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Finite Cyclicity of Graphics with a Nilpotent Singularity of Saddle or Elliptic Type[1]

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In this paper we prove finite cyclicity of several of the most generic graphics through a nilpotent point of saddle or elliptic type of codimension 3 inside C^∞ families of planar vector fields. In some cases our results are independent of the exact codimension of the point and depend only on the fact that the nilpotent point has multiplicity 3. By blowing up the family of vector fields, we obtain all the limit periodic sets. We calculate two different types of Dulac maps in the blown-up family and develop a general method to prove that some regular transition maps have a nonzero higher derivative at a point. The finite cyclicity theorems are derived by a generalized derivation–division method introduced by Roussarie.

1. INTRODUCTION

A graphic (singular cycle, limit periodic set, polycycle) of a planar vector field is an invariant set of the vector field involving regular orbits and singular points. We are interested in the graphics of generic families of vector fields depending on a small number of parameters and their cyclicity, i.e., the maximum number of limit cycles that may appear by perturbation inside the family. A simpler problem is to prove that the graphics have finite cyclicity. The question of finding the number of limit cycles which

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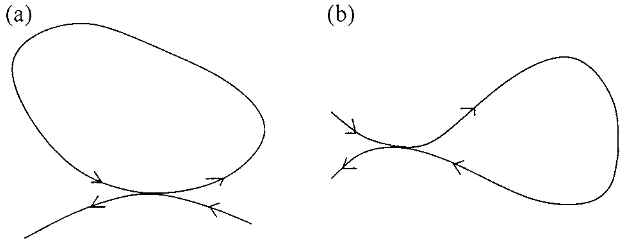


FIG. 1. Graphics through a nilpotent saddle. (a) convex, (b) concave.

appear by perturbation of a graphic in a generic family and the problem of finite cyclicity is closely related to the Hilbert–Arnold problem [1, 21]:

Hilbert–Arnold problem. Prove that for any n, the bifurcation number $B(n)$ is finite, where $B(n)$ is the maximum cyclicity of nontrivial polycycles occurring in generic n-parameter families.

A graphic of planar vector field can be elementary or nonelementary in the sense that its singular points are elementary (hyperbolic or semihyperbolic, i.e. at least one nonzero eigenvalue) or nonelementary. Some essential steps have been made toward the understanding of the bifurcation of elementary graphics through the works of Roussarie [30], Mourtada [27–29], Ilyashenko and Yakovenko [22], Dumortier *et al.* [12], Kotova and Stanzo [24], Dumortier *et al.* [7], El Morsalani [15], etc.
The graphic with a nilpotent singular point of multiplicity 2 is the cuspidal loop. Graphics through a point of multiplicity 3 are of two types:

- graphic through a nilpotent saddle (Fig. 1),
- graphic through a nilpotent elliptic point (Fig. 2).

In [14], by analytic and geometric methods based on the blowing up for the unfoldings, Dumortier *et al.* studied the simplest case, the bifurcation

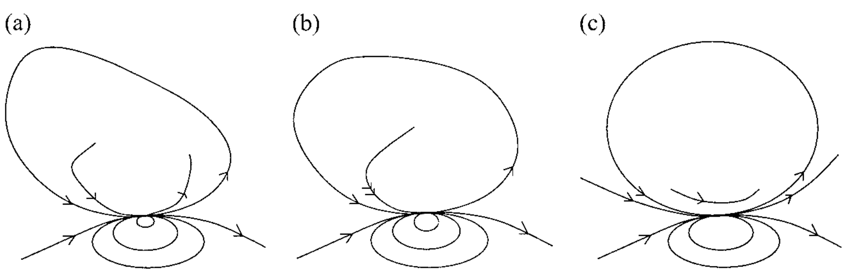


FIG. 2. Graphics through a nilpotent elliptic point. (a) Epp graphic, (b) Ehp graphic, (c) Ehh graphic.

diagram of a cuspidal loop of codimension 3. They give a complete answer for the cyclicity and bifurcation diagram up to a conjecture. From this and the complexity of the bifurcation diagram in the case of the cuspidal loop, it seems hopeless without new methods to find a complete solution to solve the similar question with triple nilpotent points. Fortunately we will show that the question of proving the finite cyclicity of a graphic is much simpler and that indeed we can give a complete answer to this question for several graphics of codimension 3 and 4. This means in particular that we do not consider the birth of small limit cycles from the singularities but only the large limit cycles which coalesce with the graphic when the parameters vanish.

In this paper, we study the finite cyclicity of graphics with a nilpotent singularity of saddle or elliptic type, i.e., the existence of a bound for the number of limit cycles which can bifurcate from such graphics. In some of the finite cyclicity theorems, we will only use the multiplicity of the nilpotent point and not its codimension, the finite cyclicity following from a global genericity assumption. The precise definition of cyclicity for a limit periodic set was given by Roussarie [30].

Definition 1.1. A limit periodic set Γ of a vector field X_{μ_0} inside a family X_μ has finite cyclicity in X_μ if there exist $N \in \mathbb{N}$ and $\varepsilon, \delta > 0$ such that any X_μ with $|\mu - \mu_0| < \delta$ has at most N limit cycles γ_i such that $\mathrm{dist}_H(\Gamma, \gamma_i) < \varepsilon$. The minimum of such N when ε and δ tend to zero is called the cyclicity of Γ in X_μ, which we denote by $Cycl(\Gamma)$.

Let X be a smooth vector field on $\mathbb{R}^2$. A singular point p (i.e., $X(p) = 0$) is said to be a triple nilpotent point of saddle or elliptic type if there is a local chart (x, y): $(\mathbb{R}^2, p) \to (\mathbb{R}^2, 0)$ in which the vector field has the form (Takens [37])

$$X = y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + dx^4 + bxy + ax^2y + ex^3y)\frac{\partial}{\partial y} + O(|(x, y)|^5),$$

where for the saddle case, $\varepsilon_1 = 1$; for the elliptic case $\varepsilon_1 = -1$, $b > 2\sqrt{2}$. Denoting the graphic with a nilpotent singularity by (X, p, Γ), we are going to study the cyclicity of Γ by considering a three-parameter unfolding X_μ of X.

To describe the different types of graphics we use a weighted blow-up $(x, y) = (r\bar{x}, r^2\bar{y})$ for the singular point. Following the convention of Kotova and Stanzo [24], we use pp to denote a graphic going out of a parabolic sector to a parabolic sector, hp to denote a graphic going out of a hyperbolic sector to a parabolic sector, and hh to denote a graphic going out of a hyperbolic sector to a hyperbolic sector. Then, a graphic through an elliptic point can be of three different types (Fig. 3):

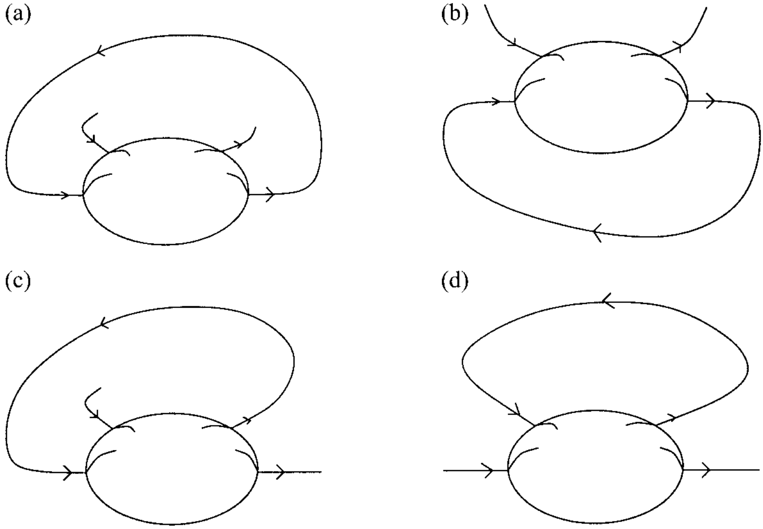


FIG. 3. Pp, hp, and hh graphics of elliptic type. (a) Epp upper, (b) Epp lower, (c) Ehp upper, (d) Ehh upper.

- pp graphic: Epp (codimension 3),
- hp graphic: Ehp (this codimension 3 type of graphic was not mentioned in [24]),
- hh graphic: Ehh (codimension 4).

Each graphic can occur in two versions: upper and lower (see one example in Fig. 3b). Although the upper and lower graphics may have different bifurcation diagrams, the proofs of their finite cyclicity are the same.

A graphic through a nilpotent saddle can be of two different types (Fig. 4): hh graphic convex and Sxhh and hh graphic concave (Sahh).

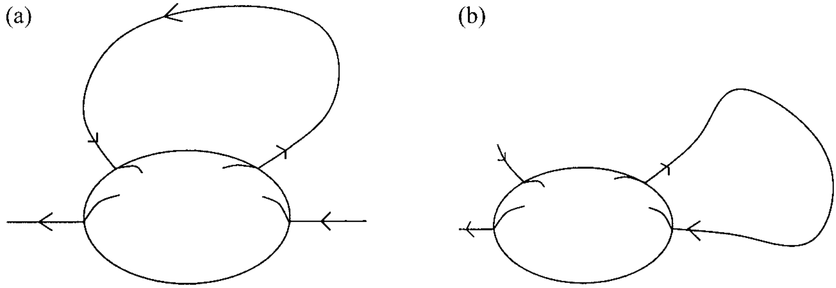


FIG. 4. Convex and concave graphics of saddle type. (a) Sxhh, (b) Sahh.

TABLE I

Main Results Concerning the Finite Cyclicity

Theorem	Graphic	Conditions	Codimension	Cyclicity
5.4	*Sxhh convex*	$P'(0) \neq 1$ $a \neq -\frac{1}{2}$ *in* (2.5) ($b \neq 0$ *in* (2.1))	4	Finite
6.3	Epp	$R^{(n)}(0) \neq 0 (n \geqslant 2)$ $a \neq \frac{1}{2}$ *in* (2.5) ($b > 2\sqrt{2}$ *in* (2.1))	$n+1$	n
6.6	Ehp	$a \neq \frac{1}{2}$ *in* (2.5) ($b > 2\sqrt{2}$ *in* (2.1))	3	Finite
6.9	Ehh	$P'(0) \neq 1$ $a \neq \frac{1}{2}$ *and* $\bar{\varepsilon}_2 \neq 0$ *in* (2.5) ($b > 2\sqrt{2}$ *and* $\varepsilon_2 \neq 0$ *in* (2.3))	4	Finite

Due to technical difficulties, we do not consider the concave graphic of saddle type. For other graphics listed in Figs. 3 and 4, we have proved four main theorems which we list in Table I.

To prove the finite cyclicity theorems listed above, one basic ingredient is the blow-up of families developed in [10] and [14]. Around that we set up a machinery which can be used for other similar graphics. Some of these tools have been introduced for the study of the cuspidal loop [14]. These tools include:

1. Normal form for a family with a nilpotent singularity: we develop a special normal form different from the classical one and allowing:

- use of the special properties of quadratic systems:
  - some transitions occur along straight lines,
  - convexity of some trajectories,
  - knowledge of the center conditions;
- easy application to graphics inside quadratic systems.

2. Blow-up of the family to allow the calculations of the passage maps near the nilpotent singularity.

3. The list of limit periodic sets appearing in the blown-up family of vector fields and which must be proved to have finite cyclicity.

4. The calculations of the different types of Dulac maps in the neighborhood of the singular points of the blown-up sphere.

5. The structure theorems on the Dulac maps allow us to prove that some compositions of a regular transition with two Dulac maps of first type at opposite points on the blown up sphere behave exactly as an affine map (Proposition 5.9). This reduces many proofs of finite cyclicity to proofs identical to the proofs of finite cyclicity for graphics in the plane with elementary singular points.

6. To derive finite cyclicity property, we consider systems whose number of solutions bounds the number of fixed points of the return map in the neighborhood of a limit periodic set under a small perturbation of the blown-up vector field. We derive bounds for the number of solutions of these systems by a generalized derivation–division method.

7. We introduce a general method to prove that some regular transitions have a nonzero higher derivative at a given point.

In a second paper [33] we will apply these results to quadratic systems and the finiteness part of Hilbert's 16th problem [11].

The paper is organized as follows: In Section 2, we develop a new general normal form unfolding the nilpotent singularity of saddle or elliptic type of codimension 3. In Section 3, we make the global blow-up for the family. In Section 4, we study two types of Dulac maps. We prove the main finite cyclicity theorems for saddle and elliptic cases respectively in Sections 5 and 6. Some details have been omitted for parts of the proof which are identical to existing proofs in the literature, but full details are written in [38].

2. NORMAL FORMS UNFOLDING THE SINGULARITY

In this section, we will first develop a new normal form unfolding the codimension 3 nilpotent singularity of saddle or elliptic type different from the standard unfolding used in [13].

We know by [36] that the germs of C^∞ vector fields at $0 \in \mathbb{R}^2$ whose 1-jet is nilpotent and 2-jet is C^∞-conjugate to a vector field with a 2-jet $y\frac{\partial}{\partial x} + \beta xy\frac{\partial}{\partial y}$ is C^∞-conjugate to a vector field with 4-jet

$$y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + dx^4 + bxy + ax^2y + ex^3y)\frac{\partial}{\partial x}, \tag{2.1}$$

where $\varepsilon_1 = 0, \pm 1$ and $a, b, c, d \in \mathbb{R}$, $b \geqslant 0$.

The codimension of the point is determined by looking at b and the quantity

$$\mathcal{Q}_{(2.1)} := 5\varepsilon_1 a - 3bd \tag{2.2}$$

associated with the 4-jet (2.1).

By [13], the vector field is C^∞-equivalent to a vector field with a 4-jet

$$y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + bxy + \varepsilon_2 x^2 y + f x^3 y)\frac{\partial}{\partial x}, \tag{2.3}$$

where $\varepsilon_{1,2} = \pm 1$ and ε_2 is a multiple of $\mathcal{Q}_{(2.1)}$.

The topological type falls into one of the following categories (Fig. 5):

(1) The saddle case: $\varepsilon_1 = 1$, any ε_2 and b (a topological saddle).

(2) The focus case: $\varepsilon_1 = -1$ and $0 < b < 2\sqrt{2}$ (a topological focus).

(3) The elliptic case: $\varepsilon_1 = -1$ and $b \geqslant 2\sqrt{2}$ (an elliptic point).

For $\varepsilon_2 = 1$ and $\varepsilon_2 = -1$, the saddle points (resp. elliptic points) have the same topological type.

For $\varepsilon_1 = 1$ (resp. $\varepsilon_1 = -1$), the nilpotent singularity is of codimension 3 if $\varepsilon_2 \neq 0$, $b \neq 0$ (resp. $\varepsilon_2 \neq 0$ $b \neq 2\sqrt{2}$); it is of codimension $\geqslant 4$ if $\varepsilon_2 = 0$, $b = 0$, or $b = 2\sqrt{2}$.

We are interested only in the vector fields with a triple nilpotent singularity of saddle or elliptic type with $\varepsilon_1 = \pm 1$ and $b > \sqrt{2}$ if $\varepsilon_1 = -1$. A family containing this singularity can be brought to [13]

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 x^3 + \lambda_2 x + \lambda_1 + y(\lambda_3 + bx + \varepsilon_2 x^2 + x^3 h(x, \lambda)) + y^2 Q(x, y, \lambda), \end{aligned} \tag{2.4}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are the parameters and $Q(x, y, \lambda)$ is C^∞ in (x, y, λ) and of arbitrarily high order in (x, y, λ). b is now a variable parameter depending on λ. For $\lambda = 0$, $b(0)$ satisfies the condition described above.

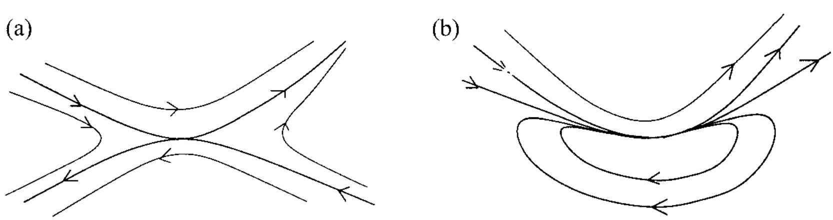


FIG. 5. The different topological types. (a) Saddle case, (b) elliptic case.

Remark 2.1. Some of the work done here will also be useful for higher codimension nilpotent saddle or elliptic singularities in the case $\varepsilon_2 = 0$ and/or $b = 0$.

In this normal form (2.4), the principal part (the remaining part on the blown-up sphere) will be cubic. We develop a new normal form for the unfolding of the nilpotent singularity of the saddle and elliptic type, so that the principal part becomes quadratic.

THEOREM 2.2. *The family* (2.4) *is* C^∞*-equivalent to*

$$\begin{aligned}\dot{X} &= Y + \mu_2 + aX^2\\ \dot{Y} &= \mu_1 + Y(\mu_3 + X + \bar{\varepsilon}_2 X^2 + X^3 h_1(X,\mu)) + X^4 h_2(X,\mu) + Y^2 Q(X,Y,\mu),\end{aligned} \tag{2.5}$$

where $\bar{\varepsilon}_2 = -a\varepsilon_1\varepsilon_2$, *and*

- *for the saddle case:* $a(0) \in (-\frac{1}{2}, 0)$;

if $a(0) = -\frac{1}{2}$, *the unfolding is of codimension* 4 *which corresponds to the case* $b(0) = 0$.

- *for the elliptic case:* $a(0) \in (0, \frac{1}{2})$;

if $a(0) = \frac{1}{2}$, *the unfolding is of codimension* 4, *type* 1, *which corresponds to the case* $b(0) = 2\sqrt{2}$ *(the two characteristic trajectories coalesce into one).*

$\mu = (\mu_1, \mu_2, \mu_3)$ *is the parameter,* $h_1(X,\mu)$, $h_2(X,\mu) = \bar{\varepsilon}_2 a + O(\mu) + O(X)$ *and* $Q(X,Y,\mu)$ *are* C^∞ *and* $Q(X,Y,\mu)$ *is of arbitrary high order in* (X,Y,μ).

Proof. In the family (2.4), we make the transformation

$$\begin{aligned}x &= m_1 + \bar{X}\\ y &= m_2 + \bar{Y} + a_2\bar{X}^2.\end{aligned} \tag{2.6}$$

To eliminate the terms $\bar{X}$, $\bar{X}^2$, $\bar{X}^3$ in the second equation for the new system, we need to solve

$$F(m_1, m_2, a_2, \lambda) := \begin{pmatrix} \lambda_2 + (b - 2a_2)\, m_2 + d_1(m_1, m_2) \\ a_2\lambda_3 + (a_2 b + 3\varepsilon_1)\, m_1 + \varepsilon_2 m_2 + d_2(m_1, m_2) \\ \varepsilon_1 + a_2 b - 2a_2^2 + d_3(m_1, m_2) \end{pmatrix} = 0. \tag{2.7}$$

For $\lambda = 0$ and $m_1 = m_2 = 0$, by (2.7) we have a equation for $a_2(0)$:

$$2a_2^2(0) - b(0)\, a_2(0) - \varepsilon_1 = 0. \tag{2.8}$$

Then

$$a_2^{\pm}(0)=\tfrac{1}{4}\,[b(0)\pm\sqrt{b^2(0)+8\varepsilon_1}\,].$$

We choose $a(0)=a^-(0)$. Note that

$$F(0,0,a_2(0),0)=0,$$

$$\det\left(\frac{\partial F(m_1,m_2,a_2,\lambda)}{\partial(m_1,m_2,a_2)}\bigg|_{(0,0,a_2(0),0)}\right)=\frac{2\varepsilon_1(a^2(0)+\varepsilon_1)(2a^2(0)+\varepsilon_1)}{a^2(0)}\neq 0.$$

So by the implicit function theorem, we solve $F(m_1,m_2,a_2,\lambda)=0$ in the neighborhood of $0\in\mathbb{R}^6$, and the solution of (2.7) can be written as

$$\begin{aligned}
a_2&=a_2(0)+O(|\lambda|)\\
m_1&=-\frac{a_2(0)\,\varepsilon_1}{2(a_2^2(0)+\varepsilon_1)}\,[\varepsilon_2\lambda_2+\varepsilon_1\lambda_3]+O(|\lambda|^2)\\
m_2&=a_2(0)\,\varepsilon_1\lambda_2+O(|\lambda|^2).
\end{aligned}$$

The family has the form

$$\begin{aligned}
\dot{\bar X}&=\bar Y+m_2+a_2\bar X^2\\
\dot{\bar Y}&=\lambda_1+m_1\lambda_2+m_2\lambda_3+O(|(m_1,m_2)|^2)\\
&\quad+\bar Y[\lambda_3+bm_1+O(|(m_1,m_2)|^2)+b_1(\lambda)\,\bar X+b_2(\lambda)\,\bar X^2+\bar X^3h_{11}(\bar X,\lambda)]\\
&\quad+\bar X^4h_{12}(\bar X,\lambda)+\bar Y^2Q_1(\bar X,\bar Y,\lambda),
\end{aligned}\tag{2.9}$$

where $h_{11}(\bar X,\lambda)$, $h_{12}(\bar X,\lambda)$, and $Q_1(\bar X,\bar Y,\lambda)$ are C^∞. Also Q_1 is of arbitrarily high order in its variables and

$$\begin{aligned}
b_1(\lambda)&=b(0)-2a_2(0)+O(|\lambda|)=-\frac{\varepsilon_1}{a_2(0)}+O(|\lambda|)\\
b_2(\lambda)&=\varepsilon_2+O(|\lambda|)\\
h_{12}(X,\lambda)&=\varepsilon_2a_2+O(\lambda)+O(X).
\end{aligned}$$

Rescaling in (2.9) by $(\bar X,\bar Y)=(X/b_1(\lambda),Y/b_1(\lambda))$, then we get a new normal form

$$\begin{aligned}
\dot X&=Y+\mu_2+aX^2\\
\dot Y&=\mu_1+Y[\mu_3+X+\bar\varepsilon_2X^2+X^3h_1(X,\mu)]+X^4h_2(X,\mu)+Y^2Q(X,Y,\mu),
\end{aligned}\tag{2.10}$$

where

$$\begin{aligned} a &= -\varepsilon_1 a_2^2 + O(\lambda) \\ \bar{\varepsilon}_2 &= \varepsilon_2 a_2^2 + O(\lambda) \\ h_2(\bar{X}, \lambda) &= -\varepsilon_1 \varepsilon_2 a_2^4 + O(\lambda) + O(\bar{X}). \end{aligned}$$

Also $\mu = (\mu_1, \mu_2, \mu_3)$ is the new parameter with

$$\begin{aligned} \mu_1 &= \frac{\varepsilon_1}{a_2(0)} \lambda_1 + O(|\lambda|^2) \\ \mu_2 &= -\lambda_2 + O(|\lambda|^2) \\ \mu_3 &= -\frac{\varepsilon_1(2a_2(0) - \varepsilon_1)}{2(a_2(0) + \varepsilon_1)} [-\varepsilon_2 \lambda_2 + 3\lambda_3] + O(|\lambda|^2). \quad \blacksquare \end{aligned}$$

PROPOSITION 2.3. *If instead of choosing $a_2^-(0)$ of* (2.8) *we choose the other root, we obtain $a \in (-\infty, -\frac{1}{2})$ (resp. $a \in (\frac{1}{2}, \infty)$) for the saddle (resp. elliptic case). For the saddle case, family* (2.5) *with $a(0) \in (-\infty, -\frac{1}{2})$ and $a(0) \in (-\frac{1}{2}, 0)$ are C^∞-equivalent. But for the elliptic case, family* (2.2) *with $a(0) \in (0, \frac{1}{2})$ and $a(0) \in (\frac{1}{2}, \infty)$ are C^∞-equivalent except for $a(0) = \frac{1}{4}$.*

Proof. Let $a(0) = a_2^+(0)$; then $a(0) \in (-\frac{1}{2}, 0)$. Consider family (2.5). Under the transformation

$$X = \hat{X}, Y = \hat{Y} + (\tfrac{1}{2} - a)\, \hat{X}^2 \tag{2.11}$$

family (2.5) becomes

$$\begin{aligned} \dot{\hat{X}} &= \mu_2 + \hat{Y} + \tfrac{1}{2} \hat{X}^2 \\ \dot{\hat{Y}} &= \mu_1 - (1 - 2a)\, \mu_2 \hat{X} + (\tfrac{1}{2} - a)\, \mu_3 \hat{X}^2 + \hat{Y}(\mu_3 + 2a\hat{X} + \varepsilon_2 \hat{X}^2 + \hat{X}^3 h_{12}) \\ &\quad + \hat{X}^4 h_{22} + \hat{Y}^2 Q_2(\hat{X}, \hat{Y}, \mu). \end{aligned} \tag{2.12}$$

To eliminate the terms $\hat{X}$, $\hat{X}^2$ in the second equation of (2.12), there exists a transformation of the form

$$\begin{aligned} \hat{X} &= \tilde{X} + \beta_1 \\ \hat{Y} &= \tilde{Y} + \beta_2 + \beta_3 \tilde{X} + \beta_4 \tilde{X}^2, \end{aligned} \tag{2.13}$$

where

$$
\begin{aligned}
\beta_1 &= \frac{1-2a}{a(4a-1)}(\varepsilon_2\mu_2 + a\mu_3) + O(|\mu|^2) \\
\beta_2 &= \frac{1-2a}{2a} + O(|\mu|) \\
\beta_3 &= -\frac{1-2a}{a(4a-1)}(\varepsilon_2\mu_2 + a\mu_3) + O(|\mu|^2) \\
\beta_4 &= \frac{1}{2a(4a-1)}[(4a-1)\,d_1 - 2\varepsilon_2^2 + 8\varepsilon_2 d_2)\,\mu_2 + 2a(4d_2\varepsilon_2)\,\mu_3] + O(|\mu|^2)
\end{aligned}
\tag{2.14}
$$

and by the transformation (2.13) and a rescaling in the variables $\tilde{X}$ and $\tilde{Y}$, family (2.12) becomes

$$
\begin{aligned}
\dot{\tilde{X}} &= \tilde{Y} + \tilde{\mu}_2 + a'\tilde{X}^2 \\
\dot{\tilde{Y}} &= \tilde{\mu}_1 + \tfrac{1}{2}\eta\tilde{X}^2 + \tilde{Y}[\tilde{\mu}_3 + \tilde{X} + \tilde{\varepsilon}_2\tilde{X}^2 + \tilde{X}^3 h_{14}(\tilde{X}, \tilde{\mu})] \\
&\quad + \tilde{X}^4 h_{24}(\tilde{X}, \mu) + \tilde{Y}^2 Q_4(\tilde{X}, \tilde{Y}, \tilde{\mu}),
\end{aligned}
\tag{2.15}
$$

where

$$
\eta = \begin{cases} 1 & \text{if } a(0) = \frac{1}{4} \\ 0 & \text{if } a(0) \neq \frac{1}{4}, \end{cases}
$$

$$
a' = \frac{1}{4a} + O(|\tilde{\mu}|) \qquad \text{and} \qquad a'(0) = \frac{1}{4a(0)} \in \left(-\infty, -\frac{1}{2}\right);
$$

also for the new parameter $(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$ we have

$$
\det\left(\frac{\partial(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3}{\partial(\mu_1, \mu_2, \mu_3)}\right)\bigg|_{\mu=0} = \frac{4a(0)(1-a(0))}{4a(0)-1} \neq 0. \tag{2.16}
$$

So for the saddle case, family (2.5) with $a(0) \in (-\frac{1}{2}, 0)$ and $a(0) \in (-\infty, -\frac{1}{2})$ are C^∞-equivalent. For the elliptic case, family (2.2) with $a(0) \in (0, \frac{1}{2})$ and $a(0) \in (\frac{1}{2}, \infty)$ is C^∞-equivalent except for $a(0) = \frac{1}{4}$. The presence of the additional term for $a(0) = \frac{1}{4}$ comes from the fact that the eigenvalues at P_3 and P_4 are $\pm\frac{1}{2}$, $\mp\frac{1}{2}$, $\mp\frac{1}{2}$; hence the linear part can have a Jordan block for the eigenvalue 1. ▮

3. GENERALITIES ON THE BLOW-UP OF THE FAMILY

3.1. *Blow-up of the Family*

Consider the normal form for the unfolding of the nilpotent singularity of saddle or elliptic type

$$X: \begin{cases} \dot{x} = y + \mu_2 + ax^2 \\ \dot{y} = \mu_1 + y(\mu_3 + x + \varepsilon_2 x^2 + x^3 h_1(x, \mu)) + x^4 h_2(x, \mu) + y^2 Q(x, y, \mu), \end{cases} \tag{3.1}$$

where $a \in [-\frac{1}{2}, 0)$ is the saddle case, $a \in (0, \frac{1}{2})$ is the elliptic case, $\mu = (\mu_1, \mu_2, \mu_3, \lambda)$ is the parameter, and $h_1(x, \mu)$, $h_2(x, \mu) = \varepsilon_1 \varepsilon_2 a + O(x)$ and $Q(x, y, \mu)$ are C^∞ and also $Q(x, y, \mu)$ has arbitrarily high order in (x, y, μ).

From now on, we denote $A = (0, \frac{1}{2})$ for the elliptic case and $A = (-\frac{1}{2}, 0)$ for the saddle case.

We are interested in this family for $a \in A$ and $(x, y, \mu) \in U \times \Lambda$, a neighborhood of $(0, 0)$ in $\mathbb{R}^2 \times \mathbb{R}^3$. Λ can be taken as $\mathbb{S}^2 \times [0, \nu_0)$. Making the change of parameters

$$\begin{aligned} \mu_1 &= \nu^3 \bar{\mu}_1 \\ \mu_2 &= \nu^2 \bar{\mu}_2 \\ \mu_3 &= \nu \bar{\mu}_3, \end{aligned} \tag{3.2}$$

where $(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3) \in \mathbb{S}^2$ and $\nu \in (0, \nu_0)$, we have a three-parameter family of vector fields in (x, y, ν) space with parameters $(a, \bar{\mu}) \in A \times \mathbb{S}^2$:

$$\hat{X}: \begin{cases} \dot{x} = y + \nu^2 \bar{\mu}_2 + ax^2 \\ \dot{y} = \nu^3 \bar{\mu}_1 + y[\nu \bar{\mu}_3 + x + \varepsilon_2 x^2 + x^3 h_1(x, \nu\bar{\mu})] \\ \qquad + x^4 h_2(x, \nu\bar{\mu}) + y^2 Q(x, y, \nu\bar{\mu}) \\ \dot{\nu} = 0. \end{cases} \tag{3.3}$$

We then make the (weighted) blow-up of the singular point of (3.3) at the origin by

$$\begin{aligned} x &= r\bar{x} \\ y &= r^2 \bar{y} \\ \nu &= r\rho, \end{aligned} \tag{3.4}$$

where $r > 0$ and $(\bar{x}, \bar{y}, \rho) \in \mathbb{S}^2$.

By the blow-up (3.4), we have a C^∞-family $\bar{X} = \frac{1}{r} \hat{X}$. For each $(a, \bar{\mu}) \in A \times \mathbb{S}^2$, $\bar{X}$ induces a 3-dimensional vector field $\bar{X}_{(a, \bar{\mu})}$ defined in the neighborhood of $\mathbb{S}^2 \times \{0\}$ with parameters $(a, \bar{\mu}) \in A \times \mathbb{S}^2$.

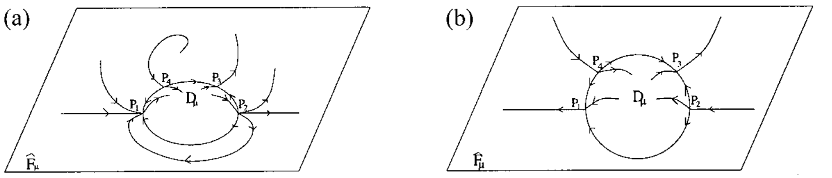


FIG. 6. The stratified set $\{r\rho=0\}$ in the blow-up. (a) Elliptic case, (b) saddle case.

A combination of (3.2) and (3.4), as in [14], for (3.1), at $(x, y, \mu_1, \mu_2, \mu_3) = (0, 0, 0, 0, 0)$, yields a global blow-up

$$\Phi\colon \mathbb{S}^2 \times \mathbb{R}^+ \times \mathbb{S}^2 \to \mathbb{R}^5$$

$$((\bar{x}, \bar{y}, \rho), r, (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)) \mapsto (x, y, \mu_1, \mu_2, \mu_3), \tag{3.5}$$

where $\bar{x}^2+\bar{y}^2+\rho^2=1$.

Because of the symmetry, as in Fig. 6, we only need to study $\bar{X}$ on $\{\rho \geqslant 0\}$ to get complete information for $((\bar{x}, \bar{y}, \rho), r, (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3))$ near $0 \in \mathbb{S}^2 \times [0, r_0) \times \mathbb{S}^2$. Note that for each $\bar{\mu}$, the foliation given by $\{v = r\rho = const\}$ is preserved by $\bar{X}_{(a, \bar{\mu})}$:

- For $\{r\rho = v\}$ with $v > 0$, the leaf is a regular manifold of dimension 2.
- For $\{r\rho = 0\}$, we get a stratified set in the critical locus. As shown in Fig. 6, there are two strata of 2-dimensional manifolds:

— $\hat{F}_{\bar{\mu}} \cong S^1 \times R^+$, the blow-up of the fiber $\mu = 0$,

— $D_{\bar{\mu}} = \{\bar{x}^2+\bar{y}^2+\rho^2=1, \rho \geqslant 0\}$.

On $\hat{F}_{\bar{\mu}} = \{\rho = 0\}$, (3.5) is just the common blow-up of the nilpotent point

$$\begin{aligned} x &= r\bar{x} \\ y &= r^2\bar{y} \end{aligned} \tag{3.6}$$

and by the blow-up (3.6), we get a vector field with four singular points P_i ($i = 1, 2, 3, 4$). P_3 and P_4 are hyperbolic saddles, while P_1 and P_2 are nodes (resp. saddles) in the elliptic (resp. saddle) case (Fig. 7).

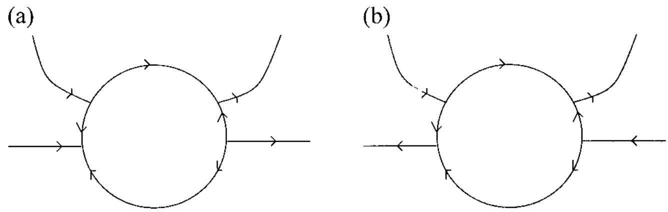


FIG. 7. Common blow-up of the nilpotent singularity. (a) Elliptic case, (b) saddle cases.

To study the objects on and near $\hat{F}_{\bar{\mu}}$, we use the "phase directional rescaling." We use charts

$$\textbf{P.R.1} \qquad \bar{x} = -1, (r_1, \rho_1, \bar{y}_1)$$
$$\textbf{P.R.2} \qquad \bar{x} = 1, (r_2, \rho_2, \bar{y}_2)$$

which cover the boundary of the half 2-sphere. In the charts **P.R.1** and **P.R.2**, by transformation ($i = 1, 2$)

$$\begin{aligned} x &= \mp r_i \\ y &= r_i^2 \bar{y}_i \\ \mu_j &= (r_i \rho_i)^{4-j}\, \bar{\mu}_j \qquad (j = 1, 2, 3) \end{aligned} \tag{3.7}$$

and after division by r_i, we get a vector field near P_i ($i = 1, 2$)

$$\bar{\mathbf{X}}_{\mathbf{P}_i} \begin{cases} \dot{r}_i = \mp(a + \bar{y}_i + \bar{\mu}_2 \rho_i^2)\, r_i \\ \dot{\rho}_i = \pm(a + \bar{y}_i + \bar{\mu}_2 \rho_i^2)\, \rho_i \\ \dot{\bar{y}}_i = \mp(1-2a)\, \bar{y}_i \pm 2\bar{y}_i^2 + \bar{y}_i[\varepsilon_2 r_i + \bar{\mu}_3 \rho_i \pm 2\bar{\mu}_2 \rho_i^2 \mp r_i^2 h_1(\pm r_i, r_i\rho_i, \bar{\mu})] \\ \qquad + \bar{\mu}_1 \rho_i^3 + r_i \bar{h}_2(\pm r_i, r_i \rho_i, \bar{\mu}) + \bar{y}_i^2 \bar{Q}_2(r_i, \rho_i, \bar{y}_i, \bar{\mu}), \end{cases} \tag{3.8}$$

where $\bar{h}_1$ and $\bar{h}_2 = a\varepsilon_2 + O(r)$ are C^∞ in $(r_i, \rho_i, \bar{\mu})$ and $\bar{Q}(r_i, \rho_i, \bar{y}_i, \bar{\mu})$ is C^∞ in $(r_i, \rho_i, \bar{y}_i, \bar{\mu})$ and of arbitrarily high order in $(r_i, \rho_i, \bar{y}_i)$.

Easily, we see that $\bar{X}_{P_1}$ has two singular points $P_1(0, 0, 0)$ and $P_4(0, 0, \frac{1-2a}{2})$, $\bar{X}_{P_2}$ has two singular points $P_2(0, 0, 0)$ and $P_3(0, 0, \frac{1-2a}{2})$. Each singularity has three real eigenvalues listed in Table II. In Fig. 8, we draw the phase portraits in the chart **P.R.1** for the elliptic and saddle case, respectively.

TABLE II

Eigenvalues at P_i ($i = 1, 2, 3, 4$)

	r	ρ	y
P_1	$-a$	a	$-(1-2a)$
P_2	a	$-a$	$(1-2a)$
P_3	$1/2$	$-1/2$	$-(1-2a)$
P_4	$-1/2$	$1/2$	$(1-2a)$

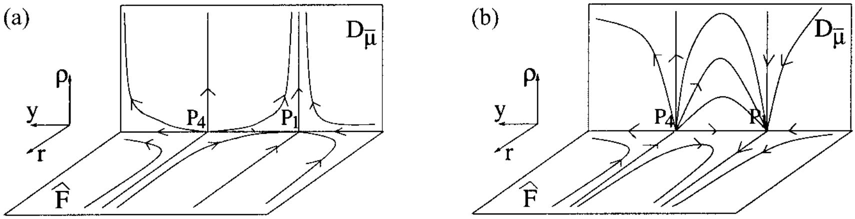


FIG. 8. The phase portraits of $\bar{X}_{P_1}$. (a) $\bar{X}_{P_1}$, the elliptic case; (b) $\bar{X}_{P_1}$, the saddle case.

To complete the phase portrait on the blown-up sphere $D_{\bar{\mu}}$, we use the family rescaling and the chart **F.R.** $\rho = 1$, $(\bar{x}, \bar{y}, r)$, yielding

$$\begin{aligned} \dot{\bar{x}} &= \bar{\mu}_2 + \bar{y} + a\bar{x}^2 \\ \dot{\bar{y}} &= \bar{\mu}_1 + (\bar{x} + \bar{\mu}_3)\,\bar{y} + r\bar{h}(\bar{x}, \bar{y}, r, \bar{\mu}) \\ \dot{r} &= 0, \end{aligned} \tag{3.9}$$

where $\bar{h}(\bar{x}, \bar{y}, r, \bar{\mu})$ is C^∞ in $(\bar{x}, \bar{y}, r, \bar{\mu})$. Especially, on $\{r = 0\}$, we have

$$\bar{X}_{\rho=1} \begin{cases} \dot{\bar{x}} = \bar{\mu}_2 + \bar{y} + a\bar{x}^2 \\ \dot{\bar{y}} = \bar{\mu}_1 + (\bar{\mu}_3 + \bar{x})\,\bar{y}. \end{cases} \tag{3.10}$$

In order to list all the possible limit periodic sets for the family, we have to study the bifurcation diagram of (3.10) for $\bar{\mu} \in \mathbb{S}^2$.

3.2. *Bifurcation Diagrams for the Family Rescaling and Limit Periodic Sets*

Following the convention of Kotova and Stanzo [24], we use **pp** to denote a graphic connecting two parabolic sectors, **hp** to denote the graphic coming out of a hyperbolic sector and connecting to a parabolic sector, and **hh** to denote the graphic connecting two hyperbolic sectors. We find the limit periodic sets after the blow-up and note that they often occur in families. The geometry of the boundary limit periodic sets is important. We hence give the complete bifurcation diagrams of system (3.10). They correspond via $\bar{y} + \bar{\mu}_2 + a\bar{x}^2 = Y$ to the bifurcation diagrams for the principal rescalings studied in [13] and [9]. Complete bifurcation diagrams have been given there except for the position of the separatrices at infinity which are better studied in the quadratic model given here.

PROPOSITION 3.1. *For the system* (3.10), *there holds*

(1) *System* (3.10) *has an invariant line* $\bar{y} = 0$ *if and only if* $\bar{\mu}_1 = 0$.

• *In the elliptic case, the curve* $\bar{\mu}_1 = 0$ *is a bifurcation curve except when there are two nodes on the line* $\bar{y} = 0$.

