactions, interactions and circumstance. Categorizing a learner based on one small aspect, usually achievement, of this multiplex is unconscionable. There are other ways of really learning about students and getting to know their learning preferences, styles and points of access. Two examples of other ways of getting to know students is: (i) Noticing: Mason (2002) has put forward a notion of "noticing" that would enable teachers to notice their students' actions, as opposed to making judgments about ability they cannot see. Through noticing, one can make discoveries about a student's learning, that takes into account the supports and surrounding environment, without sourcing the problem within the child; and (ii) Clinical interview/ dynamic assessment: The teacher or researcher interacts with each student differently, probing for understanding and modifying their own interactions based on the child's response (Ginsburg, 1997)[5].

Thus, in terms of the dimensions of equity: systemic practices of labeling operate under the façade of creating more access-*dimension of access*-but actually limit achievement both within the classroom and eventually post-schooling as well-*dimension of achievement*. The labelee's identity-*dimension of identity*-becomes tied to their access and achievement and the ability to see the world through mathematics, or mathematizing, and utilize mathematics to critique their own situation-*dimension of power*-is essentially taken from them.

[5] Because it is part of the research design, the clinical interview will be further elaborated in chapter 3.

1.1.2.2 Ability Labeling in Mathematics Education Research

Unlike researchers in other fields of education,[6] many mathematics education researchers like Skovsmose and Penteado (2011) take a positive view of ability. However, a caveat must be added because current mathematics education research trends in reform or problem-based learning act in much the same way that the categorization of inability of learners results in a bifurcation of ability and inability. The explicit construction of "good learner(s) and teacher(s)" (Chronaki, 2011, p.8) in mathematics research education also constructs the opposite-bad learners and bad teachers (Lambert, 2015).

At the same time as this trend in mathematics education research is constructing less-able learners, the research in mathematics education that does explicitly discuss learning potential or masked ability can be problematized as well. This type of research is also often constructed upon the underlying assumption of a standardized, often romanticized, way of learning mathematics[7]-or the construction of "good learners" (Chronaki, 2011, p.8). The exclusion of children with mathematics difficulties from research then leads to another type of exclusion that may be diametrically opposed to the intentions of the research in the first place. Children experiencing MDs may respond differently to reform approaches to mathematics learning and teaching (Woodward, 2006). These learners may require other important mathematical understandings and skills, i.e. procedural fluency (Gutierrez, 2002), that sometimes get ignored or their importance is downplayed (Llewellyn, 2014) in teaching. And it may be that the very understandings that get ignored are the ones needed for learners to use mathematics for equitable participation in society (Bose et al., 2011) and thus, to experience access, achievement, mathematical identity and mathematical power.

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[6] As discussed in chapter 0 research from the perspectives of psychology and neuroscience in learning mathematics are often preoccupied with delineating inherent ability.
[7] See section 2.1 for a discussion as it relates to the foregrounded theory.

Even specifically in intervention research, interventions are often at least implicitly romanticized as meeting the needs of all learners of the population they are meant for. Martin (2011), exploring racial inequities in reform and intervention research efforts, challenges this underlying inequity: "(1) if it is good for whites, then it will be good for other groups and (2) before it can be considered good for everyone, it must be considered good for whites" (p.445).

There is another problematic aspect of the research in mathematics education that does explicitly discuss learning potential or masked ability. So far, the research in differences in learning mathematics has focused mainly on inherent inequities within the system for learners belonging to non-dominant cultures (i.e. Cobb et al., 2002). I want to be clear-basing an argument for ability mainly from a cultural standpoint has had positive, equitable consequences. Cultural differences, and cultural achievement difference definitely do exist and there is an immediate concern concerning social inequity in classrooms (Gates, 2014). Importantly, these inequities do lead to a disproportionate number of learners from cultural minorities labeled, categorized and placed into special mathematics programs throughout all schooling levels, including pre-school to university (e.g. Hallahan & Mercer, 2001; Larnell, 2016). However, there is one problematic aspect of this issue that specifically concerns those experiencing mathematics difficulties. Populations of learners are unnecessarily grouped into static cultural types of learners, excluding their individual learning differences, and ignoring potential mathematics difficulties. While there are many reasons this problem occurs, a main reason is because of over-reliance on large data-sets (Gutierrez, 2012).

Similar to ability, cultural typing is another method of categorizing members of one group, the non-dominant population and what a learner should not be, versus another group, the dominant population and what a learner should aspire to be. Segregating populations of learners based on achievement and lack of achievement is problematic in its own right and leads to what Gutierrez and Dixon-Roman (2011) term 'gap gazing.' "Gap gazing… accepts a static notion of student identity, presuming that students can be reduced to a set of cultural markers, rather than recognizing they are constantly in flux, dependent upon the social structures and social relations in which they are engaged" (p.23). In gap gazing researchers focus on low achievement as a result of lack of access and deficiency as a result of a discrepancy between a 'subordinate population' and a 'dominant one.' Moreover, this type of gaze continuously operates from a deficit model, where the learner is the one that is required to improve in order to benefit from the mathematics as opposed to vice versa (Gutierrez, 2002). Kris Gutierrez (in Gutierrez, 2002, pp. 157- 158) summarized this problematic aspect as "Do I get to become a better me, or do I have to become you?"

Positioning learners as members of a non-dominant culture or possessing a disability are both "context-dependent" (Baglieri, Valle, Connor & Gallager, 2011, p.271), meaning both acts of labeling learners depend on the surrounding context. Importantly, these can be different contexts. And, while we rarely read about them in the research literature, what happens when learners from non-dominant cultures experiencing mathematical difficulties do not achieve based on the proscribed interventions? The logic of these arguments would follow that if a learner's ability is only supposed based on a cultural differences argument, and does not include innate learning differences:

- and after research-based interventions, for example culturally responsive mathematics teaching (Aguirre & del Rosario Zavala, 2013), developed specifically for social inequities are used,
- and students still have difficulty understanding mathematical concepts or following procedures;
- then because learning differences are perceived to be "outside" of social inequities inherent within the system,
- then by default it then becomes acceptable to label learners with learning differences with the other "likely" culprit at the root of the label-"low ability"
- and while more interventions, some successful, may be enacted until that learner may experience achievement for a short while, long term equitable achievement

that empowers learners past their schooling years does not really occur because the whole complexity of their difficulty is not addressed.

Thus, even though mathematics education appears to be proactive in discussions about ability relating to mathematics learning, practice, situated within broader research practices, neglects learners who experience MDs.

1.1.2.3 Ability in the Context of a Neoliberal Education

The use of mathematics through media and politics is pervasive in society. Ernest (2002) call this proliferation of mathematics the "mathematisation" (p.7) of society, where mathematics underlies almost every functioning aspect of the world around us. An understanding of mathematics thus becomes important for the fourth dimension of equity: access to participation. However, it is often in the interests of maintaining hierarchy and power structures that some members of the populace not become proficient in mathematics. "Mathematics education can function as a kind of social filter" (Aguilar et al., 2012, p.6). In our neoliberal system of education, the goal is not that all students *receive* knowledge, an expensive outcome; rather the goal is to take the less expensive route to funnel knowledge to only those deemed *worthy,* thus creating a hierarchy of knowledge bearers (Apple, 1992). This hierarchy also has the potential benefit of suppressing future dissenters who would use their knowledge to critique the system (Skovsmose, 1998).

Similar to the identity dimension of equity, Skovsmose (1998, p.197) calls this funneling, "inverse competence," where certain members of the population believe that mathematics is not their domain. As a consequence of the funnel, barriers are put in place limiting access for certain segments of the population. An example of one of these barriers is the ignoring of the aesthetic aspects of mathematics in favour of a view of mathematics as mechanistic, solitary and in the service of industry. In fact, policy efforts for educational mathematical reform, originate not from a need to create more opportunities for more students, but to create more students who will be able to perform low-level mechanistic and solitary work (Apple, 1992). Again, at the foundation of this funneling effort is the same belief that not all children have the aptitude or potential-read ability-to learn (McDermott et al., 2006). This is the paradox of the "mathematics for all" reform rhetoric (Pais & Valero, 2011, p.44): that mathematics education researchers debate (Gates, 2014) whether the funnel system creates an impossible-to-fix situation for creating opportunities for all students, when in reality the system pushes only the most elite towards becoming the bearers of knowledge. Therefore, like Gutierrez (2012) argues, access cannot be a sole source of equity. Access requires the other dimensions as well to work together and target the multiple layers of inequalities. Pais and Valero (2011) argue that instead of fighting these systemic "truths" (p.45), it is more important to acknowledge, explore, and forge ahead. Importantly, forging ahead and communal goals, may have more of an impact on promoting equity than the actual existence of equity in the system (Gutierrez, 2012).

In the next section I move on to the labeling of mathematics learning disabilities and difficulties. Extending the above review on ability labeling, I discuss the problematic aspects of labeling disabilities in mathematics, with specific attention to the special consideration and nuances of mathematics learning.

1.2 Mathematics Learning Disabilities

There are many different terms for students who experience difficulties in mathematics due to a learning disability (LD). These terms have included: acalculia, dyscalculia, mathematics disability, mathematics difficulties, arithmetic disorders and mathematics disorders. There has been no real consensus as to the definitions of each term-the meanings can vary by researcher, context or field. Additionally, these definitions of LD that are utilized tend to have two problems. First, the definitions tend to be too general and expound little on exactly what is an LD. Second, the definitions tend not to answer why one child may be diagnosed and another child not diagnosed (Kavale, Holdnack, & Mostert, 2005). However, there are some commonalities across the definitions in use that may be helpful in understanding the label (Gersten, Clarke & Mazzocco, 2007).

One commonality, which will be further discussed below is that of the relationship between the labeling of an LD and perceived classroom achievement. However, this same commonality between definitions, is a significant factor in the construction of distances between the fields of special education and psychology, and mathematics education (Sfard, 2008). In the following section the differences in definition of an MLD between mathematics education research and other fields, the diagnosis and etiology of an MLD, diagnostic tests for MLDs, and an expounding of mathematics difficulties are discussed.

1.2.1 Etiology and Diagnosis of a Mathematics Learning Disability

Sfard (2008) claims that it is differing discursive constructs and perspectives between the fields of mathematics education, and special education and psychology that have kept mathematics education researchers from actively engaging in the field. Specifically, the etiology and diagnosis of LDs in mathematics has had a lot of influence in maintaining this separation. Sfard iterates that the current discourse about LDs and mathematics stems from psychology: the nature/nurture debate, low achievement being as a result of cognitive deficiencies, and a view of knowledge acquisition and understanding as linear. Many of the current beliefs about the teaching and learning of mathematics in mathematics education research are at odds with these conceptions (e.g. Davis, 1996).

Mathematics education theorists are not the only group at issue with the diagnosing and etiology of a mathematical LD. Those doing the diagnosis, themselves, cannot agree on criteria for diagnosing. The very criteria for labeling an LD have had a controversial and inconsistent history in North America. Different states and provinces, school districts, and individual psychometers may each have different criteria for identification of an LD. Aside from different criteria, statistically the tests that diagnose LDs, because they rely on cut-off scores, are very problematic. That is, the standardized tests rely on a discrete point on a continuous “normal” model to diagnose an LD. Aside from leading to ambiguity in diagnosis, this reliance leads to two problematic outcomes. First, students who are experiencing a real mathematical difficulty may not fall within the discrete area required for a diagnosis. Thus, even though they experience real mathematical difficulties, without a diagnosis they would probably not receive support at school. Second, the problem of a discrete point on a continuous model creates an issue of over-diagnosis as well. Students may be labeled with a mathematics LD, because they fall within the pre-determined scores but may not be experiencing any mathematics difficulty.

Thorndike (1963) elaborates the many statistical problems with these diagnostic tests, including even giving parameters for "achievement." Most significant among the problems is at the foundation of these diagnostic tests is the use of a normal model. Normal models create, and are used as, a comparison to a "norm"-meaning a "normally" functioning, or achieving, child. Thus, instead of celebrating differences between children, any score that differs from the norm, especially those that differ to the left (or are lower), are considered negatively, relative to the "normally" functioning child (Davis, Sumara & Luce-Keplar, 2000). The scores, in relation to their position on the normal model, serve to put parameters around learners. If learners achieve beyond what these scores predict, that is outside or to the right of their position on the normal model, this achievement is negated with the label of "over-achieving" (Thorndike, 1963).[8] This, despite the arbitrariness of even using discrete cut-off scores on a continuous model to begin with.

Fletcher and colleagues (2013) group the various methods of diagnosis into four different categories:

(i) "aptitude- achievement discrepancy" (p.37): a discrepancy between aptitude, or measured intelligence, and achievement;

(ii) "patterns of cognitive strengths and weaknesses" (p.39): utilizes standardized tests to compare cognitive processes in relation to achievement;

[8] Note that I use an older source from 1963 to support my statement around over-achieving. This is intentional. It has been 50 years since Thorndike wrote his treatise on "over-achievement," and, yet, I have experienced this practice quite recently, with psychometrists making judgment on why certain children are achieving beyond what their scores would predict.

(iii) “low achievement method” (p.41): diagnosis is concentrated only on low achievement because of the statistical problems with labeling; and

(iv) “inadequate response to quality instruction” (p.47) with low achievement: stems from the RTI (Response to Intervention) intervention model utilized in the United States and sporadically in Canada. Essentially, in this category, students are screened and tracked before they are referred for diagnostic testing. During the screening and tracking process, learners are presented with Research-based intervention programs (see section 1.2 for a discussion on these “research-based” intervention programs), and then, only if these intervention programs do not work, and thus the learners are experiencing low achievement, are learners referred for diagnostic testing.

Fletcher and colleagues systematically problematize each of the first three approaches to diagnosis, recommending the fourth as least problematic. They felt that because of the real problem of over-diagnosis, response to intervention filters out students who may not require a diagnosis or interventions for learning. Kavale and colleagues (2005), on the other hand, argue diagnosis based on an inadequate response to intervention is no different from the other methods of diagnosis. This is because policy considerations for inadequate response to intervention are operationalized to categorize learners in order to apportion supports. Then, as a result, learners are still over-diagnosed. Since all categories have significant problems, Fletcher and his colleagues (2013) recommend diagnostic criteria that combines more than one method.

Importantly, there is still an underlying theme of low academic achievement throughout all the methods of LD diagnosis (Sfard, 2008). That is, in order to be diagnosed with an LD for each method, the child has to have low achievement “*despite”* instruction. This criterion poses a difficulty specifically for diagnosing a mathematics disability. Sfard argues, “the distinction between difficulty experienced *despite* instruction and difficulty that develops *because of* instruction is not as straightforward” (p. 23, emphasis as in the original text). Many mathematics education researchers are finding that a significant number of children experience difficulty in mathematics *because of* instruction (e.g. Adler, Ball, Krainer, Lin & Novotna, 2005; Baroody, 2011; Schoenfeld, 1988). Research is still identifying and questioning "good" mathematics teaching practice. It could then be argued that if teaching and learning mathematics is a problem across the board, not for one group of students (i.e. just students labeled with an LD), but for many, if not most students, then layering a label of "mathematical learning disability" or another similar term, would be problematic. According to Sfard (2008), it is because of this very issue that mathematics educators do not usually engage in or with mathematical LDs in mathematics education research.

Compounding the problems surrounding the labeling and identification of difficulties and achievement in mathematics, are two important issues. First, difficulties can be experienced inconsistently across mathematical domains. Difficulties can even be experienced inconsistently within the same mathematical concept (Houssart, 2004). A child may experience success with an algebraic concept one day and experience difficulty with the very same concept the next day, and vice versa. The second issue compounding the problems, is that mathematics education has shifted to problem-based, inquiry learning. This has led to a shift in the requirements of teachers and the teaching of mathematics. There has been little research around the new problem-based curriculum and pedagogy, with students experiencing difficulties. We know little of how children with difficulties in mathematics are responding to this new teaching and curricular focus (Rathmell & Gabriele, 2011). Even when changes in the curriculum are being made at a policy or researcher level, they often trickle down slowly, if at all, to the teacher. Thus, teachers may have not yet developed the expertise to account for these changes, before they are required to teach with the changes. The new focus of the problem-based/ inquiry methodologies themselves may even be problematized as well because they are often lacking in exercises that focus on practice and retention (Woodward, 2006). Practice and retention are important instructional design affordances for children experiencing mathematics difficulties. These affordances have been recommended since the 1990s (e.g. Carnine, 1997).

In terms of instruction, while many children may require significantly different approaches from each other (Baroody & Rosu, 2006), it may very well be that students with mathematics difficulties also require an interweaving of many approaches. This interweaving includes approaches from the newer focus, of inquiry, and those from a previous focus, of memorization and mastery. Thus, when Sfard (2008) iterates that difficulties arise "*because* of instruction" (p.23), that instruction may be perfectly adequate for many learners, but because of a lack of multiple approaches, or even time, could cause difficulty for some of the children (Wiebe Berry & Kim, 2008). Research has barely scratched the surface on how to integrate a variety of best practices into the teaching of mathematics (Baker, Gersten, & Lee, 2002; Ketterlin-Geller, Chard & Fien, 2008).

Another issue with the LD label is that some researchers have questioned the presence, prevalence and oppressiveness of the learning disabilities label (e.g. Skrtic, 2005). They view the learning disabilities label as a social construction with the purpose of "othering" a group of people. At the same time, different researchers view learning disabilities as a group of intrinsic traits that are inherently present regardless of societal constructs and influences, and act as barriers to learning and achievement (e.g. Hammill, Leigh, McNutt & Larsen, 1981; Swanson & Siegel, 2001). Sternberg and Grigorenko (2002) situate themselves in the middle of the two sides. They view learning disabilities as both stemming from a societal influence and as a group of intrinsic traits inherent to the child. It is interesting to note that Sternberg and Grigorenko do however stress that because of societal influence, the learning disability label is over-used and the majority of the people with the label do not actually have a learning disability. Sternberg and Grigorenko (2002) question the efficacy of testing tools and definitions that construct the parameters of "learning disability." Regardless of whether learning disabilities are as a result of internal or external factors, one of the purposes of the label is to help identify children (or adults) that are not finding academic success. It follows, then, that an extension of research into learning disabilities is to find ways to promote achievement for children whether labeled with a learning disability or not. For this reason and others (see section 1.1.4), this research project focuses on children experiencing difficulty in mathematics with and without labels.

1.2.2 Operationalizing the Term "Mathematics Difficulty" for this Study

For this study the term "mathematics difficulties" (MDs) may include those labeled with an LD, but also extends beyond the label. There are a number of reasons for this:

I. As described above, there are many issues of equity associated with labeling a person with an LD, not least of which is the association of LDs with a pathology and the medical model (Dudley-Marling, 2004). Additionally, because the mathematics of diagnosis are problematic, there are a significant number of children who do not really fit the criteria and are labeled anyway and vice versa (see Deheane, 2011, for a more complete discussion). Thus, the term mathematics difficulty in this study is not reliant on the LD label.

II. Sfard's (2008) critique that "the distinction between difficulty experienced *despite* instruction and difficulty that develops *because of* instruction is not as straightforward" (p. 23, emphasis as in the original text) has also had influence on my usage of a mathematics difficulty construct instead of an LD construct. Therefore, for this study, a learner who experiences an MD has to have persistent difficulties, not necessarily accompanied by low achievement or dependent on instruction.

III. Fletcher and colleagues (2013) discussion of the fourth method of diagnosing an LD, "inadequate response to quality instruction," as being the least problematic of all the diagnoses. If this fourth method is slightly altered through the removal of achievement as a criterion, it can then become compatible with a solution to Sfard's (2008) critique. This slight change also relieves the mathematical problems of using standardized testing for diagnosis, as they become unnecessary and incompatible for this method. Thus, the final criteria for experiencing mathematics difficulties is that the learner has not responded to quality instruction. A significant difference between the diagnosis that Fletcher and colleagues (2013) discuss and the MD concept used in this study is that the purpose of labeling difficulties in mathematics, in this study, is not diagnosis.

1.3 Zero

> *"Zero became the language of nature and the most important tool in mathematics. And the most profound problems in physics-the dark core of a black hole and the brilliant flash of the big bang-are struggles to defeat zero. Yet through all its history, despite the rejection and the exile, zero has always defeated those who opposed it. Humanity could never force zero to fit its philosophies. Instead zero shaped humanity's view of the universe-and of God."* (Seife, 2000, p.2-3).

Zero acts in much the same way to our number system as what the point does in geometry-it is the origin. The number system developed only because of the discovery of zero (Dantzig, 1930/2005; Rotman, 1987). The invention of zero thus propelled further and further developments and more abstract mathematics. Gauss (in Kaplan, 1999) laments that had zero been discovered and utilized earlier, progress of mathematics during his time would have already been more advanced. Gauss' observation about zero is well taken, however, it was the late utilization of zero and not the date of its discovery (or invention)[9] that constrained the advancement of mathematics. Subsequently, it took many years and much repression before zero became a mathematical tool available to the public. Although the use of zero in mathematics was important for the generation of new mathematics, zero had a messy history in mathematics and its integration into society was fraught with difficulties. There were many reasons for zero's difficult integration, including its inherent abstract and ambiguous (Byers, 2007) conceptual understandings, and strong religious opposition to the idea of "nothingness." Importantly, the concept of zero evolved into complex understandings and it was not one solo individual that can be credited with

[9] There are various schools of thought in mathematics claiming either mathematics is invented or that mathematics is present in the world and waiting for people to discover it. It is beyond the purview of this dissertation to discuss the debate. However, it is important to note that Lakoff and Nunez (2000), theorists utilized in this dissertation, argue for mathematics being invented. For a fuller discussion see Livio, 2009.

the invention of zero. Through the millennia, cultures and communities contributed to understandings and subsequently evolved conceptions of zero (Ifrah, 1986). Researchers (e.g. Blake & Verhille, 1985; Inhelder & Piaget, 1964; Levenson, Tsamir & Tirosh, 2007) have suggested that the difficult development and integration of zero into mathematical practice, mirrors the developmental progression of understanding zero. However, researchers have yet to map the commonalities between the two.

In what follows, I first outline Lakoff and Nunez's (2000) discussion of grounding metaphors of zero to frame different conceptions of zero. I also elaborate on the theory of conceptual blends (elaborated in Fauconnier & Turner, 2002) that underlie their grounding metaphors. Subsequently, I juxtapose zero's historical development with what research has learned about how children develop their understandings of zero, and with Lakoff and Nunez's grounding metaphors of zero. Finally, I review and categorize research up until this point on the teaching and learning of zero.

It is important to note here that zero will be discussed in many forms in this section. The zero that we have today has the benefit of all the discoveries of the different individual and connected attributes of zero that came before it. Thus, each invention or discovery of an attribute of zero is discussed below in its historical context, without the benefit of later elaborations of zero. These various attributes of zero are cumulative until we reach the more robust understanding of the zero we have today. Additionally, the purpose of the following exploration of zero is not meant to delineate a linear progression of acquiring a zero concept, as in the acquisitionist metaphor (Sfard, 1998). Instead the purpose is first to explore how ideas and relationships in, around, and about zero build on each other to construct the zero concept, and then to explore how this evolution can mirror how individuals build ideas and relationships in, around and about zero. In any case, as will be explored below, the evolution of zero was in no way a linear process-it was quite recursive.

1.3.1 Conceptual Blends and Grounding Metaphors of Zero

In discussing embodied understandings of zero and how zero came to be, Lakoff and Nunez (2000) utilize Fauconnier and Turner's (in Lakoff et al., 2000 and elaborated in Fauconnier & Turner, 2002) conceptual blends. A conceptual blend is the combining of similarities of two separate cognitive inputs into a new, third cognitive space. A blend is an inference (third space) that results from an extension of experience (one cognitive input) with the extension of a new cognitive input. Not every particularity from the two initial inputs are brought into the third space. Only those understandings of the two spaces that have similarities to each other are brought into the third space. It is here, in creating the third space and at the point where the individual has to identify similarities between two inputs, that Lakoff and Nunez (2000) introduce their theory of metaphors. They assert that in identifying the similarities, a person creates metaphors between the different cognitive inputs. These metaphors are "grounding metaphors," basic metaphors "grounded" to experience, meaning they are the metaphors related to real-world practice. Then, after creating grounding metaphors, through conceptual blending, one can blend concepts across domains to create new, more intricate metaphors. These new, more intricate metaphors are "linking metaphors." They "link" metaphors into new more abstract understandings. In their elaboration of how these metaphors are constructed, Lakoff and Nunez utilize zero as a prime example.

Zero is a polysemous mathematical term (Mamolo, 2010). That is, zero encompasses a number of different meanings, including its mathematical and everyday meanings. This characteristic of zero especially becomes evident in the multiple synonyms that people use instead of "zero." For example: "null," "oh," "naught," and "zilch" (Barrow, 2000). Byers (2007) theorizes that it was because of zero's multiple conceptions, and thus its ambiguity, that it became such a central concept to mathematics and science. Thus, because of its complexity, zero is a mathematical concept that requires first grounding metaphors as a foundation, with basic mathematical ideas at their root, and then the creation of linking metaphors across domains. Lakoff and Nunez (2000) identify four grounding metaphors at the foundation of zero understandings (p. 75-76):

- Object collection: zero is the collection without any contents. This is the zero that means "empty," or "nothing."
- Object construction: zero is constructed (or deconstructed as in the case of subtraction) as an absence of something. There were seven pencils, I took them away. Now there are zero pencils.
- Measuring stick metaphor: zero is the smallest measurement possible.
- Motion metaphor: zero is a midpoint on a symmetrical line of numbers.

All four metaphors represent different understandings, or attributes, of zero. For each specific situation, the individual can utilize any metaphor, or groups of metaphors, to help them understand zero. Thus, the usage of metaphor for the specific situation that zero presents itself has an effect on understanding. Applying a misrepresentative metaphor could create misconceptions about zero.

1.3.2 Juxtaposing History and Development of Zero: the Beginning

Growing a conceptual understanding of number is extremely difficult, number is not a physical entity, it is a conceptual construct with multiple identities and constantly evolving (Stewart, 2008). In early human history, when counting conceptions of number were just developing, people used number solely as a tool to count objects. Thus, a conceptual understanding of number arose from a need to count. If there were "no" items, there was no need to count them, and no need for a symbol to represent the absence of the need to count (Ifrah, 1986). Similarly, developmentally, since they cannot count zero items, and zero has no one-to-one correspondence, young children do not have a need for a symbolic or linguistic representation of "zero." Like in the object collection metaphor (Lakoff et al. 2000), young children view not needing to count something, meaning not needing to perform a numerical cumulative one-to-one correspondence, as the absence of a collection. Thus, not having a collection becomes a metaphor of “none” or “nothing” and one is viewed as the smallest number (Wellman & Miller, 1986). Children do not usually encounter zero formally as a symbol until later when they meet it as a placeholder for the number ten (David, 1989). It was also used as a placeholder, historically, when zero, or rather its representation, led to it first being conceived as an entity.

Historically, a symbol representing zero was first conceived as a placeholder by the Babylonians and the Mayans at around the same time. However, our usage of zero can be traced only from the Babylonians-the Mayan’s conception of zero did not have a chance to be dispersed as it never left their continent and ended when their civilization ended (Kaplan, 1999). The time of the Babylonians and Mayans marks the beginning of the discoveries that lead to the zero we have now. The Babylonians in around 3000 BC developed a place-value number system that originally had no placeholder concept of zero. It still took a millennium to introduce a placeholder of zero (Toma, 2008)[10].

Today, a placeholder in a place-value system seems a necessary invention, one we cannot do without, especially since we use the Hindu-Arabic place-value system of numbers. The number 46 would be a good example to explain the difficulties our place-value system would have without zero as a placeholder. The number 46 without a placeholder of zero could represent many numerical possibilities that include a 4, a 6, and possibly zeros: 46, 406, 460, 4006 or even 40006000… The list of possibilities is infinite. Yet, the Babylonians functioned for a very long time without the invention of a zero as a placeholder. While zero is not a necessary component of many number systems, for example Roman Numerals do not need a zero, zero is actually integral to our Hindu-Arabic number system. One specific and important affordance of positional systems like the Hindu-Arabic and Babylonian systems, compared to other systems like the Roman Numeral System, is that they allow for easy calculations (Brysbaert, 2005). And, although there were discrepancies in their calculations, the Babylonians didn’t use a placeholder because they determined the value of the numbers based on the context. However, one can imagine that

[10] Note that Kaplan (1999) claims it took until 600 BC to introduce zero as a placeholder.

there were still probably many difficulties and mistakes that occurred from only relying on a context and numbers other than zero. Finally, someone somewhere in Babylonia developed a symbol, although still not "0," a type of "punctuation mark" (p. 118), to stand for the empty space (Toma, 2008). This was an extremely important punctuation mark as Dantzig (1930/1958) contends that "no progress was possible" (p. 31) until the symbol for zero was invented.

Like our ancient ancestors, young children understand number based on counting, or the context it represents. And similar to the Babylonian invention of zero, young children will identify a zero numeral before they necessarily build the metaphors to support conceptual understanding of the symbol. Young children will also label an empty set with the words "nothing," or "none," but because they do not understand the zero concept yet, will not necessarily label the empty set with the number zero (Wellman et al. 1986). Although children may be labeling the symbol "0" with the word "zero," because zero is not yet thought of as a representing or constructing number, the object collection metaphor of zero as "none" or "nothing" is still in use. Indeed, as will be discussed later, the evolving symbols of zero for place value, first an empty space representing zero, and later various round, sometimes empty symbols, all signify the object collection metaphor of zero.

This place-value aspect of zero is especially complex:

> "There is no doubt, that as a numeral, the mathematical sign zero points to the absence of certain other mathematical signs, and not to the non-presence of any real 'things' that are supposedly independent of or prior to signs which represent them. At any place within a Hindu numeral the presence of zero declares a specific absence: namely, the absence of the signs 1,2…9 at that place. Zero is thus a sign about signs, a meta-sign, whose meaning as a name lies in the way it indicates the absence of the names 1,2,…,9." (Rotman, 1987, p.12)

As described by Rotman, zero is a meta-sign representing the absence of the other numerical signs. Rotman's analysis, however, requires expanding upon as zero is not the only placeholder and not the only numeral to represent the absence of other numerals. All numerals have the potential to be a placeholder (Blake et al., 1985). For example, taking the 46 from above, the 4 is a placeholder for 40, or 4 tens, and the 6 is a placeholder for six ones. The "4" represents the absence of the other numerals. It is not 5 tens or 1 ten or 9 tens; it is 4 tens. The same can be said of the "6." Thus, zero must have something extra, something the other numerals do not have to make it a meta-sign. While zero has the same attribute of identifying the content of a place, it also has an additional attribute of being implicit, that makes it more complex than the other numerals. As another example take the number 70. Here, the zero is an explicit placeholder. In this context, zero relates that there are no ones in the number, but at the same time, its very presence also acts as a support to the numeral 7, converting it into 70. In contrast, when a different numeral fills the ones place, for example the numeral 3, creating 73, the role of zero becomes more complex. In 73, the 3 acts as a support to the 7, and the zero now takes on an implicit role. When a number, such as 73, looks to contain no zeroes, the understandings of zero as a placeholder are still implicitly present. In this case, the role of zero acting as a support is taken on by the 3, but the numeral 7 is still really 70. Thus, there is an implicit zero. The implicit zero is not "really," as in concretely, present. The implicit zero is a metaphor based on the real-life experience of zero as a placeholder. The implicit zeros are metaphorically present, lending support to an understanding of a numeral layered onto place as having a larger value than the numeral itself. Certain numbers have implicit and explicit zeros such as the number 126,005. This number has two explicit zeros: the numeral 1 is really 100,000, the numeral 2 is really 20,000 and the numeral 6 is really 6,000. Thus, the number 126,005 has multiple implicit and explicit zeros.

Indeed, the understanding of zero as a placeholder is not yet understanding zero as a number (De Cruz, 2006). And in fact, some early number systems never progressed from using zero as a placeholder to zero as a number (Walmsley & Adams, 2006), and some never even contained a zero[11]. The learning of zero as a place holder is at the root of many

[11] For example, Roman Numerals. Egyptian mathematics was actually thought to have been stagnated for its lack of zero. See Joseph (2010) for a fuller discussion. Deheane (2011)

complex ideas including algebra. Place value understanding of zero also leads into understanding numbers of greater magnitude, as through these place-value concepts, learners are able to use the same symbols, repeatedly creating numbers of increasingly greater magnitude (Kilpatrick, Swafford & Findel, 2001).

After the Babylonians, the Greeks had the potential to continue to contribute to the development of zero, yet there seemed to be no development of zero as a number or placeholder during this time. There are scant writings about zero from the time period of the Greeks. Kaplan (1999) would like to contend that the scant writings about zero during the Greek period of mathematics were because of the secrecy of the mathematical orders of the time. It is possible that the Greeks thought about and continued to develop the mathematical properties of zero, but their orders were sworn to secrecy about publicizing them. Contrary to Kaplan's hypothesis, many scholars (i.e. Pogliani, Randic & Trinajstic, 1998; Toma, 2008) discuss two other reasons for the stagnation of the development of zero during this time. The first reason for the absence of zero in Greek writings was that their main mathematical concentration was in Euclidean geometry. And just like developmentally, as already discussed above, a young child progresses from no need to count an absence to a need to describe an absence of objects. The Greek mathematicians had no need for a number that represented absence of things. The Greeks were dealing with presence of things in their Euclidean geometry-forms and shapes (Toma, 2008). The second reason for the absence of zero in Greek writing is somewhat loosely connected to the developmental progression of zero. The Greeks had a fear of zero both because of the conflict between their religious beliefs and the "strangeness" of nothing that zero represents (Seife, 2000). In other words, the Greeks feared the conceptual blend that arose from their blending of their religious beliefs of the great abyss and nothing and the object collection metaphor for zero. Dantzig (1930/ 2005) views this lag in development as support for the idea that nothing could progress until a set representative symbol for zero was invented. In retrospect, Dantzig's contention is a significant support for a

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iterates that our place value system only became "highly efficient" (p.86) because of the invention of zero.

representative metaphor, in this case the object collection metaphor, being necessary to spur growth in understanding of zero.

1.3.3 Juxtaposing History and Development of Zero: Something for Nothing

Alexander the Great was fortuitous for the advancement of mathematics and zero, when around 400 BC he and his troops brought the concept of zero to India. It was in India where the understanding of zero advanced and with it, mathematics (Seife, 2000). In terms of zero as a placeholder, by the fifth century the Hindus began utilizing a small circle to represent an empty space in their place-value system (Wilson, 2001). The symbol representing zero as a dot began to evolve as other cultures, sometimes unconnected to the Indian scholars, adopted the metaphorical rounded outline with an empty space inside as their zero symbol (Joseph, 2010). This symbol eventually evolved into what we have today: an outline of an oval with empty space in the middle. The new symbols of zero, round on the outside and empty on the inside, represented the paradoxes of the object collection metaphor. The symbols were paradoxes because these symbols essentially represented "something" in order to represent "nothing." That is, this initial historical discovery of zero relied on the paradox of realizing zero ("nothing") first and then subsequently creating a symbol ("something") as a representation of "nothing" (Byers, 2007). The thick lines around the emptiness inside the zero demonstrates parameters being placed around nothing (Lakoff et al., 2000). The very shape of the symbol for zero "0" represents this "generation" and "creation" paradox. The "0" is shaped like an egg, a symbol of "generation" and "creation" throughout Western history (Rotman, 1987, p.60).

This oval with an empty space we have today, is our physical representation of the object collection grounding metaphor. Importantly, similar to the paradox that was inherent to its initial findings, we experience the same paradox with the representation of zero. Our concept of nothing comes before we learn the physical representation of nothing. Additionally, like the mathematics scholars of antiquity, even our learned representation of zero, at first, does not represent the conceptual meaning of zero (Wellman et al., 1986). In two task and interview experiments, Wellman and Miller (1986) explored the understanding of zero concepts in children from Kindergarten to grade three. Their findings mirror the beginnings of the historical development of understanding zero. Wellman and Miller found that young children's understanding of zero progresses through three stages. Young children first come to understand zero as a symbol, which they can recognize and label. This recognizing of zero by young children exists before they begin to think of zero as a number representing the null set. During the second phase, children begin to think of zero representing the null set and view zero as representing "none or nothing" (p.35). Finally, in the third phase, young children begin to realize connections between zero and the other natural numbers, and think of zero as representing less than one. Thus, the object collection metaphor and the symbol that represents the object collection metaphor leads to the measuring stick metaphor of zero being a number smaller than one.

1.3.4 Juxtaposing History and Development of Zero: Zero Becomes a Number

Similar to the findings by Wellman and colleagues (1986) in the progression of understanding of zero, the Jains, a religious sect in India, were instrumental in progressing zero from being thought of through the object collection metaphor-the absence of whatever it was they were counting-to zero being thought of through the measuring stick metaphor-zero as a number that is less than one. It was because of the Indian invention of a symbol for zero that the Jains were able to progress zero conceptually from being only a placeholder to also being thought of as a number (Joseph, 2010). The Jains were intrigued by large numbers, and as mentioned above, zero has special affordances for representing large numbers (Kilpatrick et al., 2001). The Jains studied space and its concepts of infinity, concepts that have a strong connection to zero (Joseph, 2010). At the same time as zero was developing into a number, or according to Dantzig (1930/2005) *because* zero was developing as a number, other numbers were also undergoing revolutionary new understandings. Numbers were now beginning to really be thought of outside of geometry (Seife, 2000). The new concept of zero was beginning to solve old problems.

Before zero, there was a problem of closure in arithmetic: there was no natural number that could represent a – a = ? (Lakoff et al., 2000). Now that zero became a number, the Indian mathematicians Brahmaghupta, Mahavira, and Bhaskara began to contemplate the special properties of zero. They began experimenting with zero in their calculations using addition, subtraction, multiplication and division (Toma, 2008). This usage of zero in computations was the beginning of conceiving zero as a result of-or party to-an act of construction (the object construction metaphor of zero). Indeed, Kaplan (1999) writes that the use of zero in calculations was "momentous" (p.70) and a "paradigm" (p.71) shift for zero. Because calculations also use the collection of objects metaphor, using zero as a result of a calculation can be problematic. The collection of objects metaphor reinforces the expectation that a result from a calculation should be an actual collection of objects. In subtraction the collection of six take away the collection of four results in the collection of two. However, the collection of six take away the collection of six results in… no collection? Thus, Lakoff and Nunez (2000) assert that in order to use zero in calculations, zero must first be constructed as a number.

As a result of the new conception of using a zero in calculations, an access point to calculations was created for the non-scholar and general population. Up until this point the non-scholar usually only performed calculations on their fingers, they did not use an abacus like the gentry. With the advancement of zero as a number, and then the resulting invention of algorithms, the average man could perform calculations without an abacus (Dantzig, 1930/2005). Performing and pondering calculations with zero solidifies zero from being only a placeholder to also being a number (Rotman, 1987). With this paradigm shift, the additive identity of zero, a + 0 = a, the multiplication identity property of zero, a x 0 = 0, and division by zero as undefined, all metaphorically based in the object construction metaphor, all became discovered.

Negative numbers, with zero as their origin (motion metaphor), was another major mathematical concept that grew out of this paradigm shift (Joseph, 2010). Negative numbers cannot even exist without zero (Aczel, 2015). The zero as point of origin metaphor propelled people to understand numbers spatially, projected onto a (number) line. This conception was highly important in developing our understanding of the "uniformity" of number (Lakoff et al., 2000). A significant attribute of zero as origin is that zero does not have one-to-one correspondence with any of the other numbers. Thus, adding an integer from one side of zero, with its symmetrical counterpart on the other side of zero, for example +6 (+) -6, balance out to the only number without a symmetrical counterpart: zero.

1.3.5 Juxtaposing History and Development of Zero: Labeling the Symbol

After the Hindu mathematicians, the Arab mathematicians were instrumental in introducing European and Asian (Kaplan, 1999) countries to zero. Through the Arab transmission of zero, the Hindu word for zero, "sunya," became the Arab translation of "sifr" (Reid, 2006). Scholars (e.g. Pogliani et al., 1998; Kaplan, 1999; Toma, 2008) believe that Fibonacci's book, Liber Abaci, with its inclusion of zero was pivotal in disseminating the concept of zero from the Arab countries to the outside world. Upon arriving in the Western world, through Liber Abaci, zero had a turbulent existence. One example includes that zero was feared because it was part of the "infidel symbols" (Blake et al., 1985, p. 45). During this time the Arabic "sifr" became "cifra," today written and pronounced as "cipher." From "cipher," zero had many different variations, finally transitioning into the Latin "zeuero." With the invention of the printing press the shape of the zero as "0" became established and the scholars strategically decided to establish the term "zuero" or "zero" as the mathematical term for zero. The lay people at the time, because of the perceived meanings of "cifra" were confusing colloquial meanings of zero with mathematical meanings. It was hoped there would be less confusion through the establishment of "zuero" and "zero" as the mathematical term over the term "cipher" (Dantzig, 1930/2005). Nevertheless, as will be discussed below, confusions between colloquial meanings and mathematical meanings of zero persist in learners today despite the establishment of the mathematical term "zero."

It was not until the 1600s that zero finally held an uncontested place (Toma, 2008) and mathematics could progress with the inclusion of this important number. From the new understanding of zero as a number, other mathematical understandings were able to develop. This included calculations with negative numbers (Pogliani et al., 1998), square roots, and variables for algebra (Kaplan, 1999), and most especially calculus. Other mathematically "strange" ideas such as zero exponents and zero in quadratic equations could also be explored as well.

1.3.6 Tying the History and Development of Zero to Schooling and Practice: Conclusion

As will be explained below, the published research literature in mathematics education, special education or psychology has few examples of instruction, or interventions around zero. And of note, all these examples of teaching are after the children have already learned calculation with zero (e.g. Levenson et al., 2007). The research that has been conducted on zero has found that every area of calculations with zero have common misconceptions, and difficulties associated with it. Some of these misconceptions are inevitable because of the way understandings are rooted in metaphors based on experience (Lakoff et al., 2000). Of interest is that some known areas of difficulty for learners associated with zero, like the learning of negative numbers, are usually not associated with zero in the research literature. Fundamental concepts of calculating with integers stem from conceptualizing zero as point of origin, yet not much research exploring negative numbers have implicated zero. Van den Heuvel-Panhuizen (2008) does remark in her Dutch curriculum guide that zero as point of origin is especially difficult for learners.

A conceptualization of zero has to include its colloquial understandings. Zero is a paradox (Byers, 2007) and historically, even after conceptual understanding of zero was discovered, there were other, colloquial ideas of zero that challenged these understandings. Historically, and still today, zero has been tied to the occult, religious fear and the void, infinity, common expressions (zero in, zero tolerance, zero-sum game…), and art (Rotman, 1987), to name a few. It was only after these ideas were either reconciled, recognized, or incorporated with mathematical zero, that revolutionary mathematical inventions like calculus were discovered.

1.4 Zero, Understanding of Zero and Zero Misconceptions

In 1969, Boyd Henry wrote an article for the teaching magazine, The Arithmetic Teacher, entitled "Zero the troublemaker." This article began by declaring the struggle of understanding zero for students in elementary school: "Every elementary school teacher will agree that the concept of zero is difficult for many children to grasp. In fact, many teachers themselves are uncomfortable when they must work with numbers involving zero" (p.365). Henry did not use research to support his statement, as there was little if any research to support his assertion at that time. In fact, Henry does not include a bibliography in his article at all. Henry's article, meant to demonstrate various mathematical properties of calculations involving zero to a teacher, marked the beginning of a concern about the teaching and understanding of zero.

