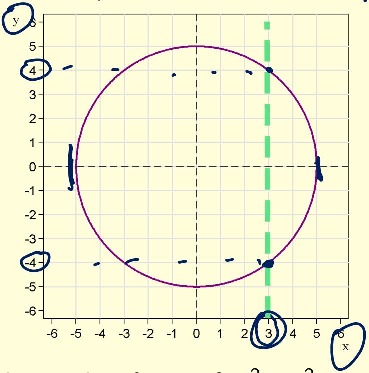
Implicit Functions



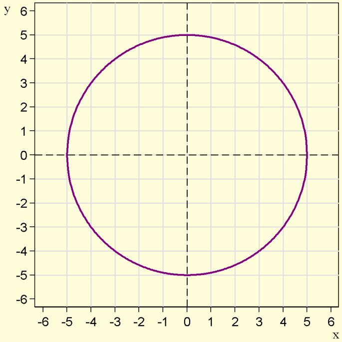
Is a circle a function? $y^2 + x^2 = 25$
vertical line Test

A circle $x^2+y^2=25$
"fails" the vertical line test.
A circle is not a function.

For some x-values, there are more than one y-value.

$$y^2+x^2=25$$
$$y^2=25-x^2$$
$$y^2=(5+x)(5-x)$$
$$y=+/-\sqrt{(5+x)(5-x)}$$

Implicit Functions



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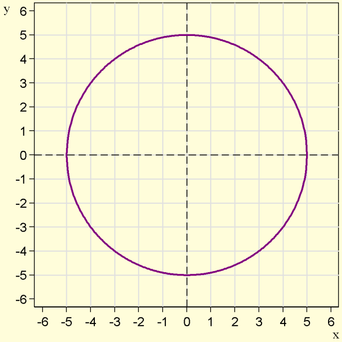
Example: $y^2+x^2=25$

- We cannot write y as an explicit function of x, yet the value of y depends on the value of x.
- The above expression defines y *implicitly* as a function of x

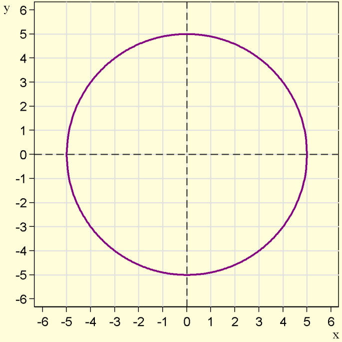
Implicit Differentiation



$$\left[y^2 + x^2 = 25 \right]$$

1. Recognize that the value of y depends on the value of x. $y = f(x)$

Implicit Differentiation



$$y^2 + x^2 = 25$$

1. Recognize that the value of y depends on the value of x.
2. Take the derivative with respect to (w.r.t.) x of both sides of the expression while applying the **Chain Rule** to y.
3. Re-arrange the expression to isolate/solve for the derivative. y'

$$\frac{dy}{dx}$$

1. Example 1: $x^2 + y^2 = 25$
   a) Use implicit differentiation to find the slope when $x = 3, y = -4$
   b) Illustrate the slope with a tangent line on at $x = 3, y = -4$ on a graph.

let $y = f(x) \Rightarrow x^2 + (f(x))^2 = 25$

$y' = f'(x)$ $\qquad \frac{d}{dx}\left(x^2 + \underline{(f(x))^2}\right) = \frac{d}{dx} 25$