• *In the saddle case, the curve* $\bar{\mu}_1 = 0$ *is a bifurcation curve precisely when there are two finite saddles on it.*

(2) *If* $a \neq \frac{1}{4}$, *the system* (3.10) *has an invariant parabola*

$$y = \left(\frac{1}{2} - a\right)x^2 + \bar{\mu}_3 \frac{1-2a}{1-4a} x + \frac{(1-2a)(2a\bar{\mu}_3^2 + (1-4a)^2\,\bar{\mu}_2}{a(1-4a)^2} \tag{3.11}$$

if and only if

$$\bar{\mu}_1 = \frac{2a(1-2a)}{(1-4a)^3}\,\bar{\mu}_3^3 + \frac{2(1-2a)}{1-4a}\,\bar{\mu}_2\bar{\mu}_3. \tag{3.12}$$

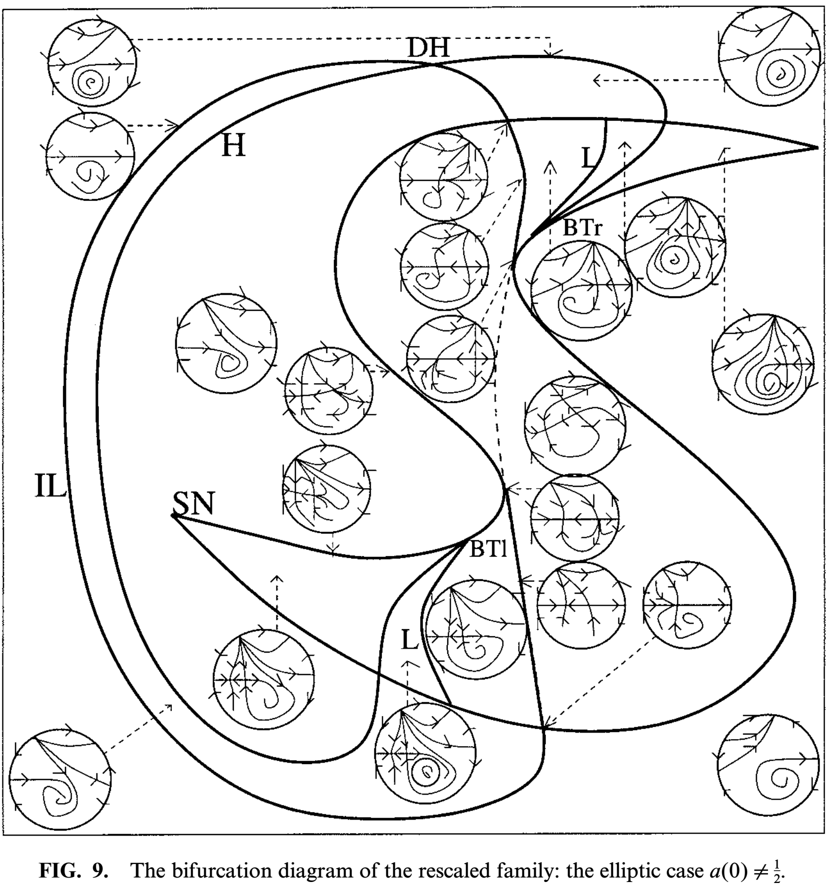


FIG. 9. The bifurcation diagram of the rescaled family: the elliptic case $a(0) \neq \frac{1}{2}$.

(3) *If* $a = \frac{1}{4}$, *the system* (3.10) *has an invariant parabola if and only if* $\bar{\mu}_3 = 0$. *For* $\bar{\mu}_3 = 0$, *the system* (3.10) *has one, two, or three invariant parabolas*

$$y = \tfrac{1}{4}x^2 + Bx + \bar{\mu}_2 + 2B^2 \tag{3.13}$$

if $27\bar{\mu}_1^2 + 16\bar{\mu}_3^3 > 0$, $= 0$, *or* < 0, *and* B *is the solution of the algebraic equation*

$$2B^3 + 2\bar{\mu}_2 B - \bar{\mu}_1 = 0.$$

Proof. Direct calculations. ∎

THEOREM 3.2. *The bifurcation diagram of* (3.10) *for the elliptic case* $(0 < a < \frac{1}{2})$ *is in Fig.* 9, *the bifurcation diagram for the saddle case*

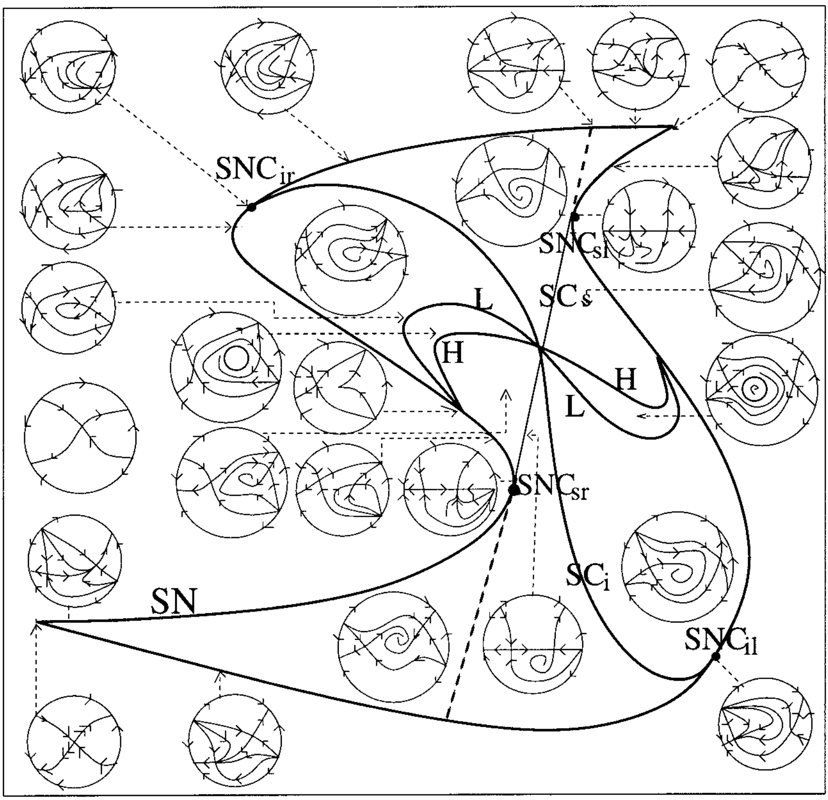
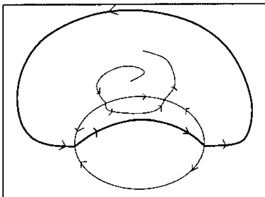
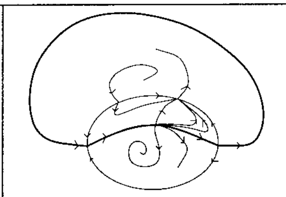
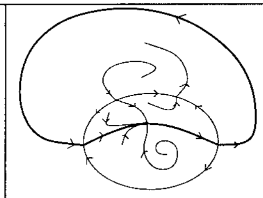


FIG. 10. The bifurcation diagram of the rescaled family: the saddle case $a(0) \neq -\frac{1}{2}$.

TABLE III

Limit Periodic Sets of pp Type for the Elliptic Case

		
graphic Epp1	family of graphics Epp2	graphic Epp3

$(-\frac{1}{2} < a < 0)$ is in Fig. 10. In particular the limit periodic sets are listed in Tables III–VII.

Note that when we study $\bar{X}_{\rho=1}$ at infinity, we use the quasi-homogeneous compactification:

$$\bar{x} = \pm \frac{1}{z} \qquad \bar{y} = \frac{u}{z^2}. \tag{3.14}$$

TABLE IV

Limit Periodic Sets of hp Type for the Elliptic Case

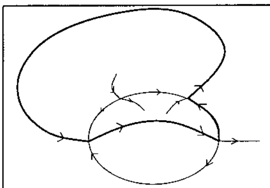
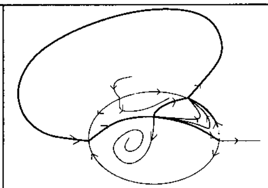
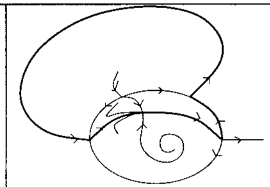
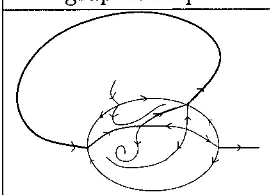
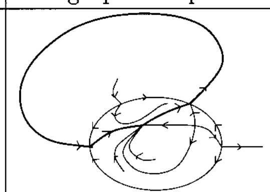
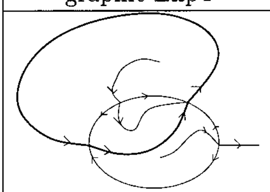
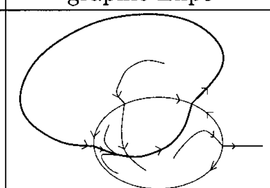
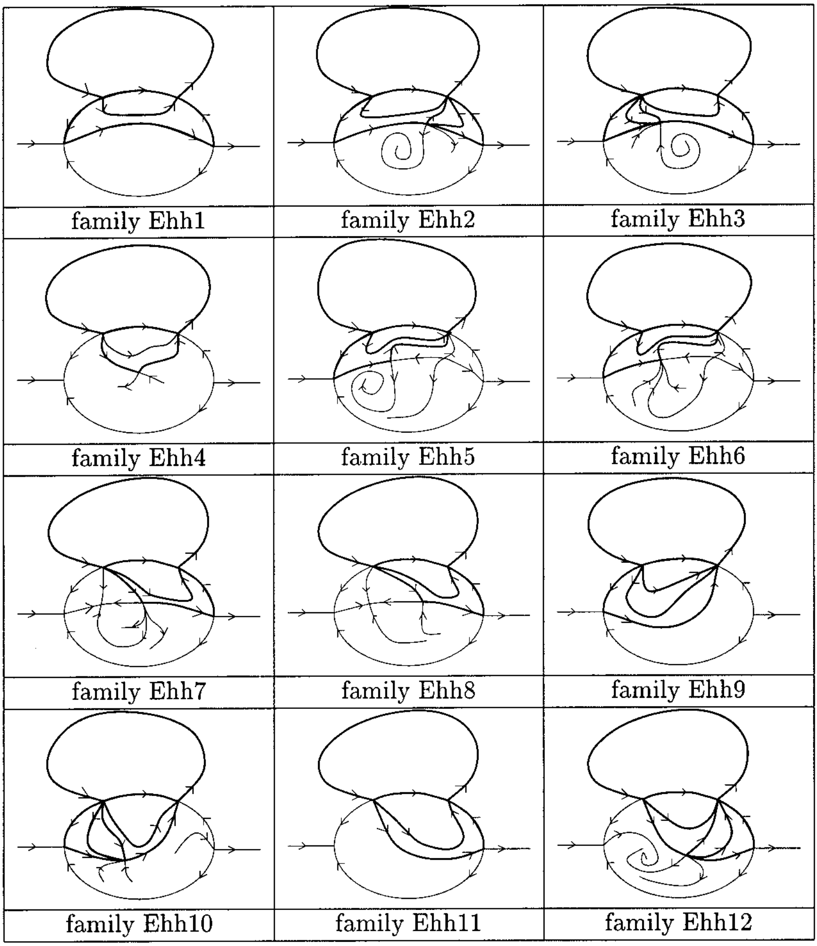
		
graphic Ehp1	graphic Ehp2a, b, c	graphic Ehp3
		
graphic Ehp4		graphic Ehp5
		
graphic Ehp6		graphic Ehp7

TABLE V

Limit Periodic Sets of hh Type for the Elliptic Case



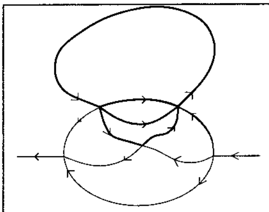
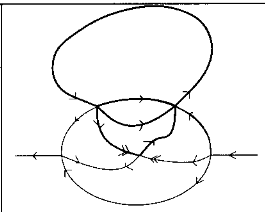
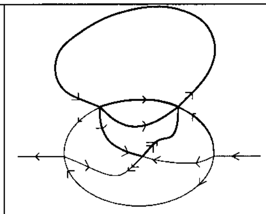
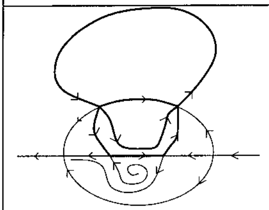
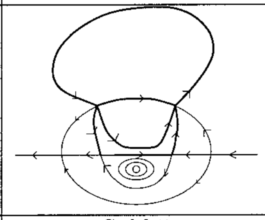
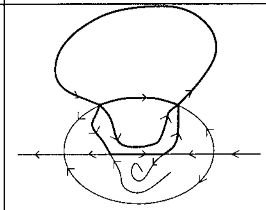
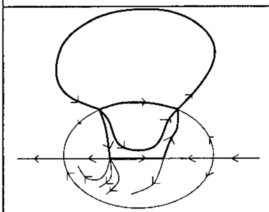
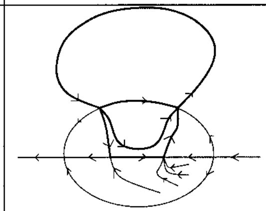
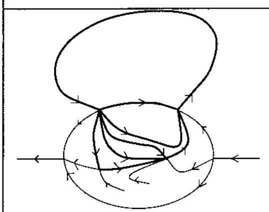
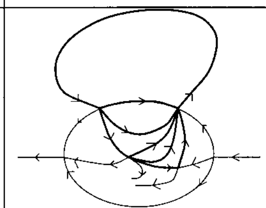
These transformations are just the transformations we used in the charts P.R.1 and P.R.2.

For the elliptic case, there are 22 limit periodic sets which fall into three types: Epp, Ehp, and Ehh. We list the 22 limit periodic sets of elliptic type in Tables III–V.

For the saddle case, there are two types of limit periodic sets: convex graphics Sxhh and concave graphics Sahh. We list all the possible limit periodic sets of saddle type in Tables VI and VII.

TABLE VI

Convex Limit Periodic Sets of hh Type for the Saddle Case

		
Sxhh1	Sxhh2	Sxhh3
		
Sxhh4	Sxhh5	Sxhh6
		
Sxhh7		Sxhh8
		
Sxhh9		Sxhh10

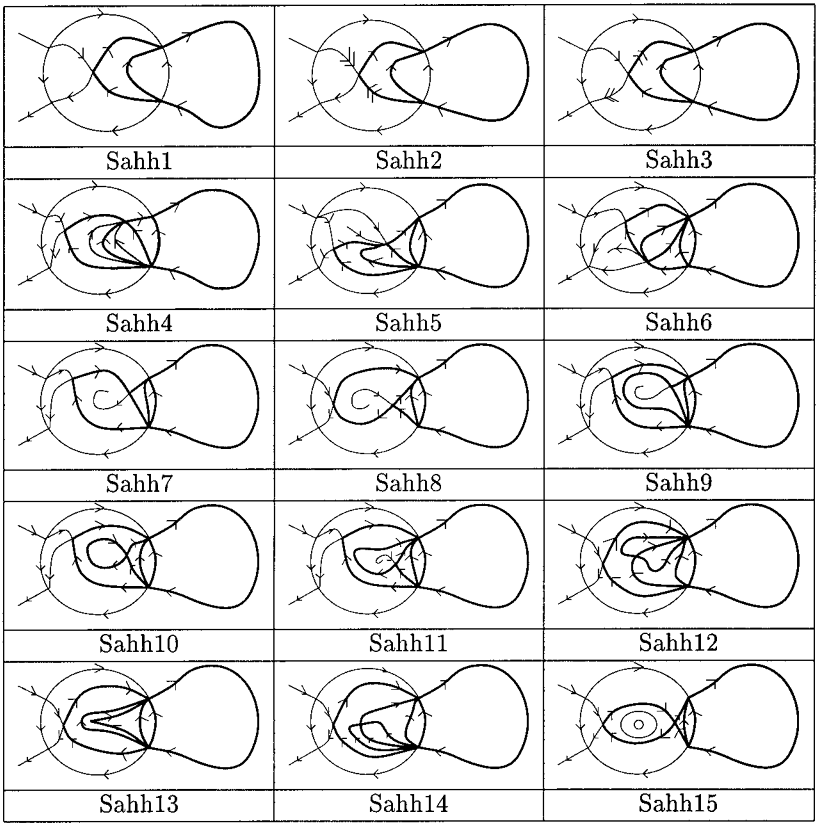
For all the families of limit periodic sets listed in the following tables, we use a to denote the upper boundary graphic, b or d to denote the intermediate graphics, and c or e to denote the lower boundary graphic.

4. THE DULAC MAPS AT THE ENTRANCE POINTS OF THE BLOWN-UP SPHERE

To study the cyclicity of the graphics after the global blow-up, we will need some basic properties of the transition maps in the neighborhood of an elementary singular point.

TABLE VII

Concave Limit Periodic Sets of hh Type for the Saddle Case



4.1. *Transition Maps near the Elementary Singular Points in the Plane*

DEFINITION 4.1. (1) A singular point is elementary if it has at least one nonzero eigenvalue. It is hyperbolic (resp. semihyperbolic) if the two eigenvalues are not on the imaginary axis (resp. exactly one eigenvalue is zero).

(2) The hyperbolicity ratio at a hyperbolic saddle is the ratio $r = -\lambda_1/\lambda_2$, where $\lambda_1 < 0 < \lambda_2$ are the two eigenvalues.

Let X_λ, $\lambda \in \Lambda$, be a C^∞ family of vector fields defined in the neighborhood of a hyperbolic saddle at the origin. We also assume that the coordinate axes are the invariant manifolds near the saddle point.

By the normal form theory, for any fixed $k \in \mathbb{N}$, up to C^k-equivalence, we can write the vector field X_λ into some explicit expressions of the normal form (cf. [20, 35]). Let r_λ be the hyperbolicity ratio of X_λ at the origin; then

- If r_0 is irrational, then, $\forall k \in \mathbb{N}$, the vector field X_λ is C^k-equivalent to

$$\dot{x} = x$$
$$\dot{y} = -r(\lambda)\, y$$

for λ in some neighborhood W of the origin in parameter space.

- If $r_0 \in \mathbb{Q}$, let $r_0 = \frac{p}{q}$, $(p, q) = 1$. Then $\forall k \in \mathbb{N}$, X_λ is C^k-equivalent to

$$\dot{x} = x$$
$$\dot{y} = y\left[-r_0 + \sum_{i=0}^{N(k)} \alpha_{i+1}(\lambda)(x^p y^q)^i\right]$$

for λ in some neighborhood W of the origin in parameter space. In particular, $\alpha_1 = r_0 - r(\lambda)$.

Let $\tilde{\Sigma}_1 = \{y = y_0\}$ and $\tilde{\Sigma}_2 = \{x = x_0\}$ be two sections transversal to the vector field X_λ (Fig. 11), where x_0, $y_0 > 0$ constant. The flow of X_λ induces a transition map $D_\lambda(., \lambda)$, also called the Dulac map

$$D_\lambda\colon \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$$

for all $\lambda \in W$.

The Dulac map is C^∞ for $x > 0$. The following theorem of Mourtada [27] describes its behavior near $x = 0$.

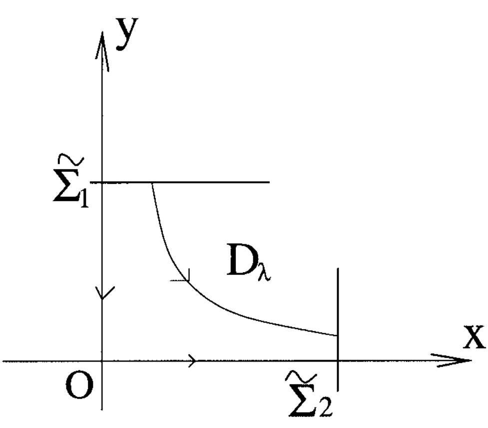


FIG. 11. Dulac map near a hyperbolic saddle.

PROPOSITION 4.2 (Mourtada). *The Dulac map D_λ can be written as*

$$D_\lambda(x) = x^{r(\lambda)}[c(\lambda) + \psi(x, \lambda)], \tag{4.1}$$

where $c(\lambda) = y_0/x_0^{r(\lambda)}$ and $\psi(x, \lambda)$ is C^∞ for $(x, \lambda) \in (0, x_0] \times W$. Furthermore, ψ satisfies the following property (I_0^∞):

$$(I_0^\infty): \qquad \forall n \in \mathbb{N}, \lim_{x \to 0} x^n \frac{\partial^n \psi}{\partial x^n}(x, \lambda) = 0 \text{ uniformly for } \lambda \in W. \tag{4.2}$$

(1) *If $r_0 \in \mathbb{Q}$, then $\psi \equiv 0$;*

(2) *If $r_0 = 1$, then the expression* (4.1) *is in general not fine enough for proving the cyclicity.*

DEFINITION 4.3 [25, 31]. The Leontovich–Andronova–Ecalle–Roussarie compensator of the vector field X_λ is defined as

$$\omega(x, \alpha_1) = \begin{cases} \dfrac{x^{-\alpha_1} - 1}{\alpha_1} & \text{if } \alpha_1 \neq 0 \\ -\ln x & \text{if } \alpha_1 = 0. \end{cases} \tag{4.3}$$

By the definition of ω, we can easily check $\omega(x, \alpha_1)$ has the following property:

PROPOSITION 4.4.

$$\omega(ab, \alpha_1) = \omega(a, \alpha_1)(1 + \alpha_1 \omega(b, \alpha_1)) + \omega(b, \alpha_1). \tag{4.4}$$

Since the Dulac map in Proposition 4.2 is not fine enough to prove the cyclicity for the case $r_0 = 1$, in [30], by using the compensator, Roussarie has an additional refinement:

PROPOSITION 4.5. *If $r_0 = 1$, then the Dulac map D_λ has a well-ordered asymptotic expansion*

$$\begin{aligned} D_\lambda(x) = {} & \alpha_1(\lambda)[x\omega + \cdots] + \beta_1(\lambda)[x + \cdots] \\ & + \alpha_2(\lambda)[x^2\omega + \cdots] + \alpha_k(\lambda)[x^k\omega + \cdots] + \psi_k(x, \lambda), \end{aligned} \tag{4.5}$$

where $\alpha_1(\lambda) = r(\lambda) - 1$, $\beta_1(0) \neq 0$, and ψ is a C^k function, k-flat with respect to $x = 0$.

4.2. *Normal Forms at the Entrance Points*

To study the Dulac maps in the neighborhood of the entrance points, the vector fields should be in the normal form.

For the saddle and elliptic cases, the family of vector fields at each point P_i $(i=1,2,3,4)$ has the same form as (3.8), the three eigenvalues not all having the same sign. Due to the special form of the family (3.8), after dividing by a C^∞ positive function, system (3.8) is linear in r and ρ. If necessary we change the time $(t \mapsto -t)$ so that we have two negative eigenvalues while the third is positive (Table II). So for the three eigenvalues at each point, there are only two possibilities

$$-1, \qquad 1, \qquad -\sigma(a)$$

or

$$1, \qquad -1, \qquad -\sigma(a),$$

where

$$\sigma(a)=\begin{cases} \left|\dfrac{1-2a}{a}\right| & \text{at } P_1 \text{ and } P_2 \\ 2(1-2a) & \text{at } P_3 \text{ and } P_4. \end{cases}$$

By exchanging the roles of r and ρ, we only need to consider one case for system (3.8) which we rewrite as

$$X_{(a,\bar{\mu})}\begin{cases} \dot{r}=-r \\ \dot{\rho}=\rho \\ \dot{\bar{y}}=-\sigma(a)\,\bar{y}+f_{(a,\bar{\mu})}(r,\rho,\bar{y}), \end{cases} \tag{4.6}$$

where

$$f_{(a,\bar{\mu})}(r,\rho,\bar{y}) = \sigma(a)\,\bar{y}+\frac{\begin{array}{r}-(1-2a)\,\bar{y}+2\bar{y}^2+\bar{y}[\varepsilon_2 r+\bar{\mu}_3\rho+2\bar{\mu}_2\rho^2-r^2h_1(r,r\rho,\bar{\mu})] \\ +\bar{\mu}_1\rho^3+r\bar{h}_2(r,r\rho,\bar{\mu})+\bar{y}^2\bar{Q}_2(r,\rho,\bar{y},\bar{\mu})\end{array}}{a+\bar{y}+\bar{\mu}_2\rho^2}$$

and the parameters $(a,\mu)\in A\times\mathbb{S}^2$, where for the saddle case $A=(-\frac{1}{2},0)$ and for the elliptic case $A=(0,\frac{1}{2})$.

PROPOSITION 4.6. *Consider the family $X_{(a,\bar{\mu})}$ in the form of* (4.6) *with parameters $(a,\bar{\mu})\in A\times\mathbb{S}^2$. Then $\forall(a_0,\bar{\mu})\in A\times\mathbb{S}^2$ and $\forall k\in\mathbb{N}$, there exists $A_0\subset A$, a neighborhood of a_0, $N(k)\in\mathbb{N}$, and a C^k-transformation*

$$\Psi_{(a,\bar{\mu})}\colon (r,\rho,\bar{y})\to(r,\rho,\psi_{(a,\bar{\mu})}(r,\rho,\bar{y})),$$

where

$$\psi_{(a,\bar{\mu})}(r,\rho,\bar{y}) = \bar{y} + o(|(r,\rho,\bar{y})|) \tag{4.7}$$

such that $\forall (a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$, *the map* $\psi_{(a,\bar{\mu})}$ *transforms* $X_{(a,\bar{\mu})}$ *into one of the following normal forms:*

- *if* $\sigma(a_0) \notin \mathbb{Q}$

$$\tilde{X}_{(a,\bar{\mu})} \begin{cases} \dot{r} = \mp r \\ \dot{\rho} = \pm \rho \\ \dot{y} = -\bar{\sigma}(a,\bar{\mu},v)\, y. \end{cases} \tag{4.8}$$

- *If* $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$

$$\tilde{X}_{(a,\bar{\mu})} \begin{cases} \dot{r} = -r \\ \dot{\rho} = \rho \\ \dot{y} = \kappa r^p + \dfrac{1}{q}\left[-p + \sum_{i=0}^{N(k)} \alpha_{i+1}(a,\bar{\mu},v)(\rho^p y^q)^i \right] y, \end{cases} \tag{4.9}$$

where $v = r\rho > 0$ *and*

$$\begin{aligned} \bar{\sigma}(a,\bar{\mu},v) &= \sigma(a) - \alpha_0(a,\bar{\mu},v) \\ \alpha_0(a,\bar{\mu},v) &= \sum_{i=1}^{N(k)} \gamma_i v^i \\ \alpha_1(a,\bar{\mu},v) &= p - \bar{\sigma}(a,\bar{\mu},v)\, q, \end{aligned} \tag{4.10}$$

where $\gamma_i(a,\bar{\mu})$, α_i, *and* κ *are smooth functions defined for* $(a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$. *Especially if* $q \geqslant 2$, $\kappa = 0$.

Proof. The proof is a straightforward application of normal form theory (see for instance [16, 20]). ▮

In order to study the cyclicity of the graphics with a nilpotent singularity of elliptic or saddle type, we only need to consider $\tilde{X}_{(a.\bar{\mu})}$ with eigenvalues $-1, 1, -\sigma(a)$ in the normal forms (4.8) and (4.9) and consider the following two types of Dulac maps (Fig. 12):

$$\begin{aligned} \Delta_{(a,\bar{\mu})} &= (d, D) \colon \Sigma \to \Pi \\ \Theta_{(a,\bar{\mu})} &= (\xi, \Xi) \colon \tau \to \Pi, \end{aligned}$$

where $\Sigma = \{r = r_0\}$, $\Pi = \{\rho = \rho_0\}$, and $\tau = \{y = y_0\}$ are sections in the normal form coordinates and r_0, ρ_0, and y_0 are positive constants.

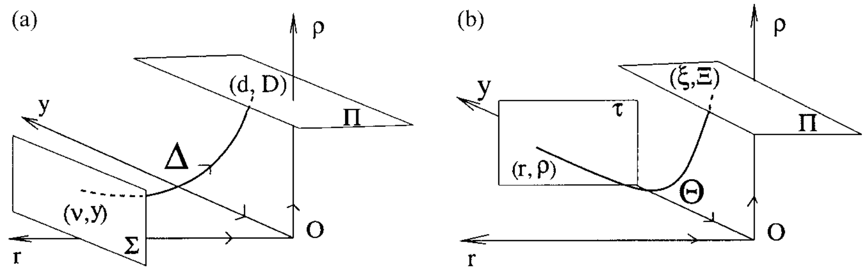


FIG. 12. Two types of Dulac map. (a) First type, (b) second type.

To simplify the notation, for all the maps and vector fields, we will drop the index $(a, \bar{\mu})$. For example, the Dulac map $\Delta(\nu, \tilde{y}_1)$ means $\Delta_{(a, \bar{\mu})}(\nu, \tilde{y}_1)$.

In the next section, we give some preparation propositions for the proof of the two main theorems about the Dulac maps.

4.3. *Preliminaries*

First we define a notation $I(i_1, i_2, ..., i_j; m, n)$:

Definition 4.7. Let $m, n, j \in \mathbb{N} \cup \{0\}$; we define

$$I(i_1 i_2 \cdots i_j; m, n) = \left\{ i_i, i_2, ..., i_j \in \mathbb{N} \cup \{0\} \;\middle|\; \begin{matrix} i_1 + i_2 + \cdots + i_j = m \\ i_1 + 2i_2 + \cdots + ji_j = n \end{matrix} \right\}.$$

Proposition 4.8. *Let $f(t, z)$ be a smooth function and consider the initial value problem*

$$\frac{dz}{dt} = f(t, z_0, z), \qquad z(0) = z_0.$$

Denote the unique solution as $z = z(t, z_0)$. Then $\forall n \in \mathbb{N}$, the nth derivative $(\partial^n z / \partial z_0^n)(t, z_0)$ satisfies the linear initial value problem

$$\frac{d}{dt}\left(\frac{\partial^n z}{\partial z_0^n}\right) = \frac{\partial f}{\partial z}\frac{\partial^n z}{\partial z_0^n} + f_n\left(t, z_0, z, \frac{\partial z}{\partial z_0}, \frac{\partial^2 z}{\partial z_0^2}, ..., \frac{\partial^{n-1} z}{\partial z_0^{n-1}}\right)$$

$$\frac{\partial^n z}{\partial z_0^n}(0) = 0,$$

where

$$f_n\left(t, z_0, z, \frac{\partial z}{\partial z_0}, \frac{\partial^2 z}{\partial z_0^2}, \ldots, \frac{\partial^{n-1} z}{\partial z_0^{n-1}}\right)$$
$$= \frac{\partial^n f}{\partial z_0^n} + \sum_{i=2}^{n} \frac{\partial^i f}{\partial z^i} \sum_{I(i_1 i_2 \cdots i_{n-1};\, i,\, n)} * \prod_{l=1}^{n-1} \left(\frac{\partial z}{\partial z_0}\right)^{i_l}$$
$$+ \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \frac{\partial^{i+j} f}{\partial z_0^i \partial z^j} \sum_{I(i_1 i_2 \cdots i_{n-1};\, j,\, n-j)} * \prod_{l=1}^{n-1} \left(\frac{\partial z}{\partial z_0}\right)^{i_l}.$$

f_n as well as all the partial derivatives are all evaluated at $(t, z_0, z(t, z_0))$ and $*$ denotes a positive integer.

Proof. By induction. ∎

PROPOSITION 4.9. *Let us consider the initial value problem*

$$\frac{dz}{dt} = f(t, z)$$
$$z(0) = 0$$

with $t \in [0, T]$. If there exist two continuous functions $A(t)$, $B(t)$ with

$$|f(t, z)| \leqslant |A(t)|\, |z| + |B(t)|, \qquad (t, z) \in [0, T] \times [0, Z_0],$$

then for $t \in [0, T]$, the solution of the initial value problem satisfies

$$|z(t)| \leqslant e^{\int_0^t |A(t)|\, dt} \int_0^t |B(t)|\, e^{-\int_0^t |A(t)|\, dt}\, dt.$$

Furthermore, if there exist constants $M_1, M_2 > 0$ such that

$$|A(t)| \leqslant M_1$$
$$|B(t)| \leqslant M_2,$$

then there exists a constant $K > 0$ such that for $t \in [0, T]$, there holds

$$|z(t)| \leqslant K M_2 t.$$

Proof. Calculations. ∎

4.4. *Dulac Map of the First Type*

We now study the first type of Dulac map $\Delta = (d, D)$. If we parameterize the sections Σ and Π by (v, y) with the obvious relation $\rho = v/r_0$ on Σ and $r = v/\rho_0$ on Π, then we have

THEOREM 4.10. *For any $a_0 \in A$ and $\bar{\mu} \in \mathbb{S}^2$, consider the family $\tilde{X} = \tilde{X}_{(a,\bar{\mu})}$ with eigenvalues $-1, 1, \sigma(a_0)$ in normal form (4.8) or (4.9). Then $\forall Y_0 \in \mathbb{R}$, there exist $A_0 \subset A$, a neighborhood of a_0, and $v_1 > 0$ such that $\forall v \in (0, v_1)$ and $(a, \bar{\mu}, y) \in A_0 \times \mathbb{S}^2 \times [0, Y_0]$, the Dulac map $\Delta(v, y) = (d(v, y), D(v, y))$ has the form*

$$\begin{aligned} d(v, y) &= v \\ D(v, y) &= \eta\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right)\right) + \left(\frac{v}{v_0}\right)^{\bar{\sigma}}\left[y + \phi\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), y\right)\right], \end{aligned} \tag{4.11}$$

where $v_0 = r_0 \rho_0 > 0$ a constant and

if $\sigma(a_0) \notin \mathbb{Q}$

$$\eta = \phi = 0;$$

if $\sigma(a_0) \in \mathbb{Q} \setminus \mathbb{N}$

$$\eta = 0;$$

if $\sigma(a_0) = p \in \mathbb{N}$

$$\eta\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right)\right) = \kappa r_0^p \omega\left(\frac{v}{v_0}, -\alpha_1\right)\left(\frac{v}{v_0}\right)^{\bar{\sigma}};$$

if $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$, $p, q \in \mathbb{N}$ and $(p, q) = 1$, then $\phi\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), y\right)$ is C^∞ and

$$\begin{aligned} \phi &= O\left(v^{\bar{p}}\omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right)\ln\frac{v}{v_0}\right) \\ \frac{\partial\phi}{\partial y} &= O\left(v^{\bar{p}}\omega^{q}\left(\frac{v}{v_0}, -\alpha_1\right)\ln\frac{v}{v_0}\right) \\ \frac{\partial^j\phi}{\partial y^j} &= O\left(v^{\bar{p}\left(1+\left[\frac{j-2}{q}\right]\right)}\omega^{q-j+1+q\left[\frac{j-2}{q}\right]}\left(\frac{v}{v_0}, -\alpha_1\right)\ln\frac{v}{v_0}\right), \qquad j \geqslant 2 \end{aligned} \tag{4.12}$$

where

$$\bar{p} = \begin{cases} q\bar{\sigma}(a, v) & \alpha_1 \geqslant 0 \\ p & \alpha_1 < 0. \end{cases} \tag{4.13}$$

Also all the partial derivatives with respect to the parameters $(a, \bar{\mu})$ are of order

$$O\left(v^{\bar{p}}\omega^{q}\left(\frac{v}{v_0}, -\alpha_1\right)\ln\frac{v}{v_0}\right).$$

Proof. The proof is similar to the corresponding proof in [14].

Since we consider the Dulac map $\Delta(v, y)$ in the invariant leaf $v = r\rho > 0$, let $v_0 = r_0\rho_0$. Then the transition time from Σ to Π is

$$t = \ln \frac{\rho}{\rho_0} = -\ln \frac{v}{v_0}.$$

The first component $d(v, y)$ is easily obtained:

$$d(v, y) = \rho_0 r_0 e^{-t} = \rho_0 r_0 \frac{v}{v_0} = v.$$

Now let us consider the second component $D(v, y)$.

(1) Case $\sigma(a_0) \notin \mathbb{Q}$. By the normal form (4.8), we directly have

$$D(v, y) = e^{-\bar{\sigma}t} y = \left(\frac{v}{v_0}\right)^{\bar{\sigma}} y.$$

(2) Case $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$, $p, q \in \mathbb{N}$, $(p, q) = 1$. Note that $r(t) = r_0 e^{-t}$, $\rho(t) = (v/r_0)\, e^t$, so by the third equation of (4.9), we can write the solution of y as

$$y(t) = e^{-\bar{\sigma}t}[y_0 + \kappa r_0^p \Omega(-\alpha_1, t) + U(t)] = e^{-\bar{\sigma}t} W(t), \tag{4.14}$$

where

$$\Omega(\alpha_1, t) = \begin{cases} \dfrac{e^{\alpha_1 t} - 1}{\alpha_1} & \alpha_1 \neq 0 \\ t & \alpha_1 = 0. \end{cases}$$

Note that if $q \geqslant 2$, then $\kappa = 0$.

A straightforward calculation shows that $U(t)$ satisfies

$$\begin{aligned} \dot{U}(t) &= g(v, t, W(t)) \\ U(0) &= 0, \end{aligned} \tag{4.15}$$

where $g(v, t, W(t)) = \sum_{i=1}^{N} (\alpha_{i+1}/r_0^{pi})\, v^{pi} e^{\alpha_1 it} W^{qi+1}(t)$. First we are going to prove that $U(t)$ is bounded for $t \in [0, |\ln v/v_0|]$.

By the definition of $W(t)$ in (4.14), there exist constants $K_1, K_2 > 0$ such that

$$|W(t)| \leqslant K_1 + K_2 \omega\left(\frac{v}{v_0}, -\alpha_1\right) + |U(t)|, \qquad t \in \left[0, \left|\ln \frac{v}{v_0}\right|\right], \tag{4.16}$$

where $K_2 = 0$ as long as $q \geqslant 2$.

We want to show that $U(t)$ is bounded, so we only need to consider the region where $|U(t)| \geqslant 1$. In such a region, by the definition of $\bar{p}$ in (4.13), there exists $K_3 > 0$ such that for $t \in [0, |\ln v/v_0|]$ with v sufficiently small

$$|g(v, t, W)| \leqslant K_3 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right) |U|^{Nq+1}. \tag{4.17}$$

Indeed, for v sufficiently small and for $t \in [0, |\ln v/v_0|]$,

$$\begin{aligned}
|g(v, t, W)| &\leqslant \sum_{i=1}^{N} \frac{|\alpha_{i+1}|}{r_0^{pi}} v^{pi} e^{\alpha_1 it} |W(t)|^{qi+1} \\
&\leqslant \sum_{i=1}^{N} \frac{|\alpha_{i+1}| v_0^{\alpha_1 i}}{r_0^{pi}} v^{\bar{p}i} |W(t)|^{qi+1} \\
&\leqslant \sum_{i=1}^{N} \frac{|\alpha_{i+1}| v_0^{\alpha_1 i}}{r_0^{pi}} v^{\bar{p}i} \left[K_1 + K_2 \omega\left(\frac{v}{v_0}, -\alpha_1\right) + |U(t)|\right]^{qi+1} \\
&\leqslant K_3 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right) |U(t)|^{Nq+1}.
\end{aligned}$$

Hence $U(t)$ stays bounded by the solution of the initial value problem

$$\begin{aligned}
\dot{Z}(t) &= K_3 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right) Z(t)^{Nq+1} \\
Z(0) &= 1.
\end{aligned}$$

So for $t \in [0, |\ln v/v_0|]$, there exist constants $K_4, K_5 > 0$ such that

$$\begin{aligned}
|U(t)| \leqslant Z(t) &= \frac{1}{\left(1 - NqK_3 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right) t\right)^{\frac{1}{Nq}}} \\
&\leqslant \left[1 - K_4 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right) \ln \frac{v}{v_0}\right] \leqslant K_5.
\end{aligned}$$

Since $|U(t)|$ is bounded, by (4.15) and (4.17), there exists a constant $K_6 > 0$ such that for $t \in [0, |\ln v/v_0|]$

$$|\dot{U}(t)| \leqslant K_6 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right).$$

So, $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$, $\forall v \in (0, v_0)$, and for $t \in [0, |\ln v/v_0|]$, we have

$$|U(t)| \leqslant K_6 v^{\bar{p}} \omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right) \left|\ln \frac{v}{v_0}\right|. \tag{4.18}$$

Substituting the transition time $t = -\ln v/v_0$ into (4.14) and letting

$$U(t)|_{t=-\ln\frac{v}{v_0}} = \phi\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), y\right),$$

then for the second component of the map Δ, we get

$$\begin{aligned} D(v, y) &= \left(\frac{v}{v_0}\right)^{\bar{\sigma}}\left[y + \kappa r_0^p \omega\left(\frac{v}{v_0}, -\alpha_1\right) + \phi\left(a, v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), y\right)\right] \\ &= \eta\left(v, \omega\left(\frac{v}{v_0}\right)\right) + \left(\frac{v}{v_0}\right)^{\bar{\sigma}}\left[y + \phi\left(a, v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), y\right)\right] \end{aligned}$$

where ϕ is C^∞ in $(a, \bar{\mu}, v, \omega(v/v_0, -\alpha_1), y)$ and uniformly bounded; i.e., for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$, $v \in (0, v_1)$, and $y \in [0, Y_0]$, we have

$$\phi\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), y\right) = O\left(v^{\bar{p}}\omega^{q+1}\left(\frac{v}{v_0}, -\alpha_1\right)\ln\frac{v}{v_0}\right).$$

Now we consider the derivatives $\partial^i\phi/\partial y^i$ for $i \geqslant 1$.
For $\frac{\partial\phi}{\partial y}$, since $\frac{\partial W}{\partial y} = 1 + \frac{\partial U}{\partial y}$, so $\frac{\partial W}{\partial y}$ satisfies

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial W}{\partial y}\right) = g_1(v, t, W(t))\frac{\partial W}{\partial y} \\ &\frac{\partial W}{\partial y}(0) = 1, \end{aligned} \tag{4.19}$$

where $g_1(v, t, W(t)) = \sum_{i=1}^N ((qi+1)\,\alpha_{i+1}/r_0^{pi})\, v^{pi} e^{\alpha_1 i t} W^{qi}(t)$.