Catterall (2005) remarked about the paucity of research in the area of conceptions of and teaching and understanding of zero. Other mathematics education areas, for example fractions (i.e. Mack, 1995), have received much more attention than this important concept. While there is some overlap, the literature up until now on the development of understanding zero and the teaching of zero in schools has followed five threads:

(a) Progression of understanding zero,
(b) Research about the difficulties and misconceptions of understanding zero,
(c) Cognitive science and neuroscience perspectives on cognition of zero and the mental number line,
(d) Understanding division involving zero, and
(e) Articles directed at teachers about teaching and understanding zero.

Category (a) has already been discussed in a previous section (1.2), and as category (a) and (b) are most pertinent to this research, what follows is a short discussion of category (b), not already discussed above.

Gelman and Gallistel (1986) note that the historical development of zero (described in section 1.2), demonstrates the difficulty and lag in acquiring the concept of zero for a child. However, they add that explicit instruction of zero is fundamental to enable younger children in "development of a true understanding of zero as a number" (p. 240). Importantly, as discussed previously, zero is a polysemous number, that is, it is imbued with multiple meanings. Often polysemous concepts and symbols in mathematics are more difficult to learn because of the constant shift the learner has to make between multiple everyday and mathematical meanings (Mamolo, 2010).

Reys and Grouws (Reys, 1974; Grouws & Reys, 1975; Reys & Grouws, 1975) looked at the conceptions and misconceptions children in grades four through eight develop about zero through a series of tasks based on division involving zero. They found that the students had learned many of their misconceptions about zero from misinformed teachers. Compounding the problem, Reys and Grouws found that some textbooks at the time, either ignored the topic completely or provided erroneous or very little information for the teachers (Reys, 1974). Considering that Rey and Grouws implicated teaching in the misconceptions they found, Wheeler and Feghali (1983) wondered what teachers really do know about zero. Wheeler and Fegali set up a series of tasks and interviews with 62 pre-service teachers (they analyze the data for 52 of the teachers) to explore their understanding of various aspects of zero. Wheeler and Feghali found that the pre-service teachers did not have an "adequate understanding" (p. 154) of zero. Misconceptions held by the pre-service teachers included not considering zero as a number. Additionally, many pre-service teachers were also unable to correctly perform calculations involving zero. Confusion with calculations with zero as a result of understanding zero as nothing is not surprising. Wheeler and Fegali recommend that teacher education focus attention on understanding and learning conceptions of zero. While some researchers (e.g. Russell &

Chernoff, 2011) have continued research with pre-service and in-service teachers around zero with the same results, there has since been little studies on intervention as recommended by Wheeler and colleague in 1983.

Catterall (2005) examined data from 100 children aged ten to eleven, analyzing how children ordered zero in relation to the other natural numbers. The children in the study had difficulty in reconciling zero as a symbol and zero as representing the null set. Other research has delved into misconceptions that children and adults have regarding zero, including conceptions that zero is not a number (Blake et. al., 1985), zero is the same as nothing (Pogliani et al., 1998), and thinking that adding zeroes as place holders to a number does not affect the magnitude of a number (Smith, Solomon & Carey, 2005). Specific topics around zero, too, have been studied to shed light on children's and adult's thinking about zero. Some examples include whether zero is even or odd (Levenson, et al., 2007) and how teachers and students understand the zero exponent (Levenson, 2012).

Related to the research involving understanding zero as nothing, learners sometimes have difficulty with zero in measurement (Clements & Stephan, 2004). Understanding of measurement follows from understandings from counting. Thus, some learners will begin measuring from the number one, not considering its one-to-one correspondence with the space before one as part of the measurement. Or in terms of the grounding metaphors of zero (Lakoff et al., 2000), learners remain only thinking about zero through the object collection metaphor and not through the measuring stick metaphor. Subsequently, during measurement, like they did in counting, the child considers zero as "nothing," and zero is ignored. While it is perfectly possible to get an accurate measurement when beginning from the number one on a measuring stick, nevertheless one misconception arises when the total length is interpreted. If beginning a measure from 1 and ending the measure at 6, then the total measure is 5. However, because the spaces are not considered in measurement-only the numerals, as Lehrer and colleagues (1999) discovered, a common misconception is that learners will say "6" as the total measure.

Misconceptions of zero in place value has received some attention. As a result of the complexity involved in understanding, described above, it follows that children may have difficulty learning zero as a placeholder, especially when it is in the middle of a number (Cady, Hopkins & Price, 2014). This difficulty may be so persistent that difficulty understanding zero as a placeholder continues into adulthood (Ball, Hill & Bass, 2005). The zero within numbers was noted by Fuson and colleagues (1997) to be difficult because of concatenation of the numbers when we speak. In saying the number 53, we first say "fifty" (50) and then "three" (3). This causes children to write 53 as 503 (fifty-three). Another common misconception with zero as a placeholder is the viewing of each numeral in a string with zero as independent of the zero. For example, when asked what happens when you remove "0" from the number 70, some learners answer that "7" is left. Meaning, they view the "7" in 70 as representing a quantity of seven and unaffected by having a zero beside it (Anthony et al., 2004). Ball (1991) found specifically with multiplication that pre-service teachers, even those with a background in mathematics, may iterate the importance of zero as a placeholder but cannot necessarily explain its significance.

Zero in decimals and zero in calculations have also been a focus of research. Zero to the right of the decimals can be a source of a number of misconceptions. Durkin and Rittle-Johnson (2015) discuss their findings that zeros to the right of a decimal may often be ignored. Learners will thus view 0.07 and 0.7 as the same number. Stacey and Steinle and colleagues (e.g. Stacey & Steinle, 1998; Stacey, Helme & Steinle, 2001a; Stacey, Helme, Steinle, Baturo, Irwin & Bana, 2001b; Pierce, Steinle, Stacey & Widjaja, 2008) have had a robust research agenda exploring student's, pre-service, and in-service teacher's misconceptions around zero and decimals. They have had numerous findings about misconceptions involving zero in decimals including finding a discrepancy between rule-based understanding of zero in decimals, and misconceptions both about shorter is larger and longer is larger decimals numbers, and conceptual underpinnings of zero in decimals.

Anthony and colleague (2004) noted that misconceptions with zero in calculation is because zero does "in fact do something" (p.40). Because of this paradox, zero is often taught as a rule to students who have mathematics difficulties. Bryant and colleagues (2006) write in a chapter on their framework of best teaching practices in mathematics for children with mathematics difficulties, about an appropriate adaptation for teaching the zero identity property of multiplication (a x 0 = 0). The adaptation is finding the pattern of a x 0 = 0 through multiple examples and memorizing the rule.

Of note is the case study conducted by Levenson (2013) of the progression of understanding of zero by one student, Sharon, between grade two and grade ten. Sharon, a high achieving student in mathematics, was interviewed four times, three about concepts of zero: (1) in grade two about multiplication and its relation to zero, (2) in grade five about even and odd numbers and their relation to zero, (3) in grade six about fractions, and (4) in grade ten about the three previous concepts. While Sharon had been performing well in mathematics in school, zero remained an obstacle for her from grade two to ten. Much of Sharon's difficulty with zero was as a result of not thinking of zero as a number with mathematical properties; instead she thought of zero as the same as nothing.

At the root of my research is a hypothesis that the reason for many of these difficulties is a gap in understanding zero as a number. Many children, and adults (e.g. Russell & Chernoff, 2011), experience difficulties calculating with zero despite researchers (e.g. Baroody, 2011) claiming that rules surrounding calculations with zero have special affordances for learners for understanding the number system. While zero as an affordance has the potential to aid learners in discovering properties of the number system, that understanding can only come after a conceptual understanding of zero and the attributes that came before. One purpose of this research, in one way similar to earlier studies in that it is conducted with children who have already learned calculations before conceptual underpinnings, is to begin to understand how the conceptual understanding of zero can lead to affordances for understanding the number system.

Chapter 2.0: Theoretical Framework

Multiple theories play multiple roles in each research project. On the one hand, there is the explicit theory, or *foreground* theory, that is used to frame the design and analysis of the research. This theory is explicit in that it is openly acknowledged in such things as the literature review and analysis. On the other hand, there is also an implicit theory, or *background* theory, that informs the implicit constructs of the research project, such as the choices the researcher makes and the questions asked and even the choice of the foreground theory (Mason & Waywood, 1996 p. 1055). Like other projects, this project has both foreground and background theories. The foreground theory for this project is the Pirie-Kieren Theory of Mathematical Understanding (PK). At the same time, and at the root of the research questions and the choice of PK as the foreground theory, are the implicit and complex concepts of mathematical knowledge and understanding. In fact, these ideas have already appeared foregrounded in the opening chapters of this dissertation. It is "knowledge" and "understanding" about mathematics that I argue mathematics education researchers can add to the research discussion about children experiencing mathematics difficulties.

Both "knowledge" and "understanding" are constructs that have been explored, analyzed and debated in mathematics education research for several decades (e.g. Brownell, 1947; Erlwanger, 1973; Skemp, 1976; Steen, 2001). Notably, once the trend in research in mathematics education moved away from behaviorist theories of learning to cognitive theories, understanding became a main focus of research (Sfard, 2008). One can argue that it is the very question of the parameters and definition of knowledge and understanding that propels much of modern mathematics education research in the first place.

In what follows, I first theorize the concepts of knowledge and understanding that are underlying this research. Because of the complexity, centrality and debated parameters of knowledge and understanding to mathematics education research, I use a multitude of networked (Bikner-Ahsbahs & Prediger, 2014) theories in order to discuss the growth of knowing and understanding. I network: Sfard's (2008) theory of Commognition, Pirie and Kieren's (1994) theory for the Dynamical Growth of Mathematical Understanding, Fauconnier and Turner's (2002) theory of Conceptual Blending, along with ideas from Enactivism (e.g. Varela, 1999) and Embodied Cognition (Lakoff et al., 2000). In the last sections, I discuss the theoretical framework that is utilized in this study: the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding.

2.1 Theorizing Growth of Understanding and Knowledge

Of the many trends in the research on mathematical understanding and knowledge, two trends stand in my foreground and have had significant influence on this project. The first trend is as a result of the initial behavioral influences on mathematics education research. Initially at the root of the research on understanding was the idea that mathematical knowledge and understanding are tangible, measurable objects that can be "acquired." Sfard (1998) termed this focus of "learning-as-acquisition," as the "acquisition metaphor" (AM). She compares the AM metaphor with a newer metaphor emerging then, in the late nineties, the "metaphor of participation" (PM). The participation metaphor takes into account the sociocultural aspects, for example the environment and the collective, of the growth in knowledge and understanding. Sfard iterates that each of these metaphors of understanding on their own cannot answer all the underlying problems of understanding. Because of this complexity both metaphors, AM and PM are needed, for better or worse, in order to theorize understanding.

This research project conceives the growth of knowledge and understanding as more complex than simply the acquisition of skills, but still includes both AM and PM in the make-up of that complexity. As described in the next sections, the main conception of understanding utilized in this study is based on PM. However, my research is an outgrowth of other research in the fields of special education and psychology. The language currently used in these fields is mostly based in AM. I thus use a hybrid of both languages in the analysis of my data. In my analysis, I will write of the dynamic nature of a learner "having" an understanding or prior knowing on which to build or grow. This language can certainly be categorized as AM, but at the same time, growth is considered through the media described below.

A second trend, underlying some research and theorizing on understanding in mathematics education research, has helped inform what this project is not. Llewellyn (2014) discusses that some theories of mathematical understanding have a fundamental problem at their core. These theories have already been defined by their construction of, and at the same time, their simultaneously seeking to locate the romanticized "naturally developing child": generator of knowledge and understanding. As per Llewellyn's example of Boaler's (e.g. 1998) research utilizing project-based learning, this type of research implicitly, or explicitly places value on modes and types of learning. Llewlleyn "argue(s) that this naturally curious child, where understanding naturally develops from experience, is a romantic fiction. Instead the child is more than this; the child has a context including (amongst other things) gender, sexuality, culture and race" (p.125). In regards to the child experiencing mathematical difficulties, this romantic fiction of a normally developing child and the resulting prioritization of modes and types of learning, in deference to other modes of learning, has aided in at worst delegitimizing these learners and at best providing them with an inadequate education. Thus, when exploring understanding throughout this project, I take great effort to both not prioritize types of learning and to accentuate the recursive process of learning.

Knowledge and understanding both describe some dynamic (Byers, 2000) mental activity combined with an action (Sierpinska, 1994). The act of understanding is quite multifaceted and paradoxical. Human understanding, unlike artificial intelligence, is elaborate and we can understand many aspects of something all at the same time (Pirie & Kieren, 1992). Furthermore, we can perform actions based on knowledge without either feeling or "possessing" the understanding required to perform the actions in the first place (Sfard, 2008). At the same time, because of our unique and shared experiences, perceptions and surroundings, knowledge is just as complex as understanding (Maturana & Varela, 1992). Knowledge is cyclical and results from action informing experience and experience informing action, "All knowing is doing and all doing is knowing" (p.26).

In a somewhat similar vein, Sfard (2008) asserts that the actions of communication between two entities, self and other or self and self, are the very actions of cognition, or as Sfard would term "commognition," or understanding. Indeed, knowledge and understanding are interrelated, one supporting and growing the other. But more importantly they both survive in a symbiotic relationship with everything around them. "Knowings" and "understandings" are very much embedded through these interactions in the moment they emerge. Knowledge and understanding are not habitants of a person's mind, but in the moment processes of these interactions, evolving the person and their environs (Proulx, 2013). In this light, learning can then be viewed as the change-or even the "possibility" (Davis, 1996, p. 192) of change-in the evolution of the person and their environs.

Because knowledge and understanding are in the moment processes through interactions, then the growing of knowledge and understanding involves one with more expertise enculturating a novice into the discourses of mathematics (Sfard, 2008). The learner, through this enculturation, creates blends (Fauconnier et al., 2002) between prior experiences and these new interactions. Conceptual blends, as already discussed in section 1.2.1, is the combining of similarities of two separate cognitive inputs into a new, third cognitive space. In such a way each new learning interaction includes the learning interactions before. In the growth of understanding, then, learners may need to revisit, or fold back in PK theory language (Pirie & Kieren, 1989), and strengthen some prior understandings in order to create new blends. Tall (2013) describes the result of the revisiting prior knowings:

> "It is not only that new contexts may require a cognitive reconstruction of existing schemas, but also that the original mathematical structure remains coherent in itself, while lying within a larger structure that has a new form of

coherence." (p.409)

In this way growth of understanding is recursive and grows and strengthens understandings at different places at the same time. This differs from the linear theories that Llewellyn (2014) describes.

Identifying knowledge and understanding is complex, as acts of understanding can be elusive. It was partly this elusiveness of understanding that led Sfard (2008) to her theory of commognition in the first place. Pirie (1988) agrees with the elusiveness of understanding. She claims that it is impossible to really understand understanding; for the paradox of understanding is that there is always more to understand. Instead of the futility of trying to understand understanding, Pirie suggests that "what we can, however, do is attempt to categorise, partition and elaborate component facets of understanding in such a way as to give ourselves deeper insights into the thinking of children" (p.2). Pirie and Kieren (1989) stress that looking at the component facets of understanding as integral to a whole interconnected and non-linear process of understanding, can better demonstrate the occurrence of understandings. Pirie and Kieren focus on the resulting interactions of knowing and understanding. In the same way, Sfard (2008), informed by the concept of "legitimate peripheral participation," from Lave and Wenger (1991), focuses on the actions and interactions of knowing and understanding. Stressing her focus on actions of understanding, Sfard changes the focus from "knowledge" (a noun) to "knowing" (the verb/action), and from "mathematics" (a noun) to "mathematizing" (the verb/action) (Sfard & McClain, 2002, p. 155). Importantly then, it is in the "doing of mathematics"-actions and discussions-that understanding can be viewed. The question then becomes how to identify the doing of mathematics?

2.1.1 The Interaction of Knowing and Understanding

"Cognition is enactively embodied" (Varela, 1999). While these words are used to support an enactivist theory of learning, Sfard (2009) and Lakoff and Nunez (2000)[12] are all proponents of embodiment as the source, means and products of understanding as well. In embodied cognition there is no mind/body duality, one represents and reinforces the other (Sriraman, & Wu, 2014). Embodiment is broad, and includes all embodied representations of mathematical thinking, including: (i) mathematical thinking practices, (ii) representational tools, such as tactile representations and thought representations such as metaphors (Gutierrez, Sangupta-Irving, & Dieckmann, 2010), as well as, (iii) physical representations, such as gestures (Radford, Edwards & Azarello, 2009b). It is thus in all our outputs and interactions that the understanding of mathematics can be viewed. For as Pirie (1996) eloquently wrote, "What mathematics is, comes out of who I am, and yet I cannot come to understand mathematics in any other way than through the fabric of my metaphorical understanding" (p.xiii).

We can now revisit the initial problem of understanding introduced in the opening paragraph of the previous section. That is, the problem of performing actions without the feeling of understanding for those actions (Sfard, 2008). In considering embodiment as a way of communicating and interacting learning, the problem is not a problem at all. This is because the performing of actions ***is*** the knowing and understanding of mathematics (Pirie & Martin, 2000). Importantly, this means that growth of learning happens as a result of the performance and interaction and not as a direct result of any prior planning or intentions (Davis, 1996). I emphasize here that learning is an in-the-moment, evolving, possibility and interaction. Thus, understanding, as an extension of learning, is not a static concept of "something understood." Understanding, too is an in-the-moment, evolving, possibility and interaction. There is always room for more understanding, as there is always room for

[12] Enactivism includes theories of embodiment. However, theories of embodiment and theories of enactivism diverge in their foundational underpinnings. Enactivism has biological factors at its root, whereas embodiment has linguistic factors (Goodchild, 2014).

more interaction. With the backgrounded concepts of understanding and knowledge theorized, I now turn to the foregrounded theory of Pirie Kieren theory of the Dynamical Growth of Mathematical Understanding.

2.2 Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding

The Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding (PK) was chosen for this project because it is chameleon-like in that it is very flexible and moldable, and can be networked successfully with other non-linear theories of learning. PK theory is chameleon-like for two main reasons: (i) it does not prioritize any ways of knowing, and (ii) the theory does not attempt to delineate understanding-only track its recursive growth. Because of these aspects of PK theory, and as long as the theories networked with it are not linear and prescriptive, it becomes a matter of aligning benchmarks and ways of identifying understandings in networking ideas. Thus, PK theory is compatible with all the background theories I have used to understand understanding.

Another important consideration in choosing PK theory specifically for this project, is the robustness of the PK model, elaborated on below. This model both effectively aids the researcher in tracking understanding and effectively demonstrates the recursive nature of understanding. This consideration, especially, is aligned with my research goal of demonstrating the recursive nature of the growth of mathematical understanding for children experiencing MDs.

2.2.1 Introducing the Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding

Pirie and Kieren were influenced by the constructivist theorists of their time, like Von Glasersfeld, who viewed understanding as a continuous process of constructing and coordinating mathematical understandings (Pirie et al., 1989) with “no pregiven prescribed ends toward which this construction strives” (Steffe & Kieren, 1994 p. 721). Applying their observations to the classroom situation, Pirie and Kieren (1994) said that it is not the intentions of the teachers that determine growing understanding, but how the child interacts with the actions or discourse provided them to grow their understanding. Informed by Von Glasersfeld's constructivist theories, Pirie and Kieren advocated an alternative to the discourse of the day that slotted understanding into static categories, and did not have a meta approach to understanding understanding. They stated a need to develop a model of understanding that could inform the process of teaching actions while still being a research tool (Pirie, 1988). They saw the role of their model as one of exploring acts of meaning making in mathematics, not in determining a fixed acquisition point of a child's mathematical knowledge (Pirie et al., 1994). While there were other theories of mathematical understanding at the time with similar ideas about understanding, the other models were not practically useful in charting a pathway of understanding (Pirie et al., 1989). Importantly, the theory that Pirie and Kieren developed had the benefit of analyzing growth throughout the entire process of growth, beginning from Primitive Knowing and continuing all the way through to Axiomatization (McClain & Cobb, 1998, p.58).

First in 1989 and again in 1994, Pirie and Kieren outlined a theory of growth in understanding that represented understanding as a dynamic recursive process:

> "The purpose of this paper has been to show a theory of the growth of mathematical understanding which is based on the consideration of understanding as a whole, dynamic, leveled but non-linear process of growth. This theory demonstrates understanding to be a constant, consistent organisation of ones knowledge structures: a dynamic process, not an acquisition of categories of knowing." (Pirie & Kieren, 1994, p.187)

The theory was not meant to supplant other theories; the purpose of the PK theory was to expand the lens of understanding of the researcher, teacher and field. At the root of this theory are:

(i) the beliefs that the process of mathematical understanding differs for each individual (Meel, 2003),

(ii) that observing understanding is not making judgments on the internal happenings of thinking-it is looking for the actions themselves to demonstrate thinking (Pirie et al., 2000). In this way these actions of understanding include the contexts, and interchanges between person-and-person and person-and-environment (Kieren, Pirie & Gordon-Calvert, 1999).

(iii) that "It seeks to emphasize the embedding of more localized ways of thinking mathematically (including intuitive ideas, concrete representations, specific ways of acting) within more formal actions" (Martin, 2008, p. 64).

Similar to their theorizing of understanding, the Pirie Kieren theory has continued to evolve and grow theoretically by including perspectives from embodied cognition (Pirie, 1996) and enactivism (Pirie et al., 2000). PK theory has also been applied to numerous areas in mathematics education research (see Martin, 2008 for a full review), including in considering collective understanding (Martin & Towers, 2015), and as part of professional development and curriculum design (Wright, 2014). I now seek to apply this model to a context of growth in understanding by those experiencing mathematics difficulties. What follows is a discussion of the PK model and levels of understanding and recursion through folding back.

2.2.2 The Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding: The Model

In this section I elaborate on the model (Figure 2.1, p.51) that Pirie and Kieren developed as a methodological tool for their theory. I explain the intentions of different aspects of the model and how concepts from their theory are embedded into their model. In order to better explain the affordances and intentions of the model, and to direct your attention to the affordance directly on the model, I have layered various symbols onto the diagram.

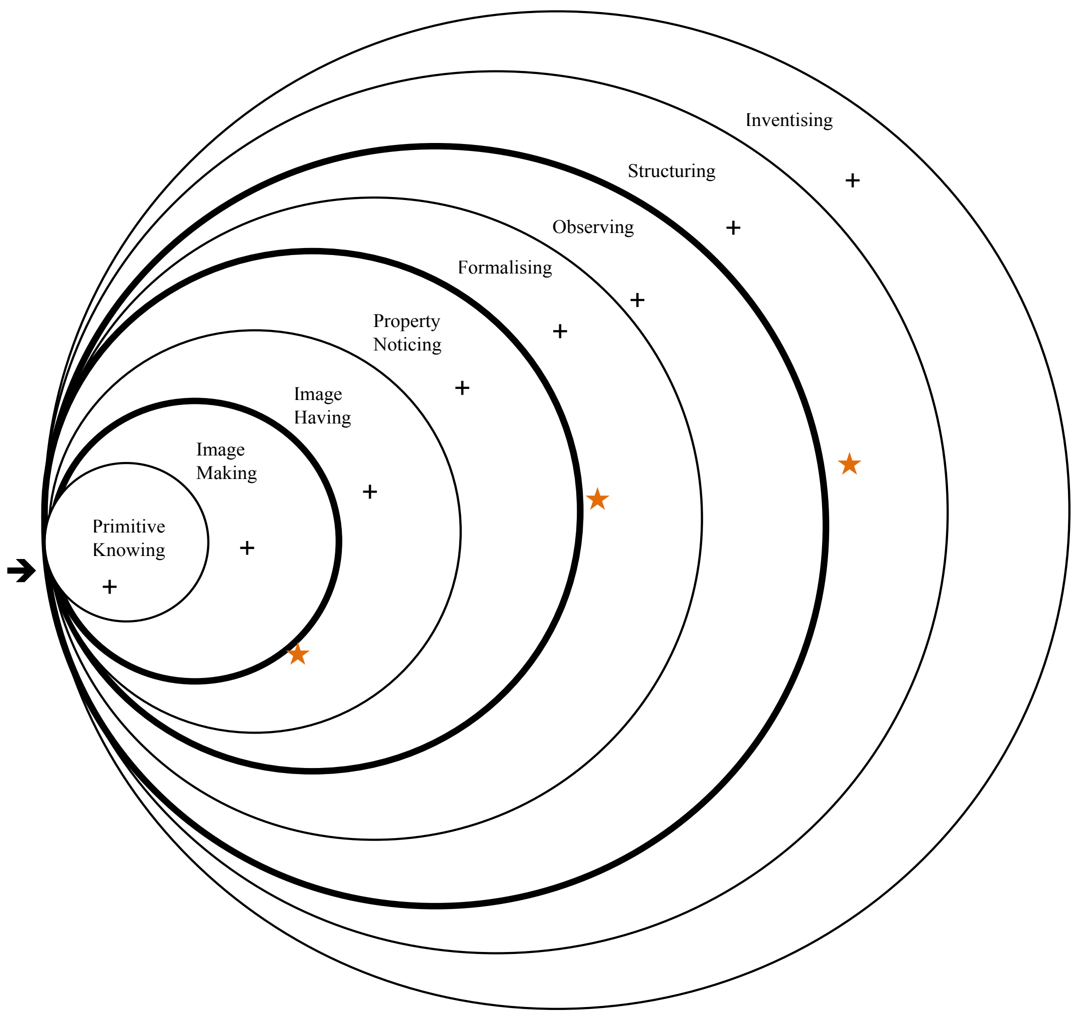

Figure 2.1 Pirie Kieren Model Of Mathematical Understanding

The PK Model for the Dynamical Growth of Understanding displays a series of eight nested circles (see +), each circle representing a mode of growth. While each circle is labeled a different mode of growth-Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising-the image of nesting represents the recursiveness of each circle. Note that the circles appear to be joined on one side (see, ➔). This joining is an artistic choice to convey the embeddedness of the circles, the "wrapping around" (Pirie et al., 1989) of each other. It further demonstrates how each mode of growth has within it the modes previous to it.

Three of the concentric circles have darker borders (see, ★). These darker borders indicate "don't need boundaries" (Pirie et al., 1994), meaning that the modes above are no longer needed to operate within the mode after the "don't need boundary." Pirie and Kieren explain the don't need boundary between Image Making and Image Having. When someone has a mathematical image of a concept, or they are at Image Having, then they no longer require the learning "actions" that led them to the growth. To further elaborate this idea, I draw on the notion of scaffolding (Wood, Bruner & Ross, 1976) as conceived by Askew (2007). There are important differences between Wood and colleagues' (1976) notion of scaffolding and PK theory. Although Askew (2007) has moved beyond the points in Wood and colleagues' (1976) theory that are contradictory to PK theory, nevertheless, since Wood and colleagues' notion of scaffolding is more prevalent, I first address the differences between Wood and colleagues' notion of scaffolding and PK theory. First, Wood and colleagues describe a hierarchical progress of skills. This hierarchy is explicitly not what Pirie and Kieren (1989, 1992 & 1994) intend with their model. It is not intentional in the construction of the model, although it may appear that way, that the eighth nested circle means a "better" or "high level" (Pirie et al., 1994, p. 172) of mathematical understanding. No layer is viewed as superior to another layer. Second, Wood and colleagues (1976) discuss scaffolding as the intentionality of a person providing support to "control those elements of the task that are initially beyond the learner" (p.90), while Pirie and Kieren (1989, 1992 & 1994) do not theorize intentionality at all. Askew (2007) views scaffolding as any support provided to a learner. Then, once the learner has progressed to where they do not need the supports anymore, the structures of the scaffolding remain in place and become part of the new knowings. The don't need barrier acts in much the same way. The growth of understandings that were developed in the circles before remain embedded in the circles after (like the supports of scaffolding), but the actions that developed the understandings do not need to be revisited.

Pirie and Kieren (1994) caution that the model, useful in operationalizing their theory, can only be a two-dimensional representation and thus, cannot fully represent all aspects of their theory. In the same way, the model is not a tool for assessment of ability or prediction for future performance. The model is instead, a descriptive tool for growth that can include growth as a result of teaching (Davis, 1996). Visually, because of the directionality and size of the model, it may seem to portray a linear mode of growth, from one direction to another (Martin et al., 2015). However, as will be discussed in further detail in the next sections, understanding grows in a non-linear fashion, moving back and forth between modes of reasoning. At the same time, pathways of understanding do begin from Primitive Knowing.

The construction of the model has not differed from its conception in 1989, however, some of the labels of modes of growth have changed (Pirie et al., 1994). Pirie and Kieren chose the names of the labels for their modes of growth with two criteria: to describe their thinking about the modes of growth, hence their use of verbs, and to connect to the colloquial usage of the terms. Through interactions at mathematics education conferences, they determined some misunderstandings that could arise from the labels and either clarified or changed them. The labels of each mode were not meant for a labeling of the child in a way that objectifies the child as an object of study. Instead, the labels were meant to describe pathways of understanding travelled through by the learner (Pirie, 1988).

2.2.3 Modes of Understanding

For this research project, I only observed actions between Primitive Knowing and Image Having. Because of this, I utilize only the first three modes of the PK theory: Primitive Knowing, Image Making and Image Having. Nevertheless, in what follows I discuss the first four modes of understanding of the PK theory. I include the fourth mode to give the reader the benefit of knowing where the boundaries are for my observations and analysis.

2.2.3.1 Primitive Knowing

The first mode of the PK Model is called "Primitive Knowing," previously called "Primitive Doing." Because knowledge can only come from knowledge, Primitive Knowing is a starting point; it is the repository for all the previous understandings that a child has accumulated and is required for the new growth. Primitive Knowings contain all the knowing, except for the existing understandings for the concept, that a learner needs in order to grow their understanding (Martin et al., 2015). Interestingly, one misconception about this level involves language. Primitive does not refer to the value of the type of knowing. In this case primitive refers to the knowing in relation to the new understandings that are about to grow. The knowledge is only primitive insofar as it has not been enriched by the new understandings for this particular context. In fact, due to the recursive nature of understanding, a child can access primitive knowing for one mathematical construct and during the same path of understanding, the new growth in understanding can become a primitive knowing for a new understanding (Pirie et al., 1994). Pirie and Kieren describe their model as having a "fractal-like quality" (p.172). Like a fractal that reveals the same patterns in smaller and smaller portions, underlying each primitive knowing is a growth of understanding that had its own primitive knowings at its foundation. The researcher learns about the primitive knowings retroactively as the child accesses them in their growing understanding (Martin, 2008).

2.2.3.2 Image Making and Image Having

The second and third modes, Image Making and Image Having, are the two modes that have images in their title. While the word 'image' conjures a notion of visual pictures, this is not necessarily the case. The word image is used to refer to 'mental objects' (Pirie et al., 1994, p.166). Mental objects can be mental visual pictures of physical representations, but they can also be non-visual representations of understandings of the learning.

The second mode of understanding is called Image Making, wherein children are actively engaged in learning and begin to create mathematical mental objects. This active engagement can be in any form of learning. And, like the images, the learning does not have to be visual in nature. The learning could be in any form-the key is the engagement. It is the activity of the active engagement that creates the mental image (Meel, 2003). In fact, Image Making will not occur without both activity and engagement. It is here, in Image Making, that the child begins to utilize their primitive knowings in ways they have not done before (Pirie et al., 1994).

When the child is able to use their mental object, or mental image, without relying on the activity that initiated the mental image, they are at the third mode of Image Having. Image Having is an abstraction of Image Making. Importantly, in order for growth to proceed from Image Having, a child would have had to have already "made" the image. "For understanding to grow, these images cannot be imposed from the outside" (Pirie et al., 1989, p.8). During this stage, the learner does not need to be engaged in the activities that produced initial understandings of the concept. The child can now use their images to anticipate and predict outcomes (Meel, 2003). An example of Image Having in a classroom context would be a child who can recognize an error they made in subtraction because the result is too big. A small child may have created mental objects during activities in school where they understood that when subtracting two numbers, the answer would have to be smaller than the first number. A child who would now anticipate that the answer to their calculation would be smaller than the original number, would be said to "have" that image because they do not need to rely on the original activities that gave them that image. It is important to note that this process of Image Making and Image Having can create erroneous mathematical images as well (Pirie et al., 1994).

2.2.3.3 Property Noticing

The next mode of understanding in the PK theory is called Property Noticing. During this mode, a child can reason with and analyze their mental image-they notice properties of the image without having to see the image. The identification of properties can then include a comparison of similarities and dissimilarities between the images (Martin, 2008). It is here that the child is able to think about their thinking. To use the child learning subtraction as an example, during Property Noticing the child can make the following statement through analyzing their mental images: "Wait a minute, all the starting numbers are larger than the numbers I am subtracting." The child would notice this property of subtraction from their mental image. They would then be able to wonder further, "what would happen if the second number was larger?" This new question would have the possibility of instigating recursion back to an inner understanding to gather more knowings, or moving the child directly into Formalising.

2.2.4 Recursion and Folding Back

Pirie (1996) wrote of the importance of new learning evolving out of present understandings: "to learn we have to engage in a dialogue between ourselves and our existing understanding" (p.xiv). As such, recursion and folding back are important foundational concepts of the PK theory. The PK Theory uses the term recursion to make explicit the nature of the relationships between the levels of understanding (Pirie et al., 2000). It is possible to be at any one level, for example Formalising, and fold back into Image Making because an understanding from Image Making is needed. As already discussed, each mode of understanding has within it all the modes below it. Thus, each mode is dependent on the previous modes, and at times may require new understandings from the previous modes for the understanding to continue to grow (Pirie et al., 1989). This movement from an upper mode to one below it, is called folding back. Folding back is the most important tool of PK theory (Meel, 2003) and imperative for the growth of understanding (Pirie et al., 1994). That being said, at the same time there are moments in one's growth of understanding where the processes that precipitated their current understanding are not necessary any more. These moments are marked by the bold black lines in the model (Davis, 1996) (see figure 2.1, p.51). During the act of folding back, the learner analyzes their previous understandings at the previous levels and determines which level would be most helpful to revisit for their continued growth of understanding. Figure 2.2 is an example of a mapping:

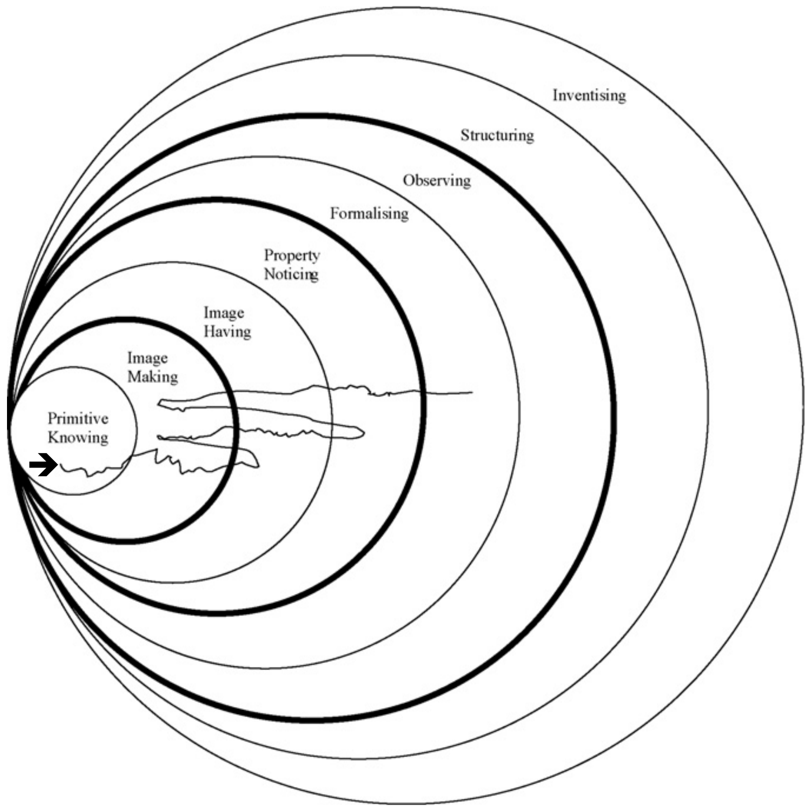

Figure 2.2 An Example of a Mapping (Martin, 2008, p. 69)

Notice the line (identified with ➔) used to track understanding moving back and forth across sections. Whoever the learner was that experienced this path, folded back: (i) at Image Having into Image Making, and (ii) at Property Noticing into Image Making. When folding back occurs, it is not a retracing of the pathway. When a person folds back, “a new kind of exploratory action takes place” (Pirie et al., 1989, p.123). The person revisits the level, but does not interact with it in the same way as they previously did. All the new understandings that they are experiencing remain with them and they can use these new understandings as they interact again at the mode they fold back to. This new interaction is said to “thicken” (Martin, 2008, p. 66) the understanding that is being reviewed at the inner level. Martin (2008) likens the thickening to a piece of paper being folded-it is the same piece of paper that is being used to layer and make “thick.” To explain, the same previous understanding is being layered and made denser. Once the understanding is thickened, it is then accessed to facilitate continued growth at the outer level.

2.2.5 Mapping

The model can be utilized as a tool to trace the growing understanding of a child. The practice of tracing growing understanding is called “mapping” (Pirie et al., 1994, p.182). Because understanding is a recursive, non-linear process, with understanding moving back and forth between layers, it can be visually mapped out as “it is observed” (p.182). Pirie and Kieren add the distinction of “as it is observed” because from the researcher’s point of view, they can only infer understanding from observing acts of the learner. The observer cannot infer understanding from what they do not see or hear. However, the choice of what to trace is the observers (Martin et al., 2015). Because constructing the path on the model is so connected to the observations, it is then also intimately tied to the choices of the observer.

An example of a mapping is shown in Figure 2.2[13] (p.57). The reader will note that the mappings in this dissertation (found in chapters 4, 5 and 6) look slightly different from the one shown in Figure 2.2. There are a few reasons for this difference:

(i)The pathway in Figure 2.2 is visually different from the pathways that are constructed for this study. Pathways will differ between individuals, thus this type of difference is inevitable between all mappings.

[13] In order to elucidate that act of mapping further, I debated whether or not to give a hypothetical example of a mapping in this section. I chose not to share a hypothetical example because, as explained above, researcher choices and observations are highly tied to the pathway outcome. Exploring the model with a hypothetical situation might obscure this very important nuance of the model. Thus, I refer the reader to the analysis chapters 4, 5, and 6, where I make I make my choices explicit around the mappings.

(ii)The pathway in Figure 2.2 stretches from Primitive Knowing to Formalising, while all three pathways in this dissertation stretch between Primitive Knowing and Image Having. This difference can be because of the way I constructed the tasks, the concepts being mapped and the strategies used to grow mathematical understanding.

(iii) Figure 2.2 has "squiggly" lines, while the mappings in this dissertation have straight lines. This difference is because of the affordance of technology and is not inherent in the model.

2.2.6 A Final Note: PK Theory and Understanding of Zero

Zero is not one idea-one entity. Zero is an evolving culmination of a network of conceptions (already outlined in the previous chapter). Thus, understanding zero is understanding the connectivity, relationships and metaphors of the underlying network. If primitive knowing is all the knowings that a learner brings to the concept, except the concept itself, then an important question is: what is primitive knowing of zero as a network of concepts? Are ideas about zero "primitive knowings," or they already constructed images? I argue that the essence of understanding zero is the interconnectivity of the underlying concepts. An idea about zero, or even unconnected multiple ideas about zero, ***is not*** the concept of zero as a network of concepts. In my mappings, therefore, I place ideas about zero into Primitive Knowing. Once connections between ideas are constructed, they can then be considered to have thickened and may possibly be considered an image.

Knowledge and understanding are complex-so is zero. It is possible for zero to be thought of in the same way that school mathematics often is-a list of skills, procedures and processes to be acquired (Lampert, 2001). In this case, then, one could say a person understands zero when they have acquired these skills, procedures and processes. But understanding and knowledge of zero is more than this Acquisition Metaphor (AM) (Sfard, 1998). An understanding of zero is understanding the relationships between the big ideas of zero, for example the grounding metaphors, already discussed in chapter 1. Significantly an understanding of zero entails the expanding of these concepts and relationships into and across other ideas in mathematics (Lampert, 2001). Some may argue that this definition of understanding zero is too general to be useful for this project. In response I argue that because understanding is ever-evolving and expanding, then what anchors the different understandings is their relationships between one another. The growth of understanding of zero occurs through various interactions, actions, relationships and representations with the underlying micro-concepts of zero. Growth then continues-or growth could happen simultaneously, recursively or even retroactively; there is no linearity to understanding-with the weaving of these concepts, building a more robust conception leading towards macro-zero.

Chapter 3.0: Methods and Methodology

This study is situated on the idea that the nature of understanding is and occurs through interactions with self, environment and other. As such, this dissertation utilizes qualitative research methods to explore the pathways of understanding of children experiencing difficulties in mathematics. Qualitative research methods allow me to critically analyze the growth of understanding and knowledge in order to gain understanding about different pathways of growth. In collecting data for this study, anything in the field could be possibly relevant (Pirie, 1996). For this reason, in order to aid my observations, I use multiple sources: video data, researcher notes, and collection of student work.

At the root of this dissertation is a concern for equity practices for those experiencing mathematics difficulties. Additionally, in designing this study and in presenting this written document of my research it is my responsibility to consider equity in order to be considerate of the community I am writing about and presenting to (Andersson & le Roux, 2017). Certain design decisions, for example the use of empathic coach (section 3.1.2) and the identification of participants (section 3.1.1), were made in light of this consideration of equity.

3.1 Research Design

There are 4 components to this study:

(i) a survey filled out by parents of participants-collecting background information;
(ii) a two-week period of observations in the mathematics classroom-acting as empathic second person (Shear & Varela, 1999) collecting contextual and background information;
(iii) one approximately 1.5 hour or two 45-minute mathematical task-based individual clinical interviews;

(iv) a series of up to five one-hour group and individual task-based intervention sessions.

For this dissertation I create a mapping, using the PK model of the third component, the mathematical task-based individual clinical interview, and utilize data from the other three components to support my analysis. The PK model is a micro-level tool that allows for a granular exploration of change.

3.1.1 Empathic Coach

Understanding as conceived in this project, and as elaborated upon in section 2.1, is a product of interactions. All entities involved in the activity are affected by the interaction. In such a way I, as participant and orchestrator, am affected by, and at the same time effect, the interactions. Thus, the term observer, or participant observer, as commonly used in the literature as one who, "seeks to uncover, make accessible, and reveal the meanings (realities) people use to make sense out of their daily lives" (Jorgensen, 1989, p.15) is not a complete description of my role. Yes, I seek to interpret. But at the same time as I observe understanding, I effect and am affected by understanding-I am both mediator (Sfard 2008) and mediatee. In these interactions, I thus see my role as "empathic resonator, with experiences that are familiar to (me) and which find in (myself) a resonant chord" (Shear et al., p. 10). In the midst of our interactions, mine and the participants, I react to those moments that resonate with me. I then respond with interventions that may or may not help both of us grow our mathematical understanding. However, at the same time, I am not experiencing the tasks in the same way as the participants. We both have our own backgrounds and experiences, and especially of note, I have orchestrated the entire situation. I try to be cognizant of my mathematical power in our interactions, and the effects it may have on their mathematical identity.