$$2x + 2f(x)f'(x) = 0$$

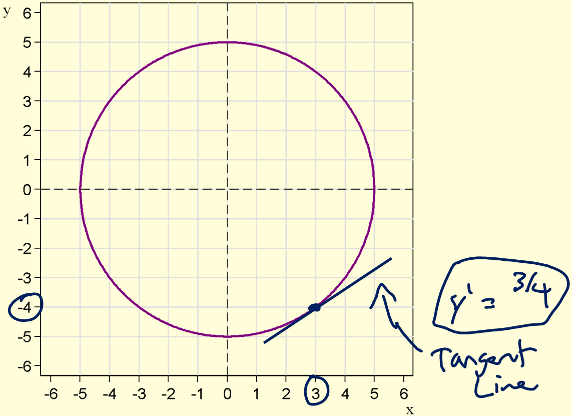
Solve for $f'(x)$

$$2f(x)\underline{f'(x)} = -2x$$

$$f'(x) = \frac{-2x}{2f(x)} = -\frac{x}{f(x)}$$

$$y' = -x/y$$

$$y' = -x/y = -\frac{3}{-4} = \underline{3/4}$$



$$x^2 + y^2 = 25$$

2. Example: $2xy = 12$

a) Find the derivative, y' using the implicit differentiation: $\left(\frac{dy}{dx}\right)$.

b) Compute the slope when $x = 6$.

c) Illustrate on a diagram.

$(2)(6)y = 12$

$y = 1$

$y = \frac{12}{2x}$

$y = 6x^{-1}$

$\boxed{\frac{dy}{dx} = -\frac{6}{x^2}}$

$\frac{dy}{dx} = y' = \frac{-6}{36} = -1/6$

(a) $2xy = 12$

$\frac{d}{dx} 2xy = \frac{d}{dx} 12$

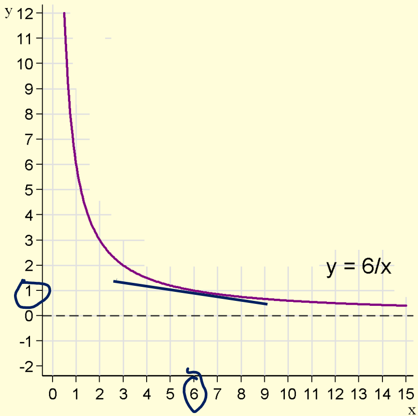
$2((1)y + y'x) = 0$

$y + y'x = 0$

$\boxed{y' = -\frac{y}{x}}$ $\Rightarrow$ $\boxed{y' = -1/6}$

(b)

$x = 6 \Rightarrow y = 1$



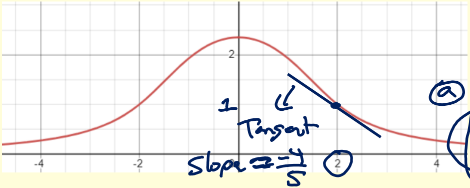
y
x
y = 6/x

Summary of Implicit Differentiation

If y is an implicit function of x,

1) let $y = f(x)$.
2) Apply the derivative operator to *both sides* of the expression.
3) Re-arrange the expression to isolate and solve for the derivative, $f'(x)$.
4) Replace $f'(x) = y'$. $y = f(x)$

Steps 1 and 4 are optional. Either do both steps 1 and 4 or do neither.



Graph created using Desmos.com/calculator

3. Example: $y^3 + 3x^2y = 13$
   a) Find y' when
   b) Compute y' when $x = 2$

ⓐ $\frac{d}{dx}\left(y^3 + 3x^2y\right) = \frac{d}{dx}13$

$3y^2y' + 3\left(2xy + y'x^2\right)$

$3y^2y' + 6xy + 3y'x^2 = 0$

$y^2y' + 2xy + y'x^2 = 0$

$y'\left(y^2 + x^2\right) = -2xy$

ⓐ $\boxed{y' = \frac{-2xy}{y^2 + x^2}}$

ⓑ $x = 2$ $y^3 + 3(2)^2y = 13$

$y = 1$ $y^3 + 12y = 13$

$(1)^3 + 12(1) = 13$

$y' = \frac{-2(2)(1)}{1^2 + 2^2} = \frac{-4}{5}$

4. Example: $y^7 - 3x = Ax^2$

Suppose A is a constant, and y is an implicit function of x.
Find the derivative of y w.r.t x.

$$\frac{d}{dx}\left(y^7 - 3x\right) = \frac{d}{dx} Ax^2$$

$$7y^6 y' - 3 = 2Ax$$

$$7y^6 y' = 2Ax + 3$$

$$y' = \frac{2Ax + 3}{7y^6}$$

Find y' (Leibniz notation: $\mathrm{d}y/\mathrm{d}x$) using implicit differentiation: $2xy = 36$