For $t \in [0, |\ln v/v_0|]$, by (4.16) and (4.18), and similar to the proof of (4.17), there exists a $\bar{K}_1 > 0$ such that

$$|g_1(v, t, W(t))| \leqslant \bar{K}_1 v^{\bar{p}} \omega^q\left(\frac{v}{v_0}, -\alpha_1\right). \tag{4.20}$$

Then, by (4.19) and (4.20) we have for $t \in [0, |\ln v/v_0|]$

$$e^{-\bar{K}_1 v^{\bar{p}}\omega^q\left(\frac{v}{v_0}, -\alpha_1\right)\left|\ln\frac{v}{v_0}\right|} \leqslant \frac{\partial W}{\partial y} \leqslant e^{\bar{K}_1 v^{\bar{p}}\omega^q\left(\frac{v}{v_0}, -\alpha_1\right)\left|\ln\frac{v}{v_0}\right|}.$$

Hence

$$e^{-\bar{K}_1 v^{\bar{p}}\omega^q\left(\frac{v}{v_0}, -\alpha_1\right)\left|\ln\frac{v}{v_0}\right|} - 1 \leqslant \frac{\partial U}{\partial y} \leqslant e^{\bar{K}_1 v^{\bar{p}}\omega^q\left(\frac{v}{v_0}, -\alpha_1\right)\left|\ln\frac{v}{v_0}\right|} - 1.$$

So there exists $\hat{K}_1 > 0$ such that for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $v \in (0, v_0)$, for $0 \leqslant t \leqslant |\ln v/v_0|$

$$\left|\frac{\partial U}{\partial y}\right| \leqslant \hat{K}_1 v^{\bar{p}} \omega^q \left(\frac{v}{v_0}, -\alpha_1\right) \left|\ln \frac{v}{v_0}\right|. \tag{4.21}$$

Thus for $\phi(v, \omega(v/v_0, -\alpha_1), y)$, we have

$$\frac{\partial \phi}{\partial y} = O\left(v^{\bar{p}} \omega^q \left(\frac{v}{v_0}, -\alpha_1\right) \ln \frac{v}{v_0}\right).$$

It is clear that the above properties on ϕ also hold for all the partial derivatives with respect to the parameters $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$.

For $\partial^i W/\partial y^i (i \geqslant 2)$, we will use induction on i. First show that for $2 \leqslant i \leqslant q+1$, there holds

$$\frac{\partial^i \phi}{\partial y^i} = O\left(v^{\bar{p}} \omega^{q+1-i} \left(\frac{v}{v_0}, -\alpha_1\right) \ln \frac{v}{v_0}\right).$$

Assume that for $2 \leqslant i \leqslant q$, we have

$$\left|\frac{\partial^i W}{\partial y^i}\right| \leqslant \hat{K}_i v^{\bar{p}} \omega^{q+1-i} \left(\frac{v}{v_0}, -\alpha_1\right) \left|\ln \frac{v}{v_0}\right|. \tag{4.22}$$

Now we turn to $\partial^{i+1} W/\partial y^{i+1}$. By Proposition 4.8, $\partial^{i+1} W/\partial y^{i+1}$ satisfies the following initial value problem

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial^{i+1} W}{\partial y^{i+1}}\right) = g_1(v, t, W) \frac{\partial^{i+1} W}{\partial y^{i+1}} + g_{i+1}\left(v, t, W, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \ldots, \frac{\partial^i W}{\partial y^i}\right) \\ &\frac{\partial^{i+1} W}{\partial y^{i+1}}(0) = 0, \end{aligned} \tag{4.23}$$

where

$$g_{i+1}\left(v, t, W, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \ldots, \frac{\partial^i W}{\partial y^i}\right) = \sum_{j=2}^{i+1} \frac{\partial^j g}{\partial W^j} \sum_{I(j_1 j_2 \cdots j_i;\, j,\, i+1)} * \prod_{k=1}^{i} \left(\frac{\partial W}{\partial y_0}\right)^{j_k}.$$

We claim that there exists a constant $\bar{K}_{i+1} > 0$ such that for $t \in [0, |\ln v/v_0|]$

$$\left|g_{i+1}\left(v, t, W, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \ldots, \frac{\partial^i W}{\partial y^i}\right)\right| \leqslant \bar{K}_{i+1} v^{\bar{p}} \omega^{q+1-(i+1)} \left(\frac{v}{v_0}, -\alpha_1\right). \tag{4.24}$$

Indeed, for $2 \leqslant j \leqslant i+1$, similar to the proof of (4.17), there exist constants $\bar{K}_{j1} > 0$ ($j = 2, 3, \ldots, i+1$) such that

$$\begin{aligned} \left|\frac{\partial^j g}{\partial W^j}\right| &= \left|\sum_{i=1}^{N(k)} j! \binom{qi+1}{j} \frac{\alpha_{i+1}}{(v_0^{\alpha_1} r_0^p)^i} v^{pi} e^{\alpha_1 i t} W^{qi+1-j}(t)\right| \\ &\leqslant \bar{K}_{j1} v^{\bar{p}} \omega^{q+1-j}\left(\frac{v}{v_0}, -\alpha_1\right). \end{aligned} \tag{4.25}$$

Note that by (4.21), we have $|\frac{\partial W}{\partial y}| \leqslant \bar{K}_0$, so by (4.25) and by the induction assumption (4.22), there exists a constant $\bar{K}_{i+1} > 0$ such that

$$\begin{aligned} &\left| g_{i+1}\left(v, t, W(t), \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \ldots, \frac{\partial^i W}{\partial y^i}\right)\right| \\ &\leqslant \sum_{j=2}^{i+1} \bar{K}_{j1} v^{\bar{p}} \omega^{q+1-j} \sum_{I(j_1 j_2 \cdots j_i;\, j, i+1)} * \prod_{l=2}^{i} \left(v^{\bar{p}} \omega^{q-(l-1)} \left|\ln \frac{v}{v_0}\right|\right)^{j_l} \\ &= \sum_{j=2}^{i+1} \bar{K}_{j1} v^{\bar{p}} \omega^{q+1-j} \sum_{I(j_1 j_2 \cdots j_i;\, j, i+1)} * \left(v^{\bar{p}} \omega^{q} \left|\ln \frac{v}{v_0}\right|\right)^{j_2+j_3+\cdots+j_i} \omega^{-j_2-2j_3-(i-1)\,j_i} \\ &= \sum_{j=2}^{i+1} \bar{K}_{j1} v^{\bar{p}} \omega^{q+1-j} \sum_{I(j_1 j_2 \cdots j_i;\, j, i+1)} * \left(v^{\bar{p}} \omega^{q} \left|\ln \frac{v}{v_0}\right|\right)^{j-j_1} \omega^{j-(i+1)} \\ &\leqslant \bar{K}_{i+1} v^{\bar{p}} \omega^{q+1-(i+1)}\left(\frac{v}{v_0}, -\alpha_1\right), \end{aligned}$$

where in the final sum the dominant term is the term with $j = j_1 = q+1$.

By (4.20), (4.24), and Proposition 4.9, for the solution of (4.23), there exists a constant $\hat{K}_{i+1}$ such that for $t \in [0, |\ln v/v_0|]$

$$\left|\frac{\partial^{i+1} W}{\partial y^{i+1}}\right| \leqslant \hat{K}_{i+1} v^{\bar{p}} \omega^{q+1-(i+1)}\left(\frac{v}{v_0}, -\alpha_1\right)\left|\ln \frac{v}{v_0}\right|, \qquad 2 \leqslant i \leqslant q. \tag{4.26}$$

Therefore, for $2 \leqslant i \leqslant q+1$, it follows from (4.22), (4.26), and by induction that we have

$$\frac{\partial^i \phi}{\partial y_0^i} = O\left(v^{\bar{p}} \omega^{q+1-i}\left(\frac{v}{v_0}, -\alpha_1\right)\left|\ln \frac{v}{v_0}\right|\right).$$

Generally, $\forall j \geqslant 2$, we can decompose j as $j-2 = lq+i$ with $0 \leqslant i \leqslant q-1$, $l \geqslant 0$. Then in the same way as for the case $l = 0$, we can prove that

$$\frac{\partial^j \phi}{\partial y_0^j} = O\left(v^{\bar{p}\,(1+[\frac{j-2}{q}])} \omega^{q-j+1+q\,[\frac{j-2}{q}]}\left(\frac{v}{v_0}, -\alpha_1\right) \ln \frac{v}{v_0}\right). \quad \blacksquare$$

Remark 4.11. In Theorem 4.10, if $q=1$, then $\forall j \geqslant 2$, we have

$$\frac{\partial^j \phi}{\partial y_0^j} = O\left(v^{(j-1)\bar{p}} \ln \frac{v}{v_0}\right).$$

Remark 4.12. By Theorem 4.10, the properties of the Dulac map Δ are valid on a compact set $\mathbb{K}_1$ of Σ. When we want to analyze a graphic intersecting Σ at (r_0, y^*), we will of course choose $\mathbb{K}_1$ so that $(r_0, y^*) \in \mathbb{K}_1$.

The following lemma will be needed later to simplify the products of the form

$$\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1} \left(\frac{v}{v_0}\right)^{-\bar{\sigma}_2} \qquad \left(\text{resp.} \left(\frac{v}{v_0}\right)^{\bar{\sigma}_3} \left(\frac{v}{v_0}\right)^{-\bar{\sigma}_4}\right)$$

appearing when we compose the Dulac map at P_1 (resp. P_3) with the inverse of the Dulac map at P_2 (resp. P_4).

LEMMA 4.13. *$\forall a \in A$,*

$$\left(\frac{v}{v_0}\right)^{\bar{\sigma}} = \left(\frac{v}{v_0}\right)^{\sigma} [1 + N(v, \ln v)], \tag{4.27}$$

where

$$N(v, \ln v) = -\gamma_1(a, \bar{\mu}, v)\, v \ln \frac{v}{v_0} + O(v^2 (\ln v)^2)$$

and $\gamma_1(a, \bar{\mu}, v)$ is defined in (4.10).

Proof. By the definition of $\bar{\sigma}$ in (4.10), we have

$$\begin{aligned}\left(\frac{v}{v_0}\right)^{\bar{\sigma}} &= \left(\frac{v}{v_0}\right)^{\sigma} \left(\frac{v}{v_0}\right)^{-\alpha_0} = \left(\frac{v}{v_0}\right)^{\sigma} \exp\left(-\alpha_0 \ln \frac{v}{v_0}\right)\\ &= \left(\frac{v}{v_0}\right)^{\sigma} \left[1 - \gamma_1 v \ln \frac{v}{v_0} + O(v^2 (\ln v)^2)\right]. \quad \blacksquare\end{aligned}$$

4.5. *Dulac Map of the Second Type*

Now we consider a Dulac map of the second type $\Theta = (\xi, \Xi)$ (Fig. 12b). If we parameterize τ by (r, ρ), Π by (v, y) with the relation $r\rho = v$, and $r = v/\rho_0$ on the two sections, respectively, then we have

THEOREM 4.14. *For any $a_0 \in A$, consider $\tilde{X}_\lambda$ with eigenvalues $-1, 1$, $-\sigma(a_0)$ in the normal form* (4.8) *and* (4.9). *Then for $r, \rho > 0$ sufficiently* *small, there exist $A_0 \subset A$, a neighborhood of a_0, and $\nu_1 > 0$ such that $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $\nu \in (0, \nu_1)$, the Dulac map $\Theta(r, \rho)$ has the form*

$$\begin{aligned} \xi(r,\rho) &= \nu \\ \Xi(r,\rho) &= \eta\left(\nu, \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right) + \left(\frac{\rho}{\rho_0}\right)^{\bar{\sigma}} \left[y_0 + \theta\left(r, \rho, \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right)\right], \end{aligned} \tag{4.28}$$

where

- *if $\sigma(a_0) \notin \mathbb{N}$, then $\eta = 0$; if $\sigma(a_0) = p \in \mathbb{N}$, then*

$$\eta(\nu, \omega(\rho, \alpha_1)) = \frac{\kappa}{\rho_0^p} \nu^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right),$$

- *if $\sigma(a_0) \notin \mathbb{Q}$, then $\theta = 0$; if $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$, then $\theta(r, \rho, \omega(\rho/\rho_0, -\alpha_1))$ is C^∞ in $(a, \bar{\mu})$ and $(r, \rho, \omega(\rho/\rho_0, -\alpha_1))$, and also satisfies*

$$\begin{aligned} \theta &= O\left(\rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\left[1 + \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]\right) \\ \rho^j \frac{\partial^j \theta}{\partial \rho^j} &= O\left(\rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right)\left[1 + \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]\Bigg), \qquad j \geqslant 1. \end{aligned} \tag{4.29}$$

which are uniformly valid for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r, \rho > 0$ sufficiently small.

Proof. $\xi(r, \rho) = \nu$ follows from the invariance of $r\rho = \nu$.

For the second component $\Xi(r, \rho)$, the transition time from τ to Π is $t = |\ln \rho/\rho_0|$. So for the case $\sigma(a_0) \notin \mathbb{Q}$ by (4.8), we directly have

$$\Xi(r,\rho) = y_0 e^{-\sigma t}\big|_{t = \left|\ln \frac{\rho}{\rho_0}\right|} = y_0 \left(\frac{\rho}{\rho_0}\right)^{\sigma}.$$

Now we consider the case $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$. Note that $r(t) = re^{-t}$ and $\rho(t) = \rho e^t$. Hence, by the third equation of (4.9), we have a first order differential equation in y

$$\dot{y} = -\bar{\sigma} y + \kappa r^p e^{-pt} + \frac{1}{q} \sum_{i=1}^{N(k)} \alpha_{i+1} (\rho e^t)^{pi} y^{qi+1}. \tag{4.30}$$

Let the solution of (4.30) with the initial value $y(0) = y_0$ be

$$y(t) = e^{-\bar{\sigma} t}[y_0 + \kappa r^p \Omega(t, -\alpha_1) + V(t)]. \tag{4.31}$$

Then $V(t)$ satisfies the initial value problem

$$\begin{aligned} \dot{V}(t) &= h(v, t, \rho, E(t)) \\ V(0) &= 0, \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} E(t) &= y_0 + \kappa r^p \Omega(t, -\alpha_1) + V(t), \\ h(v, t, \rho, E(t)) &= \sum_{i=1}^{N} \alpha_{i+1} \rho^{pi} e^{i\alpha_1 t} E^{qi+1}(t). \end{aligned} \tag{4.33}$$

So for $\Xi(r, \rho)$, substituting the transition time $t = |\ln \rho/\rho_0|$ into (4.31) and letting

$$\theta\left(r, \rho, \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right) = V(t)|_{t = |\ln \frac{\rho}{\rho_0}|}, \tag{4.34}$$

then we have

$$\Xi(r, \rho) = \eta\left(v, \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right) + \left(\frac{\rho}{\rho_0}\right)^{\bar{\sigma}} \left[y_0 + \theta\left(r, \rho, \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right)\right]$$

where $\theta(r, \rho, \omega(\rho/\rho_0, -\alpha_1))$ is C^∞ and

$$\eta\left(v, \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right) = \kappa r^p \left(\frac{\rho}{\rho_0}\right)^{\bar{\sigma}} \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right) = \frac{\kappa}{\rho_0^p} v^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right).$$

(1) Bound for $\theta(r, \rho, \omega(\rho/\rho_0, -\alpha_1))$. First we prove that $V(t)$ is bounded for $t \in [0, |\ln \rho/\rho_0|]$.

By (4.33), if we denote $M_0 = |y_0|$, then

$$|E(t)| \leqslant M_0 + |\kappa|\, r^p \Omega(t, -\alpha_1) + |V(t)|. \tag{4.35}$$

Note that for $t \in [0, |\ln \rho/\rho_0|]$, $\Omega(t, -\alpha_1) \leqslant \omega(\rho/\rho_0, -\alpha_1)$, so, if we restrict to the region $V(t) \geqslant 1$, by (4.33) and (4.35), there exists a constant $M_1 > 0$ such that

$$\begin{aligned} &|h(v, t, \rho, E(t))| \\ &\leqslant \rho^p e^{\alpha_1 t} \sum_{i=1}^{N} |\alpha_{i+1}|\, \rho^{p(i-1)} e^{\alpha_1 (i-1) t} \left[M_0 + \kappa r^p \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right) + |V(t)|\right]^{qi+1} \\ &\leqslant \rho^p e^{\alpha_1 t} \left[* \,\omega^{q+1}\left(\frac{\rho}{\rho_0}, -\alpha_1\right) |V(t)|^{q+1} + * \,|V(t)|^{2q+1} + \cdots + * \,|V(t)|^{Nq+1}\right] \\ &\leqslant M_1 \rho^p e^{\alpha_1 t} \omega^{q+1}\left(\frac{\rho}{\rho_0}, -\alpha_1\right) |V(t)|^{Nq+1}. \end{aligned} \tag{4.36}$$

Hence by (4.32) and (4.36), $V(t)$ stays bounded by the solution of the initial value problem

$$\dot{Z}(t) = M_1 \rho^p e^{\alpha_1 t} \omega^{q+1}\left(\frac{\rho}{\rho_0}, -\alpha_1\right) Z^{Nq+1}$$
$$Z(0) = 1.$$

So there exists a constant $M_2 > 0$ such that for $t \in [0, |\ln \rho/\rho_0|]$,

$$|V(t)| \leqslant Z(t) = \frac{1}{\left[1 - qNM_1 \rho^p \omega^{q+1}\left(\frac{\rho}{\rho_0}, -\alpha_1\right) \Omega(t, \alpha_1)\right]^{\frac{1}{Nq}}} \leqslant M_2. \tag{4.37}$$

Again by (4.35) and (4.37), there exists a constant $M_3 > 0$ such that for $t \in [0, |\ln \rho/\rho_0|]$,

$$|E(t)| \leqslant M_3 + \kappa r^p \omega(t, -\alpha_1). \tag{4.38}$$

We will prove that there exist constants $M_4, M_5 > 0$ such that

$$|h(v, t, \rho, E(t))| \leqslant \rho^p e^{\alpha_1 t}\left[M_4 + M_5 \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]. \tag{4.39}$$

Indeed, for the case $q = 1$, there exist M_4, M_5, and $M_{5i} > 0$ $(i = 3, 4, ..., N+1)$ such that

$$\begin{aligned}
&|h(v, t, \rho, E(t))| \\
&\quad \leqslant \sum_{i=1}^{N} |\alpha_{i+1}| \, \rho^{pi} e^{i\alpha_1 t} \, |E^{i+1}(t)| \\
&\quad \leqslant \rho^p e^{\alpha_1 t}\left[|\alpha_2| \left(M_3^2 + 2\kappa M_3 r^p \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right) + \kappa^2 r^{2p} \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right)\right. \\
&\qquad \left. + |\alpha_3| \, M_{53} + \cdots + |\alpha_{N+1}| \, M_{5(N+1)}\right] \\
&\quad \leqslant \rho^p e^{\alpha_1 t}\left[M_4 + M_5 \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right].
\end{aligned}$$

The case $q \geqslant 2$ is similar with $\kappa = 0$.

So for the solution of Eq. (4.32), for $t \in [0, |\ln \rho/\rho_0|]$, by (4.39) we have

$$
\begin{aligned}
|V(t)| &\leqslant \int_0^t |h(s, \rho, E(s))|\, ds \\
&\leqslant \int_0^t \rho^p e^{\alpha_1 s}\left[M_4 + M_5 \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right] ds \\
&= \rho^p \Omega(t, \alpha_1)\left[M_4 + \kappa M_5 r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right] \\
&\leqslant \rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\left[M_4 + \kappa M_5 r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]. \qquad (4.40)
\end{aligned}
$$

Hence for $\theta(r, \rho, \omega(\rho/\rho_0, -\alpha_1))$ given in (4.34), for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r, \rho > 0$, $r\rho = v$ sufficiently small, we have

$$
\theta\left(r, \rho, \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right) = O\left(\rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\left[1 + \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]\right).
$$

We will use induction on i ($i \geqslant 1$) to study $\partial^i \theta/\partial \rho^i$.

(2) Bound for $\frac{\partial \theta}{\partial \rho}$. By the first equation of (4.33), we have $\frac{\partial V}{\partial \rho} = \frac{\partial E}{\partial \rho}$, so $\frac{\partial E}{\partial \rho}$ satisfies the following linear equation

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial E}{\partial \rho}\right) = h_0(t, \rho, E)\frac{\partial E}{\partial \rho} + h_1(t, \rho, E) \\
&\frac{\partial E}{\partial \rho}(0) = 0,
\end{aligned} \qquad (4.41)
$$

where

$$
\begin{aligned}
h_0(t, \rho, E) &= \frac{\partial h}{\partial E}(t, \rho, E) = \sum_{i=1}^{N(k)} (qi+1)\, \alpha_{i+1} \rho^{pi} e^{\alpha_1 it} E^{qi}(t), \\
h_1(t, \rho, E) &= \frac{\partial h}{\partial \rho}(t, \rho, E) = \sum_{i=1}^{N(k)} pi \alpha_{i+1} \rho^{pi-1} e^{\alpha_1 it} E^{qi+1}(t).
\end{aligned}
$$

By (4.38) and similar to the proof of (4.39), we can prove that there exist constants $\bar{M}_{1i} > 0$ ($i = 1, 2, 3, 4$) such that

$$
\begin{aligned}
|h_0(t, \rho, E)| &\leqslant \rho^p e^{\alpha_1 t}\left[\bar{M}_{11} + \kappa \bar{M}_{12} r^p \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right] \\
&= e^{\alpha_1 t} \bar{A}(r, \rho) = A(t, r, \rho)
\end{aligned}
$$

$$|h_1(t,\rho,E)| \leqslant \rho^{p-1}e^{\alpha_1 t}\left[\bar{M}_{13}+\kappa\bar{M}_{14}r^p\omega^2\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right]$$
$$= e^{\alpha_1 t}\bar{B}(r,\rho) = B(t,r,\rho). \tag{4.42}$$

Then by (4.42) and Proposition 4.9, there exist constants $\hat{M}_{11}, \hat{M}_{12} > 0$ such that for $t \in [0, |\ln \rho/\rho_0|]$, there holds

$$\left|\frac{\partial E}{\partial \rho}\right| \leqslant \rho^{p-1}\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)\left[\hat{M}_{11}+\kappa\hat{M}_{12}r^p\omega^2\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right]. \tag{4.43}$$

So by Proposition 4.9, for $t \in [0, |\rho/\rho_0|]$, there exist a constant $\bar{M}_1, \hat{M}_{11}, \hat{M}_{12} > 0$ such that

$$\begin{aligned}
\left|\frac{\partial E}{\partial \rho}\right| &\leqslant e^{\int_0^t A(s,r,\rho)\,ds}\int_0^t B(u,r,\rho)\,e^{-\int_0^u A(s,r,\rho)\,ds}\,du \\
&= e^{\int_0^t \bar{A}(r,\rho)\,e^{\alpha_1 s}\,ds}\left[-\frac{\bar{B}(r,\rho)}{\bar{A}(r,\rho)}\int_0^t d(e^{-\int_0^u \bar{A}(r,\rho)\,e^{\alpha_1 s}\,ds})\right] \\
&= \frac{\bar{B}(r,\rho)}{\bar{A}(r,\rho)}(e^{\int_0^t \bar{A}(r,\rho)\,e^{\alpha_1 s}\,ds}-1) \\
&\leqslant \frac{\bar{B}(r,\rho)}{\bar{A}(r,\rho)}[e^{\bar{A}(r,\rho)\,\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)}-1] \\
&= \frac{\bar{B}(r,\rho)}{\bar{A}(r,\rho)}\left[e^{\xi_1}\bar{A}(r,\rho)\,\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)\right] \\
&\leqslant \rho^{p-1}\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)\left[\hat{M}_{11}+\kappa\hat{M}_{12}r^p\omega^2\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right]
\end{aligned} \tag{4.44}$$

where $\xi_1 \in (0, \bar{A}(r,\rho)\,\omega(\rho/\rho_0,\alpha_1))$, $\bar{M}_1 = e^{\bar{A}(r,\rho)\,\omega(\rho/\rho_0,\alpha_1)}$, and $\hat{M}_{11} = \bar{M}_1\bar{M}_{13}$, $\hat{M}_{12} = \bar{M}_1\bar{M}_{14}$.

So for $(a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r, \rho > 0$ sufficiently small, we have uniformly

$$\rho\frac{\partial\theta}{\partial\rho} = O\left(\rho^p\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)\left(1+\kappa r^p\omega^2\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right)\right).$$

(3) Bound for $\partial^i\theta/\partial\rho^i$, $(i \geqslant 2)$. Assume that there exist constants $\hat{M}_{j1}, \hat{M}_{j2} > 0$ such that for $2 \leqslant j \leqslant i$, there holds

$$\left|\frac{\partial^j E}{\partial \rho^j}\right| \leqslant \rho^{p-j}\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)\hat{L}_j\left(r,\omega\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right) \tag{4.45}$$

where

$$\hat{L}_j\left(r,\omega\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right)=\hat{M}_{j1}+\kappa\hat{M}_{j2}r^p\omega^2\left(\frac{\rho}{\rho_0},-\alpha_1\right).$$

Now let us consider $\partial^{j+1}E/\partial\rho^{j+1}$. By Proposition 4.8, it satisfies

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial^{i+1}E}{\partial\rho^{i+1}}\right)=h_0(t,\rho,E)\frac{\partial^{i+1}E}{\partial\rho^{i+1}}+h_{i+1}\left(t,\rho,E,\frac{\partial E}{\partial\rho},\frac{\partial^2E}{\partial\rho^2},\ldots,\frac{\partial^iE}{\partial\rho^i}\right)\\&\frac{\partial^{i+1}E}{\partial\rho^{i+1}}(0)=0,\end{aligned}\tag{4.46}$$

where

$$\begin{aligned}h_{i+1}=&\frac{\partial^{i+1}h}{\partial\rho^{i+1}}+\sum_{j=2}^{i}\frac{\partial^jh}{\partial E^j}\sum_{I(j_1j_2\cdots j_i;\,j,\,i+1)}*\prod_{k=1}^{i}\left(\frac{\partial^kE}{\partial\rho^k}\right)^{j_k}\\&+\sum_{l=1}^{i}\sum_{j=1}^{i+1-l}\frac{\partial^{j+l}h}{\partial\rho^j\partial E^l}\sum_{I(l_1l_2\cdots l_i;\,l,\,i+1-j)}*\prod_{k=1}^{i}\left(\frac{\partial^kE}{\partial\rho^k}\right)^{l_k}.\end{aligned}\tag{4.47}$$

LEMMA 4.15. *There exist constants* $\bar{M}_{i+1,1},\bar{M}_{i+1,2}>0$ *such that for* $\forall(a,\bar{\mu})\in A_0\times\mathbb{S}^2$ *and for* $r,\rho>0$ *sufficiently small, there holds*

$$|h_{i+1}(t,\rho,E)|\leqslant\rho^{p-(i+1)}e^{\alpha_1t}\omega\left(\frac{\rho}{\rho_0},\alpha_1\right)\left[\bar{M}_{i+1,1}+\kappa\bar{M}_{i+1,2}\omega\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right].\tag{4.48}$$

Proof. Let us denote the first and the second sum in (4.47) by h_I and h_{II}; i.e.,

$$h_{i+1}=\frac{\partial^{i+1}h}{\partial\rho^{i+1}}+h_I+h_{II}.\tag{4.49}$$

For $\partial^{i+1}h/\partial\rho^{i+1}$, by (4.38) and the definition of h in (4.33), there exist constants $M_{i+1,1},M_{i+1,2}>0$ such that

$$\left|\frac{\partial^{i+1}h}{\partial\rho^{i+1}}\right|\leqslant\rho^{p-(i+1)}e^{\alpha_1t}\left[M_{i+1,1}+\kappa M_{i+1,2}r^p\omega^2\left(\frac{\rho}{\rho_0},\alpha_1\right)\right].\tag{4.50}$$

Similarly, there exist constants $\tilde{M}_{j1}$, $\tilde{M}_{j2}$, $\tilde{M}_{jl1}$, and $\tilde{M}_{jl2}>0$ such that

$$\left|\frac{\partial^jh}{\partial E^j}\right|\leqslant\rho^pe^{\alpha_1t}\bar{L}_j\left(r,\omega\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right)\tag{4.51}$$

$$\left|\frac{\partial^{j+l}h}{\partial\rho^j\partial E^l}\right|\leqslant\rho^{p-j}e^{\alpha_1t}\tilde{L}_{jl}\left(r,\omega\left(\frac{\rho}{\rho_0},-\alpha_1\right)\right)\tag{4.52}$$

where

$$\bar{L}_j\left(r, \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right) = \tilde{M}_{j1} + \kappa\tilde{M}_{j2} r^p \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)$$

$$\tilde{L}_{jl}\left(r, \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right) = \tilde{M}_{jl1} + \kappa\tilde{M}_{jl2} r^p \omega\left(\frac{\rho}{\rho_0}, -\alpha_1\right).$$

So for h_I, by (4.51), (4.44), and assumption (4.45), for $t \in [0, |\ln \rho/\rho_0|]$ we have

$$\begin{aligned}
&\left|h_I\left(t, \rho, E, \frac{\partial E}{\partial \rho}, \frac{\partial^2 E}{\partial \rho^2}, \ldots, \frac{\partial^i E}{\partial \rho^i}\right)\right| \\
&\quad \leqslant \sum_{j=2}^{i} \left|\frac{\partial^j h}{\partial E^j}\right| \sum_{I(j_1 j_2 \cdots j_i;\, j,\, i+1)} * \left|\frac{\partial E}{\partial \rho}\right|^{j_1} \left|\frac{\partial^2 E}{\partial \rho^2}\right|^{j_2} \cdots \left|\frac{\partial^i E}{\partial \rho^i}\right|^{j_i} \\
&\quad \leqslant \sum_{j=2}^{i} \rho^p e^{\alpha_1 t} \bar{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i;\, j,\, i+1)} * \prod_{k=1}^{i} |\rho^{p-k} \omega \hat{L}_k(r, \omega)|^{j_k} \\
&\quad \leqslant \sum_{j=2}^{i} \rho^p e^{\alpha_1 t} \bar{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i;\, j,\, i+1)} * \rho^{p \sum_{k=1}^{i} j_k - \sum_{k=1}^{i} k j_k} \omega^{\sum_{k=1}^{i} j_k} \prod_{k=1}^{i} \hat{L}_k^{j_k}(r, \omega) \\
&\quad \leqslant \sum_{j=2}^{i} \rho^p e^{\alpha_1 t} \bar{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i;\, j,\, i+1)} * \rho^{pj-(i+1)} \omega^j \prod_{k=1}^{i} \hat{L}_k^{j_k}(r, \omega) \\
&\quad \leqslant \rho^{p-(i+1)} e^{\alpha_1 t} \sum_{j=2}^{i} \rho^{pj} \omega^j \bar{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i;\, j,\, i+1)} * \prod_{k=1}^{i} \hat{L}_k^{j_k}(r, \omega) \\
&\quad \leqslant o(\rho^{2p-1})\, \rho^{p-(i+1)} e^{\alpha_1 t}.
\end{aligned} \tag{4.53}$$

Similarly, for h_{II}, by (4.52), (4.44), and (4.45), we have

$$\begin{aligned}
&\left|h_{II}\left(t, \rho, E, \frac{\partial E}{\partial \rho}, \frac{\partial^2 E}{\partial \rho^2}, \ldots, \frac{\partial^i E}{\partial \rho^i}\right)\right| \\
&\quad \leqslant \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \left|\frac{\partial^{j+l} h}{\partial \rho^j \partial E^l}\right| \sum_{I(l_1 l_2 \cdots l_i;\, l,\, i+1-j)} * \prod_{k=1}^{i} \left|\frac{\partial E}{\partial \rho}\right|^{l_k} \\
&\quad \leqslant \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \rho^{p-j} e^{\alpha_1 t} \tilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i;\, l,\, i+1-j)} * \prod_{k=1}^{i} [\rho^{p-k} \omega \hat{L}_k(r, \omega)]^{l_k} \\
&\quad \leqslant \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \rho^{p-j+pl-(i+1-j)} e^{\alpha_1 t} \omega^l \tilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i;\, l,\, i+1-j)} * \prod_{k=1}^{i} \hat{L}_k^{l_k}(r, \omega) \\
&\quad \leqslant \rho^{p-(i+1)} e^{\alpha_1 t} \sum_{l=1}^{i} \rho^{pl} \omega^l \sum_{j=1}^{i+1-l} \tilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i;\, l,\, i+1-j)} * \prod_{k=1}^{i} \hat{L}_k^{l_k}(r, \omega) \\
&\quad \leqslant O\left(\rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right) \rho^{p-(i+1)} e^{\alpha_1 t}.
\end{aligned} \tag{4.54}$$

Then by (4.49), (4.50), (4.53), and (4.54), there exist positive constants $\bar{M}_{i+1,1}$ and $\bar{M}_{i+1,2}$ such that

$$\begin{aligned}
|h_{i+1}| &\leqslant \left|\frac{\partial^{i+1} h}{\partial \rho^{i+1}}\right| + |h_I| + |h_{II}| \\
&\leqslant \rho^{p-(i+1)} e^{\alpha_1 t}\left[M_{i+1,1} + \kappa M_{i+1,2} r^p \omega^2\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right] + o(\rho^{2p-1})\, \rho^{p-(i+1)} e^{\alpha_1 t} \\
&\quad + O\left(\rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\right) \rho^{p-(i+1)} e^{\alpha_1 t} \\
&\leqslant \rho^{p-(i+1)} e^{\alpha_1 t} \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\left[\bar{M}_{i+1,1} + \kappa \bar{M}_{i+1,2} \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]. \quad \blacksquare
\end{aligned}$$

End of proof of Theorem 4.14. Then for the initial value problem (4.46) with the estimations (4.42) and (4.48), similar to the proof (4.44), again by Proposition 4.9, for $t \in [0, |\rho/\rho_0|]$, there exist constants $\hat{M}_{i+1,1}, \hat{M}_{i+1,2} > 0$ such that for $t \in [0, |\rho/\rho_0|]$ we have

$$\left|\frac{\partial^{i+1} E}{\partial \rho^{i+1}}\right| \leqslant \rho^{p-(i+1)} \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\left[\hat{M}_{i+1,1} + \kappa \hat{M}_{i+1,2} r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right]. \tag{4.55}$$

Hence for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r, \rho > 0$ sufficiently small, we have

$$\rho^{i+1} \frac{\partial^{i+1} \theta}{\partial \rho^{i+1}} = O\left(\rho^p \omega\left(\frac{\rho}{\rho_0}, \alpha_1\right)\left(1 + \kappa r^p \omega^2\left(\frac{\rho}{\rho_0}, -\alpha_1\right)\right)\right). \quad \blacksquare$$

Remark 4.16. Although for $a_0 \in \mathbb{Q}$, the inverse of the map $\Theta(r, \rho)$ has no nice expression, it has a nice expression inside either the invariant subspace $r = 0$ or the invariant subspace $\rho = 0$. This expression is the usual expression of the Dulac map in the neighborhood of a 2-dimensional saddle [27, 30].

5. FINITE CYCLICITY OF CONVEX GRAPHICS WITH A NILPOTENT SINGULARITY OF SADDLE TYPE

5.1. *Preliminaries on the Derivatives of Regular Transition Maps*

In proving the finite cyclicity of graphics of saddle or elliptic type, we will need to calculate the derivatives of a regular transition map. First we recall briefly the formula of [2].

PROPOSITION 5.1 (ALGM). *Consider the vector field*

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{5.1}$$

Let $\Sigma = \{(x, y) = (f_1(s), g_1(s))\}$ *and* $\tilde{\Sigma} = \{(x, y) = (f_2(s), g_2(s))\}$ *be two arcs transverse to the same orbit. Let* $R(s)$ *be the transition map from* Σ *to* $\tilde{\Sigma}$. *Then*

$$R'(s) = \frac{\Delta(s)}{\tilde{\Delta}(R(s))} \exp\left(\int_0^{T(s)} div\, X(\gamma(t))\, dt\right), \tag{5.2}$$

where $T(s)$ *is the transition time from* $(f_1(s), g_1(s))$ *to* $(f_2(R(s)), g_2(R(s)))$ *along the orbit* $\gamma(t)$ *starting at* $(f_1(s), g_1(s))$ *for* $t = 0$ *and*

$$\Delta(s) = \begin{vmatrix} P(f_1(s), g_1(s)) & f_1'(s) \\ Q(f_1(s), g_1(s)) & g_1'(s) \end{vmatrix}$$

$$\tilde{\Delta}(\tilde{s}) = \begin{vmatrix} P(f_2(\tilde{s}), g_2(\tilde{s})) & f_2'(\tilde{s}) \\ Q(f_1(\tilde{s}), g_2(\tilde{s})) & g_2'(\tilde{s}) \end{vmatrix}.$$

It is not easy to use Proposition 5.1 to calculate the higher order derivatives of a regular transition map. The following proposition will be very useful.

PROPOSITION 5.2. *Consider the transition map* $R(x)$ *of the vector field* $X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ *between two arcs without contact,* $\Sigma = \{(x, y) = (x, f_1(x))\}$ *and* $\tilde{\Sigma} = \{(x, y) = (x, f_2(x))\}$, *in a region where* $Q(x, y) \neq 0$. *Let* $x = x(x_0, y_0, y)$ *be the solution with initial condition* $x(x_0, y_0, y_0) = x_0$. *Then*

$$\frac{dR}{dx_0}(x_0) = \exp\left(\int_{f_1(x_0)}^{f_2(R(x_0))} \left(\frac{P'_x Q - P Q'_x}{Q^2}\right)\Bigg|_{x = x(x_0, f_1(x_0), y)} dy\right)$$

$$\frac{1 - \left(\dfrac{P}{Q}\right)(x_0, f_1(x_0))\, f_1'(x_0)}{1 - \left(\dfrac{P}{Q}\right)(x_0, f_2(R(x_0)))\, f_2'(R(x_0))}. \tag{5.3}$$

Formulas for the first and second derivatives are given in the particular case where $x_0 = 0$ *and* $P(0, y) \equiv 0$. *Let* $y_i = f_i(0)$.

$$R'(0) = \exp\left(\int_{y_1}^{y_2} \frac{P'_x}{Q}(0, y)\, dy\right). \tag{5.4}$$

$$R''(0) = R'(0)\left[2\left(f_2'(0)\, R'(0) \left(\frac{P_x}{Q}\right)(0, y_2) - f_1'(0)\left(\frac{P_x}{Q}\right)(0, y_1) \right) + \int_{y_1}^{y_2} \left(\frac{P''_x}{Q}(0, y) - 2\frac{P'_x Q'_x}{Q^2}(0, y) \right) \exp\left(\int_{y_1}^{y} \frac{P'_x}{Q}(0, z)\, dz \right) dy \right]. \tag{5.5}$$

Proof. We rewrite the vector field into the equivalent differential equation

$$\frac{dx}{dy} = \frac{P}{Q}. \tag{5.6}$$

The solution is $x = x(x_0, f_1(x_0), y)$ with initial condition $x(x_0, f_1(x_0), f_1(x_0)) = x_0$. We have that $R(x_0) = x(x_0, f_1(x_0), f_2(R(x_0)))$. Moreover

$$\frac{\partial}{\partial y}\frac{\partial x}{\partial x_0} = \frac{\partial}{\partial x_0}\frac{\partial x}{\partial y} = \frac{\partial}{\partial x_0}\frac{P(x(x_0, f_1(x_0), y), y)}{Q(x(x_0, f_1(x_0), y), y)} = \frac{P'_x Q - P Q'_x}{Q^2}\frac{\partial x}{\partial x_0} \tag{5.7}$$

from which

$$\frac{\partial x}{\partial x_0} = \exp\left(\int_{f_1(x_0)}^{y} \frac{P'_x Q - P Q'_x}{Q^2}\, dy \right) \tag{5.8}$$

follows. Hence we can rewrite

$$\frac{dR}{dx_0}(x_0) = \exp\left(\int_{f_1(x_0)}^{f_2(R(x_0))} \left(\frac{P'_x Q - P Q'_x}{Q^2} \right)\Bigg|_{x = x(x_0, f_1(x_0), y)} dy \right) \frac{1 - \left(\frac{P}{Q}\right)(x_0, f_1(x_0))\, f_1'(x_0)}{1 - \left(\frac{P}{Q}\right)(x_0, f_2(R(x_0)))\, f_2'(R(x_0))}. \tag{5.9}$$

The second derivative of R is most easily calculated from this formula. However, the general formula is very long. In the particular case $x_0 = 0$ we get (5.4) and (5.5) for $R'(0)$ and $R''(0)$. ∎

5.2. *Generic Property of the hh Graphic of Saddle or Elliptic Type*

Graphics through a nilpotent saddle point can be of two types: convex or concave. We only consider the convex graphics. Let Γ be the convex hh graphic of saddle type (see Fig. 13a). Let Σ' be a section transverse to the

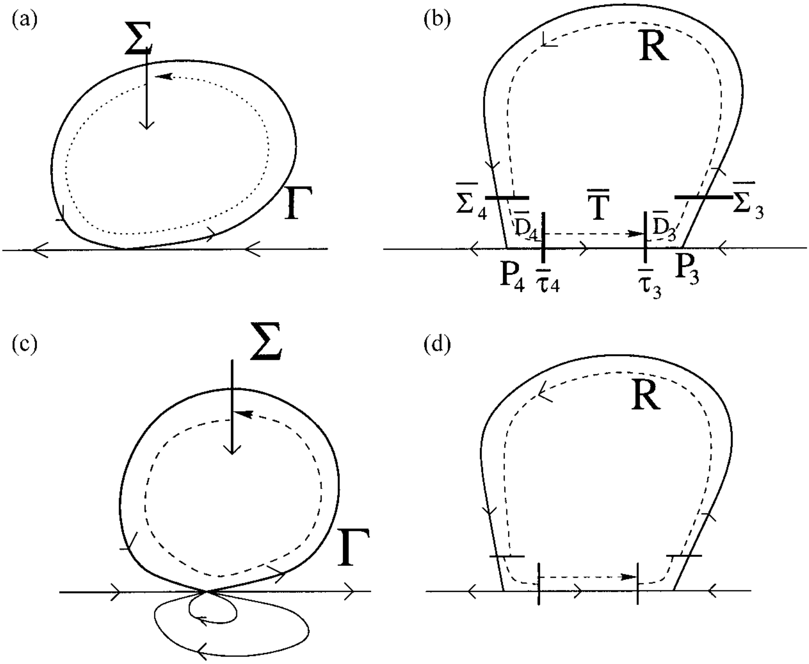


FIG. 13. The Poincaré first return map for the hh graphic of saddle or elliptic type. (a) Return map (saddle type), (b) blow-up of the graphic, (c) return map (elliptic type), (d) blow-up of the graphic.

connection Γ and parameterized by a regular C^∞ coordinate. We will consider the Poincaré first return map P: $\Sigma \to \Sigma'$ with $\Sigma \subset \Sigma'$, a neighborhood of $\Gamma \cap \Sigma'$.