Metz and Simmt (2015) explore and elaborate on Shear and Varela's (1999) role of the empathic second-person. It is the description of empathic second-person that describes my role during this project. Importantly, through the role of empathic second-person the role of researcher and teacher melds into one entity. Thus, denoting the effects and affects of the orchestrator, I am both teacher and researcher. The second-person is not the first-person directly experiencing the task, or even claiming to directly know the experience. The second-person is not the third-person either-somewhat detached from the experience, narrating observations without effecting the narrative. Instead the second-person is situated somewhere in the middle of the first- and third-person, they act as a coach. A coach recognizes the multiplex of inputs, causes, reactions, interactions and circumstance that makes the learner their own individual. And, at the same time the coach empathizes with these multiplex of inputs and acts as a coach to elicit and develop understanding. I attempt to convey this role in the mapping chapters 4 through 6 where I analyze these interactions. I include my own intentions, responses and the evolution of both through these interactions.

3.1.2 Participants

This study focuses on children who are experiencing mathematical difficulties. Because of the concerns of equity already discussed in the previous sections, the distinction for this project is that mathematics difficulties are considered an experience and not an inherent trait. As such, this study does not rely on a formal label by school or psychologists. Instead, in order to identify participants, identification includes the following criteria:

1) Persistent experienced difficulties, not necessarily accompanied by low achievement or dependent on instruction.
2) The child self-identifies as experiencing difficulties. And in order to determine if the difficulties are persistent, either the school or parents or both corroborate those difficulties.

I utilized selective sampling to procure participants for this study. In this methodology of sampling, the criteria for participants in research are mainly based on the outlined parameters of the research previous to conducting the research, and the specific constraints of time and place for the researcher. During the research, as the researcher analyzes data, they may seek out additional participants based on the developing needs of their research (Coyne, 1997).

Six girls in grade five, aged ten to eleven, participated in the study. This age group was chosen for two reasons. First of all, the Ontario elementary mathematics curriculum introduces more abstract concepts that explicitly rely on zero, such as integers, in grades seven and eight. I consequently chose a grade level, where images of zero would be less likely to be confused due to participants encountering more abstract images of zero already in a formal manner. Second, I first procured a school to participate in the project and then, in consultation with the principal, chose a grade level. The principal felt that the grade five class would be a rich source for this project because there were proportionally, compared to other classes, a higher number of students struggling in mathematics.

Ethics forms (see Appendix A) and surveys (see Appendix B) were given to all the children in the class. Six forms were returned. Three forms were from students who self-identified as having mathematics difficulties: Megan, Angela and Melissa. Parents corroborated the difficulties on all three, and the school corroborated difficulties on two: Megan and Angela. Two forms were from students who did not experience difficulties in mathematics: Debra and Rosa. Finally, one form was returned from a student who self-identified as experiencing difficulties in mathematics with corroboration from her parents. However, she also has a diagnosis of Attention Deficit Hyperactivity Disorder (ADHD): Taylor. All students who handed in their ethics forms to me participated in all aspects of the project. This was because of a concern that through explicitly identifying those experiencing mathematics difficulties, and then only interviewing those participants, I could further marginalize students who were already marginalized. The analysis section focuses on the three students who experience mathematics difficulties-Megan, Angela and Melissa. Data from the other three participants are, at times, used to support analysis and findings when appropriate. Taylor was not included as a focus of this study because characteristics of her ADHD diagnosis created significant differences between Taylor and the other participants. I feel that this dissertation, with the questions I pose, is not the avenue to explore Taylor's mathematical interactions.

3.1.3 Interviewing

Interviewing plays an important role in this study-interviews are the pivotal media through which I, together with the participants, explore growth in understanding. However, interviewing as a research tool is a fairly new development in mathematics education (Zazkis & Hazzan, 1998). Erlwanger's 1973 article of his interview about fractions with a child named Benny is one of the first instances of interviewing as research in mathematics education. While Benny demonstrated adequate performance in fractions at school, Erlwanger's interview revealed misconceptions in Benny's understanding of the concepts about fractions underlying his performance, and about mathematics in general. Erlwanger's important contribution was that he demonstrated that only through the interview process could Benny's misconceptions have been discovered.

Interviewing is an integral methodological construct of the PK theory. Because understanding is a dynamic process within the PK theory, the method that uncovers understanding should be dynamic as well. Written assessments to determine understanding are hollow and cannot explicate the underlying dynamism of understanding. Instead interviews, with their ability to allow for immediate probing questions to elicit understanding, are dynamic and can reveal growing understanding (Meel, 2003). However, not all interviews are dynamic in nature. Structured interviews that do not respond within the moment to answers being given can have the same static effect as written assessments. Semi-structured interviews, on the other hand, can be dynamic in nature while still being closely aligned to research goals.

In preparation for the semi-structured interview, I prepared questions to be asked. However, during the interview there is flexibility to the questions being asked, and I, as empathic coach, make in-the-moment decisions. I sometimes choose to ask probing questions based on responses, and sometimes decide to skip some of the questions altogether (Basit, 2010). Semi-structured interviews have the double benefit for this research of allowing me to elicit and follow growing understanding, while still helping to focus myself on my research questions.

Ginsburg (1997) calls the process of semi-structured interviews in education and psychology from a dynamic perspective, “a clinical interview.” In a clinical interview the interviewer is immersed in “exploration,” “hypothesis testing,” “establishing competence,” and “theorizing” (p.107-108). Through exploration the researcher is not confined to pre-determined notions of the thinking they assume the child has; they are able to explore the child’s responses with probing questioning tactics. At the same time as the researcher should be exploring the child’s thinking, they should keep in mind that they are trying to test a hypothesis and interviews should be redirected back to the research purpose. In establishing competence, the interviewer is constantly trying to ensure that the child is engaged and performing at their optimal level. Ensuring performance at the optimal level is important because sometimes children will not articulate a complete response, or their responses will not be representative of their thinking. In this way, the interviewer is constantly evaluating responses to decide whether the child is responding at their optimal level.

Using Semi-structured interviewing in projects concerned with mathematical understanding does have its detractors. One common problem considered to be associated with semi-structured interviewing is that the interviewer can relate the very mathematical knowledge to be tested directly to the participant through their questioning (Ginsburg, 1997). Ginsburg outlines questions and procedures that help alleviate this problem. However, this argument and others like it, do not pose a problem to interviewing for this project. I argue in section 2.2 that understanding is a product that occurs through interactions. Thus, the process of interviewing, if dynamic, is then also the process of understanding. I openly acknowledge that through the process of interviewing, and the exchange that happens, that I influence the participants understanding. But this is not a problem because that influence *is* the process of understanding. For those who disagree with this theorizing of understanding, my influence still does not pose a problem. Through this project I am not seeking to discover a static notion of what the participants know. Instead, I seek to explore the process of growing understanding. Thus, the problem is moot for this project even for those that define understanding in a different way.

3.1.4 Use of Video in Research

The use of a video camera as a primary method of data collection has become common practice in educational research. In fact, in 2005 a federal agency of the United States government assembled leading scholars to a conference to critically discuss video camera use in educational research (Derry, 2007). While there are many benefits to using video data, data collection using a video camera can also pose many problems. There are many things to consider in recording data on video (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003), including the ethics of use (Powell, Francisco & Maher, 2003). Powell and colleagues (2003) explain that video data has the potential to capture both the good and the bad, including compromising positions for participants. While the participants of the research may have given their permission to the use of data prior to collection, the compromising positions caught on video create an ethical conundrum. They raise questions of permissions and access. Although this issue did not arise during this project, it is important to note that Powell and colleagues also suggest a solution. Should such an ethical issue arise, the researcher should show the participants transcripts or video data after collection so the participants can “regive” consent.

An issue that did arise during this project, is that the camera acts in much the same way as a human observer; it can only observe what it is looking at. Thus, the use of video data has logistical problems as well. In order to overcome this issue, I followed Pirie’s (1996, p.14-15) advice to have a plan, to practice with the camera and to ask the following questions:

- "What is your subject focus?"
- "In what context do you need to set your subject?"
- "What details do you wish to record?" and
- "What equipment do you need?"

One benefit of video data is the possibility to view the data multiple times, gaining new perspective with each viewing (Pirie, 1996). Multiple viewings of the video data afford me the opportunity to review and deliberate multiple times, adding to the complexity of my analysis.

3.2 Tasks

The mathematical task-based individual clinical interviews consist of either one-1.5 hour session, after school, or two-45 minute sessions, during school hours. This session consists of seven tasks:

(1) An introductory interview
(2) Number flexibility tasks
(3) Reading and writing numbers
(4) Number line task
(5) Exploring zero tasks
(6) Exploring number relationships tasks
(7) Telling number stories

The order of the tasks mostly follows the above sequence. However, some participants did experience some changes with the order. There are two reasons for changes. First, considerations of time, especially for those participants who participate in the two-45 minute sessions. Second, some participants prior to our session discuss the tasks with other participants who have already participated in these tasks. When these new participants then excitedly discuss prospective tasks, I use their excitement to segue into that particular task, even out of order.

3.2.1 Introductory Interview

Every session begins with an introductory interview. The purpose of the introductory interviews is to: (i) explain the research to the participant, and that the focus of the project is on processes and not "the right answer," (ii) begin to develop rapport with the participant, and (iii) discuss the participant's previous experiences (identity) with mathematics. I have a script for the interviews. However, in line with semi-structured interview practice and my role as empathic observer, the actual interview often deviates from the script.

3.2.2 Number Flexibility Tasks

The purpose of the number flexibility tasks, like the introductory interview is to begin with a task with which students would have success. Included in this set are 3 tasks, all of which include linking cubes as a manipulative:

(i) decomposing number task,
(ii) hiding task, and
(iii) addition and subtraction with trains.

The decomposing number task consists of asking participants to decompose the numbers 10, 17 and 36 into multiple arrangements of two sets of numbers. Although each number can be decomposed into more than two sets of numbers, for example the number 10 can be broken down into 1, 1, 2 and 6, or it can be broken down into 1 set of 5, 1, 1, 1, 1, and 1, I purposely ask for only two sets when decomposing. I have two considerations for this focus. The first consideration is time. This task is meant to be a quick introductory task- there are too many combinations and possibilities for decomposing the numbers into more than two sets. The second consideration is because of the zero. In asking for sets of two I am able to wonder if the students will include a set of zero as a possibility for the number combinations. For all the participants I first model the decomposition of the number 5 with the sets of cubes. Importantly I do not include a set of 5 and 0 in my modeling.

The hiding task used for this study is based on an assessment, "Hiding Assessment" by Kathy Richardson (2002). Although this project follows a similar format to the Hiding Assessment, there are a few significant differences. Richardson's purpose is to assess for skills in decomposing the number ten and its relation to procedures of subtraction. Thus, the Hiding Assessment only focuses on decomposing the number 10. Also, the Hiding Assessment hides a certain number of cubes each time and asks the learners based on the cubes shown, how many are hidden. Zero is never included as a hidden number. My purpose, on the other hand is to further explore how students decompose numbers especially when faced with the possibility of zero in the decomposition.

For this task, then, there are two separate decomposing parts: one with 10 as the total and one with 20 as the total. Both parts have the number zero as a possible set for the decomposition. Because this task now creates a situation where zero is part of the decomposition, this new task is different from the previous task of "the decomposing number task." Also, the act of decomposition for this task is not as explicit as in the previous task. The participants are not asked to physically do the action of decomposition, only to identify the hidden, or decomposed, number of cubes.

The third and final task of this set, addition and subtraction with trains, is also based on an assessment by Kathy Richardson (2003): "Two-Digit Addition and Subtraction." The usage of models of rows of ten linking cubes and scattered cubes representing ones are the same between both the assessment and the task for this project. The questions are very similar as well and only slightly reworded for the ease of using them in an interview. For this project I ask for the model of 26:

- What would happen if we added 9 more?
- What would happen if we added 16 more?

And for the model of 36 cubes I ask:

- What would happen if we added 20 more?
- What would happen if we added 12 more?
- What would happen if we took 9 away?
- What would happen if we took 17 away?

However, again, Richardson's purpose and my purpose differ. Richardson's purpose is to "assess children's ability to add and subtract numbers" (p.30). I want to explore how the explicit and implicit zeroes interact with addition and subtraction and the decomposition of numbers. Thus, though the tool may be the same and the questions similar, the resulting interactions are not.

3.2.3 Reading and Writing Numbers Tasks

For the reading and writing numbers task, I ask participants to first transcribe ten numbers of up to five place values. I then ask the participants to read ten numbers of up to six place values, from cue cards. This section of tasks presents a sort of mental break between the problem solving of the last task and the problem solving for the next section. Much of this task relies on recall of rules. Importantly, it should be noted that this set of tasks has the potential to be problematic because of the risk of concatenation (Fuson et al., 1997) of the numbers. However, at the same time, through pairing the writing and reading task together with the clinical interview, this task can be a rich source of data for how participants interact with explicit and implicit zeroes.

3.2.4 Number Line Task

The number line task presents participants with an open number line with only the numbers 1 and 10 benchmarked. In a non-sequential order, I ask the participants to place various numbers on the number line. At one point, in order to support explorations of zero as a midpoint number, or the motion metaphor of zero (Lakoff et al., 2000), the participants are asked to place the zero on the number line.

3.2.5 Exploring Zero Tasks

The Exploring Zero Tasks consists of three tasks:

(1) What is Zero?,
(2) Zero Worksheet, and
(3) Which is Worth More?

The What is Zero task relies on embodied understandings of zero. It first asks the participants what they think zero is, and then asks the participants to show what zero is without verbal language.

The second task in this group, the Zero Worksheet, is based on a task from Cockburn and Parslow-Williams (2008, p.19-20). The authors and their colleagues devised a challenge for the readers of their chapter to test their understanding of zero in computations. Because the purpose of this task for this project and for the chapter differ, I made changes to the order of the questions, omitted some questions and added others. First, I regrouped similar questions with the purpose that the participant could utilize their understanding from one question to solve another. I also included only questions with zero, with one exception. When the questions move from addition and subtraction into multiplication, I left in the 1x1 question. Because multiplication and multiplicative reasoning are known and common areas of difficulty for learners experiencing mathematics difficulties (Tzur, Xin, Si, Kenney & Guebert, 2010), similar to the purpose for regrouping the questions, I left in 1x1 as a model for the possibility of applying this question to the subsequent multiplication questions with zero.

The third task, Which is Worth More?, is based on a challenge presented by Cockburn and colleague (2008) in their chapter on misconceptions of zero. For this task, I present participants with two number cards, one with 150 and the other with 105. I ask the same question as the authors, “Is the digit 0 worth more in 105 or 150?” (p.19). It is important to note that this question has no answer. Cockburn and her colleague asked a number of mathematics educators and mathematicians what they thought was the answer to the question, and received various input. This task, because of its multiple answers, was chosen for the potential it would have to deeply explore reasoning about the number zero, including about implicit and explicit zeroes.

3.2.6 Exploring Number Relationships Tasks

The Exploring Number Relationships Task consists of one worksheet with two parts: (i) Fill in the missing numbers, and (ii) True or False. The purpose of these tasks is to explore the growth that results from the participants reasoning about number relationships involving zero, and how zero can be a tool to learn about relationships.

Both the tasks used in this section are commonly employed in algebra to explore number relationships. However, many learners experience difficulties when completing these tasks because of a misunderstanding that the equals symbol identifies an answer. Difficulties may arise as well from having the sum first, for example 9=7+2, and from having addends and augends on both sides of the equal sign, for example 8+4=7+5 (Carpenter, Franke, & Levi, 2003). Questions with the potential for these common errors are purposefully incorporated into the worksheets. Then, to explore the potential of zero building relationships, some of these questions are followed by similar questions incorporating zero.

3.2.7 Telling Number Stories

Creating number stories has a lot of potential for exploring understanding of number and operations in mathematics (Graven & Coles, 2017). This task thus creates an opportunity for exploration in growth of understanding zero by itself and in relation to mathematical operations. The telling number stories task asks participants to first follow a short verbal addition story as a model and then create their own stories with specific numbers as solutions. The last of the numbers the participants are asked to create a story for, is zero. By including a progression from addition to solutions and then to zero, the learner could potentially fold back to these immediate experiences in order to support a story for zero.

3.3 Data Analysis

The method for data analysis of the video data follows Powell and colleagues (2003) seven phases of analysis that, like PK theory, are non-linear. This seven-phase method of video data analysis is especially useful for this project as they were developed specifically for studying mathematics thinking research. The seven phases are:

- viewing the video data attentively,
- describing the video data,
- identifying critical events,
- transcribing,
- coding,
- constructing storyline, and
- composing narrative.

I do not analyze the whole corpus of data that I collect (Pirie, 1996). Instead, I view and review the data looking for moments of, and movements in understanding. The purpose of the first three phases of viewing attentively, describing and identifying is to narrow down the amount of data to be analyzed into manageable parts. Thus, by the time I reach the phase of transcribing, I only transcribe the data which I intend to analyze further.

Categories of coding are extremely important to the research and are determined by my research questions (Powell, 2003), stemming from the theoretical frameworks. It is important to note that this seven-step process does not happen once; the process of analysis is dynamic in nature, and occurs multiple times. While I transcribe, I question the event and then decide to re-review parts from the critical events phase to gain more perspective. Reviewing the critical events phase then leads me to look at other data and so on. Often at the same time as I re-review data, I begin to construct codes, a storyline and a narrative for my analysis. This leads me back to the critical events phase as I explore the critical events across mappings, and so on.

In addition to the Powell phases of data analysis, I use a method first used by Towers (1996 in Pirie) and discussed in Pirie (1996). This method can be described as keeping an additive set of research notes that expand during each viewing of the data. I create an initial set of notes with my observations, questions and analysis during my first viewing of the video data. I then type these notes into an electronic file, making note of extra analysis. I then review the video and during my next set of observations I add to, comment on, modify, strike and question these original observations whilst typing new observations all in a different colour. I then continue this process of filing, modifying and new observations throughout my process of analysis, using new colours each time. In this way "the original notes are retained unaltered, creating an ever more detailed dossier that still retains the facility to return to original notions for comparison and verification" (p.10).

3.3.1 Presenting the Data: How to read Chapters 4, 5, and 6

Chapters 4 through 6 each respectively describe the growth of a zero concept for Angela, then Melissa, then Megan. In presenting the data, each chapter follows the same pattern:

- I first introduce the reader to the participant. Much of the data for this first introduction is from the initial introductory interview (see section 3.2.4.1) during the mathematical task-based interview. Here, identity and relationships to and with mathematics are explored.

- Utilizing the PK model, in the next section, I present the reader with a mapping for that participant's growth in understanding zero.

- The final introductory section, discusses a short overview of the participant's journey. The purpose of this section is to present a meta-view of the participant's growth-discussing shape, trends and connecting events.

- The bulk of the chapter and the final sections are a zooming in and out of pivotal moments of movement from the mapping. The notion of "zooming" (Lampert, 2001) is important here because each action of zooming-in takes into account that these movements are embedded in a larger story of recursion and growth. I have given each of these sections a title, reflecting the narrative analysis of the moment of growth. In these sections, transcript excerpts and analysis are not separated. Instead, because of my role as empathic observer, I interweave the analysis with the transcripts and include my own intentions, responses and the evolution of both through these interactions. My analysis does not present the data in a temporally linear fashion either. Instead, when the data supports my analysis, I draw from episodes "out of linear order." I use temporal language to cue the reader to these moments of non-linearity.

A note to reading the transcripts: Transcription excerpts appear: i) on their own, single spaced, and indented, and ii) within paragraphs of analysis themselves. These excerpts all have quotation marks around them. Transcription excerpts also appear as dialogued conversation. These excerpts indicate speaker by letter and colon, for example Angela is "A:," Melissa is "ME:," Megan is "MN:," and myself the interviewer is "I:." Also, conversation that is occurring simultaneously is noted by brackets. Finally, communication outside of dialogue, for example gestures and sighs, are noted in brackets and italics, for example: (*A reaches for a cube*).

Chapter 4.0 Angela

4.1 Introduction to Angela

> "Like sometimes when we're doing a boring lesson I'll hate it (*referring to math*). Sometimes when it's an easy lesson, then I'll like it. Sometimes we have a test coming up… and… I study. Sometimes I get it wrong, and I'm just like, I hate math… Usually I hate math."

Angela presents a complicated mathematical identity: she expresses opposing beliefs and dispositions (Martin, 2007) about mathematics. Much of the tension at the root of this complicated mathematical identity is as a result of her belief that the high amount of effort she expends attempting to do well in mathematics, should be proportionate to a high achievement level. Angela's mathematical identity is encapsulated by her mathematical difficulties; she has had persistent difficulties in school mathematics and has received outside support. At the same time as receiving support and expending a lot of energy trying to do well, Angela's achievement in school has been labored and inconsistent. However, the mathematics that Angela says she hates is not actually mathematics. Angela hates the structures of schooling (McDermott, 1996) that surround the teaching of school mathematics: testing/grades and learning procedures devoid of connections.

It is only after receiving her grades and realizing the discrepancy between her effort and achievement that Angela says she hates mathematics. At the same time, moments of mathematical insight, her "AHA moments" (Mason, Burton & Stacey, 2010), propel Angela to persevere through her frustrations. Angela relates that it is during these AHA moments that she develops more positive beliefs about mathematics. Angela describes her shifting feelings from frustration to pleasure as a result of persevering in understanding a protractor and angles:

> "But like when we did like the protractor. At first I didn't get it. So it was really hard and annoying and frustrating because I kept on like, getting the wrong

> answer when we were doing paper. Finally when I got it, then it was a little bit more funner than when I didn't like it."

When I probe as to why she thinks the protractor was frustrating at first, Angela recalls that she could not remember the rules of how the protractor worked. Like other children who experience mathematical difficulties, memorizing rules without context or meaning presents a barrier (Baroody, 2011) for Angela to performing mathematical procedures. Angela's reason for her resentment of having to do the procedure of measuring an angle repetitively without remembering the rules, is as a result of rote learning devoid of connections. Angela's description of her AHA moment, when she "got it," was as a result of understanding the connection between the conceptual and structural aspects of the protractor. Angela's difficulty arose from having memorized a set of rules in isolation.

4.2 Mapping: Angela's Pathway of Dynamical Growth

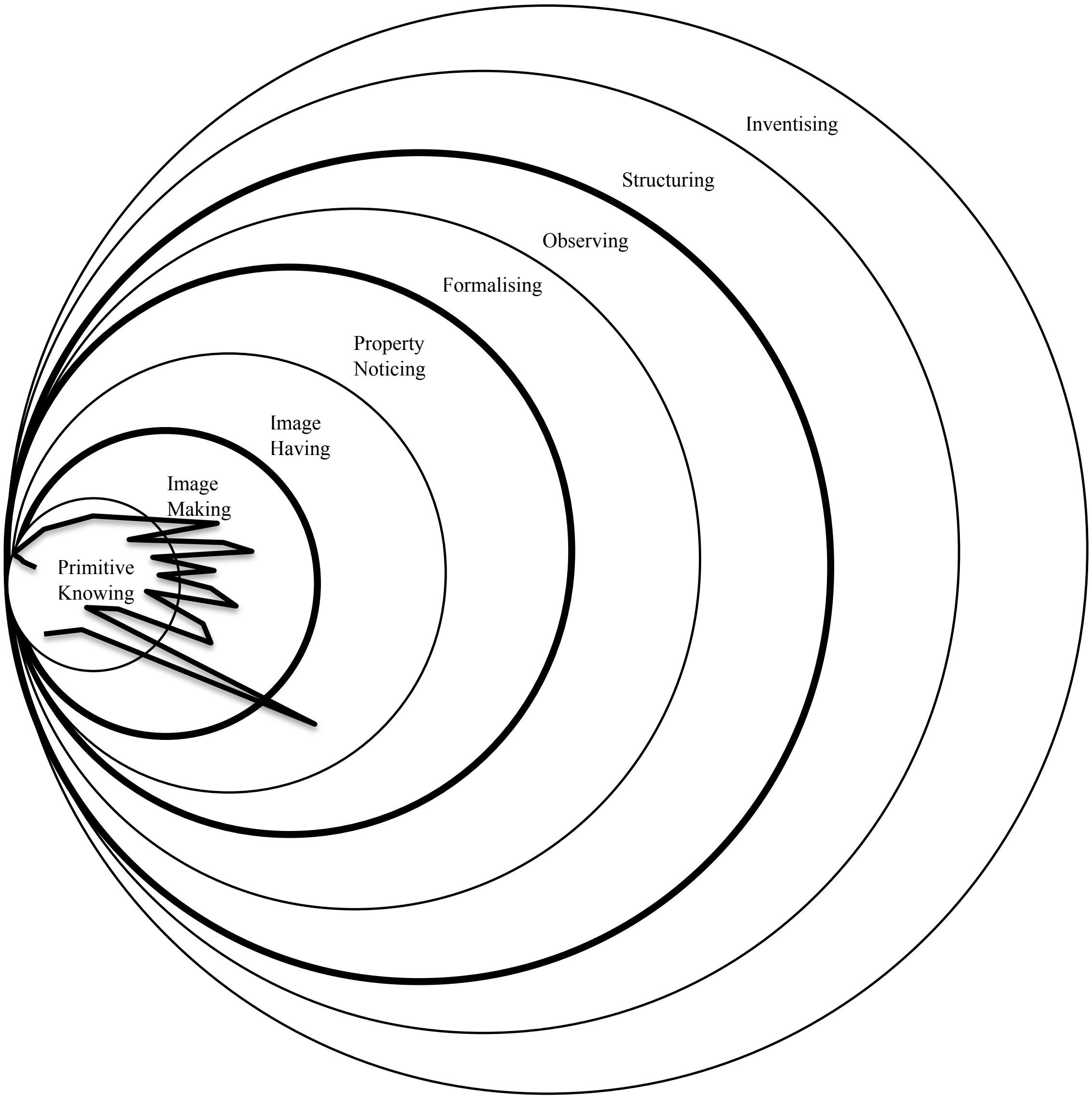

Figure 4.1 Angela's Mapping

4.3 Angela's Journey

Throughout Angela's journey of growth in understanding zero, she hovers in, out and around Primitive Knowing. Each time Angela moves into Image Making, and once into Image Having, she does not stay there for long; she quickly folds back to her previous understandings of zero. One reason for this constant recursive movement is that Angela does not usually trust her own ideas. And if developed further, often these ideas could lead to more growth. After proposing an idea that could potentially demonstrate growth toward creating images of zero, Angela will often look to my face to gauge the accuracy of her statement. Relying on adult input, such as Angela's gauging the correctness of her answer through my non-verbal cues, is a typical response by children who experience mathematical difficulties (Shih, Speer, & Babbitt, 2011). Without a supportive response from myself, Angela will deny her new understanding and revert back to her original understandings. These folding back movements are not with the intention of thickening, and are often accompanied by the words "I don't know." However, even though as an immediate response Angela reverts back to primitive knowings, the developing understandings of zero that emerge, re-emerge more quickly and strongly later during our session. At the end of our sessions together, Angela identifies my lack of response as both her most frustrating moments and her moments of learning.

4.4 Zooming In… ***The First Signs of Zero, Primitive Knowing***.

Figure 4.2 Angela Zooming In… The First Signs of Zero

Angela begins her journey with the number combination task and identifying composites of 5. She partitions the pile of blocks and identifies the composite numbers: 1 and 4 and 2 and 3. Angela pauses. I ask her, "any other ways?" I am wondering if Angela is going to recognize 5 and 0 as a possible combination for 5. Angela immediately says five and zero. I realize that zero as an existing image is a possibility; but still question which representation of zero Angela is referring to. Is this the number zero, a counting zero, a zero that represents nothing or maybe this zero is an artifact from a memorized set of combinations?

The next part of the task asks Angela to find all the composites of 10. I wonder again if Angela will recognize zero as a possibility. Again, I wait until Angela exhausts every other combination of 10, except 10 and 0. Then I ask Angela if there are any other combinations for 10. Angela systematically reiterates the combinations she already did: 1 and 9, 2 and 8… etc., but does not automatically arrive at a combination of 0 and 10. Since the 0 and 10 combination is not in Angela's immediate consciousness, I wonder if her previous immediate recognition of five and zero is because it is memorized. Or, perhaps her immediate recognition is because there are less combinations for five, and thus they are easier to be kept track of, than for ten.

This episode is one example of Angela's primitive knowing. Through the task and my interaction with Angela, I can only identify that Angela has some sort of knowing about zero in her primitive knowing. I have not learned the type of knowing that Angela has around zero. It is impossible to identify all the primitive knowing that one person has around a concept. primitive knowing is too vast and often we can only identify retroactively, through later mathematical interactions, the primitive knowings to understand or accomplish something mathematical. This primitive knowing, that 5 and 0 are a combination for five, whatever representation that zero is, is an entry point for growth in understanding of zero for Angela. Even though this primitive knowing around zero may not be at the forefront of Angela's consciousness and may be completely rote, still, an idea about zero is there to be accessed and reverted to.

A similar example of Angela's primitive knowing around zero arises during the hidden blocks task. I present 10 blocks and ask Angela to turn around while I hide a certain number of blocks. Angela then tells me how many blocks I have hidden. After three rounds of hiding different numbers of blocks, I hide zero blocks. Through hiding zero, I want to know if Angela uses the word "zero" or the word "nothing" to identify what is hidden. As

Blake and Verhille (1985) note, this distinction between the label of “nothing” and the label of “zero” is important; it conveys a distinction between the colloquial and the mathematical. Angela looks at the revealed blocks and answers “zero.” Again, Angela reveals some of her primitive knowing. This time, it is an at least implicit distinction between zero and nothing.

4.5 Zooming In… *The Implicit Overlapping Zero*

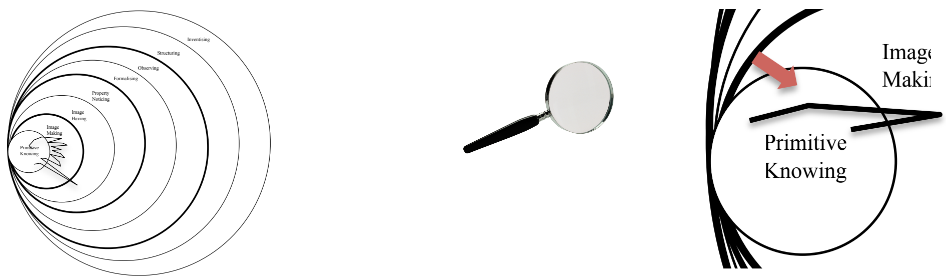

Figure 4.3 Angela Zooming In… Implicit Zero

Angela’s interaction with the writing number task reveals primitive knowings around the implicit and explicit zero in numbers of larger magnitude. In numbers with more than one place, zero can act as a placeholder and/or a support of the numerals before it. One of the numbers I ask Angela to write, 6000, is a good example. 6000 has 3 explicit zeroes, meaning all 3 zeroes are showing. Although the numeral 6 is in the number, the zeroes, acting as supports, convert the 6 into 6000. Meaning through the zeroes, the 6 represents six *groups* of a thousand and not 6 *individual* items. At the same time the zeros act as placeholders, demonstrating that no other numerals belong in that space. However, when some zeroes are implicit and explicit in the same number, for example in another number that Angela is asked to write, 3048, the role of zero becomes more complicated. 3048 has one explicit zero, the zero showing in the hundreds place. Additionally, there are two implicit zeroes. The numeral 3 is really the number 3000 and the numeral 4 is really the number 40. At the same time, 3000 and 40 overlap and share zeroes. Thus, the number 3048 has one explicit zero, at the same time as multiple implicit, overlapping zeroes.

Another complication for Angela is that the verbalization of a number makes explicit many of the implicit written zeroes (Chan, Au & Tang, 2014). One example of this complication from the numbers I ask Angela to write, is 126,002 (one hundred and twenty-six thousand and two). A few of the ways that 126,002 could be translated from verbalization to written form are: with the zeroes completely explicit as in 1002060002, or certain zeros could remain implicit as in 1260002, or certain explicit zeroes could become implicit and certain implicit zeroes could become explicit as in the way that Angela writes the number: 1002602.

The numbers that Angela is asked to write and their implicit zeroes prior to 126,002 pose no problem for Angela. 236, 3048, 12 603, 34 920, examples of the prior numbers, are all written silently and without question, usually Angela's sign of assuredness. Yet when I ask Angela to write 126,002, her immediate reply is "I don't think I know that." To be fair, although reading a number with six digits is in Angela's current curriculum, writing a number with six places only appears in the Ontario elementary curriculum (2005) in the year after the grade Angela is currently in. It is possible that Angela has not come across whole numbers with six places in her learning yet this year. And even if Angela has already encountered six places, the learning may be new and in the midst of its own pathway of growth. In these cases, the question can be asked, then, what exactly am I trying to learn by asking Angela to write a number with six places? In asking Angela to write a number with six places, I am inquiring into what happens with the implicit and explicit zeroes when Angela encounters something with which she is potentially unfamiliar? Even though Angela may have not learned six places explicitly, she may still have available primitive knowings to fold back to and access. For example, Angela could potentially explore patterns and relationships in place value in order to make a conjecture. However, in this case, when Angela encounters six place values, she does not pause to access other knowings, and instead she answers "I don't think I *know* that." Angela does not fold back, yet there is still movement. The words, "I don't think I know that," convey movement. "I don't know," has now become "I don't think I know." Angela now conveys hesitance when she is faced with something she does not know right away.

Still, I wonder of what is Angela's *knowing* referring to? Angela's comment tells me that writing the previous numbers is something she *knows*. Angela *knows* how to write 236 and 405. Reading numbers is also something that Angela *knows* how to do, as she demonstrates next in the reading numbers task. 23 555, 54 001, 340 781 and 600 045, are all numbers that Angela *knows*. But what is this knowing that Angela implies when she says "I don't think I know that"?

A deeper analysis of Angela's answer of "1002602" reveals that the number has within it both explicit and implicit zeroes. The one hundred is completely explicit, while twenty-six thousand and two retains four places for the thousands, and only has one explicit zero instead of two. Angela has no difficulty writing the numbers with five places: 12 603 and 34 920. The explicit zeroes were all in the right place. Now, while encountering a new number, one for which she has yet to internalize the rules for, the awareness of the implicit zeroes and the multiple overlapping of the implicit zeroes become confusing. The zeroes in one hundred (thousand) are disconnected from the zeroes in twenty-six (thousand). At the same time, the thousands retain four places, but the overlapping of the twenty (thousand) and six (thousand) become confused. Angela's primitive knowing encompasses conceptual ideas of implicit and explicit zero, but not their connectedness. This knowing becomes more explicit and will be thickened and moved into image making and image having in a later session when Angela and Melissa work on a task together. This will be an "AHA" moment for Angela when she will discover how it is possible for zeroes to be implicit and overlap, while certain zeroes are still explicit.

The following excerpt is a discussion between Angela and Melissa with small interjections by myself, the interviewer. The excerpt comes from one of the group task-based intervention sessions. In the task, a manipulative of overlaying strips is utilized to explicitly display the implicit zeroes.[14] The strips are different sizes and partitioned to

[14] I, as of yet do not have a source for this manipulative. At some point in my teaching career I learned of this manipulative, and have been constructing and using it since. I have since found a similar tool, but with slightly different affordances in a Montessori place

represent different place values. For example, the strips consist of: a ones strip that has one open space to write a numeral in, a tens strip with two spaces-one space to write a zero and one space to write a numeral-the ones strip fits perfectly over the zero space on the tens strip, a hundreds strip with three spaces-two for zero and one for a numeral-the tens strip fits perfectly over the two zero spaces and the ones strip fits perfectly over one zero space. One set of strips, each with the affordances of lower place values fitting onto higher place values and partitioned spaces for zero and one numeral, consists of a unit card through to a million card. This manipulative provides opportunity for the learner to confront the presence of the implicit zero. Using the number "56,020,800," figures 4.7, 4.8 and 4.9 demonstrate a specific example of how the strips have the potential to make the implicit zeroes explicit.

Figure 4.4 Example of Different Sized Strips with Partitions

value card manipulative. One difference is that while the Montessori manipulative can be overlaid, it is too clunky to have multiple overlays. Also, in the manipulative that I use, the spaces are all blank, and the learner fills in all the zeroes and numerals on the card. It is possible that the manipulative that I use is an outgrowth of the Montessori manipulative.

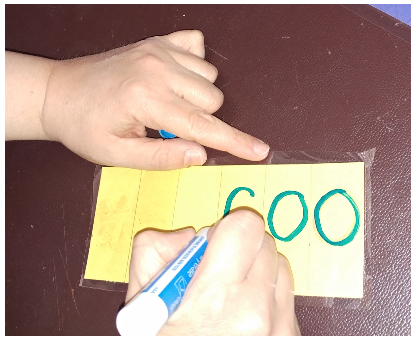

Figure 4.5 Writing the Implicit Zeroes on a Hundred Thousands Strip

Figure 4.6 Filling in the Billions Place on a Full Set of Strips

Figure 4.7 The Number: 56,020,800, The Strips Viewed Separately

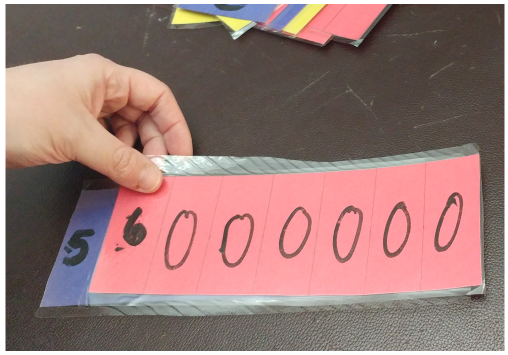

Figure 4.8 The Number: 56,020,800, Beginning to Layer the Strips

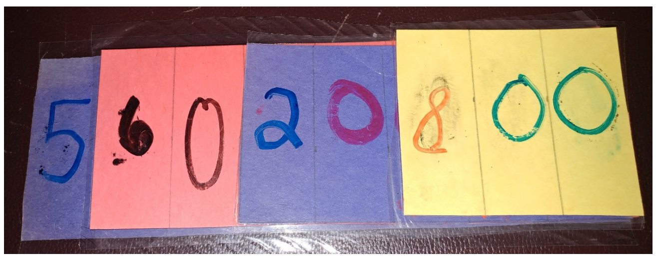

Figure 4.9 The Number: 56,020,800 Layered Strips

Angela has just confronted the implicit zero and is trying to make sense of how a numeral, the unit card, can be layered on top of the zero. This idea leads Angela to the conclusion that zero is a "big" number because it could represent anything, meaning the implicit zeroes in a number can have any numeral overlaid on top of them. In Angela's understanding this makes zero a large number. However, this understanding becomes conflicted by the idea that 5135 is larger than 5130. Angela and Melissa are working hard to reconcile this conflict:[15]

A: Kay. There's a not... kay. Over here we have a number 5, 1, 3 (*writing numbers*). 5, 1, 3. Oh sorry. yeah 5, 1, 3. And we want to add a number. But we're thinking a zero. So you put a zero and that's going to be five one three zero. So five thousand and one hundred and thirty. But let's say we wanted to add a five? That would be five hundred, five thousand, one hundred and thirty-five. And isn't a zero can be any

[15] Brackets either indicate actions, or two or more speakers speaking simultaneously.

number so I thought a zero can be bigger than a number but if... thirty-five is bigger than thirty, like this number right now is bigger than three one three zero. 'cuz that's thirty and this one's thirty-five. But how... like, I thought zero can be big? Like (I thought it was really big)...

Angela is just coming across a paradox in her reasoning: If zero is a big number because anything can go on top of the zero than why is 5135 bigger than 5130? Melissa attempts to help Angela through her paradox:

M: (Well) it's just adding five.
A: I thought like zero is like, could be... like, is a bigger number. Or could be a bigger number than...
M: Well.. it kind of could. like... if you have... kind of, if you would have like... (*reaches for another strip*) three hundred and five (*writes it down*). and then like the zero is kind of coming to represent the three hundred, and then the five's just like added.

Melissa is attempting to explain to Angela that the essence of the implicit zero is not about units, where the unit zero is bigger than any other unit number. She is explaining that the implicit zero is multiplicative. The zero(es) is attached together to change the meaning of digits in higher places.

A: Ohhh, I think I get what you're saying
M: And it's in the ones place maybe.
A: Oh, I think I get what she's saying. Like. that's like three hundred and then, three oh five. It's three hundred then (you're just adding a five to make it a little bigger)
M: (Is it the zero almost bigger, 'cuz it's in the) hundreeeeeeeeeds
A: Or there's... Or there's a five and you wanted to make it smaller so you (erase the other number). (*Whispers the last part*).
M: (Yeah because) isn't... the five is in the one's place and the zero is in the tens place so it's (kind of considered bigger).
A: (Should I explain it to...)
I: (and what's hiding behind the five?)
M: Zero.
I: So what can we say then?
A: Oh I get what she's saying. Once there is a number five three one oh, three hundred and ten, under... I mean sorry...k…three hundred and five, and under that five is a zero. Is that what she's saying?
I: Is that what you're saying Melissa?
M: yeah.

A: Under the five the zero

In this later episode, Angela's understanding of the implicit zero has moved into Image Making and then into Image Having as she makes connections between implicit zero and object construction (Lakoff et al., 2000) zero. At the beginning of this excerpt, Angela's idea that zero has the largest value of any number stems from her thinking that zero is not implicit, but flexible with the possibility that zero can be changed into any digit. Working with Melissa, Angela begins to recognize the implicitness of zero, that it is present, even though she cannot see it. Melissa tells Angela that "five's just like added." This is an act of shared Image Making: the zero is implicit, and that the five is layered on top of the zero. With this understanding Angela realizes how it is possible for 5135 to be larger than 5130.

During our original session, it seems that elements of the implicit zero and its connectedness are not yet in Angela's repertoire for numbers of larger magnitude. However, elements of the implicit zero are in Angela's primitive knowing for two-digit numbers. In this case, Angela demonstrates a particular knowing, or a limited image, of an implicit zero important for decomposing two-digit numbers (Kilpatrick et al., 2001). For the adding trains task, I present Angela with two trains of ten blocks and six non-connected blocks. I ask Angela to add 9 to the trains. Angela can use whatever tools she would like to solve the problem. Angela chooses to use her fingers to count up from 26 to 35. Then I ask Angela to add 16 to the original 26 blocks, this time Angela chooses a different strategy:

> "Well, (tapping fingers on the table each time she says a number) 6 plus 6 is 12. (looks to the side) So 12... then that's 32. And add a 10... 42."

Here, Angela decomposed the numbers 26 and 16, with their implicit zeroes, to arrive at her answer of 42. While at this point I am still unsure as to why Angela used the two different strategies, her use of the strategies is important for distinguishing between seeking a relationship with ten (Murata & Fuson, 2006)-the first addition task, and the implicit zero-the second addition task. Angela's strategy of decomposing the two-digit numbers reveals a knowing of implicit zero.

4.6 Zooming In… *What is Zero?*

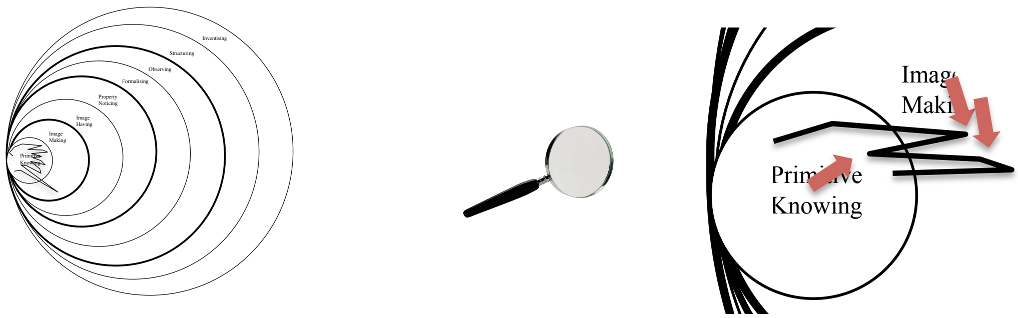

Figure 4.10 Angela Zooming In… What is Zero?