A) $\mathrm{d}y/\mathrm{d}x = \quad 0$

B) $\mathrm{d}y/\mathrm{d}x = \quad \frac{x}{y}$

C) $\mathrm{d}y/\mathrm{d}x = \quad \frac{y}{x}$

✓ D) $\mathrm{d}y/\mathrm{d}x = -\frac{x}{y}$

E) $\mathrm{d}y/\mathrm{d}x = -\frac{y}{x}$

$$\frac{d}{dx} 2xy = \frac{d}{dx} 36$$

$$\cancel{2}y + \frac{dy}{dx}\cancel{2}x = 0$$

$$y + \frac{dy}{dx}x = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

5. Example: $2w^2 - 2v = w$
   Find w' (dw/dv) using implicit differentiation:

$$\frac{d}{dv}\left(2w^2 - 2v\right) = \frac{d}{dv} w$$

$$(4w)\frac{dw}{dv} - 2 = \frac{dw}{dv}$$

$$(4w)\frac{dw}{dv} - \frac{dw}{dv} = 2$$

$$\frac{dw}{dv}(4w - 1) = 2$$

$$\frac{dw}{dv} = \frac{2}{4w-1}$$

Utility in a Two-Period Model

- Consumers enjoy utility which they receive from consumption today (c_1) and consumption in the future (c_2).
$$u(c_1) + \beta u(c_2) = U$$
- Consumers are impatient – they want consumption now rather than later. This is captured by β:
- If you reduce c_1, you save more and c_2 rises.
- So, we can say that c_2 is an implicit function of c_1.

happiness

Utility in a Two-Period Model

How much does c_2 change if we change c_1 by a (very) small amount?

$u(c_1)$

$$u(c_1) + \beta u(c_2) = \bar{U}$$

$\beta \rightarrow$ impatience

$u(c_2) \rightarrow$ function

$\bar{U} \rightarrow$ constant

$$\frac{d}{dc_1}\left(\underbrace{u(c_1)} + \beta u(c_2)\right) = \frac{d}{dc_1}\bar{U}$$

$$u'(c_1)(1) + \beta u'(c_2)\frac{dc_2}{dc_1} = 0$$

$$\beta u'(c_2)\frac{dc_2}{dc_1} = -u'(c_1)$$

$$\frac{dc_2}{dc_1} = -\frac{u'(c_1)}{\beta u'(c_2)} = -\frac{\text{gain in happiness from a little more consumption today}}{\text{gain in happiness from a little more consumption in future}}$$

impatience $\leftarrow \beta$

1

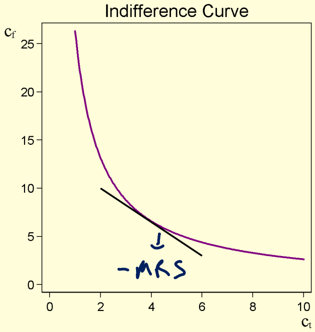
If the consumer were to decrease c1 by some small amount, how much more c2 would they need to be just as happy?

Utility in a Two-Period Model

$$\frac{dc_2}{dc_1} = -\text{MRS}$$

- The marginal rate of substitution (MRS) is how much additional c_2 the consumer needs to be indifferent towards giving up some c_1.
- The MRS is the slope of an indifference curve.
- In general,

$$\text{MRS} = \frac{\text{MU}_{c_1}}{\text{MU}_{c_2}}$$



Second Implicit derivative

$2xy = 10$

$\frac{d}{dx}(2xy) = \frac{d}{dx}10$

$2y + 2xy' = 0$

$\boxed{2xy' = -2y}$ ←

First implicit derivative $y' = -\frac{y}{x}$ ←

$\frac{d}{dx}2xy' = \frac{d}{dx}(-2y)$

The second implicit derivative tells us whether the first implicit derivative is increasing or decreasing.