PROPOSITION 5.3. *For the convex graphic of saddle or elliptic type and $\forall a \in A$, the Poincaré first return map $P(z)$ is at least C^1 for $z \geqslant 0$ and*

$$P'(0) = \gamma^* = \exp\left(\int_{-\infty}^{\infty} div\, X(\Gamma(t))\, dt\right). \tag{5.10}$$

Proof. We give the proof for the saddle case. For the elliptic case, the proof is almost the same.

Consider the system

$$\begin{aligned} \dot{x} &= y + ax^2 \\ \dot{y} &= y(x + \varepsilon_2 x^2 + x^3 h(x)) + y^2 Q(x, y), \end{aligned} \tag{5.11}$$

where $h(x)$ and $Q(x, y)$ are C^∞ and $Q(x, y) = O(|(x, y)|^N)$ for N sufficiently large. To study the dynamics near the singularity at (0, 0), we make the blow-up (3.6). Let $\bar{y} = 1$ in (3.6); we have

$$\begin{aligned} \dot{r} &= \tfrac{1}{2} r(\bar{x} + rO(|(r, \bar{x})|)) := \bar{P}(r, \bar{x}) \\ \dot{\bar{x}} &= 1 - (\tfrac{1}{2} - a)\, \bar{x}^2 + rO(|(r, \bar{x})|) := \bar{Q}(r, x). \end{aligned} \tag{5.12}$$

The system (5.12) has two singular points P_3 and P_4 and both are hyperbolic saddles (see Fig. 13b). The eigenvalues at P_3 are $(\sqrt{2(1-2a)}/(2(1-2a)), -\sqrt{2(1-2a)})$; the eigenvalues at P_4 are $(-\sqrt{2(1-2a)}/(2(1-2a)), \sqrt{2(1-2a)})$. Hence the hyperbolicity ratio at P_3 (resp. P_4) is $\sigma_3(a) = 2(1-2a)$ (resp. $1/(\sigma_3(a))$).

Take sections $\bar{\Sigma}_i = \{r_i = r_0\} (i = 3, 4)$, $\bar{\tau}_3 = \{\tilde{x}_3 = -x_0\}$, and $\bar{\tau}_4 = \{\tilde{x}_4 = x_0\}$ in the normal form coordinates in the neighborhood of P_3 and P_4, respectively. For the Dulac maps near P_3 and P_4, we have

$$\begin{aligned} \bar{D}_4(\tilde{x}_4) &= \tilde{x}_4^{\frac{1}{\sigma_3(a)}} [c_4 + \theta_4(\tilde{x}_4)] \\ \bar{D}_3(r_3) &= r_3^{\sigma_3} [c_3 + \theta_3(\tilde{r}_3)], \end{aligned} \tag{5.13}$$

where c_3, c_4 are positive constants, $c_4 c_3^{1/\sigma_3} = 1$, and $\theta_3, \theta_4 \in (I_0^\infty)$.

Then we can decompose the Poincaré first return map P as

$$P = R \circ \bar{D}_3 \circ \bar{T} \circ \bar{D}_4, \tag{5.14}$$

where $\bar{T}\colon \bar{\tau}_4 \to \bar{\tau}_3$ and $R\colon \bar{\Sigma}_3 \to \bar{\Sigma}_4$ are two regular transition maps in the normal form coordinates.

For the transition map $\bar{T}$ along $r = 0$, the two sections become

$$\begin{aligned} \bar{\tau}_4 &= \left\{ \left(r, -\frac{2}{\sqrt{1-2a}} + x_0 + O(|(r, x_0)|^2 \right) \right\} \\ \bar{\tau}_3 &= \left\{ \left(r, \frac{2}{\sqrt{1-2a}} - x_0 + O(|(r, x_0)|^2 \right) \right\}. \end{aligned}$$

Note that for the system (5.12), along $r = 0$, $div\, X|_{r=0} = -(1-2a)\, \bar{x}$ and $\bar{P}(0, \bar{x}) = 0$, so by Proposition 5.1 we have $\bar{T}'(0) = 1$. Thus we have

$$\bar{T}(r_4) = r_4 + O(r_4^2). \tag{5.15}$$

Therefore, by (5.13) and (5.15), if letting $\tilde{x}_4 = \bar{D}_3 \circ \tilde{T} \circ \bar{D}_4(\tilde{x}_3)$, then

$$\tilde{x}_4 = \tilde{x}_3 + o(\tilde{x}_3). \tag{5.16}$$

To calculate the map R, as in [7, 34, 14], we introduce two auxiliary sections $\tilde{\Sigma}_i = \{r_i = r_{00}\}$ $(i = 3, 4)$ in the normal form coordinates. Then the map R can be calculated by the decomposition

$$R = R_{40} \circ \bar{R} \circ R_{30}, \tag{5.17}$$

where $R_{30}\colon \bar{\Sigma}_3 \to \tilde{\Sigma}_3$ and $R_{40}\colon \tilde{\Sigma}_4 \to \bar{\Sigma}_4$ are regular transition maps. Similar to $\bar{T}'(0)$, for R_{30} and R_{40} we have

$$\begin{aligned} R'_{30}(0) &= \left(\frac{r_{00}}{r_0}\right)^{1-\sigma_3(a)} \\ R'_{40}(0) &= \left(\frac{r_0}{r_{00}}\right)^{1-\sigma_3(a)}. \end{aligned} \tag{5.18}$$

For the transition map $\bar{R}\colon \tilde{\Sigma}_3 \to \tilde{\Sigma}_4$, using the original coordinates $(r, \bar{x})$, the two sections are

$$\tilde{\Sigma}_3 = \left\{\left(r_{00}, \frac{2}{\sqrt{1-2a}} + \bar{x} + O(\bar{x}^2) + r_{00} O(|(r, x_0)|)\right)\right\}$$

$$\tilde{\Sigma}_4 = \left\{\left(r_{00}, -\frac{2}{\sqrt{1-2a}} - \bar{x} + O(\bar{x}^2) + r_{00} O(|(r, x_0)|)\right)\right\}.$$

So again by (5.2) in Proposition 5.1, we have

$$\bar{R}'(0) = \exp\left(\int_{-T_1}^{T_2} div\, X(\Gamma(t))\, dt\right).$$

Note that R is independent of r_{00}, so

$$\begin{aligned} R'(0) &= \lim_{r_{00} \to 0} R'_{40}(0)\, \bar{R}'(0)\, R'_{30}(0) = \lim_{r_{00} \to 0} \exp\left(\int_{-T_1}^{T_2} div\, X(\Gamma(t))\, dt\right) \\ &= \exp\left(\int_{-\infty}^{\infty} div\, X(\Gamma(t))\, dt\right). \end{aligned} \tag{5.19}$$

Thus, by (5.17), (5.18), and (5.19), we have

$$R(\tilde{x}_4) = \exp\left(\int_{-\infty}^{\infty} div\, X(\Gamma(t))\, dt\right) \tilde{x}_4 + O(\tilde{x}_4^2). \tag{5.20}$$

It follows from (5.14), (5.16), and (5.20) that there holds

$$P(\tilde{x}_3) = \exp\left(\int_{-\infty}^{\infty} div\, X(\Gamma(t))\, dt\right) \tilde{x}_3 + o(\tilde{x}_3);$$

thus we proved (5.10). ∎

5.3. *Main Theorem on the Convex Graphic of Saddle Type*

For the convex graphic of saddle type, we have

THEOREM 5.4. *A convex hh graphic through a triple nilpotent saddle of codimension 3 has finite cyclicity if the generic hypothesis $P'(0) \neq 1$ is satisfied.*

For the proof, by changing the vector field X to $-X$ if necessary, we impose

HYPOTHESIS 5.5. *The convex hh graphic with a nilpotent saddle is attracting:*

$$[\mathrm{H}]: \quad P'(0) = \gamma^* < 1. \tag{5.21}$$

After the global blow-up in Section 3.1, for the convex graphic through a triple nilpotent saddle, we get a total of 10 families of convex graphics: Sxhh1, Sxhh2, ..., Sxhh10 (see Table VI). For each family Sxhh$i$ ($i = 1, 2, \ldots, 10$), the graphics fall into three groups:

- the upper boundary graphic: Sxhh$i$a ($i = 1, 2, \ldots, 10$);
- the intermediate graphics: Sxhh$i$b ($i = 1, 2, \ldots, 10$), Sxhh9d, and Sxhh10d;
- the lower boundary graphics: Sxhh$i$c ($i = 1, 2, \ldots, 10$), Sxhh9e, and Sxhh10e.

To prove the finite cyclicity of the convex graphic with a nilpotent saddle, we have to prove that all the graphics listed above have finite cyclicity.

Notation 5.6. For convenience in the notation, in the remainder of Section 5 and in Section 6, let r_0, ρ_0, and y_0 be positive constants. We will always use

$$\begin{aligned}
\Sigma_i &= \{r_i = r_0\}, && i = 1, 2, 3, 4\\
\Pi_i &= \{\rho_i = \rho_0\}, && i = 1, 2, 3, 4\\
\tau_i &= \{\tilde{y}_i = y_0\}, && i = 1, 2\\
\tau_i &= \{\tilde{y}_i = -y_0\}, && i = 3, 4
\end{aligned} \tag{5.22}$$

to denote the sections in normal form coordinates $(r_i, \rho_i, \tilde{y}_i)$ in the neighborhood of the four singular points P_i ($i = 1, 2, 3, 4$).

We begin with the upper boundary graphics.

5.4. *The Upper Boundary Graphics have Finite Cyclicity*

PROPOSITION 5.7. *For the convex hh graphics of saddle type or hh graphics of elliptic type, under the generic assumption, all the upper boundary graphics have cyclicity one.*

Proof. As shown in Fig. 14a, to prove the upper boundary graphic of saddle type, we study the Poincaré first return map defined on the section Σ_4:

$$P: \Sigma_4 \longrightarrow \Sigma_4.$$

We can factorize it as

$$P = R \circ \Theta_3 \circ \bar{T}_{43} \circ \Theta_4^{-1}, \tag{5.23}$$

where Θ_4 and Θ_3 are the second type of Dulac maps in the neighborhood of P_4 and P_3, respectively; $\bar{T}_{43}$ and R are the regular transition maps.

At P_3, the eigenvalues are $(1, -1, \sigma_3(a))$, where for $a \in (-\infty, 0)$, $\sigma_3(a) = 2(1-2a) > 0$. By the normal form discussion in Proposition 4.6, depending on whether $a_0 \notin \mathbb{Q}$ or $a_0 \in \mathbb{Q}$, the vector field near P_3 has the normal form of (4.8) or (4.9) with $\sigma = \sigma_3$. Correspondingly, we use β_i $(i = 1, 2, \ldots, N(k))$ instead of using α_i to make the distinction. In particular, $\beta_1 = p_3 - \bar{\sigma}_3(a)\, q_3$.

By Theorem 4.14, the second type Dulac map $\Theta_3 = (\xi_3, \Xi_3)\colon \tau_3 \to \Sigma_3$ has the expression

$$\xi_3(r_3, \rho_3) = v$$

$$\Xi_3(r_3, \rho_3) = \eta_3\left(v, \omega\left(\frac{r_3}{r_0}, \beta_1\right)\right) + \left(\frac{r_3}{r_0}\right)^{\bar{\sigma}_3}\left[y_0 + \theta_3\left(r_3, \rho_3, \omega\left(\frac{r_3}{r_0}, -\beta_1\right)\right)\right], \tag{5.24}$$

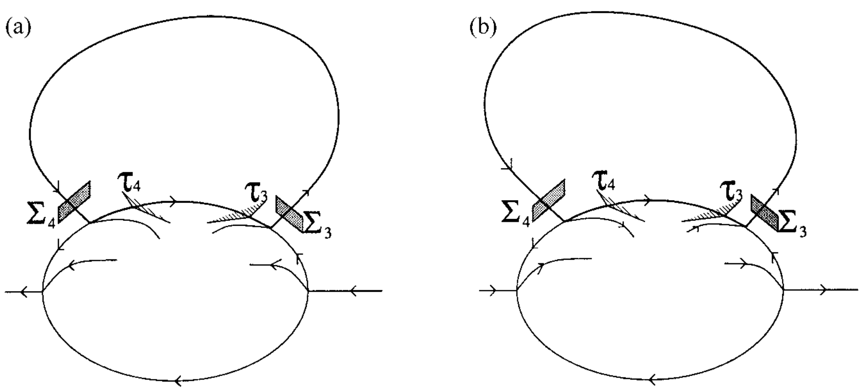


FIG. 14. Upper boundary graphics of (a) saddle and (b) elliptic type.

where $\eta_3(v, \omega(r_3/r_0, \beta_1)) = (\kappa_3/r_0^{p_3})\, v^{p_3}\omega(r_3/r_0, \beta_1)$ and $\theta_3(r_3, \rho_3, \omega(r_3/r_0, -\beta_1))$ satisfies the property (4.29). Due to the symmetry, the Dulac map $\Theta_4: \tau_4 \to \Sigma_4$ has the same form as Θ_3 in (5.24). Note that both $\bar{\sigma}_3$ and $\bar{\sigma}_4$ satisfy Lemma 4.13.

We calculate the transition $\bar{T}_{43}: \tau_4 \to \tau_3$ using the polar coordinates $(\bar{x}, \bar{y}) = (r\cos\theta, r^2\sin\theta)$ in the chart F.R.. Then we have

$$(1+\sin^2\theta)\,\dot{r} = r\cos\theta(a\cos^2\theta + \sin^2\theta + \sin\theta) + O(r^2)$$

$$(1+\sin^2\theta)\,\dot{\theta} = \sin\theta((1-2a)\cos^2\theta - 2\sin\theta) + O(r)$$

or

$$\frac{dr}{d\theta} = -r\,\frac{\cos\theta(a\cos^2\theta+\sin^2\theta+\sin\theta)}{\sin\theta((1-2a)\cos^2\theta-2\sin\theta)} + O(r^2).$$

Making the translation $\theta = \bar{\theta} + \frac{\pi}{2}$, then

$$\frac{dr}{d\bar{\theta}} = r\,\frac{\sin\bar{\theta}(a\sin^2\bar{\theta}+\cos^2\bar{\theta}+\cos\bar{\theta})}{\cos\bar{\theta}((1-2a)\sin^2\bar{\theta}-2\cos\bar{\theta})} + O(r^2) = f(\bar{\theta})\,r + O(r^2). \quad (5.25)$$

Note that $f(-\bar{\theta}) = -f(\bar{\theta})$ and the two symmetric sections τ_3 and τ_4 correspond to the two symmetric positions $\bar{\theta} = \bar{\theta}_0$ and $\bar{\theta} = -\bar{\theta}_0$. So integrating (5.25) from $\bar{\theta}_0$ to $-\bar{\theta}_0$ gives that for $v = 0$

$$r_3 = r_4\exp\left(\int_{-\theta_0}^{\theta_0} f(\bar{\theta})\,d\bar{\theta}\right) + O(r_4^2) = r_4 + O(r_4^2). \quad (5.26)$$

Let

$$\hat{T} = \Theta_3 \circ \bar{T}_{43} \circ \Theta_4^{-1}. \quad (5.27)$$

Easily we have $\hat{T}_1(v, \tilde{y}_4) = v$. Now we calculate the first derivative of $\hat{T}_2$. Note that by (5.24), we have

$$\frac{\partial}{\partial r_3}\,\Xi_3(r_3, \rho_3) = \left(\frac{r_3}{r_0}\right)^{\bar{\sigma}_3-1}\left[y_0 + \frac{\beta_1\kappa_3}{r_0^{\bar{\sigma}_3-1}}\,\rho_3^{p_3} + \hat{\theta}_3\left(r_3, \rho_3, \omega\left(\frac{r_3}{r_0}, -\beta_1\right)\right)\right]. \quad (5.28)$$

For $\Xi_4^{-1}(v, \tilde{y}_4)$, we have

$$\frac{\partial}{\partial\tilde{y}_4}\,\Xi_4^{-1}(v, \tilde{y}_4) = \frac{1}{\left(\frac{r_4}{r_0}\right)^{\bar{\sigma}_4-1}\left[y_0 + \frac{\beta_1\kappa_4}{r_0^{\bar{\sigma}_3-1}}\,\rho_4^{p_3} + \hat{\theta}_4\left(r_4, \rho_4, \omega\left(\frac{r_4}{r_0}, -\beta_1\right)\right)\right]}. \quad (5.29)$$

Hence by (5.27), (5.28), (5.26), (5.29), and Lemma 4.13, we have that $\hat{T}_2$ is at least C^1 and

$$\hat{T}'_2(0, 0) = 1. \tag{5.30}$$

We calculate the transition map R in the chart P.R.3. We have $R_1(v, \tilde{y}_3) = v$. For the second component R_2, as in Proposition 5.3, by using the auxiliary sections and formula of [ALGM] in Proposition 5.2, we obtain

$$R'_2(0, 0) = \gamma^*. \tag{5.31}$$

It follows from (5.23), (5.30), and (5.31) that we have

$$\det P(0, 0) = \gamma^*.$$

By Hypothesis 5.5, $\gamma^* < 1$. Hence the first return map P has at most one fixed point; i.e., $Cycl(Shhia) \leqslant 1$, $i = 1, 2, \ldots, 10$.

In the above proof for the saddle case, we only use the fact that $1 - 2a > 0$. For the elliptic case with $a \in (0, \frac{1}{2})$, the same proof gives that the upper boundary hh graphic of elliptic type has finite cyclicity 1. ▮

5.5. *Intermediate and Lower Boundary Graphics*

Let Γ be any intermediate or lower boundary graphic of the 10 families. To study its cyclicity, as shown in Fig. 15, take sections Π_3 and Π_4 (as defined in (5.22)) in the normal form coordinates $(r_i, \rho_i, \tilde{y}_i)$ $(i = 3, 4)$. We are going to study the displacement map

$$L = R^{-1} - T \colon \Pi_4 \to \Pi_3 \tag{5.32}$$

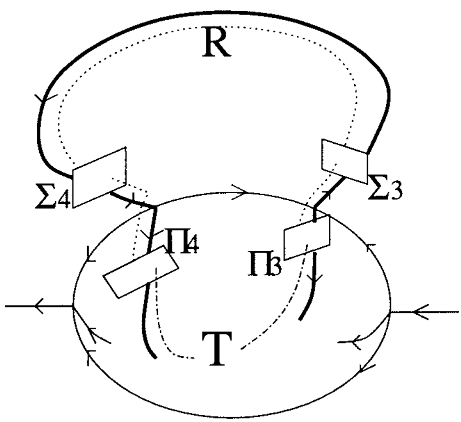


FIG. 15. Transition map for the intermediate hh graphics of saddle type.

or the displacement map

$$\mathscr{L} = R - T^{-1}\colon \Pi_3 \to \Pi_4 \tag{5.33}$$

where $R\colon \Pi_3 \to \Pi_4$ is the transition map along the regular orbit in the normal form coordinates and $T\colon \Pi_4 \to \Pi_3$ is the transition map passing through the blown-up singularity. Then by the derivation–division method introduced by Roussarie in [30], we study the number of small roots of $L = 0$ or $\mathscr{L} = 0$. The maximum number of roots bounds the cyclicity.

Remark 5.8. Unless otherwise stated, due to the existence of an invariant foliation, we focus our attention only on the second component.

We begin with the transition map R. Obviously $R_1(v, \bar{y}_3) = v$. The second component $R_2(v, \bar{y}_3)$ is *almost affine* (the two passages near P_3 and P_4 have a "funneling effect" [14]).

PROPOSITION 5.9. *For any $k \in \mathbb{N}$ and $\forall a_0 \in A = (-\frac{1}{2}, 0)$, there exist $A_0 \subset A$, a neighborhood of a_0, and $v_1 > 0$ such that $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $\forall v \in (0, v_1)$, $R_2(v, \tilde{y}_1)$ is C^k and*

(1) *If $a_0 \notin \mathbb{Q}$*

$$R_2(v, \tilde{y}_3) = m_{340}(v) \left(\frac{v}{v_0}\right)^{-\sigma_3} + (\gamma^* + O(v \ln v))\, \tilde{y}_3 + \sum_{j=2}^{k} O(v^{j-1})\, \tilde{y}_3^j + O(v^k \tilde{y}_3^{k+1}).$$

(2) *If $a_0 \in \mathbb{Q}$*

$$R_2(v, \tilde{y}_3) = \gamma_{340}\left(v, \omega\left(\frac{v}{v_0}, -\beta_1\right)\right) + \sum_{i=1}^{k} \gamma_{34i}\left(v, \omega\left(\frac{v}{v_0}, -\beta_1\right)\right) \tilde{y}_3^i + O(\tilde{y}_3^{k+1}), \tag{5.34}$$

where $m_{340}(0) = 0$, and

$$\gamma_{340} = m_{340}(v) \left(\frac{v}{v_0}\right)^{-\bar{\sigma}_4} + \kappa_3 r_0 (\gamma^* - 1)\, \omega\left(\frac{v}{v_0}, -\beta_1\right) + \kappa_3 O\left(v^{\bar{\sigma}_3} \omega^2 \left(\frac{v}{v_0}, -\beta_1\right)\right)$$

$$\gamma_{341} = \gamma^* + O\left(v \ln \frac{v}{v_0}\right) + O\left(v^{\bar{p}_3} \omega^{q_3} \left(\frac{v}{v_0}, -\beta_1\right) \ln \frac{v}{v_0}\right)$$

$$\gamma_{34j} = O\left(v \ln \frac{v}{v_0}\right) + O\left(v^{\bar{p}_3 (1 + [\frac{j-2}{q_3}])} \omega^{q_3 + 1 - j - q_3 [\frac{j-2}{q_3}]} \left(\frac{v}{v_0}, -\beta_1\right) \ln \frac{v}{v_0}\right),$$

$$j \geqslant 2.$$

Also $R_2^{-1}(v, \tilde{y}_4)$ is C^k and has precisely the same form as R_2.

Proof. We limit ourselves to the second case: $a_0 \in \mathbb{Q}$. Decompose the transition map R as

$$R := \Delta_4^{-1} \circ R_{34} \circ \Delta_3,$$

where $\Delta_j \colon \Pi_j \to \Sigma_j$ $(j = 3, 4)$ are the two Dulac maps of the first type in the normal form coordinates near P_3 and P_4, respectively, and $R_{34} \colon \Sigma_3 \to \Sigma_4$ is the regular transition map with second component

$$R_{342}(v, \tilde{y}_3) = m_{340}(v) + \sum_{i=1}^{k} m_{34i}(v)\, \tilde{y}_3^i + O(\tilde{y}_3^{k+1}), \tag{5.35}$$

where $m_{340}(0) = 0$ and $m_{341}(0) = \gamma^* + O(v)$.

The system near P_3 (resp. P_4) has the form (4.8) or (4.9) with $\sigma = \sigma_3(a)$ (resp. $\sigma = \sigma_4(a)$). By Theorem 4.10, the Dulac maps Δ_i $(i = 3, 4)$ have second components

$$D_i(v, \tilde{y}_i) = \eta_i\left(v, \omega\left(\frac{v}{v_0}, \beta_1\right)\right) + \left(\frac{v}{v_0}\right)^{\bar{\sigma}_i}\left[\tilde{y}_i + \psi_i\left(v, \tilde{y}_i, \omega\left(\frac{v}{v_0}, \beta_1\right)\right)\right], \tag{5.36}$$

where η_i and ψ_i have the same property as in (4.12).

Let

$$\hat{y}_4 = \left(\frac{v}{v_0}\right)^{-\bar{\sigma}_4}\left[D_4(v, \tilde{y}_4) - \eta_4\left(v, \omega\left(\frac{v}{v_0}, \beta_1\right)\right)\right]. \tag{5.37}$$

Then by (5.36) and (5.37), we have

$$\hat{y}_4 = \tilde{y}_4 + \psi_4\left(v, \omega\left(\frac{v}{v_0}, -\beta_1\right), \tilde{y}_4\right). \tag{5.38}$$

Let

$$\tilde{y}_4 = \hat{y}_4 + \bar{\phi}_4\left(v, \omega\left(\frac{v}{v_0}, -\beta_1\right), \hat{y}_4\right)$$

be the inverse of (5.38). Then $\bar{\phi}_4$ has the same property as ψ_4.

If we use $\hat{y}_4$ as the variable, we can express the $y-$component of Δ_4^{-1} as

$$D_4^{-1}(v, \tilde{y}_4) = \hat{y}_4 + \bar{\phi}_4\left(v, \omega\left(\frac{v}{v_0}, -\beta_1\right), \hat{y}_4\right). \tag{5.39}$$

Hence, for the second component of transition map $\bar{R}$, by (5.36), (5.35), (5.39), and Lemma 4.13, a straightforward calculation gives the result. ▮

The following proposition will serve to treat the intermediate graphics while the lower boundary graphics will require ad hoc methods in each case.

PROPOSITION 5.10. *Assume that we have a convex hh graphic Γ of saddle or elliptic type shown in Fig. 15. Let*

$$T\colon \Pi_4 \to \Pi_3$$

be the transition map along the connection in the chart F.R. Then if T satisfies one of the following conditions:

- *T is the identity while the graphic is generic (i.e., $\gamma^* < 1$);*
- *$T'_2(0, 0)$ is sufficiently small or $T'_2(0, 0)$ is sufficiently large;*
- *$T_2(0, \tilde{y}_4)$ is nonlinear of order n,*

then Γ has finite cyclicity.

Proof. We consider the displacement map L or its inverse defined in (5.32). By Proposition 5.9, the second component $R_2(v, \tilde{y}_3)$ of R is almost affine, yielding the results. Γ has cyclicity $\leqslant 1$ in the first two cases and cyclicity $\leqslant n$ in the third. ▮

It seems a priori difficult to show that a transition map is nonlinear. For all the cases, we will deal with families of graphics. This allows an interesting observation which we state in the following proposition.

PROPOSITION 5.11. *It is possible to choose normalizing coordinates near P_3 and P_4 such that $\tilde{y}_i(0, \rho_i, y_i)$ $(i = 3, 4)$ is analytic.*

Proof. We modify the normalization process. For both the saddle or elliptic cases, the vector field near P_3 can be written as

$$\begin{aligned} \dot{r} &= r \\ \dot{\rho} &= -\rho \\ \dot{y} &= -\sigma_3(a)\, y + h(a, \bar{\mu}, r, \rho, y), \end{aligned} \tag{5.40}$$

where $h(a, \bar{\mu}, r, \rho, y) = o(|r, \rho, y|)$ and for both the saddle $A = (-\frac{1}{2}, 0)$ or the elliptic $A = (0, \frac{1}{2})$ case, we have $\sigma_3(a) = 2(1 - 2a) > 0$.

Let us consider (5.40) for $r = 0$. Then we get

$$\begin{aligned} \dot{\rho} &= -\rho \\ \dot{y} &= -\sigma_3(a)\, y + h(a, \bar{\mu}, 0, \rho, y). \end{aligned} \tag{5.41}$$

For the subfamily (5.41), the tuple of eigenvalues $(-1, -\sigma_3(a))$ is in the Poincaré domain, the subfamily has no (resp. one) resonant monomial when $\sigma_3(a_0) \notin \mathbb{N}$ (resp. $\sigma_3(a_0) \in \mathbb{N}$). Hence there exists an analytic map

$$Y = y + \hat{\phi}(\rho, y) \tag{5.42}$$

which brings the family (5.41) into the normal form

$$\begin{aligned} \dot{\rho} &= -\rho \\ \dot{Y} &= -\sigma_3(a)\, Y + \kappa_3 \rho^{p_3}, \end{aligned} \tag{5.43}$$

where $p_3 = \sigma_3(a_0)$, and, if $\sigma_3(a_0) \notin \mathbb{N}$, then $\kappa_3 = 0$.

Applying the map (5.42) to the original family (5.40) brings the system to the form

$$\begin{aligned} \dot{r} &= r \\ \dot{\rho} &= -\rho \\ \dot{Y} &= -\sigma_3(a)\, Y + \kappa_3 \rho^{p_3} + rH(a, \bar{\mu}, r, \rho, Y). \end{aligned} \tag{5.44}$$

For the system (5.44), by Proposition 4.6, $\forall (a, \bar{\mu}) \in A \times V$, there exists a C^k map of the form

$$\tilde{y} = Y + r\bar{\phi}(r, \rho, Y) \tag{5.45}$$

which brings system (5.44) into the normal form (4.8) or (4.9). ∎

Corollary 5.12. Analytic extension principle. *Assume that $\forall a_0 \in A$ and $\bar{\mu}_0 \in V$ $(V \subset \mathbb{S}^2)$, we have a family of graphics of the saddle (convex) or elliptic type which only differ by a segment joining two nodes (Fig. 15). Let Γ be any intermediate graphic in the family. Then $\forall (a, \bar{\mu}) \in A \times V$, the normal form coordinates $\tilde{y}_3$, $\tilde{y}_4$ can be taken so that $\tilde{y}_i(0, \rho, y_i)$ is analytic. Take sections Π_3 and Π_4 in the normal form coordinates in the neighborhood of P_3 and P_4, respectively. Let $\Gamma \cap \Pi_4 = (0, \tilde{y}_4^*)$. Consider the transition map associated with the graphic Γ*

$$\begin{aligned} T\colon \Pi_4 &\to \Pi_3 \\ (v, \tilde{y}_4) &\mapsto (v, T_2(v, \tilde{y}_4)). \end{aligned}$$

If $\forall (a, \bar{\mu}) \in A \times V$, $T_2(0, \tilde{y}_4)$ is nonlinear in the neighborhood of $\tilde{y}_4^*$, then its analytic extension in its extension domain $\mathscr{I}$ in $\mathbb{R}$ is nonlinear at any particular value of $\tilde{y}_4 \in \mathscr{I}$ for $(a, \bar{\mu}) \in A \times V$.

Proof of Theorem 5.4. There are 10 families of convex graphics of saddle type (Table VI). We have proved in Proposition 5.7 that all the upper boundary graphics have finite cyclicity, so we need to prove that in each family all the intermediate and lower boundary graphics have finite cyclicity.

For each lower boundary graphic, we will study the number of roots for the corresponding displacement map $L = R^{-1} - T$ or $\mathscr{L}$: $R - T^{-1}$ defined in (5.32) or (5.33). Let Γ be any intermediate graphic of the corresponding family and let $T: \Pi_4 \to \Pi_3$ be the transition map associated with the graphic Γ in the chart F.R. We will apply Proposition 5.10 to the graphic Γ. Usually if the criterion that T is nonlinear is used, the starting point is chosen near the lower boundary graphic.

The map R satisfies Proposition 5.9 and R_2 is almost affine. For the transition map T, since $r = 0$ is invariant in the chart F.R., then $T_1(0, \tilde{y}_3) = 0$. We will go over all 10 families of graphics by considering the second component $T_2(0, \tilde{y}_3)$ or its inverse.

For each family of the graphic Sxhh$i$ ($i = 1, 2, \ldots, 10$), we use $V_i \subset \mathbb{S}^2$ to denote the set of $\bar{\mu}$ in which the family Sxhh$i$ exists.

(1) Family Sxhh1. As shown in Fig. 16a, family Sxhh1 has a lower boundary graphic Sxhh1c which passes through a hyperbolic saddle point in the chart F.R. Using the good properties of R_2 the proof is almost the same as for the finite cyclicity of a homoclinic loop except for the following points.

Let $\lambda_0(\bar{\mu}_0)$ be the hyperbolicity ratio at the saddle point. If $\lambda(\bar{\mu}_0) \neq 1$, we find *Cycl*(*Sxhh*1*c*) $\leqslant 1$. If $\lambda(\bar{\mu}_0) = 1$ and $a \neq -\frac{1}{2}$, we need to calculate the first saddle quantity of the saddle point of system (3.10). Using the formula in [7], we find it is

$$\alpha_{02} = \frac{2a(a-1)(1+2a)^2}{a(4a-1)\,\bar{\mu}_{30}^2 - (1+2a)^2\,\bar{\mu}_{20}}\,\bar{\mu}_{30}. \tag{5.46}$$

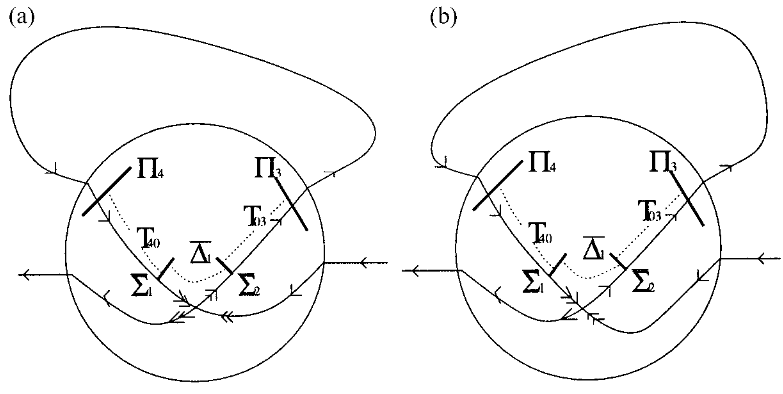


FIG. 16. Transition map T for the family (a) Sxhh1 and (b) Sxhh2.

Then we have two cases

• If $\bar{\mu}_{30} \neq 0$: The lower boundary graphic Sxhh1c is studied as a homoclinic loop for which the first saddle quantity is nonzero. Then $Cycl(Sxhh1c) \leqslant 3$. Moreover, near the lower boundary graphic $T_2''(0, \tilde{y}_4) \neq 0$, T_2 is nonlinear in $\tilde{y}_4$. By Proposition 5.10, we have that $Cycl(Sxhh1b)$ is finite.

• If $\bar{\mu}_{30} = 0$: In this case $\lambda(\bar{\mu}_0) = 1$ implies that the system (3.10) is symmetric with respect to the y-axis. The lower boundary graphic is treated as a homoclinic loop for which the first derivative is nonzero. Hence $Cycl(Sxhh1c) \leqslant 2$.

(2) Families Sxhh2 and Sxhh3. For the family Sxhh2, system (3.10) has a semihyperbolic saddle on Sxhh2c (Fig. 16b). Consider the map $\bar{\Delta} = (\bar{d}, \bar{D})$: $\bar{\Sigma}_1 \to \bar{\Sigma}_2$. In this case for $v = 0$, $\bar{D}$ is the stable center transition near the semihyperbolic saddle. Then by [12], $\forall i_1, i_2 \in \mathbb{N}$, $\forall (a, \bar{\mu}) \in A_0 \times V_{20}$, and $v > 0$ sufficiently small, we have

$$\frac{\partial^{i_1} \bar{D}}{\partial \hat{x}^{i_1}}(v, \hat{x}) = O(\hat{x}^{i_2}). \tag{5.47}$$

So for T_2, by (5.47) we have $T_2'(0, \tilde{y}_4) \to 0$ which gives $Cycl(Sxhh2c) \leqslant 1$ and the nonlinearity of T_2, hence the finite cyclicity of $Cycl(Sxhh2b)$.

By changing $(x, t) \mapsto (-x, -t)$, similar to family Sxhh2, the result holds for the family Sxhh3.

(3) Families Sxhh4, Sxhh5, and Sxhh6. For the families Sxhh4, Sxhh5, and Sxhh6, the corresponding lower boundary graphic has a saddle connection $\mathcal{S}_1\mathcal{S}_2$. At $\mathcal{S}_1$ and $\mathcal{S}_2$, the hyperbolicity ratios are

$$\mathcal{S}_1 : \lambda_1 = \frac{\bar{\mu}_3 - \sqrt{-\frac{\bar{\mu}_2}{a}}}{2a\sqrt{-\frac{\bar{\mu}_2}{a}}}; \qquad \mathcal{S}_2 : \lambda_2 = \frac{-2a\sqrt{-\frac{\bar{\mu}_2}{a}}}{\bar{\mu}_3 + \sqrt{-\frac{\bar{\mu}_2}{a}}}.$$

Families Sxhh4 and Sxhh6 correspond to $\bar{\mu}_{30} \neq 0$; i.e., $\lambda_1\lambda_2 \neq 1$. The proof is exactly the same as for a polycycle with two hyperbolic saddles [28]; yielding their cyclicity is at most 2.

The family Sxhh5 exists if and only if $\bar{\mu} = (0, 1, 0)$. Then system (3.10) in the chart F.R. is symmetric with a center (Fig. 17). For the intermediate graphics, we easily see that, for $v = 0$, the transition map T is the identity, yielding by Proposition 5.10 that the graphic Sxhh5b has finite cyclicity.

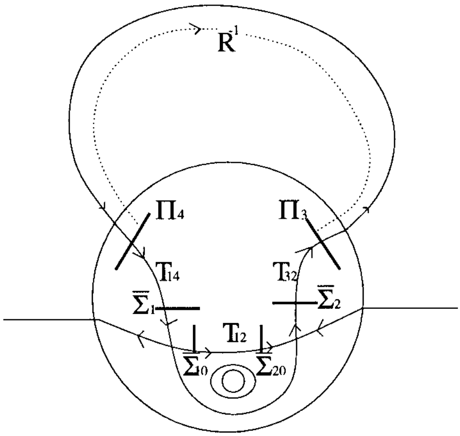


FIG. 17. Transition map T for the family Sxhh5.

Now we consider the graphic Sxhh5c. The hyperbolicity ratios λ_1 and λ_2 satisfy $\lambda_1 = -\frac{1}{2a} > 1$, $\lambda_2 = -2a < 1$ and $\lambda_1 \lambda_2 = 1$. By Proposition 4.2, the Dulac maps defined in the neighborhood of the two saddles can be written as

$$\begin{aligned} \bar{D}_1(v, x_1) &= x_1^{\lambda_1}(1 + \bar{\phi}_1(v, x_1)) \\ \bar{D}_2^{-1}(v, x_2) &= x_2^{\frac{1}{\lambda_2}}(1 + \bar{\phi}_2(v, x_2)), \end{aligned} \tag{5.48}$$

where $\bar{\phi}_i(v, x_i)$ $(i = 1, 2)$ satisfies (I_0^∞) for $(a, \bar{\mu}) \in A_0 \times V_{51}$, $v \in (0, v_1)$, and x_i sufficiently small.

Consider the displacement map

$$\begin{aligned} &L: \bar{\Sigma}_1 \longrightarrow \bar{\Sigma}_{20} \\ &L = \bar{T}_{12} \circ \bar{\Delta}_1 - \bar{\Delta}_2^{-1} \circ \bar{T}_{32} \circ R^{-1} \circ \bar{T}_{14}, \end{aligned} \tag{5.49}$$

where

- $\bar{T}_{12}: \bar{\Sigma}_{10} \to \bar{\Sigma}_{20}$, $\bar{T}_{122}(v, y_1) = m_{120}(v) + (1 + O(v))\, y_1 + O(y_1^2)$;
- $\bar{T}_{14}: \bar{\Sigma}_1 \to \Pi_4$, $\bar{T}_{14}(v, x_1) = m_{140}(v) + m_{141}(v)\, x_1 + O(x_1^2)$;
- $R^{-1}: \Pi_4 \to \Pi_3$, By Proposition 5.9 we have $R_2^{-1}(v, \tilde{y}_4) = m_{340}(v) + (\frac{1}{\gamma^*} + O(v))\, \tilde{y}_4 + O(\tilde{y}_4^2)$;
- $\bar{T}_{32}: \Pi_3 \to \bar{\Sigma}_2$, $\bar{T}_{322}(v, \tilde{y}_3) = m_{320}(v) + m_{321}(v)\, \tilde{y}_3 + O(\tilde{y}_3^2)$ and $m_{321}(0)\, m_{141}(0) = 1$ because of the symmetry of the system (3.10).

Then a straightforward calculation gives

$$\begin{aligned} L_2(v, x_1) = \bar{\varepsilon}_1(v) + x_1^{\lambda_1}(1+\bar{\phi}_{11}(v, x_1)) \\ -[\bar{\varepsilon}_2(v)+\bar{\gamma}^*(v)\, x_1 + O(x_1^2)]^{\frac{1}{\lambda_2}}(1+\bar{\phi}_{21}(v, x_1)) \end{aligned} \tag{5.50}$$

where $\bar{\gamma}^*(v) = (m_{321}(v)\, m_{141}(v)/\gamma^*) + O(v) \neq 1$ and $\bar{\phi}_{11}, \bar{\phi}_{21} \in (I_0^\infty)$.