I ask Angela the question, "what do you think zero is?" and our resulting exchange lasts 39 seconds. While 39 seconds seems like a short period of time, during those 39 seconds a lot of recursive movement, or change, happens with Angela's understandings surrounding zero. Most of this change occurs in Angela's thickening of her primitive knowings.

Because of the density and frequency of movement, I have chosen, for this section, to change the presentation of the narrative. In what follows, I first zoom in through an overview of the movements occurring in these 39 seconds. In this zooming I share the entire transcript of the 39 seconds with the reader. I then zoom in further, analyzing sections of Angela's utterances and actions. To indicate where I am in my analysis, with each new zooming in, I begin with an italicized quote from the transcript that I am analyzing.

The First Zooming In:

I: What do you think zero is?
A: A number that is like... iiiiis like 10 hands (*shows ten fingers*). It's 10 like, pieces. Zero is zero pieces (*when she says zero makes thumb and pointer fingers in the shape of a "0"*). Or is zero just... a number that's nothing?
I: What does that mean?
A: ummm like it's um like 1 is 1, like 1 thing. zero is nothing.

I:	Ok
A:	Cuz if you have like 1 piece of paper, that's 1. But If you have zero, like... (*looks to the side*), there's no zero... Like you can't say you have zero paper if you have paper. 'Cuz like zero isn't a number. Well (*looks up*) zero is a number, but it's nothing.

Zooming In Deeper- A number that is like... iiiiis like 10 hands (shows ten fingers). It's 10 like, pieces. zero is zero pieces (when she says zero makes thumb and pointer fingers in the shape of a "0"). Or is zero just... a number that's nothing?

My question about zero causes Angela to begin to interrogate her concept of number. Like the historical progression of number (Toma, 2008), Angela's concept of number is based on counting and identifying tangible, physical quantities. Thus, Angela's immediate answer refers to the physical-showing ten fingers to represent ten. And then Angela's answer refers to the container metaphor-making a zero sign with her pointer finger and thumb to represent zero. Both ten and zero, because their understanding stems from actions of counting, have physical representations to Angela. These tangible representations of number versus her prior understanding of zero as nothing, or the absence of tangibility, initiates a cognitive conflict (Movshovitz-Hadar & Hadass, 1990).

Angela now begins to question her concept of number as only a tangible entity containing nothing: "Or is zero just… a number that's nothing?" Angela uses the word "or," meaning either zero is a container or it is a number. It would seem on first analysis that the word "or" here indicates a possible recursion in the growth of understanding the connections between zero concepts. However, this "or" is the beginning of a movement forward because it initiates a cognitive conflict. And, in trying to explain the cognitive conflict, Angela begins to think about connections between zero as container and zero as number. As will be further elaborated in the next sections, this cognitive conflict essentially propels Angela into Image Making of the connections while thickening her understanding of zero as a container.

Zooming In Deeper- ummm like it's um like 1 is 1, like 1 thing. zero is nothing.

In attempting to figure out the meaning of “a number that’s nothing,” Angela now folds back, and in two separate statements revisits her conception of the physicality of number and the nothingness of zero. In fact, throughout her journey Angela spends a lot of time in Primitive Knowing-interrogating and thickening her prior understandings.

One is something. Zero is nothing. Something and nothing are not reconciled yet. Angela moves backward to dip into Primitive Knowing, and then makes shifting movements within that space. These shifting movements are movements of thickening as Angela now has to define for herself what is that something of one, and what is that nothing of zero, and if these ideas can really be connected. The thickening is occurring right now only because of the conflict that Angela is experiencing. The conflict forces Angela to revisit, interrogate and rebuild her previous understandings, leading to new images of zero and number. Because Angela is simultaneously thickening as she is building new images, this folding back that Angela is doing is not to the same place she was before when she was at the ten hands and zero pieces place.

Angela’s understanding of nothing and number are changing: thickening and moving, converging within Primitive Knowing. The integration of these ideas, for Angela, has become important to understanding the multiple roles that zero plays in mathematics, and in creating an image of zero as a mathematical object. Angela has to remain in Primitive Knowing to reconcile her two apparently opposing ideas. She has to thicken each understanding in order to integrate them. Once they are thickened, Angela can move into Image Making and connect her understandings.

Zooming In Deeper-Cuz if you have like 1 piece of paper, that's 1. But If you have zero, like... (looks to the side), there's no zero...

Because of the cognitive conflict Angela is experiencing, her upward and sideways movement of thickening is different than her movement to and from different levels of understanding. Zero is a paradox (Byers, 2007), and reconciling this paradox is difficult. There is a tension in Primitive Knowing as prior understandings about the different elements of the paradox compete with each other. It is this tension that is shifting the thickening understandings sideways and upwards.

Figure 4.11 Zooming in Further: Shifting Sideways and Upwards in Primitive Knowing

Here, Angela revisits the counting, tactile explanation for number. However, she is not only revisiting this prior understanding, she is attempting to integrate it together with her concept of zero being nothing. In her first movement, discussed in the previous paragraph, Angela says “1 is 1, like 1 thing. Zero is nothing,” as two separate statements. These two separate statements state two prior knowings that have yet to be integrated. Now Angela shifts sideways as she explores her primitive knowing-two seemingly incompatible ideas. Angela is persevering in attempting to integrate the two opposing ideas, answering how can there be something from nothing? With the tentative words “there’s no zero,” Angela pauses for a moment in her movement. She reiterates the paradox, gathering her prior knowing. Because the cognitive conflict has still not been reconciled, this pause only serves to propel Angela into further collecting her primitive understanding.

Zooming In Deeper-there's no zero... Like you can't say you have zero paper if you have paper. ‘Cuz like zero isn't a number. Well (looks up) zero is a number, but it's nothing.

“There’s no zero” illustrates the tension that Angela experiences while attempting to reconcile two contradictory previous knowings. To better see Angela’s tension, we can replace the word zero with the word “nothing,” a word that represents how Angela views zero, in Angela’s reasoning statement. We can see that, “Like you can’t say you have zero paper if you have paper,” means to Angela: you can’t say you have nothing if you have something. Angela concludes that there are irreconcilable differences between zero numbers: “ ‘Cuz like zero isn’t a number.” However, in direct conflict with this conclusion, from somewhere in Angela’s prior experience, she knows that zero is a number. Angela is moving upwards toward Image Making here. With the last statement, “Well zero is a number, but it’s nothing,” Angela has certainly not reconciled her tension and does not have an image of zero as a number. Angela has, however, found a momentary peace with the possibility that zero can be nothing and a number at the same time. This is a movement into Image Making as Angela toys with the notion that zero can have more than one role: as a number, and as nothing.

4.7 Zooming In*... Zero is nothing but...*

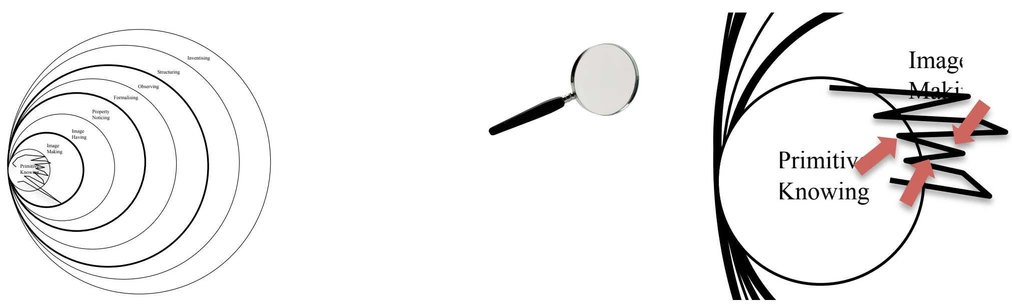

Figure 4.12 Angela Zooming In… Zero is nothing but…

I show Angela two cards, one with 105 written on it, and the other with 150 written. I then ask Angela a question which she should think is preposterous, especially considering her understanding of zero as nothing. “Is zero worth more in this number (105) or this number (150)?” The way the question is worded implies the possibility that zero has value. To be sure, Angela does not immediately engage with this idea. Her first and immediate response, like other times before, is “I don’t know.” However, this time, this stance of not knowing is only momentary. There is only a slight pause, just enough for a period between thoughts, between the "I don't know" and an engagement with the question. Angela tells me why she doesn't know as she explains her reasoning:

"I don't know. 'Cuz…"

In continuing immediately after saying "I don't know," Angela is becoming more comfortable expressing her thoughts, and is relying less on me to ascertain their "correctness."

Even though Angela is quick to share her reasoning, I still wonder why she does not have a problem with the question itself. Why does Angela not simply respond, "They are the same-they are both nothing"? In entering this task, I expect this answer considering that the knowings that Angela has so far displayed around zero has had mostly to do with zero being nothing. Maybe Angela does not question the question because of my authority. Angela associates me with a teacher knowledgeable in mathematics. Therefore, she may be thinking, why would someone knowledgeable in mathematics ask an impossible question? However, one or two problematic results would occur if it were only the case that Angela accepts this question because of my authority. Skipping reasoning about the question, Angela would either not know what primitive knowings to gather, or, like Angela did with the previous paradox, she would perseverate in previous understandings without discretion. Consequently, instead of requiring some sort of intervention from me, Angela immediately engages in reasoning an answer to the paradox of one zero being worth more than another zero. Thus, another explanation is that Angela's tension with the previous paradox, of "*Well zero is a number, but it's nothing,*" and the simultaneous thickening and movement she experienced, has enabled her to encounter a new paradox without rejecting the paradox outright. Thus, there has been an additional change as a result of the last paradox. Angela experienced a thickening, a shift with regard to paradoxes of zero. Because of this change, Angela's interactions, and therefore her pathway during this paradox is significantly different than for the previous paradox. This time, Angela does not spend time interrogating the paradox itself, and attempting to conjoin two seemingly disparate ideas, instead she makes quick movements into Primitive Knowing to support her developing Image Making. There are quick back and forth movements between Image Making and Primitive Knowing, where she does not stay in either for long.

In order to make the following excerpt more lucid for the reader, I break it down, inserting visuals of the numbers that Angela points to, and gestures she makes. I was not given permission by Angela to include images from the video in this dissertation.

“I don't know. 'Cuz this number is 50 (*Pointing to the 50 in 150*)

Figure 4.13 This Number is 50

so, it's not like... and in this number-(*points to the 150*).

Figure 4.14 This number is 150

Oh (*with certainty*)! This one because like, because then it makes it 50. If you didn't have the zero because then it would be 15 (*uses hand to cover the zero*).

Figure 4.15 “15” Because the Zero is Covered

if you didn't have the... (*Uses hand to cover the zero in the other card-looks confused),*

Figure 4.16 This is Also 15 When the Zero is Covered

but also... this number is just like putting the number in the middle 'cuz there's nothing there. It's just 5. (*points to the zero in 150*). 150, zero.

Figure 4.17 There's Nothing There

Yeah but this number (*points to 105*) would also be 15.

Figure 4.18 Revisiting 15 in the 105

(*covers the zeros on both cards*)... I don't know maybe not...

Figure 4.19 Both Zeroes Covered

it would be one and five because there has to be something in the middle. (*pointing to the 150*). This is at the end."

Figure 4.20 This Zero is at the End

Here, in order to decide which zero is worth more, Angela strategizes using her primitive knowing to explore isolating the zero, and then, removing the zero. Angela's interactions with this exploration lead her to Image Making, making connections around zero as a placeholder that (a) zero has a relationship with the surrounding numbers, and (b) zero depends on its relationship with the surrounding numbers for zero's identity. In constructing her images, Angela is also thickening knowings around the implicit zero.

Angela's first strategy is to use her primitive knowing of decomposing numbers to isolate the "50" from 150. Similar to the previously discussed decomposition task, Angela is utilizing her primitive knowing of the implicit and explicit zeroes up until the tens place. This strategy of dipping back into Primitive Knowing to retrieve the implicit zero up until the tens place, leads Angela to begin Image Making about implicit and explicit zero's relationship with the surrounding numbers. At the same time, because Angela has not yet thickened her knowing to include the implicit zero in other place value places, another cognitive conflict arises. Angela then removes the explicit zero first from 150. Angela is left with the two numerals, 1 and 5 to make fifteen. This satisfies Angela until she realizes that removing the explicit zero from 105, leaves her with the same numerals, 1 and 5 to make fifteen. The two identical numbers of 15 cause a cognitive conflict for Angela. This conflict then propels Angela to immediately dip back into Primitive Knowing. She gathers the same understandings as before, implicit and explicit zeroes in two place values, but now, uses her primitive knowings in a different way. This time, Angela focuses on the 105. Different than the 150, the 105 has a zero in the middle, in the tens place. This slight difference, affords Angela to reason that even with the removal of the explicit zero, there is still something there: "*it would be one and five because there has to be something in the middle*." The one and five could not join together because there is still something in the tens place. She explains that even if the (explicit) zero is removed, something has to be positioned between the one and five. The explicit zero to Angela in 105 is different than in 150. Thus, she reasons that the numerals "1" (one hundred) and "5" (fifty) in 150, can be 15 because they are beside each other, but the one (hundred) and five in 105 cannot be fifteen because something has to be placed in the middle between them.

As Angela plays and moves between the different explicit zeroes in 105 and 150, she is on the cusp of Image Making. Angela is making images of the relationships between the explicit zero and zero. To be sure, Angela has not yet constructed an image around the implicit zero, she is not even considering this zero-in Angela's realm of possibility 150 can be 15. At the same time, Angela's conjecture as to why 105 cannot be 15, has the potential to similarly apply to 150. However, the reasoning for 105 relies on the explicit zero, and the reasoning for 150 relies on the implicit zero.

4.8 Zooming In… *Zero images.*

Figure 4.21 Angela Zooming In… Zero images

Similar to a previous task when I ask Angela what zero is, Angela again experiences a conflict between something and nothing. However, the pathway that Angela takes this time is very different from the pathway she takes when she first encounters this paradox of something from nothing. The difference now is that Angela has already constructed images and has thickened, but not reconciled, her understandings during the previous, "What is Zero?" task. Thus, when Angela encounters the something from nothing paradox in a new context, she remains in Image Making while dipping into Primitive Knowing to support her burgeoning images. This pathway is the opposite of what occurs during the "What is Zero?" task. While reasoning what zero is, during the "What is Zero?" task, Angela situates herself primarily in Primitive Knowing, while dipping into Image Making.

Angela is working on the zero sheet when she comes to her next quandary. The question on the page asks Angela to write an answer to 0 – 1 = ____. Having not encountered integers formally in school, Angela does not have a prior learned rule as a primitive knowing. Angela is unsure of the answer. Therefore, she produces two options. First, with an inflection in her voice, Angela iterates that zero would be the answer. Then, Angela pauses and corrects her answer to be one. I subsequently ask Angela to explain her reasoning for both the zero and the one answer. As Angela explains her reasoning as to why she chooses one as the answer, she oscillates back to zero again being the answer.

The idea that zero is a possibility as an answer for Angela, is not surprising. Zero would be the most likely answer because Angela is experiencing difficulty with the zero paradox of something and nothing. Thus, Angela's reasoning:

"Oh no it's zero because how do you take away one if you have nothing there?"

makes perfect sense. Hence, even though partly incorrect, it is remarkable in light of her difficulty with the paradox that Angela even contemplates one as an answer to the question 0 - 1. The remarkability lies in the idea that nothing does not always have to preclude something in order to arrive at the answer of one. The idea that nothing does not always have to preclude something is a foundational relationship for constructing zero as a mathematical object. Accordingly, with the answer of one, Angela begins by situating herself in Image Making. However, Angela does not remain there very long.

When I ask Angela to clarify why she chose the answers zero and one, she immediately dips back into Primitive Knowing and to thinking about the paradox of something from nothing. At the same time as Angela is reasoning about the paradox, she is trying to remember her support for one being an answer. Angela cannot remember:

> "Because... zero.... take away 1 (*motioning with hands*)...you're gonna have nothing here and you take away...1... (*looks to the side*). Oh no it's zero because how do you take away one if you have nothing there? But I was thinking zero also because... maybe it's... zero. I don't know."

Angela is not satisfied with zero as an answer. She is experiencing a conflict, as Angela knows that she had a fleeting reasoning that allowed her to arrive at one as an answer. Importantly, even though Angela cannot identify her own reasoning, she has moved from the paradox in Primitive Knowing to Image Making with this tension. There is an alternative to the paradox of something from nothing somewhere in Angela's reasonings. The fact that she cannot access it at this moment is inconsequential to the idea that movement, and thus, change has occurred. This change is the beginnings of a construction of an image-Image Making-of zero as a mathematical object. Even though Angela is revisiting the paradox, the paradox is not as absolute as before. There is still some awareness, represented by her confusion, that the paradox is reconcilable.

4.9 Zooming In… *Leveraging zero.*

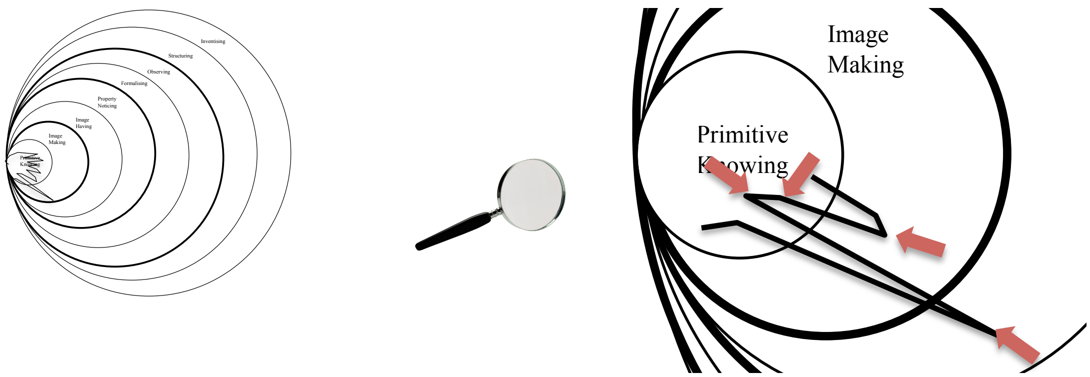

Figure 4.22 Angela Zooming In… Leveraging Zero

Angela has just encountered 8 = 10 - 2 on the "True or False" page of questions. Interacting with an equation that has one number on the left side of the equals sign and an expression on the right side of the equals sign is new for Angela. This is similar to other learners (Falkner, Levi, & Carpenter, 1999). Often, a "reversed order" for an equation can pose problems for learners because (1) many tend to view the equals sign as meaning "the answer," and (2) although only a convention, because equations in school are often written on the left side and a single numeral on the right side, many learners implicitly deduce a rule that equations can only be written in this one way. Thus, these misconceptions can become embedded in Primitive Knowing. However, this isn't the first example in these pages where Angela happens upon this "reverse" type of question. Angela has already encountered 9 = 7 + ☐. As such, she has already developed a strategy to help her through the question. This strategy is now ready to be accessed for this new question. Angela uses the strategy of subtraction as the inverse of addition to help her understand whether 8 = 10 - 2 is true or false.

> “I don't know. No because I was thinking... ten take away 8 equals 2. No? No- 10 take away 2 equals 8. So that's true.”

Immediately after answering true beside the equation 8 = 10 - 2, Angela moves onto the next question. She encounters the equation: 0 + 8 = 10 - 2. Although very similar to the previous question, 0 + 8 = 10 – 2 poses a problem for Angela.

> “Zero plus eight equals ten... take away 2. I don't get that one. (shakes head)”

With the addition of zero on one side of the equation, Angela finds she is at a loss as to how to figure out if the equation is true or false. I wonder about zero’s role in causing Angela’s difficulty. Although the only difference between the two equations is that the second equation has zero added to the left side, there is no absolute reason to conclude that it is the zero that prevents Angela from using her previously learned, and so far successful, strategy of subtraction as the inverse of addition. Not only is zero being introduced here, but so is an expression on both sides of the equal sign. Had the equation changed to 1 + 7 = 10 - 2, Angela might still be experiencing difficulty. While it cannot be concluded that zero was the cause of Angela’s new difficulty, however, it is the zero that helps Angela find a solution.

After Angela’s pause when she says she doesn’t know, and after encouragement from me to continue, Angela dips back into Primitive Knowing and attempts to use her previously learned strategy:

> “8. I'm just going to ignore that zero. eight plus... eight equals ten take away 2... Ten take away 2 equals eight. (*looks at me*)... No eight take away... No (*shakes head yes*) ten take away two equals eight (*Sounds sure- she sits up but still looks to me before she writes- I don't make a change in expression*)... No (*falls back in her chair*). Something else equals ten... I don't know (*shrugs*).”

Angela ignores the zero and then, like before, uses the strategy of subtraction as the inverse of addition. Attempting to do exactly what she has just done with the previous equation of

8 = 10 - 2. Angela's strategy should, and in fact does, lead to a correct answer. However, Angela is experiencing tension that does not allow her to be satisfied using this strategy. Instead, Angela looks to me for validation of her answer. When I do not give validation, Angela iterates "something else equals ten," and then gives up. Once Angela ignores the zero, there is no difference between the two equations. However, Angela is still experiencing a tension with the same equation- so much so that she gives up. Angela's explicit language is similar in both cases: (See Table 4.1, p.115)

Table 4.1 compares Angela's output in reasoning the two similar equations. Side by side, Angela's language in the two situations almost mirror each other. The main difference first lies in Angela's encounter with zero and then in her gestures and reactions that indicate a reliance on myself, the empathic coach here. Once Angela "ignores" the zero, then like in the first instance where Angela has developed strategies, Angela should not feel a need to rely on me anymore.

Ignoring the zero for Angela would not be a problem if zero means "nothing." However, throughout her journey of growth so far, Angela has thickened concepts around zero. Zero is not just "nothing" anymore. Zero has grown to include other relationship and conceptual aspects, including the idea that zero is a number and that zero has a relationship to the numbers around it. With these concepts thickening, it is not as easy to ignore zero anymore.

Thus far, throughout her reasoning, Angela remains in Primitive Knowing. She is gathering her knowings about zero as nothing, zero as a number and equations and subtraction as the inverse of addition. These ideas are all bubbling at the surface of Angela's reasoning; I wonder if an intervention from me can push these ideas into Image Making. I wonder if I can ask Angela a question that will cause her to think about integrating the disparate ideas. I reason that Angela must have learned the addition identity property of zero, ($a + 0 = a$), that when zero is added to any number the result is the original number. As discussed in section 1.3.1, the identity property is usually taught

Table 4.1 Comparison of Responses to 8 = 10 – 2 and 0 + 8 = 10 - 2

	8 = 10 - 2	**0 + 8 = 10 - 2**
Entry	I don't know.	I don't get that one
Response	(no pause immediately reasons out an answer)	(long pause- requires encouragement)
Deals with zero		I'm just going to ignore that zero
Reasoning the equation	No because I was thinking... ten take away 8 equals 2	eight plus... eight equals ten take away 2... Ten take away 2 equals 8.
Response	No? (Intrinsic- does not wait for a response from myself)	(*looks at me*) (Extrinsic- waits for a response from myself)
Second Reasoning	No- 10 take away 2 equals 8.	No 8 take away... No (*shakes head yes*) 10 take away 2 equals 8
Response	So that's true	(*Sounds sure- she sits up but still looks to me before she writes- I don't make a change in expression*)... No (*falls back in her chair*). Something else equals ten... I don't know (*shrugs*)
Result	**Immediately moves on**	**Stuck**

as a memorized rule/mnemonic for learners experiencing mathematics difficulties (Bryant et al., 2006). However, with the addition identity property of zero, zero acts as both:

1) a number to be acted upon - it is situated in an equation, and there are symbols around it-requiring action[16]- the object construction metaphor (Lakoff et al., 2000), and
2) nothing-it has no effect on the other number.

I ask Angela what she thinks about 8 = 10 - 2 and 0 + 8 = 10 - 2. In so doing, I have called attention to the idea that the two questions are similar. Angela responds:

> "(*looks at the questions*) Oh. So this one (*pointing to the 0+8=10-2*) is the same... Yeah so it's true. Because it's the same. Just this one doesn't have zero because 8 (moving pencil back and forth between the 2 questions) equals 10 take away 2 (*sounds sure but looks at me*). (*Writes true beside the question*)."

Angela looks at both questions and her tension is immediately resolved. The questions are similar, the only difference is a zero. In the moment Angela writes true beside her answer, I wonder if Angela is in Primitive Knowing and thickening her understandings or is she in Image Making, beginning to build images of the relationships between the ideas of zero. Has Angela gone back to zero as nothing and therefore she can ignore it, or has she begun to integrate the addition identity property of zero? It is difficult to tell right now, but with the next question of 0 + 8 = 8 - 0 and Angela's reasoning moving into Image Having, I realize that Angela is in Image Making here. Although now building in Angela's understandings are these other personalities of zero, Angela still has the addition identity property of zero in her primitive knowing. Thus, Angela concludes that both the questions are the same, and if something is true for one question then it is true for the other. Through the identification of similarities and the reasoning of zero's role, Angela is building an image of how the different properties of zero that make up zero as a mathematical object, operate with each other.

Angela now encounters her next quandary: 0+8=8-0:

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[16] Note that in the story problem, Angela's primitive knowings around addition, subtraction and number are focused on action.

A: Now... zero... ignore the zero. Eight equals eight take away zero. I don't get that one.
I: (Think it through)
A: (Eight equals) eight take away zero (*Hand on her forehead*). Eight plus eight equals zero... (*looks to the side and says quickly*) eight take away eight equals zero.
I: Ok.
A: That's true, you take away 8 that's 0.

Upon encountering this now slight difference of 8 - 0, Angela's entry into the problem is similar to her entry into the two previous ones: "I don't know," and "I don't get that one." Thinking these words signify the same block to thinking as last time, I attempt to intervene with the words "Think it through." As Angela iterates "equals eight," at the same time as my intervention, I realize that these words of "not knowing" do not represent a barrier, a stop to thinking, or a not knowing how to proceed. Instead, I realize these words sometimes are only an artefact of their original intention, and indeed propel Angela to continue engaging with the problem before her.

Angela, presently in Image Having, integrates different ideas about zero together, and utilizes them at the same time. First, Angela is now able to ignore the first zero (from 0 + 8) because of the addition identity property of zero, she used in the last equation. This property is now not in conflict with the idea that zero also means "nothing." Next, Angela uses her strategy of subtraction as the inverse of addition to decide if the equation is true or false. She attempts to add eight and eight to get zero. Angela immediately realizes that she gets the wrong answer and therefore her strategy does not work. Angela now has to switch strategies. In order to switch strategies, zero has to take on yet another personality aside from the ones Angela has already used: zero has to be the result of a mathematical expression. Importantly, Angela does not express any tension integrating another conceptual aspect of zero to solve her problem. Angela then realizes that 8 – 8 = 0 and her problem is solved. For the equation 0 + 8 = 8 - 0 Angela has integrated many ideas about zero, without having to collect the Image Making that led to these ideas in the first place. Angela has integrated: the identity property of zero, zero as being the result of an action, zero as nothing, and zero as the result of a mathematical equation.

4.10 Zooming In… *Telling stories*

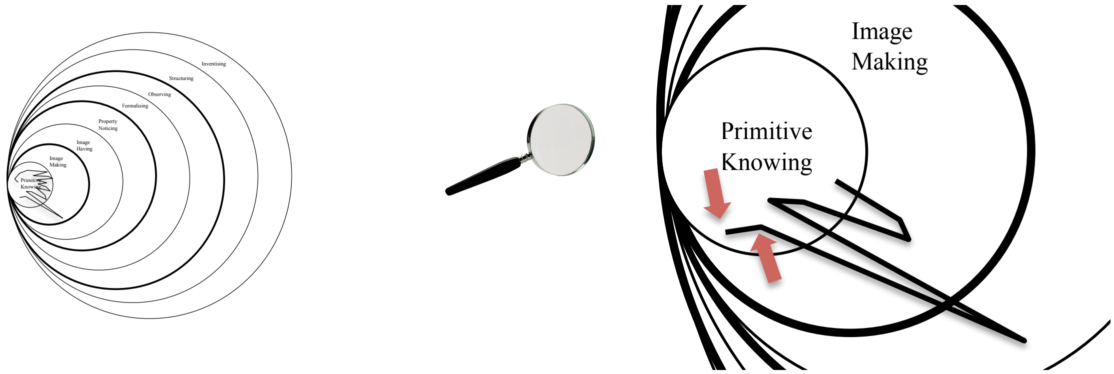

Figure 4.23 Angela Zooming In… Telling Stories

Angela and I tell each other equation stories. I ask Angela to tell me a story where the answer is 8. Angela tells me a story of using flowers for a project, 4 pink and 4 red:

> "ummm… There was once a girl. And… she went to, she was doing a project. And she was making a picture of garden. So she went to her garden and picked flowers… and…she only had pink and red ones and she really liked the red ones. Soooo… where, wait it has to equal 5, 8 (*I confirm it is 8*). So she wanted to do at each, at each corner of the poster 2, so 2 pink, 2 pink, 2 red, 2 red, so she needed 4 of each colour. So 4 pink and 4 red equals 8."

The equation of 4 + 4 = 8 in Angela's story is very explicit, and based on an action. Angela is joining together-an action-2 pinks and 2 pinks and 2 reds and 2 reds to make 4 of each colour. Angela then joins together-another action-the 4 pink and 4 red to make 8. I wonder, especially after all the previous movement into Image Making and Image Having, what type of story Angela will tell for zero. I have reason to wonder because I am asking Angela to look at a new relationship of zero. Telling a story with 0 as an answer not only creates a context for actions with zero, but also creates a situation where zero becomes the result of an action in context. At this point, I wonder if Angela will make a story for zero based on action as well? Will the images that Angela has created so far, be extended to a context for zero? After Angela tells her eight story, I ask Angela to tell a story where the answer is zero:

> “There was this girl and she was going to do something for her project. So she went to the flower and... I mean she went to the garden ...and then... She picked 3 flowers, and she realized she didn't need any, so... she picked zero.”

The story that Angela tells for zero is similar in content to the story she tells for eight. There is a project and there are flowers. However, the mathematical underpinnings of the numbers within each story are very different.

The first story uses the numbers 4 and 4 as mathematical objects to be operated upon and translated into the equation: 4 flowers + 4 flowers = 8 flowers. The second story uses numbers differently. First the girl in the story picks 3 flowers. These 3 flowers, tangible entities, are like the 4 flowers, also tangible entities, in the first story. However, these 3 flowers, tangible entities, now magically disappear when the girl “realized she didn't need any.” The flowers disappear in much the same way that zero disappears when it is ignored.

Angela begins her zero story in much the same way she begins her eight story. However, unlike the number eight, Angela does not yet have an image of zero as a mathematical object, integrating its different aspects. Thus, in order to finish the story, Angela dips back into Primitive Knowing. First, Angela accesses an aspect of zero as nothing: the girl realizes she doesn’t need flowers. At the same time, Angela knows she needs the aspect of zero as a number in order to finish her story. Thus, she dips back into Primitive Knowing again. This time Angela accesses zero as a counting number, “she picked zero.”

Here, I end the narrative about Angela and her growth in understanding. There is an interesting juxtaposition that I would like to now call to the reader’s attention. Angela experiences mathematics difficulties, and, as discussed in chapters 0 and 1, the proscribed method of teaching learners experiencing difficulties is with rote, procedural tasks that promote fluency. This rote, mechanistic type of learning is often used instead of the type of learning that Angela experiences here, during our sessions. I cannot argue that Angela actually receives this type of rote instruction. Instead, I argue that learners experiencing mathematics difficulties, like Angela, are likely to experience rote, mechanistic instruction in mathematics. I juxtapose this type of experience of rote instruction next to the experience Angela has problem-solving in our sessions. I am specifically visiting this juxtaposition here with Angela, and not with the other participants. The other participants also experience mathematics difficulties and also participate in the problem-solving experiences. I focus on Angela because her affectual responses and reactions to exploring mathematical ideas beyond their mechanistic meanings, her excitement at her "AHA" moments, make this juxtaposition between the experience of the different instructional methods most pronounced. Angela appreciates the feelings that the act of problem-solving-not knowing and then knowing-offer her. At the end of all our sessions, Angela gives me a thank you card and tells me the worst and best parts of our sessions together is when I do not give her the answer-when I make her think the mathematics through. Additionally, Angela's mother who usually waits for Angela in her car after our sessions, leaves her car to approach me on the last day of our sessions. She, too, thanks me and confirms that Angela's involvement in the sessions is a positive experience for her. In light of Angela and the positive feelings that result for her from her participation in problem-solving, can it be equitable to only provide a mechanistic, rote learning for learners experiencing mathematics difficulties?

Chapter 5.0 Melissa

5.1 Introduction to Melissa

Melissa's teacher would not tell you that Melissa has a mathematics difficulty. Her achievement in class this year is above the class average. However, Melissa tells a different story. Melissa says that regardless of her achievement, until this year mathematics has always been a struggle. When learning a new concept, Melissa will often require repetition and extra time as "sometimes it takes her a few times to catch on."[17] As a result, Melissa relates, she disliked mathematics in previous years and did not want to "do math." Melissa does not outwardly present as someone who has experienced barriers to mathematics. In class, Melissa does not demonstrate the typical aversion or disdain for mathematics; and in our sessions, even when describing her difficulties, Melissa retains her smile, and her tone and body language seem upbeat. Thus, I am surprised at Melissa's claims of a struggle in mathematics and of not liking math. However, how Melissa presents and what she experiences are two different things. Melissa's barrier to learning mathematics is invisible- it happens outside of the observations and accountability practices of teachers.

The concept of struggle in mathematics education research is interesting. Researchers (e.g. Hiebert & Grouws, 2007) extol the virtues of a struggle while learning mathematics. Hiebert and Grouws (2007) list only two classroom practices that aid in the growth of mathematical understanding, one of which is engaging students in struggle, termed "productive struggle" (p.391). In my own previous research (Ruttenberg-Rozen, 2016), I have found that encountering unsettling experiences in mathematics or struggling through a mathematical problem can be beneficial to mathematics learning, specifically for someone experiencing mathematics difficulties. It seems odd, then, that Melissa's struggle is a barrier to her mathematics learning. However, Melissa's struggle is different than the productive struggle that Hiebert and Grouws (2007) describe. Melissa's struggle cannot be

[17] Mathematical background sheet filled in by parent.

termed productive, because for her there is not a positive affective outcome. For Melissa, the learning, grade, or other intrinsic or extrinsic "positive" outcome she might receive, does not justify the amount of struggle she experiences.

Some might argue that Melissa should not be included in a study focused on children experiencing mathematics difficulties. As discussed in section 1.1.1, often in special education research, whether someone has a mathematics difficulty is answered by the question: How far below a threshold, in one form or another, is their achievement level? This year Melissa's achievement is actually above the various thresholds used to determine a mathematics difficulty. Consequently, Melissa would not be considered by these identification means to have a difficulty in mathematics. At the same time, Melissa's struggle causes her to experience many barriers to full participation in mathematics. Through including Melissa in this study, I wish to make a distinction between the difficulties experienced and, thus, identified by schooling practices of mathematics, including assessment, and difficulties experienced in the larger field of mathematics. It is important to note that the content and processes of school mathematics is only a small fraction of what mathematics includes. The difficulties that Melissa is experiencing is not part of what is assessed and observed through the teaching of school mathematics, rather it is a part of the larger context of processes (Isoda & Katagiri, 2012) and content of mathematics. During our sessions, and expanded upon below, I notice that Melissa often has strong images around supporting conceptions in mathematics, without having an image of the conception itself. Two especially strong examples of this are explored in section 5.5 with the ruler and number line, and section 5.6 with infinity. These understandings are often in conflict with her learned understandings from school. For Melissa, this type of understanding and conflict does not allow her to create meta-relationships across concepts. Instead Melissa compartmentalizes each knowing. This compartmentalization of knowings creates little opportunity for thickening of prior knowings and revisiting understandings. Thus, Melissa is a good example of how a mathematics difficulty may present itself in invisible ways outside of scholastic achievement.

In explaining the difference in her struggle with mathematics between the years prior and now, Melissa tells of her move this past summer from the United States to Canada. As a result, Melissa feels that she is both ahead now in mathematics, and that mathematics is more to her “level.” I am curious what Melissa means by “being ahead” and at “level.” Melissa explains that she has more motivation now, the pace is slower and the curriculum feels like a review that is “like going up to one more tiny thing of it every time we learn something.” It is, thus, possible that Melissa’s change in circumstances this year is at least partly responsible for her positive disposition that I am still observing, despite her struggles, towards mathematics. However, this supposition only accounts for Melissa’s positive disposition. Despite her struggles, Melissa has never received school organized academic help for mathematics, or been on an IEP (individual education plan) in mathematics. Additionally, while Melissa describes a more positive experience with mathematics this year, she does not preclude the potential for difficulty in the future (Sfard & Prusak, 2005).

5.2 Mapping: Melissa's Pathway of Dynamical Growth

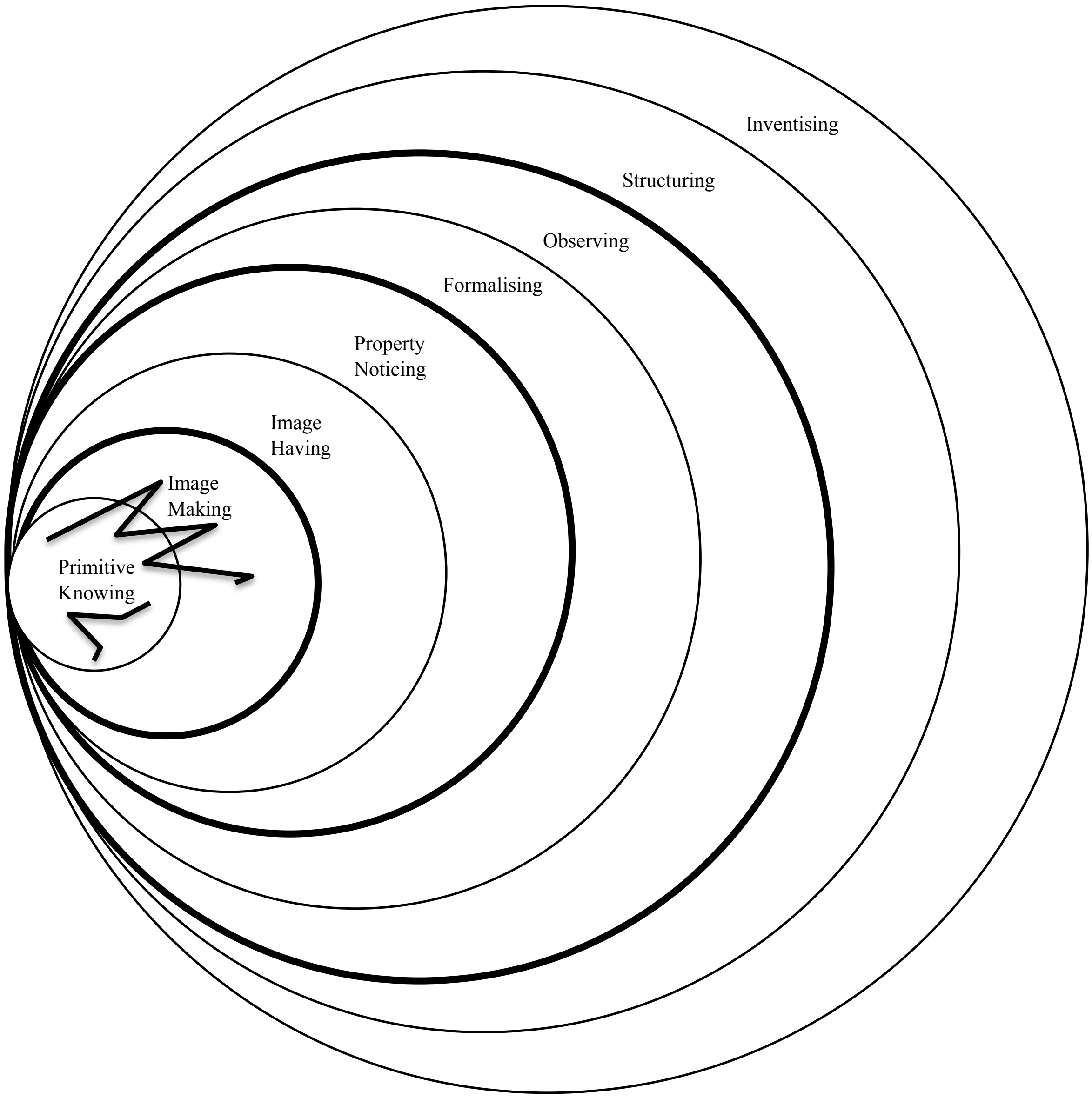

Figure 5.1 Melissa's Mapping

5.3 Melissa's Journey

Melissa's pathway is similar and different from the other learners experiencing mathematics difficulties in this study. Like the other learners, Melissa hovers around Primitive Knowing throughout our sessions. Unlike the other learners, it is much more difficult for me, as the empathic second person (Metz et al., 2015) to track Melissa's change. One reason may be that Melissa has had different prior experiences with some of the tasks than the other learners. For example, Melissa does not indicate any explicit disequilibrium with the "missing numbers" page, whereas the other learners[18] all relate that at least one of the examples are new. As a result, some of the tasks may not be structured enough to produce conflict for Melissa.

Another reason for the difficulty I find in identifying change, is that when Melissa does experience conflict, she never really outwardly remains in a state of confusion or conflict. This is not because Melissa reconciles her conflict. At the first sign of conflict, Melissa often revisits her primitive knowing with what seems to be an outward assuredness. Melissa does this even when her primitive knowing is too underdeveloped, or erroneous, to aid her in solving her problem. At times, I attempt to redirect Melissa to fold back towards her Primitive Knowings. This is not always a successful intervention, as Melissa does not usually question and attempt to thicken her primitive knowings. Thus, for Melissa, thickening is not occurring in the same way as for the others. And because Melissa's journey hovers around Primitive Knowing, thickening then, is an important marker of change. This is not to say that change has not occurred, as will be explored below Melissa's movement and pathway surely conveys change. I am arguing that because of Melissa's disposition towards learning, outward markers for change are more difficult to identify and thickening of prior knowings is not as robust-not that change is absent.

[18] Includes those participants in the study not experiencing mathematics difficulties.

5.4 Zooming In… *Just Ten There Is Nothing Else*

Figure 5.2 Melissa Zooming In… Just Ten

Melissa is decomposing ten. She starts from nine and one, making a pile of nine cubes and one cube, and then progresses to eight and two, also representing the pairs by two piles of cubes. Melissa continues in this way through to five and five and then back to one and nine. After one and nine, Melissa makes a pile of ten cubes and declares it to be ten:

ME: uhhh 4 and ummm... (*looks to the side and wiggles fingers*) 6. 3 and 7. I did that one. 2 and 8...uhhh 1 and 9 and (*looks to the side*) 10 (*laughs and makes hands go out*).

With her laugh and hand motion, Melissa has conveyed to me both that she has noticed that ten is different than the other numbers, and that she is finished decomposing. Although Melissa notices that ten is different, she has yet to indicate zero as the corresponding pile to ten. I wonder if I prompt her, if Melissa will then consider the zero:

I: 10 and...