$\frac{d}{dx}2xy' = \frac{d}{dx}(-2y)$

$\cancel{2}(y' + xy'') = -\cancel{2}y'$

$2y' + xy'' = 0$

$\boxed{\text{Replace } y' = \frac{-y}{x}}$

$2\left(\frac{-y}{x}\right) + xy'' = 0$

$xy'' = -2\left(\frac{-y}{x}\right)$

$y'' = \frac{2y}{x^2}$ second implicit derivative

Second Implicit derivative

$\cancel{xy^2 = 10x}$ $x^2 + y^2 = 25$

Find the second implicit derivative, y'', and simplify so that only y, x and any constants appear on the right-hand side of your equation.

Second Implicit derivative

$$x^2 + y^2 = 25$$

Find the second implicit derivative, y'', and simplify so that only y, x and any constants appear on the right-hand side of your equation.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25$$

$$\cancel{2}x + \cancel{2}yy' = 0$$

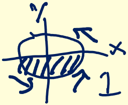
$$yy' = -x$$

$$\boxed{y' = -x/y}$$

$$\frac{d}{dx}\left(x + yy' = \frac{d}{dx}0\right)$$

$$1 + y'y' + yy'' = 0$$

$$y'' = -1 - (y')^2$$



$$y'y' = (y')^2$$

$$1 + y'y' + yy'' = 0$$

Replace $y' = -x/y$

$$1 + \left(\frac{-x}{y}\right)^2 + yy'' = 0$$

$$yy'' = -1 - \left(\frac{-x}{y}\right)^2$$

$$\rightarrow yy'' = -1 - \left(\frac{x}{y}\right)^2$$

$$\rightarrow (-1)^2\frac{x^2}{y^2}, \quad (-1)^2 = 1$$

$$y'' = \frac{-1}{y} - \frac{x^2}{y^3}$$

$$y'' = -\left(\frac{1}{y} + \frac{x^2}{y^3}\right)$$

$$\boxed{y'' = -\left(\frac{y^2 + x^2}{y^3}\right)}$$

A Function within an Expression

Example 1: $4g(x) + y^3 = 2$
Find y'. Note that $g(x)$ is a function.

$$\frac{d}{dx}\left(4g(x) + y^3\right) = \frac{d}{dx} 2$$

$$4g'(x) + 3y^2 y' = 0$$

$$y' = \frac{4g'(x)}{3y^2}$$

Find y'' Quotient

$$\frac{d}{dx} y' = \frac{d}{dx}\left(\frac{4g'(x)}{3y^2}\right)$$

$$y'' = \frac{4g''(x)3y^2 - 6yy'(4)g'(x)}{9y^4}$$

$$y'' = \frac{12y^2 g''(x) - 24y g'(x)\frac{4g'(x)}{3y^2}}{9y^4}$$

$$y'' = \frac{12y^2 g''(x)}{9y^4} - \frac{96(g'(x))^2 y}{9y^4 3y^2}$$

$$y'' = \frac{12y^2 g''(x)}{9y^4} - \frac{32(g'(x))^2}{9y^5}$$

$$y'' = \frac{12y^3 g''(x) - 32(g'(x))^2}{9y^5}$$

$$y'' = \frac{4(3y^3 g''(x) - 8(g'(x))^2)}{9y^5}$$

Functions within Functions

Example 2:

① $g(h(x)) - yx - y^2 = 0$

② $h(x) = 2x^2 + 4$

Find y' where g, h are functions.

$\frac{d}{dx}\left(g(h(x)) - yx - y^2\right) = \frac{d}{dx} 0$

$g'(h(x))h'(x) - y - y'x - 2yy' = 0$

$g'(h(x))h'(x) - y = y'x + 2yy'$

$g'(h(x))h'(x) - y = y'(x + 2y)$

$\frac{g'(h(x))h'(x) - y}{x + 2y} = y'$

g is a general form function while h is specified.