By (5.50),

$$\begin{aligned} L_2'(v, x_1) = \lambda_1 x_1^{\lambda_1 - 1}(1+\bar{\phi}_{12}(v, x_1)) \\ -\frac{1}{\lambda_2}[\bar{\varepsilon}_2(v)+\bar{\gamma}^*(v)\, x_1 + O(x_1^2)]^{\frac{1}{\lambda_2}-1}(\bar{\gamma}^*(v)+\bar{\phi}_{22}(v, x_1)), \end{aligned}$$

where $\bar{\phi}_{12}, \bar{\phi}_{22} \in (I_0^\infty)$. $L_2'(v, x_1)$ has the same number of small roots $x_1 > 0$ as

$$\begin{aligned} L_{21}(v, x_1) = (\lambda_1\lambda_2)^{\frac{\lambda_2}{1-\lambda_2}} x_1^{\frac{(\lambda_1-1)\lambda_2}{1-\lambda_2}}(1+\bar{\phi}_{13}(v, x_1)) \\ -[\bar{\varepsilon}_2(v) - \bar{\gamma}^*(v)\, x_1 + O(x_1^2)](\bar{\gamma}^{*\frac{\lambda_2}{1-\lambda_2}}(v)+\bar{\phi}_{23}(v, x_1)). \end{aligned} \tag{5.51}$$

Let $\bar{\beta}_1 = (1-\lambda_1\lambda_2)/(1-\lambda_2)$. For the term $x_1^{(\lambda_1-1)\lambda_2/(1-\lambda_2)}$, we make the following development

$$x_1^{\frac{(\lambda_1-1)\lambda_2}{1-\lambda_2}} = x_1^{1-\bar{\beta}_1} = x_1(1+\bar{\beta}_1\bar{\omega}),$$

where $\bar{\omega} = \omega(x_1, \bar{\beta}_1)$.

By (5.51), we then have

$$\begin{aligned} L_{21}'(v, x_1) = (\lambda_1\lambda_2)^{\frac{\lambda_2}{1-\lambda_2}}(1-\bar{\beta}_1+\bar{\beta}_1(1-\bar{\beta}_1)\,\bar{\omega})(1+\bar{\phi}_{14}(v, x_1)) \\ -[\bar{\gamma}^{*1+\frac{\lambda_2}{1-\lambda_2}} + O(x_1)](1+\bar{\phi}_{24}(v, x_1)) \end{aligned} \tag{5.52}$$

which has the same number of zeroes as

$$L_{22}(v, x_1) = 1-\bar{\beta}_1 - \hat{\gamma}^*(v) + O(v) + \bar{\beta}_1(1-\bar{\beta}_1)\,\bar{\omega} + O(x_1), \tag{5.53}$$

where $\hat{\gamma}^*(v) = \bar{\gamma}^{*1/(1-\lambda_2)}/(\lambda_1\lambda_2)^{\lambda_2/(1-\lambda_2)}$ and $\hat{\gamma}^*(0) = \gamma^*$.

Let $L_{23} = L_{22}/\bar{\omega}$; then

$$L_{23} = \frac{1-\bar{\beta}_1-\hat{\gamma}^*(v)+O(v)}{\bar{\omega}} + \bar{\beta}_1(1-\bar{\beta}_1) - \frac{O(x_1)}{\bar{\omega}}.$$

If we differentiate L_{23} and let $L_{24} = \bar{\omega}^2 x_1^{1+\bar{\beta}_1} L_{23}'$, then

$$L_{24} = [-1+\bar{\beta}_1+\hat{\gamma}^*(v)+O(v)] + O(x_1).$$

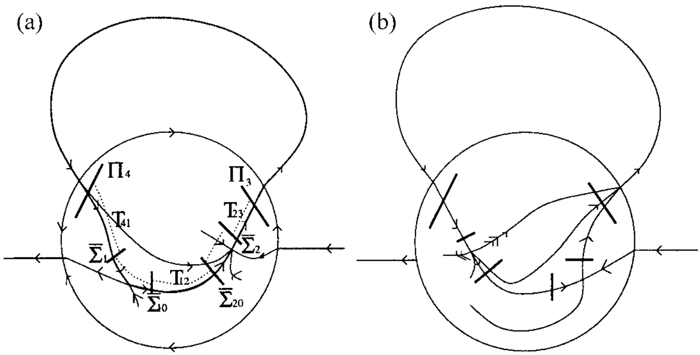


FIG. 18. Transition map T for the families (a) Sxhh9 and (b) Sxhh10.

Since $\hat{\gamma}^*(0) = \gamma^* \neq 1$, then $\forall(a, \bar{\mu}) \in A_0 \times V_{51}$ and for $\nu > 0$ sufficiently small, we have that L_{24} does not vanish. So $L = 0$ has at most three roots which gives $Cycl(Sxhh5c) \leqslant 3$.

(4) Families Sxhh7 and Sxhh8. The proof is exactly the same as for the finite cyclicity of a graphic through a saddle node and a hyperbolic saddle of the same attracticity. As in [12], the cyclicity is $\leqslant 1$.

(5) Families Sxhh9 and Sxhh10. As shown in Fig. 18a, the family Sxhh9 has two subfamilies of graphics: intermediate graphics Sxhh9b and Sxhh9d and two boundary graphics Sxhh9c and Sxhh9e.

First note that the graphic Sxhh9c that passes by an attracting saddle node has the same structure as the graphic Sxhh2c, so we only need to consider the lower boundary graphic Sxhh9e. The proof of its finite cyclicity is the same as that of a graphic through a hyperbolic saddle and a saddle node with central transition (see [12]).

For the intermediate graphics Sxhh9d, the transition map T along the graphic can be factorized as two regular transition maps and a central transition map. Obviously, $T_2(\nu, \tilde{y}_4)$ has a first derivative which is small; thus $Cycl(Sxhh9d)$ is finite.

For family Sxhh10 we change $(x, t) \mapsto (-x, -t)$. ∎

6. FINITE CYCLICITY OF GRAPHICS WITH A NILPOTENT SINGULARITY OF ELLIPTIC TYPE

6.1. *Finite Cyclicity of pp Graphics of Elliptic Type.*

In Table III, we have three families of pp graphics of elliptic type: Epp1, Epp2, and Epp3. All the pp-graphics have no return map. For the passage near the blown-up sphere $r=0$ we have system (3.10) in the chart F.R. Let V_i $(i=1,2,3)$ be the set of parameters in which Epp$i$ exists.

The following proposition will be important in proving the finite cyclicity of pp and hh graphics of elliptic type.

PROPOSITION 6.1. *Let S_2 be the second component of the transition map $S\colon \Pi_1 \to \Pi_2$ in the normal form coordinates. Then $\forall (a, \bar{\mu}) \in A \times V_{I_1}$ and $\nu > 0$ sufficiently small, we have*

$$\begin{aligned}
\frac{\partial S_2}{\partial \tilde{y}_1}(0,0) &= \exp\left(\frac{\pi \bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\right) + O(\rho_0) \\
\frac{\partial^2 S_2}{\partial \tilde{y}_1^2}(0,0) &= \frac{1}{a(1-2a)}\, e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\left[1 - e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\right] + O(\rho_0).
\end{aligned} \tag{6.1}$$

Proof. The transition map S can be factorized as

$$S = \Psi_2|_{\Pi_2} \circ \Phi_{02}|_{\Pi_2} \circ \bar{S} \circ \Phi_{10}|_{\Pi_1} \circ \Psi_1^{-1}|_{\Pi_1}, \tag{6.2}$$

where

(1) Ψ_1 and Ψ_2 are the C^k-coordinate changes normalizing the vector fields (3.8) at P_1 and P_2 respectively,

$$\Psi_1^{-1}|_{\Pi_1} \colon \begin{cases} r_1 = \dfrac{\nu}{\rho_0} \\ \bar{y}_1 = b_{11}\tilde{y}_1 + b_{12}\tilde{y}_1^2 + O(\tilde{y}_1^3) \end{cases}$$

$$\Psi_2|_{\Pi_2} \colon \begin{cases} r_2 = \dfrac{\nu}{\rho_0} \\ \tilde{y}_2 = b_{21}\bar{y}_2 + b_{22}\bar{y}_2^2 + O(\bar{y}_2^3) \end{cases}$$

where b_{1i} and $b_{2i}(i=1,2)$ are functions of r_i, ρ_i, respectively. On the sections Π_1 and Π_2, we have $\rho_1 = \rho_2 = \rho_0$, so on $r=0$, we have

$$b_{11} = 1 + \frac{\bar{\mu}_3}{a}\rho_0 + O(\rho_0^2)$$

$$b_{12} = -\frac{1}{a(1-2a)} + O(\rho_0)$$

$$b_{21} = 1 + \frac{\bar{\mu}_3}{a}\rho_0 + O(\rho_0^2)$$

$$b_{22} = \frac{1}{a(1-2a)} + O(\rho_0).$$

(2) Φ_{10} and Φ_{02} are coordinate changes between charts P.R.1 and F.R. and F.R. and P.R.2, respectively. On the corresponding sections, they are linear:

$$\Phi_{10}|_{\Pi_1}: \begin{cases} \bar{x} = -\dfrac{1}{\rho_0} \\ \bar{y} = \dfrac{\tilde{y}_1}{\rho_0^2}, \end{cases} \qquad \Phi_{02}|_{\Pi_1}: \begin{cases} r_2 = \dfrac{v}{\rho_0} \\ \tilde{y}_2 = \rho_0^2 \bar{y}. \end{cases}$$

(3) $\bar{S}$: $\{\bar{x} = -x_0\} \to \{\bar{x} = x_0\}$ is the transition map in the original coordinates $(\bar{x}, \bar{y})$ in the chart F.R., where $x_0 = 1/\rho_0$.

For $\bar{\mu} \in V_{I_1}$, system (3.10) has no singular points on the invariant line $\bar{y} = 0$, so $\bar{S}_2$ is a C^k regular transition map

$$\bar{S}_2(v, \bar{y}) = m_0(v) + m_1(v)\,\bar{y} + m_2(v)\,\bar{y}^2 + O(\bar{y}^3), \tag{6.3}$$

where $m_0(0) = 0$. For the coefficients $m_1(v)$ and $m_2(v)$, by Proposition 5.2, we have

$$m_1(0) = \exp\left(\int_{-\bar{x}_0}^{\bar{x}_0} \frac{\bar{x} + \bar{\mu}_3}{a\bar{x}^2 + \bar{\mu}_2}\, d\bar{x}\right) = \exp\left(\frac{2\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}} \arctan \frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}\right),$$

$$m_2(0) = m_1(0) \int_{-\bar{x}_0}^{\bar{x}_0} -\frac{2(\bar{x} + \bar{\mu}_3)}{(a\bar{x}^2 + \bar{\mu}_2)^2} \exp\left(\int_{-\bar{x}_0}^{\bar{x}} \frac{\bar{x} + \bar{\mu}_3}{a\bar{x}^2 + \bar{\mu}_2}\, d\bar{x}\right) d\bar{x} = m_1(0)\, I_{12}(\bar{x}_0),$$

where

$$I_{12}(\bar{x}_0) = \frac{e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}} \arctan \frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}_0^2 + \bar{\mu}_2)^{\frac{1}{2a}}} \int_{-\bar{x}_0}^{\bar{x}_0} \frac{-2(\bar{x} + \bar{\mu}_3)\, e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}} \arctan \frac{a\bar{x}}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}^2 + \bar{\mu}_2)^{2 - \frac{1}{2a}}}\, d\bar{x}.$$

By L'Hospital's rule,

$$\lim_{\bar{x}_0 \to \infty} I_{12}(\bar{x}_0)(a\bar{x}_0^2 + \bar{\mu}_3) = \frac{2}{1 - 2a}\left(1 - e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\right).$$

So

$$I_{12}(\bar{x}_0) = \frac{2}{a(1 - 2a)}\left(1 - e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\right) \frac{1}{\bar{x}_0^2} + o\left(\frac{1}{\bar{x}_0^2}\right).$$

Therefore, for $\bar{x}_0 = \frac{1}{\rho_0}$ and $\rho_0 > 0$ small, we have

$$m_1(0) = e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}} + O(\rho_0)$$

$$m_2(0) = \frac{2}{a(1 - 2a)}\, e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\left(1 - e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\right) \rho_0^2 + o(\rho_0^2)\}.$$

Then by (1), (2), (3), and (6.2), we have

$$\frac{\partial^2 S_2}{\partial \tilde{y}_1^2}(0,0)=\frac{1}{a(1-2a)}\, e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\left[1-e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\right]+O(\rho_0). \quad \blacksquare$$

Let Γ be any pp graphic in the family. To prove its finite cyclicity, as shown in Fig. 19, we take sections Σ_1 and Σ_2 in normal form coordinates in the neighborhood of P_1 and P_2, respectively. We study the displacement maps

$$L=R^{-1}-T \qquad \text{or} \qquad \mathscr{L}=R-T^{-1}, \tag{6.4}$$

where $R\colon \Sigma_2 \to \Sigma_1$ is the regular transition map along the regular orbit and $T\colon \Sigma_1 \to \Sigma_2$ is the transition passing through the blown-up nilpotent elliptic singularity.

For the transition map T, similar to Proposition 5.9, the passage from P_1 to P_2 has a *funneling effect*, i.e., its second component T_2 is almost affine.

PROPOSITION 6.2. *There exists $\varepsilon_0>0$ such that for any $k\in\mathbb{N}$, $a_0\in(0,\frac{1}{2})$, there exist $A_0\subset(0,\frac{1}{2})$, a neighborhood of a_0 such that for $\forall(a,\bar{\mu})\in A_0\times V_{I_3}$, $T_2'(0,0)$ is sufficiently small, while for $(a,\bar{\mu})\in A\times V_{I_2}$, $T_2^{-1}(0,0)$ is sufficiently small. For any $(a,\bar{\mu})\in A_0\times V_{I_1}$ and $v>0$ sufficiently small, the second component $T_2(v,\bar{y})$ of T is C^k and*

$$\begin{aligned} T_2(v,\tilde{y}_1)={}&\gamma_{120}\left(v,\omega\left(\frac{v}{v_0},-\alpha_0\right)\right)+\sum_{i=1}^{k}\gamma_{12i}\left(v,\omega\left(\frac{v}{v_0},-\alpha_0\right)\right)\tilde{y}_1^i\\ &+O\left(v^{\bar{p}_1\left(1+\left[\frac{k-2}{q_1}\right]\right)}\omega^{q_1+1-k-q_1\left[\frac{k-2}{q_1}\right]}\left(\frac{v}{v_0},-\alpha_0\right)\ln\frac{v}{v_0}\,\tilde{y}_1^{k+1}\right) \end{aligned} \tag{6.5}$$

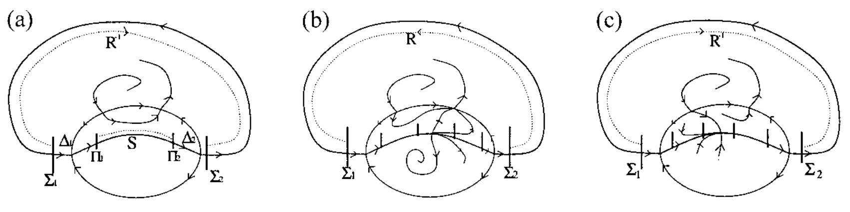


FIG. 19. Displacement maps for pp graphics. (a) Epp1, (b) Epp2, (c) Epp3.

where

$$\begin{aligned}\gamma_{120} &= m_{120}(v)\left(\frac{v}{v_0}\right)^{-\bar{\sigma}_2} + \kappa_1 r_0^{p_1}(1-m_{121}(v))\,\omega\left(\frac{v}{v_0},\alpha_1\right)\\ &\quad + O\left(\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\omega^2\left(\frac{v}{v_0},-\alpha_0\right)\right)\\ \gamma_{121} &= m_{121}(v) + O\left(v\ln\frac{v}{v_0}\right) + O\left(\left(\frac{v}{v_0}\right)^{\bar{p}_1}\omega^{q_1}\left(\frac{v}{v_0},-\alpha_0\right)\ln\frac{v}{v_0}\right)\\ \gamma_{12i} &= O\left(v\ln\frac{v}{v_0}\right) + O\left(v^{\bar{p}_1(1+[\frac{i-2}{q_1}])}\omega^{q_1+1-i-q_1[\frac{i-2}{q_1}]}\left(\frac{v}{v_0},-\alpha_0\right)\ln\frac{v}{v_0}\right),\\ &\quad i \geqslant 2\end{aligned}$$

and $m_{120}(0) = 0$, $m_{121}(0) = \exp(\pi\bar{\mu}_3/\sqrt{a\bar{\mu}_2})$.

THEOREM 6.3. *We consider a pp graphic with a triple nilpotent elliptic point of any codimension. If the second component R_2 of the regular transition map R has its nth derivative nonvanishing, then $Cycl(Epp) \leqslant n$.*

Proof. There are three types of pp limit periodic sets through a nilpotent elliptic point. We write

$$R_2(v,\tilde{y}_1) = \sum_{i=0}^{k} \tilde{\gamma}_i(v)\,\tilde{y}_1^i + o(\tilde{y}_1^k), \tag{6.6}$$

where $\tilde{\gamma}_0(0) = 0$ and $\tilde{\gamma}_1(0), \tilde{\gamma}_n(0) \neq 0$.

So for the displacement map L in (6.4), we have $L_1(v,\tilde{y}_1) = 0$ and

$$L_2(v,\tilde{y}_1) = \sum_{i=0}^{k}\left[\gamma_{12i}\left(v,\omega\left(\frac{v}{v_0},-\alpha_0\right)\right) - \tilde{\gamma}_i(v)\right]\tilde{y}_1^i + O(\tilde{y}_1^{k+1}). \tag{6.7}$$

For the graphic Epp3, $\gamma_{121}(v,\omega)$ sufficiently small, so we have

$$\frac{\partial L_2}{\partial \tilde{y}_1}(v,\tilde{y}_1) = \gamma_{121}\left(v,\omega\left(\frac{v}{v_0},-\alpha_0\right)\right) - \tilde{\gamma}_1(v) + O(\tilde{y}_1) \neq 0,$$

which gives $Cycl(Epp3) \leqslant 1$. Similarly $Cycl(Epp2) \leqslant 1$.

For the graphic Epp1, if we choose $k \geqslant n$, then by (6.5) and (6.7), $\forall (a, \bar{\mu}) \in A_0 \times V_{I_1}$ and $\forall v \in (0, v_1)$, there holds

$$\begin{aligned}
&\frac{\partial^n L_2}{\partial \tilde{y}_1^n}(v, \tilde{y}_1)\\
&\quad = -n!\, \tilde{\gamma}_n(v) + \left(v \frac{v}{v_0}\right) + O\left(v^{\bar{p}_1\,(1+[\frac{n-2}{q_1}]} \omega^{q_1 1+1-n+q_1 [\frac{n-2}{q_1}]}\left(\frac{v}{v_0}, -\alpha_0\right) \ln \frac{v}{v_0}\right)\\
&\qquad + O(\tilde{y}_1)\\
&\quad \neq 0.
\end{aligned}$$

So by Rolle's theorem, for any $(a, \bar{\mu}) \in A_0 \times V_{I_1}$ and $\forall v \in (0, v_1)$, $L_2(v, \tilde{y}_1) = 0$ has at most n small roots in the neighborhood of $\tilde{y}_1 = 0$; i.e., $Cycl(Epp1) \leqslant n$. ▮

PROPOSITION 6.4. *In Theorem 6.3, for the transition map R we assumed that for $n \geqslant 2$, $R_2^{(n)}(0, 0) \neq 0$. This assumption is intrinsic.*

Proof. The ideas come from [17]. By saying that the assumption $R_2^{(n)}(0, 0) \neq 0$ $(n \geqslant 2)$ is intrinsic we mean that this property depends neither on the choices of coordinate changes which bring the system near P_1 and P_2 to normal forms nor on the choice of the sections parallel to the coordinate axes in the normal form coordinates.

In the coordinates $(r_1, \rho_1, \tilde{y}_1)$, the system near P_1 has the normal form (4.8) or (4.9). The Dulac map $\Delta_1: \Sigma_1 \to \Pi_1$ has the form (4.11). Assume that by an another "nearly-identity" change of coordinates, we bring the system near P_1 into the same normal form with coordinates $(r_1, \rho_1, \tilde{\tilde{y}}_1)$. Let $\tilde{\Sigma}_1 = \{r_1 = r_{10}\}$ and $\tilde{\Pi}_1 = \{\rho_1 = \rho_{10}\}$ be two sections parametrized by the new normal form coordinates $\tilde{\tilde{y}}_1$ and let $\tilde{\tilde{\Delta}}_1 = (\tilde{\tilde{d}}, \tilde{\tilde{D}}_1)$ be the Dulac map $\tilde{\Sigma}_1 \to \tilde{\Pi}_1$ in the new normal form coordinates. Then $\tilde{\tilde{\Delta}}_1$ has the same form as Δ_1 in (4.11), and we should have

$$\begin{aligned}
\tilde{\tilde{\Delta}}_1(v, \tilde{\tilde{y}}_1) &= \hat{\Phi}_{11} \circ \Delta_1 \circ \tilde{\Phi}_{11}(v, \tilde{\tilde{y}}_1) \qquad \text{or}\\
\tilde{\tilde{\Delta}}_1 \circ \tilde{\Phi}_{11}^{-1}(v, \tilde{y}_1) &= \hat{\Phi}_{11} \circ \Delta_1(v, \tilde{y}_1),
\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}
\tilde{\Phi}_{11}^{-1}(v, \tilde{y}_1) &= (v, \tilde{\phi}_{11}^{-1}): \tilde{\Sigma}_1 \to \Sigma_1\\
\hat{\Phi}_{11}(v, \tilde{y}_1) &= (v, \hat{\phi}_{11}): \Pi_1 \to \tilde{\Pi}_1
\end{aligned}$$

are the compositions of coordinate changes and C^k regular transitions, respectively. Let

$$\begin{aligned}\tilde{\phi}_{11}^{-1}(v, \tilde{y}_1) &= \sum_{j=0}^{k} \tilde{m}_{11j}(v)\, \tilde{y}_1^j + O(\tilde{y}_1^{k+1}) \\ \hat{\phi}_{11}(v, \tilde{y}_1) &= \sum_{j=0}^{k} \hat{m}_{11j}(v)\, \tilde{y}_1^j + O(\tilde{y}_1^{k+1}),\end{aligned} \tag{6.9}$$

where $\tilde{m}_{111}(0) > 0$ and $\hat{m}_{111}(0) > 0$.

We only consider the most difficult case $a_0 \in \mathbb{Q} \cap A$. Substituting (6.9) and the expressions for Δ_1, $\tilde{\tilde{\Delta}}_1$ into the second equation of (6.8), we have

$$\begin{aligned}&\eta_1\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right)\right) + \left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\left[\tilde{\phi}_{11}^{-1}(v, \tilde{y}_1) + \phi_1\left(v, \omega\left(\frac{v}{v_0}\right), \tilde{\phi}_{11}^{-1}(v, \tilde{y}_1)\right)\right] \\ &\quad = \bar{\eta}_1\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right)\right) \\ &\qquad + \sum_{j=0}^{k} \hat{m}_{11j}(v)\left[\bar{\eta}_1\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right)\right)\right. \\ &\qquad \left. + \left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\left(\tilde{y}_1 + \bar{\phi}_1\left(v, \omega\left(\frac{v}{v_0} - \alpha_1\right), \tilde{y}_1\right)\right)\right]^j \\ &\qquad + O(\tilde{y}_1^{k+1}).\end{aligned} \tag{6.10}$$

Equating the coefficient of monomial $\tilde{y}_1^j$ on both sides of (6.10), we get a series of equations about $\hat{m}_{11j}$ and $\tilde{m}_{11j}$. Then for $j = 2, 3, \ldots, k$, we have

$$\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\left[\tilde{m}_{11j}(v) + o\left(\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\right)\right] = \hat{m}_{11j}(v)\left(\frac{v}{v_0}\right)^{j\bar{\sigma}_1} + o\left(\left(\frac{v}{v_0}\right)^{(j-1)\bar{\sigma}_1}\right). \tag{6.11}$$

Then by (6.11), for $2 \leqslant j \leqslant k$, we have

$$\tilde{m}_{11j}(v) = \left(\frac{v}{v_0}\right)^{\bar{\sigma}_1} \hat{m}_{11j}(v) + o\left(\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\right).$$

Therefore we get

$$\tilde{m}_{11j}(0) = 0, \qquad j = 2, 3, \ldots, k. \tag{6.12}$$

Let

$$\tilde{\Phi}_{22}(v, \tilde{y}_2) = (v, \tilde{\phi}_{22}) \colon \tilde{\Sigma}_2 \to \Sigma_2$$

be the corresponding composition of coordinate change and a C^k regular transition map. If we denote

$$\tilde{\phi}_{22}^{-1}(v, \tilde{y}_2) = \tilde{y}_2 + \sum_{j=2}^{k} \tilde{m}_{22j}(v)\, \tilde{y}_2^j + O(\tilde{y}_2^{k+1}),$$

then similar to (6.12), we get

$$\tilde{m}_{22j}(0) = 0, \qquad j = 2, 3, ..., k. \tag{6.13}$$

Let $\tilde{\tilde{R}}$: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ be the transition map in the new normal form coordinates; then we have

$$\tilde{\tilde{R}} = \tilde{\Phi}_{22}^{-1} \circ R \circ \tilde{\Phi}_{11}^{-1}. \tag{6.14}$$

It follows from (6.12), (6.13), and (6.14) that one finds a constant $\bar{C} > 0$ such that $\tilde{\tilde{R}}_2^{(n)}(0, 0) = \bar{C} R_2^{(n)}(0, 0)$ by which we finish the proof. ∎

Remark 6.5. In the new normal form coordinates $(r_i, \rho_i, \tilde{\tilde{y}}_1)$ $(i = 1, 2)$, the second component of the transition map T is still almost affine.

6.2. *Finite Cyclicity of hp Graphics of Elliptic Type*

THEOREM 6.6. *A hp graphic with a triple nilpotent singularity for which* $a \neq \frac{1}{2}$ *has finite cyclicity.*

Proof. We consider the concave hp graphic. We will study the cyclicity of all the graphics listed in Table IV.

(1) Graphics Ehp1, Ehp2c, and Ehp3. As shown in Fig. 20, take sections τ_2 and Π_2 (Notation 5.6). We study the displacement map defined on τ_2,

$$\begin{aligned} L&: \tau_2 \to \Pi_2 \\ L &= \tilde{T} - \hat{T}, \end{aligned} \tag{6.15}$$

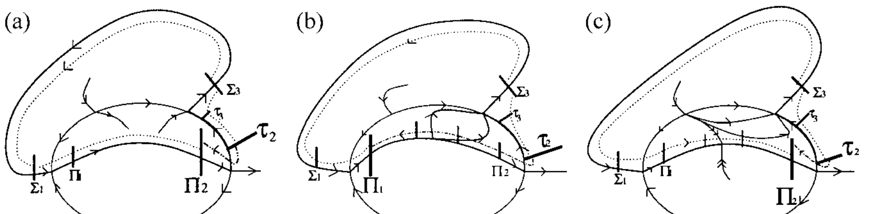


FIG. 20. Displacement maps for graphics (a) Ehp1, (b) Ehp2c, and (c) Ehp3.

where $\tilde{T}$ is the transition map along the graphic and $\hat{T} = \Theta_2$ is the second type of Dulac map near P_2.

On τ_2, the coordinates are (r_2, ρ_2) with $r_2\rho_2 = \nu$, $\nu > 0$ small and invariant. We want to cover a domain $|r_2| < \varepsilon$, $|\rho_2| < \varepsilon$, where $\varepsilon > 0$ small. Then $\nu \leqslant \varepsilon^2$. So $\forall u \in (0, 1)$, on the curve $\nu = u\varepsilon^2$, we have $r_2\rho_2 = u\varepsilon^2$. Therefore $r_2, \rho_2 \in (u\varepsilon, \varepsilon)$. Let

$$r_2 = \nu^{1-d}, \qquad \rho_2 = \nu^d. \tag{6.16}$$

We then parameterize the section τ_2 using the coordinates $(\nu, d) \in (0, \varepsilon^2) \times \mathscr{I}_\nu$, where $\mathscr{I}_\nu = (\frac{\ln \varepsilon}{\ln \nu}, \frac{\ln u\varepsilon}{\ln \nu}) \subset (0, 1)$ and $\nu = u\varepsilon^2$.

To prove the finite cyclicity of the graphics, we are going to prove that the two functions $\tilde{T}_2(\nu, d)$ and $\hat{T}_2(\nu, d)$ have different convexity, i.e., $\tilde{T}''_2(\nu, d) < 0$ and $\hat{T}''_2(\nu, d) > 0$, which will yield $Cycl(Ehp1, Ehp3) \leqslant 2$.

We calculate $\hat{T}''_2(\nu, d)$ first. Using coordinates (ν, d) on section τ_2, for $\hat{T} = \Theta_2 = (\xi_2, \Xi_2)$, by Theorem 4.14, we have

$$\Xi_2(\nu, d) = \eta_2\left(\nu, \omega\left(\frac{\nu^d}{\rho_0}, \alpha_1\right)\right) + \nu^{\bar{\sigma}_2 d}\left[l_1 + \theta_2\left(\nu, \nu^d, \omega\left(\frac{\nu^d}{\rho_0}, -\alpha_1\right)\right)\right] \tag{6.17}$$

where $l_1 = y_0/\rho_0^{\bar{\sigma}_1} > 0$, $\eta_2(\nu, \omega(\nu^d/\rho_0, \alpha_1)) = (\kappa_3/\rho_0^{p_1}\nu^{p_1})\, \omega(\nu^d/\rho_0, \alpha_1)$, and $\theta_2(\nu, \nu^d, \omega(\nu^d/\rho_0, -\alpha_1))$ is C^∞. Also $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$, for $d \in (0, 1)$ and $\nu > 0$ sufficiently small, we have uniformly

$$\frac{\partial^i \theta_2}{\partial d^i}\left(\nu, \nu^d, \omega\left(\frac{\nu^d}{\rho_0}, -\alpha_1\right)\right) = O\left(\nu^{p_1 d}\omega\left(\frac{\nu^d}{\rho_0}, \alpha_1\right)(\ln \nu)^i\right), \qquad i \geqslant 0. \tag{6.18}$$

We also have

$$\frac{\partial}{\partial d}\eta_2\left(\nu, \omega\left(\frac{\nu^d}{\rho_0}, \alpha_1\right)\right) = -\frac{\kappa_1}{\rho_0^{p_1-\alpha_1}}\, \nu^{p_1}\nu^{-\alpha_1 d} \ln \nu = -\frac{\kappa_1}{\rho_0^{\bar{\sigma}_1}}\, \nu^{\bar{\sigma}_1 d}\nu^{p_1(1-d)} \ln \nu. \tag{6.19}$$

Note that $\forall d \in (\frac{\ln \varepsilon}{\ln \nu}, \frac{\ln u\varepsilon}{\ln \nu})$, $\nu^{1-d} \in (u\varepsilon, \varepsilon)$ and $\nu^{p_1(1-d)} \in (0, \varepsilon^{p_1})$, so if we differentiate $\hat{T}_2(\nu, d)$ twice with respect to d, then we have

$$\hat{T}''_2(\nu, d) = \nu^{\bar{\sigma}_2 d}(\ln \nu)^2\left[\bar{\sigma}_2^2 l_1 + \alpha_1\frac{\kappa_1}{\rho_0^{\bar{\sigma}_2}}\, \nu^{p_1(1-d)} + \hat{\theta}_{21}\left(\nu, \nu^d, \omega\left(\frac{\nu^d}{\rho_0}, -\alpha_1\right)\right)\right], \tag{6.20}$$

yielding $\forall (\nu, d) \in (0, \varepsilon^2) \times \mathscr{I}_\nu$ with $\varepsilon > 0$ sufficiently small, $\hat{T}''_2(\nu, d) > 0$.

Now, for $\tilde{T}_2(\nu, d)$, we make the decomposition

$$\tilde{T} = S \circ \Delta_1 \circ R \circ \Theta_3 \circ V, \tag{6.21}$$

where

- Δ_1 is the first type of Dulac map near P_1. It satisfies Theorem 4.10 with $\sigma=\sigma_1(a)$,
- Θ_3 is the second type of Dulac map near P_3. It satisfies Theorem 4.14 with $\sigma=\sigma_3(a)$: Using coordinates (v, r_3) on the section τ_3 defined in normal form coordinates by $\{\tilde{y}_3=-y_0\}$, the second component of Θ_3 is

$$\Xi_3(v, r_3)=\eta_3\left(v, \omega\left(\frac{r_3}{r_0}, \beta_1\right)\right)+r_3^{\bar{\sigma}_3}\left[-l_3+\theta_3\left(v, r_3, \omega\left(\frac{r_3}{r_0}, -\beta_1\right)\right)\right], \tag{6.22}$$

where $l_3=y_0/r_0^{\bar{\sigma}_3}>0$ and $\theta_3(v, r_3, \omega(r_3/r_0, -\beta_1))$ satisfies a similar property as θ_2 in (6.18).

- $S\colon \Pi_1\to\Pi_2$ is the transition map defined in Proposition 6.1 with S_2 in (6.3),
- $R\colon \Sigma_3\to\Sigma_1$ is a C^k regular transition map

$$R_2(v, \tilde{y}_3)=m_{310}(v)+m_{311}(v)\,\tilde{y}_3+O(\tilde{y}_3^2), \tag{6.23}$$

where $m_{310}(0)=0$ and $m_{311}(0)>0$,

- $V\colon \tau_2\longrightarrow\tau_3$ is a C^k regular transition map which can be written as

$$\begin{aligned} V_1(r_2, \rho_2)&=r_2[m_{231}+O(|(r_2, \rho_2)|)]\\ V_2(r_2, \rho_2)&=\rho_2[\hat{m}_{231}+O(|(r_2, \rho_2)|)], \end{aligned} \tag{6.24}$$

where $m_{231}(0), \hat{m}_{231}(0)>0$ are constants.

Let

$$r_3=v^{1-d}[m_{231}+O(|(v^d, v^{1-d})|)]. \tag{6.25}$$

Then for the transition map $\tilde{T}$, by (6.21) and using coordinates (v, d) on the section τ_2, a straightforward calculation gives

$$\begin{aligned} \tilde{T}_2(v, d)=&\ \delta_{00}(v)+\delta_{01}(v)\,v^{\bar{\sigma}_1+p_3}\omega(r_3, -\beta_1)(1+O(v^{p_3}\omega(r_3, -\beta_1)))\\ &+\delta_{11}(v)\,v^{\bar{\sigma}_1}r_3^{\bar{\sigma}_3}[1+\theta_{31}(v, r_3, \omega(r_3, -\beta_1))], \end{aligned} \tag{6.26}$$

where

$$\begin{aligned}\delta_{00}(v) &= m_0(0)+m_{310}(v)\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}+O\left(\left(\frac{v}{v_0}\right)^{\bar{\sigma}_1}\omega\left(\frac{v}{v_0}, -\alpha_1\right)\right)\\ \delta_{01}(v) &= m_1(v)\, m_{311}(v) \qquad (6.27)\\ \delta_{11}(v) &= -\frac{l_3 m_1(v)\, m_{311}(v)}{v_0^{\bar{\sigma}_1}}<0,\end{aligned}$$

$m_1(v)$ is large for Ehp2c and small for Ehp3.

Note that if $q_3=1$, $p_3-\beta_1=\bar{\sigma}_3$ and $v^{1-d}v^{p_3}v^{-(1-d)(1+\beta_1)}=v^{p_3 d}v^{\bar{\sigma}_3(1-d)}$. A first derivative of $\tilde{T}_2(v,d)$ gives

$$\begin{aligned}\tilde{T}_2'(v,d) &= -v^{\bar{\sigma}_3(1-d)}\delta_{11}(v)\, v^{\bar{\sigma}_1}\ln v(1+O(v^d, v^{1-d}))\\ &\left[\bar{\sigma}_3 m_{231}^{\bar{\sigma}_3}+O(v^{p_3 d})+\theta_{33}\left(v, v^d, \omega\left(m_{231}\frac{v^{1-d}}{r_0}, -\beta_1\right)\right)\right]. \qquad (6.28)\end{aligned}$$

where θ_{33} has the same property as θ_{31}.

Therefore for $\tilde{T}_2''(v,d)$, we have

$$\begin{aligned}\tilde{T}_2''(v,d) &= v^{\bar{\sigma}_1}v^{\bar{\sigma}_3(1-d)}\delta_{11}(v)(\ln v)^2\,(1+O(v^d, v^{1-d}))\\ &\left[\bar{\sigma}_3 m_{231}^{\bar{\sigma}_3}+O(v^{p_3 d})+\theta_{34}\left(v, v^d, \omega\left(m_{231}\frac{v^{1-d}}{r_0}, -\beta_1\right)\right)\right] \qquad (6.29)\end{aligned}$$

where $\delta_{11}(v)<0$. So $\forall (v,d)\in(0,\varepsilon^2)\times\mathcal{I}_v$ with $\varepsilon>0$ sufficiently small, $\tilde{T}_2''(v,d)<0$.

Remark 6.7. For the hp graphics Ehp1, Ehp3, and Ehp2c considered above, we studied the displacement maps defined on the section τ_2 which is transverse to the passage from P_2 to P_3 along the equator. Since $v=r_2\rho_2$ is invariant, on τ_2 we have $\rho_2=v/r_2$. So it is the passage from P_2 to P_3 along the equator which forces the two functions $\tilde{T}_2$ and $\hat{T}_2$ to have different convexity. Similar phenomenon happens on the passage from P_1 to P_4. Therefore, if a graphic contains exactly one of these two passages and has a structure similar to that of Ehp1, then it has finite cyclicity 2.

(2) Cyclicity of graphic Ehp2a. As shown in Fig. 21a, Ehp2a is a hp graphic through a repelling saddle node. The composition of T_{31} with the two Dulac maps of type one is similar to the passage near an attracting saddle node. Hence the proof is exactly the same as for a graphic through two saddle nodes, one with central transition and one with center-unstable transition and uses the Khovanskii procedure [15, 38].

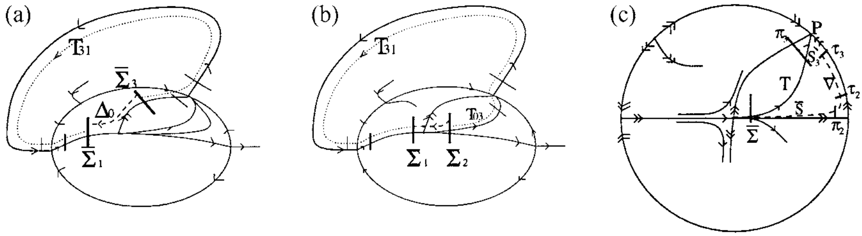


FIG. 21. Displacement maps for graphics (a) Ehp2a, (b) Ehp2b, and (c) map T.

(3) Cyclicity of graphic Ehp2b. As in Fig. 21b, let $\bar{\Sigma}_1 = \{\tilde{x} = -x_0\}$ and $\bar{\Sigma}_2 = \{\tilde{x} = x_0\}$ be two sections transversal to the graphic Ehp2b, and consider the displacement map

$$\begin{aligned} L&: \bar{\Sigma}_2 \to \bar{\Sigma}_1 \\ L &:= \bar{\Delta}_0 - \bar{T}_{31}, \end{aligned} \tag{6.30}$$

where $\bar{\Delta}_0(v, \tilde{x}) = (\bar{d}_0, \bar{D}_0)$: $\bar{\Sigma}_2 \longrightarrow \bar{\Sigma}_1$ is the central transition near the saddle node in the normal form coordinates $(\tilde{x}, \tilde{y})$, and

$$\bar{D}_0(v, \tilde{y}) = m(\hat{\mu}_2)\, \tilde{y}, \qquad \lim_{\hat{\mu}_2 \to 0} m(\hat{\mu}_2) = 0. \tag{6.31}$$

$\bar{T}_{31}$ is the transition along the flow of the graphic which can be factorized as

$$\begin{aligned} \bar{T}_{31}&: \bar{\Sigma}_2 \to \bar{\Sigma}_1 \\ &(v, \tilde{x}) \to (v, \bar{T}_{312}) \\ \bar{T}_{31} &= \bar{T}_{10} \circ \Delta_1 \circ R \circ \Delta_3 \circ \bar{T}_{03}, \end{aligned} \tag{6.32}$$

where especially $\bar{T}_{03}$: $\bar{\Sigma}_2 \to \Pi_3$ is a regular transition map in normal form coordinates which can be written as

$$\begin{aligned} \bar{T}_{031}(v, \tilde{y}) &= \frac{v}{\rho_0} \\ \bar{T}_{032}(v, \tilde{y}) &= m_{030}(v) + \sum_{i=1}^{n} m_{03i}(v)\, \tilde{y}^i + O(\tilde{y}^{n+1}), \end{aligned}$$

where $m_{030}(0) = 0$.