With my prompt I am inviting Melissa to revisit the idea about ten being alone, in comparison to the other pairs of numbers. At the same time, I am conveying an expectation that ten might have a corresponding number:

ME: and just 10 because you can't really do 10 plus someth...uhhh 10 plus zero (*laughs*).

Melissa's initial response, "just ten because you can't really do 10 plus someth(ing)," reveals primitive knowings that Melissa has around decomposing numbers and addition. When Melissa is decomposing ten and choosing a number, she is doing a "plus" movement, or action, to arrive at the corresponding number. When Melissa iterates the pair of 7 and 3, she is making two piles, one of seven cubes and then the creation (plus) of another pile of three cubes. When Melissa is unsure what number corresponds to 4, she wiggles her fingers to help her think about 6. In Melissa's Primitive Knowing, addition is action between something and something.

The second part of Melissa's statement, "10 plus someth...uhhh 10 plus zero," is very interesting. I wonder about that partial "something" followed by the "plus" and then the "zero." Before elaborating on Melissa's statement, I would like to answer a problematic aspect that the reader might have considered to have arisen at this moment. Through explaining why she has stopped decomposing at ten, Melissa comes to the realization that zero is a number that can be added to ten. Where did this concept come from? Could it have come from me now? Or maybe the zero came from our previous discussions about the project and goals of the project? It is possible that Melissa purposefully accesses zero as a mathematical object in her Primitive Knowing because of a previous conversation we have had around this study. Melissa and I have previously discussed that the purpose of the study I am conducting is to look at the way learners think about zero. It is possible that this discussion put zero at the forefront of Melissa's consciousness, especially since I, the one who is studying conceptions of zero, am the one to prompt Melissa to think further. However, it does not matter whether or not our conversations have led Melissa to consider zero by way of my prompt. It would be impossible to distinguish this anyways, as the very act of studying something, changes that thing (Davis, 1996). Thus, whether this movement results from one of our conversations or from Melissa on her own is irrelevant. What is relevant is Melissa's movement at this moment- from "you can't really do 10 plus someth(ing)" to "10 plus zero." The starting point of the movement is the primitive knowing of the idea that you cannot do an action, or add, with nothing. Therefore, Melissa cannot consider a corresponding number to ten, because she cannot do an action between something and nothing. In this case, the movement is predicated on the idea that only a number, conceptualized as a mathematical object, when zero is something other than nothing, could be acted upon. With this discovery, Melissa moves into Image Making. In Image Making, Melissa is constructing zero into a mathematical object that can be acted upon-Melissa is now creating an image of the action of ten plus zero. At the same time, Melissa is thickening her understanding of how numbers are decomposed. An underlying knowing of decomposing numbers is that the decomposed number, in this case 10, includes the numbers it is decomposed into. For example, in this case the number 10 has within it 9, 8, 7, 6, 5, 4, 3, 2, 1 and now 0 as well.

5.5 Zooming In… *Zero At The Beginning*

Figure 5.3 Melissa Zooming In… Zero At The Beginning

In introducing the number line task, I ask Melissa if she has had experience with number lines. Melissa describes the number line as having "a benchmark of numbers." She notes that the number line I give her begins at 1 and ends at 10. Then Melissa describes what she means by a benchmark of numbers:

> "So as you go along the number line it will be like zero point one (0.1), zero point two (0.2), zero point three (0.3), and then you'll get to one. One point one (1.1), one point two (1.2)."

I note that there are a few interesting points about the way that Melissa is describing the number line. Melissa tells me that the number line I give her begins at one, yet in describing her benchmark numbers, Melissa begins with decimal numbers to the tenth before one. At the same time, Melissa does not begin her benchmark numbers at zero either. Instead Melissa begins just after zero, at 0.1.

Similar to the decomposing task, Melissa has various images around zero as a number in her Primitive Knowing, without having actual zero as a number at the forefront of her consciousness. For example, the number line utilizes the motion metaphor of the grounding metaphors of zero (Lakoff et al., 2000). This means that zero is viewed as the symmetric center on the number line. A supporting primitive knowing of zero on the number line can be the concept that decimal numbers without units-meaning there are no units in the ones place-are still greater than zero. Even without mentioning zero, Melissa has alluded to this primitive knowing. And, later in her journey, when I ask Melissa to place a 0 on the number line, as discussed further on in this section, she places it to the left of 0.1. Melissa does this despite the common misconception that decimal numbers without units are smaller than zero (Stacey, et al., 2001b). Common with this misconception is a learner placing decimal numbers without units: for example, 0.5, 0.03 and 0.127, all to the left of the zero on the number line, and even in successive order. Melissa does not place her decimal numbers according to this common misconception, yet at the same time zero is not at the forefront of her consciousness.

Again, like in the decomposing task, the zero is noticeably absent. Again, as before with the decomposing task, I wonder where zero is. I also wonder about other supporting knowings of zero that Melissa may have images of. For example: (1) the spatial relationship between the numbers on the number line, each one being equidistant to the ones before, and (2) the changing spatial relationship between the numbers relative to the overall space. Melissa does not mention these two knowings at all, but considering that Melissa places the decimals in the appropriate place while still omitting the zero, I wonder about these three supporting images as well. In order to probe these knowings, I continue with the number line task.

I ask Melissa to place the number five on the number line. Melissa moves her thumb and pointer along the line. It seems that she is apportioning spaces for the numbers around five. Melissa moves her fingers close to ten and then a few spaces backwards, before she decides where to place the five: near the midpoint between ten and one, closer to one. I am curious as to how Melissa is thinking about deciding where to put the five. Is Melissa using the supporting knowing that the numbers on a number line are equidistant from each other? I ask Melissa to explain to me what she is thinking. She explains that the number line page is on top of a sheet of graph paper. Melissa is using the graph paper to help her determine where "the middle is" to place the five. Earlier, before beginning this task, Melissa and I noticed that the table Melissa would be writing on was not smooth and full of bumps. Melissa had placed a pile of papers under the papers she would be writing on. At the top of the pile is a sheet of graph paper. Certainly, the intention of the graph paper is not to use it as a tool. And, importantly without a recognition of the relationship between space and place on the number line, the graph paper would not necessarily be an accurate tool. However, Melissa, recognizing that a knowing of the number line is that the numbers are each equidistant, utilizes the graph paper to help her.

I now ask Melissa to place the two onto the number line. This time, Melissa does not pause to measure before placing the two on the number line. Melissa does pause to think, however, after placing the two. She now judges the five to be too close and decides to shift the five to the left. The five is now visibly to the left of the midpoint. Melissa pauses again and takes a breath:

ME: These are, like very weird places (*laughs*).
I: Are the lines throwing you off in the back also?
ME: No, it's not this. It's ummm… where the one is. If the one was heerrrrre (*points to space right next to the left arrow on the number line*)…
I: It would be easier?
ME: Right.

Melissa is experiencing difficulty placing the numbers on the number line. One reason for this difficulty is she is not viewing the number line, with its dual arrows, as continuous. The number line as continuous is a supporting image of the motion metaphor of zero. Rather than the location where I have placed the one, somewhere in the middle of the number line, Melissa would prefer to have the one in the first available space beside the left arrow. Noticeably, again, Melissa does not mention the zero. Instead, Melissa wants to move the one into that first place.

Melissa's primitive knowing of the number line as discreet is causing a conflict. Yet, although Melissa is experiencing a conflict, she is still working within the parameters of the task I have presented-Melissa has not physically moved the one to where she would prefer. While Melissa indicates she is experiencing a conflict, at the same time she is not independently engaging with the conflict in order to reconcile her understandings. Previously, while in Melissa's classroom, I observed geometry lessons explicitly covering continuous and discrete lines. Thus, I make a note that Melissa does potentially have a prior experience with continuous lines and arrow symbols that could support an engagement with her conflict.

Thinking about how to intervene, I decide to ease the conflict for now by making Melissa consider the line as discreet. Considering the line as discreet still leaves room for some conflicts, but I will eventually re-enter the conflict of the line being continuous when I ask Melissa to place the zero on the number line. In this way, utilizing the notion of continuous, I can observe Melissa's ideas around zero, and if and how zero can help reconcile her conflict. I ask Melissa if it would help her to cover the empty space between the one and the left arrow. Melissa tentatively agrees. With the empty space covered with my hand, Melissa uses her pencil to apportion out the numbers:

ME: So… 2…3…4…4 (*taps on page where the 5 was before she moved it*)…5, 6, 7, 8, 9. There's not enough room.
I: huh.
ME: Because it has to go point (*apportioning space with her finger on the number line*)1, point 2, point 3, point 4, point 5, point 6, (point 7, point 8…)
I: (Like a ruler?)
ME: point 9, 2.

I: So you are thinking about spaces for the decimal?
ME: (*Nods head.)*
I: And all these have to have spaces for the decimal?
ME: (*looks down at number line*).
I: Ok. So is there another place you would like to move it then, if that's too close?
ME: No. I think that's good.

Initially, when Melissa speaks about the decimals on the number line, I have some wonderings about her knowings around some of the supporting images of zero. As I delve deeper with Melissa, I begin to tease out some of her understandings about these knowings and other supporting knowings as well. Melissa has incorporated the knowing of the spatial relationship between the numbers on the number line, each one being equidistant to the ones before, in her prior knowings. Demonstrating this knowing, Melissa painstakingly, using her fingers first, then pencil to signify space, apportions out the spaces for each individual number. At the same time, Melissa does not yet have an image of the changing spatial relationship between the numbers on the number line relative to the overall space, in her primitive knowing. Thus, Melissa concludes, there is not enough room between the 1 and 10 for all the numbers to be equidistant. For Melissa, all the numbers includes decimal numbers in the tenths place in between each whole number. Each decimal number also requires a physical space on the number line, similar to a ruler. Melissa is using a measuring stick metaphor to explain her reasoning. At this point, much like I wonder about the continuous nature of the number line, I could also wonder about Melissa's knowings vis-à-vis the continuous nature of decimals. There are really an infinite number of possible decimal combinations in the space between 0 and 1. However, I decide not to take that direction because a literal translation of the measuring stick metaphor, to a centimeter ruler where the tenths are millimeters, can be a likely explanation of Melissa's usage of the tenths.

Melissa's metaphor of a ruler for the number line comes into direct conflict with the supporting knowing of the number line and its changing spatial relationship between its numbers relative to the overall space. The number line is continuous, infinite, and the actual space between the numbers is often irrelevant to these conceptual underpinnings. In a number line, the placement of numbers is arbitrary, as long as the numbers are sequential and equidistant. An important defining concept of the ruler, on the other hand, is that the space between numbers is static no matter the overall measure of the ruler. The space between the numbers represents a convention of measurement. A ruler is a tool. It is discreet and has a finite space for the numbers, and although any measurement tool has the potential to be theoretically infinite, the ruler is practical, and practically it cannot be infinite. The measuring stick metaphor that Melissa is using is similar to the measuring stick metaphor of zero. At the same time, the metaphor is still different because zero is not present here yet.

Melissa's conflicting conception between a number line and a ruler furthers her cognitive conflict. Thus, wanting to both probe further and to intervene into the conflict, I ask Melissa if she would like to move any of the numbers on the number line. I am thinking that the movement in space I am offering to Melissa, could help to bring out some ideas about relative space. Melissa declines my offer, indicating a complacency with where the numbers are placed. I am surprised that Melissa opts not to continue because just moments ago Melissa expresses a real conflict with the placing of the numbers, and this conflict has yet to be resolved. I wonder what happened to the conflict Melissa just experienced. Why is Melissa so complacent to move on when there should be a nagging doubt about the numbers? Having lost an opportunity to explore Melissa's reasoning about the measurement metaphor, I continue hoping that Melissa's conflict will return when I ask her to place new numbers on the number line. We continue with placing numbers on the number line. I leave the option open for Melissa to move numbers around if she chooses.

I ask Melissa to place the 6 on the number line. Melissa, without pause, places the 6 to the right of the 5, approximately equidistant to the space between the 2 and 3. As Melissa places more numbers on the number line, I am still waiting for her to revisit her conflict. I am wondering where that conflict went? Why is Melissa so complacent now? I am wondering, as I ask Melissa to place the rest of the numbers, if there is a way for me to intervene in order to bring the conflict back? I ask Melissa to place the 0 on the number line. I anticipate that through encountering zero, Melissa will have to revisit her conflict. Placing the zero should come into direct conflict with Melissa's understanding of the number line as discreet. After all, the number line Melissa is working with appears to begin at 1 and end at 10.

ME: (*looks closely at paper- about to write on the end, near the left arrow, and moves hand back and forth twice between next to the arrow and next to one*)... well, like, does it want me to start from (*puts palm down on paper at end near left arrow*) like he, where it would be here. Or like if it was an actual one starting here. Probably. (*writes zero down lightly near the left arrow)*. here.

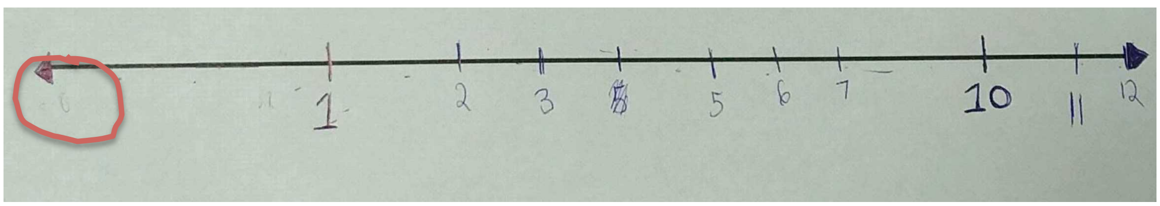

Figure 5.4 Melissa Tentatively Writes a Zero Beside the Left Arrow

I: So what were you thinking when you put the zero there? Because you had a question, the... I was about to ask you what you meant when you put it (there).

ME: (Well) I was thinking. Like... because if it was here (*pointing to beside the arrow*) it would originally be at the start (*emphasizing start*) of it. So I'm thinking if this is... kind of this like, this is the one. So technically here (*draws in a zero equidistant from the one, draws over it two times while talking*) would kind of be the start where the zero would be.

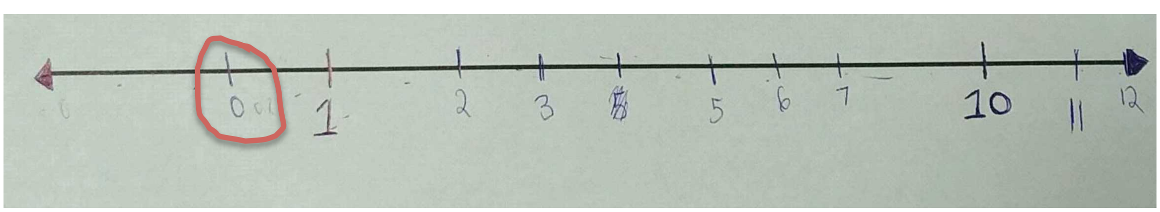

Figure 5.5 Melissa's Second Zero

I: OK. So, Oh. So you mean the start of the (number line)?
ME: (number line)
I: Uh Huh. What do you think would go here then? (*pointing to the space between where the 0 next to the left arrow is and the one*).
ME: (*Shakes head*) Nothing. It would be because zero point zero one, lalalala. But really if it was like a real... with like the one here (*motions to a space closer to the left arrow*), then it would be the zero here. (*points to next to the left arrow*). This is the starting of the line. Like here (*points to next to the left arrow*).

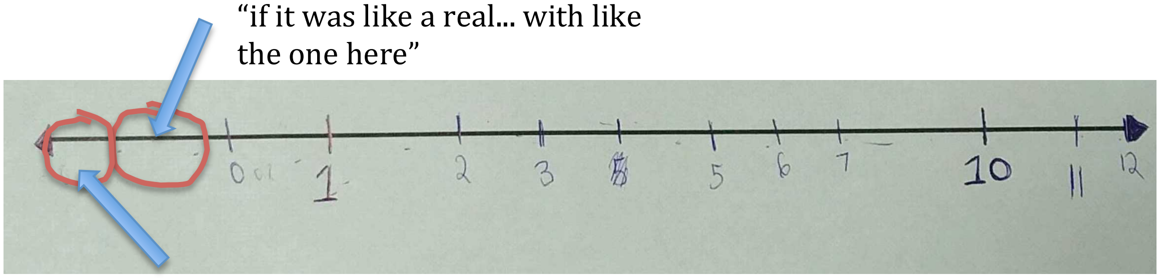

Figure 5.6 If It Were a Real Number Line

Placing the zero on the number line does indeed come into direct conflict with Melissa's understanding of the number line as discreet, and does cause Melissa to reengage with the spatial issues she experienced. This second time is different, however, because the problem is not an internal cognitive conflict anymore for Melissa. Instead, Melissa is shifting the cause of the problem to the activity itself, "well, like, does it want me to start from…" and, "if it was like a real..." Since Melissa determines that the number line has issues, it cannot be "real" like the number lines from her prior experience. Subsequently, Melissa's interactions with the task are now based on what she perceives is "wanted" from her by the activity. Along with the shift of onus comes a shift in focus on the mathematics. In the first instance of this conflict, Melissa's language centres around the mathematical conceptual underpinnings of the number line. In the first instance, Melissa is talking about space and the numbers being equidistant. Now, Melissa is focusing on the peripheral, non-mathematical supports of the task, thinking about what the task wants her to do.

I want to help Melissa shift away from the constructs of the page and back to her reasoning surrounding the mathematics and spatial issues from before. For that reason, I decide to not engage Melissa's shift of onus of the problem, but instead to call her attention back to the zero. In this way, I am hoping that Melissa will engage with the underlying mathematical ideas that are causing her conflict. Contrary to Melissa's primitive knowing of the space between numbers on a number line being equidistant, there is a lot of space between where Melissa would like to place the zero, to the far left, and where the number one is. I am wondering that if I point out this discrepancy, a concept of which she was approximately a minute ago very concerned about, it will reignite the internal conflict for Melissa. When I do point out the conflict, "What do you think would go here then?", I do not create disequilibrium for Melissa. Instead, Melissa says "nothing" and then reiterates the decimal numbers, that in the ruler metaphor, would be situated between zero and one. Melissa does not visually or linguistically convey a problem with her resolution to my question. Melissa's utilization of the decimal numbers as support to the issue of there being a large amount of space between zero and one is problematic, because Melissa had just previously apportioned out the spaces to be equidistant. The space between the zero near the left arrow and one is significantly larger than the space for the apportioned other numbers. In fact, Melissa can fit in three equidistant numbers into the space between zero and one. Instead, Melissa, considering what the task wants from her instead of the underlying mathematics, is ready to proceed placing numbers onto the number line. We continue on and Melissa does not pause or show any conflict before placing the seven beside the six and approximately equidistant to the space between the two and three.

During this short exchange Melissa reveals primitive knowings around mathematical zero. At the beginning of the task, as Melissa discusses the decimal numbers before one, I wonder where the zero is and if Melissa has an image of zero as a midpoint marker of the symmetry of the number line. I find out that this concept is complicated in Melissa's primitive knowing and is very much tied to her ruler metaphor of the number line.

To support her reasoning when faced with conflict Melissa utilizes a myriad of primitive knowings during this task including:

Numbers: Decimal numbers represent parts of the whole in between each whole number; Decimal numbers represent something larger than the number before; and Zero is a marker of the beginning of the whole numbers.

Number Line: There is a spatial relationship between the numbers on the number line, each one being equidistant to the ones before; The number line contains whole numbers and rational numbers; and the number line is similar to a ruler.

Yet, at the same time Melissa does not make the interconnections between these knowings that would allow for thickening and change.

5.6 Zooming In… *What is zero?*

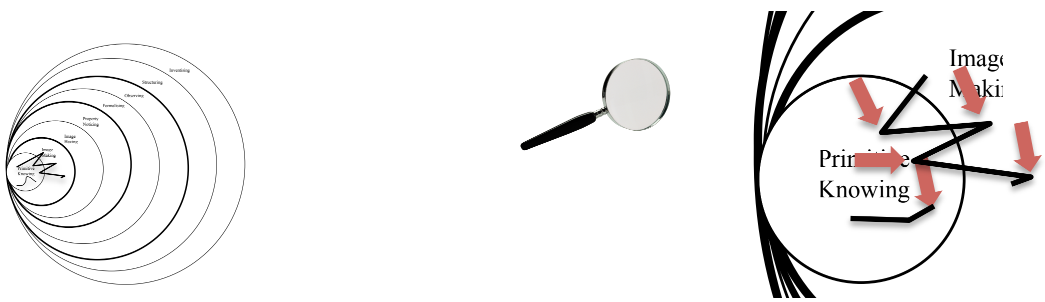

Figure 5.7 Melissa Zooming In… What is Zero?

I ask Melissa what she thinks zero is. Melissa's immediate reply is "a number." The intention of the question is to ask Melissa to think deeply about zero, and I do not have to wait long for this to occur. Without prompting, Melissa immediately revises her answer and dips into, then out of, Image Making in the process.

> "Infinity. The zeroes... (*looking up*). Infinity would beeee (*hand goes up, opens, like about to grasp*)... I feel like because... (*moving hand like she is grabbing something*). Zero's nothing (*puts hand down*)."

In this episode Melissa begins sentences orally, then finishes them with gestures. Thus, watching and analyzing Melissa's gestures (Radford, 2009a) are important to help me understand what she is trying to convey about her understanding of zero. In trying to relate the abstractness of zero, Melissa reaches for infinity. Although infinity and zero are similar in abstractness and in their late acceptance in the history of numbers (Pogliani et al., 1998), infinity and zero stand at two opposite and extreme ends of the spectrum. Melissa experiences a conflict as she tries to blend the two disparate ideas into one image. Blending requires making a connection between one input to another input. I wonder what the connection is that Melissa is attempting to make.

For infinity, Melissa pauses and she lifts her hand into an open grasp, as if she is waiting to catch something. Infinity is number without end, a number that can't be caught. Then, faced with the narrowness of "zero as nothing," Melissa grasps the zero and drops her hands. There is at first a fluidity and then a finality to Melissa's movements as she iterates that zero is nothing. While I am still wondering about Melissa's blend and subsequent gestures, Melissa leaves infinity and drops back to Primitive Knowings about decimal numbers and the number line. I do not interrupt Melissa's movement into decimal numbers to ask her about infinity. Through the number line task, I have learned that Melissa can set aside her conflict without resolving it. I wait for Melissa to revisit the idea of infinity on her own because if I intervene I run the risk of ending the conflict.

At present Melissa is now trying to create a blend between zero and decimal numbers:

ME: (*looking up*)...It's like...if you...it's the number before one. (*laughs*). It's like...it's... because it is a number, so technically it's the first number. Unless you would do like...like (*moves mouth*) point nine?
I: (*nodding head*)
ME: Or point nine
I: And point nine would be the first number?
ME: Well…

I: If it wasn't for zero?
ME: No. Probably point zero. (*looking around confused*). (*laughs*)… Wait one...
I: What do you think point zero means?
MA: ...zero (says it likes she is sure). Yeah. But like...It's just nothing. (Laughs).

At the same time as Melissa iterates that zero is a number, she is starting to create an image of what the concept of zero as a number means. Melissa knows sequentially, zero comes before one. We have already encountered zero before one during the number line task, and Melissa did not experience any conflict with this notion. Yet now Melissa's language is tentative. She uses the word "technically," as a modifier for zero being "the first number." Zero as the first number is not the only concept Melissa is tentative about, Melissa is tentative about point nine as well. She mouths "point nine" before she articulates the words "point nine." Zero is the first number, "technically," and it is a potential contradiction to her image that decimal tenths also come before one. Point nine comes before one and so does zero. Melissa has not created a blend between these two ideas yet. Zero before one and the possibility of decimal tenths before one are separate concepts for Melissa. Zero and decimal tenths now need to be reconciled.

At this moment I have a lot of questions about Melissa's thinking. I am still wondering about infinity, and now about Melissa's tentativeness about zero being the first number, as well as zero's conflict with decimal tenths being numbers before one. I decide that since Melissa has just experienced the conflict between decimal numbers and zero, I would take that avenue of questioning.

From our previous discussions around the number line, I know that Melissa has the other decimal tenths in her frame of reference in addition to point nine. I wonder what would happen with Melissa's thinking if I draw out the other decimal numbers that come before point nine. Will Melissa get to zero or will she continue from point one on to decimal hundredths? Thus, I ask Melissa if, "point nine would be the first number?" adding, "if it wasn't for zero?" I immediately regret this addition. The phrase, "If it wasn't for" could create a hierarchy where only if zero was not there could decimal tenths be the first number. I worry because previously, during the number line task, Melissa has been ready to push her thinking aside in favor of what she thinks the task wants from her. There is, thus, a possibility that we could lose the chance, again, of exploring Melissa's conflict. However, I am relieved because Melissa rejects my addition, and instead follows her original thinking. Melissa continues the concept of decimal numbers before one towards its logical conclusion-point nine continued to the end is point zero.

Point zero confuses Melissa and she laughs. It is this moment, that Melissa enters Image Making. Her paradox of the decimals and zero can begin to be blended into an image of "nothing." I want to confirm Melissa's movement into Image Making and I ask Melissa to explain the point zero. To Melissa, point zero and zero are the same thing because they are both "nothing." With her laugh and confusion, Melissa is in Image Making.

I now want to revisit the idea that initiates Melissa's mention of infinity, when she attempted to make a connection between zero and infinity. What connection is Melissa making between zero and infinity? And how is zero viewed through the lens of this connection? I broach the topic with Melissa:

I: …So You said the word infinity, I'm curious what you meant by infinity.
MA: Because you can't really make anything out of it. Really... anything you do won't work. Like especially with math. Like if you do like times zero, it will always be zero. If you doooo...um plus zero, it will always be like the number that it was. Nothing will change if you use a zero.
I: Uh huh. Ever? Nothing changes?
MA: Well I don't know about in division. I don't know because if you do like 15 divided by zero, then it is zero. (*Looks to the side*) Right? Yeah. (*looking to both sides*)
I: How do you think about infinity?
MA: ... 'cuz I was thinking that like it's nothing. Like we don't exactly know what infinity is. What is zero? I don't know (*laughing*).
I: (*laughing*). So zero's like infinity. Something kind of out there?
MA: (*nodding head*)

Melissa is dipping back to some of the images in her Primitive Knowing that she used for the decomposing numbers task. In this primitive knowing, numbers are meant to be acted upon, and more than that the action of acting upon creates identifiable change. In Melissa's knowings, zero does not create any change because of the identity properties of addition (a+0=a) and multiplication (ax0=0). And, because "we don't exactly know what infinity is," infinity cannot be acted upon or create a change either. Once this connection is made between zero and infinity, Melissa reasons that since she does not know infinity she cannot know zero.

With these statements, I wonder where Melissa is on her path of growth of understanding. Melissa laughs when she states her growing realizations aloud, so Melissa's awareness of the reasoning behind her connection between zero and infinity seems new. This connection could be a growing image of zero. At the same time Melissa's movement could be within Primitive Knowing as she thickens two prior knowings.

Earlier, when Melissa first makes her claim about zero and infinity being similar, she cannot support her claim with neither verbal nor gestured images. Faced with this difficulty, Melissa reverts back to zero as nothing. Now, here in our subsequent exchange, Melissa supports her reasoning. This new support would indicate that Melissa has begun to build an image around zero, and that she is not in the process of folding back to her Primitive Knowings. The difficulty with this is that Melissa is stuck in a place of abstractness, of debilitating conflict-what is infinity? what is zero?-which prevents her from initiating or constructing an image (being in Image Making). The only thing Melissa can do from this state of being is to go back to zero as nothing. This zero as nothing is not the same as the previous zeroes as nothing in primitive knowing. This zero does not signify the absence of something or the null result of an action. This zero as nothing signifies something too abstract to think about, something Melissa does not know. Thus I wonder, if Melissa is not in Image Making nor in Primitive Knowing, where is she?

Through explaining the connection between zero and infinity, Melissa is thickening these concepts. This type of thickening indicates that Melissa is situated in Primitive Knowing. However, when did Melissa fold back? When did Melissa go back and collect her prior knowings? Melissa has taken a pause in her reasoning and jumped backward to Primitive Knowing. This type of movement, to Primitive Knowing without folding back could pose a difficulty in mathematical reasoning, like the one that Melissa experiences now with the obtrusiveness of zero and infinity. Melissa's path around this understanding disconnects. She has nothing to gather and no tools to build upon because the idea is just too abstract. I wonder about the belabored process we took to uncover this jump and its possible connection to the difficulties Melissa experiences. Might this type of jump into Primitive Knowing play a role in that difficulty?

5.7 Zooming In… *Zero minus one*

Figure 5.8 Melissa Zooming In… Zero minus One

Melissa is now working on the zero question sheet. She accurately fills in the answer of 1 for the first three questions: 1+0, 0+1 and 1-0. Melissa laughs and remarks, "they're all the same thing." Then Melissa arrives at the fourth question: 0-1=___, and stops:

MA: one minus zero... what? I'm really confused. Is this like... you can't do (*stresses the word do*) zero minus one.
I: Ok, so what would you write there?
ME: (*shrugging and laughing*) I don't know. Zero. (*funny look on her face*). (*Writes zero*). But that's not zero.
I: What's not zero?
ME: This.

Zero minus one is a new concept that Melissa has yet to encounter in a formalized way. It is possible, that Melissa, like other learners (Bishop, Lamb, Pierson, Whitacre, Schappelle & Lewis, 2014), has had prior experience with negative numbers, but she is not folding back to access that experience for this question. Melissa is noticeably confused and exclaims, "what." She is confused because zero minus one is not something "you can *do*,"- she cannot perform an action of subtraction with zero as the starting point. Again, like in previous tasks, Melissa has expressed a problem with the "actions" of calculations on zero. Recognizing that Melissa is experiencing a conflict and knowing that she has previously become stuck in Primitive Knowing, I wonder if I can intervene to help Melissa move forward. I ask Melissa what she thinks should be written as the answer to 0-1. Melissa says zero should be written as the answer, "but that's not zero." Melissa answers that zero should be written on the line but does not believe the answer to be zero. Yet Melissa writes a "0" on the line and moves on.

I immediately see that Melissa is interpreting my question, similar to the way she interacted with the number line, thinking about what the task, or in this case I, wants her to answer. As Melissa writes a zero as the answer, she acknowledges a problem. However, that problem is now superseded by the authority of having an answer. The act of writing an answer seems to be of great importance to Melissa because with this act, Melissa's outward confusion completely disappears. Melissa still acknowledges the conflict, but all the outward signs that would suggest a need to reconcile this specific conflict dissipate. Even later, when I revisit 0 - 1 with Melissa, she still demonstrates none of her initial confusion, even while discussing her confusion.

I want to take a moment to describe the moving on that Melissa has done even after expressing her confusion. This moving on in the midst of a conflict has now become a pattern and is becoming important in understanding Melissa's understanding. Immediately after writing the zero symbol, Melissa slightly lifts her pen off the paper and places it opposite to the next question. The slight move happens fast. It seems like there is no thought, no reason on Melissa's part to hesitate from moving on. From the outside it is impossible to tell that Melissa has just experienced any conflict at all. Melissa is simply beginning to process the next question. Indeed, there are slight changes in the way Melissa interacts with the questions as she continues. Melissa now reads the questions aloud, and she places emphasis on the zeroes, while making gestures with her hands. Melissa's inflection and gestures while reading the questions tells me that she is still experiencing confusion. However, the inflections and gestures are not there for me to respond to. Melissa gives me no time to respond to them, she is quickly moving through the questions despite her confusion.

After Melissa finishes the rest of the page without reference to her conflict, I want to revisit that conflict in order to understand her interaction better:

I: Ok, so I'm curious. You weren't sure what to put here (*points to the 0-1*) How come you put zero? What made you decide to put (zero)?

ME: (Oh) because I was thinking if you have nothing and you subtract it by 1 (*writes the North American algorithm of zero on top minus one underneath:)*

$$\begin{array}{r} 0 \\ \underline{-1} \end{array}$$

ME: (*continues*) you are not going to gain anything. It might as well be zero (*Writes zero as the answer to her algorithm*)(*laughs*).

Melissa's reasoning for why she put 0 as her answer to 0-1 draws on the same primitive knowing she used for reasoning how infinity and zero are similar:

> "Because you can't really make anything out of it. Really... anything you do won't work. Like especially with math. Like if you do like times zero, it will always be zero. If you doooo...um plus zero, it will always be like the number that it was. Nothing will change if you use a zero.

For Melissa, zero is the appropriate answer in two situations. The first situation is when the answer needs to display a lack of positive change-or "gain." In other words, one role for zero is that it represents an action that is not necessary or did not need to be taken in the first place. For the second situation, zero has a role as the default answer-"it might as well be zero." For Melissa, zero is the answer when she cannot work something out, or another answer is too abstract to reason.

Thus I have an answer to my earlier wonderings about the disappearance of Melissa's confusion and her ignoring of her own objection at the writing of zero as an answer to 0-1. Zero is not "the answer," zero is an answer to be reverted to. When Melissa moves on after writing "0," she does not suppress her conflict, she appeases her conflict with a default answer.

I realize that because of Melissa's appeasement, focusing on "the answer" as an intervention is not a good strategy to help Melissa move into Image Making. I decide to probe the idea of negative numbers to see if it will be helpful for Melissa. I do this even though I am still unsure whether Melissa has interacted with negative numbers before. At the same time, I also wonder whether the difficulty that Melissa is having in performing an action (subtraction) on zero will be magnified with negative numbers because they are even less than zero. I ask Melissa to gather her primitive knowings about any numbers less than zero.

I: Ok. So if you don't...OK...So what's less than zero?
ME: (*quietly*) Nothing... Well, then you get into like negative numbers.
I: So...
ME: But that's not what (*looks at me with a confused look on her face*)...zero minus... (*Says half excitedly*) Well, because... it's like... zero minus one if you ha... (*writing 0-1 in the typical algorithm again*) had (*stresses the word had*) like... one hundred minus one and then you'd be able to carry and stuff. And you'd be like I can't do this (*crossing out the zeros changing to nines, and the one to zero*). Then you... you can't do this. Then you can. So you go 10, nine, ten then its ninety-nine.

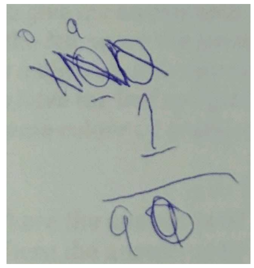

Figure 5.9 Melissa Carrying Zero

I: Uh huh

ME: But it would just zero minus one, you can't actually do it.

Note there is nothing to make into "10"

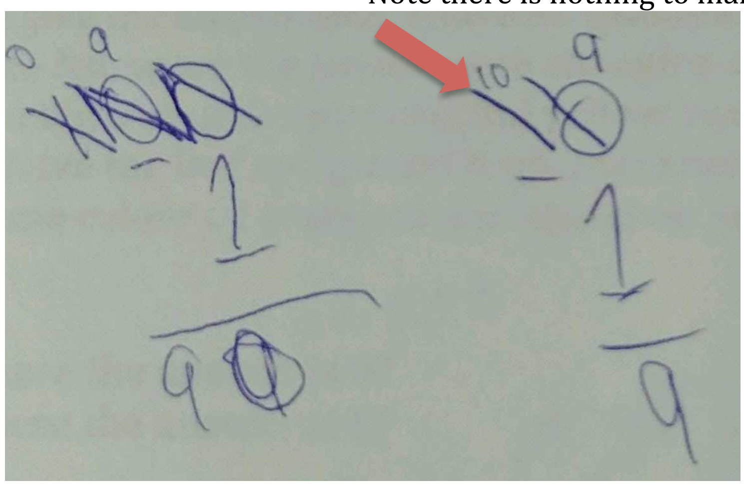

Figure 5.10 You Can't Actually Carry Nothing

There is a conception of negative numbers in Melissa's primitive knowing. I am still unsure what that conception is. In whatever way Melissa conceives of negative numbers, she still discounts them in favor of a practice she has probably done hundreds of times in her school-life- "carrying." Melissa discounts negative numbers as a possibility, not because she extends the problems with zero to negative numbers, as I had wondered about earlier, but because the construction of the equation conflicts with her understanding of the standard algorithm. Having zero by itself on top of a subtraction algorithm means her next step is looking to the numeral beside it to borrow from. And, unlike her example of the

100-1, there is no other digit to borrow from. No action can be taken. Melissa is essentially blocked from creating an image because of her learned rules and practices of schooling.

5.8 Zooming In… *Zero stories*

Figure 5.11 Zooming In… Melissa's Zero Stories

After giving Melissa an example of an equation story, I ask her to tell me a story about 2 + 3. Melissa does not pause to plan the story, instead she eagerly begins with a narrative of a trip to a store to buy candies. Melissa relates of buying two of one type of candy, and three of another type, creating the action of combining (Carpenter & Moser, 1984) of 2+3. I then ask Melissa to tell me a story where the answer is "8." With this new question type, I wonder if and how Melissa's reasoning will change with the change in the demands of the task. In asking Melissa to create a story around 2 + 3, I am determining the "action" Melissa performs to get her answer. I have also given Melissa the parts (2 and 3) of which she can choose to create a whole (5). Now, in asking Melissa to tell me a story with eight as an answer, I leave both the action and the parts open for Melissa to choose.

Melissa relates a story about saving money for college:

> "Oh, OK ummm...Ok. IIIII waaas... saving up for my college. And I needed 8 more dollars. I had one four and one four. Would that make eight? Now would I have enough money? Something like that."

Melissa's number story has all the individual elements of the task's query: it has the parts (4 and 4), it has an action (more = addition), and it has the answer (8). However, with this slight change in the demands of the task, Melissa experiences difficulty coordinating all the elements together. Because the change in the demands of the task create a difficulty for Melissa, I wonder if I should continue the progression from eight and move on to zero at all. What would adding in the complexity of zero now afford our understanding? Would zero even actually add in another complexity? And, if zero does add complexity, then how would Melissa's construction change? Can zero create an opportunity for change? Thus, I decide to ask Melissa to create another story with zero as the answer:

> "I was looking in my fridge. And there was nothing in the fridge. How many things were in my fridge? Zero (*Motions the hole with her thumb and finger when she says zero*)."

From nothing in the fridge, Melissa constructs a balance/equality of nothing = zero. Thus, contemplating zero as an answer does change the way that Melissa interacts with the elements of the number sentence. It also changes the elements themselves. With eight as an answer, Melissa discusses action: "I needed," "saving up," and constructs the equation of 4 + 4 = 8. When asked for a story with zero as an answer, much like her example with eight, Melissa begins the story with action: "I was looking." However, that action leads to nothing: "there was nothing in the fridge." Because the action of looking in the fridge results in nothing, the action becomes unconnected to the equation. Looking in the fridge is a preamble to introduce and explain the equation for zero, 0 = 0. I am not surprised about the balance that Melissa constructs, as she has been cogitating around (in)action, nothing and zero since the beginning of our session. I am curious about this balance, but am unsure about what questions to ask Melissa that would further explain her thinking. Instead I ask, Melissa for another story with zero as an answer:

> "Zero? Ok... ummm. I was looking for paintbrushes at Michaels. I went in and I asked the person where I could find them. She said, "Look over there." And then I went and I saw that... ummm. I oh... Here's a harder one, OK. I had fifteen dollars and 35 cents. There were two paint... there was one paint brush and it

> costed... umm... No, no. There were threeeee paint brushes but each of them costed fifteen dollars and 36 cents. (*laughs*). I had 35 cents also so ummm, how many paint brushes was I able to get? Zero."

With this story, Melissa shows a slightly new way she is viewing zero. Melissa is still constructing a balance, but this time the balance is an inequality: 15.35 < 15.36, and not balanced at all. Nevertheless, to Melissa the result of this inequality is zero. There is a shift in action here. The inequality, with its unequal balance, is a form of action. The action still does not involve zero, but this time it leads to zero. I think about the change that is occurring in Melissa's reasoning. Like some of the other changes that occur for Melissa, the change is not movement through the spheres of understanding, nor is the change thickening. The change is slight and moves towards Image Making but still stays in Primitive Knowing. There is no making of an image just yet, it is the action of moving towards an image. Our first session ends here. However, I have a plan for Melissa for our next session. I want to intervene to help her move into Image Making. I know, though, that I have to tread carefully. Melissa's movements are fragile and she (me, the exercise…) can create a barrier to movement quite easily.

Chapter 6.0 Megan

6.1 Introduction to Megan

> "Much teaching leaves the pupils dependent not on publicly established systems of knowledge (if such exist) but on quite trivial preconceptions set up arbitrarily either on the spur of the moment, or when the teacher planned the lesson during the previous evening. This reduces the part played by the pupils to a kind of guesswork in which they try to home in upon the teacher's signals about what kind of answer is acceptable."
> (Barnes, 1976, p.179)

Megan experiences persistent mathematics difficulties in school[19]. While Megan's teachers are aware of the difficulty that Megan experiences, she has also developed elaborate schooling coping strategies (Folkman & Lazarus, 1988) to compensate for her difficulties. Like the students described by Barnes (1976) in the quote above, much of Megan's coping mechanisms revolve around determining the "most acceptable" answer, that is, the answer the teacher wants from her. During our sessions together, Megan figures out that if she comes early, while I am setting up the tasks and cameras, she can look at my notes and attempt to find "correct" answers for the questions I am about to ask. Prior to each session Megan also inquires of her friends about the questions I ask them and the answers they give. In the classroom, I observe Megan studying the teacher's facial expression carefully both before she initiates an answer to a question, and after giving an answer. Depending on what she interprets from her observations, sometimes Megan will change her answer. In class, Megan also surreptitiously glances at her neighbors when she seems to be unsure about what to do. In the two weeks I spent as an empathic second person in Megan's classroom I never witnessed her initiate a question.

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[19] I note that, for reasons already outlined in the methodology section, although I chose not to delve into working memory in this study, I noted on numerous occasions that working memory might possibly be contributing to Megan's difficulties. Indeed, I incorporated small intervention strategies for Megan, such as encouraging her to write down numbers, revisiting recent ideas, and chunking. These strategies, common accommodations for working memory difficulties, often enabled Megan to continue when she faced an immediate barrier in her problem solving.

Because of Megan's reliance on me, I become tentative in my interactions with her. I ask less questions than I want to, especially at the beginning of our interactions. I wonder before each initiation how Megan will perceive my question or input. I fear that the very asking of a question, or making of a comment, may trigger Megan to negate her own thinking in favor of what she perceives I want.

Another coping mechanism for Megan is ignoring her intuitions and holding steadfastly to (sometimes what appears to be arbitrary) mathematical rules, even in the face of competing intuitions. Megan remarks during one session that something during one of the tasks, that does not fit in with one of her rules, is "decoration" with the purpose of "mixing (her) up." Because Megan's coping mechanisms rely on these peripheral coping supports, she experiences mathematical concepts passively and often does not initiate questions or connect patterns in the world around her. At one point in our sessions I become worried that I am about to go over-time. I ask Megan to confirm the time that school ends. Megan answers that she does not know. And when probed further, Megan cannot approximate an answer nor does she indicate an interest in an answer. Without the noticing of patterns, mathematics for Megan can be a "magical" subject where there is often no rhyme or reason to why things work. For Megan, mathematics is mostly about arbitrary rules that are applied regardless of what her thinking tells her.

Megan's past history and current experiences of success and difficulty, her beliefs toward mathematics, her coping mechanisms and her future expectations (Skovsmose, 2005) all interlap cyclically and dynamically, reinforcing and growing the other to construct Megan's mathematical dispositions. These dispositions sometimes come into conflict with each other throughout our sessions. Megan's conflicting dispositions come to the fore when the task we are working on does not fit in with her preconceived rules. And, as will be explored below, I sometimes purposefully, as an intervention strategy, push Megan into these conflicts.