$h(x) = 2x^2 + 4$

$h'(x) = 4x$

$y' = \frac{g'(2x^2+4)4x - y}{x + 2y}$

Functions within Functions

general form

$2x+y$ is specified

Example 3: $g(2x+y) - x^2 + y^2 = 0$

Find y' where g is a function

$$\frac{d}{dx}\left(g(2x+y) - x^2 + y^2\right) = \frac{d}{dx}0$$

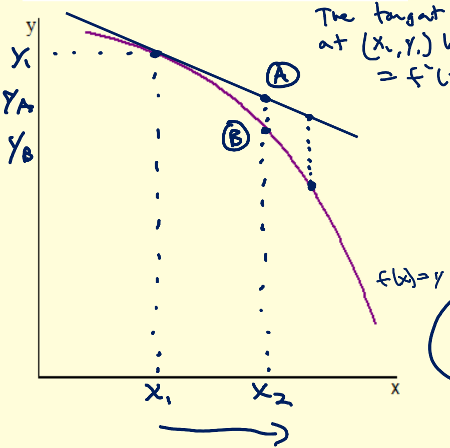
$$g'(2x+y)(2+y') - 2x + 2yy' = 0$$

$$2g'(2x+y) + g'(2x+y)y' + 2yy' = 2x$$

$$y'\left(g'(2x+y) + 2y\right) = 2x - 2g'(2x+y)$$

$$y' = \frac{2x - 2g'(2x+y)}{g'(2x+y) + 2y} = \frac{2\left(x - g'(2x+y)\right)}{g'(2x+y) + 2y}$$

Approximating Change With the Derivative



The tangent line at (x_1, y_1) has slope $= f'(x)$

A $\quad y_A = f(x) + \underbrace{f'(x)\overbrace{(x_2 - x_1)}^{\text{The change in } x}}_{\text{The change in } y}$

Approximation using the derivative.

Linear approximation.

$y^3 + 3x^2y = 13$

B $\quad y_B = f(x_1 + (x_2 - x_1))$

$= f(x_2)$

Approximating Change With the Derivative

$$y = f(x)$$

- Find the derivative: $\dfrac{dy}{dx} = y'$

Approximating Change With the Derivative

$$y = f(x)$$

- Find the derivative: $\dfrac{\mathrm{d}y}{\mathrm{d}x}$
- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ $\quad \dfrac{dy}{dx}(dx) = f'(x)(dx)$

Approximating Change With the Derivative

$$y = f(x)$$

- Find the derivative: $\dfrac{dy}{dx}$
- $\dfrac{dy}{dx} = f'(x)$
- $dy = f'(x)dx$

$\dfrac{dy}{dx}(dx) = f'(x)(dx)$

dx = change in x

dy = approximate change in y

Approximating Change With the Derivative

$$y = f(x)$$

- Find the derivative: $\frac{\mathrm{d}y}{\mathrm{d}x}$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$
- $\mathrm{d}y = f'(x)\mathrm{d}x$
- where $\mathrm{d}x$ is the change in x, $f'(x)\mathrm{d}x$ is the approximate change in y.

where $dx = x_2 - x_1$

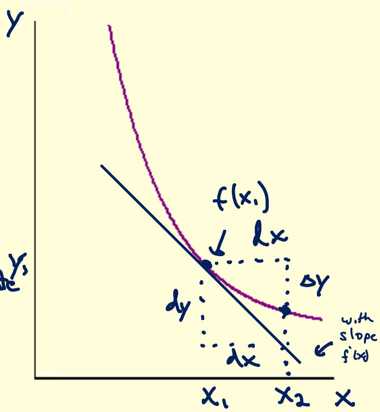
approximate

$dy = f'(x)(x_2 - x_1)$

actual

$\Delta y = f(x_1 + (x_2 - x_1)) - f(x_1)$

$\Delta y = f(x_1 + dx) - f(x_1)$



Example 1: Bond Prices and Interest Rate Risk

- Stocks reward investors with dividends.
- However, Bonds promise to pay a fixed amount.
- Fixed amounts make bond prices vulnerable to unexpected price inflation and changes in market interest rates more generally.
- **Duration** is a useful way to measure the sensitivity of a bond to a change in market interest rates. [a]

[a] Market interest rates are (partially) set by central banks like the Bank of Canada or U.S. Federal Reserve.