LEMMA 6.8. *We consider the vector field*

$$\begin{aligned} \dot{x} &= y + ax^2 \\ \dot{y} &= y(x+1) \end{aligned} \tag{6.33}$$

with a saddle node at the origin and a singular point P at infinity given by $(u, z) = (\frac{1-2a}{2}, 0)$, where $(u, z) = (y/x^2, \frac{1}{x})$ (Fig. 21c). Let $(\tilde{x}, \tilde{y})$ be normal coordinates near the origin and $(\tilde{u}, \tilde{z})$ be normal coordinates near P. Then the transition map

$$T: \{\tilde{x} = x_0\} \to \{\tilde{z} = z_0\}$$

is nonlinear at any point $\tilde{y}_0$ of $\{\tilde{x} = x_0\}$; i.e., $\forall \tilde{y}_0$, there exists $n \geqslant 2$ such that

$$\frac{d^n T}{d\tilde{y}^n}(\tilde{y}_0) \neq 0. \tag{6.34}$$

Proof. The proof is very similar to that in Proposition 5.11. The argument lies essentially in the fact that it is possible to choose normalizing coordinates near the saddle node so that the intersection of the section $\bar{\Sigma}_2$ with $v = 0$ is an analytic section in the original coordinates. This highly nontrivial fact was explained to us by Y. Ilyashenko. The proof will appear in [8]. The Appendix contains a statement of the results. Also as in Proposition 5.11 we can suppose that the section Π_3 is analytic. Then it suffices to prove that the transition map $T: \bar{\Sigma}_0 \to \pi_3$ is nonlinear at one point, where $\bar{\Sigma}_0 = \{\tilde{x} = x_0\}$. This will be done by considering the asymptotic expansion of T near $\tilde{y} = 0$ on the lower boundary graphic (Fig. 21c). Then

$$T = S_3 \circ \bar{V} \circ \bar{\Theta}_2^{-1} \circ \bar{S}, \tag{6.35}$$

where $\bar{S}$ and $\bar{V}$ are regular. Since we are in the invariant subspace $r = 0$, then by Remark 4.16 we have

$$\bar{\Theta}_2^{-1}(\tilde{y}_2) = \begin{cases} \tilde{y}_2^{\frac{1}{\sigma_1}} & \text{if } \ \sigma_1(a_0) \notin \mathbb{Q} \\ \tilde{y}_2^{\frac{1}{\sigma_1}} \left[1 + \sum_{i=1}^{k} \alpha_i \tilde{y}_2^{ip} \omega(\tilde{y}_2, \alpha_1) + o(\tilde{y}_2^{kp}) \right] & \text{if } \ \dfrac{1}{\sigma_1(a_0)} = \dfrac{p}{q} \in \mathbb{Q}, \end{cases} \tag{6.36}$$

where $\alpha_1 = p/q - 1/\sigma_1$. A direct calculation yields that

$$S_3(\rho_3) = \frac{1}{\rho_3^{\sigma_3}} (-C_1 + C_2 \kappa_3 \ln \rho_3), \tag{6.37}$$

where C_1, C_2 are positive constants and $\kappa_3 = 0$ if $\sigma_3(a_0) \notin \mathbb{N}$.

By (6.35)–(6.37), it is then clear that T sends a neighborhood of $\tilde{y} = 0$ to a neighborhood of ∞. No affine maps can have this property. Hence T is nonlinear at each point in a neighborhood of $\tilde{y} = 0$. By analytic extension, it is nonlinear at every point of $\bar{\Sigma}_0$. ∎

End of proof of Theorem 6.6. By Lemma 6.8, there exists $n \geqslant 2$ such that $\partial^n T_{032}/\partial \tilde{y}^n(0,0) = \tilde{m}_{03n} \neq 0$. Then for the transition map $\bar{T}_{31}$, we have

$$\begin{aligned}\bar{T}_{312}(v, \tilde{y}) &= \gamma_{310}\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), \omega\left(\frac{v}{v_0}, -\beta_1\right)\right) \\ &\quad + \left(\frac{v}{v_0}\right)^{\bar{\sigma}_1+\bar{\sigma}_3}\left[\sum_{i=1}^{n} \gamma_{31i}\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), \omega\left(\frac{v}{v_0}, -\beta_1\right)\right)\tilde{y}^i + O(\tilde{y}^{n+1})\right].\end{aligned} \tag{6.38}$$

where for $v > 0$ sufficiently small, $\gamma_{31n}(0) = * \, \tilde{m}_{03n} \neq 0$.

Now consider the displacement map $L := \bar{T}_{31} - \Delta_0$. Obviously $L_1(v, \tilde{y}) = 0$; for L_2, it follows from (6.30), (6.31), and (6.32) that we have

$$\begin{aligned}L_2(v, \tilde{y}) &= -m(\hat{\mu}_2)\, \tilde{y} + \gamma_{310}\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), \omega\left(\frac{v}{v_0}, -\beta_1\right)\right) \\ &\quad + \left(\frac{v}{v_0}\right)^{\bar{\sigma}_1+\bar{\sigma}_3}\left[\sum_{i=1}^{n} \gamma_{31i}\left(v, \omega\left(\frac{v}{v_0}, -\alpha_1\right), \omega\left(\frac{v}{v_0}, -\beta_1\right)\right)\tilde{y}^i + O(\tilde{y}^{n+1})\right].\end{aligned}$$

Derivating L_2 with respect to $\tilde{y}$ n times, we have

$$\tilde{L}_n(v, \tilde{y}) = \left(\frac{v}{v_0}\right)^{-(\bar{\sigma}_1+\bar{\sigma}_3)} L_2^{(n)}(v, \tilde{y}) = * \, \tilde{m}_{03n}(0) + O(v) + O(\tilde{y}) \neq 0. \tag{6.39}$$

So $L = 0$ has at most n small roots; i.e., *Cycl*(*Ehp*2*b*) $\leqslant n$.

(4) Cyclicity of Ehp4, Ehp5, Ehp6, and Ehp7. Note that passing from Π_3 to Π_1 is like passing an attracting saddle node. So for Ehp4, the proof is the same as that of the finite cyclicity of a graphic with a saddle and a saddle node [38].

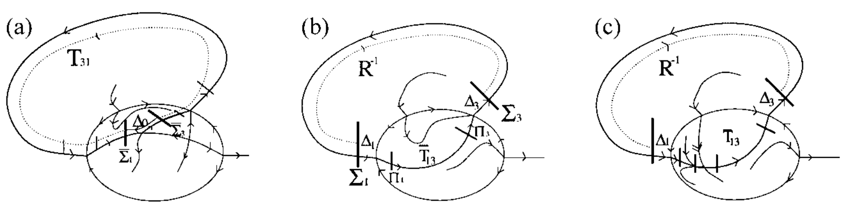


FIG. 22. Displacement maps for graphics (a) Ehp4, (b) Ehp6, and (c) Ehp7.

For the limit periodic sets Ehp5, Ehp6, and Ehp7, as shown in Fig. 22b and 22c, since the return map can be written as a composition of regular transition maps and maps with derivatives sufficiently small, we get $Cycl(Ehp5, Ehp6, Ehp7) \leqslant 1$. ∎

6.3. *Finite Cyclicity of hh Graphics of Elliptic Type*

In this section, we study the 12 families of hh graphics listed in Table V. We state the main result in Section 6.3.1 and give a generalized Rolle's theorem in Section 6.3.2. The main theorem is proved in Sections 6.3.3 and 6.3.4.

6.3.1. *Main theorem on the hh graphics of elliptic type.*

THEOREM 6.9. *An hh graphic through a triple nilpotent elliptic point of codimension 3 has finite cyclicity if the generic hypothesis $P'(0) \neq 1$ is satisfied.*

For the proof, by changing the family X to $-X$ if necessary, we impose

Hypothesis 6.10. The hh graphic with a nilpotent elliptic point is attracting:

$$[H]: \quad P'(0) = \gamma^* < 1. \tag{6.40}$$

In Table V, there are 12 families of hh graphics of elliptic type: Ehh$i$ ($i = 1, ..., 12$).

By Proposition 5.7, all the upper boundary graphics in the 12 families have finite cyclicity 1. We are going to prove that all the lower boundary and intermediate graphics have finite cyclicity. The proof is long and split into several subsections.

6.3.2. *Generalized Rolle's theorem and a transition map.* We will have to study the number of intersection points of two planar curves, hence the following generalization of Rolle's theorem (in the spirit of Khovanskii's method) is useful.

THEOREM 6.11 (Generalized Rolle's theorem). *Let $\mathscr{D} = (x_1, x_2) \times (y_1, y_2)$. Let $F(x, y)$, $G(x, y)$ be two functions continuous on $\bar{\mathscr{D}}$ and smooth in $\mathscr{D}$. Assume that in $\mathscr{D}$, $F'_x(x, y)$, $F'_y(x, y) \neq 0$. Denote the number of intersections of $F(x, y) = 0$ and $G(x, y) = 0$ in the region $\mathscr{D}$ by $\#(F, G)$ and let*

$$J[F, G](x, y) = F'_y(x, y)\, G'_x(x, y) - F'_x(x, y)\, G'_y(x, y).$$

Then

$$\#(F, G) \leqslant 1 + \#(F, J[F, G]).$$

Proof. First note that if $\forall (x, y) \in \mathcal{D}$, $F(x, y) \neq 0$, then $\#(F, G) = 0$, and the conclusion holds.

Assume that there exists a point $(x_0, y_0) \in \mathcal{D}$ such that $F(x_0, y_0) = 0$. Since $F(x, y)$ is smooth and $F_y(x, y) \neq 0$, by the implicit function theorem, there exists $\hat{\varepsilon}_0 > 0$ such that $F(x, y) = 0$ defines a unique smooth curve: $y = f(x)$, in $(x_0 - \hat{\varepsilon}_0, x_0 + \hat{\varepsilon}_0)$. As $F'_y(x, y) \neq 0$, the function $y = f(x)$ can be extended both ways to the boundaries of the region. Let $[x_3, x_4]$ be the maximum interval in which $y = f(x)$ is defined. Then $x_1 \leqslant x_3 \leqslant x_4 \leqslant x_2$.

The curve $y = f(x), x \in [x_3, x_4]$ is the unique branch defined by $F(x, y) = 0$ in the region $\mathcal{D}$. Indeed, if $x_4 < x_2$, since $F'_x(x, y), F'_y(x, y) \neq 0$, so either $F'_x(x, y)\, F'_y(x, y) > 0$ or $F'_x(x, y)\, F'_y(x, y) < 0$. In the first case, then for $x \in [x_3, x_4]$, $f'(x) = -F'_x(x, f(x))/F'_y(x, f(x)) < 0$, yielding $f(x_4) = y_1$. Therefore, $\forall (x, y) \in (x_4, x_2] \times [y_1, y_2]$ there holds

$$\begin{aligned} F(x, y) &= F(x, y) - F(x_4, y_1) \\ &= [F(x, y) - F(x, y_1)] + [F(x, y_1) - F(x_4, y_1)] \\ &= F'_y(x, \bar{y})(y - y_1) + F'_x(\bar{x}, y_1)(x - x_4) \neq 0, \end{aligned}$$

where $\bar{x}$ and $\bar{y}$ are between x, x_4 and y, y_1, respectively. The case $F'_x(x, y)\, F'_y(x, y) < 0$ is similar. So $\forall (x, y) \in ([x_1, x_3) \cup (x_4, x_2]) \times [y_1, y_2]$, $F(x, y) \neq 0$.

Let $g(x) = G(x, f(x))$. Then we turn to study the number of roots of $g(x) = 0$ for $x \in [x_3, x_4]$. Since

$$g'(x) = \left.\frac{J[F(x, y), G(x, y)]}{F'_y(x, y)}\right|_{y = f(x)},$$

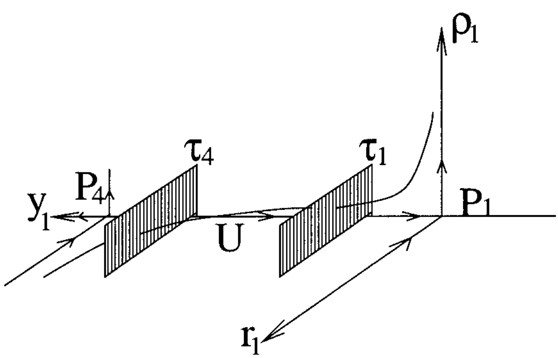


FIG. 23. The transition map $U\colon \tau_1 \to \tau_4$.

by Rolle's theorem,

$$\begin{aligned}\#(F, G) &\leqslant 1+\#(g'(x), 0)\\ &= 1+\#(J[F, G](x, y), F(x, y))\\ &= 1+\#(F, J[F, G]). \quad \blacksquare\end{aligned}$$

We will use Theorem 6.11 for a pair of functions F, G in a region depending on ν.

To study the cyclicity of the family Ehh1, we use Proposition 6.1, and we also need the transition map U in Fig. 23 to be nonlinear.

PROPOSITION 6.12. *Let $U=(U_1, U_2)\colon \tau_1 \to \tau_4$ be the transition map along $r_1=\rho_1=0$ in the normal form coordinates (see Fig. 23). If $a \neq \frac{1}{3}, \frac{1}{4}$, then*

$$\begin{aligned}U_1(r_1, \rho_1) &= r_1[m_{141}+m_{142}r_1+m_{143}\rho_1+O(|(r_1, \rho_1)|^2)]\\ U_2(r_1, \rho_1) &= \rho_1[\hat{m}_{141}+\hat{m}_{142}r_1+\hat{m}_{143}\rho_1+O(|(r_1, \rho_1)|^2)].\end{aligned} \tag{6.41}$$

Also $\forall (a, \bar{\mu}) \in A \times \mathbb{S}^2$,

$$\begin{aligned}\frac{\partial^2 U_1}{\partial r_1^2}(0, 0) &= 2m_{142} = * \, \varepsilon_2\\ \frac{\partial^2 U_2}{\partial \rho_1^2}(0, 0) &= 2\hat{m}_{143} = * \, \bar{\mu}_3.\end{aligned} \tag{6.42}$$

Furthermore, if $m_{142} \neq 0$, then $\hat{m}_{142} \neq 0$; if $\hat{m}_{143} \neq 0$, then $m_{143} \neq 0$.

Proof. The map U is a regular transition map along the invariant line $\{r_1=0\} \cap \{\rho_1=0\}$. Since $r_1=0$ and $\rho_1=0$ are invariant, we can write $U=(U_1, U_2)$ in the form of (6.41) and calculate the derivatives $\partial^i U_1/\partial r_1^i(0, 0)$, in the plane $\rho_1=0$ (resp. $\partial^i U_2/\partial \rho_1^i(0, 0)$, in the plane $r_1=0$).

We begin with the derivatives with respect to r_1. In the plane $\rho_1=0$, the system (4.6) becomes

$$\frac{dr_1}{d\bar{y}_1} = \frac{-(a+\bar{y}_1)\, r_1}{-(1-2a)\, \bar{y}_1+2\bar{y}_1^2-\bar{y}_1 r_1(\varepsilon_2+r_1\bar{h}_1)+r_1\bar{h}_2} = \frac{\bar{P}_1(r_1, \bar{y}_1)}{\bar{Q}_1(r_1, \bar{y}_1)}, \tag{6.43}$$

where $\bar{h}_1$ and $\bar{h}_2$ are C^∞ functions and $\bar{h}_2=\varepsilon_2 a+O(r_1)$.

We are going to do the calculations using the system (4.6) in the original coordinates $(r_1, \rho_1, \bar{y}_1)$.

In the neighborhood of P_1, the system has the form (4.6). By the normal form change (4.7), the system (4.6) is in the normal form (4.9) or (4.8). In the plane $\rho_1=0$, if $a \neq \frac{1}{4}, \frac{1}{3}$, the section $\tau_1=\{\tilde{y}_1=y_0\}$ becomes

$$\tau_1'\colon \quad \bar{y}_1 := g_{11}(r_1) = d_{10}(y_0)+d_{11}(y_0)\, r_1+O(r_1^2), \tag{6.44}$$

where

$$d_{10}(y_0) = y_0 + O(y_0^2), \qquad d_{11}(y_0) = \varepsilon_2 \left[\frac{a}{1-3a} + O(y_0) \right].$$

Similarly, in the coordinates $(r_1, \rho_1, \bar{y}_1)$, the section $\tau_4 = \{\tilde{y}_4 = -y_0\}$ becomes

$$\tau_4': \quad \bar{y}_1 := g_{14}(r_1) = d_{40}(y_0) + d_{41}(y_0)\, r_1 + O(r_1^2), \tag{6.45}$$

where

$$d_{40}(y_0) = \frac{1-2a}{2} - y_0 + O(y_0^2), \qquad d_{41}(y_0) = \varepsilon_2 \left[\frac{8a(1+4a)}{3-4a} + O(y_0) \right].$$

Then by Proposition 5.2, we have

$$\frac{\partial U_1}{\partial r_1}(0,0) = \exp\left(\int_{g_{11}(0)}^{g_{14}(0)} \frac{a+\bar{y}_1}{\bar{y}_1(1-2a-2\bar{y}_1)}\, d\bar{y}_1 \right) = \frac{\left(\frac{1-2a}{2}\right)^{\frac{1+2a}{2(1-2a)}}}{y_0^{\frac{1+2a}{2(1-2a)}}}(1+O(y_0)). \tag{6.46}$$

Now we calculate $\partial^2 U_1/\partial r_1^2(0,0)$. Since $\bar{P}_1(0, \bar{y}_1) = 0$, by Proposition 5.2 we have

$$\frac{\partial^2 U_1}{\partial r_1^2}(0,0) = \frac{\partial U_1}{\partial r_1}(0,0)[PI_1 + PI_2 + PI_3], \tag{6.47}$$

where

$$\begin{aligned} PI_1 &= 2g_{14}'(0)\frac{\partial U_1}{\partial r_1}(0,0)\left(\frac{\bar{P}_{1r_1}}{\bar{Q}_1}\right)(0, g_{14}(0)) \\ &= \frac{4\varepsilon_2 a(1+4a)}{3-4a}\left(\frac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}} \frac{1+O(y_0)}{y_0^{\frac{3-2a}{2(1-2a)}}} =* \frac{1+O(y_0)}{y_0^{\frac{3-2a}{2(1-2a)}}} \\ PI_2 &= -2g_{11}'(0)\left(\frac{\bar{P}_{1r_1}}{\bar{Q}_1}\right)(0, g_{11}(0)) \\ &= -\frac{\varepsilon_2 a^2(1+O(y_0))}{(1-3a)(1-2a)\, y_0} =* \, sign(1-3a)\, \varepsilon_2 \frac{1+O(y_0)}{y_0} \end{aligned} \tag{6.48}$$

and

$$PI_3 = \int_{g_{11}(0)}^{g_{14}(0)} \left[\frac{\bar{P}''_{1r_1}}{\bar{Q}_1}(0, \bar{y}_1) - 2\frac{\bar{P}'_{1r_1}\bar{Q}'_{1r_1}}{\bar{Q}_1^2}(0, \bar{y}_1) \right] \exp\left(\int_{g_{11}(0)}^{u} \frac{\bar{P}'_{1r_1}}{\bar{Q}_1}(0, u)\, du \right) d\bar{y}_1$$
$$= -\frac{* \, \varepsilon_2 (1 + O(y_0))}{y_0^{\frac{a}{1-2a}}} \int_{d_{10}(y_0)}^{d_{40}(y_0)} \frac{(a + \bar{y}_1)(\bar{y}_1 - a)}{\bar{y}_1^{\frac{2-5a}{1-2a}} \left(\frac{1-2a}{2} - \bar{y}_1 \right)^{\frac{5-8a}{2(1-2a)}}}\, d\bar{y}_1. \tag{6.49}$$

Since

$$\lim_{y_0 \to 0} \frac{\displaystyle\int_{d_{10}(y_0)}^{d_{40}(y_0)} \frac{(a + \bar{y}_1)(\bar{y}_1 - a)}{\bar{y}_1^{\frac{2-5a}{1-2a}} \left(\frac{1-2a}{2} - \bar{y}_1 \right)^{\frac{5-8a}{2(1-2a)}}}\, d\bar{y}_1}{\dfrac{1}{y_0^{\frac{3-4a}{2(1-2a)}}}} = -\frac{8a(1+4a)}{3-4a} \left(\frac{1-2a}{2} \right)^{\frac{3a-1}{1-2a}},$$

then

$$\int_{d_{10}(y_0)}^{d_{40}(y_0)} \frac{(a + \bar{y}_1)(\bar{y}_1 - a)}{\bar{y}_1^{\frac{2-5a}{1-2a}} \left(\frac{1-2a}{2} - \bar{y}_1 \right)^{\frac{5-8a}{2(1-2a)}}}\, d\bar{y}_1 = -* \frac{1 + O(y_0^{\frac{1+2a}{2(1-2a)}})}{y_0^{\frac{3-4a}{2(1-2a)}}}, \tag{6.50}$$

so, by (6.49) and (6.50), for PI_3, we have

$$PI_3 = * \, \varepsilon_2 \frac{1 + O(y_0^{\frac{1+2a}{2(1-2a)}})}{y_0^{\frac{3-2a}{2(1-2a)}}}. \tag{6.51}$$

Therefore, it follows from (6.47), (6.46), (6.48), and (6.51) that we have

$$\frac{\partial^2 U_1}{\partial r_1^2}(0, 0) = * \, \varepsilon_2 \frac{1 + O(y_0) + O\,(y_0^{\frac{1+2a}{2(1-2a)}})}{y_0^{\frac{2}{1-2a}}}. \tag{6.52}$$

So, if we take $y_0 > 0$ small, then by (6.52), $\partial^2 U_1 / \partial r_1^2(0, 0) = * \, \varepsilon_2$.

Now we prove that $\partial^2 U_2 / \partial \rho_1^2(0, 0) \neq 0$. In the plane $r_1 = 0$, the system (4.6) becomes

$$\frac{d\rho_1}{d\bar{y}_1} = \frac{(a + \bar{y}_1 + \bar{\mu}_2 \rho_1^2)\, \rho_1}{-(1-2a)\, \bar{y}_1 + 2\bar{y}_1^2 + \bar{\mu}_3 \rho_1 \bar{y}_1 + 2\bar{\mu}_2 \rho_2^2 \bar{y}_1 + \bar{\mu}_1 \rho_1^3} = \frac{\hat{P}_1(\rho_1, \bar{y}_1)}{\hat{Q}_1(\rho_1, \bar{y}_1)}. \tag{6.53}$$

We still use the system (4.6) in the original coordinates $(r_1, \rho_1, \bar{y}_1)$ to do the calculations.

In the plane $r_1 = 0$, if $a \neq \frac{1}{4}, \frac{1}{3}$, the section $\tau_1 = \{\tilde{y}_1 = y_0\}$ becomes

$$\hat{\tau}_1\colon \quad \bar{y}_1 := \hat{g}_{11}(r_1) = \hat{d}_{10}(y_0) + \hat{d}_{11}(y_0)\,\rho_1 + O(\rho_1^2), \tag{6.54}$$

where

$$\hat{d}_{10}(y_0) = y_0 + O(y_0^2), \qquad \hat{d}_{11}(y_0) = \frac{\bar{\mu}_3}{a} + O(y_0).$$

Similarly, in the coordinates $(r_1, \rho_1, \bar{y}_1)$, on $r_1 = 0$ the section $\tau_4 = \{\tilde{y}_4 = -y_0\}$ becomes

$$\hat{\tau}'_4\colon \quad \bar{y}_1 = \hat{g}_{14}(\rho_1) = \hat{d}_{40}(y_0) + \hat{d}_{41}(y_0)\,\rho_1 + O(\rho_1^2), \tag{6.55}$$

where

$$\hat{d}_{40}(y_0) = \frac{1-2a}{2} - y_0 + O(y_0^2), \qquad \hat{d}_{41}(y_0) = -\frac{1-2a}{1-4a}\,\mu_3 + O(y_0).$$

Then, for $\partial U_2/\partial\rho_1(0, 0)$, by Proposition 5.2, we have

$$\frac{\partial U_2}{\partial \rho_1}(0, 0) = \frac{y_0^{\frac{1+2a}{2(1-2a)}}(1 + O(y_0))}{\left(\dfrac{1-2a}{2}\right)^{\frac{1+2a}{2(1-2a)}}}. \tag{6.56}$$

Now we calculate $\partial^2 U_2/\partial\rho_1^2(0, 0)$. By Proposition 5.2, since $\hat{P}_1(0, \bar{y}_1) = 0$, we have

$$\frac{\partial^2 U_2}{\partial \rho_1^2}(0, 0) = \frac{\partial U_2}{\partial \rho_1}(0, 0)[\widehat{PI}_1 + \widehat{PI}_2 + \widehat{PI}_3], \tag{6.57}$$

where

$$\widehat{PI}_1 = 2\hat{g}'_{14}(0)\,\frac{\partial U_2}{\partial \rho_1}(0, 0)\left(\frac{\hat{P}_{1\rho_1}}{\hat{Q}_1}\right)(0, \hat{g}_{14}(0)) = \frac{\bar{\mu}_3\, y_0^{\frac{6a-1}{2(1-2a)}}(1 + O(y_0))}{(1-4a)\left(\dfrac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}}} \tag{6.58}$$

$$\widehat{PI}_2 = -2\hat{g}'_{11}(0)\left(\frac{\hat{P}_{1\rho_1}}{\hat{Q}_1}\right)(0, \hat{g}_{11}(0)) = \frac{2\bar{\mu}_3(1 + O(y_0))}{(1-2a)\, y_0}$$

and

$$\widehat{PI}_3 = \int_{\hat{g}_{11}(0)}^{\hat{g}_{14}(0)} \left[\frac{\hat{P}''_{1\rho_1}}{\hat{Q}_1}(0, \bar{y}_1) - 2\frac{\hat{P}'_{1\rho_1}\hat{Q}'_{1\rho_1}}{\hat{Q}_1^2}(0, \bar{y}_1)\right] \exp\left(\int_{\hat{g}_{11}(0)}^{u} \frac{\hat{P}'_{1\rho_1}}{\hat{Q}_1}(0, u)\, du\right) d\bar{y}_1$$
$$= -\frac{\bar{\mu}_3\, y_0^{\frac{a}{1-2a}}(1+O(y_0))}{2\left(\frac{1-2a}{2}\right)^{\frac{1}{2(1-2a)}}} \int_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)} \frac{a+\bar{y}_1}{\bar{y}_1^{\frac{2-3a}{1-2a}}\left(\frac{1-2a}{2}-\bar{y}_1\right)^{\frac{3-8a}{2(1-2a)}}}\, d\bar{y}_1. \tag{6.59}$$

Since

$$\lim_{y_0\to 0} \frac{\displaystyle\int_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)} \frac{a+\bar{y}_1}{\bar{y}_1^{\frac{2-3a}{1-2a}}\left(\frac{1-2a}{2}-\bar{y}_1\right)^{\frac{3-8a}{2(1-2a)}}}\, d\bar{y}_1}{\dfrac{1}{y_0^{\frac{1-a}{1-2a}}}} = \frac{a(1-2a)}{(1-a)\left(\frac{1-2a}{2}\right)^{\frac{3-8a}{2(1-2a)}}},$$

thus

$$\int_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)} \frac{a+\bar{y}_1}{\bar{y}_1^{\frac{2-3a}{1-2a}}\left(\frac{1-2a}{2}-\bar{y}_1\right)^{\frac{3-8a}{2(1-2a)}}}\, d\bar{y}_1 = \frac{2a\left[1+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]}{(1-a)\left(\frac{1-2a}{2}\right)^{\frac{1-4a}{2(1-2a)}} y_0^{\frac{1-a}{1-2a}}}. \tag{6.60}$$

By (6.59) and (6.60), we have

$$\widehat{PI}_3 = \frac{2a\bar{\mu}_3}{(1-2a)\, y_0}\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]. \tag{6.61}$$

It follows from (6.57), (6.58), and (6.61) that we have

$$\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0) = \frac{\partial U_2}{\partial \rho_1}(0,0)\left[\frac{\bar{\mu}_3\, y_0^{\frac{6a-1}{2(1-2a)}}(1+O(y_0))}{(1-4a)\left(\frac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}}} + \frac{2\bar{\mu}_3(1+O(y_0))}{(1-2a)\, y_0}\right.$$
$$\left. + \frac{2a\bar{\mu}_3\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]}{(1-2a)\, y_0}\right]$$
$$= \frac{(1+2a)\,\bar{\mu}_3}{(1-2a)\, y_0}\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]. \tag{6.62}$$

By the invariance of $v = r_1\rho_1 = r_2\rho_2$ and (6.41), we have

$$r_2\rho_2 = r_1\rho_1[m_{141} + m_{142}r_1 + m_{143}\rho_1 + O(|(r_1, \rho_1)|^2)]$$
$$[\hat{m}_{141} + \hat{m}_{142}r_1 + \hat{m}_{143}\rho_1 + O(|(r_1, \rho_1)|^2)]. \tag{6.63}$$

Equating the coefficients of terms of r_1 and ρ_1 respectively on both sides of (6.63), we have

$$\begin{aligned} m_{141}\hat{m}_{142} + \hat{m}_{141}m_{142} &= 0 \\ m_{141}\hat{m}_{143} + \hat{m}_{141}m_{143} &= 0. \end{aligned} \tag{6.64}$$

Since $m_{141}\hat{m}_{141} = 1$, if $m_{142} \neq 0$, then $\hat{m}_{142} \neq 0$; if $\hat{m}_{143} \neq 0$, then $m_{143} \neq 0$. ∎

COROLLARY 6.13. *For the transition map $V: \tau_2 \to \tau_3$ in the normal form coordinates, if $a \neq \frac{1}{3}, \frac{1}{4}$, then*

$$\begin{aligned} \frac{\partial V_1}{\partial r_2}(0,0) &= \frac{\partial U_1}{\partial r_1}(0,0) \\ \frac{\partial^2 V_1}{\partial r_2^2}(0,0) &= \frac{\partial^2 U_1}{\partial r_1^2}(0,0) \\ \frac{\partial V_2}{\partial \rho_2}(0,0) &= \frac{\partial U_2}{\partial \rho_1}(0,0) \\ \frac{\partial^2 V_2}{\partial \rho_2^2}(0,0) &= \frac{\partial^2 U_2}{\partial \rho_1^2}(0,0). \end{aligned} \tag{6.65}$$

Proof. Just note that in the plane $\rho_2 = 0$, the system is the same as system (6.43) except for the sign of the term $r_2\bar{h}_1$ which does not influence the first and second derivatives. ∎

6.3.3. *Lower boundary hh graphics of elliptic type.* Among the 12 lower boundary graphics, Ehh1c, Ehh2e, and Ehh3e pass through both passages P_2P_3 and P_1P_4 along the equator (Remark 6.7) and require a special treatment. Indeed in general an explicit formula does not exist for the inverse of the second type of Dulac map. We will replace the study of zeroes of the displacement map by the study of the zeroes of a system of two variables using generalized Rolle's theorem.

To prove the finite cyclicity of the graphic Ehh1, we give the following lemma.

LEMMA 6.14. *For the system in the neighborhood of P_3, if $\sigma_3 = \frac{1}{n}$, $n \in \mathbb{N}$, then the first saddle quantity is nonzero for the 2-dimensional system on $\rho = 0$ as soon as $\varepsilon_2 \neq 0$.*

For the system in the neighborhood of P_1, if $\sigma_1 = 1$ ($a_0 = \frac{1}{3}$) or $\sigma_1 = 2$ ($a_0 = \frac{1}{4}$), then for the 2-dimensional system on $r = 0$, the first saddle quantity is $\alpha_2 = * \, \bar{\mu}_{30}$.

Proof. The proof relies on the fact that the system in the neighborhood of P_3 is simple, allowing a control on the normalizing process up to the determination of the first saddle quantity. See Appendix A.2 for the calculations. ∎

THEOREM 6.15. *The generic graphic Ehh1c through a nilpotent elliptic point of codimension 3 has finite cyclicity. For the generic graphics Ehh2e and Ehh3e, Cycl(Ehh2e, Ehh3e) $\leqslant$ 2.*

Proof. As shown in Fig. 24, take transversal sections Π_i, $\Sigma_{i+2} (i = 1, 2)$, and $\tau_j (j = 1, 2, 3, 4)$ in the charts P.R. 1, 2, 3 and P.R.4 respectively (sections were defined in 5.6). Similar to what we have done in Theorem 6.6 for the graphic Ehp1 in (6.16), we take

$$r_1 = \nu^{1-c}, \qquad \rho_1 = \nu^c$$

on the section τ_1 and

$$r_2 = \nu^{1-d}, \qquad \rho_2 = \nu^d$$

on the section τ_2 with $(\nu, c) \in (0, \varepsilon^2) \times \mathscr{I}_\nu$ and $(\nu, d) \in (0, \varepsilon^2) \times \mathscr{I}_\nu$, where $\mathscr{I}_\nu = (\frac{\ln \varepsilon}{\ln \nu}, \frac{\ln u\varepsilon}{\ln \nu}) \subset (0, 1)$ and $u \in (0, 1)$, and $\nu = u\varepsilon^2$.

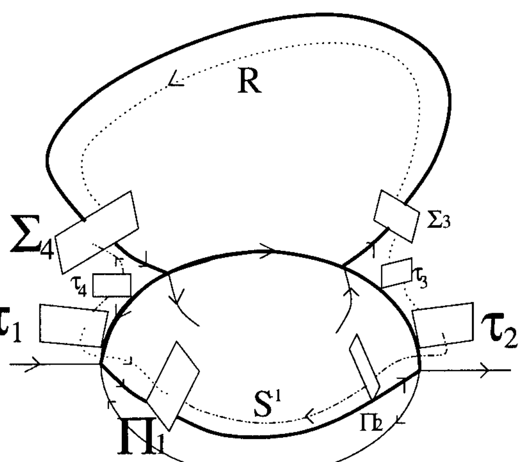


FIG. 24. The displacement maps defined on τ_1 and τ_2.

To study the cyclicity of the lower boundary graphics, we are going to study the displacement maps defined on the sections τ_1 and τ_2 respectively with images in Π_1 and Σ_4. Limit cycles are given by roots of the system

$$\begin{aligned} T_1(v, c) &= T_2(v, d) \\ T_4(v, c) &= T_3(v, d). \end{aligned} \tag{6.66}$$

To study their number, we use Theorem 6.11 in a region $(c, d) \in \mathscr{I}_v \times \mathscr{I}_v$ depending on v with $v > 0$ sufficiently small.

The proof will go in several steps.

(1) Developing the transition maps T_i ($i = 1, 2, 3, 4$).

(1.1) Transition map T_1. The map $T_1: \tau_1 \to \Pi_1$ is the second type of Dulac map near P_1. By Theorem 4.14, for $r = v^c$ and $\rho = v^{1-c}$, it has an expression similar to (6.17) with $\bar{\sigma}_3 = \bar{\sigma}_1$. Hence

$$T_{12}(v, c) = \eta_1\left(v, \omega\left(\frac{v^c}{\rho_0}, \alpha_1\right)\right) + v^{\bar{\sigma}_1 c}\left[l_1 + \theta_1\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \tag{6.67}$$

where $l_1 = y_0/\rho_0^{\bar{\sigma}_1} > 0$; $\eta_1(v, \omega(v^c/\rho_0, \alpha_1)) = (\kappa_1/\rho_0^{p_1})\, v^{p_1}\, \omega(v^c/\rho_0, \alpha_1)$ and $\theta_1(v, v^c, \omega(v^c/\rho_0, -\alpha_1))$ is C^∞ and satisfies the same properties as θ_2 in (6.18).

(1.2) Transition map T_4. The transition map $T_4: \tau_1 \to \Sigma_4$ can be factored as

$$T_4 = \Theta_4 \circ U,$$

where

- $U: \tau_1 \to \tau_4$ is the regular transition map defined in Proposition 6.12. In the coordinate c on τ_1, the first component U_1 of the map U can be written as

$$U_1(v^{1-c}, v^c) = v^{1-c}[m_{141} + m_{142}(v)\, v^{1-c} + m_{143}(v)\, v^c + O(v^{2(1-c)}, v^{2c})], \tag{6.68}$$

where by Proposition 6.12, $m_{141}(0) \neq 0$, $m_{142}(0) = *\ \varepsilon_2$ and $m_{143}(0) = *\ \bar{\mu}_{30}$.

- $\Theta_4: \tau_4 \to \Sigma_4$ is the second type of Dulac map near P_4 which satisfies Theorem 4.14 with $\bar{\sigma} = \bar{\sigma}_4$. By (6.68), we have

$$r_4 = v^{1-c}(m_{141} + O(v^c, v^{1-c})). \tag{6.69}$$

Let $m_4 = m_{141}/r_0$. Using (6.69) we have

$$\omega\left(\frac{r_4}{r_0}, \beta_1\right) = \omega\left(\frac{m_{14}}{r_0} v^{1-c}(1+O(v^c, v^{1-c})), \beta_1\right)$$
$$= \omega(m_4 v^{1-c}, \beta_1) + O(v^{c-(1-c)\beta_1}, v^{(1-c)(1-\beta_1)}). \qquad (6.70)$$

So by (6.68), (4.28), and (6.70) and by Lemma 4.13 for $\bar{\sigma}_4$, we have

$$T_{42}(v, c) = \eta_4(v, \omega(r_4, \beta_1))$$
$$+ v^{\bar{\sigma}_3(1-c)}[m_{41}(v) + m_{42} v^{1-c} + m_{43} v^c + O(v^{2(1-c)}, v^{2c})$$
$$+ \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, -\beta_1))], \qquad (6.71)$$

where $m_{41} = (y_0/r_0^{\bar{\sigma}_3})(m_{141}(v))^{\bar{\sigma}_3} = l_4 m_{141}(v)$, $m_{42}(0) = * \, m_{142}(0) = K_{142}\varepsilon_2$, and $m_{43}(0) = * \, m_{143}(0) = K_{143}\bar{\mu}_{30}$ and K_{142}, K_{143} are nonzero constants; in $\eta_4(v, \omega(r_4/r_0, \beta_1))$ we still keep r_4 as a function of c in (6.68); also $\theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, -\beta_1))$ is C^k and satisfies

$$\frac{\partial^i \theta_{41}}{\partial c^i} = O(v^{p_3(1-c)}\omega(m_4 v^{1-c}, \beta_1)(\ln v)^i) \qquad i \geqslant 0. \qquad (6.72)$$

(1.3) Transition map T_2. The transition map $T_2\colon \tau_2 \to \Pi_1$ can be factored as

$$T_2 = S^{-1} \circ \Theta_2(r_2, \rho_2),$$

where

- $\Theta_2\colon \tau_2 \to \Pi_2$ is the second type of Dulac map near P_2. Using the coordinates (v, d) on the section τ_2 is given in (6.17). $\eta_2(v, \omega(v^d/\rho_0, \alpha_1)) = (\kappa_1/\rho_0^{p_1}) v^{p_1}\omega(v^d/\rho_0, \alpha_1)$. Also $\theta_2(v, d, \omega(v^d/\rho_0, -\alpha_1))$ is C^∞ and satisfies the same properties as θ_1 in (6.18).
- $S^{-1}\colon \Pi_2 \to \Pi_1$ is a C^k regular transition map defined in Proposition 6.1. We can write its second component as

$$S_2^{-1}(v, \tilde{y}_2) = m_{210}(v) + m_{211}(v)\,\tilde{y}_2 + m_{212}(v)\,\tilde{y}^2 + O(\tilde{y}_2^3), \qquad (6.73)$$

where $m_{210}(0) = 0$ and $m_{211}(0) \neq 0$ and by Proposition 6.1, for Ehh1c, $m_{212}(0) = * \, \bar{\mu}_{30}$, while for Ehh2e (resp. Ehh3e) $m_{211}(0)$ is small (resp. large) .

By (6.73) and Lemma 4.13 for $\bar{\sigma}_2$ in Θ_2, for the transition map T_2, we have

$$T_{22}(v,d)=m_{20}\left(v,\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)$$
$$+v^{\bar{\sigma}_1 d}\left[m_{21}\left(v,\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)+m_{22}\left(v,\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)v^{\bar{\sigma}_1 d}\right.$$
$$\left.+\theta_{211}\left(v,v^d,\omega\left(\frac{v^d}{\rho_0},-\alpha_1\right)\right)\right] \tag{6.74}$$

where

$$m_{20}\left(v,\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)=m_{210}(v)+\kappa_1\left[\frac{m_{211}(v)}{\rho_0^{\bar{\sigma}_1}}v^{p_1}\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right.$$
$$\left.+O\left(v^{2p_1}\omega^2\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)\right]$$
$$m_{21}\left(v,\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)=m_{211}(v)\,l_1+\kappa_1 O\left(v^{p_1}\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)$$
$$m_{22}\left(v,\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right)=m_{212}(v)\,l_1^2+\kappa_1 O\left(v^{p_1}\omega\left(\frac{v^d}{\rho_0},\alpha_1\right)\right),$$

and θ_{211} is C^k and has the same properties as θ_1 in (6.18).

(1.4) The transition map T_3. Transition map T_3: $\tau_2\to\Sigma_4$ can be factored as

$$T_3=R\circ\Theta_3\circ V,$$

where

- V: $\tau_2\to\tau_3$ is the regular transition map defined in Corollary 6.13. Using the coordinates (v,d) on section τ_3, we have

$$V_1(v^{1-d},v^d)=v^{1-d}[\bar{m}_{141}+\bar{m}_{142}(v)\,v^{1-d}+\bar{m}_{143}(v)\,v^d+O(v^{2(1-d)},v^{2d})], \tag{6.75}$$

where by Corollary 6.13 we have $\bar{m}_{14i}(0)=m_{14i}(0)$ $(i=1,2,3)$;

- Θ_3: $\tau_3\to\sigma_3$ is the second type of Dulac map near P_3 which satisfies Theorem 4.14 with $\sigma_3(a_0)=2(1-2a_0)$;
- the regular transition map R: $\Sigma_3\to\Sigma_4$ is C^k and can be written as

$$R_2(v,\tilde{y}_3)=m_{340}(v)+\sum_{j=1}^{N_2}m_{34j}(v)\,\tilde{y}_3^j+O(\tilde{y}_3^{N_2+1}), \tag{6.76}$$

where by Hypothesis 6.10, we have $m_{341}(0)<1$.