During our first session, Megan expresses her overall view of mathematics. Megan relates that mathematics can be divided into two sections: geometry-easy and likable-and the rest of mathematics–hard:

> "I like how some of the math is like shapes, and we can like count the shapes size and the vertices and yeah, (*whispers*) and I don't like about math is that it takes such a long time to just get one little part."

For Megan, the two sections of mathematics are not relative in size. The hard part, that "takes such a long time to get one little part," makes up a disproportionate amount of Megan's experience. Hard is also intertwined with many of Megan's mathematics difficulties. Notably, alongside Megan's coping strategies, she applies aspects of geometric reasoning, her declared strength, especially symmetry, to areas beyond geometry even when the connections are not explicit (Nunes, Bryant, & Watson, 2009). For example, Megan reasons with symmetry in the decomposing number task and in a later session when she revisits telling stories using zero.

6.2 Mapping: Megan's Pathway of Dynamical Growth

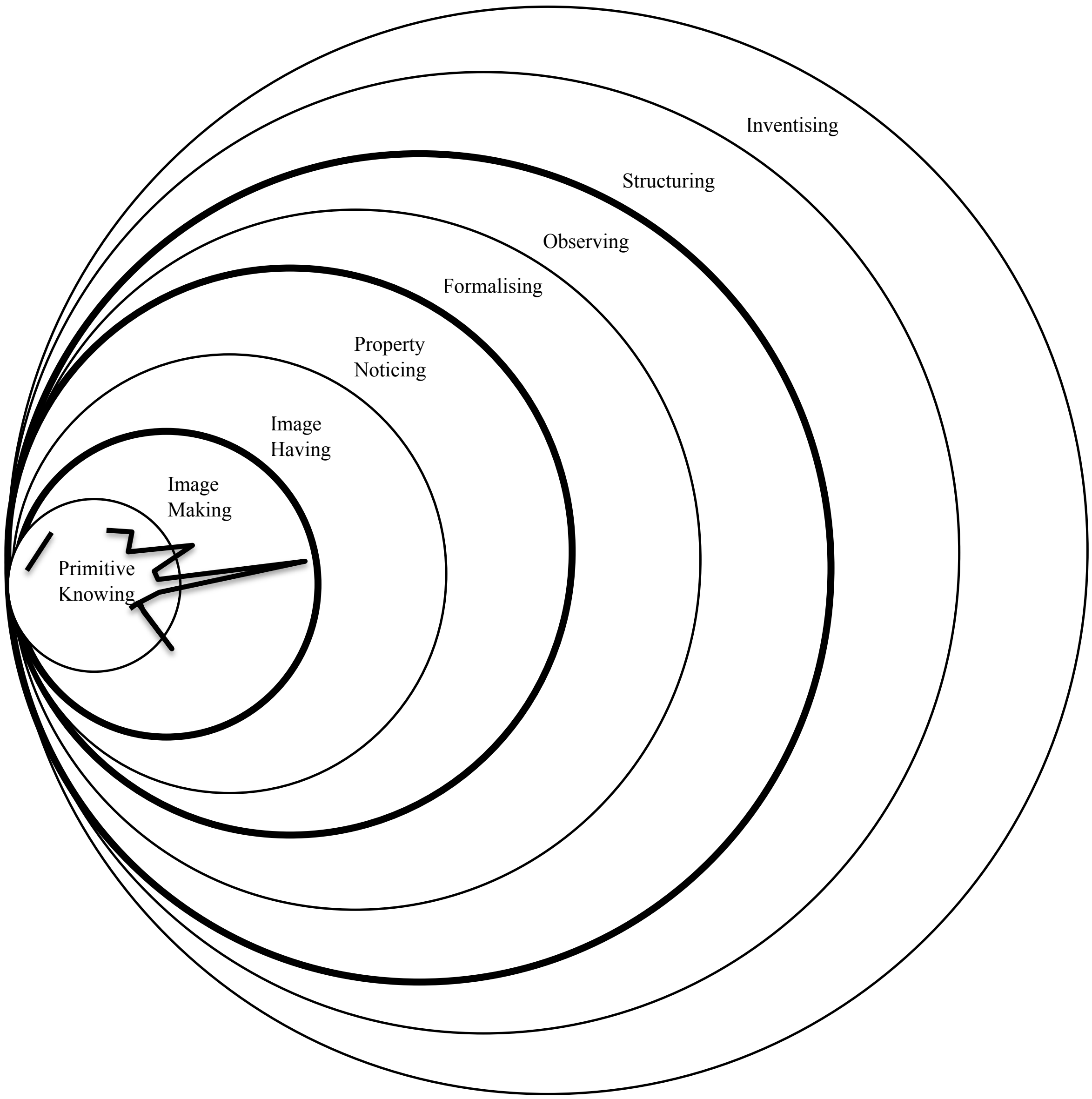

Figure 6.1 Megan's Mapping

6.3 Megan's Journey

The mapping of Megan's journey of growth in understanding is situated in and around Primitive Knowing with bursts of mathematical insight emanating outwards. The bursts do not begin immediately when we begin our session. It takes a while for Megan to initiate those bursts. This lag in time is partly because it takes me a while to figure out how to engage with Megan. The interactions between Megan and I, throughout our sessions, are different than between myself and the other participants. Especially at the beginning of our sessions, and sporadically all the way through, Megan often relies on her coping mechanism of trying to gauge an answer from my expression or notes. It takes me a while to figure out how to circumvent her reliance on me. One way I do this is when she looks at the page to see what I am writing, I stop to specifically invite Megan into my train of thought. When Megan inevitably answers that she accepts my invitation, I explicitly share my thoughts with her. I explain the details about what I am thinking, and discuss a strength of hers that I am noticing. Thus, it is not until near the end of our first session that Megan's first burst occurs. That first burst is not a small feat. That first movement is pivotal, and afterwards, Megan is in constant movement between Primitive Knowing, Image Making and sometimes Image Having.

6.4 Zooming In… *Two piles*

Figure 6.2 Megan Zooming In… Two Piles

Megan finishes the decomposing ten section of the decomposing number task. She relies heavily on symmetry for each decomposition of ten, creating multiple symmetrically placed piles of blocks (e.g. 2+2+2+2+2, 1+1+1+1…). I note that Megan is very purposeful with her symmetry. Because symmetry of zero on the number line is another important conceptual understanding underlying zero, I wonder if and how Megan may use this understanding for zero. I also wonder if this knowing may be leveraged later if need be, to help Megan understand a conception of zero. I next explore symmetry with prime numbers and zero with Megan. I ask Megan to decompose the number 17, but this time to use only two piles. Thus, my intention is to create a conflict for Megan.

Megan begins with all 17 cubes in one pile in front of her. She moves one cube to the right and then one to the left continuously until she is left with one cube in her hand. She shifts the right pile to the side and says, "one pile," then Megan shifts the left pile to the side and says "two piles," finally she places the one left over cube in the middle of the piles and says "number one." I ask Megan how much is in each pile. Later, when I review the video recording of our session, I can see Megan count all the cubes in one pile and begin to count two of the cubes in the second pile. Megan assuredly announces "seven and seven." However, in person I cannot hear Megan's answer and ask her to repeat herself. In repetition, Megan's answer then becomes tentative.

> "Seven, (*points to the right pile*) and then I'm guessing seven here (*points to the left pile*), because seven... (*pauses to count the left pile by twos*). Yeah. (*looks at interviewer*) I'm pretty sure. Yeah."

The fact that there are really eight cubes in each of Megan's piles is inconsequential here. It is common for all people, at times, to make small numerical errors especially between close numbers (Deheane, 2011), in this case 7 and 8. Of interest is Megan's tentativeness to give an answer for the second pile only after I ask her to repeat herself. Megan's usage of symmetry in this case is strategic. Once she knows the number in one pile, there is no need to count the cubes in the second pile. Because of her motions of symmetry, making one-to-one correspondence between the two piles, the two piles are the same amount. Through asking Megan to repeat her answer, I am giving her a cue to question her strategy and thus, her understanding. Notably, Megan begins to defend that there are seven in the second pile: "because seven…" However, this does not last long, as still questioning herself, Megan pauses to count and then look at me for verification.

After asking Megan about the two piles, she first moves the middle cube into the right pile, noting while studying my expression that the number in each pile is now eight and seven. Then in finding another way to create two piles from 17, Megan transfers one cube from the right pile into the left pile and again while still focusing on my expression, notes the number in the piles as seven and eight. After these two, Megan pauses and changes her focus:

MN: (*moves one pile towards the second pile and looks at the Interviewer*). Can I do that? Or...
I: Yeah. You can definitely do that.
MN: (*looks excited*) OK.
I: So what's your 2 piles?
MN: seven... This is one (*motions in circular motion with hand*) and then... (*Interviewer nods*) seventeen in one.
I: Ok Good. So what would the number sentence be for this one?
MN: (*big sigh*) seventeen. (*looks to the right*) uhhhh... this one. Would it be 1 +1 +1 +1... or 17 is 10+17.
I: Or it could be 17 plus...
MN: (*long pause, then looks at the interviewer*) 10 plus 7.

It could seem from this transcript that Megan does not understand what she has just done. There is obviously a disconnect between the representation of piles on the table and the number sentence that Megan is asked to attach to the representations. However, I disagree with the conclusion that Megan does not understand, as Megan's comments do not reflect on her understanding at all. The more likely culprit is that Megan is not problem solving the problem at hand, but instead problem solving the answer that "I want." As I say my leading statement of "Or it could be 17 plus…," and Megan answers, "10 plus 7," I immediately see my error in asking Megan to create a number sentence. Davis (1996) describes a similar occurrence between himself and a student, where he eventually told his student the answer he wanted. However, when Davis asked his student the exact same question immediately after she gave her "correct response," she reverted back to her original misunderstanding. Similarly, I try to lead Megan into Image Making for zero: "or it could be 17 plus…" However, I am too far ahead and Megan is too focused on me. I realize that I have to be very careful with Megan. It is going to be very difficult to separate her thinking about the mathematics from her thinking about pleasing me.

6.5 Zooming In… *Oh zero*

Figure 6.3 Megan Zooming In… Oh Zero

For the hiding cubes task, there are ten cubes on the table. I ask Megan to close her eyes and I hide cubes. Each time I hide the cubes, Megan guesses the number hidden. Then I ask Megan to close her eyes again. I make a lot of noise hiding cubes, but I don't hide any of the cubes. I am looking to see how Megan says "0." Does Megan say "zero," "none," or "nothing"? When Megan looks at the pile, her first response is one. She looks at my face, to check her answer. In order to build confidence, I try very hard to have my facial and bodily expressions remain the same. I can tell from Megan's own facial and bodily responses that I am succeeding. Like for the other hidden cubes, I ask Megan how she knows the answer is one. With my question, whatever strategy Megan used to figure out there was one cube hiding is lost now. Instead Megan takes the question as a cue to check her answer. Megan immediately moves the cubes two by two and counts:

"There are two, four, six, eight…"

Megan pauses and looks at me, she has her finger on the last two cubes in the pile. There is another long pause as Megan slowly drags the cubes towards the ones she has just counted. Megan smiles and says:

"Oh, zero!"

There is an excitement in Megan's voice as she realizes there are zero cubes hiding under my hand.

Megan has utilized the word "zero" and not "none" or "nothing" to describe the absence of cubes under my hand. The labeling of zero as "zero" is significant and is in Megan's primitive knowing.

I pause here for a moment, as I want to revisit an idea around primitive knowing of zero I introduce during Melissa's journey. In my telling of Melissa's journey, I posit that the primitive knowing of zero I am labeling, may be a result of our interactions around zero. Thus, some would not consider those knowings identified as primitive knowings to be Primitive Knowing at all (see section 5.4). After arguing that understanding is about interactions, I then dismiss this possible influence on Melissa as irrelevant because any gaze on anything changes that thing anyway (Davis, 1996). This argument is about all knowledge and is true for Melissa, Megan and everyone in any interaction. Still, I want to juxtapose this argument next to what is occurring for Megan here. Surely my interaction with Megan changes her responses. I already witness this in the previous instance where Megan dismisses her own thinking in favor of molding herself to my needs. And, notably, I will witness this molding again and again throughout our interactions. However, there is something that makes this instance different. There is a realness in the moment of Megan's realization of "Oh, zero." In this moment, I see Megan's thinking unencumbered by her need to please me. Megan is excited because she discovers something. Someone watching this episode can argue that at the moment that Megan says "zero," she is still giving me the "correct answer" to the question I ask. Thus, a conclusion might be that this is just another instance of Megan giving in to what she perceives I want. Of course, it is true that "zero" is the correct answer to my question about how many blocks are hiding. Aside from the tone of Megan's voice there is an important difference here. In the moment that Megan realizes zero cubes, she does not look to me for approval. There is no outward sign of Megan molding her answer to what she thinks I want.

This episode of change then, can be contrasted with the episode described in the previous section. For the decomposing number task, I use a strategy that furthers Megan's reliance on myself for reasoning mathematically. This strategy is of course unsuccessful for her. The result is that Megan becomes stuck trying to find the answer I want and her mathematical thinking is redirected to conforming to my thinking. In this instance, on the other hand, my intervention is slight and focused on Megan's reasoning and not the answer. Although I again inadvertently refocus Megan on the "correct answer," still I have not usurped her mathematical reasoning. My intervention here, thus, does not remove Megan's thinking independence.

Megan's paths within Primitive Knowing for these two growths are thus different. For the decomposing number task, Megan's path does not lead into another learning. At least not another learning that I can see. This path is stymied, for now. Instead Megan's path starts anew from the hiding number task, whereas after this task, Megan's path of growth continues.

6.6 Zooming In… *Zero with different rules*

Figure 6.4 Megan Zooming In… Different Rules

Megan is presented with three cube trains of ten and one cube train of six (36). I ask Megan to tell me how many she thinks there will be all together if we add twenty more cubes. Twenty is an interesting number because it has an explicit zero as a place holder. Twenty has explicitly two tens, thus, an easy way to complete this task, and a way based on a rote rule many children learn (Fuson, 1990), is to just count up two tens from the thirty. Unsurprisingly, this is the strategy that Megan uses as she assuredly explains:

"10, 20, 30, 40, 50, 6 (*taps finger for each tens number*)."

The rote rule allows the counter to ignore the "6" in 36, and thus the implicit zero. Ignoring the six and implicit zero, the counter can skip count forty-six, then fifty six. However, Megan does not ignore the six. She decomposes the thirty-six into "30," with its now explicit zero, and "6." After skip counting, Megan then adds the six back onto fifty.

Here, Megan demonstrates a knowing of the implicit zero in her primitive knowing. The problem I just gave Megan has the augend (36) with an implicit zero and the addend (20) with an explicit zero. I wonder if this knowing is rule-based and if there might be some flexibility to Megan's knowing. I wonder what would happen if I ask Megan to complete an addition problem where both the addend and augend have implicit zeroes. Will Megan fold back to the rule she just used in order to separate the numbers and make the implicit zero explicit? I ask Megan to add a new number to the thirty-six cubes:

"What would happen if we add 12 more?"

This time Megan is more tentative. She looks to the side and slowly her smile disappears. Megan begins with the same rule she uses for adding twenty to thirty-six. Beginning with ten, Megan counts by tens. Then, for each subsequent ten, similar to when she adds twenty, Megan taps each train of blocks, mouthing the words, "ten, twenty, thirty." However, unlike when she adds twenty, Megan does not continue on from the thirty. She stops. This time the rule does not work for Megan. She has to shift strategies to counting on using her fingers.

Megan counts the fingers on her right hand twice to signify adding ten. Beginning with her pinky finger, Megan slowly taps her thumb to each finger on her right hand, counting up from thirty-six. After going through the four fingers on her hand, Megan begins with her pinkie again. She taps her four fingers again. Because Megan's thumb is being used to tap her fingers and not as a counter finger, she taps 8 times instead of 10 and counts to 44. Megan very tentatively responds with "44" as her answer. Again, Megan's rule falls apart.

As elaborated in chapter 1, researchers (e.g. Bryant et al., 2006) have advocated the rote learning of rules for zero as an appropriate intervention for difficulties with various mathematical concepts. In contrast to this recommendation, Megan experiences difficulty with her rote rules. In Megan's engagement with this task, she utilizes two different rote rules: (i) when adding two, two-digit numbers, one with an explicit zero, one only needs to add the numbers in the tens column; and (ii) there are five fingers and to count ten one needs to count one hand twice. Both rules here are based on the affordances of a base ten system. And both rules fall apart, albeit in different ways. The first rule falls apart because it is not flexible (McGowen & Tall, 2013) enough to allow Megan to fold back and refine it for a new instance. At the same time, Megan is unaware that her second rule does not work for her. I could intervene here and ask Megan questions that might push her into thinking about her rules.

There are a number of instances where I could intervene or answer one of my wonderings through choosing to ask Megan questions about her reasoning. Another instance I choose not to answer my wondering is when Megan shifts from adding twelve to thirty-six to adding ten to thirty-six. Despite the understandings of implicit and explicit zero that this could imply, I am afraid to ask questions about her shift. Megan molds herself to what she thinks my needs are. I wonder what I would be implying about getting a right answer if I ask Megan about her shift?

6.7 Zooming In… *Commas, Spaces and Zeroes*

Figure 6.5 Megan Zooming In… Commas, Spaces and Zeroes

I ask Megan to write down the numbers I am reading. Megan is swift in her writing. She does not ask me to repeat any numbers and she seems to require little time to process the numbers I am reading. We move quickly through the two- and three-digit numbers. These are all written correctly, with the explicit zeroes in the appropriate place. After the three-digit numbers we move onto the four-digit numbers, beginning with six thousand. Six thousand has two concepts that I specifically watch out for with Megan: (1) in six thousand all the zeroes are explicit, and (2) now with four places there is a convention of placing a comma after every three numbers starting from the right. Considering Megan's reliance on rules in our previous interaction, I wonder if and how Megan will interact with the comma rule and the explicit zeroes.

For 6000, Megan writes "600,000." Megan has included the comma in the appropriate place, but at the same time there are two extra zeroes in her answer. I am not sure what to make of the "600,000" that Megan writes. In my research for the literature review, I did not come across this type of error. Still, a likely culprit might be concatenation (Fuson, 1990). However, because there are no implicit zeroes in the number, I am having a difficult time justifying a concatenation argument. I wonder where the extra zeroes come from. Maybe Megan could be experiencing difficulty with the explicit zeroes. Now, in retrospect, after viewing our interaction multiple times, looking at the number Megan writes-600,000-I notice how symmetrical it is with the comma balancing an equal number of places on both sides. Maybe Megan has added the extra zeroes to make the number symmetrical? Zero itself, is a wonderfully symmetric number, having an infinite number of lines of symmetry. I wonder about the comma rule, and if Megan is applying it rigidly-three numbers comma, three numbers-regardless of the context. The comma convention does represent an important underlying understanding of the base-ten place value system. The comma after specifically three places signifies a pattern implicit in the base-ten number system. The first three places of ones, tens, and hundreds are repeated after every comma, in every three places. For example, the second set of numbers, or the thousands group, has a thousand for the ones, a **ten**-thousand for the tens and a **hundred** thousand for the hundreds. Van de Walle and colleagues (2017) call this affordance "the triples system" (p.204).

I continue with the next number. The next number has both explicit and implicit zeroes: three thousand forty-eight (3048). Megan writes "300,048." Megan is carefully maintaining the rule for commas. The way Megan writes 3048 as "300,048," leaves me to remain wondering if she is experiencing a difficulty with explicit zeroes, or she is over-applying symmetry to the comma convention. Concatenating may still be a logical explanation for this number, as three thousand, "3000," forty-eight, "48" concatenated becomes "300048."

The next number I read is one thousand seven (1007). Megan writes "100,007." The number Megan writes is indeed symmetrical on both sides of the comma. Whether Megan is concatenating or creating symmetry is still difficult to determine just yet, as it is possible that Megan accidentally put four zeroes instead of three. I wait to see if there is a pattern with the numbers.

It is only when I ask Megan to write nine thousand two hundred eighteen (9218), that Megan shares her first comment: "Wow." Megan does not even look up when saying, "wow." There is no pause. Megan immediately writes "900,018." That there is no two (hundred) included in Megan's answer does not make me wonder. To be sure, there are three pieces of information in the instruction to write 9218. There is the nine thousand, the two hundred and the eighteen. And Megan does say, "wow." It is possible that with three pieces of information Megan simply let one piece go. Even though I am in the midst of a delicate balance between giving Megan space to think and intervening, I regret not asking Megan to revisit the two hundred. A discussion around the absence of the two hundred might have the potential to either problematize the symmetry that Megan is creating, or, alternatively, the revisiting might bring concatenation to the forefront.

I wonder again if Megan is confusing the rule for the commas with the actual place value, or if Megan is concatenating the number? How is Megan viewing the added zeroes? Are they arbitrary, and can therefore simply be added to create the symmetry? I wonder which pattern will show when we move onto numbers with five places. I imagine when Megan writes numbers with five places I will begin to find answers to my wonderings.

I ask Megan to write twelve thousand, six hundred and three (12,603). Megan writes: 120,603-there is one extra zero, creating symmetry around the comma again. It seems that Megan must be creating symmetry around the zero because this number concatenated would be: 12000603. I ask Megan to write another five-digit number (34,920). Megan adds the same zero and the same symmetry-340,920. I wonder if and how Megan's (mis)use of zeroes for symmetry of place value, impacts her reading of the numbers. Does Megan make a connection between the comma and the place-value pattern the comma makes explicit?

One by one I show Megan cards and I ask her to read the numbers. None of these cards have commas. For the five- and six-digit cards, the cards have spaces instead of commas. I wonder how Megan will interact with these cards. What will happen to Megan's symmetry? Will Megan add extra zeroes and commas to make sense of the cards? I imagine, based on our interactions so far, that Megan will probably not add any marks to the cards if I do not explicitly tell her she is allowed. Then how, I wonder, will Megan reason through these obstacles to symmetry?

I present Megan with cards to read beginning with two- and three-digit numbers and ending with six-digit numbers. Megan does not pause and speaks with confidence as she reads the two-, and three-digit numbers. Then, as each place value is added to each card, Megan simultaneously progresses through modes of understanding. The following narrative is unique amongst our other experiences for three reasons:

(1) As Megan reads, I do not intervene in any way for any of the numbers.
(2) Megan often gauges my verbal and non-verbal cues for approval. Yet, here she does not look to me for approval-at all. Thus, at least from the outside, Megan moves through the tasks unencumbered by perceptions of my wants.
(3) At other times, I observe Megan distancing herself, from tasks, especially in group situations. In our meetings this often occurs when Megan is faced with a problematic aspect of the mathematics together with no clear way, verbal or non-verbal, of determining her own accuracy. Here, even though Megan experiences difficulty with the task and is unsure of her accuracy, Megan does not distance herself from the problem, in fact she persists with the problem.

As a result of the above three points I get a clear view of growth for Megan. Because of this view I have again (see Chapter 4, section 4.6) chosen to change the presentation of the narrative. In what follows, I first zoom in through presenting a table (Table 6.1) of the numbers and utterances. I then zoom in further, discussing Megan's responses and movement. To indicate where I am in my analysis, I italicize the title of each new section of zooming in. Each of these sections begin with the number from the card and quote from the transcript that I am analyzing.

The First Zooming In:

Table 6.1 Megan Reads Numbers

Number	Megan Reads
6000	"Six thousand"
2394	"Twoooo…two thousand and… two thousand three hundred and ninety-four."
3008	"Three… three thousand and eight."
23 555	"Twenty-three thousand, five hundred and fifty five"
54 001	"Fifty-four thousand and one"
340 789	"Three thousand forrr… and… three thousand and … three thousand forty, seven hundred and eighty-nine"
645 000	"Six thousand and forty-five… thousand"

Zooming in Deeper- 6000, 2394 and 3008, "Six thousand." "Twoooo…two thousand and… two thousand three hundred and ninet- four." "Three… three thousand and eight."

A change occurs when I present Megan with the second of the four-digit numbers. Notably, the first four-digit number I present to Megan is six thousand, the same four-digit number she writes as "600,000." And notably Megan reads the number fluidly as "six thousand." Then, as Megan reasons the next 2 four-digit numbers, there is a change in her fluidity. She pauses to think about these numbers as she reads them, hesitating, reiterating and using the word "and." On the four-digit cards I am using to prompt Megan, there are no indicators of symmetry, like a space or comma. A space or a comma would explicitly signal a movement from the first set of triples onto the next set of triples. Through reiterating the first numeral twice, Megan is acknowledging the symmetry and making this movement on her own. She is demarcating these numerals as occupying space in the thousands section, different from the first set of triples. This movement of demarcation is also a movement of folding back into Primitive Knowing as Megan is accessing rules, symmetry and triples. Megan's movement is enabled by the inherent symmetry in the place value before her.

Still, I am perplexed as to why Megan does not separate out the "6" from six thousand. Why does Megan need to demarcate the two and the three thousands, but not the six thousand? I Consider the affordances of the number "6000" and Megan's previous interactions with symmetry for her writing of this number. Six thousand is exactly the way it is concatenated. In the number "6000," there are no implicit zeroes. The pattern, then, is explicit-three explicit zeroes are in the first triple then the numeral "6." Still, Megan writes the number six thousand as "600,000." I want to revisit my analysis of the "600,000" that Megan writes again in light of this current apparent contradiction. I wait to revisit my analysis because there is a larger narrative about symmetry at play. I first wait to consider Megan's interaction with the five- and six- digit numbers.

Zooming in Deeper- 23 555, and 54 001, "Twenty three-thousand, five hundred and fifty-five." "Fifty-four thousand and one."

The five-digit number cards, I now show Megan, are different from the four-digit number cards in that they have an additional affordance. For these cards the triples are separated by a space. As Megan begins to read these five- digit numbers, she is accessing her knowings of place value rules and the symmetry of the triples. The five-digit numbers are read fluidly-Megan does not reiterate nor pause after reading the first numeral. She is still in Primitive Knowing.

Zooming in Deeper- 340 789, and 645 000, "Three thousand forrr… and… three thousand and … three thousand forty, seven hundred and eighty-nine." "Six thousand and forty-five… thousand."

I now present Megan with six-digit numbers. I show Megan two cards, each with a symmetrical placement of a triple on either side of a space. There is a wonderful symmetry to these six-digit numbers and I wonder how Megan is going to use the symmetry. Because of Megan's usage of symmetry so far, I do not wonder *if* Megan will leverage the symmetry-only *how* she will do it. Throughout our time Megan consistently draws from this strength and I would be surprised if she did not use symmetry, here, in this instance. The convention of reading theses six-digit numbers dictates the reading as almost the same on both sides of the space. The only difference is that the name of the triples- “thousand”- is said after the second set of triples. Working through this convention around symmetry, Megan moves into Image Making.

The first number I show Megan is 340 789. Megan gathers her previous knowings, and begins applying them to the new set: “Three thousand forrr…” Megan is trying to make sense of the “340.” She hesitates, drops the forty, then adds a conjunction, and, then hesitates yet again: “… and… three thousand and…” I wonder about Megan’s placement of the word thousand. I know from experience with other learners that there is a problematic aspect in this convention of symmetry and place value. If every set of triples has a title, for example the thousands, millions and quadrillions, then what is the very first triple, around the units, called? I have experienced learners calling the first triple the “hundreds” set. Then my students logically reason that none of the other sets can have the title “hundreds” in them. I wonder if this is why Megan is avoiding the word “hundred.” Yet, I cannot ask her. Asking Megan might deflect her attention onto me. I do not want to do this because Megan has moved into Image Making. She is working hard to create an image around this new place on the thousands side of the triples. I want Megan to persevere here in her Image Making so I do not risk interrupting her process. Megan decides the forty goes with the three thousand and gives her answer of, “three thousand forty, seven hundred and eighty-nine.”

Immediately we move onto the next number. Megan begins where she left off during her Image Making with the last card. Like the “340” (thousand), the 645 becomes “six thousand and forty-five.” Then Megan hesitates. She is stuck because she needs to contemplate what to do with these last three zeroes. The contemplation does not last long and Megan labels them thousand: “Six thousand and forty-five… thousand.”

While Megan is in the middle of Image Making, I make the decision to stop and move onto the next task. In retrospect, stopping when Megan is just on the cusp of creating an image seems like an odd thing to do. It seems almost anti-climatic: Megan works so hard, and yet I do not orchestrate a "final" means of acquiring an image. Since I am tracking growth, it would make sense to follow Megan's process to the conclusion of creating an image. Moving from Primitive Knowing to Image Making is growth in understanding. Moving on is still in line with my research questions and goals of the project, thus, from that perspective it is ok to stop in the midst of Image Making. However, I may have a problem of equity, I stop right before Megan has access (Gutierrez, 2012) to a new image. I will be meeting with Megan repeatedly over the next while and I do plan to build on Megan's images. I want to carefully plan a task that will build on her growth so far. Still, with further tasks, it is possible that Megan's Image Making may not grow into actual images she can work with, or Image Having. This is ok because growth is embedded, recursive, and continuous (Pirie et al., 1994) and not defined based only on a clear acquired product. If one were to rely solely on the acquisition metaphor (Sfard, 1998), then Megan not reaching Image Having would be a problem. The acquisition metaphor creates binaries of understandings, a learner has acquired something or they have not. In this view, Megan will have an image or she won't. In reality Megan, being in Image Making, has an "almost image." This almost image is still tangible, she still has access to it and can manipulate it.

6.8 Zooming In… *Micro-Moments and Jumps*

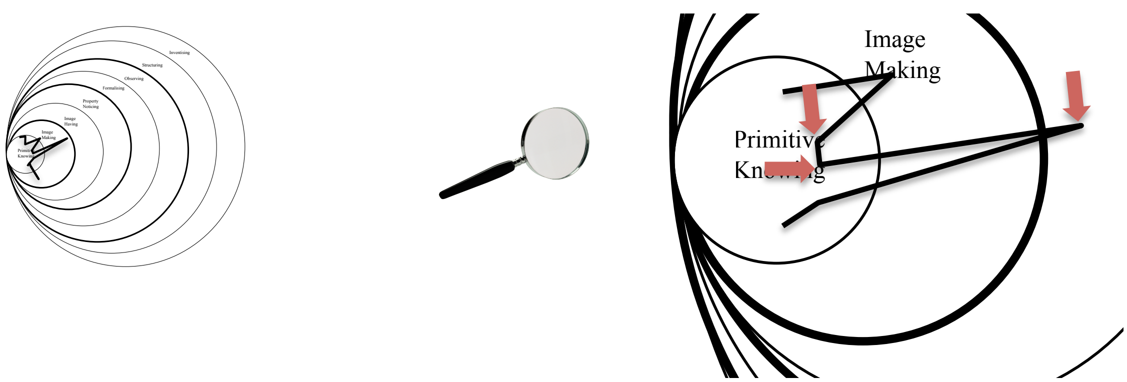

Figure 6.6 Megan Zooming In… Micro-Moments and Jumps

It is a new day, and I am preparing the video cameras before my session with Megan. Megan arrives early. In front of my seat sit my notes, open to the first of today's task. I see Megan focused on reading my notes (bold as in the original):

> "**Zero This one is for language- do they regularly use the word "zero" or nothing**.
>
> "What is zero?"
> "Is zero a number? Why? Why not?"
>
> Can you show me zero without using language? What is zero to you- you can show me with your hands your face, whatever, but save your language for explaining later."

Megan keeps her eyes pasted on my notes. I do not interrupt her. We have only recently established a rapport, and part of that rapport is the openness and sharing of what I am doing. As I sit down, I do acknowledge that Megan has seen my notes. This acknowledgement is strategic on my part as I am quite aware that I am continuing to develop trust with Megan. Thus, I try to acknowledge in a positive way without judgment. I also want to support Megan's use of this strategy. I see Megan's strategy of actively discovering what is expected of her as a strength. The strategy can be an important survival skill in the classroom, when experiencing difficulties. I am quite curious as to how Megan's anticipation strategy and newfound knowledge will play out in our session. Admittedly, I also worry that this strategy can deter the element of surprise, and thus, learning for Megan. So far, with the other learners in the study, I often rely on creating conflict in order to initiate or promote growth in understanding. I am curious how Megan is leveraging her strategy when she encounters new mathematical ideas. I wonder about the effect it has on her growth in understanding.

I ask Megan what she thinks zero is. Megan has just read my notes about zero and nothing, and is prepared to answer. Indeed, Megan is even looking at my notes as she answers:

"Ok, so zero is a number, but it also means nothing."

This statement, of zero being both nothing and a number, is far more than Megan's preparation for an attempt to appease me. There is a second part coming; her statement needs an explanation. Megan cannot prepare her explanation from my notes-nowhere in my notes does it say why zero is both a number and nothing. Thus, even though Megan does not have the element of surprise because of her strategy of anticipation, growth still occurs. This growth occurs through her explanation as she reasons aloud about this dual nature of zero. As Megan is explaining, she propels herself into thickening primitive knowings. Megan signifies the move into thickening by shifting her attention away from my notes and towards me. Additionally, even when looking at me, Megan does not have the outward signs of looking for my approval, for example inflection in her voice or a quizzical expression. It is through this act of explaining that Megan gathers her primitive knowings. She then uses these knowings to spring back outwards into Image Having. This is the second time (see section 6.5) I have now seen Megan's growth catalyzed through seeking an explanation to an answer she anticipates. In what follows I present the transcript of our interaction and analysis of this growth. In order to highlight the embeddedness of Megan's acts of collecting and movement, I include each previous excerpt of transcript in the next excerpt. I use italics and bold font to indicate the newly, added transcript.

Megan begins her explanation why zero is also a number by gathering Previous Knowings:

MN: Ok, so zero is a number, but it also means nothing. ***(looks at I) Because you have to start somewhere and zero is where you start.***

Megan is gathering the idea that since numbers are sequential then there has to be a beginning-a starting point of all the numbers. Thus, zero has to be a number because it is a starting point of the sequence of numbers. Megan has to thicken this knowing before moving on:

MN: Ok, so zero is a number, but it also means nothing. (looks at I) Because you have to start somewhere and zero is where you start. ***Then if you didn't have zero, you would start at one and one would be the lowest.***

Megan's understanding of zero as starting point becomes thickened as she considers why zero is the starting point for the numbers, and why the number one is not the starting point. She reasons that one would be the starting point only if zero did not exist. With this reasoning, Megan is beginning to thicken her understanding of the sequence of numbers to include the arbitrariness of this starting point convention (Zazkis & Rouleau, 2018). This understanding of zero as a starting point is also at the root of the measuring stick and motion metaphors of zero. Thus, with this thickening Megan has the potential to venture into either one of these metaphors.

MN: Ok, so zero is a number, but it also means nothing. (looks at I) Because you have to start somewhere and zero is where you start. Then if you didn't have zero, you would start at one and one would be the lowest.
I: **Aha.**
MN: **Zero is on the number line, so...**

Megan is invoking a shared image from a previous encounter we have had around the number line. As Megan invokes this shared image, I realize she has moved into Image Having. Interestingly, the actual construction of the images she is using (Image Making) has occurred in small micro-spaces, during various previous encounters. Megan has made images during these micro-spaces and now she is gathering them together into a new image. These micro-spaces, the source of the Image Makings, and the resulting actual Image Makings are outlined in Table 6.2.

Table 6.2 Megan's Micro-Spaces of Image Makings

Micro-Space	Source	Image Making
I mention an open number- line strategy (Gravemeijer, 1994)	Megan is experiencing difficulty decomposing tens	Number lines can be open
Megan learns the convention that a line with an arrow on either side is continuous	Observed Classroom Lesson	Lines can be continuous
Megan compares the number line I give her to the ones she is "used to."	I present Megan with a number line with the zero away from the beginning of the line	The zero does not have to be at the beginning of a number line.
Megan connects the open number line to continuous lines	I present Megan with a number line with arrows at each end	The number line can be continuous

There is a spike in Megan's pathway as she moves from Primitive Knowing into Image Having. This spike is deceptive though. Megan has not simply jumped from Primitive Knowing to Image Having. During small micro-moments, Megan carefully curates understandings (Image Makings) and deposits them into her Primitive Knowing. These moments of image making are not visible as moments of image making at the time they are discussed or performed-they are small moments, that seem to not require much attention. On their own, each moment could not be labeled as a moment of image making. These moments seem inconsequential at the time-they are "asides" to the tasks-and certainly not indicative of any major movement in growth. Notwithstanding their seeming negligibility, Megan hangs on to the micro-moments, depositing them in her Primitive Knowing, until there is opportunity[20] to gather these micro-moments together.

[20] The word "opportunity" here was carefully deliberated upon and purposefully chosen. It is not meant to convey an agency on Megan's part. Megan might have collected, deposited and connected the moments purposefully. She also might not have been purposeful in her actions. Regardless of her agency, Megan performs all these small movements and they result in an explicit moment of growth.

After gathering all these images, Megan makes a statement of which she has a high probability of its correctness-after all, she read the answer in my notes. Megan is then unencumbered by the anticipation of failure that may result in an incorrect answer. This state of being unencumbered becomes the catalyst for Megan connecting her various knowings around the number line and zero. With these connections, Megan can then deliberate and grow to Image Having.

6.9 Zooming In… *Zero is a Number, But…*

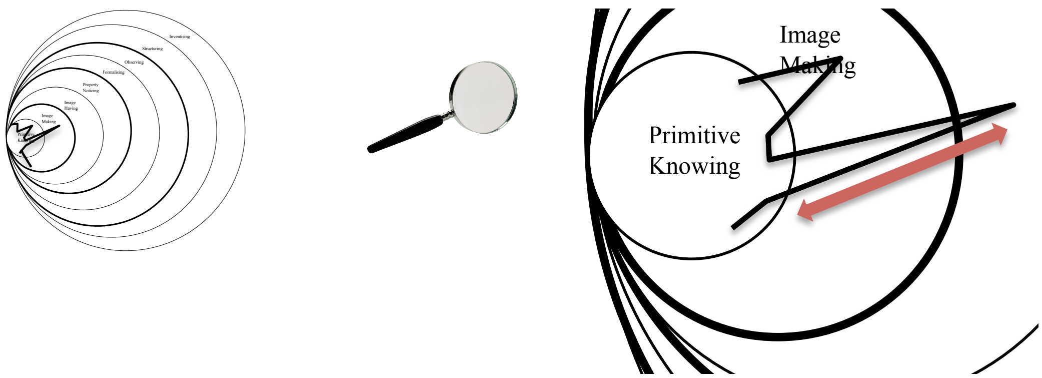

Figure 6.7 Megan Zooming In… Zero is a Number, But…

Megan is working her way through the zero question sheet. The first two questions involve adding zero to one: 1 + 0 = _____, and 0 + 1 = _____. Megan fills these in quickly. She does not show concern with the presence of zero in an equation. Then Megan moves to the next question: 1 – 0 = ______. She pauses, writes “0,” and moves on to the next question. The next question is 0 – 1 = _____, and Megan follows the same pattern of pause, write and move on for this question as well. I wonder about these pauses that Megan is making:

I: OK, I am going to stop you, because you stopped to think. There. (*points to 0 - 1*) and I want (to know)

MN: (Yeah because) zero... you can't minus...zero (*whispers*). So it's one. (*Writes a “1” over the zero in 0-1 and then writes a “1” over the zero in 1-0*).

I: So it stays the same? because you can't minus from zero? (*MN nods*).

Two important things occur in this vignette, first Megan states a rule around zero, and second Megan's verbal reasoning causes her to re-evaluate her answer and thinking.

Megan's answer of being unable to subtract from zero does not surprise me. She has yet to formally encounter negative numbers in her schoolwork. Yet, Megan does have some understanding of negative numbers in her primitive knowing. During a later session, I group Megan with Rosa, to work on a task introducing the concept of zero as a midpoint on a number line. I present them with a number line, on the computer, that goes from "-20" to (+) "20." The objective of the task is to move an icon backwards (subtraction) and forwards (addition) over the line. Notably, Megan is undeterred by the negative numbers and is able to operate within them. At the same time, I cannot quite discern the extent of Megan's primitive knowing around negative numbers. There is a history between Megan and Rosa that I am not privy to. Throughout the entire session Megan defers to Rosa for all reasoning, and refuses to talk about her thinking in front of Rosa. When I gently press Megan to verbalize her reasoning, she says "I wasn't actually thinking." Megan essentially "shuts down" in front of Rosa. Thus, I can see that Megan has some experience with negative numbers, but with Rosa present I cannot engage her in any way to neither just determine the knowing nor to push the knowing.

As Megan changes her answer from zero to one, I note that yet again, verbalizing her thinking has pushed Megan into a change in her own understanding. This happens again as I redirect Megan to the question 1 – 0 =______.

I: You also stopped to think over... you changed this one (*points to 1 - 0*).
MN: Yeah I changed this one.
I: OK. So why did you change it?
MN: Because... zero, is a number but, like, you can't minus zero? kind of? So it's going to be the same number.
I: Ok.
MN: MN: (*hand over mouth*) I'm pretty sure.
I: Ok. So you said you, you were pretty sure. So I'm kind of thinking of a question that's going to ask... what are you thinking when you say I am pretty sure? cuz you're not.

You're almost sure. Just not completely sure. So what are you thinking about? (What's making you pretty sure)?

MN: (Because I learned) something and it said like zer... ummm you can't minus zero so it has to stay one. And I don't know what like that subject was for but it was a type of math, I don't know what. So, I just guessed.

Megan has constructed some sort of image around zero as a number: "Zero is a number." "But" that image is conflicting with a learned rule about zero: "you can't minus zero." Consequently, although zero is a number, and numbers can be operated on and calculated with, zero also has a competing rule that means it cannot be used to subtract. Megan has come into conflict with repeating rules before-but this time is different. Notably, whereas in our prior conversations Megan holds steadfastly to her rules in the face of her competing reasoning (image), this time Megan notes that she is "pretty sure" she is correct. Megan is not holding as steadfastly to her rule about subtracting in the face of her image that zero is a number. This is a movement of growth for Megan, separate than her growth around zero. This movement shows she is beginning to consider her own intuition and reasoning, her own constructed images, in the face of conflicting memorized rules. Because the source of this growth and the subsequent pathway are not around zero, I do not map them on Megan's mapping. There is though, a simultaneous growth around zero that can be mapped.

Megan is "pretty sure" and admits to making a "guess." These are words that leave potential open for change, thickening and growth. These words are very different than conflicting ideas to her rules being "decoration" with the purpose of "mixing (her) up." Thus, Megan is at the outer edges of her images and knowings around zero as a number here. So far, when faced with a competing knowing, Megan protects her current images and knowings from thickening. It is almost as if she places a barrier separating what she knows and understands from her memorized rules. Prior to this moment, I have yet to encounter Megan allowing those outer edges to be permeable for thickening when she is confronted with a competing memorized rule. I am not arguing that Megan's image, in her primitive knowing now, is being thickened. I am arguing that the supports that enable her to thicken the image are growing, or that the barrier is weakening. Thus, Megan's dispositions (Martin, 2007) towards her own understanding are changing. There are two pathways that can be mapped from here: (1) Megan's growth in her dispositions, and (2) Megan's development of understanding zero as a number.

At the same time that I do not map the affective pathway, I later note that the two pathways are dependent on each other. In later sessions with Megan, I attempt to continue both pathways. All these sessions are conducted in small groups. Similar to the session with Rosa, described above, in each session Megan defers her own thinking in favor of the other group members-no matter who they are. Like with Rosa, Megan essentially "shuts down" with all other learners. Megan's thickening is dependent on her growing disposition toward accepting and valuing her own thinking.