Example 1: Bond Prices and Interest Rate Risk

- Let's consider a zero-coupon bond:

TVM

→ one payment

Example 1: Bond Prices and Interest Rate Risk

- Let's consider a zero-coupon bond:
- pays a lump sum face value, F, in $\underline{t}$ years.

maturity or time-to-maturity

Example 1: Bond Prices and Interest Rate Risk

Relative rate of change

- Let's consider a zero-coupon bond:
- pays a lump sum face value, F, in t years.
- can be bought and sold today at price, P.
- is affected by a market interest rate, r, which is continuously compounded.

$$P = Fe^{-rt}$$

a) Find the relative rate of change in the bond price with respect to the market interest rate, r. Interpret your result.

$$\frac{dP/dr}{P}$$

initial price

$$\frac{dP}{dr} = Fe^{-rt}(-t)$$

$$P = Fe^{-rt}$$

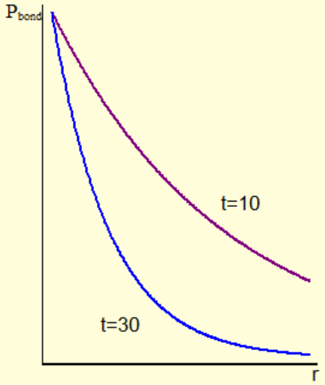
Duration

$$\frac{dP/dr}{P} = \frac{-t\cancel{F}\cancel{e^{-rt}}}{\cancel{F}\cancel{e^{-rt}}} = -t$$

① $\uparrow r \Rightarrow \downarrow$ Bond price

$\downarrow r \Rightarrow \uparrow$ Bond pric

② The size of change in the price of the Bond depends on the time to maturity.



P bond
t=10
t=30
r

Example 1: Bond Prices and Interest Rate Risk

$$P = Fe^{-rt}$$

- b) Use your answer in part a to approximate the relative rate of change in the price of bond if the market interest rate increases by 1 percent.
- c) Use your answer in part a to approximate the relative rate of change in the price of bond if the market interest rate decreases by 1 percent.

$$\frac{dP/dr}{P} = -t$$

$$\left(\frac{dP/\cancel{dr}}{P}\right)\cancel{dr} = -t\,dr$$

$$\boxed{\frac{dP}{P} = -t\,dr}$$

b) $dr = 0.01 \Rightarrow \frac{dP}{P} = -t(0.01)$

$= -\underline{0.01t}$

c) $dr = -0.01$

$$\frac{dP}{P} = -t(-0.01)$$

$$= 0.01t$$

Example 1: Bond Prices and Interest Rate Risk

Consider a bond that will pay \$1,000 in 10 years with a market interest rate of 2%.

d) Calculate the bond price, P.

e) Calculate the **actual** relative rate of change in the bond price if the market rate increases 1 percent.

f) Calculate the **actual** relative rate of change in the bond price if the market rate decreases 1 percent.

g) How do your answers compare with your answers in parts a,b?

(d) $P = Fe^{-rt}$

$P_{.02} = \$1000e^{-.02(10)}$

$= 1000e^{-.2}$

$P_{.02} = 819.73$

(e) $\dfrac{P_{.03} - P_{.02}}{P_{.02}}$ $\quad P_{.03} = 1000e^{-.03(10)}$

$P_{.03} = 740.82$

$(740.82 - 818.73)/818.73 = -0.095$

or -9.5%

f) Calculate the **actual** relative rate of change in the bond price if the market rate decreases 1 percent.

g) How do your answers compare with your answers in parts a,b?

$P_{.01} = 1000e^{-.01(10)}$

$P_{.01} = 904.84$ $\quad P_{.02} = 818.73$

f

$\frac{904.84 - 818.73}{818.73} = 0.105$

10.5%

g

$\frac{P_{.03} - P_{.02}}{P_{.02}} = -9.5\%$

$\frac{P_{.01} - P_{.02}}{P_{.02}} = 10.5\%$

actual

estimate/approximation

$\frac{dP_{.02}}{dr_{.01}} = -10(.01)$

$= -10\%$

$\frac{dP_{.02}}{dr_{-.01}} = -10(-.01)$

$= +10\%$

Example 1: Bond Prices and Interest Rate Risk