So it follows from (6.75), (4.28), and (6.76) that the transition map T_3 satisfies

$$
\begin{aligned}
T_{32}(v, d) = m_{30}(v, \omega(r_3, \beta_1)) + v^{\bar{\sigma}_3(1-d)} \Big[& m_{31}(v, \omega(m_4 v^{1-d}, \beta_1)) \\
& + m_{32}(v)\, v^{1-d} + m_{33} v^d + O(v^{2(1-d)}, v^{2d}) \\
& + \sum_{j=1}^{N_1} \bar{m}_{34j} v^{\bar{\sigma}_3(1-d)\, j} + O(v^{(N_1+1)\, \bar{\sigma}_3(1-d)}) + \theta_{31}(v, v^{1-d}, \omega(m_4 v^{1-d}, \beta_1)) \Big],
\end{aligned}
\tag{6.77}
$$

where θ_{31} is C^k and satisfies the properties of θ_{41} in (6.72) and

$$
\begin{aligned}
m_{30} &= m_{340}(v) + \kappa_3 \left[\frac{m_{341}(v)}{r_0^{p_3}} v^{p_3} \omega\left(\frac{r_3}{r_0}, \beta_1\right) + O\left(v^{2p_3} \omega^2\left(\frac{r_3}{r_0}, \beta_1\right)\right)\right] \\
m_{31} &= \frac{\bar{m}_{141}^{\bar{\sigma}_3} m_{341}\, y_0}{r_0^{\bar{\sigma}_3}} + \kappa_3 O(v^{p_3} \omega(m_4 v^{1-d}, \beta_1)), \qquad m_{31}(0) \neq 0 \\
m_{32} &= \frac{\bar{\sigma}_3 \bar{m}_{142}(v)\, y_0}{r_0^{\bar{\sigma}_3} \bar{m}_{141}^{1-\bar{\sigma}_3}} + \kappa_3 O(v^{p_3} \omega(m_4 v^{1-d}, \beta_1)), \qquad m_{32}(0) = *\, K_{142} \varepsilon_2 \\
m_{33} &= \frac{\bar{\sigma}_3 \bar{m}_{143}(v)\, y_0}{r_0^{\bar{\sigma}_3} \bar{m}_{141}^{1-\bar{\sigma}_3}} + \kappa_3 O(v^{p_3} \omega(m_4 v^{1-d}, \beta_1)), \qquad m_{33}(0) = *\, K_{143} \bar{\mu}_{30} \\
\bar{m}_{341} &= \frac{\bar{m}_{141}^{2\bar{\sigma}_3} y_0^2 m_{342}}{r_0^{2\bar{\sigma}_3}} + \kappa_3 O(v^{p_3} \omega(m_4 v^{1-d}, \beta_1)), \qquad \bar{m}_{341}(0) = *\, m_{342}(0)
\end{aligned}
$$

and in $m_{30}(v, \omega(r_3/r_0, \beta_1))$ we still keep r_3 as the function of d in the expression (6.75).

To get the cyclicity of Ehh1c and Ehh3e, we are going to apply Theorem 6.11 to study $\#(F, G)$ of the following system

$$
\begin{aligned}
F_v(c, d) &:= T_{12}(v, c) - T_{22}(v, d) = 0 \\
G_v(c, d) &:= T_{42}(v, c) - T_{32}(v, d) = 0
\end{aligned}
\tag{6.78}
$$

for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $(c, d) \in \mathscr{D}_v$, where $\mathscr{D}_v$ is a square whose size depends on v. With $v = u\varepsilon^2$ and $u \in (0, 1)$, then $\mathscr{D}_v = \mathscr{I}_v \times \mathscr{I}_v$.

(2) Functions $F_v(c, d)$ and $G_v(c, d)$ satisfy the conditions of Theorem 5.3.2. For $0 < v < \varepsilon^2$, $F_v(c, d)$ and $G_v(c, d)$ are continuous on $\bar{\mathscr{D}}_v$ and smooth in $\mathscr{D}_v$. Note that $\forall c \in (\frac{\ln \varepsilon}{\ln v}, \frac{\ln u\varepsilon}{\ln v})$, we have $v^{1-c} \in (u\varepsilon, \varepsilon)$, hence $v^{p_1(1-c)} \in (0, \varepsilon^{p_1})$. So for $0 < v < \varepsilon^2$ and $\forall (c, d) \in \mathscr{D}_v$ with $\varepsilon > 0$ sufficiently small, by (6.19), a first derivation gives

$$
\begin{aligned}
\frac{\partial}{\partial c} F_v(c,d) &= -\frac{\kappa_1}{\rho_0^{\bar{\sigma}_1}} v^{\bar{\sigma}_1 c} v^{p_1(1-c)} \ln v + v^{\bar{\sigma}_1 c} \ln v \left[\bar{\sigma}_1 l_1 + \theta_1 \left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right) \right. \\
&\quad \left. + \frac{1}{\ln v} \frac{\partial}{\partial c} \theta_1 \left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \\
&= v^{\bar{\sigma}_1 c} \ln v \left[\bar{\sigma}_1 l_1 + l_{11} v^{p_1(1-c)} + \theta_{11} \left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, \alpha_1\right)\right)\right] \\
&\neq 0 \qquad (6.79)
\end{aligned}
$$

where $l_{11} = -\kappa_1/\rho_0^{\bar{\sigma}_1}$, $\theta_{11}(v, v^c, \omega(v^c/\rho_0, \alpha_1))$ is C^k and satisfies the properties (6.18). Since for $x > 0$ sufficiently small, $(x^{p_1}\omega(x, \alpha_1))' = x^{p_1 - 1}[\bar{\sigma}_1 \omega(x, \alpha_1) - 1] > 0$, for θ_{11} with $c \in \mathscr{I}_v$, we have the estimation $\theta_{11}(v, v^c, \omega(v^c/\rho_0, \alpha_1)) = O(\varepsilon^{p_1}\omega(\varepsilon, \alpha_1))$.

Similar for $0 < v < \varepsilon^2$ and $\forall (c, d) \in \mathscr{D}_v$ with $\varepsilon > 0$ sufficiently small, we have

$$
\begin{aligned}
&\frac{\partial}{\partial d} F_v(c,d) \\
&\quad = -v^{\bar{\sigma}_1 d} \ln v \left[\bar{\sigma}_1 l_2 + l_{21} v^{p_1(1-d)} + l_{22}(v)\, v^{\bar{\sigma}_1 d} + \theta_{23} \left(v, v^d, \omega\left(\frac{v^d}{\rho_0}, -\alpha_1\right)\right)\right] \\
&\quad \neq 0 \qquad (6.80)
\end{aligned}
$$

where $l_2(0) = l_1 m_{211}(0) \neq 0$, $l_{21} = -\kappa_1 m_{211}/\rho_0^{\sigma_1}$, and $l_{22} = 2\bar{\sigma}_1 l_1^2 m_{212}$. Also for Ehh1c, $l_{22}(0) = *\, m_{212}(0) = *\, \bar{\mu}_{30}$. For θ_{23}, it is C^k and satisfies the properties (6.18).

By (6.79) and (6.80), for $0 < v < \varepsilon^2$ and $\forall (c, d) \in \mathscr{D}_v$ with $\varepsilon > 0$ sufficiently small, $F_v(c, d)$ and $G_v(c, d)$ satisfy the conditions of Theorem 6.11. So we have

$$
\#(F, G) \leqslant 1 + \#(F, J[F, G]). \qquad (6.81)
$$

(3) Calculation of $\#(F, J[F, G])$. To calculate $\#(F, J[F, G])$, we have to calculate $J[F, G] = \frac{\partial F}{\partial c}\frac{\partial G}{\partial d} - \frac{\partial F}{\partial d}\frac{\partial G}{\partial c}$.

Note that for the case $q_3 = 1$, $\bar{\sigma}_3 + \beta_1 = p_3 = 1$, so

$$
v v^{-\beta_1(1-c)} = v^{1-\beta_1(1-c)} = v^{(1-\beta_1)(1-c)} v^{1-(1-c)} = v^c v^{\bar{\sigma}_3(1-c)}.
$$

Therefore, for $\eta_4(v, \omega(r_4/r_0, \beta_1))$, by (6.68) and similar to (6.19) we have

$$\begin{aligned}
&\frac{\partial \eta_4}{\partial c}\left(v, \omega\left(\frac{r_4}{r_0}, \beta_1\right)\right) \\
&\quad = \frac{\kappa_3}{r_0}\, v\, \frac{\partial}{\partial c}\, \omega\left(\frac{r_4}{r_0}, \beta_1\right) = \frac{\kappa_3}{r_0}\, v \left[-\left(\frac{r_4}{r_0}\right)^{-1-\beta_1}\right] \frac{\partial r_4}{\partial c} \\
&\quad = \frac{\kappa_3}{m_4^{\bar{\sigma}_1}}\, v^{\bar{\sigma}_3(1-c)} v^c \ln v [1 + \hat{\bar{m}}_{142}(v)\, v^{1-c} + \hat{\bar{m}}_{143}(v)\, v^c + O(v^{2(1-c)}, v^{2c})].
\end{aligned} \tag{6.82}$$

So by (6.71), (6.77), and (6.82) and direct derivation, we have

$$\begin{aligned}
&\frac{\partial}{\partial c} G_v(c, d) \\
&\quad = v^{\bar{\sigma}_3(1-c)} \ln v [\bar{\sigma}_3 l_4 + \bar{l}_{411} v^{p_3 c} + \bar{l}_{422}(v)\, v^{1-c} + \bar{l}_{423}(v)\, v^c \\
&\qquad + O(v^{2(1-c)}, v^{2c}) + \theta_{42}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_1))] \\
&\frac{\partial}{\partial d} G_v(c, d) \\
&\quad = -v^{\bar{\sigma}_3(1-d)} \ln v \Bigg[\bar{\sigma}_3 l_3 + \bar{l}_{311} v^{p_3 d} + \bar{l}_{322}(v)\, v^{1-d} + \bar{l}_{323}(v)\, v^d + O(v^{2(1-d)}, v^{2d}) \\
&\qquad + \sum_{j=1}^{N_1} \bar{l}_{33j} v^{\bar{\sigma}_3(1-d)\, j} + O(v^{(N_1+1)\,\bar{\sigma}_3(1-d)}) + \theta_{32}(v, v^{1-d}, \omega(m_4 v^{1-d}, -\beta_1))\Bigg]
\end{aligned} \tag{6.83}$$

where $\theta_{42}(v, v^{1-c}, \omega(m_4 v^{1-c}, -\beta_1))$ and $\theta_{32}(v, v^{1-d}, \omega(m_4 v^{1-d}, -\beta_1))$ are C^k and satisfy the properties (6.72); $l_4 = (y_0/r_0^{\bar{\sigma}_1})\, m_{141}^{\bar{\sigma}_3}$, $l_3 = m_{341} l_4$, and $l_3(0)\, l_4(0) \neq 0$. Also

$$\begin{aligned}
\bar{l}_{311}(0) &= *\, \kappa_3 \\
\bar{l}_{411}(0) &= *\, \kappa_3 \\
\bar{l}_{331}(0) &= *\, m_{342}(0) \\
\bar{l}_{422}(0) &= *\, m_{142}(0) = *\, K_{142} \varepsilon_2 \\
\bar{l}_{423}(0) &= *\, m_{143}(0) = *\, K_{143} \bar{\mu}_{30} \\
\bar{l}_{322}(0) &= *\, m_{341}(0)\, \bar{l}_{422}(0) = *\, m_{341} K_{142} \varepsilon_2 \\
\bar{l}_{323}(0) &= *\, m_{341}(0)\, \bar{l}_{423}(0) = *\, m_{341} K_{143} \bar{\mu}_{30}.
\end{aligned}$$

Let

$$G_1(v,c,d) := \frac{\bar{\sigma}_3 v^{\bar{\sigma}_3}}{\bar{\sigma}_1} \frac{J[F,G](v,c,d)}{\dfrac{\partial G}{\partial c}\dfrac{\partial G}{\partial d}}.$$

It follows from (6.79), (6.80), and (6.83) that

$$G_1(v,c,d) = v^{(\bar{\sigma}_1+\bar{\sigma}_3)\,c}[1+\bar{h}_1(v,c)] - v^{(\bar{\sigma}_1+\bar{\sigma}_3)\,d}\left[\frac{m_{211}(v)}{m_{341}(v)} + \bar{h}_2(v,d)\right].$$

Then for $0 < v < \varepsilon^2$ and $\forall (c,d) \in \mathscr{D}_v$ with $\varepsilon > 0$ sufficiently small, the equation $G_1(v,c,d) = 0$ is equivalent to equation $G_2(v,c,d) = 0$, where

$$G_2(v,c,d) := v^c[1+h_1(v,c)] - v^d[\gamma_1(v) + h_2(v,d)], \tag{6.84}$$

where $\gamma_1(v) = (m_{211}(v)/m_{341}(v))^{1/(\bar{\sigma}_1+\bar{\sigma}_3)}$, and

$$\begin{aligned}
h_1(v,c) = {} & \bar{\gamma}_{111} v^{p_1(1-c)} + \bar{\gamma}_{311} v^{p_3 c} \\
& \bar{\gamma}_{142} v^{1-c} + \bar{\gamma}_{143} v^c + O(v^{2(1-c)}, v^{2c}) \\
& + H_1\left(v, v^c, v^{1-c}, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right), \omega(m_4 v^{1-c}, -\beta_1)\right) \\
h_2(v,d) = {} & \bar{\gamma}_{211} v^{p_1(1-d)} + \bar{\gamma}_{411} v^{p_3 d} \\
& + \bar{\gamma}_{232} v^{1-d} + \bar{\gamma}_{233} v^d + O(v^{2(1-d)}, v^{2d}) \\
& + \sum_{j=1}^{N_1} \bar{\gamma}_{34j} v^{\bar{\sigma}_3(1-d)\,j} + O(v^{(N_1+1)(1-d)\,\bar{\sigma}_3}) + \gamma_{22}(v)\, v^{\bar{\sigma}_1 d} \\
& + H_2\left(v, v^d, v^{1-d}, \omega\left(\frac{v^d}{\rho_0}, -\alpha_1\right), \omega(m_4 v^{1-d}, -\beta_1)\right)
\end{aligned}$$

where

$$\begin{aligned}
&\bar{\gamma}_{111}(0) = * \kappa_1, \qquad \bar{\gamma}_{211}(0) = * \kappa_1 \\
&\bar{\gamma}_{311}(0) = * \kappa_3, \qquad \bar{\gamma}_{411}(0) = * \kappa_3 \\
&\bar{\gamma}_{22}(0) = * m_{212}(0) \\
&\bar{\gamma}_{341}(0) = * m_{342}(0) \\
&\bar{\gamma}_{232}(0) = * \bar{\gamma}_{142}(0) = * K_{142}\varepsilon_2 \\
&\bar{\gamma}_{233}(0) = * \bar{\gamma}_{143}(0) = * K_{143}\bar{\mu}_{30}.
\end{aligned}$$

Also H_1 and H_2 are C^k and

$$H_1 = O\left(v^{p_1 c}\omega\left(\frac{v^c}{\rho_0}, \alpha_1\right), v^{p_3(1-c)}\omega(m_4 v^{1-c}, \beta_1)\right)$$

$$H_2 = O\left(v^{p_1 d}\omega\left(\frac{v^d}{\rho_0}, \alpha_1\right), v^{p_3(1-d)}\omega(m_4 v^{1-d}, \beta_1)\right).$$

Similar to what we did in **(2)** with the functions $F_v(c, d)$ and $G_v(c, d)$, one can check that for $0 < v < \varepsilon^2$ and $\forall (c, d) \in \mathscr{D}_v$ with $\varepsilon > 0$ sufficiently small, $G_2(v, c, d)$ and $F_v(c, d)$ satisfy the conditions of Theorem 6.11. Hence we have

$$\#(F, G) \leqslant \#(G_2, J[F, G_2]) + 2. \tag{6.85}$$

(4) Calculation of $\#(G_2, J[F, G_2])$. Let

$$G_3(v, c, d) := -\bar{\sigma}_1 l_1 \frac{J[F, G_2](v, c, d)}{G'_{2c} G'_{2d}}.$$

Then a straightforward calculation gives

$$G_3(v, c, d) = v^{(\bar{\sigma}_1 - 1)c}[1 + h_{31}(v, c)] - v^{(\bar{\sigma}_1 - 1)d}[\gamma_2(v) + h_{32}(v, d)], \tag{6.86}$$

where $\gamma_2(v) = m_{211}(v)(m_{341}(v)/m_{211}(v))^{1/\bar{\sigma}_1 + \bar{\sigma}_3}$.

By (6.84), if for $0 < v < \varepsilon^2$ and $\forall (c, d) \in \mathscr{D}_v$ with $\varepsilon > 0$ sufficiently small, $G_2(v, c, d) \neq 0$, then $\#(G_2, J[F, G_2]) = 0$ and we already finish the proof. Otherwise, similar to the proof in the Theorem 6.11, $G_2(v, c, d) = 0$ defines a unique connected curve which satisfies

$$v^c = v^d \frac{\gamma_1(v) + h_2(v, d)}{1 + h_1(v, c)}. \tag{6.87}$$

By iterating the relation (6.87), the unique curve defined by $G_2(v, c, d) = 0$ can be written as

$$v^c = v^d[\gamma_1(v) + h_0(v, d)], \tag{6.88}$$

where

$$\begin{aligned} h_0(v, d) = {} & \bar{\gamma}_{001} v^{p_1(1-d)} + \bar{\gamma}_{003} v^{p_3 d} \\ & + \bar{\gamma}_{012} v^{1-d} + \bar{\gamma}_{013} v^d + O(v^{2(1-d)}, v^{2d}) \\ & + \sum_{j=1}^{N_1} \bar{\gamma}_{03j} v^{\bar{\sigma}_3(1-d) j} + O(v^{(N_1+1)(1-d)\bar{\sigma}_3}) \\ & + \bar{\gamma}_{02}(v)\, v^{\bar{\sigma}_1 d} + H_0\left(v, v^d, v^{1-d}, \omega\left(\frac{v^d}{\rho_0}, -\alpha_1\right), \omega(m_4 v^{1-d}, -\beta_1)\right) \end{aligned}$$

where

$$\bar{\gamma}_{001}(0) = *\,\kappa_1, \qquad \bar{\gamma}_{003}(0) = *\,\kappa_3$$
$$\bar{\gamma}_{02}(0) = *\,m_{212}(0)$$
$$\bar{\gamma}_{031}(0) = *\,m_{342}(0)$$
$$\bar{\gamma}_{012}(0) = *\,m_{142}(0)(m_{211}(0) - m_{341}(0)) = *\,K_{142}\varepsilon_2$$
$$\bar{\gamma}_{013}(0) = *\,m_{143}(0)(m_{211}(0)\, m_{341}(0)) = *\,K_{143}\bar{\mu}_{30}.$$

Also H_0 is C^k and

$$H_0 = O\left(v^{p_1 d}\omega\left(\frac{v^d}{\rho_0}, \alpha_1\right), v^{p_3(1-d)}\omega(m_4 v^{1-d}, \beta_1)\right). \tag{6.89}$$

If we substitute (6.88) into $G_3(v, c, d)$ and let

$$g(v, d) = \gamma_1^{1-\bar{\sigma}_1} v^{1-\bar{\sigma}_1} G_3(v, c, d)|_{(6.88)},$$

then a straightforward calculation gives

$$\begin{aligned} g(v, d) = {} & \gamma(v) + \delta_{01} v^{p_1(1-d)} + \delta_{03} v^{p_3 d} \\ & + \delta_2 v^{1-d} + \delta_3 v^d + O(v^{2(1-d)}, v^{2d}) \\ & + \sum_{j=1}^{N_1} \delta_{4j} v^{\bar{\sigma}_3(1-d)\,j} + O(v^{(N_3+1)(1-d)\,\bar{\sigma}_3}) \\ & + \delta_1(v)\, v^{\bar{\sigma}_1 d} + H\left(v, v^d, v^{1-d}, \omega\left(\frac{v^d}{\rho_0}, -\alpha_1\right), \omega(m_4 v^{1-d}, -\beta_1)\right), \end{aligned} \tag{6.90}$$

where

$$\gamma(v) = 1 - (m_{211}^{\bar{\sigma}_3}(v)\, m_{341}^{\bar{\sigma}_1}(v))^{\frac{1}{\bar{\sigma}_1 + \bar{\sigma}_3}}, \tag{6.91}$$

and

$$\delta_{01}(0) = *\,\kappa_1, \qquad \delta_{03}(0) = *\,\kappa_3$$
$$\delta_1(0) = *\,m_{212}(0) = *\,\bar{\mu}_{30}$$
$$\delta_2(0) = *\,m_{142}(0)(m_{211}(0) - m_{341}(0)) = *\,K_{142}\varepsilon_2$$
$$\delta_3(0) = *\,m_{143}(0)(m_{211}(0) - m_{341}(0)) = *\,K_{143}\bar{\mu}_{30}$$
$$\delta_{41}(0) = *\,m_{342}(0).$$

Also H is C^k and satisfies (6.89).

In order to get the cyclicity of Ehh2e, Ehh3e, and Ehh1c, we will study the number of roots of the equation $g(v, d)=0$ for $d \in (0, 1)$ and $v>0$ sufficiently small.

(5) *Cycl(Ehh2e, Ehh3e)* $\leqslant 2$. For the graphic Ehh2e (resp. Ehh3e), since $m_{211}(0)$ can be sufficiently small (resp. large), so $\gamma(0) \to 1$ (resp. is sufficiently large). Hence for $(a, \bar{\mu}) \in A_0 \times V_{I_2}$ (resp. $(a, \bar{\mu}) \in A_0 \times V_{I_3}$), and $\forall v \in (0, v_0)$ and $d \in (0, 1)$, we have $g(v, d) \neq 0$. Therefore, $\#(G_2, J[F, G_2]) = 0$ and by (6.85), we have $\#(F, G) \leqslant 2$, i.e., *Cycl(Ehh2e, Ehh3e)* $\leqslant 2$.

(6) Cyclicity of Ehh1c when $m_{211}(0) \leqslant 1$ (this contains the case $\bar{\mu}_{30}=0$). For the graphic Ehh1c, by (6.91), we have

$$\gamma(0)=1-(m_{211}^{\bar{\sigma}_3}(0)\, m_{341}^{\bar{\sigma}_1}(0))^{\frac{1}{\bar{\sigma}_1+\bar{\sigma}_3}}. \tag{6.92}$$

By Hypothesis 6.10 we have $m_{341}(0)<1$, so if $m_{211}(0) \leqslant 1$, then by (6.92) we have $\gamma(0) \neq 0$; i.e., *Cycl(Ehh1c)* $\leqslant 2$.

For the graphic Ehh1c with $m_{211}(0)>1$ (which implies $\bar{\mu}_{30} \neq 0$), we will study the equation $g(v, d)=0$ with $0<v<\varepsilon^2$ and $d \in \mathscr{I}_v$ for $a_0 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$ and $a_0 \in (0, \frac{1}{2}) \cap \mathbb{Q}$ in (7) and (8), respectively.

We have

$$\begin{aligned} \delta_2(0) &= *\, m_{142}(0)[m_{211}(0)-m_{341}(0)] = *\, K_{142}\varepsilon_2 \neq 0 \\ \delta_3(0) &= *\, m_{143}(0)[m_{211}(0)-m_{341}(0)] = *\, K_{143}\bar{\mu}_{30} \neq 0. \end{aligned} \tag{6.93}$$

(7) Cyclicity of Ehh1c when $m_{211}(0)>1$ ($\bar{\mu}_{30}>0$): Case $a_0 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$. For $a_0 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$, the function $g(v, d)$ in (6.90) can be simplified to

$$\begin{aligned} g(v, d) = {} & \gamma(v)+\delta_1(v)\, v^{\bar{\sigma}_1 d}+o(v^{\bar{\sigma}_1 d}) \\ & +\delta_2(v)\, v^{1-d}+\delta_3(v)\, v^{d}+O(v^{2(1-d)}, v^{2d}) \\ & +\sum_{j=1}^{[\frac{1}{\bar{\sigma}_3}]} \delta_{4j} v^{\bar{\sigma}_3(1-d)\, j}+O(v^{([\frac{1}{\bar{\sigma}_3}]+1)\, \bar{\sigma}_3(1-d)}). \end{aligned} \tag{6.94}$$

Let

$$DD\colon \begin{cases} g_0(v, d)=\dfrac{1}{\ln v}\dfrac{\partial}{\partial d}\, g(v, d) \\[2ex] g_j(v, d)=\dfrac{v^{\bar{\sigma}_3(1-d)\, j}}{\ln v}\dfrac{\partial}{\partial d}(g_{j-1}(v, d)\, v^{-\bar{\sigma}_3(1-d)\, j}), \qquad j=1, \ldots, \left[\dfrac{1}{\bar{\sigma}_3}\right]. \end{cases} \tag{6.95}$$

Then after $[1/\bar{\sigma}_3]+1$ steps of successive derivation and division in (6.95), we get

$$g_{[\frac{1}{\bar{\sigma}_3}]}(v, d) = \bar{\delta}_1(v)\, v^{\bar{\sigma}_1 d} + o(v^{\bar{\sigma}_1 d}) + \bar{\delta}_2(v)\, v^{1-d} + \bar{\delta}_3(v)\, v^d + o(v^{2(1-d)}, v^{2d}), \tag{6.96}$$

where $\bar{\delta}_1(0) = *\, m_{212}(0) = *\, \bar{\mu}_{30} \neq 0$, and by (6.93), $\bar{\delta}_2(0) = *\, \delta_2(0) \neq 0$.
We introduce a lemma.

Lemma 6.16. *Consider the equation*

$$L(v, d) = \bar{l}_1(v)\, v^{\bar{\sigma}_1 d} + o(v^{\bar{\sigma}_1 d}) + \bar{l}_2(v)\, v^{1-d} + o(v^{1-d}) + \bar{l}_3(v)\, v^d + o(v^d)$$

for $v \in (0, \varepsilon^2)$ *and* $d \in (\frac{\ln \varepsilon}{\ln v}, \frac{\ln u\varepsilon}{\ln v})$ *with* $u \in (0, 1)$, $v = u\varepsilon^2$, *and* $\varepsilon > 0$ *sufficiently small. If* $\bar{\sigma}_1 > 1$ *and* $\bar{l}_2(0)\, \bar{l}_3(0) \neq 0$ *or* $\bar{\sigma}_1 < 1$ *and* $\bar{l}_1(0)\, \bar{l}_2(0) \neq 0$, *then* $L(v, d) = 0$ *has at most* 1 *solution.*

Proof. For the case $\bar{\sigma}_1 > 1$, $L(v, d)$ can be simplified to

$$L(v, d) = \bar{l}_2(v)\, v^{1-d} + o(v^{1-d}) + \bar{l}_3(v)\, v^d + o(v^d).$$

Note that $\bar{l}_2(0)\, \bar{l}_3(0) \neq 0$, so we have two possibilities:

- if $\bar{l}_2(0)\, \bar{l}_3(0) > 0$, then $L(v, d) \neq 0$;
- if $\bar{l}_2(0)\, \bar{l}_3(0) < 0$, then $L'_d(v, d) \neq 0$, thus $L(v, d) = 0$ has at most one solution.

The case $\bar{\sigma}_1 < 1$ is similar. ∎

End of proof of Theorem 6.15. In this case, note that we have $\bar{\delta}_1(0)\, \bar{\delta}_2(0)\, \bar{\delta}_3(0) \neq 0$ and $\bar{\sigma}_1 \neq 1$, so applying Lemma 6.16 to the function in (6.96), we conclude that $g_{[1/\bar{\sigma}_3]}(v, d)$ has at most one root. Hence $g(v, d) = 0$ has at most $[1/\bar{\sigma}_3]+2$ roots, yielding $Cycl(Ehh1c) \leqslant [1/\bar{\sigma}_3]+4$.

(8) Cyclicity of Ehh1c when $m_{211}(0) > 1$ ($\bar{\mu}_{30} > 0$): Case $a_0 \in (0, \frac{1}{2}) \cap \mathbb{Q}$. Then $\sigma_1(a_0), \sigma_3(a_0) \in \mathbb{Q}$. For $i = 1, 3$, let $\sigma_i(a_0) = p_i/q_i$, $p_i, q_i \in \mathbb{N}$ and $(p_i, q_i) = 1$; thus we have three subcases:

(8.1) Case $a_0 \in (0, \frac{1}{2}) \cap \mathbb{Q} \setminus \{\frac{1}{3}, \frac{2n-1}{4n}, n \in \mathbb{N}\}$. Note that in this case $1/\sigma_3 \notin \mathbb{N}$ and $\sigma_1 \neq 1$; therefore Eq. (6.90) can be reduced to

$$\begin{aligned} g(v, d) = {} & \gamma(v) + \delta_1(v)\, v^{\bar{\sigma}_1 d} + o(v^{\bar{\sigma}_1 d}) + \delta_3(v)\, v^d + o(v^d) \\ & + \sum_{j=1}^{q_3} \delta_{4j} v^{\bar{\sigma}_3(1-d)\, j} + \delta_2(v)\, v^{1-d} + o(v^{1-d}). \end{aligned} \tag{6.97}$$

Applying the DD process (6.95) $[1/\sigma_3]+1$ steps to the function $g(v, d)$ in (6.97), we get

$$g_{p_1, q_3}(v, d)=\begin{cases} l\hat{\delta}_1(v)\, v^{\bar{\sigma}_1 d}+o(v^{\bar{\sigma}_1 d})+\hat{\delta}_2(v)\, v^{1-d}+o(v^{1-d}) & \text{if} \quad \sigma_1<1 \\ \hat{\delta}_3(v)\, v^{d}+o(v^{d})+\hat{\delta}_2(v)\, v^{1-d}+o(v^{1-d}) & \text{if} \quad \sigma_1>1, \end{cases}$$

where $\hat{\delta}_i(0)\neq 0$ $(i=1, 2, 3)$.

Then by Lemma 6.16 we obtain that $g_{p_1, q_3}(v, d)=0$ has at most one solution. Hence $g(v, d)=0$ has at most $[1/\bar{\sigma}_3]+4$ solutions; i.e., Ehh1c has finite cyclicity.

(8.2) Case $a_0=\frac{1}{4}$. In this case, we have $\sigma_1(\frac{1}{4})=2$ and $\sigma_3(\frac{1}{4})=1$. By Lemma 6.14, the first saddle quantity at P_1 is $\alpha_2=*\,\bar{\mu}_{30}\neq 0$, the first saddle quantity at P_3 is $\beta_2\neq 0$.

For the second type of Dulac map near P_3, by Theorem 4.14, we have

$$\begin{aligned} &\theta_4\left(r_4, \rho_4, \omega\left(\frac{r_4}{r_0}, -\beta_1\right)\right) \\ &\quad =\beta_2 K_3 r_3 \omega\left(\frac{r_3}{r_0}, \beta_1\right)+O\left(v\omega^2\left(\frac{r_4}{r_0}, -\beta_1\right)\omega\left(\frac{r_4}{r_0}, \beta_1\right)\right). \end{aligned} \tag{6.98}$$

Then the function in (6.90) has the form

$$\begin{aligned} g(v, d)&=\gamma(v)+\delta_{11}(v)\, v^d+\delta_{12}(v)\, v^{2d} \\ &\quad+\delta_1(v)\, v^{2d}\omega\left(\frac{v^d}{\rho_0}, \alpha_1\right)+O\left(v^2\omega^2\left(\frac{v^d}{\rho_0}, -\alpha_1\right)\omega\left(\frac{v^d}{\rho_0}, \alpha_1\right)\right) \\ &\quad+\delta_{21} v^{1-d}+\delta_2(v)\, v^{1-d}\omega(m_4 v^{1-d}, \beta_1) \\ &\quad+O(v\omega^2(m_4 v^{1-d}, -\beta_1)\, \omega(m_4 v^{1-d}, \beta_1)) \end{aligned} \tag{6.99}$$

where $\delta_{11}(v)$ can vanish, $\delta_1(0)=*\,\alpha_2(\frac{1}{4})=*\,\bar{\mu}_{30}\neq 0$, and $\delta_2(0)=*\,\beta_2\neq 0$.

By applying the standard division–derivation method to the function $g(v, d)$ in (6.99), we can kill the terms $\gamma(v)$ and v^d. Then $g(v, d)=0$ has at most four roots plus the number of roots of

$$\begin{aligned} g_1(v, d)&=\hat{\delta}_{12} v^{2d}+\hat{\delta}_1(v)\, v^{2d}\omega\left(\frac{v^d}{\rho_0}, \alpha_1\right)+O\left(v^2\omega^2\left(\frac{v^d}{\rho_0}, -\alpha_1\right)\omega\left(\frac{v^d}{\rho_0}, \alpha_1\right)\right) \\ &\quad+\hat{\delta}_{21}(v)\, v^{1-d}+\hat{\delta}_2(v)\, v^{1-d}\omega(m_4 v^{1-d}, \beta_1) \\ &\quad+O(v\omega^2(m_4 v^{1-d}, -\beta_1)\, \omega(m_4 v^{1-d}, \beta_1)) \end{aligned}$$

where $\hat{\delta}_i(0)=*\,\delta_i(0)\neq 0$ $(i=1, 2)$.

Let $g_2(v, d) = v^{2d}/\ln v \frac{\partial}{\partial d} (v^{-2d} g_1(v, d))$; then

$$g_2(v, d) = -\rho_0^{\alpha_1} \tilde{\delta}_1 v^{\bar{\sigma}_1 d} + O\left(v^2 \omega^2 \left(\frac{v^d}{\rho_0}, -\alpha_1 \right) \omega \left(\frac{v^d}{\rho_0}, \alpha_1 \right) \right)$$
$$- [(3+\beta_1)\, \omega(m_1 4 v^{1-d}, \beta_1) + 1]\, \tilde{\delta}_2(v)\, v^{1-d}$$
$$+ O(v\omega^2(m_4 v^{1-d}, -\beta_1)\, \omega(m_4 v^{1-d}, \beta_1)).$$

Let $g_3(v, d) = v^{1-d}/\ln v \frac{\partial}{\partial d} (v^{-(1-d)} g_2(v, d))$; then

$$g_3(v, d) = \bar{\delta}_1 v^{\bar{\sigma}_1 d} + O\left(v^2 \omega^2 \left(\frac{v^d}{\rho_0}, -\alpha_1 \right) \omega \left(\frac{v^d}{\rho_0}, \alpha_1 \right) \right)$$
$$+ \bar{\delta}_2(v)\, v^{\bar{\sigma}_3 d} + O(v\omega^2(m_4 v^{1-d}, -\beta_1)\, \omega(m_4 v^{1-d}, \beta_1)),$$

where $\bar{\delta}_1(0) = -\rho_0^{\alpha_1} \tilde{\delta}_1(0) \neq 0$, $\bar{\delta}_2(0) = (3+\beta_1)\, m_4^{-\beta_1} \tilde{\delta}_2(0) \neq 0$.

Again applying Lemma 6.16 to the function $g_3(v, d)$ we conclude that $g_3(v, d) = 0$ has at most one root, so for $a_0 = \frac{1}{4}$, $Cycl(Ehh1c) \leqslant 9$.

(8.3) Case $a_0 = \frac{2n-1}{4n}$, $n \in \mathbb{N}$, $n \neq 1$. In this case, we have $\sigma_1(a_0) = \frac{2}{2n-1} < 1$ and $\sigma_3(a_0) = \frac{1}{n}$.

Since $\sigma_1 < 1$, this case is similar to the case (8.2), but simpler. The function in (6.90) has the form

$$g(v, d) = \gamma(v) + \delta_1(v)\, v^{\bar{\sigma}_1 d} + o(v^{\bar{\sigma}_1 d}) + \sum_{j=1}^{n-1} \delta_{2j} v^{\bar{\sigma}_3 (1-d) j} + \delta_{22}(v)\, v^{1-d}$$
$$+ \delta_2(v)\, v^{1-d} \omega(m_4 v^{1-d}, \beta_1) + O(v\omega^2(m_4 v^{1-d}, -\beta_1)\, \omega(m_4 v^{1-d}, \beta_1)), \tag{6.100}$$

where $\delta_1(0) = *\, m_{212}(0) = *\, \bar{\mu}_{30} \neq 0$, and by Lemma 6.14, $\delta_2(0) = *\, \beta_2 \neq 0$.

After killing the terms $\gamma(v)$ and $v^{\bar{\sigma}_3(1-d) j}$ $(j = 1, 2, ..., n-1)$ by the derivation–division process, then similar to the process in (8.2), we obtain that $Cycl(Ehh1c) \leqslant n+3$.

(8.4) Case $a_0 = \frac{1}{3}$. Note that in this case, $\sigma_1(\frac{1}{3}) = 1$, $\sigma_3(\frac{1}{3}) = \frac{2}{3}$. Then the function in (6.90) has the form

$$g(v, d) = \gamma(v) + \delta_{11}(v)\, v^d + \delta_1(v)\, v^d \omega \left(\frac{v^d}{\rho_0}, \alpha_1 \right)$$
$$+ O\left(v\omega^2 \left(\frac{v^d}{\rho_0}, -\alpha_1 \right) \omega \left(\frac{v^d}{\rho_0}, \alpha_1 \right) \right)$$
$$+ \delta_{21} v^{\bar{\sigma}_3 (1-d)} + \delta_2(v)\, v^{1-d} + o(v^{1-d})$$

where $\delta_1(0) = * \alpha_2$, α_2 is the saddle quantity for the 2-dimensional system near P_1 on $r=0$. By Lemma 6.14,$\alpha_2 = -* \bar{\mu}_{30} \neq 0$. Then, similar to the case (8.3), we get the finite cyclicity of Ehh1c. ▮

Next we study the rest of the lower boundary graphics of Ehh families using the following remark:

Remark 6.17. The system (3.10) is invariant under the transformation

$$(-t, -x, -\bar{\mu}_1, -\bar{\mu}_3) \mapsto (t, x, \bar{\mu}_1, \bar{\mu}_3) \tag{6.101}$$

so the families Ehh7 and Ehh8 can be obtained from the families Ehh5 and Ehh6 and the families Ehh11 and Ehh12 can be obtained from the families Ehh9 and Ehh10. We will only need to deal with families Ehh5, Ehh6, Ehh9 and Ehh10 as long as we do not use Hypothesis 6.10: $\gamma^* < 1$.

THEOREM 6.18. *For the families Ehh*4, …, *Ehh*12, *all the lower boundary graphics have finite cyclicity.*

Proof. For the family Ehh4, the proof is the same as for the family Sxhh1 in the saddle case, so by Theorem 5.4, Ehh4c has finite cyclicity.

To prove the finite cyclicity of graphics Ehh5c, Ehh6c, Ehh9c, and Ehh10e, take sections τ_1 and Σ_3 as defined in Notation 5.6. We are going to study the displacement map defined on the section τ_1,

$$L = \hat{T} - \tilde{T}\colon \tau_1 \to \Sigma_3, \tag{6.102}$$

where $\hat{T}$ is the transition map through the blown-up singularity and $\tilde{T}$ is the transition map along the regular orbit. Similar to the graphic Ehp1c, on the section τ_1, we will use coordinates (v, c) with $c \in \mathscr{I}_v$.

(1) Lower boundary graphics Ehh9c and Ehh10e. The lower boundary graphics Ehh9c and Ehh10e are treated exactly as graphics Ehp1, Ehp2c and Ehp3 in Section 6.2 (see also Remark 6.7).

(2) Lower boundary graphics Ehh5c and Ehh6c. Taking sections τ_4, Σ_4, and Σ_3 in the normal form coordinates (Notation 5.6 and Fig. 25), the transition map $\tilde{T}$ can be calculated by the decomposition

$$\tilde{T} = R^{-1} \circ \theta_4 \circ U,$$

where $U\colon \tau_1 \to \tau_4$ is the regular transition map defined in Proposition 6.12 with expression (6.41), $\Theta_4\colon \tau_4 \to \Sigma_4$ is the second type of Dulac map near

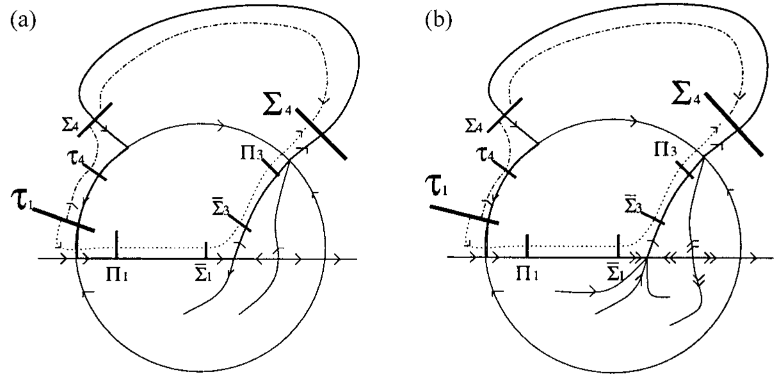


FIG. 25. Lower boundary graphics (a) Ehh5c and (b) Ehh6c: Displacement maps.

P_4, and $R^{-1}\colon \Sigma_4 \to \Sigma_3$ is the inverse of the transition map R defined in (6.76). Then a straightforward calculation gives

$$\begin{aligned}\tilde{T}_2(v, c) &= \tilde{m}_{130}(v) + \tilde{m}_{131}(v)\, \eta_4(v, \omega(r_4, \beta_1)) + O(v^{2p_3(1-c)}\omega^2(m_4 v^{1-c}, \beta_1)) \\ &\quad - m_{13}(v)\, v^{\bar{\sigma}_3(1-c)}[1 + \theta_4(v, v^{1-c}, \omega^2(m_4 v^{1-c}, \beta_1))],\end{aligned} \tag{6.103}$$

where $r_4 = v^{1-c}[m_{141} + O(v^c, v^{1-c})]$, $m_4 = m_{141}/r_0$, and $m_{13}(0) > 0$.

Since the graphics Ehh5c and Ehh6c pass through a saddle and a saddle node, respectively, the transition map $\hat{T}$ may not be C^2, and the graphics need a special treatment.