6.10 Zooming In… *More Gatherings*

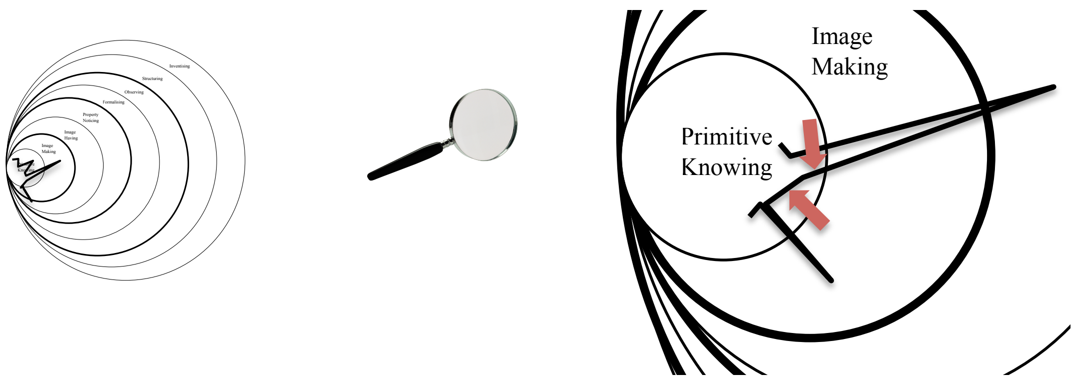

Figure 6.8 Megan Zooming In… More Gatherings

Before I begin tidying up the previous task, I place the two cards, with 105 and 150, from the task "Which is worth more?" on the table in front of us. As I am tidying up the papers, I notice Megan studying the cards. I have given Megan time to anticipate. I wonder what Megan is going to do with that time. For previous tasks, Megan uses the time to plan the answer to what she thinks I want. I wonder if Megan is going to do that again. I imagine not-there are no instructions or questions that go along with the cards I place on the table. It would be extremely difficult to anticipate the question I am about to ask, especially since the question "which zero is worth more?" is not a typical question one might get asked. This question rests on the concept that zero is worth *something*. In the same vein, Megan has just experienced a movement in her dispositions around mathematics. Thus, I wonder will that movement transfer here when she is confronted with having to anticipate something unfamiliar? I begin with my script:

> "So this is a weird question. You ready?"

At this moment I expect Megan to nod her head, or convey some other gesture or verbalization indicating she is ready for the question. Instead, Megan smiles and shares her anticipated answer/observation:

> "Zero just moves."

Again, Megan is thickening her dispositions around knowing mathematics. There are at least two gatherings that Megan has to perform in order to come to the seemingly confident response of "zero just moves:"

- Gathering her various experiences during her time with me, Megan anticipates that the task is going to revolve around zero.
- Megan is sharing an observation as an answer. This observation is of the inherent structure of the task and not an action to perform-she is given no instruction. Thus, Megan is gathering knowings that conceptual underpinnings of mathematical ideas exist.

I acknowledge Megan's observation and ask the question of the task, "Which zero is worth more?" Now I wonder if and how Megan will use her observation to answer the question. The question is functional and moves Megan's attention away from the zeros towards the function, or the worth, of the entire numbers. Megan indicates the "150" card as worth more. And, indeed her reasoning revolves around the quantity of the entire number:

> "Because of there's one hundred (*points to the one in 150*) and fifty (*points to the 5*) and this one (*points to the 0 in 105*) is a hundred and five."

Megan looks to the side immediately after her response. I confirm Megan's reasoning in different words back to her:

> "So it's worth more (*points to 150*) because it's a bigger number?"

Megan continues to look to the side, while I re-voice her reasoning, and through her subsequent nod. The nod, indicates her agreement with my interpretation of her answer. However, while Megan is looking to the side and nodding she is still revisiting her reasoning. I am unsure whether my reinterpretation initiates the revisiting, or if revisiting her own thinking is now another evolution of Megan's growing dispositions towards mathematics. In any case, I view Megan's nod as permission to go on. I begin to explain why the question is a weird question, when Megan interrupts me:

> "(No, uh actually) zero. They're both worth almost the same. Because this (*points to the 0 in 150*) is 0 and this is 0 (*points to the 0 in 105*). And then the numbers have to count, the only big numbers here are these 2 (*pointing to 1 and 5 on* the *150 card*). And these 2 (*points to the 1 and 5 on the 105 card*) are bigger than this one (*points to the 0 on the 105 card*). So none of them are bigger because they're both the same amount."

Megan's first answer takes into account only the magnitude of the numbers as a whole. 150 has a larger magnitude than 105. Thus, Megan answers that the zero in 150 is bigger than the zero in 105. At some point during her reasoning, Megan feels a need to revisit her understanding. This need initiates a gathering of ideas about place value from Primitive Knowing. Megan gathers partial ideas, or maybe, rather, Megan partially gathers ideas-it is difficult to tell-around breaking down the numbers into their places. These are partial ideas because Megan is considering only the individual numerals and does not explicitly consider the value of each numeral she is comparing. With this new gathered idea, Megan now looks to the numbers on the cards that are around zero. Both cards have a “1” and a “5,” “big numbers,” according to Megan. She reasons that both the one and five are bigger than zero. Therefore, the zero on both cards must be worth the same.

6.11 Zooming In… *Zero Potential*

Figure 6.9 Megan Zooming In… Zero Potential

Megan is working on the “Fill in the missing numbers” page. After the first two questions:

$$2 + \boxed{3} = 5$$

$$9 = 7 + \boxed{2}$$

Figure 6.10 Two Plus What Equals Five

the page proves to be a challenge for Megan. The sheet quickly moves to two addends on either side of the equal sign. For 8 + 4 = ☐ + 5, Megan places a “12” in the empty box:

$$8 + 4 = \boxed{12} + 5$$

Figure 6.11 The Answer is 12

In placing a twelve in the box, Megan has just demonstrated a common misconception about the equals sign, that it means “the answer” (Knuth, Stephens, McNeil & Alibali, 2006). As Megan moves on to the next question, I try to bring her back to the twelve. I am not surprised that Megan writes twelve in the box; I am surprised that the “+5” has not given Megan pause to think further about the problem. I wonder what and if Megan is thinking about the “+5,” and why she seems to be ignoring this step. I ask Megan:

> “So,...So what's that 5 there for? What are you supposed to do with that 5?”

Even while I am in the midst of asking Megan to revisit the five, she continues to focus on the next question: 5 + 3 = ☐ + 3. She counts on her fingers and places an “8” in the empty box. While Megan is writing the “8” she shakes her head and answers my question:

> “I don’t know.”

Even the tone of Megan’s answer conveys her utter lack of concern at the “+5.” Her “I don’t know” is said in a laissez-faire manner. Even once I point out the problem, she does not feel a need to stop and think. Megan explains that:

> “It's just decoration.”

Students with mathematics difficulties are sometimes taught to focus on key words when problem solving (Clement & Bernhard, 2005), or that extraneous information might be “decoration” when they are working out word problems. I wonder if this is where Megan has built her image of decoration from? Wherever this image arose, Megan’s “decoration” argument can be quite problematic as it has the potential to stymie any conflict she may experience, as it has done here. In effect this idea has effectively shut Megan down. Megan does not need to think about the conflicting idea of “+5” because it is “decoration.” I want to probe more. But since Megan is feeling absolutely no conflict and is reasoning the “extraneous numbers” away, I worry that continued probing might make me lead Megan to the answer “I want.” I have done this before, earlier in our interactions (see section 6.4), where the effect of my intended intervention was to end Megan’s path. Leading Megan to the answer “I want” would be problematic at this point, as she is moving away from a need to gauge my perceptions with every answer. I decide not to intervene at the moment as any interventions would probably stymie Megan’s pathway, again.

Megan remains in this state of equilibrium until the last question on the page, $20 - 0 = 6 + \square$, where Megan encounters both a positional change with the empty space, and a zero. Megan has just argued for the subtraction of zero page, during the previous task, that, “zero, is a number but, like, you can't minus zero?” Now Megan is faced with subtraction of zero yet again. Megan stares at the question:

MN: I don't get this one.
I: Ok, what are you thinking about, what are you thinking?
MN: Twenty minus zero equals six...So when it's six plus something and then it doesn't say the answer. So you don't know.
I: Hmmm. Does twenty minus zero equal six?

Megan’s explanation is in a matter of fact tone. To her, there should be an answer where the “6” is on the page, but the page, of which she has given agency, is not putting an answer in the right place: “it doesn’t say the answer.” Because the problem is still situated within something outside of her control-the page or “it,” Megan is not in the midst of a conflict yet. I wonder if I question Megan further if she will begin to see the contradiction of her answers in the empty spaces. To get Megan to think about the conflict, I turn her reasoning into a question: “Does twenty minus zero equal six?”

MN: I don't know (shrugs). yeah (says quickly then hesitates)...ummm 20, 19, 17 (touching a finger with each count). Oh. (crosses out the 6 and writes 20) That's 20. And 20...plus...(looks at my face)I don't know. Is it 20 plus zero still?

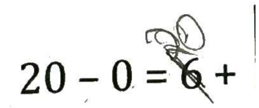

Figure 6.12 Crossing Out the Six

Megan interprets my question as an imperative to calculate. She ignores the zero and begins subtracting six from twenty. As Megan subtracts the numbers, she realizes her focus should be on the twenty and the zero. Megan decides to fix the page. She crosses out the “6” and writes “20” in its place. Now Megan is looking for an answer to put in the empty box. However, there is no equals sign to the right of her new equation, and there is no number to the right of that imaginary equals sign. 20 + □ = a number, does not exist on the page. Consequently, instead of focusing to the right of her new equation, Megan is forced back to the left side-the 20 – 0 =.

MN: (writes 0 and stares at it).
I: So 20 - 0 equals 20 + 0?
MN: (whispers) twenty plus... yeah I think it's zero.

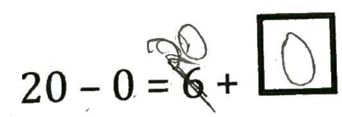

Figure 6.13 The Answer is Zero

Megan places a zero in the empty box. She looks unsure. Megan is building an image around zero here. Zero is acting as a balance (motion metaphor), and zero is being acted upon (object construction metaphor). I decide to push Megan’s thinking by reading the new equation, with the second zero, aloud to her. Megan accepts the zero as an answer. However, I wonder why Megan is whispering. Is she still unsure of something? Maybe it is the zero because “you can’t minus zero.” Notably Megan has veered completely away from her previous reasoning around extraneous information and “decoration.” There is no segue between her thinking before and now. It is almost as if this is a completely new experience, unconnected to the experience before.

I see Megan's hesitance and want to engage her in thinking about the zero. I begin by supporting Megan's thinking. My intention is to move on to ask questions about the zero, but Megan surprises me by revisiting the "decoration":

I: That's a true statement. That would make sense.
MN: But why did they put a six here?

Surprisingly Megan remains conflicted because of the extraneous six. Megan crossed the six out because it did not make sense to her. Instead she placed a twenty-the answer to the first equation on the line, 20 – 0. But Megan is still focused on its presence, and the agency of the "they" who put the six on the page. Megan is giving some unknown entity agency for the equations she is asked to complete. This same unknown entity probably makes "decorations" on the page. Thus, wanting to focus Megan on the purposefulness of the six and away from decorations, I attempt to transfer the mysterious agency to myself. I am hoping that the purposefulness of our interactions will be translated to the task at hand. This backfires. There are many reasons that could explain the backfire. It is not the backfire that is particularly interesting right now, but rather it is that Megan gives more clarity as to how she is understanding what she perceives as extraneous information:

I: That's a good question! Why do you think I made this up. So I'm the tricky one. Why do you think I put a 6 there?
MN: To mix me up.

The person creating the extraneous information, in this case me, does not have good intentions. We purposefully create barriers "to mix her up." It is possible that my use of the word "tricky" leads Megan to her conclusion. But, regardless of whether I lead Megan to say "to mix me up" or not, her conclusion, like the "decoration," remains situated in the peripheral surroundings of school mathematics and not in the mathematics itself. Again, Megan seems satisfied with this explanation and her conflict virtually disappears. There is no more mention of the six, as I must have put it there as a barrier to her success. Megan has come full circle, back to her original thinking about the worksheet.

Postscript to Chapters 4 to 6

I end my narratives of Angela's, Melissa's and Megan's journey here with one caveat. They all begin their journey and end their journey in Primitive Knowing. Without the kind of micro-lens of PK used in the zooming in, one might erroneously conclude that there has not been any growth. One can imagine one of the participants, for example Angela, being assessed at the beginning point to her journey and at the end point, as children experiencing mathematics difficulties so often are- the "pre-" and "post-" test. One can imagine then, the assessor concluding that no growth has occurred, or that the mathematics has not been mastered, or whatever it is assessors conclude when finding the dubious finding that a learner is still at their starting point. The assessor would be wrong. As elaborated by Pirie and Kieren (1992; 1994), Mathematical ideas are complex and they take a lot of movement, back and forth and across in order to grow. Growth happens recursively. Importantly, growth is what occurs between the beginning and end points-it is not defined by the beginning and end points. Each participant experienced growth in between the beginning and end points. Since Angela is our current example, in between her beginning and end points, she has experienced the growth of zero as nothing and zero as a number coinciding with each other.

Chapter 7.0 Findings and Implications

This dissertation explored growth in understanding around zero by learners experiencing difficulties in mathematics. In this section, I first remind the reader of my research questions and then use them to frame the findings resulting from this research:

4) What are the images and prior knowings that children experiencing difficulties in mathematics have about zero?
5) What is the process of change, the growth of understanding that each child passes through?
    a. Are there commonalities between the processes and mappings of developing understandings?
6) How do images and previous understandings about zero thicken through interactions and mathematizing around mathematical tasks?
    a. What specific conceptual areas about zero thicken through mathematizing?
    b. Does specific intervention interrupt pathways and initiate change? And if so how?

7.1 Findings-What are the Images and Prior Knowings that Children Experiencing Difficulties in Mathematics Have About Zero?

Zero is an important conceptual idea in mathematics, however as discussed in Chapter 1, there is not much focus on zero in the curriculum. Learners also experience many problems around learning and image making of zero. Zero is a paradox (Byers, 2007), and reconciling this paradox is difficult. It was, therefore, unsurprising that similar to earlier findings from other studies (e.g. Reys et al., 1975; Wheeler et al., 1983), zero posed a conceptual difficulty for all the participants in this study. A new perspective resulting from this study was, therefore, not that zero poses a difficulty, but how difficulties around zero might pose that difficulty and the lost potential of education to mediate these difficulties.

The developmental progression of zero can indeed be mapped onto the historical progression of zero. Researchers (e.g. Blake et al., 1985; Inhelder et al., 1964) had suggested this mapping, but had yet to map the commonalities between the two. In section 1.2, I mapped the commonalities between what we know of the historical progression of zero and what has been researched around the learning development of zero. Notably, historically the concept of zero as a number was developed before zero's development in calculations and using zero to extend the number system, for example integers. As Kaplan (1999) writes, the use of zero in calculations was a paradigm shift for zero. And more importantly, the use of zero in calculations extends conceptual ideas of the number system (Lakoff et al., 2000; Levenson et al., 2007). It is difficult to study ideas resulting from the extension of a number system without an understanding of the pivotal idea that extends that system. Thus, we learn from history the potential problem of teaching calculations with zero, before children really understand zero as a number.

This problematic aspect of calculating with zero before understanding the concept of zero was clearly demonstrated when Megan and Melissa interacted with the zero question page and when Angela told her number story around zero. Megan initially wrote the answer to 1 – 0 as zero because, "you can't minus...zero," and when Melissa was confused with 0 – 1 she tried to misapply the rule of "regrouping" and utilized zero as a catch all default answer. Finally, Angela had a difficult time even constructing a number sentence with zero as a calculated result, relying instead on an inequality. In light of this finding, it would stand to reason then, that contrary to recommendations by researchers (e.g. Bryant et al., 2006), memorizing the rules for the zero identity property and other zero computations would not aid in understanding procedures for computations. Before learners memorize, it is important that they create zero as a mathematical object, otherwise, like for all three participants, the memorized rules could create barriers to further understanding.

7.1.1 Extension of Zero Findings - Implicit Zero

Lakoff and colleague (2000) list four metaphors for understanding zero (p.75-76):

- Object collection: zero is the collection without any contents. This is the zero that means “empty,” or “nothing.”
- Object construction: zero is constructed (or deconstructed as in the case of subtraction) as an absence of something. There were seven pencils, I took them away. Now there are zero pencils.
- Measuring stick metaphor: zero is the smallest measurement possible.
- Motion metaphor: zero is a midpoint on a symmetrical line of numbers.

Another finding of this project is that there is a fifth grounding metaphor of zero. This fifth grounding metaphor of zero was found through analysis of the historical alignment of the development of zero to the developmental analysis of zero, as well as how the participants in the study interacted with zero in place value. In what follows, I construct an argument for the fifth grounding metaphor of zero utilizing both sources of data.

These above four grounding metaphors are at the root of understanding zero. However, zero is much more complex than just these four ideas. Subsequently, more abstract understandings around zero, termed the conceptual metaphors, build off from these grounding metaphors. Lakoff and Nunez (2000) arrived at the four grounding metaphors through a method called Mathematical idea analysis. In order to perform this analysis, they asked the following important questions (p.8):

- “How much of mathematical understanding makes use of the same kinds of conceptual mechanisms that are used in the understanding of ordinary, nonmathematical domains?
- Are the same cognitive mechanisms used to characterize ordinary ideas also used to characterize mathematical ideas?

- If yes, what is the biological or bodily grounding of such mechanisms?”

All these questions can also be asked around the implicit zero. First of all, implicit and explicit information is foundational to understanding (e.g., Dienes & Perner, 1999). Thus, the first two questions of the mathematical idea analysis are answered in the affirmative. Answering the third question, the big ideas of zero as a placeholder and specifically implicit zero is that:

(a) Even though zero is not seen, or explicit, it is still present. This is demonstrated through concatenation (Fuson et al., 1997) of the numbers.
(b) Place holder zero has a relationship with the surrounding numbers.
(c) Zero depends on its relationship with the surrounding numbers for zero’s identity

Before even beginning to explicate the idea of implicit and explicit zero with the data from the participants, a sound argument can be constructed for the implicit zero grounding metaphor from just the historicity of the progression of zero understanding. As discussed in section 1.2, The first conceptual idea about zero to be discovered was as a place-holder for a place value system. Thus, if as demonstrated in section 1.2, the developmental trajectory of zero follows the historical development of zero, and zero as a place-holder was the first understanding to be discovered of zero, then it only makes sense that ideas around place-holder zero would be grounding metaphors of zero, and not linking metaphors of zero. The question is then, can the grounding metaphors of zero be mapped onto this concept of place value zero, and do the grounding metaphors cover all the metaphors of the concepts of place value zero? In order to attempt to map these ideas together, I next explore the conceptual underpinnings of place holder zero. I begin the exploration through a historical lens, as it is here that the conceptualization of place holder zero first occurred. Additionally, I want to support my historical argument that the concepts underlying place holder zero should be grounding metaphors as well. At the end of these sections, I attempt to map the current grounding metaphors of zero onto implicit and explicit zero.

In this study I have separated place holder zero into the explicit zero and implicit zero. Historically, the Babylonians first used zero only explicitly-to mark an empty space. The explicit zero, translated to our number system, is the zero that can be seen explicitly in the number; for example, the number “3000” has three explicit zeroes. The Babylonian number system was a sexagesimal system and the numbers were not combined as in our Hindu Arabic system. For example, to mark the number “96,” the Babylonians would place one wedge in the first column to mean sixty, three crescents in the second column, each representing ten, to mean thirty, and six small wedges in the third column to mean six (Ifrah, 1986, p.179). To figure out the number, one would add 60 + 10 + 10 + 10 + 1 + 1 + 1 + 1 + 1 +1 to find “96.” Up until their explicit zero, the only way to show the difference between “96” and “66” was to leave a space between the wedges for the sixty and the wedges for the six ones. As soon as we started combining numbers into one string of numerals, implicit zeroes became necessary to understand that the numerals in the string were symbolic representations of much larger numbers than the numerals themselves indicated. The implicit zero, is the zero that is implied in a number with a string of numerals; for example, the number “3652” has three implicit zeroes belonging to the three thousand-one under the six, one under the five and one under the two. In the same vein, the implicit zero under the numerals “5” and “2” also belong to the “6” to make it six hundred. Because place holder zero has these two early conceptions for zero, implicit and explicit zeroes, the grounding metaphors of zero should be able to be tied to both these understandings of place holder zero.

Of the four metaphors, the object collection metaphor can be most clearly mapped onto the explicit place-value zero. After all, it could be reasoned that the explicit zero as a place holder marks an empty container for there being no numeral in the space. And in support of this idea, historically the signifier for that empty space eventually became a symbol for an empty container (Wilson, 2001). Thus, the object collection metaphor can be blended with explicit place value zero. At the same time, implicit place value zero does not map onto the object collection metaphor since it implicitly occupies the same space as a collection. Metaphorically the implicit zero cannot be a container for the space; instead the zero shares the space with whatever other numeral is there.

The object construction metaphor may have a tenuous connection to the explicit and implicit zero. Like in the object construction metaphor, one can imagine that explicit zero can be deconstructed in that space as a result of having taken something away. However, this metaphor would be more in line for calculations resulting from the explicit or implicit zero, and not a grounding metaphor for either zero directly. The other two grounding metaphors, measuring stick metaphor and motion metaphor, do not have even a tenuous connection to either implicit or explicit place value zero.

In the study, the participants had difficulty with implicit zero in different ways. Angela specifically experienced difficulty with the grounding metaphors of zero. The metaphors, that were supposed to create opportunities for understanding, actually posed barriers to understanding. Since that was the case, it must be that the four grounding metaphors of zero do not apply to implicit and explicit zero.

Angela's difficulty with the implicit zero was most apparent in the episode described in section 4.5. As Angela reasoned about the implicit zero, she thought of zero as the largest possible number because it could take on any numeral. However, Angela was faced with a conflict. If her reasoning was correct then why was 5135 larger than 5130? Angela was attempting to apply both the object collection metaphor and the object construction metaphor to the implicit zero, both to no avail. Zero was not a collection; it did not gather the numerals-it sat under the numerals, or side-by-side with the numerals. Thus, her reasoning that zero was a collection for all the numerals fell apart. Additionally, the object construction metaphor constructs zero into the absence of quantity. Angela attempted to construct zero into the abundance of numerals, which she then equated with quantity.

Hence, her conclusion that 5130 should be larger than 5135 was based on grounding metaphors that did not apply to the ideas she was growing.

It was not until Angela and Melissa created a shared understanding around a metaphor for shared space, that Angela was able to add this knowing to her understandings. Consequently, a possible grounding metaphor for the implicit zero is something around shared space. At the same time, this grounding metaphor of shared space does not completely describe the relationship between implicit zero and the surrounding numbers that Angela was trying to articulate.

I note here, that all along I have been utilizing a new metaphor for zero-that of implicitness and explicitness. Although one might argue that implicit and explicit are not necessarily metaphors. I constructed these terms as metaphors as I explained the complex and abstract underlying conceptual understandings of these zeroes. And because of the nature of understanding and knowing, these metaphors were actually co-constructed as I took into account my reader, and you, the reader, are interacting with the product. In this case, any term I would have used instead of implicit, such as hidden or shared, would have become a metaphor for the understanding of implicit zero.

7.2 Findings - What Are the Commonalities Between the Processes of Change, the Growth of Understanding, That Each Child Passes Through?

This project found three types of similarities between the mappings (figure 7.1) of the participants:

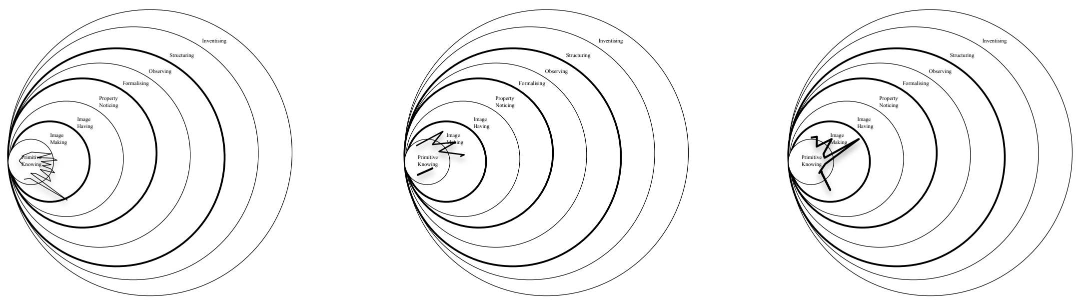

Figure 7.1 Angela's, Melissa's, and Megan's Mappings

First, all three participants hovered around Primitive Knowing and made recursive growth movements back and forth between Primitive Knowing and often Image Making, and sometimes Image Having. Second, all three mappings began and ended in the same space- Primitive Knowing. The final commonality was that two of the mappings had holes in them.

7.2.1 Commonalities Between Mappings: Primitive Knowings

The participants in this research constantly folded back to and thickened their primitive knowings. Thus, Primitive Knowing and thickening were central ideas and a major finding of this project is the commonalities between participants and how they performed their actions of folding back and thickening their primitive knowings. Each participant constantly folded back to their Primitive Knowings and used them:

i) as a foundation to anchor their growth, and/or
ii) as a comparative for their new growth.

Sometimes when Primitive Knowings acted as anchors, the thickening was about prepping a Primitive Knowing to create a blend extending from it. I note here that the study was designed around this integration; it is therefore unsurprising that a blend between zero as a number and zero as nothing would be created. The new finding here, is that learners folded back to thicken their blends before preparing them for the third space where the blend occurred. One example of this type of thickening is Angela. In answering what she thinks zero is, she spent a lot of time in Primitive Knowing gathering ideas about zero as a number and zero as nothing. Angela wanted to blend the two ideas into a third space:

> “*Or is zero just... a number that's nothing?*”

But she had not thickened each of the ideas enough to create the third space yet. As Angela gathered knowings, she also thickened her knowings. Once she thickened her knowings of zero as nothing, and the properties of zero as a number, she was able to then choose commonalities between ideas and create the actual blend that zero is a number and zero is nothing.

Melissa created a third space ready for a blend in Primitive Knowing also during her explanation of what zero is. Melissa had two separate existing ideas, one about decimal numbers and one about zero. Melissa had to thicken her understanding about decimal numbers before decimal numbers and zero could co-exist in the same space. The thickening happened as Megan considered “point zero.” Melissa noted that 0.9 would eventually become 0.0-which is the same as zero. Melissa’s space, where she thickened her understanding, was the moment after her expectation that 0.9 was in a separate realm than zero. The moment that Melissa said 0.0, she had created a blend between decimal numbers and zero. The space for the blend could only have happened after she thickened her Primitive Knowings.

Sometimes the moments of folding back utilized by the participants were to repeatedly revisit the same knowings. Participants would fold back and gather, and sometimes thicken, only some of what they needed. They would then need to revisit the Primitive Knowings again to gather and thicken more understandings around the same idea. Sometimes the repeated revisiting of the same knowing was because the knowing was a strength that could be leveraged, like Megan and her preference for symmetry. Other times the repeated revisiting was because the knowing required more thickening, correcting or changing that only one visit would allow. These Primitive Knowings were too embedded to visit and make changes only once.

All three participants required repeated revisiting of zero as “nothing” in order to thicken their understandings. This type of revisiting, especially of zero, might be viewed as something that was embedded into the study and therefore unsurprising, and an artefact of the design. However, repeatedly revisiting the same understanding of zero was not embedded into the design of the study at all. The design of the study explored many different concepts of zero, for example, implicit zero (reading and writing numbers task), decomposed zero (decomposing number task), and zero as midpoint (number line task). None of these tasks actually required the participant to fold back to zero as nothing. It was the participants who went back to zero as nothing. Additionally, any of the participants could have revisited this understanding only once, thickened it enough to include new ideas about zero and not have had to revisit zero as nothing again. Yet they repeatedly went back to it, and some more than others:

Table 7.1 Revisiting Zero as Nothing

Participant	Tasks Where Zero is Revisited	# of times revisits zero as nothing
Angela	What is Zero? Which is Worth More? Zero Worksheet, Fill in the Missing Numbers Telling Number Stories	12
Melissa	Decomposing Number Task, What is Zero? Zero Worksheet, Telling Number Stories	8
Megan	What is Zero? Zero Worksheet	3

The idea of zero as nothing, especially for Angela and Melissa, was so embedded that it required repeated revisiting, thickening and gathering.

Some times the Primitive Knowings that participants revisited were the ones associated with memorized procedures and rules. Generally, these knowings, although also very embedded as well, were inflexible (McGowan et al., 2013) and flimsy in that there were not very many, if any at all, mathematical understandings supporting these knowings. And because of their flimsiness and lack of support, the participants had hardly any awareness, if at all, when they broke down. Consequently, because the knowings were so unconnected it was easy for these rules to become "catch-alls" to explain away misunderstandings or obstacles. In this way, these "catch-alls" became connected to many things. One example was Melissa and her rule of carrying. In Melissa's Primitive Knowing was a knowing about a procedure for when there are zeroes in the minuend of subtraction problems. Melissa had in her Primitive Knowing that subtracting zero was something "you can't do." Melissa demonstrated her knowing of what you can "do" in section 5.7:

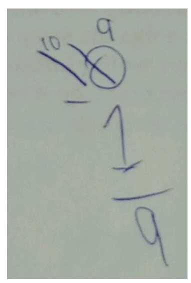

Figure 5.9 Melissa Carrying Zero

Melissa crossed out the one and "carried" it to the zero in the units place. This Knowing worked well for the number one hundred as the minuend. Now, this knowing was confronted with only the number zero in the minuend and no other numerals. Melissa then used this knowing to say why one cannot be subtracted from zero:

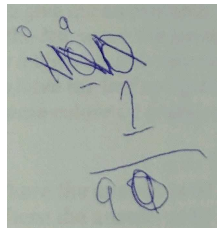

Figure 7.2 Application of Knowing of Subtracting From Zero

This knowing, of carrying when subtracting from zero, was quite flimsy for Melissa. The knowing did not have more embedded connections, for example to positional value and decomposing numbers (Fuson et al., 1997), or conceptual underpinnings of subtraction (Hiebert & Wearne, 2009). If the knowing had had more embedded connections, then at some point while Melissa was revisiting the knowing she would have gathered some of the other embedded conceptions that went along with it. The only knowing Melissa gathered was that 0 – 1 did not work was because the procedure could not be performed on it. Melissa's entire explanation revolves around the procedure and no other ideas. This explanation was in stark contrast to when she gathered other more supported knowings. For example, the ruler with the number line, and infinity with zero.

Megan also talked about an unsupported, "catch all" knowing after she wrote "one" as an answer to 1 – 0:

> "you can't minus zero so it has to stay one. And I don't know what, like, that subject was for but it was a type of math, I don't know what. So, I just guessed."

It would make sense that this unsupported knowing grew out of Megan's experiences with division. It is possible that she learned that a number cannot be divided by zero. Wherever this rule arose from, it remained sitting, unsupported in Megan's knowings, waiting to be gathered when Megan will determine the possibilily of a connection, no matter how tenuous. In essence, this unsupported knowing, because it was unsupported was a "catch-all."

7.2.1.1 Extension of Findings Around Primitive Knowings: The Movement Within Primitive Knowing

Another finding around what occurs in Primitive Knowing from this project was that there can be recursive movements within the mode of Primitive Knowing. This finding was demonstrated with Angela. Angela confronted tension with zero (section 4.6) and as a result experienced shifts and movements within Primitive Knowing. These shifts and movements were because of acts of thickening of competing understandings that were *related to zero*, but *outside of the connections* of the network of zero. Zero specifically brought these shifts to the fore because some of the understandings in its network are paradoxical (Byers, 2007) to each other. As Angela experienced, prior unconnected and opposed knowings can compete against each other. In the midst of this tension, Angela remained in Primitive Knowing, she did not move into Image Making, she could not until the competing knowings were at least somewhat thickened. The type of thickening that Angela had to accomplish was different than other acts of thickening. This thickening was in preparation for a reconciliation of a tension between two ideas **within** Primitive Knowing. This preparation could not be static, it was full of tension, and therefore, recursive movement occurred within Primitive Knowing. In her thickening, while in Primitive Knowing, Angela did not reconcile the two competing ideas. In any case, one could argue that the act of reconciliation would mean drawing connections between the knowings, and would therefore be a growth within Image Making.

7.2.2 Commonalities Between Mappings: Beginning and Ending in the Same Space

All three learners began their journey and ended their journey in the same space-in Primitive Knowing (see Figure 7.1). Importantly, what this dissertation clearly shows is that growth is what happens in between the beginning and end points-growth is not defined by the beginning and end points. This idea is in contrast to the theories and ideas around categorizing and labeling learners with mathematics difficulties discussed in chapters 0 and 1. As already discussed, the very labeling of a disability or difficulty, as the case may be, is mostly because of an overreliance on identifying beginning and endpoints of knowing and understanding.

Without the kind of micro-lens used in the "zoomings in" in chapters 4, 5, and 6, one might erroneously conclude that there has not been any growth for any of the participants. Relating this to common practices now, one can imagine any one of the participants being assessed at the beginning point to her journey and then again at the end point, as children experiencing mathematics difficulties so often are. And then the assessor concluding that no growth has occurred, or that the mathematics has not been mastered, or whatever it is assessors conclude when finding the dubious finding that a learner is still at their starting point. The assessor would be wrong. Growth occurred between the starting and end point, not at the end point.

In any case, beginning and end points on a model of growth are discretionary. Growth is not absolute because it is continuous and not discrete. Pathways of growth came before the first marker and pathways of growth will come after the last marker. The tracker/observer only chooses where on the continuous model they will start and end tracking growth. The point at the beginning and the point at the end of my analysis indicates a choice by me, the researcher; it does not indicate a beginning nor an end of the whole pathway. At the same time as I constructed a picture of the participants through this dissertation, each participant had multiple pathways before this study and they continue on after this study. This idea is inherent to PK theory, as Primitive Knowings would not exist without prior pathways.

Each participant experienced a number of growths between those beginning and end points. The complete description of each of the participants' growths can be found in their respective chapters. Instead, in the following paragraphs, to support my argument that growth is what happens between the beginning and end points, I track only the starting point and ending point of each of the participants. I provide an example of a growth that occurs, relative to the start and end point, in between the two points.

Angela began her journey (section 4.4) in this study with a concept of zero as a number side by side with a concept of zero as nothing in her Primitive Knowing. Angela ended her journey (section 4.10) in this study in Primitive knowing, again revisiting zero as a number and zero as nothing. However, Angela's understandings around zero and nothing are now more robust then they were to begin with. Now, she has in her Primitive Knowing that zero as nothing and zero as a number can coincide beside each other.

Similarly, Melissa began her journey (section 5.4) in this study in Primitive Knowing, visiting the idea of decomposing numbers with zero. She was beginning to consider the object construction metaphor of zero. Melissa then began to move into Image Making utilizing this metaphor. Melissa ended her journey (section 5.7) in this study again in Primitive Knowing. At the end of her journey, Melissa was revisiting the object construction metaphor of zero. At the same time as she has started and ended in the same place, Melissa has begun to develop the measuring stick metaphor for zero.

Megan, too ended in the same place she started. Megan began her journey (section 6.5) in this study iterating a distinction between zero and nothing. Megan ended her journey (section 6.11) in this study back in Primitive Knowing, as she focused on the agency of the worksheet instead of the mathematics. The comparison for Megan's growth can best be seen in the moment before she ended back in Primitive Knowing. Just before this, Megan was using the affordances of zero to begin to build an image of equations and variables.

7.2.3 Commonalities Between Mappings: Holes in the Mappings

An interesting commonality and finding was that Both Megan's and Melissa's mappings had holes. A hole occurred in Megan's mapping when just after creating two piles for seventeen, I asked her what the number sentence would be for her piles. I led Megan to an answer that subsequent questioning showed she had no meaning for. At the point of leading Megan to an answer, we were communicating from different spaces. I was communicating around the mathematical symbolism of the actions Megan had just taken, and Megan was communicating around the symmetrical symbolism of the actions she had just taken. As discussed in section 2.2, "Knowings" and "understandings" are very much embedded through interactions in the moment they emerge. Knowledge and understanding are not habitants of a person's mind, but in the moment processes of these interactions, evolving the person and their environs (Proulx, 2013). Consequently, a gap was created because we, the two participants of the in the moment processes, were communicating from different spaces.

Melissa's mapping also had a hole when after Melissa realized that point zero is zero, I brought her back to infinity as zero. Infinity was too abstract for Melissa to explore on her own and she became stuck in the abstractness of her knowing of infinity. At that point, Melissa jumped back from Image Making to Primitive Knowing. The hole in Melissa's mapping did not occur because she alone could not comprehend infinity, but because our interaction around the idea of infinity and zero stopped. Melissa was thus left to jump back alone to her Primitive Knowings. The interaction between Melissa and myself that did occur:

MA: ... 'cuz I was thinking that like it's nothing. Like we don't exactly know what infinity is. What is zero? I don't know (*laughing*).
I: (*laughing*). So zero's like infinity. Something kind of out there?
MA: (*nodding head*)

was situated after the jump back to Primitive Knowing and not in the Image Making where Melissa had just been before. Specifically, the interaction was a clarification of where Melissa was in Primitive Knowing.

In both cases Megan and Melissa experienced barriers to further growth, that might be interpreted as negative. Consequently, it might seem from the above descriptions that holes in mappings are a negative consequence of miscommunication. To clarify, because growth is continuous and recursive, the knowing at the edge of the hole, in Megan's case the symmetry of the piles and in Melissa's case the idea of infinity, still have the possibility to be revisited later. Holes are a natural consequence of communication and the process of growth. Similar to the communication between Megan and myself in that moment, it cannot be that two people will always be in the same space when communicating their knowings and understandings. And similar to Melissa and myself in that moment, not all knowings have to be explored in the moment. Because of time constraints, I made a conscious choice not to engage with Melissa in her Image Making at that moment.

The hole in both the mappings occurred not because of anything intrinsic to the participant or the researcher, but because of the interactions around the knowing. The hole in the mapping itself did not even necessarily signify a barrier, but only conveyed a moment in time when the communicators involved in "knowings" and "understandings" situated themselves in different spaces, or had different intentions. Growth occurred after the hole for both Megan and Melissa. And, because of the continuousness of growth, growth occurred after any hole in a mapping.

In light of this idea around holes in mappings and the continuousness of growth, I would like to revisit and explain some of my previous wonderings, especially because these wonderings might give credence to the idea that holes in mappings are barriers to knowing and not natural consequences of inter-communication. The wonderings I am referring to were around Melissa's difficulties and whether they could at least be partially caused by jumps to Primitive Knowing without folding back, similar to the one that caused the hole in her mapping. In order to analyze and account for anything in this study, the communication around thinking had to be explicit, that is, shared between two people, the participant and myself. I could not analyze anything I did not see (Pirie et al., 1994). I could only question and wonder and try to devise questions or tasks to make the implicit become explicit. My entire role in this study was to find ways to make the implicit, explicit. Because of this role, I orchestrated a lot of what might have remained implicit, explicit. My wondering was based on this role I had, but also that communication of knowing and growth of knowing also occurs at an internal level. This happens through the communication that occurs with oneself during self-talk (Sfard, 2008). My wonderings were, therefore, inquiring into what might happen for Melissa when there is not someone with her in the role of making her implicit, explicit. Could some of her difficulties only be the result of a lack of "other" providing access? This type of hole did not occur when Melissa explored concepts with her peers during the intervention sessions[21]. This juxtaposition, between holes and continuous lines in mappings, and in the moment interactions with peers, would be an area for further research, especially around those experiencing MDs.

7.3 Findings-Do Specific Interventions Interrupt Pathways and Initiate Change?

The words "specific intervention" can be thought of in more than one way. One way to think of "specific intervention" is as a prescriptive strategy that intervenes and makes change. I think this may have been the one I was thinking about when I originally asked the question. However, the finding that resulted from this study actually looked at a very different aspect of intervention, one more in line with the recursiveness of the knowledge and understanding I was studying in the first place. I found that the intervention is not a tangible strategy that one can do, but a space or juncture where access to the mathematics and change becomes possible. Still, the expectation of "intervention" as a strategy plays an important role in these spaces, and in what follows I explicate these spaces and their role in intervening with learners experiencing mathematical difficulties.

Because the spaces occur in small moments and are the result of micro-looks, I call these junctures "micro-spaces of growth." I have chosen to use this new label of "micro-space" for two reasons:

(i) As will be elaborated upon below, these micro-spaces are different from the larger changes typically reported in research.

[21] See section 4.5 for an example of Melissa and Angela exploring the value of the implicit zero

(ii) I have not found adequate language to describe the difference between small changes and large changes. This is important because the whole underlying purpose of this research is to demonstrate that change happens on a small scale, and how change happens. And, as discussed in chapter 0, language presents a difficulty especially for this population of learners. I therefore felt it important to delineate non-deficit language associated with changes on a small scale.

These micro-spaces exist and are sandwiched between expectation and result. Expectations of, for, about and within learning and growth play an important role in this dissertation. There are expectations that surround success and achievement-Angela, Melissa and Megan all have expectations around their success with the tasks. They also attempt to gauge my expectations of the task and their success by looking to my expressions. There are also expectations of mathematics: Angela expects to hate mathematics, Melissa expects to struggle with mathematics, and Megan expects mathematics to take a long time to understand. Then there are the expectations that occur within learning, just before a micro-space of growth occurs.

All three learners experienced growth through, and sometimes during, micro-spaces. Micro-spaces are those small, short moments in-between anticipation, or expectation of a result and that result. These micro-moments are not the large AHA moments, "moment(s) of insight on the heels of lengthy, and seemingly fruitless, intentional effort " (Liljedahl, 2005, p.220), that mathematics education researchers describe about themselves (e.g. Tzur, 2001), about mathematicians and students (e.g. Liljedahl, 2005), and recommend as a strategy in problem solving (e.g. Mason et al., 2010). Instead, micro-spaces of growth are small and answer the question of what growth on a small scale looks like.

These small moments in the micro-spaces are not solitary moments-they are moments realized through shared understandings, verbally and through gestures. These are the moments that have the potential to resist an expectation or anticipation. Sometimes that resistance results in a cognitive conflict (Movshovitz-Hadar et al., 1990), and sometimes that resistance results in a detour of thinking: where the learner gathers, plants and thickens understandings. These micro-spaces are not always resolved spaces. Sometimes growth, or movement, occurs without a resolution. This type of growth in the micro-space remains in limbo, ready to be gathered again. Often, the micro-spaces are in Primitive Knowing, where the learners fold back to gather prior knowings. And sometimes connections are drawn between the spaces and the micro-spaces move into Image Making. Other times, connections are drawn within the moments themselves.

Micro-spaces often require space between the time of creation and reflection. The spaces are often too small to identify at the time they occur. And to make it even more difficult to identify, the relationships between the micro-spaces are not always temporally linear. Consequently, sometimes these micro-spaces can only be identified retroactively through considered reflection. One way of recognizing these spaces is through the learner. Sometimes the learner will suddenly present a mathematical image, and then through their reflection they retrace their various micro-spaces of collecting and gathering knowings and understandings. At the same time, micro-spaces can also be identified by the observer through the repeated revisiting of these small moments in light of later occurrences. Two tools in the methodology for this project allowed for the identification of the small spaces: (i) Powell and colleagues (2003) seven phases of data collection, where data is identified revisited often, and coded, and (ii) the PK model, because of its affordances to track changes on a small scale, makes many of these moments explicit and therefore was a useful tool in illuminating the micro-moments. After using these tools, these moments can be turned into a narrative of growth.

In this study the micro-spaces were either fluid or inflexible spaces. When a micro space was fluid, the space between expectation and result was respondent to interventions allowing for subsequent growth to occur. In fact, whenever growth occurred, it occurred in tandem with the creation of a fluid space. In this type of space, the outcome was not determined before the encounter. This makes the fluid micro-space more complicated in comparison to the inflexible micro-space. In fluid space there is more potential for tension because growth is not pre-determined and may result from a cognitive conflict (see section 7.3.1). Note the depiction of fluid micro-space in Figure 7.3. The arrows move in every direction, indicating the potential that interventions have for growth. The space also has dotted lines surrounding it, indicating the permeability and flexibility of the space:

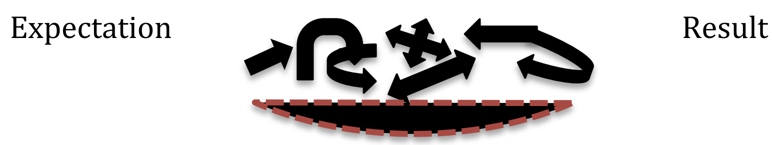

Figure 7.3 Fluid Micro-Space Between Expectation and Result

The inflexible micro-space does not have much variation, as the outcome of an interaction in fixed micro-space is fixed. Because of this, the space is not respondent to interventions. Note in Figure 7.4 that there is one linear arrow between expectation and result, and there are no lines indicating the permeability around the space. Instead the space conveys a denseness.

Figure 7.4 Inflexible Micro-Space Between Expectation and Result

This type of micro-space was resistant to change because reasoning in this space tended to be circular and within parameters that may not make sense mathematically. A prime example of the inflexible micro-space was with one of the new findings of this research: contrary to recommendations by researchers (e.g. Bryant et al., 2006), memorizing the rules for the zero identity property, and other zero computations, did not aid in understanding the procedures for computations nor in understanding zero. Instead the memorized rule without meaning created an inflexible micro-space resistant to intervention.

Melissa's rule of carrying and borrowing when subtracting from zero (section 5.7), created an inflexible micro-space as she tried to explain that the only reason one could not subtract one from zero was that there was nothing to carry from. In this case zero was not a numerical entity anymore, it was something that needed to be replaced by a number carried over from the other numbers. In the end, Melissa was satisfied by her reasoning and did not express the need to reconcile this problem further. There was no intervention because the pre-determined inflexible space created by the memorized rule engulfed the potential for intervention.

In terms of a memorized rule for zero Megan iterates (section 6.9):

> "(Because I learned) something and it said like zer... ummm you can't minus zero so it has to stay one. And I don't know what like that subject was for but it was a type of math, I don't know what. So, I just guessed.

Once Megan brought forth her rule, wherever it came from, she did not need to think anymore. She retrieved her rule, applied it and her inflexible micro-space ended.

That micro-space between expectation and result was not interrogated by any of the participants for memorized rules-it became predetermined, inflexible and resistant to intervention. The space was so inflexible, that if the result was actually different than the participant's expectations, then they identified their expectations as being incorrect-not the result. This means that the micro-spaces resulting from their memorized rules were static, and already determined before the encounter.

As the examples above demonstrate, the results after the expectations did not make sense mathematically. However, instead of there being space for reasoning further mathematically, each participant negated any misgivings I tried to supplant and moved on. As a result, the participants gave credence to their overall ideas despite their suspect results. In what follows I expand on the different fluid micro-spaces of growth. I do not include every example as it would be impossible. Fluid micro-spaces occurred at every pivot of growth in the mappings. Instead I include a few illustrative examples of each fluid micro-space.

7.3.1 Micro-Spaces Through Time, Cognitive Conflict, Detours of Thinking and Unresolved Spaces

Melissa experienced the type of micro-space resulting from a cognitive conflict during the number line task (see section 5.6). She had two separate ideas: zero comes before one and so do decimal tenths. Melissa had not created a blend between these ideas yet-zero before one and decimals tenths before one did not exist on the same number line yet for Melissa. Thus, she does not expect these ideas to converge into one concept on the same number line. Melissa's micro-space of growth was the moment immediately after her expectation that following point nine to its logical conclusion would be a separate conception than zero being a starting point on the number line. The moment Melissa said her conclusive statement of "point zero," she had already experienced her cognitive conflict and created the blend that point zero ***is*** zero. Notably Melissa's conclusion was still tentative. "Point zero" was said with an inflection-she still questioned her result. This was because Melissa was still in Image Making at this point. The micro-space of growth is just between her expectation that zero and point zero are separate entities on the number line, and her result that they could be the same.

Angela also experienced the same type of micro-space resulting from a cognitive conflict when she was asked to explain what zero is. Angela explained zero is a number, and she explained that zero is also nothing. Angela conveys her expectation that zero can be both a number and nothing at the same time. However, in the moment after her expectation was voiced, in the moment of her micro-space of growth, Angela experienced a conflict when she could not explain how zero can both be a number and be nothing. This is the moment just before Angela came to the result that she did not have an image of zero as a number and zero as nothing, together. Consequently, Angela detoured: gathering, planting and thickening as she oscillated between many moments of expectations and results. For example, the following comment took Angela only fourteen seconds to say. But during those fourteen seconds, she was moving back and forth between three different expectations and results.

> "Cuz if you have like 1 piece of paper, that's 1. But If you have zero, like... (*looks to the side*), there's no zero... Like you can't say you have zero paper if you have paper. 'Cuz like zero isn't a number. Well (*looks up*) zero is a number, but it's nothing."

There are three moments of expectation in these short 14 seconds: i)1 piece of paper has one-to-one correspondence, so zero must have one-to-one correspondence as well, ii)zero does not have one-to-one correspondence so it is not a number, and iii)zero is a number.

A lot of growth occurred for Angela in this small space. Angela thought she already had that third space (Fauconnier et al., 2002) for the blend where zero as a number exists together with zero as nothing. However, as she oscillated between expectation and result she began realizing that she could not reconcile her ideas into a blend just yet. Angela was still gathering concepts that support that third space while she oscillated through these small micro-spaces between expectation and result. Thus, the third space she thought she had, was actually created here in the moment just after her final expectation: "zero is a number, but it's nothing." At the same time, Angela did not plant anything in the third space yet. She only gathered ideas surrounding zero as a number and zero as nothing and their potential to exist together in the same space. This was a micro-moment of growth that remained somewhat unresolved. Ideas were gathered, but there was no resolution. Angela will require all the growth that has the potential to occur after this moment in order to fill the third space and create an image. Importantly the growth that can happen after this moment is not a necessary condition of the micro-moment of growth in the first place.

Angela may never fill her space thus she may never create that image. This non-resolution does not negate the growth that has just occurred.

Similarly, Melissa also experienced similar moments of oscillating and gathering, planting and thickening as she detoured around her explanation of what zero is. Many of Melissa's expectations were not voiced, but were communicated through her gestures:

> "Infinity. The zeroes... (*looking up*). Infinity would beeee (*hand goes up, opens, like about to grasp*)... I feel like because... (*moving hand like she is grabbing something*). Zero's nothing (*puts hand down*)."

Melissa's first expectation was that a connection could be drawn between zero and infinity. This expectation then got put aside as Melissa looked up and entered her first micro-space. She had an expectation around zeroes: "The zeroes…". Melissa gathered something as she looked up, though I am still not sure what that was because I was only able to notice the gathering and not the contents of the space. Whatever the gathering was, it caused Melissa to oscillate towards gathering a definition for infinity. This was Melissa's third expectation- that infinity can be defined. Next, Melissa in her new micro-space, tried to physically grasp that definition of infinity, however the definition remained elusive. In that moment before realization, Melissa experienced a micro-moment of growth around infinity. This growth did not further spur Melissa on to collect and plant understandings, instead she came to the conclusion that "Zero's nothing."

Megan's experience of explaining what she thinks zero is, was a prime example of the gathering and planting of micro-spaces of growth that occurs over time. This example also points to the retroactivity that is involved in identifying those moments. In section 6.8, I present a full discussion of Megan's micro-moments and her gathering of understandings. Here, I quickly review the occurrence and connect Megan's activity of collection with the aforementioned examples.

In order to create an explanation for why zero is a number, Megan collected four relatively temporally distant moments. Each of these moments were planted in Primitive Knowing, at different times, for future use:

i) number lines can be open-micro-moment in a short exchange in the beginning of one of our sessions,
ii) lines can be continuous-micro-moment during class,
iii) zero does not have to be at the beginning of a number line- micro-moment during one of our previous exchanges, and
iv) the number line can be continuous-A blend of two micro-moments, one from during class and one from one of our sessions.

I was privy, that is I was physically present, for all these spaces of growth, and importantly, my primary role during these moments was to observe growth. Yet each of these micro-spaces of growth remain unnoticed by myself until the moment Megan gathers them all together to explain her reasoning.

The space between expectation and result was then an important moment for all three learners. It is then also within all these micro-spaces, Angela's, Mellissa's, and Megan's, the time in-between, that potential for equity existed.

Thus, a major finding of this research is that there is a space where pathways have the potential to be interrupted. In the next section I connect this potential interruption to interventions that may initiate change.

7.3.2 Interventions For Micro-Spaces

The small space in between, the micro-space of growth, is important in this study in two ways. First, it is in the micro-space that potential for change occurs, and second, it is in the micro-space that decisions are made and the potential for interventions can be realized.

Interventions are typically prescriptive strategies that intervene before a difficulty to prevent that difficulty or after the difficulty has occurred to mitigate the difficulty (e.g., Gersten, Jordan & Flojo, 2005). Instead, I conceive of intervention as more dynamic with the purpose of activating the micro-space. I utilize a lens of equity to explore micro-spaces as a space of intervention. I use three of Gutierrez's (2012) dimensions of equity-access, identity and power to explore the dynamic space of intervention between expectation and result. I begin my discussion of expectations, micro-spaces and equity, with Gutierrez's dimension of power. I specifically explore power in relation to the positioning (Wagner et al., 2012) of the learner and their growth, and in relation to the very noticing (Mason, 2002) of the micro-space where the growth resides. Note positioning as a concept is utilized a lot in research, with different meanings. For this study I am influenced by Wagner and Herbl-Eisenmann's (2009) conception of positioning as narratives surrounding actions within social structures. I first utilize the story of David, discussed in chapter 0, to explicate these ideas. I then discuss these ideas in relation to the data from this study.

If we revisit the David episode that led to this dissertation to begin with, in light of the finding of micro-spaces, we can see how power, specifically positioning (Wagner et al., 2012), played out in the micro-moment between expectation and result. Because of David's diagnosis and the perception of David's lack of ability in mathematics, he was positioned as someone not able to do mathematics. The expectation then was that he could not think or do advanced mathematics. I was present for the result of David's moment of growth, when he explained "64" as his answer to 16 x 4. The principal, on the other hand, was not physically present for David's moment of growth nor the result of that growth. She did, however, have her own moment between expectation and result. As I explained the moment to the principal, she expected to hear that David could not reason abstractly in order to do the multiplication. The result was, from my description, that David did indeed reason abstractly and do the multiplication. That micro-space between her expectation and the result had tremendous potential for equity. But equity was not to be. The prevailing discourse remained (Siegler, 1998) and like the teachers' reactions in Houssart (2004), David was repositioned as unable and his moment as an anomaly. Two opposing actions went into this outcome: the noticing of the space and the negating of the space. Seemingly paradoxically if I had not noticed the space, the space could not be negated.

7.3.2.1 Interventions For Micro-Spaces-Power

Power is an important silent operative in the potential for interventions. Because of the power an "other" possesses, the moment of an other noticing that micro-space of growth is a very vulnerable time for the learner. The other could or could not participate in an interaction for understanding. The other could or could not accept new growth. The other could or could not surpass their own expectations to experience their own micro-space of growth. The participants in the study seemed to recognize their own vulnerability, as each enacted many different strategies to gauge my approval, most commonly looking to my expressions.

Megan especially, had elaborate actions to protect her vulnerability, including checking in with her peers and reading my notes. During the decomposing number task, Megan experienced a vulnerable moment when I asked her to repeat an answer I didn't hear. Through later viewing of the video data, I saw a notable difference between Megan's answer before I asked her to repeat herself-a statement of knowing-and after-a tentative statement of possibly knowing, using words like "I'm guessing." In protecting her vulnerability, Megan often turned her expectations towards me and my wants instead of the mathematics. As a result, her micro-spaces revolved around peripherals to mathematics and not the mathematics. She talked about what the worksheets wanted from her, as if the worksheets had their own agency. For Megan, I specifically noted that her thickening of her Prior Knowings was dependent on whether she could essentially overcome her vulnerability and accept her own thinking.

Angela had the least amount of actions, and notably her mapping has more movements than Melissa's and Megan's. At the same time, Angela's growth and path is different as well because she used her actions to protect her vulnerability, just before her micro-space, or just after-often saying “I don’t know.” These words had the effect of stopping her expectation in its tracks. Once Angela would say the words, “I don’t know,” there could be no result and. therefore, no micro-space of growth. Through Angela iterating the words like “I don’t know” after a result, she would end up limiting the extension of her micro-space. Notably, toward the end of our time together, Angela still said “I don’t know,” but these words did not have the limiting effect they did at the beginning of our sessions. These three words became an artefact, a habit, of her vulnerability. They remained even though their outcome did not.

In light of the discussion on interventions, equity, micro-spaces and power, I want to take a moment here and explore an idea mentioned in section 6.5, that of “thinking independence.” The context for this idea was Megan and her need to please and anticipate my needs. At one point, Megan had a moment of unencumbered thought that I termed thinking independence in my analysis. This was a moment of thinking independence because Megan was unencumbered by her need to please me. Independence then, in this case did not mean thinking separately, or unconnectedly, or disjointedly from my input. I have already written in section 2.2 about the inter-connectedness of all knowings. Instead thinking independence meant thinking that occurs without the worry of the learners’ vulnerability of an other’s expectations. Tying this back to the reconception of interventions: in this dynamic intervention, the intervention (all the interactions that promoted this independence) was an activation of the dynamic micro-space that allowed for growth (the opening of the micro-space and loosening of Megan’s need to please me).

Although Melissa, Angela and Megan eventually stopped attempting to gauge my acceptance of their answers, and stopped saying “I don’t know,” their vulnerability never really disappeared. As long as there will be someone with expectations noticing their space, they will remain vulnerable. Even though I was supposed to be impartial as a researcher, I could never be truly impartial. I, even as empathic coach, am an other-with my own expectations.

The retroactivity of observations and identification of micro-spaces has another layer of expectations layered onto them. In my role as empathic coach (Metz et al., 2015), combining researcher and teacher, I also had my own expectations. As a researcher, in order to build on previous research and situate this study, I conducted a literature review. This act, of building onto and building up knowings of previous research, led to my expectations about possible outcomes for this project. In my second identity, as teacher, I also had expectations. These expectations came from years of experience of working with, reflecting on, and planning for learning with children. Both these types of expectations often led to the wonderings discussed throughout the analyses of the mappings. Di Martino and Baccaglini-Frank (2017, p.43) note a dichotomy between expectations of a teacher and expectations of a researcher:

> "While the unexpected was of interest to the researchers from the very beginning of the project, for the teachers this was not the case; what teachers seemed to attend to was an evaluation of the didactical significance of the unexpected, of its being connected to a significant goal, and of its being related to something done in class (and therefore being responsible for it)."

My teacher and researcher expectations also all occur in micro-spaces, between some sort of initializer or expectation, be it question, thought or task, and the eventual response to the initializer. In each of these moments, I too have the possibility for interventions. Thus, each kind of my own expectations both has the potential to create a micro-space of growth for myself, and also creates a constant vulnerability for each of the learners in the study. And with each vulnerability comes a potential for positioning the learner.

7.3.2.2 Interventions For Micro-Spaces-Identity

Closely attached to this vulnerability and also impacting interventions in the micro-space is the dimension of identity. Identity is an amalgamation of narratives both about the self and about mathematics and is, like positioning, constantly constructed and reconstructed. Identity often results from positioning (Heyd-Metzuyanim, 2017). In this case, identity can play out in one or both of two ways in the micro-space: expectations through the eyes of the observer, constructing and reconstructing another's identity, and expectations through the eyes of the learner, constructing and reconstructing their own identity.

After the first sessions, I began pairing the participants in small groups of two to four for their intervention sessions. I did not plan too far in advance as I wanted to ensure the compatibility of groupings. With all Megan's group interactions, no matter who she was paired with, or how many people she was grouped with, she played a subservient role to the others in the group. The others in the group, whoever they were, seemed to accept and even acknowledge their roles. Often times the others simply took over-be it taking over the thinking process or filling out worksheets. And, like Heyd-Metzuyanim (2013), I found myself drawn into that role as well. Megan dutifully followed along, not once questioning the role of the other, or her own. This subservient role, meant that I could not observe Megan participating in interactions promoting growth in mathematics. Notably, I did observe the reconstruction of Megan's micro-spaces around identity, where Megan expected to be subservient and the result of her subservience. Each of those moments was another expectation on all of our parts, Megan's, the other participants, and my own. Instead of there being potential for growth, we all kept reaffirming her identity. In a dynamic space, this reaffirmation is also an intervention, although not a positive one in this case. We dynamically created the space for Megan to reaffirm her identity.

Interestingly in the one-to-one situations between Megan and me, this type of reaffirmation did not occur. There are three reasons for this. First, because I had not seen Megan interact at a small group level, I did not have any expectations (yet) as to how she would act with me. Second, the focus of our sessions was only on Megan. Megan tried to be subservient to my wants at the beginning of our sessions, but eventually this disappeared as I made a conscious effort to create a space for intervention and move her away from that response. Third, my focus in the group sessions was fractured. I was observing multiple people at the same time. I was also looking for those micro-spaces of growth around zero.

In retrospect, it was quite easy to focus in on everyone but Megan, because that was where the most obvious growth was occurring.

Melissa and Angela both had their identities constructed in the micro-space as well. Melissa experienced difficulties in mathematics and saw herself as one who experienced difficulties. Her identity though, was not supported by the expectations of her teacher or the school. I, too was surprised that Melissa experienced difficulties, especially considering her upbeat manner when describing them. Melissa's barrier to learning mathematics was consequently invisible-it happened outside of the expectations, and accountability practices of teachers. Angela discussed her identity at our very last session where she said her most frustrating moments were from the probing that I had done. At the same, Angela also admitted it was in these moments when she learned a lot. Angela already had an identity as a "learner of mathematics," and those micro-spaces of growth in our sessions reconstructed this identity for her. At the end of our sessions together, Angela gave me a thank you card she had created. Angela was thankful for the understandings she learned. Her identity as learner of mathematics was expanded. In contrast to Megan, Angela's intervention created a positive space for mathematics.

7.3.2.3 Interventions For Micro-Spaces-Access

Because the learner is constantly positioned and re-positioned, the dimension of access, then, plays a role in the equitable potential of interventions in micro-spaces. As already explored in section 2.2, understanding occurs through activity, or communication with an other. Then access to understanding and knowing also occurs through activity, or communication with an other. Thus, there were a number of points in this study where interventions resulted in participants gaining access to their own micro-spaces of growth through our interactions. Sometimes the access was gained just from participants voicing their space of growth out loud, and other times the access was gained through me re-voicing understandings, or from my questioning and probing of the learner's thinking. Sometimes one, two or all three types of the aforementioned access could be realized through one exchange. The commonality across all the three different points of access was that all three points, whatever their origin, became shared understandings.

When Angela encountered 0 - 1 on the zero worksheet, she was unsure what the answer should be. Angela decided first that zero should be the answer to the question, and then she decided that one would be the answer. Angela did not share her thinking with me, and we had no shared understanding yet. I, therefore, probed Angela's understandings as to why she chose zero first as the answer and then she chose one. My (intervention) question, asking Angela to clarify her answers, caused her to enter and re-enter micro-spaces of growth. Angela's thinking became explicit and shared. As a result, she created access for me, and herself at the same time:

> "Because... zero..... take away 1 (*motioning with hands*)...you're gonna have nothing here and you take away...1... (*looks to the side*). Oh no it's zero because how do you take away one if you have nothing there? But I was thinking zero also because... maybe it's... zero. I don't know."

As Angela explained her moments that led to zero, she veered back to zero as the answer again. My intervention of asking Angela to revisit her answer of one had in effect given her access to renegotiate her answer again. When Angela was explaining zero she had an expectation that one was the answer, and she expected to be able to negate her reasoning for zero as the answer, and then to reason one as the answer. However, something occurred between Angela's expectation that one was the answer and then her answer of zero again. That something occurred because Angela had access to her reasoning. At the end of her communication, Angela became vulnerable again. Up until the point of Angela's vulnerability, she gave me access to her moments, and in so doing she gave access to herself as well.

An example of giving access to the micro-space through re-voicing or paraphrasing was an exchange I had with Melissa around the number line. Melissa was perplexed as to why the one was in the middle of the number line that I presented to her. Melissa felt the one should have been at the beginning of the number line. She attempted to explain her reasoning, that I then subsequently intervened and paraphrased in the form of questions. I asked Melissa if she meant that the decimal numbers needed to go before the whole numbers, like on a ruler (section 5.5). By voicing the metaphor of "ruler," for Melissa's understanding, I had provided access for Melissa to her own micro-space. We, in turn, developed a shared understanding with shared language about decimal numbers and the ruler. This micro-space though could only be identified retroactively. That shared understanding did not come to the fore until later, as described in section 5.6, when Melissa used these spaces she had gathered-decimal numbers and the ruler-to explain how zero is the first number on the number line. In this later episode, I intervened in the micro-space by probing Melissa with a question about her thinking that caused her to extend her thinking to its logical conclusion:

"No. Probably point zero. (*looking around confused*). (*laughs*)… Wait one..."

The intervention resulted in a micro-space being created here, already discussed above in section 7.1.1. What is important is that just through iterating "point zero," Melissa gave herself access to that space.

Interestingly, access for Megan happened in her explanations of why she was answering what she was answering. These explanations arose from her need to please me with the answers I wanted. Through elaborate strategies Megan was able to anticipate the answer "I wanted," but not her explanation of that answer. It is in those spaces that we built shared understanding and Megan gained access to her space. A good example of this is when Megan says that zero is a number and nothing. In order to explain her answer, Megan had to draw on previous knowings and in doing so created shared understandings. Essentially in this case, Megan performed her own intervention by creating access to her micro-space.

An especially strong example of probing and questioning giving access to reasoning occurred with Megan. For the equation 8 + 4 = ☐ + 5, Megan put a “12” in the empty box. Attempting to intervene and create some disequilibrium, I asked Megan what she thought the “+5” was there for. This question seemed to essentially backfire, as Megan dismissed the “+5” as decoration. Here again, I only realized retroactively that my question about the five gave Megan access for a later micro-space when she encountered 20 – 0 = 6 + ☐ . I had originally thought that because I had not immediately identified a micro-space of growth for Megan, she had not experienced an intervention. However, although unidentifiable at the time, Megan had experienced an intervention. A micro-space of growth was created, but could only be identifiable at a later time. As Megan tackled the problem, she changed the equation to 20 – 0 = 20 + 0. Megan should have been satisfied with her answer, especially given her reliance on mathematics on the page being “decoration.” However, Megan was not satisfied and she wondered out loud:

“But why did they put a six here?”

Although Megan outwardly dismissed my question about the “+5,” she had still gathered it. And because Megan gathered my question, she had access to a potential new micro-space and growth, making it an intervention. I write “potential” here because, importantly, that potential was not realized in our time together. Importantly, just because the potential was not realized, does not negate its occurrence.

7.4 Implications

The first implication is that this project demonstrated growth on a small scale, and a myriad of ways in which that growth occurs. Acknowledging that growth also happens on a small scale (Siegler, 1998) is important for repositioning learners with mathematics difficulties from “unable” to “able.” All the learners in the study experienced growth, and all that growth occurred on a small scale. At the same time, growth and understanding for learners experiencing mathematics difficulties is complicated. Growth and understanding are complicated not because learners experiencing mathematics difficulties need linear, step-by-step progressions-growth is recursive. Nor is it because learners experiencing mathematics difficulties have a finite amount of potential-all learners have an infinite amount of potential. Growth and understanding are complicated because the process of learning is complicated. In chapters 0 and 1, I explored the way that current writings, pedagogical guides and research reporting deal with difficulties in mathematics-that is, if they even deal with difficulties in mathematics. That writing is often uncomplicated with uncomplicated recommendations: do these things to identify the learner with difficulties, assess these things to find gaps in knowings, and do these strategies to fix these gaps. And the recommendations are circular-for each difficulty in knowing or understanding start the process over again. And if the gaps still are not fixed, then, well, start the process again. This is not a system that promotes growth-growth is not prescriptive and it cannot be "fixed." Growth is a process, it is a natural movement, and the role of the educator is to facilitate that movement-not to prescribe the movement. At this point I am going to pause for a moment and confess something to you. This is not the first iteration of implications. For my initial writings I produced a list of recommendations. Yes, I spent an entire project exploring how growth is facilitated and not prescripted and my initial implications were about the prescription of growth. That is how ingrained prescription is (Gallagher, 2005). Thus, the most important implication of this research is that it provides evidence of growth through facilitation for learners experiencing mathematics difficulties.

This research also provides descriptive evidence of what facilitation looks like, the tensions involved in facilitating, and a redefinition of interventions through facilitation. In chapters 0 and 1, I have already explicated the typical nature of teaching, learning and interventions for learners experiencing difficulties. Much of the pedagogies utilized presently for learners experiencing mathematical difficulties are behavioral in nature (Lambert, 2015). Behavioral pedagogies by nature are not partnerships like the ones the participants in this study experienced. Behavioral pedagogies are one-sided actions masked as interactions. Like my list of recommendations behavioral pedagogies are things you ***do to*** another person. They are lists of strategies, best practices or the like that will intervene. These lists imply linear causality-if you do A, then B will be fixed or overcome-and are highly problematic. They are problematic because what happens if you do A, and B does not occur (see chapter 2 for a discussion around why understanding is more complex than this linear progression)? If B does not occur, then the problem has to be located in the child. These behavioral pedagogies are so highly problematic and yet, they are at the root of the whole intervention field. The implications from this research suggests otherwise-that learning occurs with an other(s). Also, that learning occurs without a concentration on behaviorist pedagogies. Learning occurs with and through an other(s). Throughout this research, it was the interactions that led to growth and it was also interactions that led to the holes in mappings. For instance, for Megan's hole, I led her to an answer for which she had not done Image Making or Image Having. The difference between the growth and the holes was the difference between pedagogies that ***do with*** (growth) and ***do to*** (hole). It was not just the holes that highlighted the difference between behavioral and interactional pedagogies. The entire project and all the recorded growth rested on these interactions. Throughout the analysis, I made my decisions around interactions explicit and posited how growth was encouraged and discouraged through these interactions. I have learned that these decisions occur in the moment (Mason & Davis, 2013) and are themselves subject to the same movements of recursivity as all other communications and interactions of knowledge and understanding are. Thus, facilitation is a shift in focus from ***doing to***, to ***doing with***, for those experiencing mathematics difficulties.

An implication related to facilitating growth versus prescribing growth is the significance of aligning interventions with the facilitation of growth. If growth is not linear, neither should the interventions. Interventions need to reflect the process of growth they are meant to facilitate, and they should not re-hash linear, behavioral pedagogies that pose barriers to growth. This dissertation rethought of interventions as an activation of the dynamic micro-space that allowed for growth. Interventions were not specific prescriptive and tangible strategies, they were actions of creating spaces. This is an important implication because in schooling interventions go hand-in-hand with difficulties-one is often dependent on the other (McDermott, 1996). Added to this is that interventions themselves often create more difficulties (Gallagher, 2005). Interventions then need to be rethought in light of growth.

One of the ways this project rethought about interventions was through access. Giving the learner access to their own spaces is just as important as identifying the spaces in the first place. Access is about 'opportunity to learn.' And while I cannot make any conclusion about achievement from this present study, Gutierrez (2012) has already stated that achievement results from, or is built on that access. Thus, access is quite important for achievement. This means that the observing other really has an imperative to share access to the learner's own micro-spaces. I am arguing both for explicit communication of the micro-space to the learner, and I am arguing for the discursive practices of exchange and probing that creates potential for more micro-spaces to occur. This too, is a different implication for interventions than current modes of intervention that are usually behavioral in nature (Lambert, 2015). Three points of access found to be important in this study:

I. Sharing-the learners share an understanding with an "other," building shared understanding around the space
II. Re-voicing spaces-the learner's micro-space of growth is summarized, re-voiced, and shared with the learner, building shared understanding around the space.
III. Questioning and probing-the learner's micro-space is questioned and probed leading to new micro-spaces, building shared understanding around the space.

Prior understandings in mathematics is an area that remains underexplored. A lot of what happened in this study occurred around prior understandings. Two implications arise from this occurrence and subsequent analysis of the data: i) an expansion of our understanding of what happens when learners access their prior understandings and how those prior understandings support further learning, and ii) an extension of the concept of

Primitive Knowing for the Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding (Pirie et al., 1994). Prior understandings were extremely resilient. Even when erroneous, prior understandings were not discounted, or attempted to be "overwritten" by the participant. Prior understandings played a vital role in growth both whether they were "correct" or "mistaken." The results of this study indicate a new hypothesis for further study that the best way to eradicate erroneous knowings is not to eradicate them at all. The best way may very well be to thicken the prior knowings until they have been blended and integrated with other knowings, meaning the erroneous knowings would essentially be thickened into something new.

The explanation for the implication of an extension to Primitive Knowing in PK theory is an expansion of the argument found in section 2.2.6, around my usage of an evolved conception of Primitive Knowing. Here I develop the argument further. The Pirie Kieren Theory for the Dynamical Growth of Mathematical Understanding was useful for this project especially because of its language and focus on mathematical understanding, its inherent tools for exploring micro-changes, and because it is adaptable to use with other theories of understanding and growth. At the same time that PK theory was useful in exploring zero with learners experiencing mathematical difficulties, there was one aspect of the theory, namely Primitive Knowing, that presented a barrier and required expanding upon. For this project, the mode of Primitive Knowing was expanded to include complex and connected understandings. This expansion resulted in a new finding-the shifts and movements that occur within Primitive Knowing.

Pirie and Martin (2000) describe Primitive Knowing as "all the previously constructed knowledge, outside of the topic, that students bring to the learning of a topic" (p.129). This description is problematic to operationalize for both the mathematical concept explored, zero, and for the population who participated in this study, those experiencing mathematics difficulties. First of all, in section 2.2.6 I argued that zero is an evolving culmination of a network of conceptions, and as such, understanding of zero is also a connected network of conceptions. These network of connections are connected to and draw from multiple mathematical concepts. In light of this, the "outside the topic" that Pirie and Martin refer to, becomes very problematic and is difficult to operationalize. At the foundation of this problem is that any attempt to delineate a separation between "the topic of zero" and "the topics around zero" leads to a reductionism of the complexities of the network of interconnectivities between these understandings, within and around the concept. Secondly, the concept of Primitive Knowing, as outlined by Pirie and Martin can be further problematized for learners experiencing MDs. For this group, knowings that they bring to the topic can sometimes be elusive and difficult to access. I have argued for much of chapter 1 that learners experiencing MDs are often unjustly categorized, sometimes because of the elusivity of their prior knowings (see Houssart, 2004 and the "maths fairies"). There were therefore also considerations of equity for this project, in expanding the term Primitive Knowing.

Because understanding of zero is operationalized for this project as an interconnected network, then the primitive knowings of this connected zero had to be operationalized as: all the previously constructed knowledge *related to zero*, *outside of the connections*, that students bring to the learning of zero. This broader operationalization of Primitive Knowings-related to the concept, outside of the connections-also allowed for both a non-deficit perspective on elusive knowings and a new finding (see previous section) around the shifts and movements that can occur within Primitive Knowing.

7.5 Conclusion

Earlier this year I was discussing the evolution of my dissertation with a classroom full of about thirty Ontario teacher candidates and another teacher. I stood at the front of the room telling my story. Of course, my dissertation story is intertwined with my story with David-so in the midst of my dissertation story, I told our story as well. Right after I finished speaking, as I was putting my things away, the other teacher remarked quite happily that thank goodness stories like what happened to David do not happen anymore. No sooner had the words left her mouth than I was approached by four teacher candidates.

The teacher candidates wanted advice because they had experienced similar narratives in their placements, to the one David and I experienced. One by one they told their stories of being told to ignore, not work with, or not focus on a child because that child had learning difficulties and did not have potential. There were variations in the stories- but to me they were all saying the same thing-there are many more Davids.

There were a number of findings from this research and a number of implications extending from those findings. Prior Knowings are important-we can begin to understand how prior knowings support growths in understanding; access is important-there are different ways we can create access; labeling is problematic-because it is something we do to another and not with; interventions need to be just as active and recursive as growth is …however the first place to start, and the only way growth can occur is through partnership.

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Appendix A (Consent Forms)

York University
Faculty of Education

Feb. 18, 2016

My name is Robyn Ruttenberg-Rozen and I am a doctoral candidate in the Faculty of Education at York University. I am currently engaged in a research study that aims to understand more about the difficulties students experience in mathematics and interventions that will help them. We are seeking volunteers to participate in this study and this document gives you some brief information about the study and what you might be asked to do if you agree for your child to take part.

Study Name:
Strengthening the mathematical understanding of children: Creating images of zero and number sense concepts around zero

Researchers:
Principal Investigator: Robyn Ruttenberg-Rozen
Supervisor: Dr. Lyndon Martin, Faculty of Education, York University

Purpose of the research:
This study brings together two understudied areas in mathematics education: children who experience difficulties in mathematics, and the number zero. Through this exploration we will analyze the understandings that children have about zero and the mathematical concepts that stem from understandings of zero and possible teaching strategies and activities that help children to understand different mathematical ideas and number sense concepts about zero.

In mathematics education research zero is a known area of difficulty for many students. Misconceptions regarding zero, for example that zero represents nothing and is not a number, can persist into university and adulthood. These misconceptions can be barriers to success in mathematics and can prevent students from understanding more advanced concepts.

Two primary goals of this study are to explore how children interact with zero as a number with mathematical properties and as a mathematical concept, and to examine different activities and strategies that may enable the children to build richer understandings of zero and number sense concepts.

What you will be asked to do in the research:

If your child participates in this study he/she will be observed in the classroom as he/she participates in the regular activities of the class during their mathematics lessons.

We may take photocopies and/or photographs of your child's work. We may also invite your child to participate in a set of video-taped problem-solving tasks interview sessions. The purpose of the initial session will be to introduce number sense concepts that may be missing and/or correct misconceptions and determine what interventions may be appropriate. After the initial session, interventions will be developed specifically for your child based on analysis of the first session. Following the development of the interventions, will be a set of 5-8 sessions implementing the interventions. In these sessions your child will be asked to work on mathematics problems and tasks, sometimes utilizing different manipulatives. These problem-solving sessions will also be audio- and videotaped and your child may be asked questions such as "How did you reach that answer" or "Why did you add those two numbers together?" in order to help us better understand your child's mathematical thinking process.

The researchers will be available to discuss any aspect of the research at any point during the study. A written report will be made available to you at the conclusion of the study. If you wish to receive a copy of the final report of the study you must check the appropriate box in the signature section of this form. You will be responsible for ensuring that you provide ongoing contact details to us until such time as the report is prepared.

Risks and discomforts:

If you allow your child to participate in the study, and if your child will be interviewed, you are agreeing to her being video- and audio-taped for data collection purposes, and you are given a choice how those clips may be used. You can choose separately whether to allow clips to be used in scholarly presentations or publications and/or in the researchers' teaching.

The videotaped data will be transferred to a password-protected external digital storage device and the storage device will be kept in a locked cupboard.

Video and audio data, selected for potential use in conference presentations, papers, classroom teaching etc., may be retained under secure conditions, and will be securely archived after January 1 2030 but not destroyed. If you do agree to the public use of these clips, your child will always be referred to by a pseudonym. Still, there remains a chance that your child could be recognized by a member of the audience. Also, with the popularity of cell phone cameras, there is some risk that if data clips featuring your child are shown in scholarly presentations, an audience member could make a personal recording of some or all of the presentation, and make this material viewable online.

Benefits of the research and benefits to you:

Participating in this study may have no direct benefit to your child, as the strategies and activities may not be effective for your child. It is possible that the strategies and activities may be effective and that she may find the participation in research and the interview process interesting. The research will contribute to the body of knowledge about difficulties students experience in mathematics and mathematical interventions that may enable access to mathematical concepts.

Voluntary participation:

Your child's participation in the research is completely voluntary and participants may choose to stop participating at any time. A participant's decision not to continue participating will not influence their relationship or the nature of their relationship with researchers or with staff of York University either now or in the future.

Withdrawal from the study:

Your child's participation in this study is completely voluntary. We ask that you discuss the study with your child and ensure that they are happy to participate. We will also provide information directly to your child and seek their assent before starting the study. If you allow your child to participate and later change your mind, you may withdraw him/her from the study at any time without giving a reason by contacting Robyn Ruttenberg-Rozen (contact details above).

Your child may stop participating in the study at any time, for any reason, if you so decide. Your child may also withdraw him/herself from the study by making his/her wishes known to the research team. Your or your child's decision to stop participating, or to refuse to answer particular questions, will not affect their relationship with the researchers, York University, or any other group associated with this project. In the event that you withdraw your child from the study, all associated data collected will be immediately destroyed wherever possible.

Confidentiality:

If you give consent for your child to participate in the study, my supervisor and I are the only people who will have direct access to identifying data (such as videotapes, transcripts that contain identifying information, etc.). Information that may identify your child (such as your child's name) will be stored separately. Pseudonyms will be used in all publications and presentations based on the data, and will be used to identify all raw data (for example, pseudonyms rather than real names will be used to store and file data). When not in use, data will be kept securely locked at York University and/or stored in password-protected computer files. Videotaped data is complex and can require considerable time to analyze fully. Hence, data will be retained for as long as any of the research team requires access to it for this or potential follow-up projects.

It is important to recognize that you may still give consent for your child to participate in the study even if you do not give permission for the video and audio clips to be used publicly. In this case, videotaped/audiotaped data may still be used to generate written documents (with your child identified only by pseudonym), but the research team will be the only people to view/hear the videotapes/audiotapes themselves.

Confidentiality will be provided to the fullest extent possible by law.

Questions about the research?

If you have questions about the research in general or about your child's role in the study, please feel free to contact Robyn Ruttenberg-Rozen or Dr. Lyndon Martin or the Graduate Program in Education Office.

This research has been reviewed and approved by the Human Participants Review Sub-Committee (York University's Ethics Review Board) and conforms to the standards of the Canadian Tri-Council Research Ethics guidelines. If you have any questions about this process, or about your child's rights as a participant in the study, please contact the Senior Manager & Policy Advisor for the Office of Research Ethics, York University.

Legal Rights and Signatures:

There are several options for you to consider if you decide to allow your child to take part in this research. You can choose all, some, or none of them. Please check "Yes" or "No" for each item.

I grant permission for excerpts of videotapes in which my child appears to be used publicly in the researchers' teaching.	Yes: ___ No: ___
I grant permission for audio clips featuring my child's voice to be used publicly in the researchers' teaching.	Yes: ___ No: ___
I grant permission for audio clips featuring my child's voice to be used publicly in scholarly presentations and conferences.	Yes: ___ No: ___
I grant permission for the researchers to retain the videotapes in which my child appears for future analysis consistent with the objectives of this research study.	Yes: ___ No: ___
I wish to have the videotapes in which my child appears destroyed once this study is complete.	Yes: ___ No: ___
I wish to receive a copy of the final report of the study [or I wish to be notified of the parent meeting at which results of the study will be presented].	Yes: ___ No: ___

Your signature on this form indicates that you 1) understand to your satisfaction the information provided to you about your child's participation in this research project, 2) agree for your child to participate as a research subject, and 3) have discussed the study with your child and that they are happy to participate.

I consent for my child to participate in *Strengthening the mathematical understanding of children: Creating images of zero and number sense concepts around zero* conducted by Robyn Ruttenberg-Rozen. I have understood the nature of this project and wish to participate. I am not waiving any of my legal rights by signing this form. My signature below indicates my consent.

Participant's Name: (please print) ________________________________

Parent or Guardian's Name: (please print) ________________________________

Relationship to Student: (please print) ________________________________

Parent or Guardian's Signature ________________________________ Date: __________

Researcher's Name: (please print) ________________________________

Researcher's Signature: ________________________________ Date: __________

Pupil-Consent Forms

Invitation to Participate in a Research Study

"*Creating images of zero and number sense concepts around zero*"

My name is Robyn Ruttenberg-Rozen and I am a doctoral candidate in the Faculty of Education at York University. I am currently engaged in a research study that aims to understand more about the difficulties that children have in understanding certain areas of mathematics and how to help them. We are seeking volunteers to participate in this study and this document gives you some brief information about the study and what you might be asked to do if you take part.

If you agree to participate in the study, you may be videotaped during a series of sessions as you participate in interviews, interact with researchers and with the mathematics materials and problems.

If your parent or guardian has given permission for you to take part in the study they will be asked to sign a consent form that gives more information about what will happen to the data we collect, and how your anonymity will be protected. The form also gives you some choices about how the data will be used. You will be free to withdraw from the study at any time without giving a reason.

If you need more information, please contact Robyn Ruttenberg-Rozen by email.
Thank you for considering this invitation.

If you are happy to participate in *Creating images of zero and number sense concepts around zero* conducted by Robyn Ruttenberg-Rozen (and if your parent or guardian also agrees) then please sign below. You are free to withdraw from this research project at any time. You should feel free to ask for clarification or new information throughout your participation.

Participant's Name: (please print) ____

Participant's Signature: ____ Date: ___________

Researcher's Name: (please print) ____

Researcher's Signature: ____ Date: ___________

Appendix B (Survey)

Dear Parents or Guardian,
If you have given permission for your child to participate in the interview and mathematical tasks, we would like to understand your child's history with mathematics. Understanding your child's history will help us to create tasks for your child. Below are 5 optional questions about the mathematical background of your child. Your answers to these questions will be kept confidential and will not be shared with the school or your child's classroom teacher. The only people that will have access to your answers will be myself and my supervisor. You may decline to answer any or all of the questions. To ensure confidentiality, when you have completed this form, please place it in the enclosed envelope, seal it and return it with your child to school.

Name of student:____

Age:____

Date of Birth:____

1. What do you think your child's experience with mathematics has been in school?

   Decline to answer ☐

2. What kinds of difficulties and challenges has your child experienced with mathematics?

   Decline to answer ☐

3. Does your child use extra help resources (i.e. after-school program, tutor, in school special help, etc.) to help with mathematics learning?

____
____
____
____

Decline to answer ☐

4. Has your child been diagnosed with a learning disability, AD(H)D, or another disability that may impede their mathematics learning? If your child has been diagnosed with AD(H)D or a learning disability, is there a type (i.e. inattention, hyperactive, working memory, language… etc.) associated with the diagnosis?

____
____
____
____

Decline to answer ☐

5. Does your child have an IEP (Individual Education Plan) for mathematics?

____
____
____
____

Decline to answer ☐