Let us see Ehh5c first. As shown in Fig 25, in the normal form coordinates in the neighborhood of the saddle point, take sections $\bar{\Sigma}_1 = \{\tilde{x} = -x_0\}$ and $\bar{\Sigma}_3 = \{\tilde{y} = y_0\}$ and let $\lambda(\bar{\mu}_0)$ be the hyperbolicity ratio of the saddle point. Then for the transition map

$$\bar{\Delta}_0 = (\bar{d}_0, \bar{D}_0)\colon \bar{\Sigma}_1 \to \Sigma_3$$

its second component $\bar{D}_0(v, \tilde{y})$ can be written in the form of Mourtada (Proposition 4.2).

Take sections Π_1 and Π_3 as defined in Notation 5.6 in the normal form coordinates; then the transition map $\hat{T}$ has the decomposition

$$\hat{T} = \Delta_3 \circ \bar{T}_{03} \circ \bar{\Delta}_0 \circ \bar{T}_{10} \circ \Theta_1, \tag{6.104}$$

where

- $\Theta_1\colon \tau_1 \to \Pi_1$ is the second type of Dulac map near P_1 which satisfies Theorem 4.14 with $\sigma = \sigma_1$,

- $\bar{T}_{10}\colon \Pi_1 \to \bar{\Sigma}_1$ and $\bar{T}_{03}\colon \bar{\Sigma}_3 \to \Pi_3$ are C^k regular transition maps with

$$\bar{T}_{102} = m_{100}(v) + m_{101}(v)\,\tilde{y} + m_{102}(v)\,\tilde{y}^2 + O(\tilde{y}^3)$$
$$\bar{T}_{032} = m_{030}(v) + m_{031}(v)\,\tilde{x} + m_{032}(v)\,\tilde{x}^2 + O(\tilde{x}^3),$$

- $\Delta_3\colon \Pi_3 \to \Sigma_3$ is the first type of Dulac map near P_3 which satisfies Theorem 4.10 with $\sigma = \sigma_3$.

(2.1) Case $\lambda(\bar{\mu}_0) > 1$. Let

$$\tilde{y}_1 = \frac{\kappa_1}{\rho_0^{p_1}} v^{p_1}\omega\left(\frac{v^c}{\rho_0}, \alpha_1\right) + v^{\bar{\sigma}_1 c}\left[l_1 + \theta_1\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \tag{6.105}$$
$$\tilde{y}_3 = m_{030}(v) + m_{031}\tilde{y}_1^{\lambda}[\beta_0 + \phi_0(v, \tilde{y}_1)],$$

where $l_1 = y_0/\rho_0^{\bar{\sigma}_1} > 0$, $m_{030}(0) = 0$, $m_{031}(0) \neq 0$, and $\phi_0(v, \tilde{y}_1) \in (I_0^\infty)$. Then the second component $\hat{T}_2(v, c)$ can be written as

$$\hat{T}_2(v, c) = \kappa_3\rho_0^{p_3}\omega\left(\frac{v}{v_0}, -\beta_3\right)\left(\frac{v}{v_0}\right)^{\bar{\sigma}_3} + \left(\frac{v}{v_0}\right)^{\bar{\sigma}_3}\left[\tilde{y}_3 + \phi_3\left(v, \tilde{y}_3, \omega\left(\frac{v}{v_0}, -\beta_3\right)\right)\right] \tag{6.106}$$

where $\tilde{m}_{031}(0) > 0$ and $\tilde{\phi}_{031}$ is C^k and satisfies the property (6.18).

Consider the displacement defined in (6.102). By (6.103) and (6.106), a first derivation of $L_2(v, c)$ gives

$$\begin{aligned}
L_2'(v, c) &= \hat{T}_2'(v, c) - \tilde{T}_2'(v, d)\\
&= \left(\frac{v}{v_0}\right)^{\bar{\sigma}_3}\left[1 + \frac{\partial\phi_3}{\partial\tilde{y}_3}\left(v, \tilde{y}_3, \omega\left(\frac{v}{v_0}, -\beta_3\right)\right)\right]\\
&\quad [m_{031}(v)\,\lambda\tilde{y}_1^{\lambda-1}(1 + \phi_{01}(v, \tilde{y}_1))]\\
&\quad v^{\bar{\sigma}_1 c}\ln v\left[\bar{\sigma}_1 l_1 + O(v^{p_1(1-c)}) + \theta_{11}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right]\\
&\quad - v^{\bar{\sigma}_3(1-c)}\ln v[m_{13}(v)\,\bar{\sigma}_3 + O(v^{p_3 c})\\
&\quad + \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3))],
\end{aligned} \tag{6.107}$$

where $\phi_{01} \in (I_0^\infty)$, and θ_{11}, θ_{41} satisfy (6.18), and also $\partial\phi_3/\partial\tilde{y}_3 = O(v^{p_3}\omega^{q_3}(\frac{v}{v_0}, -\beta_3)\ln\frac{v}{v_0})$. $L_2'(v, c)$ has the same number of roots as

$$\begin{aligned}
L_{21}(v, c) &= \frac{v^{-\bar{\sigma}_3(1-c)}}{\ln v} L_2'(v, c)\\
&= -m_{13}(v)\,\bar{\sigma}_3 + O(v^{p_3 c}) + \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3)) + O(\tilde{y}_1^{\lambda-1}).
\end{aligned} \tag{6.108}$$

Since $m_{13}(0) \neq 0$, so $L_{21}(v, c) \neq 0$. Thus $L_2(v, c) = 0$ has at most one root; i.e., $Cycl(Ehh5c) \leqslant 1$.

(2.2) Case $\lambda(\bar{\mu}_0) < 1$. In this case, $L_2'(v, c)$ has the same number of roots as

$$\begin{aligned}
\bar{L}_{21}(v, c) \\
= v^{\frac{(\bar{\sigma}_1+\bar{\sigma}_3)c}{1-\lambda}} \left[1 + \frac{\partial \phi_{31}}{\partial \tilde{y}_3} \left(v, \tilde{y}_3, \omega\left(\frac{v}{v_0}, -\beta_3\right)\right)\right] \\
\left[(\bar{\sigma}_1 l_1)^{\frac{1}{1-\lambda}} + O(v^{p_1(1-c)}) + \theta_{12}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \\
- \tilde{y}_1(1+\phi_{11}(v, \tilde{y}_1))[(m_{13}\bar{\sigma}_3)^{\frac{1}{1-\lambda}} + O(v^{p_3 c}) + \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3))].
\end{aligned} \tag{6.109}$$

Then

$$\begin{aligned}
\bar{L}_{21}'(v, c) = v^{\frac{(\bar{\sigma}_1+\bar{\sigma}_3)c}{1-\lambda}} \ln v \left[1 + O\left(v^{p_3} \omega^{q_3}\left(\frac{v}{v_0}, -\beta_3\right) \ln \frac{v}{v_0}\right)\right] \\
\left[\frac{(\bar{\sigma}_1+\bar{\sigma}_3)(\bar{\sigma}_1 l_1)^{\frac{1}{1-\lambda}}}{1-\lambda} + O(v^{p_1(1-c)}) + \theta_{13}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \\
- v^{\bar{\sigma}_1 c} \ln v \left[l_1 \bar{\sigma}_1 + O(v^{p_1(1-c)}) + \theta_{14}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \\
(1+\phi_{03}(v, \tilde{y}_1)) \\
[(m_{13}(v)\,\bar{\sigma}_3)^{\frac{1}{1-\lambda}} + O(v^{p_3 c}) + \theta_{43}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3))],
\end{aligned} \tag{6.110}$$

which has the same number of roots as $L_{22}(v, c) = -(v^{\bar{\sigma}_1 c}/\ln v)\, \bar{L}_{21}'(v, c)$. Note that

$$\begin{aligned}
L_{22}(v, c) = l_1 \bar{\sigma}_1 (m_{13}(v)\,\bar{\sigma}_3)^{\frac{1}{1-\lambda}} [1 + \phi_{03}(v, \tilde{y}_1)] \\
\left[1 + O(v^{p_1(1-c)}) + \theta_{14}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right] \\
[1 + O(v^{p_3 c}) + + \theta_{43}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3))] + O(v^{\frac{(\bar{\sigma}_1\lambda+\bar{\sigma}_3)c}{1-\lambda}}) \\
\neq 0;
\end{aligned} \tag{6.111}$$

hence, $L_2(v, c) = 0$ has at most two roots, yielding $Cycl(Ehh5c) \leqslant 2$.

(2.3) Case $\lambda(\bar{\mu}_0) = 1$. In this case, for the second component $\hat{T}_2$ of $\hat{T}$ defined in (6.104), letting

$$\tilde{y} = m_{100}(v) + m_{101} v^{p_1} \omega\left(\frac{v^c}{\rho_0}, \alpha_1\right) + m_{11}(v)\, v^{\bar{\sigma}_1 c}\left[1 + \theta_{11}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, \alpha_1\right)\right)\right] \tag{6.112}$$

and using the refinement of Roussarie [30] for $\bar{T}_{03} \circ \bar{\Delta}_0$, then a straightforward calculation gives

$$\hat{T}_2(v, c) = \alpha_{00}(v) + O\left(v^{\bar{\sigma}_3}\omega\left(\frac{v}{v_0}, -\beta_3\right)\right)$$
$$+ v^{\bar{\sigma}_3}[\alpha_{11}(v)\, \tilde{y}\omega(\tilde{y}, \alpha_{11}) + \alpha_{22}(v)\, \tilde{y} + O(v^{\bar{\sigma}_3}\tilde{y}^2\omega(\tilde{y}, \alpha_{11}))] \tag{6.113}$$

where $\alpha_{00}(0) = 0$, $\alpha_{22}(0) \neq 0$.

Then the first derivation of $L_2(v, c)$ gives

$$L_2'(v, c) = v^{\bar{\sigma}_3}[\bar{\alpha}_{11}(v)\, \omega(\tilde{y}, \alpha_{11}) + \bar{\alpha}_{22}(v) + O(v^{\bar{\sigma}_3}\tilde{y}\omega(\tilde{y}, \alpha_{11}))]$$
$$\left[m_{11}(v)\, v^{\bar{\sigma}_1 c} \ln v \left(1 + O(v^{p_1(1-c)}) + \theta_{11}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right)\right]$$
$$- v^{\bar{\sigma}_3(1-c)} \ln v[m_{13}(v)\, \bar{\sigma}_3 + O(v^{p_3 c}) + \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3))], \tag{6.114}$$

where $\bar{\alpha}_{11}(v) = \alpha_{11}(1 - \alpha_{11})$ and $\bar{\alpha}_{22} = \alpha_{22} - \alpha_{11}$ with $\bar{\alpha}_{22}(0) \neq 0$.

Denote

$$L_{0j}(v, c) = 1 + O(v^{p_1(1-c)}, v^{p_3 c})$$
$$+ \bar{\theta}_{1j}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right) + \bar{\theta}_{4j}(v, v^{1-c}, \omega(m_4 v^{1-c}, -\beta_3)) > 0,$$
$$j \geqslant 1.$$

where the θ_{1j} and θ_{4j} will have similar properties to those of θ_{11} and θ_{41}, respectively.

Then the equation $L_2'(v, c) = 0$ has the same number of roots as

$$L_{22}(v, c)$$
$$= \frac{v^{-(\bar{\sigma}_1 + \bar{\sigma}_3)c} L_2'(v, c)}{\omega(\tilde{y}, \alpha_{11}) \ln v \left(1 + O(v^{p_1(1-c)}) + \theta_{11}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right)}$$
$$= m_{11}(v)\, \bar{\alpha}_{11}(v) + m_{11}(v) \frac{\bar{\alpha}_{22}(v)}{\omega(\tilde{y}, \alpha_{11})} - \frac{\bar{\sigma}_3 m_{13}(v)\, v^{-(\bar{\sigma}_1 + \bar{\sigma}_3)c} L_{01}(v, c)}{\omega(\tilde{y}, \alpha_{11})}. \tag{6.115}$$

The number of roots of the equation $L_{22}(v, c) = 0$ is at most one plus the number of roots of

$$L_{23}(v, c) = \frac{v^{(\bar{\sigma}_1+\bar{\sigma}_3)c}\tilde{y}^{1+\alpha_{11}}\omega^2(\tilde{y}, \alpha_{11})\, L'_{22}(v, c)}{(\bar{\sigma}_1+\bar{\sigma}_3)\, m_{13}(v)\, L_{02}(v, c)\ln v}$$
$$= \tilde{y}^{1+\alpha_{11}}\omega(\tilde{y}, \alpha_{11}) + m_{11}(v)\, v^{\bar{\sigma}_1 c} L_{02}(v, c) + O(v^{(2\bar{\sigma}_1+\bar{\sigma}_3)c}). \tag{6.116}$$

Let

$$L_{24}(v, c) = \frac{L'_{23}(v, c)}{\bar{\sigma}_1 v^{\bar{\sigma}_1 c}\ln v\left[1 + O(v^{p_1(1-c)}) + \theta_{11}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right]}.$$

Then

$$L_{24}(v, c) = m_{11}(v)[1 + (1+\bar{\alpha}_{11})\,\tilde{y}^{\bar{\alpha}_{11}}\omega(\tilde{y}, \alpha_{11})] + \bar{\sigma}_1 L_{03}(v, c) + O(v^{(2\bar{\sigma}_1+\bar{\sigma}_3)c}) > 0, \tag{6.117}$$

where the term $\tilde{y}^{\bar{\alpha}_{11}}\omega(\tilde{y}, \alpha_{11})$ is positive and sufficiently large. Therefore, $L_2(v, c) = 0$ has at most three roots which gives $Cycl(Ehh5c) \leqslant 3$.

Now let us study the graphic Ehh6c. In the decomposition of $\hat{T}$, the second component of the transition map $\bar{\delta}_0 = (\bar{d}_0, \bar{D}_0)$ satisfies (5.47).

Still letting $\tilde{y}$ be defined as in (6.112), and also letting

$$\tilde{y}_3 = m_{030}(v) + O(\tilde{y}^{i_2}), \qquad i_2 \geqslant 2,$$

then a first derivation of $L_2(v, c)$ gives

$$L'_2(v, c) = \left(\frac{v}{v_0}\right)^{\bar{\sigma}_3} O(\tilde{y}^{i_2-1})\left[1 + \frac{\partial\phi_3}{\partial\tilde{y}_3}\left(v, \tilde{y}_3, \omega\left(\frac{v}{v_0}, -\beta_3\right)\right)\right]$$
$$v^{\bar{\sigma}_1 c}\ln v\left[\bar{\sigma}_1 l_1 + O(v^{p_1(1-c)}) + \theta_{11}\left(v, v^c, \omega\left(\frac{v^c}{\rho_0}, -\alpha_1\right)\right)\right]$$
$$- v^{\bar{\sigma}_3(1-c)}\ln v[m_{13}(v)\,\bar{\sigma}_3 + O(v^{p_3 c}) + \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3))],$$

which has the same number of roots as

$$L_{21}(v, c) = \frac{L'_2(v, c)\, v^{\bar{\sigma}_3(c-1)}}{\ln v}$$
$$= -m_{13}(v)\,\bar{\sigma}_3 + O(v^{p_3 c}) + \theta_{41}(v, v^{1-c}, \omega(m_4 v^{1-c}, \beta_3)) + O(\tilde{y}^{i_2-1}) \neq 0.$$

Therefore, $L(v, c) = 0$ has at most one small root, i.e., $Cycl(Ehh6c) \leqslant 1$. ∎

6.3.4. *Intermediate graphics of the Ehh families.*

THEOREM 6.19. *Under the generic assumption, all the intermediate hh graphics of elliptic type of the* 12 *families Ehh*1*, Ehh*2*, ..., Ehh*12 *have finite cyclicity.*

Proof. Let Γ be any of the intermediate hh graphics of elliptic type of the 12 families. Similar to the intermediate concave graphics of saddle type, take sections Π_3 and Π_4 as defined in (5.22) in the normal form coordinates in the neighborhood of P_3 and P_4, respectively. Consider the map

$$T\colon \Pi_3 \to \Pi_4$$

defined in Proposition 5.10. We are going to discuss the transition map $T_2(0, \tilde{y}_3)$ in the chart F.R. on $r=0$. By taking $r_3=0$ and $r_4=0$ in the normal forms in the neighborhood of P_3 and P_4 in Proposition 4.6, we obtain the normal forms in the neighborhoods of P_3 and P_4 in the chart F.R. on $r=0$,

$$\begin{aligned} \dot{\rho}_i &= (-1)^i \rho_i \\ \dot{\tilde{y}}_i &= (-1)^i \sigma_3(a)\, \tilde{y}_i + (-1)^{i+1} \kappa_3 \rho_i^{p_3}, \end{aligned} \tag{6.118}$$

where $i=3, 4$ and if $a \neq \frac{1}{4}$ then $\kappa_3 = 0$.

Let $\pi_i = \{\rho_i = \rho_0\}$ $(i=3, 4)$ be the two sections in the chart F.R. on $r=0$ parametrized by the normal form coordinate $\tilde{y}_i$. Then we are reduced to study the 1-dimensional transition map

$$T_2(0, \tilde{y}_4)\colon \pi_4 \to \pi_3$$

or its inverse. We will verify that for each family, the corresponding map $T_2(0, \tilde{y}_4)$ or its inverse satisfies one of the sufficient conditions listed in Proposition 5.10.

(1) Family Ehh1. Let Γ be any intermediate graphic of the family Ehh1. Since the systems (6.118) $(i=3, 4)$ exist globally, so the map T_2 exists globally on π_4 and not only in the neighborhood of $\pi_4 \cap \Gamma$. We are going to prove that $T_2(0, \tilde{y}_4)$ is either the identity or nonlinear. By Proposition 5.11, to prove the nonlinearity of T_2, it suffices to prove that it is nonlinear at one point on π_3. To do this, as shown in Fig. 26, we take line sections $\tau_4 = \{\tilde{y}_4 = -y_0\}$ and $\tau_3 = \{\tilde{y}_3 = -y_0\}$ in the normal form coordinates, chosen such that any intermediate graphic of the family intersects either τ_i or π_i inside the neighborhood of P_i $(i=3, 4)$ respectively. Then over some subinterval of π_4 the map T_2 can be factorized as

$$T_2 = S_3 \circ \hat{T}_2 \circ S_4^{-1}, \tag{6.119}$$

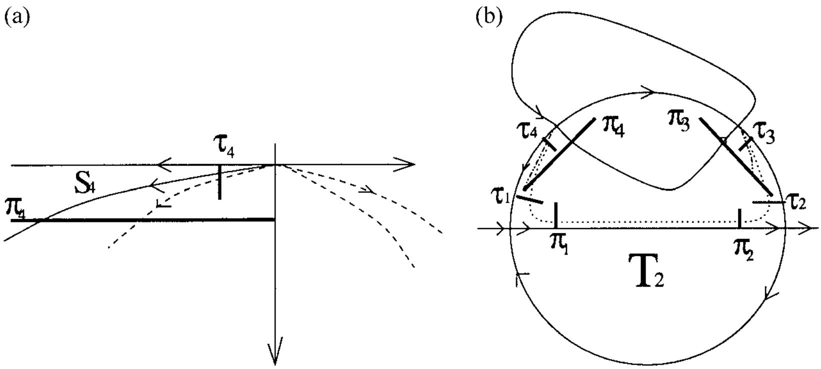


FIG. 26. Transition map T for the intermediate graphic of family Ehh1. (a) Regular transition, (b) T_2.

where as shown in Fig. 26a, S_i: $\tau_i \to \pi_i$ $(i = 3, 4)$ are the regular transition maps in the normal form coordinates in the neighborhood of P_3 and P_4, respectively, and $\hat{T}_2$: $\tau_4 \to \tau_3$ is the transition map which is in particular defined near the lower boundary graphic Ehh1c.

We first calculate S_3 and S_4^{-1}. Due to the easy form of system (6.118), the transition S_3 can be directly calculated by integration and

$$S_3(0, \rho_3) = \frac{1}{\rho_3^{\sigma_3}} [-C_1 + C_2 \kappa_3 \ln \rho_3], \tag{6.120}$$

where C_1 and C_2 are positive constants. Let $v_i = 1/\tilde{y}_i$ $(i = 3, 4)$. If we parameterize the section π_i by v_i, then by (6.120), we have

$$S_i\colon v_i = \frac{\rho_i^{\sigma_3}}{-C_1 + C_2 \kappa_i \ln \rho_i}. \tag{6.121}$$

In particular, the transition map S_4 sends the points on section τ_4 in the positive neighborhood of 0 to the points on the section π_4 at infinity.

Remark 6.20. Although the normal form is only valid in the neighborhood of P_3 and P_4, the systems (6.118) $(i = 3, 4)$ exist globally.

Note that if $\bar{\mu}_{30} = 0$, then $\hat{T}_2$ is the identity since the system is symmetric. In the case $\bar{\mu}_{30} \neq 0$ we now turn to the calculation of $\hat{T}_2$. As shown in Fig. 26b, there are two saddles P_1 and P_2 at infinity in the chart F.R. on $r = 0$, so $\hat{T}_2$ can be calculated by the following decomposition

$$\hat{T}_2 = V_2 \circ \bar{D}_2 \circ S_2 \circ \bar{D}_1 \circ U_2^{-1}. \tag{6.122}$$

For the components of $\hat{T}_2$ in (6.122), we have

- U_2 and V_2 are the regular transition maps defined in Proposition 6.12 and Corollary 6.13. By (6.41) and (6.65), we have

$$
\begin{aligned}
U_2^{-1}(0, \rho_4) &= m_{41}\rho_4 + m_{42}\rho_4^2 + O(\rho_4^3) \\
V_2(0, \rho_2) &= \frac{1}{m_{41}}\left[\rho_2 - \frac{m_{42}}{m_{41}^2}\rho_2^2 + O(\rho_2^3)\right].
\end{aligned}
\tag{6.123}
$$

Also by Proposition 6.12 and Corollary 6.13, we have $m_{42} \neq 0$ since $\bar{\mu}_{30} \neq 0$.

- $\bar{D}_1$ and $\bar{D}_2$ are Dulac maps in the neighborhood of the infinite singular points P_1 and P_2

$$
\begin{aligned}
&\bar{D}_1 \colon \tau_1 \to \pi_1 \\
&\bar{D}_2 \colon \pi_2 \to \tau_2
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{D}_1(0, \rho_1) &= \begin{cases} \rho_1^{\sigma_1}(\beta_{10} + \phi_{11}(0, \rho_1)) & \text{if } \sigma_1 \neq 1 \\ \beta_{10}\rho_1 + \alpha_1\rho_1\omega_1[1 + \cdots] + \alpha_2\rho_1^2\omega_1[1 + \cdots] + \cdots & \text{if } \sigma_1 = 1 \end{cases} \\
\bar{D}_2(0, \tilde{y}_2) &= \begin{cases} \tilde{y}_2^{\frac{1}{\sigma_1}}(\bar{\beta}_{10} + \bar{\phi}_{11}(0, \tilde{y}_2)) & \text{if } \sigma_1 \neq 1 \\ \bar{\beta}_{10}\tilde{y}_2 + \alpha_1\tilde{y}_2\omega_2[1 + \cdots] + \alpha_2\tilde{y}_2^2\omega_2[1 + \cdots] + \cdots & \text{if } \sigma_1 = 1, \end{cases}
\end{aligned}
\tag{6.124}
$$

where $\omega_1 = \omega(\rho_1, \alpha_1)$ and $\omega_2 = \omega(\tilde{y}_2, \alpha_1)$, ϕ_{11}, $\bar{\phi}_{11}$ satisfy (I_0^∞).

- S_2 is the second component of the transition map S defined in Proposition 6.1 and satisfies (6.1).

It follows from (6.122), (6.123), (6.124), and (6.1) that we have

- Case $\sigma_1 \neq 1$.

$$
\hat{T}_2(0, \rho_4) = m_1\rho_4 + \hat{m}_{42}\rho_4^2 + \hat{m}_2\rho_4^{1+\sigma_1} + \hat{\phi}_1(\rho_4, \omega(\rho_4, \alpha_1)),
\tag{6.125}
$$

where $\hat{m}_2 = * \, S_2''(0) = * \, \bar{\mu}_{30} \neq 0$ and $\hat{m}_{42} = * \, m_{42} \neq 0$; also $\hat{\phi}_1(\rho_4, \omega(\rho_4, \alpha_1))$ is C^∞ and satisfies I_0^∞.

- Case $\sigma_1 = 1$.

$$
\begin{aligned}
\hat{T}_2(0, \rho_4) = m_1\rho_4 &+ \alpha_1\tilde{m}_2\rho_4\omega(\rho_4, \alpha_1)[1 + \cdots] \\
&+ \alpha_2\tilde{m}_1\rho_4^2\omega(\rho_4, \alpha_1)[1 + \cdots] + \cdots,
\end{aligned}
\tag{6.126}
$$

where $\tilde{m}_1 \neq 0$. For the case $\sigma_1 = 1$ $(a = \frac{1}{3})$, by the formula in [7], we have the first saddle quantity $\alpha_2 = * \, \bar{\mu}_3 \neq 0$ (see (8.4) in the proof of Theorem 6.15).

So for both cases $\sigma_1 \neq 1$ and $\sigma_1 = 1$, if $\bar{\mu}_3 \neq 0$, $\hat{T}_2(0, \rho_4)$ is nonlinear in ρ_4.

Now we show that the map $T_2(0, \tilde{y}_4)$ is nonlinear if $\bar{\mu}_{30} \neq 0$. Indeed, by (6.119), we have

$$S_4 = T_2^{-1} \circ S_3 \circ \hat{T}_2. \tag{6.127}$$

If $T_2(0, \tilde{y}_4)$ were linear in $\tilde{y}_4$, i.e., $T_2(0, \tilde{y}_4) = \hat{b}\tilde{y}_4$ ($\hat{b} \neq 0$), then by (6.127) and (6.121), we should have

$$S_4(0, \rho_4) = \frac{(\hat{T}_2(0, \rho_4))^{\sigma_3}}{\hat{b}[-C_1 + C_2\kappa_3 \ln \hat{T}_2(0, \rho_4)]},$$

which is a contradiction to (6.121) for all the cases of σ_3, κ_3, and κ_4.

Thus all the intermediate graphics of the family Ehh1 have finite cyclicity.

(2) Family Ehh3. As in Fig. 27a, we have a family of intermediate graphics Ehh3b, Ehh3c, and Ehh3d. Note that Ehh3d is similar to the graphic Ehh6c while Ehh2d is similar to the graphic Ehh10e. As in Theorem 6.18, we conclude that $Cycl(Ehh3c) \leqslant 1$ and $Cycl(Ehh3d) \leqslant 2$. To study the cyclicity of the graphic Ehh3b, we study the transition map T_2 defined on π_4 in the neighborhood of the graphic Ehh3c.

From the form of T_2, we have $\lim_{\tilde{y}_4 \to 0} T_2(0, \tilde{y}_4) = -\infty$. Hence T_2 maps $(0, \infty)$ to $(-\infty, \infty)$. Since $T_2(0, \tilde{y}_4)$ is analytic and bijective, it has to be nonlinear in $\tilde{y}_4$, thus any intermediate graphic Ehh3b has finite cyclicity.

(3) Family Ehh2. Its finite cyclicity follows from Remark 6.17.

(4) Family Ehh4. The proof is exactly the same as for the family Sxhh1 of saddle type.

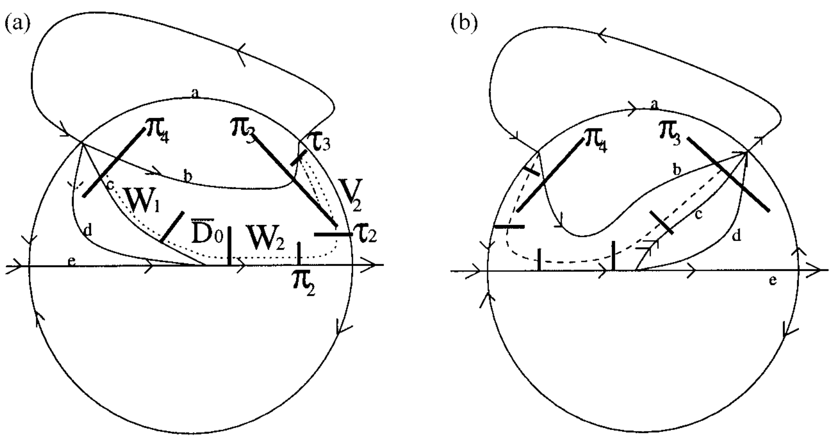


FIG. 27. Transition map T for the intermediate graphics of (a) Ehh3 and (b) Ehh2.

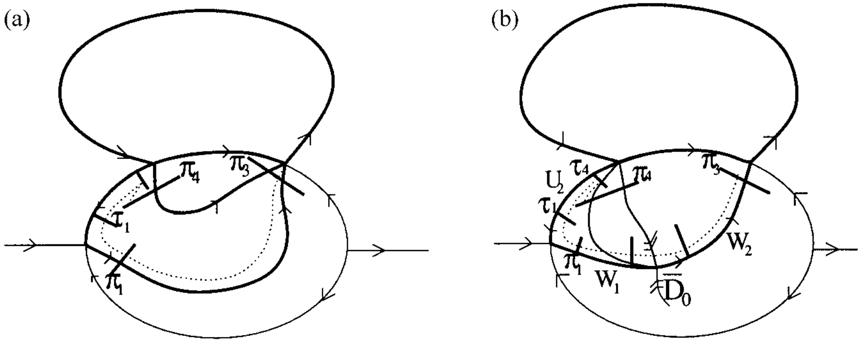


FIG. 28. Transition map T for the families (a) Ehh9b and (b) Ehh10d.

(5) Families Ehh9, Ehh10, Ehh11, and Ehh12. By Remark 6.17, we only need to consider the intermediate graphics for the family Ehh9 and Ehh10. We could have proved directly that Ehh10d has cyclicity $\leqslant 1$, but the proof given here will work simultaneously for Ehh9b and Ehh10d.

As shown in Fig. 28b, the corresponding transition map T_2 can be factored as

$$\begin{aligned} T_2 &= \hat{T}_2 \circ S_4^{-1} \\ \hat{T}_2 &= W \circ \bar{D}_1 \circ U_2, \end{aligned} \tag{6.128}$$

where U_2, S_4 are are given in (6.121) and (6.123), respectively, $\bar{D}_1\colon \tau_1 \to \pi_1$ is given in (6.124), and the map $W\colon \pi_1 \to \pi_3$ is a C^k map

$$W(v, y) = \bar{m}_0(v)\, y + o(y),$$

where $\bar{m}_0(v) > 0$ and $\bar{m}_0(v)$ small for Ehh10d.

Then a straightforward calculation gives

- Case $\sigma_1 \neq 1$.

$$\hat{T}_2(0, \rho_4) = \tilde{m}_0(\bar{\mu})\, \rho_4^{\sigma_1} + o(\rho_4^{\sigma_1}), \tag{6.129}$$

where $\tilde{m}_0 = *\, \bar{m}_0$.

- Case $\sigma_1 = 1$ $(a = \frac{1}{3})$.

$$\begin{aligned} \hat{T}_2(0, \rho_4) = \bar{m}_0(\bar{\mu})[&\gamma_1 \rho_4 + \alpha_1 \rho_4 \omega(\rho_4, \alpha_1)[1 + \cdots] \\ &+ \alpha_2 \rho_4^2 \omega(\rho_4, \alpha_1)[1 + \cdots] + \cdots], \end{aligned} \tag{6.130}$$

where we have the saddle quantity $\alpha_2 = *\, \bar{\mu}_{30} \neq 0$ since $\bar{\mu}_{30} \neq 0$ for these graphics.

Let $v_4 = 1/\tilde{y}_4$, we parameterize section π_4 by v_4 and denote $\tilde{T}_2(0, v_4) = T_2(0, \tilde{y}_4)$. We claim that the map $\tilde{T}_2(0, v_4)$ is nonlinear in v_4 in the neighborhood of $v_4 = 0$. Indeed, if $\tilde{T}_2(0, v_4) = \hat{b}v_4$ ($\hat{b} \neq 0$), by (6.128) we have $\tilde{T}_2 \circ S_4 = \hat{T}_2$:

- Case $\sigma_1 \neq 1$.

$$b\,\frac{\rho_4^{\sigma_3}}{-C_1 + C_2 \ln \rho_4} = \bar{m}_0(\bar{\mu})\,\gamma_1(v)\,\rho_4^{\sigma_1} + o(\rho_4^{\sigma_1}).$$

- Case $\sigma_1 = 1$.

$$b\,\frac{\rho_4^{\sigma_3}}{-C_1 + C_2 \ln \rho_4} = \gamma_0(v) + \bar{m}_0(\bar{\mu})[\gamma_1(v)\,\rho_4 + \alpha_1 \rho_4 \omega(\rho_4, \alpha_1)[1 + \cdots]$$
$$+ \alpha_2 \rho_4^2 \omega(\rho_4, \alpha_1)[1 + \cdots] + \cdots].$$

Since $\sigma_1 = \frac{1-2a}{a}$, $\sigma_3 = 2(1-2a)$, and $\forall a \in (0, \frac{1}{2})$, $\sigma_1 \neq \sigma_3$, the above equations are impossible, whether $\bar{m}_0(v)$ is small (Ehh10d) or moderate (Ehh9b).

Ehh10b and Ehh10c are treated exactly as Sxhh2b and Sxhh2c in the saddle case.

(6) Families Ehh5, Ehh6, Ehh7, and Ehh8. By Remark 6.17, we only need to study families Ehh5 and Ehh6. The family Ehh6 is similar to Ehh2. As shown in Fig. 29, the lower boundary graphic Ehh5c passes through two saddle points. Similar to the case of Ehh 2 $\lim_{\tilde{y}_4 \to -\infty} T_2(0, \tilde{y}_4) = 0$. Hence T_2 maps $(-\infty, \infty)$ to $(0, \infty)$, yielding that T_2 is nonlinear.

Altogether, we have proved that all the intermediate graphics of the Ehh type have finite cyclicity. ∎

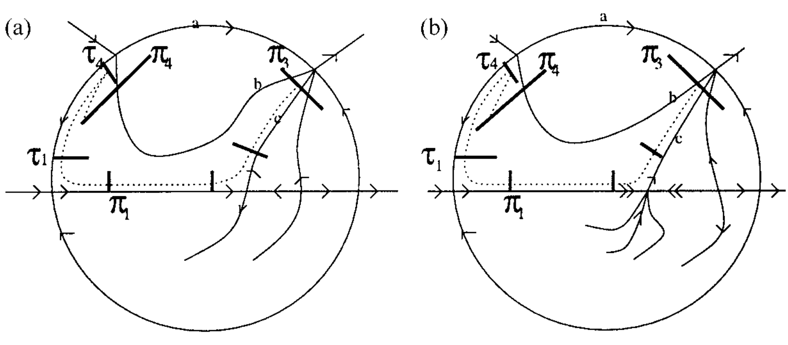


FIG. 29. Transition map T for the families (a) Ehh5b and (b) Ehh6b.

APPENDIX

A.1. *Normal Form for a Saddle Node* [8]

THEOREM A.1. *Consider a real analytic germ of a saddle node vector field on* $(\mathbb{R}^2, 0)$ *with one zero and one negative eigenvalue. Then it is* C^∞ *orbitally equivalent to its normal form*

$$v_0 = z^{\mu+1}(1+az^\mu)^{-1}\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}.$$

The equivalence may be taken analytic outside the stable manifold.

THEOREM A.2. *For any* C^∞ *unfolding of a germ from Theorem* 1 *there exists a finitely smooth orbital equivalence with the polynomial normal form*

$$v_\varepsilon = P(\varepsilon, z)\frac{\partial}{\partial z} + z^{\mu+1}(1+a(\varepsilon)\, z^\mu)^{-1}\frac{\partial}{\partial z} - w\frac{\partial}{\partial w},$$

with $P(\varepsilon, z) = \sum_{i=0}^{\mu-1} b_i(\varepsilon)\, z^i$ *where* $b_i(0) = 0$. *For the critical parameter value this equivalence is analytic outside the stable manifold of the saddle node germ.*

The proof uses the sectorial normalization theorem [19, 26]).

A.2. *Proof of Lemma* 6.14

Proof. By (3.8) with $i = 2$, after a translation $y = y_2 - \frac{1-2a}{2}$, the system on $\rho = 0$ in the neighborhood of P_4 can be written as

$$\begin{aligned} \dot{r} &= r \\ \dot{y} &= -\sigma_3\, y - \frac{8ay^2}{1+2y} + \varepsilon_2 r + O(r^2), \end{aligned} \tag{6.131}$$

where $\sigma_3 = 2(1-2a)$. In the case $\sigma_3 = \frac{1}{n}$, $n \in \mathbb{N}$, we have $a = \frac{2n-1}{4n}$.

By the linear transformation $y = z + (\varepsilon_2/(3-4a))\, r$, system (6.131) becomes

$$\begin{aligned} \dot{r} &= r \\ \dot{z} &= -\frac{1}{n}\, z - \frac{8az^2}{1+2z} + \frac{\bar{\varepsilon}_2 z(1+z)}{(1+2z)^2}\, r + O(r^2), \end{aligned} \tag{6.132}$$

where $\bar{\varepsilon}_2 = 16a\varepsilon_2/(4a-3)$. For convenience, instead of expressing a in terms of n, we still keep a in the higher order terms.

By normal form theory (see for instance [16, 20]), we will obtain the normal form of (6.131):

$$\dot r = r$$
$$\dot Z = -\frac{1}{n} Z + \sum_{i=1}^{k} \beta_{i+1} (rZ^n)^i Z, \tag{6.133}$$

where β_2, the coefficient of the term rZ^{n+1}, is the first saddle quantity.

In order to obtain the normal form (6.133) from system (6.132), we rewrite system (6.132) as

$$\dot r = r$$
$$\dot z = -\frac{1}{n} z - 8a \sum_{i=0}^{\infty} (-2)^i z^{i+2} + \bar\varepsilon_2 \left[z - \sum_{i=0}^{\infty} (-2)^i (i+3) z^{i+2} \right] r + O(r^2). \tag{6.134}$$

To prove $\beta_2 \neq 0$, we are going to apply the normal form theory to system (6.134). The proof goes in two steps. For any $n \in \mathbb{N}$, we will first kill the terms $rz, rz^2, ..., rz^n$. In the second step, we get rid of the nonresonant part $8a \sum_{i=0}^{\infty} (-2)^i z^{i+2}$.

(1) Kill the terms $rz, rz^2, ..., rz^n$ successively.

(1.1) Kill the term rz first. Let $z = z_1 + r\bar\varepsilon_2 z_1$. Then by (6.134), we obtain the equation of z_1,

$$\dot z_1 = -\frac{1}{n} z_1 - 8a \sum_{j=0}^{\infty} (-2)^j z_1^{j+2} - r \sum_{j=0}^{\infty} (-2)^j c_{1j} z_1^{j+2} + O(r^2),$$

where $c_{1j} = \bar\varepsilon_2 [8a(j+1) + (j+3)] \neq 0$ and all the coefficients c_{1j} have the same sign as $\bar\varepsilon_2$.

Note that if $n = 1$, the coefficient of the resonant term rz_1^2 is $c_{10} \neq 0$. Then the first step stops here.

(1.2) Let $n \geqslant 2$. Assume that by $n-2$ steps of near-identity transformation of the form $z_{k-1} = z_k + b_k r z_k^k$, $k = 2, ..., n-1$, we get rid of the terms $rz_1^2, rz_1^3, ..., rz_1^{n-1}$ and obtain the equations of z_n

$$\dot r = r$$
$$\dot z_n = -\frac{1}{n} z_n - 8a \sum_{j=0}^{\infty} (-2)^j z_n^{j+2} - r \sum_{j=n-2}^{\infty} (-2)^j c_{nj} z_n^{j+2} + O(r^2), \tag{6.135}$$

where for $j \geqslant n-2$, $c_{nj} \neq 0$ and they have the same sign as ε_2.

(1.3) Kill the nonresonant term rz_n^n in (6.135). Let $z_n = w + b_n r w^n$, where $b_n = -nc_{n,n-2}(-2)^{n-2}$; then

$$\dot{w} = -\frac{1}{n} w - 8a \sum_{j=0}^{\infty} (-2)^j w^{j+2} - c_{n+1,n} r w^{n+1} + rO(w^{n+2}) + O(r^2), \tag{6.136}$$

where

$$\begin{aligned} c_{n+1,n} &= -[16ab_n + (-2)^{n-1} c_{n,n-1}] \\ &= (-2)^{n-2} [16anc_{n,n-2} + 2c_{n,n-1}] \neq 0. \end{aligned}$$

Therefore, we bring system (6.134) into the form

$$\begin{aligned} \dot{r} &= r \\ \dot{w} &= -\frac{1}{n} w - \frac{8w^2}{1+2w} - c_{n+1,n} r w^{n+1} + rO(w^{n+2}) + O(r^2). \end{aligned} \tag{6.137}$$

(2) Remove the nonresonant part $-8w^2/1+2w$.
By

$$\frac{dw}{-\frac{w}{n} - \frac{8aw^2}{1+2w}} = \frac{dZ}{-\frac{Z}{n}}$$

we can solve for Z:

$$Z = w(1+4nw)^{\frac{1-2n}{2n}}. \tag{6.138}$$

So if we make the change of coordinate (6.138), we bring the system (6.137) into

$$\begin{aligned} \dot{r} &= r \\ \dot{Z} &= -\frac{1}{n} Z - c_{n+1,n} r Z^{n+1} + rO(Z^{n+2}) + O(r^2). \end{aligned} \tag{6.139}$$

Hence we get that the first saddle quantity $\beta_2 = -c_{n+1,n} \neq 0$. ∎

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